

Hyperbolic Rank of Products

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Abstract

Generalizing [BrFa] we prove the existence of a bilipschitz embedded manifold of pinched negative curvature and dimension $m_1 + m_2 - 1$ in the product $X := X_1^{m_1} \times X_2^{m_2}$ of two Hadamard manifolds $X_i^{m_i}$ of dimension m_i with pinched negative curvature.

Combining this result with [BuySch] we prove the additivity of the hyperbolic rank for products of manifolds with pinched negative curvature.

1 Introduction

In [BrFa] the authors proved that the product $X := H^{m_1} \times \dots \times H^{m_k}$ of hyperbolic spaces H^{m_i} admits an embedding of $Y := H^{m_1 + \dots + m_k - k + 1}$ in X that is quasi-isometric in the sense that the Riemannian distance functions d_X and d_Y on X and Y are related via

$$d_X|_Y \leq d_Y \leq \alpha \cdot d_X|_Y + \beta \quad ,$$

with constants $\alpha, \beta \in \mathbb{R}^+$, $\alpha > 1$.

In [Foe] this was generalized to the case of Riemannian products of certain types of warped products, that were shown to admit bilipschitz embeddings of warped products of negative sectional curvature.

In this paper we further generalize this observation proving the

Theorem 1 *Let $(X_i^{m_i}, g_i)$, $i = 1, \dots, k$, be Hadamard manifolds of pinched negative sectional curvature $-b_i^2 \leq K(X_i^{m_i}, g_i) \leq -a_i^2$. Then their Riemannian product $(X = \prod_{i=1}^k X_i^{m_i}, g_X)$ admits a bilipschitz embedding of a Hadamard manifold (Y^n, g_Y) of pinched negative sectional curvature $-b^2 \leq K(Y^n, g_Y) \leq -a^2$ and dimension $n := m_1 + \dots + m_k - k + 1$.*

The existence result of Theorem 1 implies the additivity of the hyperbolic rank for products of Hadamard manifolds with pinched negative curvature (compare [BuySch]).

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Given a metric space X consider all locally compact $CAT(-1)$ Hadamard spaces Y quasi-isometrically embedded into X and let

$$\text{rank}_h X := \sup_Y \dim \partial_\infty Y$$

over all such Y , where $\dim \partial_\infty Y$ is the topological dimension of the boundary of Y . This invariant was introduced by Gromov ([Gr]) in a slightly stronger form, where he called it the hyperbolic corank. As explained in [BuySch] we will call it the hyperbolic rank.

Combining Theorem 1 with the result of [BuySch] we obtain

Theorem 2 *Let $X = X_1 \times \dots \times X_k$ be the product of Hadamard manifolds with pinched negative curvature, then*

$$\text{rank}_h X = \sum_{i=1}^k \text{rank}_h X_i.$$

2 Proof of Theorem 1

It is convenient to have the following

Definition 1 *Let (X, g_X) be a Riemannian manifold. A function $f : X \rightarrow \mathbb{R}$ is said to have pinched convex levels, if there are constants $0 < c_1 \leq c_2$ such that for any unit speed geodesic $c : (-\eta, \eta) \rightarrow Y$ with $\dot{c}(0) \perp \text{grad} f(c(0))$:*

$$c_1 \leq (f \circ c)''(0) \leq c_2.$$

We now prove the

Lemma 1 *Let (Y, g_Y) be a complete Riemannian manifold with*

1) $|K| \leq \Lambda,$

such that there exists a function $f : Y \rightarrow \mathbb{R}$ satisfying

2) $\| \text{grad} f \| \equiv 1,$

3) *f has pinched convex levels and*

4) $K(\sigma_p) < -\delta$ *for all planes σ_p with $\text{grad} f(p) \subset \sigma_p$.*

Then (Y, g_Y) is lipshitz equivalent to a Riemannian manifold of pinched negative sectional curvature.

