EQUIVALENCES OF HIGHER DERIVED BRACKETS

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Abstract. This note elaborates on Th. Voronov's construction [V1, V2] of L_∞-structures via higher derived brackets with a Maurer–Cartan element. It is shown that gauge equivalent Maurer–Cartan elements induce L_∞-isomorphic structures. Applications in symplectic, Poisson and Dirac geometry are discussed.

1. Introduction

In [V1] Th. Voronov showed that a Maurer–Cartan element in a graded Lie algebra which is split into an abelian subalgebra $\mathfrak{a}$ and another subalgebra $\mathfrak{p}$ induces an L_∞-structure on the abelian subalgebra $\mathfrak{a}$ in terms of higher derived brackets.

This has interesting applications, e.g., in Poisson geometry—especially in view of quantization—where Voronov's construction yields an L_∞-structure on the exterior algebra of sections of the normal bundle of every submanifold (this structure being flat if and only if the submanifold is coisotropic) [OP, CF]. A choice of embedding of the normal bundle is however involved. It is therefore important to understand how Voronov's construction depends on this choice. Ultimately this requires understanding how morphisms of graded Lie algebras influence the induced L_∞-structures.

It is not difficult to see that morphisms respecting the splittings induce morphisms of the induced L_∞-algebras (see subsection 2.3). In the application at hand, this implies that a linear automorphism of the normal bundle induces an L_∞-automorphism (see Remark 4.4). However, more general diffeomorphisms of the normal bundle do not correspond to such automorphisms.

The central result of this paper is that gauge equivalences of Maurer–Cartan elements respecting the graded Lie subalgebra $\mathfrak{p}$ induce L_∞-automorphisms. We discuss this *i*) in the formal setting (Theorem 3.1) and *ii*) in case the gauge equivalence is really a flow (Theorem 3.2). We get an explicit flow, see equations (16) and (17), of L_∞-algebra automorphisms defined on the same existence interval.

From this we deduce that the L_∞-algebra structure for a submanifold of a Poisson manifold is canonical up to L_∞-automorphisms (see Section 4). As a corollary, an isomorphism class of flat L_∞-algebras is canonically associated to every regular Dirac manifold (existence of a flat L_∞-structure was proved in [CZ]). For the special case of presymplectic manifolds see [OP].

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In [V2] it is shown how to extend the original construction to Maurer–Cartan elements in the graded Lie algebra of derivations respecting the graded Lie subalgebra $\mathfrak{p}$. In the present paper we take into account both constructions [V1] and [V2].

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2. Higher derived bracket formalism

We review the higher derived bracket formalism introduced by Th. Voronov in [V1, V2] and explain the problem of finding 'induced automorphisms' in this setting.

2.1. **Preliminaries.** Let V be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$ (or any other field of characteristic 0); i.e., V is a collection $\{V_i\}_{i\in\mathbb{Z}}$ of vector spaces V_i over $\mathbb{R}$. Homogeneous elements of V of degree $i \in \mathbb{Z}$ are the elements of V_i. We denote the degree of a homogeneous element $x \in V$ by $|x|$. When speaking of linear maps or morphisms, we assume throughout that grading is preserved.

The nth suspension functor $[n]$ from the category of graded vector spaces to itself is defined as follows: given a graded vector space V, $V[n]$ denotes the graded vector space given by the collection $V[n]_i := V_{n+i}$.

One can consider the tensor algebra $T(V)$ associated to a graded vector space V which is a graded vector space with components

$$T(V)_m := \bigoplus_{k\geq 0} \bigoplus_{j_1+\cdots+j_k=m} V_{j_1} \otimes \cdots \otimes V_{j_k}.$$

$T(V)$ naturally carries the structure of a cofree coconnected coassociative coalgebra given by the deconcatenation coproduct:

$$\Delta(x_1 \otimes \cdots \otimes x_n) := \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).$$

There are two natural representations of the symmetric group Σ_n on $V^{\otimes n}$: the even one which is defined by multiplication with the sign $(-1)^{|a||b|}$ for the transposition interchanging a and b in V and the odd one by multiplication with the sign $-(-1)^{|a||b|}$ respectively. These two actions naturally extend to $T(V)$. The fix point set of the first action on $T(V)$ is denoted by $S(V)$ and called the graded symmetric algebra of V while the fix point set of the latter action is denoted by $\Lambda(V)$ and called the graded skew–symmetric algebra of V. The graded symmetric algebra $S(V)$ inherits a coalgebra structure from $T(V)$ which is cofree coconnected coassociative and graded cocommutative.

Definition 2.1. A differential graded Lie algebra $(\mathfrak{h}, [\cdot,\cdot])$ is a graded vector space $\mathfrak{h}$ equipped with a linear map $[\cdot,\cdot]\colon \mathfrak{h}\otimes\mathfrak{h} \to \mathfrak{h}$ satisfying the following conditions:

- graded skew-symmetry: $[x, y] = -(-1)^{|x||y|}[y, x]$,
- graded Jacobi identity: $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$,

for all $x \in \mathfrak{h}_{|x|}$, $y \in \mathfrak{h}_{|y|}$ homogeneous and $z \in \mathfrak{h}$.

Let V be a graded vector space together with a family of linear maps

$$\{m^n\colon S^n(V) \to V[1]\}_{n\in\mathbb{N}}.$$

Given such a family one defines the associated family of Jacobiators

$$\{J^n : S^n(V) \to V[2]\}_{n\geq 1}$$

by

$$(1) \quad J^n(x_1 \cdots x_n) := \\ = \sum_{r+s=n} \sum_{\sigma\in(r,s)-\text{shuffles}} \operatorname{sign}(\sigma)\, m^{s+1}(m^r(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)})$$

where sign(·) is the Koszul sign, i.e., the one induced from the natural even representation of Σ_n on $S^n(V)$, and (r,s)-shuffles are permutations σ of $(1,\ldots,r+s)$ such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(n)$.

Definition 2.2. A family of maps $(m^n : S^n(V) \to V[1])_{n\in\mathbb{N}}$ defines the structure of an L_∞-algebra on the graded vector space V whenever the associated family of Jacobiators vanishes identically.

This definition is essentially the one given in [V1]. We remark that this definition deviates from the more traditional notion of L_∞-algebras in two points. The early definitions used the graded skew–symmetric algebra over V instead of the graded symmetric algebra as part of the definition. The transition between these two settings uses the so called décalage-isomorphism

$$\begin{array}{rccc} \operatorname{dec}^n: & S^n(V) & \to & \Lambda^n(V[-1])[n] \\ & x_1\cdots x_n & \mapsto & (-1)^{\sum_{i=1}^n (n-i)|x_i|} x_1\wedge\cdots\wedge x_n. \end{array}$$

More important is the fact that we also allow a 'map' $m_0 : \mathbb{R} \to V[1]$ as part of the structure given by an L_∞-algebra. This piece can be interpreted as an element of V_1. In the traditional terminology m_0 was excluded from the standard definition. Relying on a widespread terminology, we call structures with $m_0 = 0$ 'flat'. Observe that in a flat L_∞-algebra m_1 is a differential.

2.2. V-algebras and induced L_∞-structures.

Definition 2.3. We call the triple $(\mathfrak{h}, \mathfrak{a}, \Pi_\mathfrak{a})$ a V-algebra (V for Voronov) if $(\mathfrak{h}, [\cdot,\cdot])$ is a graded Lie algebra, $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{h}$ – i.e. $\mathfrak{a}$ is a graded vector subspace of $\mathfrak{h}$ and $[\mathfrak{a},\mathfrak{a}] = 0$ – and $\Pi_\mathfrak{a} : \mathfrak{h} \to \mathfrak{a}$ is a projection such that

$$(2) \qquad \Pi_\mathfrak{a}[x,y] = \Pi_\mathfrak{a}[\Pi_\mathfrak{a} x, y] + \Pi_\mathfrak{a}[x, \Pi_\mathfrak{a} y]$$

holds for every $x, y \in \mathfrak{h}$.

