Pre-Poisson submanifolds

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Abstract

In this note we consider an arbitrary submanifold C of a Poisson manifold P and ask whether it can be embedded coisotropically in some bigger submanifold of P. We define the classes of submanifolds relevant to the question (coisotropic, Poisson-Dirac, pre-Poisson ones), present an answer to the above question and consider the corresponding picture at the level of Lie groupoids, making concrete examples in which P is the dual of a Lie algebra and C is an affine subspace.

1   Introduction

In this note we wish to give an analog in Poisson geometry to the following statement in symplectic geometry. Recall that (P, Ω) is a symplectic manifold if Ω is a closed, non-degenerate 2 form and that a submanifold $\hat{C}$ is called coisotropic if the symplectic orthogonal $T\hat{C}^\Omega$ of $T\hat{C}$ is contained in $T\hat{C}$. The statement is: if $i\colon C \to P$ is any submanifold of a symplectic manifold (P, Ω), then there exists some symplectic submanifold $\tilde{P}$ containing C as a coisotropic submanifold iff $i^*\Omega$ has constant rank. The submanifold $\tilde{P}$ is obtained taking any complement $R \subset TP|_C$ of $TC + TC^\Omega$ and "extending C along R". Further there is a uniqueness statement "to first order": if $\tilde{P}_1$ and $\tilde{P}_2$ are as above, then there is a symplectomorphism of P fixing C whose derivative at C maps $T\tilde{P}_1|_C$ to $T\tilde{P}_2|_C$. This result follows using techniques similar to those used by Marle in [13], and relies on a technique known as "Moser's path method".

The above result should not be confused with the theorem of Gotay [9] that states the following: any presymplectic manifold (i.e. a manifold endowed with a constant rank closed 2-form) can be embedded coisotropically in some symplectic manifold, which is moreover unique up to neighborhood equivalence. The difference is that Gotay considers an abstract presymplectic manifold and looks for an abstract symplectic manifold in which to embed; the problem above fixes a symplectic manifold (P, Ω) and considers only submanifolds of P.

In this note we ask:

1) Given an arbitrary submanifold C of a Poisson manifold (P, Π), under what conditions does there exist some submanifold $\tilde{P} \subset P$ such that

    a) $\tilde{P}$ has a Poisson structure induced from Π

    b) C is a coisotropic submanifold of $\tilde{P}$?

2) When the submanifold $\tilde{P}$ exists, is it unique up to neighborhood equivalence (i.e. up to a Poisson diffeomorphism on a tubular neighborhood which fixes C)?

We will see in Section 4 that a sufficient condition is that C belongs to a particular class of submanifolds called pre-Poisson submanifolds. In that case we also have uniqueness: if $\tilde{P}_1$ and $\tilde{P}_2$ are as above, then there is a Poisson diffeomorphism of (a tubular neighborhood of C in) P fixing C which takes $\tilde{P}_1$ to $\tilde{P}_2$. When the Poisson structure on P comes from a symplectic form Ω, the pre-Poisson submanifolds of P are exactly the submanifolds for which the pullback of Ω has constant rank; hence we improve the "uniqueness to first order" result in the symplectic setting mentioned above to uniqueness in a neighborhood of C.

Since the above question is essentially about when an arbitrary submanifold can be regarded as a coisotropic one, we want to motivate in Section 2 why coisotropic submanifolds are interesting at all. In Section 3 we will describe the submanifolds of P which inherit a Poisson structure; these are the "candidates" for $\tilde{P}$ as above. Then in Section 5 we will present a non-trivial example: we consider as Poisson manifold P the dual of a Lie algebra $\mathfrak{g}$, and as submanifold C either a translate of the annihilator of a Lie subalgebra or the annihilator of some subspace of $\mathfrak{g}$. Finally in Section 6 we recall how to a Poisson manifold one can associate symplectic groupoids and investigate what pre-Poisson submanifolds correspond to at the groupoid level, discussing again the example where P is the dual of a (finite dimensional) Lie algebra. All manifolds appearing in this note are assumed to be finite dimensional.