Proof of Lemma 1:

First note that condition 2) implies that Y splits into a product $Y = \mathbb{R} \times N$ such that

- a) $\mathbb{R} \times p$ is orthogonal to $N^t := t \times N$ for all $t \in \mathbb{R}$ and $p \in N$ and \mathbb{R} lies totally geodesic in Y , i.e. locally the metric is given by

$$g_Y = \begin{pmatrix} r & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{ij}(t, y^k) & \\ 0 & & & \end{pmatrix}, \quad r \in \mathbb{R}$$

Now 1) and 3) guarantee that

- b) the Riemannian embedding $N \rightarrow (Y, g_Y)$ is strictly convex in the sense, that its second fundamental form tensor is strictly positive definite, i.e. $\forall B \in T_p N$ with $\|B\| = 1$:

$$II(B, B) > \epsilon \quad \text{and}$$

- c) the intrinsic sectional curvature's absolute value $|K^N(\Delta)|$ of any tangent plane Δ that is tangent to N is bounded by some real number C_1 :

$$|K^N(\Delta)| < C_1,$$

From 1) and 4) it further follows that

- d) the sectional curvature $K(\Sigma)$ of any tangent plane Σ spanned by a vector tangent to N and the coordinate vector $\frac{\partial}{\partial t}|_t$ is bounded by negative constants $-\delta', -\delta$:

$$-\delta' < K(\Sigma) < -\delta.$$

Finally 1) can be rewritten as

- e) the norm of the curvature tensor R^{g_Y} of g_Y being bounded:

$$\|R\| := \sup_{\|A_i\| \leq 1} \left\{ \langle R_{A_1 A_2 A_3, A_4} \rangle \right\} < C_2.$$

We now stretch the metric into the direction of $grad f$ by some factor. Thus consider locally for $\lambda \neq 0$ the metric

$$h^\lambda = \begin{pmatrix} \frac{r}{\lambda^2} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{ij}(t, y^k) & \\ 0 & & & \end{pmatrix}.$$

In the following we indicate all objects coming from the metric h^λ with an additional subscript λ .

Let Γ be an arbitrary tangent plane in TY spanned by the two h^λ -orthonormal vectors

$$A + B = a \frac{\partial}{\partial t} \Big|_{(t, y^k)} + b^i \frac{\partial}{\partial y^i} \Big|_{(t, y^k)} \quad \text{and} \quad C = c^i \frac{\partial}{\partial y^i} \Big|_{(t, y^k)}.$$

The sectional curvature $K^\lambda(A + B, C)$ is given via

$$\begin{aligned} K^\lambda(A + B, C) &= \langle R_{A+C}^\lambda A, C \rangle_\lambda + 2 \cdot \langle R_{A+C}^\lambda B, C \rangle_\lambda \\ &\quad + \langle R_{B+C}^\lambda B, C \rangle_\lambda. \end{aligned} \quad (1)$$

Let Γ_{ij}^k denote the Christoffel symbols of g_Y and $\tilde{\Gamma}_{ij}^k$ those of h^λ . It is obvious that $\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k$ for $k \neq 1$ and $\tilde{\Gamma}_{ij}^1 = \lambda^2 \Gamma_{ij}^1$. Further note that $\Gamma_{1j}^1 = 0 = \tilde{\Gamma}_{1j}^1$. Write $R_{\partial^i \partial^j} \partial^k = \sum_l R_{ijk}^l \partial^l$, thus it follows immediately from

$$R_{ijk}^l = \frac{\partial \Gamma_{ik}^l}{\partial y^j} - \frac{\partial \Gamma_{jk}^l}{\partial y^i} + \Gamma_{ik}^h \Gamma_{jh}^l - \Gamma_{jk}^h \Gamma_{ih}^l$$

that $\langle R_{A+C}^\lambda A, C \rangle_\lambda = \langle R_{A+C} A, C \rangle$ and $\langle R_{A+C}^\lambda B, C \rangle_\lambda = \langle R_{A+C} B, C \rangle$. With that, equation (1) turns into

$$\begin{aligned} K^\lambda(A + B, C) &= K(A, C) \lambda^2 \langle A, A \rangle_\lambda \langle C, C \rangle_\lambda \\ &\quad + 2 \cdot \lambda \langle R_{A+C} B, C \rangle \\ &\quad + K^\lambda(B, C) \langle B, B \rangle_\lambda \langle C, C \rangle_\lambda. \end{aligned} \quad (2)$$

Since $g_Y|_N = h^\lambda|_N$ the intrinsic curvature of N is independent of the particular choice of λ . The Gauss equation yields with $\tilde{B} = \mu B$, $\langle \tilde{B}, \tilde{B} \rangle_\lambda = 1$:

$$\begin{aligned} &K^\lambda(\tilde{B}, C) \\ &= K^N(\tilde{B}, C) - \left[II^\lambda(\tilde{B}, \tilde{B}) II^\lambda(C, C) - II^\lambda(\tilde{B}, C) II^\lambda(\tilde{B}, C) \right] \\ &= K^N(\tilde{B}, C) - \lambda^2 \cdot \left[II(\tilde{B}, \tilde{B}) II(C, C) - II(\tilde{B}, C) II(\tilde{B}, C) \right] \\ &\leq C_1 - \lambda^2 \cdot \epsilon^2, \end{aligned}$$

where the inequality follows from b).

