BLOWING-UP!

MARKUS BRODMANN

1. Introduction *Blowing-up* is a fundamental concept of *Algebraic Geometry* and a subject which can be approached in quite different ways. Roughly speaking, *blowing-up an algebraic variety* X means finding a surjective map $\pi : Y \to X$ such that $\pi^{-1}(p)$ consists of a single point for all points $p \in X$ which avoid a given closed set $Z \subseteq X$. That Z is closed in the Zariski topology means that Z is the intersection of sets of zeros of polynomials. To introduce our main concept, we shall start by *blowing-up a disc*

$$D := \{(x, y) \in \mathbb{R}^2 \,\big|\, x^2 + y^2 \leq r^2\} \subseteq \mathbb{R}^2 \tag{1.1}$$

of radius $r > 0$ *with respect to two polynomials*

$$f = f(x, y),\ \ g = g(x, y) \in \mathbb{R}[x, y] \tag{1.2}$$

which define the set Z. Then we use *stereographic projection* to embed the obtained surface into a *solid torus* $\mathbb{T} \subseteq \mathbb{R}^3$. By means of this embedding we *visualize* our *blowing-up of the disc* D. We then briefly explain how the notion of *blowing-up* extends to (affine) *algebraic varieties* and discuss its basic properties and its significance, presenting an example.

2. Blowing-Up the Disc We use the previously introduced notation and assume that the *center*, that is the set

$$Z := \{(x, y) \in D \,\big|\, f(x, y) = g(x, y) = 0\} \tag{2.1}$$

of zeros of both f and g in D, is finite. Consider the *projective line*

$$\mathbb{P}^1 = \{(u : v) \,\big|\, (u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}\}. \tag{2.2}$$

We first describe in a rather informal way how the *blowing-up* $Bl_D(f, g) = Bl(f, g)$ *of the disc* D *with respect to the polynomials* f *and* g is obtained:

(2.3) *Replace each point* $p \in Z$ *by a projective line. To insert this projective line into the disc* D *instead of* p*, the "punctured disc"* $D \setminus Z$ *has to be deformed appropriately. This deformation is governed by the polynomials* f *and* g*.*

This description explains why one speaks of "blowing-up": each of the points $p \in Z$ is "blown-up" to a full projective line.

3. A More Formal Definition For those readers who wish to see a more formal description of the blowing-up $Bl(f,g)$, we shall now give a proper definition. We first consider the map

$$\varepsilon : D \setminus Z \to \mathbb{P}^1,\ ((x,y) \mapsto (f(x,y) : g(x,y))). \tag{3.1}$$

The essential point here is that the map is only defined away from the center since only there at least one of the polynomial values $f(x,y), g(x,y)$ is nonzero. The blowing-up $Bl(f,g)$ of D with respect to f and g is defined as the closure in the *Zariski topology* of the graph of ε. This graph can be pictured as the subset of $D \setminus Z \times \mathbb{P}^1$ made up of the points in $D \setminus Z$ and their images in $\mathbb{P}^1$ under the map ε. Thus

$$Bl(f,g) := \overline{\{(p,\varepsilon(p)) \,\big|\, p \in D \setminus Z\}} \subseteq D \times \mathbb{P}^1, \tag{3.2}$$

where $\overline{\bullet}$ denotes the formation of closure with respect to the Zariski topology in $D \times \mathbb{P}^1$. To see how this definition matches the informal description (2.3), we consider the *projection map*

$$\pi : Bl(f,g) \to D\ ((p,(u:v)) \mapsto p), \tag{3.3}$$

the *exceptional fibre* and the *open kernel* which are respectively defined by

$$E(f,g) := \pi^{-1}(Z) \subseteq Bl(f,g) \text{ and } \overset{\circ}{B}l(f,g) := Bl(f,g) \setminus E(f,g). \tag{3.4}$$

We then get the following commutative diagram, in which $\tilde{\varepsilon}$ is bijective and $\pi\restriction$ denotes the restriction of π.

$$\begin{tikzcd}
\overset{\circ}{B}l(f,g) \arrow[d, hook] & & D \setminus Z \arrow[ll, "\cong", "\tilde{\varepsilon}"'] \arrow[d, hook] \\
E(f,g)\dot{\cup}\overset{\circ}{B}l(f,g) = Bl(f,g) \arrow[rr, two heads, "\pi"] & & D = (D \setminus Z)\dot{\cup}Z \\
E(f,g) \arrow[u, hook] \arrow[rr, two heads, "\pi\restriction"] & & Z \arrow[u, hook]
\end{tikzcd} \tag{3.5}$$