2   Coisotropic submanifolds

A manifold P is called *Poisson manifold* if it is endowed with a bivector field $\Pi \in \Gamma(\Lambda^2 TP)$ satisfying $[\Pi, \Pi] = 0$, where $[\bullet, \bullet]$ denotes the Schouten bracket on multivector fields. Let us denote by $\sharp \colon T^*P \to TP$ the map given by contraction with Π. The image of $\sharp$ is a singular integrable distribution on P, whose leaves are endowed with a symplectic structure that encodes the bivector field Π. Hence one can think of a Poisson manifold as a manifold with a singular foliation by symplectic leaves.

Alternatively P is a Poisson manifold if there is a Lie bracket $\{\bullet, \bullet\}$ on the space of functions satisfying the Leibniz identity[1] $\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$. The Poisson bracket $\{\bullet, \bullet\}$ and the bivector field Π determine each other by the formula $\{f, g\} = \Pi(df, dg)$. In this note we will use both the geometric and algebraic characterization of Poisson manifolds.

Symplectic manifolds (P, Ω) are examples of Poisson manifolds: the map $TP \to T^*P$ given by contracting with Ω is an isomorphism, and (the negative of) its inverse is the sharp map of the Poisson bivector field associated to Π. Symplectic manifolds are exactly the Poisson manifolds whose symplectic foliation consists of just one leaf.

A second standard example, which will be used in Section 5, is the dual of a Lie algebra $\mathfrak{g}$, as follows. A linear function v on $\mathfrak{g}^*$ can be regarded as an element of $\mathfrak{g}$; one defines the Poisson bracket on linear functions as $\{v_1, v_2\} := [v_1, v_2]$, and the bracket for arbitrary functions is determined by this in virtue of the Leibniz rule. Duals of Lie algebras are exactly the Poisson manifolds whose Poisson bivector fields are linear. The symplectic foliation of $\mathfrak{g}^*$ is given by the orbits of the coadjoint action; the origin is a symplectic leaf, and unless the $\mathfrak{g}$ is an abelian Lie algebra the symplectic foliation will be singular. We will discuss this example in more detail in Section 5.

A submanifold C of a Poisson manifold P is called *coisotropic* if $\sharp N^*C \subset TC$. Here N^*C (the conormal bundle of C) is defined as the annihilator of TC, and the singular distribution $\sharp N^*C$ on C is called *characteristic distribution*. Notice that if the Poisson structure of P comes from a symplectic form Ω then the subbundle $\sharp N^*C$ is just the symplectic orthogonal of TC, so we are reduced to the usual definition of coisotropic submanifolds in the symplectic case. If a submanifold C intersects the symplectic leaves $\mathcal{O}$ of P cleanly, then C is coisotropic iff each intersection $C \cap \mathcal{O}$ is a coisotropic submanifold of the symplectic manifold $\mathcal{O}$. In algebraic terms we have the following characterization: a submanifold C is coisotropic iff $I_C := \{f \in C^\infty(P) : f|_C = 0\}$ is a Poisson subalgebra of $(C^\infty(P), \{\bullet, \bullet\}, \cdot)$.

In the following we want to motivate the naturality and importance of coisotropic submanifolds.

- Graphs of Poisson maps are coisotropic:

  Proposition 2.1 (Cor. 2.2.3 of [15]). *Let $\Phi \colon (P_1, \Pi_1) \to (P_2, \Pi_2)$ be a map between Poisson manifolds. Φ is a Poisson map (i.e. $\Phi_*(\Pi_1) = \Pi_2$) iff its graph is a coisotropic submanifold of $(P_1 \times P_2, \Pi_1 - \Pi_2)$.*