Thus equation (2) takes the form

$$\begin{aligned} K^\lambda(A + B, C) &\leq -\delta \cdot \lambda^2 \cdot \langle A, A \rangle_\lambda \langle C, C \rangle_\lambda \\ &\quad + (C_1 - \lambda^2 \cdot \epsilon^2) \cdot \langle B, B \rangle_\lambda \langle C, C \rangle_\lambda \\ &\quad + 2\lambda \cdot C_2 \\ &\leq -\lambda^2 \min\{\delta, \epsilon^2\} + C_1 + 2\lambda C_2. \end{aligned} \quad (3)$$

Hence for $\lambda \geq \frac{C_2 + \sqrt{C_2^2 + C_1 \cdot \min\{\delta, \epsilon^2\}}}{\min\{\delta, \epsilon^2\}}$, h^λ makes (Y, h^λ) a Riemannian manifold of negative sectional curvature.

$|K^{(Y, g_Y)}| < \Lambda$ and the Gauss equation yield an upper bound ϵ'^2 for

$$\left| II(\tilde{B}, \tilde{B})II(C, C) - II(\tilde{B}, C)II(\tilde{B}, C) \right| \leq \epsilon'^2$$

and analogue to equation (3) one finds

$$K^\lambda(A + B, C) \geq -\lambda^2 \max\{\delta', \epsilon'^2\} - C_1 - 2\lambda C_2. \quad (4)$$

Furthermore it is obvious that h^λ is Lipschitz equivalent to g_Y .

q.e.d.

Applying Lemma 1 we can prove the following

Proposition 1 *Let (X_i, g_{X_i}) , $i = 1, 2$, be Riemannian manifolds satisfying conditions 1) – 4) of Lemma 1. Then the natural Riemannian embedding (Y, g_Y) of $Y = \mathbb{R} \times N_1 \times N_2$ in $(X, g_X) = (X_1, g_{X_1}) \times (X_2, g_{X_2})$, with*

$$i: (t, y^k, \tilde{y}^k) \longrightarrow (t, y^k, t, \tilde{y}^k),$$

is Lipschitz equivalent to a Riemannian manifold of pinched negative sectional curvature.

Proof of Proposition 1:

As in the proof of Lemma 1 we first note that conditions 1) – 4) imply the following:

X_i is a product manifold $X_i = \mathbb{R} \times N_i$ of the real numbers \mathbb{R} and a n_i -dimensional manifold N_i and the metric g_{X_i} is such that

- i) $\mathbb{R} \times p$ is orthogonal to $N_i^t := t \times N_i$ for all $t \in \mathbb{R}$ and $p \in N_i$, and \mathbb{R} lies totally geodesic in X_i :

$$g_{X_1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & g_{ij}(t, y^k) & & \\ 0 & & & \end{pmatrix}, \quad g_{X_2} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & \tilde{g}_{ij}(t, \tilde{y}^k) & & \\ 0 & & & \end{pmatrix}.$$

- ii) the Riemannian embeddings $N_i \longrightarrow (X_i, g_{X_i})$ are strictly convex in the sense, that their second fundamental form tensors II^i are strictly positive definite, i.e. $\forall A_i \in T_p N_i$ with $\|A_i\|_i = 1$:

$$II^i(A_i, A_i) > \epsilon_i,$$

- iii) the intrinsic sectional curvatures' absolute values $|K^{N_i}(\Delta^i)|$ of any tangent plane Δ^i that is tangent to N^i is bounded by some real number C_1^i :

$$|K^{N_i}(\Delta^i)| < C_1^i,$$

- iv) the sectional curvature $K^{x_i}(\Sigma^i)$ of any tangent plane Σ^i spanned by a vector tangent to N_i and the coordinate vector $\frac{\partial}{\partial t^i}|_{t^i}$ is bounded by negative constants