A slightly more involved calculation – recommended to those readers with a basic background in algebraic geometry – shows that

$$E(f,g) = Z \times \mathbb{P}^1 = \dot{\bigcup}_{p\in Z}\{p\} \times \mathbb{P}^1,\ \text{hence } Bl(f,g) = \overset{\circ}{B}l(f,g)\dot{\cup}\dot{\bigcup}_{p\in\mathbb{Z}}\{p\} \times \mathbb{P}^1 \tag{3.6}$$

So, the "deformation" of (2.3) is performed by the map $\tilde{\varepsilon} : D \setminus Z \overset{\cong}{\to} \overset{\circ}{B}l(f,g)$ and $Bl(f,g)$ is obtained by inserting the *exceptional line* $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1$ instead of the point p into the deformed copy $\overset{\circ}{B}l(f,g)$ of $D \setminus Z$ for all $p \in Z$.

4. Embedding and Visualization It is well known that the projective line $\mathbb{P}^1$ may be viewed as a circle by means of a stereographic projection. In doing so, we may consider $D \times \mathbb{P}^1$ as a solid torus $\mathbb{T} \subseteq \mathbb{R}^3$ in 3-space. Now the blowing-up

$Bl(f,g) \subseteq D\times\mathbb{P}^1$ takes the form of a surface contained in the torus $\mathbb{T}$ and can therefore be visualized. To be more formal, we fix $R > r$ and consider the diffeomorphism

$$\mu : D \times \mathbb{P}^1 \xrightarrow{\approx} \mathbb{T},\ \mu((x,y),(u:v)) = (x,(R-y)\cos\alpha,(R-y)\sin\alpha) \tag{4.1}$$

$$\text{with} \quad \alpha = \begin{cases} 2\ \arctan\left(\frac{u}{v}\right), & \text{if } v \neq 0, \\ \pi & \text{if } v = 0, \end{cases} \tag{4.2}$$

for all $(x,y) \in D$ and all $(u:v) \in \mathbb{P}^1$. Thus we can visualize the *diffeomorphic image* $\mu(Bl(f,g)) \subseteq \mathbb{T} \subseteq \mathbb{R}^3$ of $Bl(f,g)$ in $\mathbb{T}$. The exceptional lines now appear as circles in our solid torus $\mathbb{T}$.
Performing this with $f = x$ and $g = y$, we get a *Möbius strip* (s. Figure 1). Choosing $f = x^2$ and $g = y^2$, we get a (curved) *double Whitney umbrella* (s. Figure 2). Readers who want to know more about these two particular surfaces could have a look at [2].

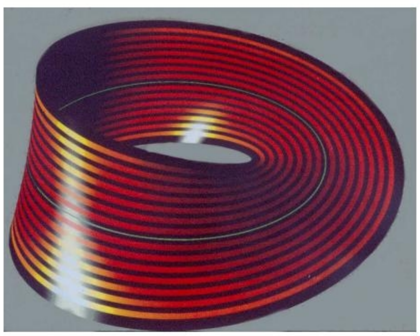

Figure 1

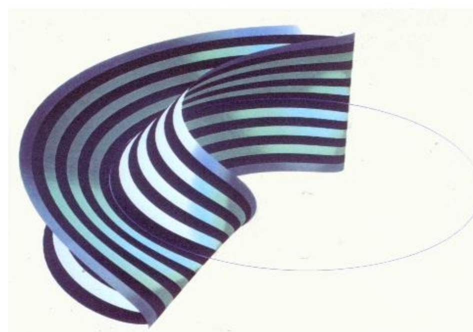

Figure 2

5. Blowing-Up Algebraic Varieties To explain the notion of blowing-up for algebraic varieties, let k be an algebraically closed field and let $X \subseteq k^n$ be an *affine algebraic variety.* Hence X is the intersection of the sets of zeros of some polynomials $f_1,\dots,f_r \in k[x_1,\dots,x_n]$ in the indeterminates $x_1,x_2,\dots,x_n$ and with coefficients in k. Finally, let