- Certain canonical quotients of coisotropic submanifolds are Poisson manifold: define $F_C := \{f \in C^\infty(P) : \{f, I_C\} \subset I_C\}$, the Poisson normalizer of I_C. It is a Poisson subalgebra of $C^\infty(P)$, and $I_C \subset F_C$ is a Poisson ideal. Further notice that F_C consists exactly of the functions on P whose differentials annihilate

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[1]In this case one says that $(C^\infty(P), \{\bullet, \bullet\}, \cdot)$ forms a Poisson algebra.

the characteristic distribution $\sharp N^*C$. Hence we have the following statements about the quotient of C by the characteristic distribution:

Proposition 2.2. F_C/I_C *inherits the structure of a Poisson algebra. Therefore* $\underline{C} := C/\sharp N^*C$, *if smooth, inherits the structure of a Poisson manifold so that* $C \to \underline{C}$ *is a Poisson map.*

Given any Poisson algebra A, one can ask whether it admits a deformation quantization, i.e. if it is possible to deform the commutative multiplication "in direction of the Poisson bracket" to obtain an associative product. Remarkable work of Kontsevich [11] showed that this is always possible if A is the algebra of functions on a smooth Poisson manifold. The Poisson algebras F_C/I_C provide natural and interesting instances of Poisson algebras which usually cannot be regarded as algebras of functions on a smooth manifold; the problem of their deformation quantization has been considered in [5, 4].

- Last, a coisotropic submanifold C gives rise to a Lie subalgebroid of the Lie algebroid associated to P. Recall that a *Lie algebroid* is a vector bundle $E \to P$ with a Lie bracket $[\bullet, \bullet]$ on its space of sections and a bundle map $\rho\colon E \to TP$ satisfying $[e_1, fe_2] = \rho(e_1)f \cdot e_2 + f[e_1, e_2]$; standard examples are tangent bundles and Lie algebras. Every Poisson manifold P induces the structure of a Lie algebroid on its cotangent bundle T^*P: the bracket is given by $[df, dg] = d\{f, g\}$ and the bundle map $T^*P \to TP$ by $-\sharp$. We have

  Proposition 2.3 (Cor. 3.1.5 of [15])**.** *If $C \subset P$ is coisotropic then the conormal bundle N^*C is a Lie subalgebroid of T^*P.*

3   Poisson-Dirac and cosymplectic submanifolds

In virtue of the question asked in the introduction it is necessary to determine which submanifolds $\tilde{P}$ of a Poisson manifold (P, Π) inherit a Poisson structure. Notice that, unlike symplectic forms, it is usually not possible to restrict a Poisson bivector field to a submanifold and obtain again a bivector field. However it is possible to view a Poisson bivector field as a Dirac structure [7], and Dirac structures restrict to (usually not smooth) Dirac structures on submanifolds. This point of view led to the definition below, which we phrase without reference to Dirac structures.

We first make the following remark, in which $(\mathcal{O}, \Omega)$ denotes a symplectic leaf of P and $\tilde{P} \subset P$ some submanifold: the linear subspace $T_p\tilde{P} \cap T_p\mathcal{O}$ of $(T_p\mathcal{O}, \Omega_p)$ is a symplectic subspace iff $\sharp N_p^*\tilde{P} \cap T_p\tilde{P} = \{0\}$. In this case $T\tilde{P}_p$ is endowed with a bivector field $\tilde{\Pi}_p$, obtained essentially inverting the non-degenerate form $\Omega_p|_{T_p\tilde{P} \cap T_p\mathcal{O}}$. Now we can make sense of the following definition (Cor. 11 of [8]):

Definition 3.1. A submanifold $\tilde{P}$ of P is called *Poisson-Dirac submanifold* if $\sharp N^*\tilde{P} \cap T\tilde{P} = \{0\}$ and the induced bivector field $\tilde{\Pi}$ on $\tilde{P}$ is a smooth.