$$-\delta'_i < K^{X_i}(\Sigma^i) < -\delta_i,$$

- v) the norms of the curvature tensors R^{X_i} of g_{X_i} are bounded:

$$\|R^{X_i}\| := \sup_{\|A_j\| \leq 1} \{R_{A_1 A_2 A_3 A_4}^{X_i}\} < C_2^i.$$

We now establish that the Riemannian embedding (Y, g_Y) of $Y = \mathbb{R} \times N_1 \times N_2$ in $(X, g_X) = (X_1, g_{X_1}) \times (X_2, g_{X_2})$, determined by

$$i: (t, y^k, \tilde{y}^k) \longrightarrow (t, y^k, t, \tilde{y}^k),$$

satisfies the requirements a) - e) of Lemma 1:

- a) The pullback metric $g_Y := i^*g_X$ on Y is

$$g_Y = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & & \\ \vdots & g_N(t, y^k, \tilde{y}^k) & & \\ 0 & & & \end{pmatrix}. \quad (5)$$

- b) Let $V = \tilde{B} + \tilde{C}$ be the unique decomposition of an arbitrary vector $V \in T_{(p,q)}N$ into the sum of the lifts \tilde{B}, \tilde{C} of vectors $B \in T_p N_1$ and $C \in T_q N_2$ onto $T_{(p,q)}N$.

With $\Pi^1(B, B) = 2\Pi^Y(\tilde{B}, \tilde{B})$, $\Pi^2(C, C) = 2\Pi^Y(\tilde{C}, \tilde{C})$ and $\Pi^Y(\tilde{B}, \tilde{C}) = 0$ it follows

$$\begin{aligned} \frac{\Pi^Y(\tilde{B} + \tilde{C}, \tilde{B} + \tilde{C})}{g_Y(\tilde{B} + \tilde{C}, \tilde{B} + \tilde{C})} &= \frac{\Pi^Y(\tilde{B}, \tilde{B}) + \Pi^Y(\tilde{C}, \tilde{C})}{g_Y(\tilde{B} + \tilde{C}, \tilde{B} + \tilde{C})} \\ &= \frac{1}{2} \frac{\Pi^1(B, B) + \Pi^2(C, C)}{g_Y(\tilde{B} + \tilde{C}, \tilde{B} + \tilde{C})} \\ &\geq \frac{1}{2} \min \left\{ \frac{\Pi^1(B, B)}{g_{N_1}(B, B)}, \frac{\Pi^2(C, C)}{g_{N_2}(C, C)} \right\} \\ &= \frac{1}{2} \min \{ \epsilon_1, \epsilon_2 \} =: \epsilon. \end{aligned}$$

- c) Consider a tangent plane Γ in TY at $\{t, p, q\} \in Y$ that is tangent to N^t . But N^t is the Riemannian product $N^t = N_1^t \times N_2^t$. Therefore it is

$$|K^N(\Gamma)| \leq \max \{C_1^1, C_1^2\} =: C_1.$$

d) Let $\tilde{B}, \tilde{C} \in T_{(t,p,q)}Y$ be as in b), thus tangent to N , and such that $\tilde{B} + \tilde{C}$ has unit length in (Y, g_Y) and set $\tilde{A} := \frac{1}{\sqrt{2}} \frac{\partial}{\partial t} \Big|_{(t,p,q)}$. Thus \tilde{A} and $\tilde{B} + \tilde{C}$ are orthonormal in (Y, g_Y) . Recall that $n_i := \dim N_i$, $i = 1, 2$. Coordinates (t, y^i) in X_1 and (t, \tilde{y}^k) in X_2 naturally induce coordinates in $X = X_1 \times X_2$ and thus $1 + n_1 + n_2$ coordinate functions (t, y^i, \tilde{y}^k) on Y . In the following the various indices are $i, j \in \{2, 3, \dots, n_1 + 1\}$ and $k, l \in \{n_1 + 2, \dots, n_1 + n_2 + 1\}$. With that one calculates

$$R_{11ij}^Y = 2R_{11ij}^{X_1}, \quad R_{11kl}^Y = 2R_{11kl}^{X_2} \quad \text{and} \quad R_{11ik}^Y = 0,$$