$$g_i : X \to k,\ i = 0,\dots,r \tag{5.1}$$

be *regular functions* on X, that is functions given by polynomials. Again, similarly as in (2.1), we define the *center of the blowing-up* as the intersection of the sets of zeros

$$Z := \{p \in X \,\big|\, g_i(p) = 0 \text{ for } i = 0,\dots,r\} \tag{5.2}$$

of the functions $g_0,g_1,\dots,g_r$. Then, as in (3.1), we consider the map

$$\varepsilon : X \setminus Z \to \mathbb{P}^r_k, \quad (p \mapsto (g_0(p) : \dots : g_r(p))) \tag{5.3}$$

to the *projective r-space* $\mathbb{P}^r_k := \{(u_0 : \ldots : u_r) \mid (u_0, \ldots, u_r) \in k^{r+1} \setminus \{(0, \ldots, 0)\}\}$. Finally, as in (3.2), we define the *blowing-up of X with respect to the functions* $g_0, \ldots, g_r$ as the closure in the Zariski topology of the graph of ε:

$$(5.4) \qquad Bl_X(g_0, \ldots, g_r) := \overline{\{(p, \varepsilon(p)) \mid p \in X \setminus Z\}}.$$

Again we have a canonical *projection morphism* $\pi : Bl_X(g_0, \ldots, g_r) \to X$ (cf (3.3)) and thus may apply the concepts defined in (3.4) to our new situation from which we get an adapted version of the diagram (3.5). On the other hand, what is stated in (3.6) finds no straightforward generalisation. In fact, the *exceptional fibre* $E_X(g_0, \ldots, g_r) := \pi^{-1}(Z)$ may now become a very complicated object although it is a so-called *Cartier divisor* in $Bl_X(g_0, \ldots, g_r)$. Finally, by a standard gluing process one may extend the notion of blowing-up to *arbitrary algebraic k-varieties X*. In fact, the notion of blowing-up may be extended even to *schemes*, the basic objects of contemporary algebraic geometry.

6. Why Blowing-Up? *Proper birational morphisms* between (irreducible) algebraic varieties are given by a blowing-up. This aspect makes blowing-up a fundamental tool of *Birational Algebraic Geometry*, primarily in the case of surfaces, [1].

Another equally important aspect of blowing-up is its *resolving effect on singularities*. We illustrate this effect by an example: In the unit disc D we consider the curve K given by $y^2 - x^2 + 2x^2 = 0$, a curve which is singular at $p = (0,0)$ (s. Figure 3). We pull back the curve K to the blowing-up $Bl(x,y)$ of D with respect to x and y, which leads to the *non-singular curve* (s. Figure 4)

$$(6.1) \qquad K' = \overline{\pi^{-1}(K \setminus \{p\})} \subseteq Bl(x,y).$$

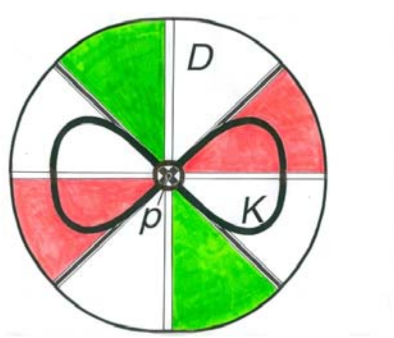

Figure 3

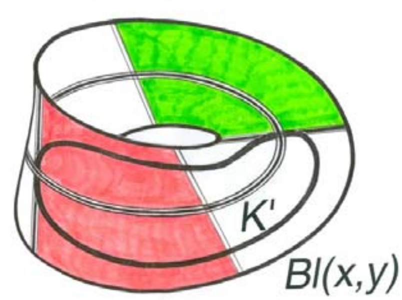

Figure 4

So we have "blown away" the singularity of the curve K. By a fundamental theorem of Hironaka [3], this phenomenon occurs in general if $k = \mathbb{C}$:

(6.2) *Each complex algebraic variety X admits a non-singular blowing-up.*

On the other hand, it is still open whether this result holds for algebraic varieties which are defined over a field k of positive characteristic.

References

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University of Zürich, Mathematics Institute, Winterthurerstrasse 190, 8057 Zürich, Switzerland - *e-mail:* brodmann@math.uzh.ch