Instead of condition (2) one can require that $\mathfrak{h}$ splits into $\mathfrak{a}\oplus\mathfrak{p}$ as a vector space where $\mathfrak{p}$ is also a graded Lie subalgebra of $\mathfrak{h}$. In terms of the projection, $\mathfrak{p}$ is given by the kernel of $\Pi_\mathfrak{a}$.

A derivation E of degree n of a graded Lie algebra $\mathfrak{h}$ is a linear map $E : \mathfrak{h} \to \mathfrak{h}[n]$ that satisfies $E[x,y] = [E(x),y] + (-1)^{n|x|}[x,E(y)]$ for all $x \in \mathfrak{h}_{|x|}$, $y \in \mathfrak{h}$. A derivation E is called inner if there is an element $z \in \mathfrak{h}$ such that $E = [z,\cdot]$.

Definition 2.4. Let $(\mathfrak{h}, \mathfrak{a}, \Pi_\mathfrak{a})$ be a V-algebra and E a derivation of $\mathfrak{h}$ that can be written as a sum $E = \hat{E} + \check{E}$ such that

- $\Pi_\mathfrak{a}\hat{E}\Pi_\mathfrak{a} = \Pi_\mathfrak{a}\hat{E}$ (in terms of $\mathfrak{p} := \operatorname{Ker}\Pi_\mathfrak{a}$ this is equivalent to $\hat{E}(\mathfrak{p}) \subset \mathfrak{p}$),
- $\check{E}$ is an inner derivation.

Such a derivation E is called adapted. We will denote the graded Lie algebra of adapted derivations by $\operatorname{Der}(\mathfrak{h}, \mathfrak{a}, \Pi_\mathfrak{a})$.

With the help of an adapted derivation $E = \hat{E} + [P, \cdot]$ of degree k one can define higher derived brackets on $\mathfrak{a}$:

$$D_E^n\colon \quad \mathfrak{a}^{\otimes n} \to \mathfrak{a}[k], \quad x_1 \otimes \cdots \otimes x_n \mapsto \Pi_{\mathfrak{a}}[[\ldots[E(x_1), x_2], \ldots], x_n] \tag{3}$$

for every $n > 0$. For $n = 0$ we set $D_E^0 := \Pi_{\mathfrak{a}} P$. It is easy to check that all these maps are graded commutative; namely,

$$D_E^n(x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n) = = (-1)^{|x_i||x_{i+1}|} D_E^n(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n)$$

for every $1 \leq i \leq n-1$.

Observe that for $\Pi_{\mathfrak{p}} := \mathrm{id} - \Pi_{\mathfrak{a}}$ one can write E as $E = (\hat{E} + [\Pi_{\mathfrak{p}} P, \cdot]) + [\Pi_{\mathfrak{a}} P, \cdot]$, where $\hat{E} + [\Pi_{\mathfrak{p}} P, \cdot]$ is also a derivation respecting $\mathfrak{p}$, and one obtains the same higher derived brackets. So we can always assume without loss of generality that E is the sum of a derivation respecting $\mathfrak{p}$ and an inner derivation by some element of $\mathfrak{a}$.

We restrict the higher derived brackets constructed from an adapted derivation E to the symmetric algebra $S(\mathfrak{a})$ and obtain a family of maps $\{D_E^n\colon S^n(\mathfrak{a}) \to \mathfrak{a}[1]\}_{n \in \mathbb{N}}$.

In [V1] it is proven that the Jacobiators of the higher derived brackets $\{D_E^n\colon S^n(\mathfrak{a}) \to \mathfrak{a}[1]\}_{n \in \mathbb{N}}$ for $E = [P, \cdot]$ purely inner and of odd degree are given by the higher derived brackets associated to the inner derivation associated to $\frac{1}{2}[P, P]$:

$$J^n_{[P,\cdot]} = D^n_{[\frac{1}{2}[P,P],\cdot]}. \tag{4}$$

From (4) it follows that all Jacobiators vanish identically if we assume that $[P, P] = 0$ holds. Elements P of degree 1 that satisfy $[P, P] = 0$ are called Maurer–Cartan elements of $\mathfrak{h}$. Observe that $[\frac{1}{2}[P, P], \cdot] = [P, \cdot] \circ [P, \cdot]$.

In [V2] the case where E is a derivation preserving $\mathfrak{p}$ is considered and it is proved that for such E of odd degree

$$J_E^n = D_{E \circ E}^n \tag{5}$$

holds. We remark that for an odd derivation E, $E \circ E = \frac{1}{2}[E, E]$ is also a derivation (of even degree).

This immediately implies that the Jacobiators for any adapted derivation E of odd degree satisfies equation (5): We assume $E = \hat{E} + [P, \cdot]$ for $P \in \mathfrak{a}$. One computes

$$J_E^n = J_{\hat{E}}^n + D^n_{[\hat{E}(P), \cdot]}$$

and using equation (5) for $J_{\hat{E}}^n$ one obtains that equation (5) holds for all adapted derivations too. Hence we obtain the following theorem which is a slight variation of similar statements given in [V1] and [V2]:

Theorem 2.5 (Voronov)**.** *Let $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ be a V-algebra and $E = \hat{E} + [P, \cdot]$ a Maurer–Cartan element in $\mathrm{Der}(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$. Then the family of higher derived brackets associated to E,*

$$\{D_E^n\colon S^n(\mathfrak{a}) \to \mathfrak{a}[1]\}_{n \in \mathbb{N}},$$

equips $\mathfrak{a}$ with the structure of an L_∞-algebra in the sense of Definition 2.2.

We remark that the higher derived brackets depend not only on E as a derivation but also on the choice of an element for the inner derivation. Assume $E = \hat{E} + [P, \cdot]$ and $E = \hat{E}' + [P', \cdot]$. The two families of derived brackets for the two decompositions only differ by their 0-ary operations. In the following we will always assume that the adapted derivation E comes along with a fixed element P such that $\hat{E} + [P, \cdot]$ is the decomposition of E.

Example 2.6. Let A be a graded commutative algebra and $\mathrm{Der}(A)$ its graded Lie algebra of derivations. Consider $\mathfrak{h}$ equal to $S_A(\mathrm{Der}(A)[-1])[1]$ or to its formal completion $\hat{S}_A(\mathrm{Der}(A)[-1])[1]$. As A itself is a $\mathrm{Der}(A)$-module, the graded space $\tilde{\mathfrak{h}} := A[1] \oplus \mathrm{Der}(A)$ inherits a graded Lie algebra structure. Since $\tilde{\mathfrak{h}}[-1]$ generates $\mathfrak{h}[-1]$ as a graded commutative algebra over A, one can extend the Lie bracket uniquely by requiring it to be a graded derivation; namely, one makes $\mathfrak{h}[-1]$ into a Gerstenhaber algebra. Set $\mathfrak{a} := A[1]$ and observe that $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ is a V-algebra. Thus, a Maurer–Cartan element induces an L_∞-structure on $A[1]$ with the additional property that the derived brackets are multiderivations with respect to the multiplication in A. Such a structure was called P_∞ (P for Poisson) in [CF].

A very special example is when $A = C^\infty(M)$ for a smooth manifold M. In this case, $\mathfrak{h} = \mathcal{V}(M)[1] := \Gamma(M, \Lambda TM)[1]$ and the Lie bracket on $\mathfrak{h}$ is the Schouten–Nijenhuis bracket of multivector fields. A Maurer–Cartan element is in this case the same as a Poisson bivector field, and the induced P_∞-structure is just an ordinary Poisson structure. More general P_∞-structures are obtained for M a graded manifold.