and therefore

$$\begin{aligned} K^Y(\tilde{A}, \tilde{B} + \tilde{C}) &= \langle R_{\tilde{A}\tilde{B}}^Y \tilde{A}, \tilde{B} \rangle_Y + \langle R_{\tilde{A}\tilde{C}}^Y \tilde{A}, \tilde{C} \rangle_Y \\ &= 2 \langle R_{A_1 B}^{X_1} A_1, B \rangle_{X_1} + 2 \langle R_{A_2 C}^{X_2} A_2, C \rangle_{X_2} \\ &\stackrel{(*)}{=} K^{X_1}(A_1, B) \langle B, B \rangle_{X_1} + K^{X_2}(A_2, C) \langle C, C \rangle_{X_2}. \end{aligned}$$

Note, that in $(*)$ we made use of $\langle A_i, A_i \rangle_{X_i} = \frac{1}{\sqrt{2}}$, $i = 1, 2$.

With that we finally achieve

$$-\delta' := \min \{-\delta'_1, -\delta'_2\} \leq K^Y(\tilde{A}, \tilde{B} + \tilde{C}) \leq \max \{-\delta_1, -\delta_2\} =: -\delta.$$

e) In order to show

$$\|R^Y\| := \sup_{\|A_i\| \leq 1} \left\{ \langle R_{A_1 A_2}^Y A_3, A_4 \rangle_Y \right\} < C_2$$

we emphasize that the sectional curvature's absolute value $|K^Y|$ is bounded: Let therefore Γ be an arbitrary tangent plane in TY spanned by the two g_Y -orthonormal vectors

$$A + B = a \frac{\partial}{\partial t} \Big|_{(t,y^k)} + b^i \frac{\partial}{\partial y^i} \Big|_{(t,y^k)} \quad \text{and} \quad C = c^i \frac{\partial}{\partial y^i} \Big|_{(t,y^k)}.$$

For the sectional curvature of Γ one has

$$\begin{aligned} K^Y(\Gamma) &= \underbrace{\langle R_{AC}^Y A, C \rangle_Y}_A + 2 \underbrace{\langle R_{AC}^Y B, C \rangle_Y}_B \\ &\quad + \underbrace{\langle R_{BC}^Y B, C \rangle_Y}_C. \end{aligned}$$

Thus it suffices to show that the terms A), B) and C) are bounded:

A) $\langle R_{AC}^Y A, C \rangle_Y$ is bounded due to d).

B) Denote by $II^{X_1 \times X_2}$ the second fundamental form tensor of Y in $X_1 \times X_2$. With $II^{X_1 \times X_2}(A, B) = 0 = II^{X_1 \times X_2}(A, C)$ the Gauss equation yields

$$\langle R_{AC}^Y B, C \rangle_Y = \langle R_{AC}^Y B, C \rangle_{X_1 \times X_2} = \langle R_{AC}^{X_1 \times X_2} B, C \rangle_{X_1 \times X_2}.$$

Thus it follows from condition $v)$ that B is also bounded.

C) From $|K^{g_{X_i}}| < \Lambda_i$ and $iii)$ it follows from the Gauss equation that

$$\left| II^i(v_i, v_i)II^i(w_i, w_i) - II^i(v_i, w_i)II^i(v_i, w_i) \right|$$

are bounded by some constants for all g_{X_i} -orthonormal systems $\{v_i, w_i\}$, which together with $ii)$ implies the existence of constants $\tilde{\epsilon}_i$ such that

$$\frac{II^i(v_i, v_i)}{g_{X_i}(v_i, v_i)} < \tilde{\epsilon}_i.$$

Hence we find as in $b)$ $II^Y(B, B) \leq \tilde{\epsilon}$ for all $B \in TY$ tangent to N of norm $\|B\|_{g_Y} = 1$. From

$$\epsilon \leq II^Y(B, B) \leq \tilde{\epsilon}$$

for all $B \in TY$ tangent to N of norm $\|B\|_{g_Y} = 1$ one achieves

$$\epsilon^2 \leq II^Y(B, B)II^Y(C, C) - II^Y(B, C)II^Y(B, C) \leq \tilde{\epsilon}^2$$

for all g_Y -orthonormal systems $\{B, C\}$.

Therefore it follows from the Gauss equation and condition $c)$ that the term C is also bounded.

Now $a) - e)$ ensure that we can apply Lemma 1. Thus the validity of Proposition 1 follows.

q.e.d.