In this case the bivector field is automatically integrable (Prop. 6 of [8]), so that $(\tilde{P}, \tilde{\Pi})$ is a Poisson manifold. Equivalently (Def. 4 of [8]) $\tilde{P}$ is a Poisson-Dirac submanifold if it admits a Poisson structure for which the symplectic leaves are (connected) intersections with the symplectic leaves $\mathcal{O}$ of P and so that the former are symplectic submanifolds of the $\mathcal{O}$. Notice the the inclusion $\tilde{P} \to P$ is usually not a Poisson map; it is iff $\tilde{P}$ is a Poisson submanifold, i.e. a smooth union of symplectic leaves.

A submanifold $\tilde{P}$ satisfying $T\tilde{P} \oplus N^*\tilde{P} = TP|_{\tilde{P}}$ is called *cosymplectic submanifold*. One can show that the induced bivector field $\tilde{\Pi}$ on $\tilde{P}$ is automatically smooth, hence cosymplectic submanifolds are always Poisson-Dirac submanifolds. The Poisson bracket on a cosymplectic submanifold $\tilde{P}$ is computed as follows: $\{\tilde{f}_1, \tilde{f}_2\}_{\tilde{P}}$ is the restriction to $\tilde{P}$ of $\{f_1, f_2\}$, where the f_i are extensions of $\tilde{f}_i$ to P such that $df_i|_{\sharp N^*\tilde{P}} = 0$.

If the Poisson structure on P comes from a symplectic 2-form, then the Poisson-Dirac and cosymplectic submanifolds are just the symplectic submanifolds.

4 Coisotropic embeddings in Poisson-Dirac submanifolds

Now we determine under what conditions on a submanifold $i\colon C \to P$ there exists a Poisson-Dirac submanifold $\tilde{P} \subset P$ so that C is coisotropic in $\tilde{P}$. We saw in the introduction that, when the Poisson structure on P comes from a symplectic form Ω, a sufficient and necessary condition is that $ker(i^*\Omega)$, which in terms of the Poisson tensor is $TC \cap \sharp N^*C$, has constant rank. In the general Poisson case however $TC \cap \sharp N^*C$, even when it has constant rank, might not be a smooth distribution on C. In the symplectic case $ker(i^*\Omega)$ has constant rank iff $TC + TC^\Omega$ has constant rank, and it turns out that this is the right condition to generalize to the Poisson case. This motivates

Definition 4.1 (Def. 2.2 of [6])**.** A submanifold C of a Poisson manifold (P, Π) is called *pre-Poisson* if the rank of $TC + \sharp N^*C$ is constant along C.

Such submanifolds were first considered in [1, 2].
We have

Theorem 4.2. *[Thm. 3.3 of [6]] Let C be a pre-Poisson submanifold of a Poisson manifold (P, Π). Then there exists a cosymplectic submanifold $\tilde{P}$ containing C such that C is coisotropic in $\tilde{P}$.*

Remark 4.3. The cosymplectic submanifold $\tilde{P}$ above is constructed by taking any complement $R \subset TP|_C$ of $TC + \sharp N^*C$ and "extending C along R".

There are submanifolds C which are not pre-Poisson but which still admit some Poisson-Dirac submanifold $\tilde{P}$ in which they embed coisotropically. This happens

for example if C has trivial intersection with the symplectic leaves of P (and the symplectic foliation of P is not regular): in this case $\tilde{P} := C$ is a Poisson-Dirac submanifold, the induced Poisson bivector field being zero.

However, if we ask that the submanifold $\tilde{P}$ be not just Poisson-Dirac but actually cosymplectic, then C is necessarily a pre-Poisson submanifold, and $\tilde{P}$ is constructed as described above (Lemma 4.1 of [6]).

The following are elementary examples of pre-Poisson submanifolds and of cosymplectic submanifolds in which they embed coisotropically. In section 5 we will give less trivial examples; see also Section 5 of [6].

Example 4.4. When C is a coisotropic submanifold of P, the construction of Thm. 4.2 delivers $\tilde{P} = P$ (or more precisely, a tubular neighborhood of C in P).