2.3. **Morphisms.** Suppose now one is given an automorphism Φ of the graded Lie algebra $\mathfrak{h}$, i.e., a bijective map $\Phi\colon \mathfrak{h} \to \mathfrak{h}$ that is is degree-preserving and satisfies $\Phi([x, y]) = [\Phi(x), \Phi(y)]$ for all $x, y \in \mathfrak{h}$. If E is a derivation of odd degree, so is $\tilde{E} := \Phi \circ E \circ \Phi^{-1}$. Suppose $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ is a V-algebra. One obtains two families of maps $\{D^n_E\}_{n\in\mathbb{N}}$ and $\{D^n_{\tilde{E}}\}_{n\in\mathbb{N}}$ that define L_∞-algebra structures on $\mathfrak{a}$. The question arises under which circumstances these two L_∞-structures are related.

The answer is straightforward as long as the automorphism Φ respects the splitting. More generally, let $\Phi\colon (\mathfrak{h}_1, \mathfrak{a}_1, \Pi_{\mathfrak{a}_1}) \to (\mathfrak{h}_2, \mathfrak{a}_2, \Pi_{\mathfrak{a}_2})$ be a morphism of V-algebras, that is, a morphism of graded Lie algebras $\mathfrak{h}_1 \to \mathfrak{h}_2$ satisfying $\Pi_{\mathfrak{a}_2} \circ \Phi = \Phi \circ \Pi_{\mathfrak{a}_1}$. Equivalently, $\Phi(\mathfrak{a}_1) \subset \mathfrak{a}_2$ and $\Phi(\mathfrak{p}_1) \subset \mathfrak{p}_2$, with $\mathfrak{p}_i = \operatorname{Ker} \Pi_{\mathfrak{a}_i}$. We say that $E_i = \hat{E}_i + [P_i, \cdot] \in \mathrm{Der}(\mathfrak{h}_i, \mathfrak{a}_i, \Pi_{\mathfrak{a}_i})$, $i = 1, 2$, are Φ-related if $E_2 \circ \Phi = \Phi \circ E_1$ and $P_2 - \Phi(P_1) \in \operatorname{Ker} \Pi_{\mathfrak{a}_2}$. Then

$$D^n_{E_2}(\Phi(x_1) \otimes \cdots \otimes \Phi(x_n)) = \Phi \circ D^n_{E_1}(x_1 \otimes \cdots \otimes x_n). \tag{6}$$

Thus, if E_1 and E_2 are Maurer–Cartan elements, Φ defines a linear morphism of L_∞-algebras $\mathfrak{a}_1 \to \mathfrak{a}_2$.

For $E_1 = [P_1, \cdot]$ an inner derivation, one may define $E_2 = [P_2, \cdot]$ with $P_2 = \Phi(P_1)$. Observe that E_1 and E_2 are Φ-related and that E_2 is Maurer–Cartan if E_1 is so.

However the requirement on Φ to respect the splittings is far too restrictive in general. In the next Section we will show that the conditions under which a family of automorphisms of $\mathfrak{h}$ induce isomorphisms of the corresponding L_∞-algebras on $\mathfrak{a}$ are much weaker.

Example 2.7. The V-algebras described in Example 2.6 for A concentrated in degree 0 (e.g., A the algebra of functions of a smooth manifold) have the additional property that the splittings respect the degrees (namely, the abelian subalgebra

$\mathfrak{a} = A[1]$ and the kernel of the projection are concentrated in degree -1 and in nonnegative degrees, respectively). So every graded Lie algebra morphism between such V-algebras is automatically a V-morphism.

Example 2.8. Let A_1 and A_2 be graded commutative algebras and $\phi\colon A_1 \to A_2$ an isomorphism. One can extend ϕ to an isomorphism of graded Lie algebras $\Phi\colon \tilde{\mathfrak{h}}_1 := A_1[1] \oplus \mathrm{Der}(A_1) \to \tilde{\mathfrak{h}}_2 := A_2[1] \oplus \mathrm{Der}(A_2)$ by $\Phi(a) = \phi(a)$ for $a \in A_1$ and $\Phi(X) = \phi \circ X \circ \phi^{-1}$ for $X \in \mathrm{Der}(A_1)$. This can be uniquely extended to an isomorphism $\tilde{\Phi}\colon \mathfrak{h}_1[-1] \to \mathfrak{h}_2[-1]$ of graded commutative algebras, which is also an isomorphism of V-algebras $(\mathfrak{h}_1, A_1) \to (\mathfrak{h}_2, A_2)$ (with the canonical projections $\mathfrak{h}_1 \to A_1$ and $\mathfrak{h}_2 \to A_2$ respectively). If we have $\tilde{\Phi}$-related Maurer–Cartan elements, then ϕ is an isomorphism of P_∞-algebras. For example, ϕ may be the pushforward of a diffeomorphism between smooth manifolds or more generally between graded manifolds.

3. Induced automorphisms

Let $(\mathfrak{h}, \mathfrak{a}, \Pi_\mathfrak{a})$ be a V-algebra and $E = \hat{E} + [P, \cdot]$ a Maurer–Cartan element in $\mathrm{Der}(\mathfrak{h}, \mathfrak{a}, \Pi_\mathfrak{a})$. We denote $\mathrm{Ker}\,\Pi_\mathfrak{a}$ by $\mathfrak{p}$ throughout.

The space of Maurer–Cartan elements is invariant under the adjoint action of the Lie algebra $\mathrm{Der}_0(\mathfrak{h}, \mathfrak{a}, \Pi_\mathfrak{a})$. Such an action is called infinitesimal gauge transformation. The aim of this Section is to show that integrated gauge transformations preserving $\mathfrak{p}$ induce L_∞-automorphisms. We do this in the formal and in the analytical setting.

In the formal setting we introduce a formal parameter t and consider the V-algebra $(\mathfrak{h}[[t]], \mathfrak{a}[[t]], \Pi_{\mathfrak{a}[[t]]})$ where we use the obvious $\mathbb{R}[[t]]$-linear extensions of all structure maps. Suppose m_t is a derivation of $\mathfrak{h}[[t]]$ of degree 0. This derivation can uniquely be integrated to an automorphism ϕ_t of $\mathfrak{h}[[t]]$.

In the analytical setting the situation is instead as follows: Suppose m_t is a family of degree 0 derivations of $\mathfrak{h}$ for $t \in I$ where $I \subset \mathbb{R}$ is a compact interval (without loss of generality we will assume that $I = [0, 1]$). We assume that there is a flow ϕ_t that integrates m_t for all $t \in I$.

In both the formal and the analytical setting the flow equation reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(z) &= m_t \circ \phi_t(z), \\ \phi_0 &= \mathrm{id}, \end{aligned} \tag{7}$$

with the difference that in the formal setting it has to hold for all $z \in \mathfrak{h}[[t]]$ while in the analytical setting it has to hold for all $z \in \mathfrak{h}$ and all $t \in I$.

We will further assume that

$$\Pi_{\mathfrak{a}[[t]]} m_t \Pi_{\mathfrak{a}[[t]]} = \Pi_{\mathfrak{a}[[t]]} m_t \tag{8}$$

in the formal setting, and

$$\Pi_\mathfrak{a} m_t \Pi_\mathfrak{a} = \Pi_\mathfrak{a} m_t, \qquad \forall t \in I, \tag{9}$$

in the analytical setting.