Corollary 1 *Let (X_1, g_{X_1}) and (X_2, g_{X_2}) be two Hadamard manifolds with pinched negative sectional curvature $-b_i^2 \leq K^{X_i} \leq -a_i^2 < 0$, $a_i, b_i \in \mathbb{R}$, $i = 1, 2$. Then their Riemannian product admits a Riemannian hypersurface that is bilipschitz to a Riemannian manifold (Y, g_Y^λ) of negative sectional curvature bounded by $-b^2 \leq K^{g_Y^\lambda} \leq -a^2 < 0$ for appropriate $a, b \in \mathbb{R}$.*

Proof of Corollary 1:

Write the factors in horospherical coordinates and apply Proposition 1.

q.e.d.

Note that the additional boundary condition ensures that the horospheres' intrinsic curvature is also bounded ([BuKa]). It is for that reason that our method of prove requires this extra condition.

Since (Y, g_Y^λ) is bilipschitz to (Y, g_Y) it suffices to prove that the Riemannian embedding (Y, g_Y) is bilipschitz to (X, g_X) .

Denote the Riemannian length function on Y , that is induced by the metric $g_Y := j^*g_X$ on Y , by d_Y and that on X , induced by the Riemannian product metric g_X , by d_X . With that we find the inequalities

$$d_X(j(p), j(q)) \stackrel{a)}{\leq} d_Y(p, q) \stackrel{b)}{\leq} (2\sqrt{2} + 2) \cdot d_X(j(p), j(q)) \quad \forall p, q \in Y.$$

While inequality a) merely is a consequence of the fact that (Y, g_Y) is a Riemannian submanifold of (X, g_X) , inequality b) requires more attention. Note that the idea for the following construction is due to Brady and Farb ([BrFa]). We denote the various canonical projections as follows:

$$\begin{array}{ll} \pi_i : X \longrightarrow X_i & t : X_i \longrightarrow \mathbb{R} \\ \eta : Y \longrightarrow N & \eta_i : X \longrightarrow N_i \end{array}.$$

Consider an arbitrary differentiable curve $c : [t_p, t_q] \longrightarrow X$ connecting $j(p) \in X$ with $j(q) \in X$. The idea is to construct a curve $\tilde{c} : [\alpha, \omega] \longrightarrow Y$ that connects $p \in Y$ with $q \in Y$, whose Riemannian length $L_Y(\tilde{c})$ in (Y, g_Y) is bounded by a constant times the Riemannian length $L_X(c)$ of c in (X, g_X) .

Therefore we consider the projections $c_i := \pi_i \circ c$ of c to the factors X_i , $i = 1, 2$, that connect $p_i := \pi_i(j(p))$ with $q_i := \pi_i(j(q))$. The further projections $t_i \circ c_i$ are continuous, thus the set $\mathcal{K} := \cup_{i=1,2} (t_i \circ c_i)([t_p, t_q]) \subset \mathbb{R}$ is compact and therefore takes its maximum $(t_{i_0} \circ c_{i_0})(b)$ for some $b \in [t_p, t_q]$, $i_0 \in \{1, 2\}$.

The continuous and piecewise differentiable curve \tilde{c} in Y we are going to follow from p to q consists of three differentiable segments v_1 , γ and v_2 as follows:

- v_1 has constant projection $\eta \circ v_1$ to $N_1 \times N_2$, that is given through $\eta \circ v_1 \equiv (\eta_1(p_1), \eta_2(p_2))$, while its projection to \mathbb{R} is $t \circ v_1 = (t_{i_0} \circ c_{i_0})|_{[t_p, b]}$.
- γ is the curve keeping its projection to \mathbb{R} constant: $(t \circ \gamma) \equiv (t_{i_0} \circ c_{i_0})(b)$, while varying along $N_1 \times N_2$ with $\eta(\gamma) = (\eta_1(c_1), \eta_2(c_2))$.
- v_2 again has constant projection $\eta \circ v_2$ to $N_1 \times N_2$ that is $\eta \circ v_2 \equiv (\eta_1(q_1), \eta_2(q_2))$. Its projection to \mathbb{R} is $t \circ v_2 = (t_{i_0} \circ c_{i_0})|_{[b, t_q]}$.