Example 4.5. When C is just a point x then the construction of Thm. 4.2 delivers as $\tilde{P}$ any slice through x transversal to the symplectic leaf $\mathcal{O}_x$.

Example 4.6. If $C_1 \subset P_1$ and $C_2 \subset P_2$ are pre-Poisson submanifolds of Poisson manifolds, the cartesian product $C_1 \times C_2 \subset P_1 \times P_2$ also is, and if the construction of Thm. 4.2 gives cosymplectic submanifolds $\tilde{P}_1 \subset P_1$ and $\tilde{P}_2 \subset P_2$, the same construction applied to $C_1 \times C_2$ (upon suitable choices of complementary subbundles) delivers the cosymplectic submanifold $\tilde{P}_1 \times \tilde{P}_2$ of $P_1 \times P_2$.

The following lemma will be useful in Section 5:

Lemma 4.7. *Let P_1, P_2 be Poisson manifolds and $f : P_1 \rightarrow P_2$ be a submersive Poisson morphism. If $C \subset P_2$ is a pre-Poisson submanifold then $f^{-1}(C)$ is a pre-Poisson submanifold of P_1. Further, if $\tilde{P}_2$ is a cosymplectic submanifold containing C as a coisotropic submanifold, then $f^{-1}(\tilde{P}_2)$ is a cosymplectic submanifold containing $f^{-1}(C)$ as a coisotropic submanifold.*

Proof. Let $y \in C$ and $x \in f^{-1}(y)$. Since

$$f_*(\sharp N_x^*(f^{-1}(C))) = f_*(\sharp f^*(N_y^* C)) = \sharp N_y^* C$$

it follows that the restriction of f_* to $T_x(f^{-1}(C)) + \sharp N_x^*(f^{-1}(C))$ has image $T_y C + \sharp N_y^* C$, whose rank is independent of $y \in C$ by assumption. Since the kernel of this restriction, being $T_x(f^{-1}(y))$, also has constant rank, it follows that $f^{-1}(C)$ is pre-Poisson.

Further it is clear that f_* maps a complement R_x of $T_x(f^{-1}(C)) + \sharp N_x^*(f^{-1}(C))$ in $T_x P_1$ isomorphically onto a complement R_y of $T_y C + \sharp N_y^* C$ in $T_y P_2$, so that $R_x + T_x(f^{-1}(C))$ is the pre-image of $R_y + T_y C$ under f_*. Using Remark 4.3 this proves the second assertion. $\square$

Now we address the uniqueness issue:

Theorem 4.8. *[Thm. 4.4 of [6]] Let C be a pre-Poisson submanifold (P, Π), and $\tilde{P}_0$, $\tilde{P}_1$ cosymplectic submanifolds that contain C as a coisotropic submanifold. Then, shrinking $\tilde{P}_0$ and $\tilde{P}_1$ to a smaller tubular neighborhood of C if necessary, there is a Poisson diffeomorphism Φ of P taking $\tilde{P}_0$ to $\tilde{P}_1$ and which is the identity on C.*

5 Duals of Lie algebras

In this subsection $\mathfrak{g}$ will always denote a finite dimensional Lie algebra. We saw in Section 2 that its dual $\mathfrak{g}^*$ is a Poisson manifold, whose Poisson bracket on linear functions (which can be identified with elements of $\mathfrak{g}$) is given by $\{g_1, g_2\} := [g_1, g_2]$. In the sequel we will need the notion of adjoint action of G on $\mathfrak{g}$, which is $Ad_g v := \frac{d}{dt}|_0 g \cdot exp(tv) \cdot g^{-1}$. Its derivative at the identity gives the Lie algebra action of $\mathfrak{g}$ on itself by $ad_w v := \frac{d}{dt}|_0 Ad_{exp(tw)} v = [w, v]$. We will also consider the (left) actions Ad^* and ad^* on $\mathfrak{g}^*$ obtained dualizing; the orbits of the coadjoint action Ad^* are exactly the symplectic leaves of the Poisson manifold $\mathfrak{g}^*$.