In the formal setting, it follows that the automorphism ϕ_t satisfies $\Pi_{\mathfrak{a}[[t]]} \circ \phi_t \circ \Pi_{\mathfrak{a}[[t]]} = \Pi_{\mathfrak{a}[[t]]} \circ \phi_t$, while in the analytical setting the equation

$$\Pi_\mathfrak{a} \circ \phi_t \circ \Pi_\mathfrak{a} = \Pi_\mathfrak{a} \circ \phi_t, \qquad \forall t \in I,$$

is satisfied under the additional assumption that the only solution to the Cauchy problem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_t &= \Pi_{\mathfrak{a}} m_t \lambda_t,\\ \lambda_0 &= 0,\end{aligned} \tag{10}$$

is $\lambda_t = 0$ for all $t \in I$. Equivalently, the condition on ϕ_t may be written as

$$\phi_t(\mathfrak{p}[[\mathfrak{t}]]) = \mathfrak{p}[[\mathfrak{t}]] \tag{11}$$

and

$$\phi_t(\mathfrak{p}) = \mathfrak{p}, \qquad \forall t \in I, \tag{12}$$

respectively.

Finally, in the formal setting, we define $E_t := \phi_t \circ \hat{E} \circ \phi_t^{-1} + [\phi_t(P), \cdot]$ and consider the associated higher derived brackets $\{D^n_{E_t}\}_{n\in\mathbb{N}}$. Since ϕ_t satisfies (11), E_t is an adapted derivation with $E_t \circ E_t = 0$. Hence we have two L_∞-algebra structures on $\mathfrak{a}[[t]]$: one is the tautological extension of $\{D^n_E : S^n(\mathfrak{a}) \to \mathfrak{a}[1]\}_{n\in\mathbb{N}}$, which we denote by $\mathfrak{a}[[t]]_0$, while the other one is the one associated to $\{D^n_{E_t}\}_{n\in\mathbb{N}}$, which we denote by $\mathfrak{a}[[t]]_t$. In the analytical setting we consider the one-parameter family of Maurer–Cartan elements $E_t := \phi_t \circ \hat{E} \circ \phi_t^{-1} + [\phi_t(P), \cdot]$ and the associated family of higher derived brackets $\{D^n_{E_t}\}_{n\in\mathbb{N}}$. We denote the space $\mathfrak{a}$ equipped with the L_∞-algebra structure defined by the family of maps $\{D^n_{E_t}\}_{n\in\mathbb{N}}$ by $\mathfrak{a}_t$.

The aim of this Section is to show that, in the formal setting or under the condition of uniqueness of solutions to (10) in the analytical setting, these L_∞-algebra structures are naturally L_∞-isomorphic. Namely:

Theorem 3.1. *Let $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ be a V-algebra and E a Maurer–Cartan element in* $\mathrm{Der}(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$. *Let ϕ_t be the automorphism of $\mathfrak{h}[[t]]$ generated by a derivation m_t of $\mathfrak{h}[[t]]$ of degree* 0 *which satisfies* (8)*. Then the L_∞-algebras $\mathfrak{a}[[t]]_0$ and $\mathfrak{a}[[t]]_t$ are naturally L_∞-isomorphic.*

Theorem 3.2. *Let $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ be a V-algebra and E a Maurer–Cartan element in* $\mathrm{Der}(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$. *Assume that ϕ_t is a family of automorphisms of $\mathfrak{h}$ generated by a one-parameter family of degree* 0 *derivations m_t satisfying condition* (9) *and suppose that equation* (10) *has a unique solution. Then the L_∞-algebras $\{\mathfrak{a}_t\}_{t\in I}$ are all naturally L_∞-isomorphic.*

The rest of this Section is devoted to the proof of the two Theorems. We also get an explicit formula, see (16) and (17), for the L_∞-automorphism. Each component of this automorphism is a polynomial in ϕ_t. So the formula makes sense for every endomorphism of $\mathfrak{h}$. It is tempting to conjecture that for every graded Lie algebra automorphism respecting $\mathfrak{p}$, it defines an L_∞-automorphism (or even an L_∞-morphism for every graded Lie algebra endomorphism and a pair of related Maurer–Cartan elements).

3.1. **Infinitesimal considerations.** We briefly review a description of L_∞-algebras, equivalent to the one given in Definition 2.2, which goes back to Stasheff [St]. We remarked before that the graded commutative algebra $S(V)$ associated to a graded vector space V is a cofree coconnected graded cocommutative coassociative coalgebra with respect to the coproduct Δ inherited from $T(V)$. A linear map $Q\colon S(V) \to S(V)$ that satisfies $\Delta \circ Q = (Q \otimes \mathrm{id} + \mathrm{id} \otimes Q) \circ \Delta$ is called a coderivation of $S(V)$. By cofreeness of the coproduct Δ it follows that every linear map from $S(V)$ to V can be extended to a coderivation of $S(V)$ and that every coderivation Q is uniquely determined by $\mathrm{pr} \circ Q$ where $\mathrm{pr}\colon S(V) \to V$ is the natural projection. So there is a one-to-one correspondence between families of linear maps $\{m^n\colon S^n(V) \to V[1]\}_{n\in\mathbb{N}}$ and coderivations of $S(V)$ of degree 1. Moreover, the graded commutator equips $\mathrm{Hom}(S(V), S(V))$ with the structure of a graded Lie algebra and this Lie bracket restricts to the subspace of coderivations of $S(V)$. Odd coderivations Q that satisfy $[Q, Q] = 0$ are in one-to-one correspondence with families of maps whose associated Jacobiators (see formula (1)) vanish identically. Consequently, Maurer–Cartan elements of the space of coderivations of $S(V)$ correspond exactly to L_∞-algebra structures on V. Since $Q \circ Q = \frac{1}{2}[Q, Q] = 0$, Maurer–Cartan elements of the space of coderivations are exactly the codifferentials of $S(V)$.

We remark that the approach to L_∞-algebras outlined above makes the notion of L_∞-morphisms especially transparent: these are just coalgebra morphisms that are chain maps between the graded symmetric algebras equipped with the codifferentials that define the L_∞-algebra structures.

In particular, we can interpret the L_∞-algebra structure on $\mathfrak{a}[[t]]$ as a codifferential $Q(t)$ of $S(\mathfrak{a}[[t]])$. In the analytical setting, we interpret the one-parameter family of L_∞-algebras $\{a_t\}_{t\in I}$ as a one-parameter family of codifferentials $Q(t)$ of $S(\mathfrak{a})$.

Next we consider the family of maps $\{D^n_{m_t}\}_{n\in\mathbb{N}}$ defined using the formulae for the higher derived brackets given in (3). As explained before, we can interpret this family of maps as a coderivation of the coalgebra $S(\mathfrak{a}[[t]])$ in the formal setting and as a one-parameter family of coderivations of the coalgebra $S(\mathfrak{a})$ in the analytical setting. We denote this coderivation (or family of coderivations respectively) by $M(t)$.