The length of $\tilde{c} := v_2 * \gamma * v_1$ is the sum

$$L_Y(\tilde{c}) = L_Y(v_1) + L_Y(\gamma) + L_Y(v_2).$$

From (5) it directly follows, that

$$\begin{aligned} L_Y(v_m) &= \sqrt{2} \cdot L_{(\mathbb{R}, dt^2)}((t_{i_0} \circ c_{i_0})|_{I_m}) \\ &\leq \sqrt{2} \cdot L_{(\mathbb{R}, dt^2)}(t_{i_0} \circ c_{i_0}) \\ &\leq \sqrt{2} \cdot L_X(c), \end{aligned} \tag{6}$$

where $I_1 = [t_p, b]$ and $I_2 = [b, t_q]$.

Now we know that the natural diffeomorphisms $\phi_i^{tt'} : N_i^t \rightarrow N_i^{t'}$ are length contracting for $t \leq t'$. With that and the particular choice of b it is

$$\begin{aligned}
L_Y(\gamma) &= \int_{t_p}^{t_q} \sqrt{g_Y(\gamma', \gamma')} dt \\
&= \int_{t_p}^{t_q} \left[g_Y \Big|_{N^{t_0}} \left((\eta \circ \gamma)', (\eta \circ \gamma)' \right) \right]^{\frac{1}{2}} dt \\
&= \int_{t_p}^{t_q} \left[g_Y \Big|_{N_1^{t_0}} \left((\eta_1 \circ c)', (\eta_1 \circ c)' \right) + g_Y \Big|_{N_2^{t_0}} \left((\eta_2 \circ c)', (\eta_2 \circ c)' \right) \right]^{\frac{1}{2}} dt \\
&\leq \int_{t_p}^{t_q} \left[g_Y \Big|_{N_1^t} \left((\eta_1 \circ c)', (\eta_1 \circ c)' \right) + g_Y \Big|_{N_2^t} \left((\eta_2 \circ c)', (\eta_2 \circ c)' \right) \right]^{\frac{1}{2}} dt \\
&\leq 2 \cdot L_X(c). \tag{7}
\end{aligned}$$

where $t_0 := (t_{i_0} \circ c_{i_0})(b)$.

Thus with (6) and (7) we can conclude that for an arbitrary curve c in X connecting two points $j(p), j(q) \in j(Y) \subset X$ there exists a curve \tilde{c} in Y connecting p and q with

$$L_Y(\tilde{c}) \leq (2\sqrt{2} + 2) L_X(c).$$

Thus the required inequality b) follows by the definitions of the Riemannian length functions.

Finally note that the generalization to products of finitely many Hadamard-manifolds of pinched negative sectional curvature is straight forward using once again that products of quasi-isometric maps are quasi-isometric.

q.e.d.

3 Proof of Theorem 2

Recall the definition of the hyperbolic rank as given in [BuySch]:

Definition 2 *Let X be a metric space. Then the hyperbolic rank, $\text{rank}_h X$, of X is defined via*

$$\text{rank}_h X := \sup_Y \dim \partial_\infty Y,$$

where the supremum is taken over all locally compact $CAT(-1)$ Hadamard spaces Y quasi-isometrically embedded into X and $\partial_\infty Y$ denotes the topological dimension of the boundary of Y .

In [BuySch] the authors proved the subadditivity of this hyperbolic rank with respect to products of Hadamard manifolds of pinched negative sectional curvature. This together with the following Lemma, that states the corresponding superadditivity, yields the proof of Theorem 2.

Lemma 2 *The hyperbolic rank, $rank_h$, is superadditive with respect to Riemannian products of Hadamard manifolds (X_i, g_i) of pinched negative sectional curvature, i.e., for the Riemannian product (X, g) of the (X_i, g_i) , $i = 1, \dots, k$, one has*

$$rank_h(X, g_X) \geq \sum_{i=1}^k rank_h(X_i, g_i).$$

Proof of Lemma 2:

From the Morse' quasi-isometric lemma it easily follows that $rank_h(X_i) = \dim X_i - 1$. From Theorem 2 we further conclude $rank_h X \geq -k + \sum_{i=1}^k \dim X_i$ and thus

$$rank_h X \geq -k + \sum_{i=1}^k \dim X_i = \sum_{i=1}^k (\dim X_i - 1) = \sum_{i=1}^k rank_h(X_i, g_i).$$

q.e.d.

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