It is known that if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then its annihilator $\mathfrak{h}^\circ$ is a coisotropic submanifold of $\mathfrak{g}^*$ (also see Prop. 5.1 below). We want to look at two generalizations: the first considers affine subspaces obtained translating $\mathfrak{h}^\circ$; the second one is obtained weakening the condition that $\mathfrak{h}$ be a subalgebra.

The following proposition says that any translate of the annihilator of a Lie subalgebra is a pre-Poisson submanifold:

Proposition 5.1. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and fix $\lambda \in \mathfrak{g}^*$. Then the affine subspace $C := \mathfrak{h}^\circ + \lambda$ is always pre-Poisson, and it is coisotropic iff λ is a character of $\mathfrak{h}$ (i.e. by definition $\lambda \in [\mathfrak{h}, \mathfrak{h}]^\circ$).*

Proof. The restriction $f \colon \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map because $\mathfrak{h}$ is a Lie subalgebra. Every point ν of $\mathfrak{h}^*$ is a pre-Poisson submanifold (see Ex. 4.5), hence by Lemma 4.7 its pre-image $f^{-1}(\nu)$ (which will be a translate of $\mathfrak{h}^\circ$) is pre-Poisson. Notice that by Lemma 4.7 we also know that, for any slice $S \subset \mathfrak{h}^*$ transverse to the H-coadjoint orbit through ν, $f^{-1}(S)$ is a cosymplectic submanifold containing coisotropically $f^{-1}(\nu)$. Further from the proof of Lemma 4.7 it is clear that $f^{-1}(\nu)$ is coisotropic in $\mathfrak{g}^*$ iff $\{\nu\}$ is coisotropic in $\mathfrak{h}^*$, i.e. if ν is a fixed-point of the H-coadjoint action or equivalently $\nu|_{[\mathfrak{h}, \mathfrak{h}]} = 0$. $\square$

Remark 5.2. An alternative proof of Prop. 5.1 can be given using Lemma 5.4 below. Indeed any $x \in C$ can be written uniquely as $y + \lambda$ where $y \in \mathfrak{h}^\circ$. Notice that $ad_h^*(y) \in \mathfrak{h}^\circ$ for all $h \in \mathfrak{h}$, because $\langle ad_h^*(y), \mathfrak{h} \rangle = \langle y, [h, \mathfrak{h}] \rangle$ vanishes since $\mathfrak{h}$ is a subalgebra. Hence

$$T_x C + \sharp N_x^* C = \mathfrak{h}^\circ + \{ad_h^*(y) + ad_h^*(\lambda) : h \in \mathfrak{h}\} = \mathfrak{h}^\circ + ad_{\mathfrak{h}}^*(\lambda),$$

which is independent on the point x. From the first computation above (applied to λ instead of y) it is clear that $ad_{\mathfrak{h}}^*(\lambda) \in \mathfrak{h}^\circ$ iff $\lambda \in [\mathfrak{h}, \mathfrak{h}]^\circ$.

Example 5.3. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. In a suitable basis the Lie algebra structure is given by $[e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$. The symplectic leaves of $\mathfrak{g}^*$ are given essentially[2] by the connected components of level sets of the Casimir function

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[2]The cone is the union of 3 leaves, one being the origin.