Lemma 3.3. *$M(t)$ satisfies the ordinary differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(t) = M(t) \circ Q(t) - Q(t) \circ M(t). \tag{13}$$

Proof. The formula for $Q(t)$ as a coderivation is

$$Q(t)(x_1 \otimes \cdots \otimes x_n) = \\ = \sum_{r+s=n} \sum_{\sigma\in(r,s)-\text{shuffles}} \mathrm{sign}(\sigma) D^r_{E_t}(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}) \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

As a consequence of (7) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(t)(x_1 \otimes \cdots \otimes x_n) = \\ = \sum_{r+s=n} \sum_{\sigma\in(r,s)-\text{shuffles}} \mathrm{sign}(\sigma) D^r_{[m_t,E_t]}(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}) \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

It is convenient to introduce an auxiliary parameter τ of degree 1 and to consider the $\mathbb{R}[\tau]/\tau^2$-modules $\mathfrak{h}[[t]][\tau]/\tau^2$ and $\mathfrak{h}[\tau]/\tau^2$. We extend the graded Lie bracket linearly by the rule $[\tau x, y] = \tau[x, y]$. From Voronov's result (4) it follows that

$$J^n_{E_t+\tau m_t} = D^n_{(E_t+\tau m_t)\circ(E_t+\tau m_t)} = D^n_{\tau[m_t,E_t]} = \tau D^n_{[m_t,E_t]}.$$

Therefore the family of maps $\{(\frac{\partial}{\partial\tau}|_{\tau=0}J^n_{E_t+\tau m_t})\}_{n\in\mathbb{N}}$ corresponds to the coderivation $\dot{Q}(t)$. We claim that $M(t)\circ Q(t)-Q(t)\circ M(t)$ also corresponds to

$$\{(\frac{\partial}{\partial\tau}|_{\tau=0}J^n_{E_t+\tau m_t})\}_{n\in\mathbb{N}},$$

which proves the Lemma. To verify the claim it suffices to use the definition (1) of the Jacobiators,

$$\begin{aligned}&(J^n_{E_t+\tau m_t})(x_1\otimes\cdots\otimes x_n)=\\ &=\sum_{r+s=n}\sum_{\sigma\in(r,s)-\text{shuffles}}\operatorname{sign}(\sigma)D^{s+1}_{E_t+\tau m_t}(D^r_{E_t+\tau m_t}(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)}),\end{aligned}$$

and to compute

$$\begin{aligned}&(\frac{\partial}{\partial\tau}\bigg|_{\tau=0}J^n_{E_t+\tau m_t})(x_1\otimes\cdots\otimes x_n)=\\ &=\sum_{r+s=n}\sum_{\sigma\in(r,s)-\text{shuffles}}\operatorname{sign}(\sigma)D^{s+1}_{m_t}(D^r_{E_t}(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)})\\ &-\sum_{r+s=n}\sum_{\sigma\in(r,s)-\text{shuffles}}\operatorname{sign}(\sigma)D^{s+1}_{E_t}(D^r_{m_t}(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)})\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(n)}).\end{aligned}$$

It is straightforward to see that the first term corresponds to $M(t)\circ Q(t)$ whereas the second term corresponds to $-Q(t)\circ M(t)$. □

3.2. **Integration to automorphisms.** We now consider the flow of $M(t)$, namely, the solution to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U(t)&=M(t)\circ U(t),\\ U(0)&=\mathrm{id}.\end{aligned}\tag{14}$$

This is equivalent to the following family of equations on the family of maps $\{U^n(t)\}_{n\in\mathbb{N}}$ corresponding to $U(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}U^n(t)(x_1\otimes\cdots\otimes x_n)=\sum_{\sigma\in\Sigma_n}\operatorname{sign}(\sigma)\sum_{k\geq1}\sum_{l_1+\cdots+l_k=n}\frac{1}{k!l_1!\cdots l_k!}\tag{15}$$

$$D^k_{m_t}\Big(U^{l_1}(t)(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(l_1)})\otimes\cdots\otimes U^{l_k}(t)(x_{\sigma(l_1+\cdots+l_{(k-1)}+1)}\otimes\cdots\otimes x_{\sigma(n)})\Big)$$

together with the initial conditions $U^1(0)=\mathrm{id}$ and $U^n(0)=0$ for $n\neq1$.

Proposition 3.4. *The Cauchy problem* (14) *has a unique solution. The solution has the property* $U^0\equiv0$.

Proof. That there exists a unique solution for $U(t)$ in the formal setting is seen as follows: first we assume that we already found (unique) expressions for $U^m(t)$, $m<n$. We want to construct $U^n(t)$. We expand it with respect to the formal parameter t: $U^n(t):=\sum_{r\geq0}U^n_rt^r$. Condition $U(0)=\mathrm{id}$ determines the term U^n_0 (it is 0 for $n\neq0$ and $\mathrm{id}_{\mathfrak{a}}$ for $n=1$). Next suppose we know U^n_v for all $v<w$. If we expand equation (15) with respect to the formal parameter t and consider the term of order $t^{(w-1)}$ we obtain an explicit expression for U^n_w in terms of U^m for $m<n$ and U^n_v for $v<w$. So U^n_w is uniquely determined by these factors. Hence we can find uniquely determined U^n_w for all $w\geq0$ successively and consequently construct

U^n. We remark that assumption (8) implies $U^0(t) = 0$. This completes the proof in the formal setting.

In the analytical setting we first assume that we have found a family of automorphisms $U(t)\colon S(\mathfrak{a}) \to S(\mathfrak{a})$ integrating the one-parameter family of coderivations $M(t)$, i.e., solving equation (14) for $t \in I$. As before, equation (14) is equivalent to the family of equations (15) for all $n \geq 0$, $x_1, \dots, x_n \in \mathfrak{a}$ and $t \in I$. Moreover $U(t) = \mathrm{id}$ is equivalent to $U^1(0) = \mathrm{id}_{\mathfrak{a}}$ and $U^n(0) = 0$ for $n > 1$. By assumption (9) we can consistently set $U^0(t) = 0$.

Using uniqueness of solutions of (10) one deduces that a solution to (14) with $U^0(t) = 0$ is unique, too: Suppose we have two solutions satisfying (14) given by the family of maps $\{U^n(t)\}_{n\geq 1}$ and $\{\tilde{U}^n(t)\}_{n\geq 1}$. We consider $\delta U^n(t) := U^n(t) - \tilde{U}^n(t)$. It follows that $\delta U^1(t)$ satisfies (10), hence $U^1(t) = \tilde{U}^1(t)$. Now assume we know that $U^k(t) = \tilde{U}^k(t)$ for all $k < n$. Equation (15) implies that $\delta U^n(t)$ satisfies (10) too, so $U^n(t) = \tilde{U}^n(t)$. By induction if follows that the two solutions coincide. It remains to prove that such a family of automorphisms $U(t)$ exists for all $t \in I$ under the condition (9). We inductively define a family of maps

$$\{U^n(t)\colon S^n(\mathfrak{a}) \to \mathfrak{a}\}_{n\geq 1}$$

that corresponds to an automorphism of $S(\mathfrak{a})$ that satisfies (14). (From now on we will suppress the t dependence of the maps $U^n(t)$ and simply write U^n instead.) For $n = 1$ we define

$$U^1(x) := \Pi_{\mathfrak{a}}\phi_t(x). \tag{16}$$

For $n \geq 1$ we set

$$\begin{aligned} U^n(x_1 \otimes \cdots \otimes x_n) := \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma) \sum_{k\geq 1} \sum_{\mu_1+\cdots+\mu_k=n-1} \frac{1}{nk!\mu_1!\cdots\mu_k!} \\ \Pi_{\mathfrak{a}}[[\cdots[\phi_t(x_{\sigma(1)}), U^{\mu_1}(x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(\mu_1+1)})], \cdots], \\ U^{\mu_k}(x_{\sigma(\mu_1+\cdots+\mu_{(k-1)}+2)} \otimes \cdots \otimes x_{\sigma(n)})] \end{aligned} \tag{17}$$

By this formula U^n is defined recursively for all $n \geq 1$.

Lemma 3.5. *The family of maps $\{U^n\colon S^n(\mathfrak{a}) \to \mathfrak{a}\}_{n\geq 1}$ defined by* (16) *and* (17) *satisfies equation* (15) *for all $n \geq 1$, $x_1, \dots, x_n \in \mathfrak{a}$ and $t \in I$.*

The proof is in the Appendix. That $U(0) = \mathrm{id}_{S(\mathfrak{a})}$ can be seen easily: First observe that $U^1 = \mathrm{id}_{\mathfrak{a}}$ for $t = 0$. Moreover, all U^n for $n > 1$ vanish at $t = 0$ since each term contains the Lie bracket between two elements of $\mathfrak{a}$ which is an abelian Lie subalgebra. □

Using equation (13) one easily deduces that $Z(t) := Q(t) \circ U(t) - U(t) \circ Q(0)$ satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Z(t) &= M(t) \circ Z(t), \\ Z(0) &= 0. \end{aligned} \tag{18}$$

In the formal setting one immediately proves that $Z(t) = 0$ is the unique solution to (18) (under assumption (8)). In the analytical setting one first computes $Z^0 = \Pi_{\mathfrak{a}}\phi_t P - \Pi_{\mathfrak{a}}\phi_t\Pi_{\mathfrak{a}}P$ (recall that our Maurer–Cartan element is $E = \hat{E} + [P, \cdot]$) which vanishes because of (12). Now one can apply the same arguments as in the proof of uniqueness for $U(t)$ and one obtains that $Z(t) = 0$.

By definition of $Z(t)$, $Z(t) = 0$ is equivalent to

$$Q(t) \circ U(t) = U(t) \circ Q(0) \tag{19}$$

which means that $U(t)$ defines an L_∞-isomorphism. This completes the proof of Theorems 3.1 and 3.2.

4. Applications

We describe an application of Theorem 3.2 in the framework of Poisson geometry. Out of it applications in symplectic and Dirac geometry follow.

Let M be a smooth finite-dimensional manifold. As noticed by Oh and Park in [OP], if M is a Poisson manifold, the space of sections of the exterior algebra of the normal bundle of a submanifold of a certain class (namely, a coisotropic submanifold) carries the structure of a flat L_∞-algebra. The same structure was found in [CF] as the semi-classical limit of a certain topological quantum field theory called the Poisson Sigma model; the L_∞-algebra structure was derived not only for coisotropic submanifolds but for every submanifold of M (coisotropic submanifolds are special in so far as they are exactly those whose associated L_∞-algebras are flat). We now briefly recall the construction in [CF], which makes use of graded manifolds and Voronov's higher derived brackets.

4.1. **Submanifolds and V-algebras.** Given a smooth manifold M, the space of multivector fields $\mathcal{V}(M)[1] := \Gamma(M, \Lambda TM)[1]$ carries the structure of a graded Lie algebra where the graded Lie bracket is given by the Schouten–Nijenhuis bracket which we denote by $[\cdot, \cdot]$; see Example 2.6.

Let S be a submanifold. Its normal bundle NS is by definition the quotient of the restriction T_SM of TM to S by TS. Set $A := \Gamma(S, \Lambda NS)$ and $\mathfrak{a} := A[1]$ as in Example 2.6. By restricting a multivector field to S and then projecting it to its normal components, we get a projection $\Pi_{M;\mathfrak{a}} \colon \mathcal{V}(M)[1] \to \mathfrak{a}$.

Denote the vanishing ideal of S by $I(S) := \{f \in \mathcal{C}^\infty(M) |\ f|_C = 0\}$. The inclusions $i_{nm} : I^m(S) \hookrightarrow I^n(S)$ for $m \geq n$ equip the collection $\mathcal{V}(M)/I^n(S)\mathcal{V}(M)$ with the structure of a projective system and we define the Gerstenhaber algebra of multivector fields on a formal neighbourhood of S in M by

$$\mathcal{V}(M, S) := \varprojlim \mathcal{V}(M)/I^n(S)\mathcal{V}(M).$$

The space $\mathfrak{h}_{M,S} := \mathcal{V}(M, S)[1]$ inherits both the structure of a graded Lie algebra and a projection $\Pi_{M,S;\mathfrak{a}}$ onto $\mathfrak{a}$. As we will shortly see, it also has the structure of a V-algebra though not in a canonical way.

Thus, a Maurer–Cartan element of $\mathfrak{h}_{M,S}$ induces an L_∞-structure on $\mathfrak{a}$. Observe that the class $[\pi]$ in $\mathfrak{h}_{M,S}$ of a bivector field π on M is a Maurer–Cartan element if and only if the restrictions to S of $[\pi, \pi]$ and all its derivatives vanish. In this case, we say that π is Poisson in a formal neighbourhood of S. Moreover, $\Pi_{M,S;\mathfrak{a}}[\pi]$ vanishes if and only if $\pi_x(\alpha, \beta) = 0\ \forall x \in S, \forall \alpha, \beta \in N_x^*S$. In this case S is called a coisotropic submanifold.

We now explain how to induce a V-structure on $\mathfrak{h}_{M,S}$ using a choice of embedding $\sigma \colon NS \hookrightarrow M$ with $\sigma|_S = \mathrm{id}_S$. Regard A as a graded commutative algebra and set $\mathfrak{h} := \hat{S}_A(\mathrm{Der}(A)[-1])[1]$ with the V-algebra structure of Example 2.6. We now claim that $\mathfrak{h}$ is isomorphic, though noncanonically, to $\mathfrak{h}_{M,S}$. To do this, we observe that A is the algebra of functions on the graded manifold $N^*[1]S$. So $\mathfrak{h}[-1]$ is the formally completed Gerstenhaber algebra of multivector fields on $N^*[1]S$. By the Legendre mapping theorem [R], this is canonically isomorphic to the formally completed Gerstenhaber algebra of multivector fields on the graded manifold $N[0]S$ which is the same as the Gerstenhaber algebra $\mathcal{V}(NS,S)$ of multivector fields on a formal neighbourhood of S in NS. Finally, the choice of embedding σ yields an isomorphism between $\mathcal{V}(NS,S)$ and $\mathcal{V}(M,S)$, and so an isomorphism $\hat{\sigma}\colon \mathfrak{h} \to \mathfrak{h}_{M,S}$.

Two different choices of embeddings yield an automorphism of the graded Lie algebra $\mathfrak{h}$. We will see in the next subsection that the assumption of Theorem 3.2 are respected, so the effect of a change of embedding may be understood easily now.

Remark 4.1. A simpler construction, avoiding graded manifolds, is that of [C]. It starts with the observation that an embedding σ yields a section $\tilde{\sigma}\colon \mathfrak{a} \to \mathfrak{h}_{M,S}$ with the property that $\tilde{\sigma}(\mathfrak{a})$ is an abelian subalgebra. Let $\mathfrak{p} := \operatorname{Ker}\Pi_{M,S;\mathfrak{a}}$ and $\iota_{\mathfrak{p}}$ its inclusion map into $\mathfrak{h}_{M,S}$. We then have the isomorphism $\tilde{\sigma}\oplus\iota_{\mathfrak{p}}\colon \mathfrak{a}\oplus\mathfrak{p} \to \mathfrak{h}_{M,S}$. This induces a V-algebra structure on $\mathfrak{a}\oplus\mathfrak{p}$. Notice however that the Lie bracket on $\mathfrak{a}\oplus\mathfrak{p}$ depends on the choice of embedding. Hence this simpler construction, while perfectly fine for inducing L_∞-structures on $\mathfrak{a}$, is not suitable for the application of Theorem 3.2 and so for discussing the effect of a change of embedding.

Remark 4.2. As already remarked, the induced L_∞-structure is flat if and only if S is a coisotropic submanifold. In this case, one can show [OP, CF, C] that the unary operation does not depend on the choice of embedding and is the Lie algebroid differential associated to the conormal bundle of S as a Lie subalgebroid of the cotangent bundle of M.

4.2. **Uniqueness of the induced L_∞-structure.** It is well-known from differential topology (see [H] for instance) that any two tubular neighbourhoods of S in M are isotopic. For our purposes this can be expressed as follows: For any two embeddings σ_0 and σ_1 of NS into M, there is a family V_t, $t\in I=[0,1]$, of open neighbourhoods of S in M, a family of diffeomorphisms $\psi_t\colon V_0\to V_t$ and a family of embeddings $\sigma_t\colon NS\to M$, such that $\psi_0=\mathrm{id}_{V_0}$, $\psi_t|_S=\mathrm{id}_S$, and $\psi_t\circ\sigma_0=\sigma_t$ in an open neighbourhood of S. The pushforward ψ_{t*} of multivector fields defines an automorphism of $\mathfrak{h}_{M,S}$ which we denote by $\hat{\psi}_t$. Denoting by $\hat{\sigma}_t$ the isomorphism $\mathfrak{h}\to\mathfrak{h}_{M,S}$ induced by σ_t, we then get $\hat{\psi}_t\circ\hat{\sigma}_0=\hat{\sigma}_t$. Let $\phi_t:=\hat{\sigma}_t^{-1}\circ\hat{\sigma}_0=\hat{\sigma}_0^{-1}\circ\hat{\psi}_t^{-1}\circ\hat{\sigma}_0$. Let $Z_t := -\frac{\mathrm{d}}{\mathrm{d}t}\psi_t$ as a vector field in an open neighbourhood of S and $\hat{Z}_t$ its class in $\mathfrak{h}_{M,S}$. Then equation (7) is satisfied with $m_t=[\hat{\sigma}_0^{-1}(\hat{Z}_t),\cdot]$. Observe that $Z_t|_S$ is tangent to S. Using the explicit formula for the Legendre mapping, it is easy to verify that this implies condition (9). Finally, uniqueness of solutions of equation (10) follows from the uniqueness of flows generated by vector fields on graded manifolds (in view of the canonical isomorphism between $\mathfrak{h}$ and $\mathcal{V}(NS,S)$ this is in this case just the uniqueness of flows generated by vector fields on NS). So all assumptions of Theorem 3.2 hold and one concludes:

Theorem 4.3. *The L_∞-algebra structures constructed on $\mathfrak{a}$ with the help of two different embeddings of NS into M as tubular neighbourhoods of S are L_∞-isomorphic.*

Remark 4.4. In case one changes the tubular neighbourhood by acting on NS via a vector bundle automorphism, there is a simpler proof by applying the construction in Example 2.8: in fact the vector bundle automorphism induces an automorphism of $A:=\Gamma(S,\Lambda NS)$, and the natural extension to an automorphism of $\mathfrak{h}:=\hat{S}_A(\mathrm{Der}(A)[-1])[1]$ also relates the two associated Maurer–Cartan elements.

Consequently the induced L_∞-algebras on $\mathfrak{a} := A[1]$ are L_∞-isomorphic, and the L_∞-isomorphism is linear.

Theorem 4.3 immediately implies the following

Corollary 4.5. *Let (M_1, π_1) and (M_2, π_2) be two Poisson manifolds and S_1, S_2 submanifolds of M_1 and M_2 respectively. Assume $\psi\colon M_1 \to M_2$ is a Poisson diffeomorphism that maps S_1 to S_2. Then the isomorphism classes of the two L_∞-algebras associated to S_1 and S_2 coincide.*

Proof. Fix an embedding of NS_1 into M_1. The diffeomorphism ψ induces a bundle isomorphism between NS_1 and NS_2 and using this identification we obtain an embedding of NS_2 into M_2. Hence ψ allows us to identify the two V-algebras associated to S_1 and S_2. Moreover, the Maurer–Cartan elements associated to π_1 and to π_2 also get identified via ψ. So the two induced L_∞-algebras are L_∞-isomorphic. By Theorem 4.3 other choices of embeddings of NS_1 and NS_2 into M_1 and M_2, respectively, will not affect the isomorphism classes of the two L_∞-algebras. □

4.3. **Presymplectic manifolds.** Let S be a finite-dimensional smooth manifold. A two-form ω on S may be regarded as a bundle map $\omega^\sharp\colon TS \to T^*S$ by $\omega^\sharp_x(v) := \omega_x(v, \ )$. If ω is closed and $\omega^\sharp$ has constant rank, S is called a presymplectic manifold. If the rank is maximal (i.e., $\omega^\sharp$ is bijective), then S is called a symplectic manifold. A symplectic manifold is also a Poisson manifold with Poisson bivector field obtained by inverting the symplectic two-form. A coisotropic submanifold in a symplectic manifold gets the structure of a presymplectic submanifold by restricting the symplectic form.

Let (S, ω) be a presymplectic manifold. Then $\mathcal{F}_\omega := \operatorname{Ker}\omega^\sharp$ is an integrable distribution. Thus, the de Rham differential descends to the quotient

$$\Omega_{\mathcal{F}_\omega} := \Omega(S)/\{\alpha \in \Omega(S) : i_X\alpha = 0\ \forall X \in \Gamma(S, \mathcal{F}_\omega)\}$$

called the foliated de Rham complex. Also observe that $\Omega_{\mathcal{F}_\omega} = \Gamma(S, \Lambda\mathcal{F}^*_\omega)$.

Corollary 4.6 (Oh–Park)**.** *The foliated de Rham complex $\Omega_{\mathcal{F}_\omega}$ of a presymplectic manifold (S, ω) carries a flat L_∞-structure, unique up to L_∞-automorphisms, with first operation the de Rham differential.*

See [OP] for a different proof.

Proof. By a theorem of Gotay [G], every presymplectic manifold (S, ω) may be embedded into some symplectic manifold (M, Ω) as a coisotropic submanifold with $\omega = \iota^*\Omega$, where $\iota\colon S \to M$ is the embedding. Moreover, $\Omega^\sharp$ establishes an isomorphism of $\mathcal{F}_\omega$ with N^*S. So the construction in the first part of this Section yields the desired flat L_∞-structure.

Gotay also proves that this coisotropic embedding is unique up to neighbourhood equivalence: namely, for every two coisotropic embeddings of S, there exist symplectomorphic neighbourhoods of S. Applying Corollary 4.5, we get uniqueness. □

4.4. **Regular Dirac structures.** Let S be a smooth manifold. Sections of $TS \oplus T^*S$ may be endowed with the Courant bracket [Cour] which is the skew–symmetrization of the Dorfman bracket [Dorf] given by

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2] = [X_1, X_2] \oplus (L_{X_1}\xi_2 - i_{X_2}d\xi_1)$$

and with the symmetric nondegenerate pairing $\langle X_1 \oplus \xi_1, X_2 \oplus \xi_2 \rangle = i_{X_1}\xi_2 + i_{X_2}\xi_1$. A subbundle L of $TS \oplus T^*S$ is called a Dirac structure if it is maximally isotropic with respect to the pairing and sections of L are closed under the Courant bracket. Examples of Dirac structures are graphs of Poisson bivector fields.

A Dirac structure (S, L) is called regular if $\mathcal{F}_L := L \cap TS$ has constant rank. Examples of regular Dirac structures are graphs of presymplectic forms. Coisotropic submanifolds of a Poisson manifold with regular characteristic distribution get an induced regular Dirac structure. Since $\mathcal{F}_L$ is an integrable distribution, one can define the foliated de Rham complex $\Omega_{\mathcal{F}_L}$. We then have the following generalization of Corollary 4.6:

Corollary 4.7. *The foliated de Rham complex $\Omega_{\mathcal{F}_L}$ of a regular Dirac manifold (S, L) carries a flat L_∞-structure, unique up to L_∞-automorphisms, with first operation the de Rham differential.*

Notice that the existence part is already contained in [CZ].

Proof. It is shown in [CZ] that, canonically up to neighbourhood equivalences, the total space of $\mathcal{F}_L^*$ can be given a Poisson structure such that the zero section is coisotropic with induced Dirac structure equal to L. In particular the Poisson structure establishes an isomorphism $N^*S \to \mathcal{F}_L$. □

Appendix. Proof of Lemma 3.5

We prove that the family of maps $\{U^n\colon S^n(\mathfrak{a}) \to \mathfrak{a}\}_{n\geq 1}$ defined by equations (16) and (17) satisfies the family of relations given by equation (15) — again we suppress the t dependence of $U^n(t)$. The proof we give works inductively: It is easy to check that $U^1(a_1) := \Pi_{\mathfrak{a}}\phi_t(a_1)$ satisfies $\dot{U}^1 = \Pi_{\mathfrak{a}} m_t \circ U^1$, which is equation (15) for $n = 1$.

Suppose we verified that equation (15) holds for all U^k, $k < n$. We show that this implies that equation (15) is satisfied for n, too. The definition of U^n by equation (17) implies

$$\dot{U}^n(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma\in\Sigma_n} \mathrm{sign}(\sigma) \sum_{k\geq 1} \sum_{\mu_1+\cdots+\mu_k=n-1} \frac{1}{nk!\mu_1!\cdots\mu_k!}$$
$$\Big(\Pi_{\mathfrak{a}}[[\cdots[[m_t\phi_t, U^{\mu_1}], U^{\mu_2}], \cdots], U^{\mu_k}] + k\Pi_{\mathfrak{a}}[[[\cdots[\phi_t, U^{\mu_1}], \cdots], U^{\mu_{(k-1)}}], \dot{U}^{\mu_k}]\Big),$$

where we suppressed the arguments $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. The first term comes from deriving ϕ_t, the second one from deriving one of the factors U^k with $k < n$ in the formula for U^n. We denote the two terms by A^n and B^n respectively. A^n contains terms of the form $[[[m_t\phi_t, U^{\mu_1}], U^{\mu_2}], \ldots]$ where we can first use that m_t is a derivation and then successively apply the graded Jacobi identity (see Definition 2.1) and obtain

$$A^n = \sum_{\sigma\in\Sigma_n} \mathrm{sign}(\sigma) \sum_{k\geq 1} \sum_{r+s=k} \sum_{\substack{\alpha_1+\cdots+\alpha_r+\\ +\beta_1+\cdots+\beta_s=n-1}} \frac{1}{nr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}$$
$$\Pi_{\mathfrak{a}}[([[\cdots[m_tU^{\alpha_1}, U^{\alpha_2}], \cdots], U^{\alpha_r}]), ([[\cdots[\phi_t, U^{\beta_1}], \cdots], U^{\beta_s}])].$$

Next we apply equation (2) which leads to

$$A^n = \Big(\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{r+s=k} \sum_{\substack{\alpha_1+\cdots+\alpha_r+\\ +\beta_1+\cdots+\beta_s=n-1}} \frac{1}{nr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}$$

$$D^{r+1}_{m_t}(U^{\alpha_1}\otimes\cdots\otimes U^{\alpha_r}\otimes \Pi_{\mathfrak{a}}[[\cdots[\phi_t,U^{\beta_1}],\cdots],U^{\beta_s}]\Big) -$$

$$-\Big(\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{r+s=k} \sum_{\substack{\alpha_1+\cdots+\alpha_r+\\ +\beta_1+\cdots+\beta_s=n-1}} \frac{1}{nr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}$$

$$\Pi_{\mathfrak{a}}[([[\cdots[\phi_t,U^{\alpha_1}],\cdots],U^{\alpha_r}]),\Pi_{\mathfrak{a}}([[\cdots[m_tU^{\beta_1},U^{\beta_2}],\cdots],U^{\beta_s}])]\Big).$$

We claim that the following two identities hold: the first is

$$\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{\mu_1+\cdots+\mu_k=n-1} \frac{1}{n(k-1)!\mu_1!\cdots\mu_k!}$$

$$\Pi_{\mathfrak{a}}[[[\cdots[\phi_t,U^{\mu_1}],\cdots],U^{\mu_{(k-1)}}],\dot{U}^{\mu_k}]) =$$

$$= \sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{r+s=k} \sum_{\substack{\alpha_1+\cdots+\alpha_r+\\ +\beta_1+\cdots+\beta_s=n-1}} \frac{1}{nr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}$$

$$\Pi_{\mathfrak{a}}[([[\cdots[\phi_t,U^{\alpha_1}],\cdots],U^{\alpha_r}]),\Pi_{\mathfrak{a}}([[\cdots[m_tU^{\beta_1},U^{\beta_2}],\cdots],U^{\beta_s}])],$$

which means that B^n cancels with the second term in the expression for A^n given above; the second is

$$\Big(\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{r+s=k} \sum_{\substack{\alpha_1+\cdots+\alpha_r+\\ +\beta_1+\cdots+\beta_s=n-1}} \frac{1}{nr!s!\alpha_1!\cdots\alpha_r!\beta_1!\cdots\beta_s!}$$

$$\big(D^{r+1}_{m_t}(U^{\alpha_1}\otimes\cdots\otimes U^{\alpha_r}\otimes \Pi_{\mathfrak{a}}[[\cdots[\phi_t,U^{\beta_1}],\cdots],U^{\beta_s}])\big)\Big)$$

$$= \sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{l_1+\cdots+l_k=n} \frac{1}{k!l_1!\cdots l_k!} D^k_{m_t}(U^{l_1}\otimes\cdots\otimes U^{l_k})$$

which means that the first term in the expression for A^n is equal to the expression from equation (15) which we would like to obtain.

The first identity is straightforward to check: By the induction hypothesis, equation (15) is satisfied for $k<n$, so we can plug in the expression for $\dot{U}^{\mu_k}$ on the left-hand side of the identity. This immediately leads to the expression on the right-hand side. To prove the second identity, we first use the recursive definition of U^n (see formula (17)) on the left-hand side of the identity to arrange the terms of the form $\Pi_{\mathfrak{a}}[[\cdots[\phi_t,U^{\beta_1}],\cdots],U^{\beta_s}]$ into some U^{β}. We arrive at

(20)

$$\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{r\geq 1} \sum_{\alpha_1+\cdots+\alpha_r=n} \frac{1}{n(r-1)!(\alpha_1-1)!\alpha_2!\cdots\alpha_r!} D^r_{m_t}(U^{\alpha_1}\otimes\cdots\otimes U^{\alpha_r}).$$

It remains to prove that this map is equal to

$$\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{k\geq 1} \sum_{l_1+\cdots+l_k=n} \frac{1}{k! l_1! \cdots l_k!} D^k_{m_t}(U^{l_1}\otimes\cdots\otimes U^{l_k}). \tag{21}$$

We give the construction of a third map for which it is easy to show that it is equal to both map (20) and map (21). Assume one is given n distinguishable objects and r 'boxes' where there are w_j boxes that can contain exactly l_j of the objects, $1 \leq j \leq k$ ($0 < l_1 < \cdots < l_k$ and $w_1 + \cdots + w_k = r$). We label this situation by $(r|(l_1, w_1), \ldots, (l_k, w_k))$. We assume that boxes that contain the same number of objects are indistinguishable. The number of different ways to put the n objects into these boxes is given by

$$\frac{n!}{w_1! \cdots w_k! (l_1!)^{w_1} \cdots (l_k!)^{w_k}}.$$

Consider

$$\sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma) \sum_{r\geq 1} \sum_{(r|(l_1,w_1),\ldots,(l_k,w_k))} |\text{ways to put } n \text{ objects into these boxes}| \left(D^r_{m_t}(U^{l_1}\otimes\cdots\otimes U^{l_1}\otimes\cdots\otimes U^{l_k}\otimes\cdots\otimes U^{l_k})\right).$$

It is straightforward to check that this map is equal to map (20) and to map (21).

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