$\nu_1^2 + \nu_2^2 - \nu_3^2$ (where ν_i is just e_i viewed as a linear function on $\mathfrak{g}^*$), and they consist of a family of two-sheeted hyperboloids, the cone $\nu_1^2 + \nu_2^2 - \nu_3^2 = 0$ and a family of one-sheeted hyperboloids [3]. $C := \{(0,t,t) : t \in \mathbb{R}\} \subset \mathfrak{g}^*$ is contained in the cone and is clearly a coisotropic submanifold; indeed it is the annihilator of the Lie subalgebra $\mathfrak{h} := span\{e_1, e_2 - e_3\}$ of $\mathfrak{g}$. If we translate C by an element in the annihilator of $[\mathfrak{h}, \mathfrak{h}] = \mathbb{R}(e_2 - e_3)$ we obtain an affine line contained in one of the hyperboloids, which hence is lagrangian in there and therefore coisotropic in $\mathfrak{g}^*$. If we translate C by any other $\lambda \in \mathfrak{g}^*$ we obtain a line that intersects transversely the hyperboloids, so at every point of such a line C' we have $TC' + \sharp N^*C' = T\mathfrak{g}^*$, showing that C' is pre-Poisson.

Before considering the case when $\mathfrak{h}$ is *not* a subalgebra of $\mathfrak{g}$ we need a

Lemma 5.4. *Let $C \subset \mathfrak{g}^*$ be an affine subspace obtained by translating the annihilator of some linear subspace $\mathfrak{h} \subset \mathfrak{g}$. Then $\sharp N_x^* C = ad_{\mathfrak{h}}^*(x) := \{ad_h^*(x) : h \in \mathfrak{h}\}$ for all $x \in C$.*

Proof. N_x^*C is given by the differentials at x of the functions $h \in \mathfrak{h} \subset C^\infty(\mathfrak{g}^*)$. Now for any $g \in \mathfrak{g}$ we have

$$\langle \sharp d_x h, g \rangle = d_x g(\sharp d_x h) = \{h, g\}(x) = \langle [h, g], x \rangle = \langle ad_h^*(x), g \rangle,$$

i.e. $\sharp d_x h = ad_h^*(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we consider the case when $\mathfrak{h}$ is just a linear subspace of $\mathfrak{g}$ and $\mathfrak{h}^\circ \subset \mathfrak{g}^*$ its dual. Since the Poisson tensor of $\mathfrak{g}^*$ vanishes at the origin we have $T(\mathfrak{h}^\circ) + \sharp N^*(\mathfrak{h}^\circ) = T(\mathfrak{h}^\circ)$ at the origin, so $\mathfrak{h}^\circ$ is pre-Poisson iff it is coisotropic (i.e. if $\mathfrak{h}$ is a Lie subalgebra). However the open subset C of $\mathfrak{h}$ on which $T(\mathfrak{h}^\circ) + \sharp N^*(\mathfrak{h}^\circ)$ has maximal rank will be pre-Poisson. Further, shrinking C if necessary, we can find a subspace $R \subset \mathfrak{g}^*$ (independent of $x \in C$) with $R \oplus (T_x C + \sharp N_x^* C) = \mathfrak{g}^*$ for all $x \in C$. For example we can construct such an R at one point $\bar{x}$ of C, and since transversality is an open condition, R will be transverse to $TC + \sharp N^*C$ in a neighborhood of $\bar{x}$ in C. By Thm. 4.2 an open subset $\tilde{P}$ (containing C) of the subspace $\mathfrak{p}^\circ := R \oplus C$ is cosymplectic. If we *assume* that $\sharp N_y^* \tilde{P}$ is independent of the footpoint $y \in \tilde{P}$ then we are in the situation of the following proposition.

Proposition 5.5. *Let $\mathfrak{p}$ be a linear subspace of $\mathfrak{g}$ such that an open subset $\tilde{P} \subset \mathfrak{p}^\circ$ is cosymplectic and $\mathfrak{k}^\circ := \sharp N_y^* \tilde{P}$ is independent of $y \in \tilde{P}$. Then $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Hence, whenever $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $(\mathfrak{k}, \mathfrak{p})$ forms a symmetric pair [10].*

Proof. The fact that $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ follows from $\mathfrak{k}^\circ \oplus \mathfrak{p}^\circ = \mathfrak{g}^*$, which holds because $\tilde{P}$ is cosymplectic. Recall that given functions f_1, f_2 on $\tilde{P}$, the bracket $\{f_1, f_2\}_{\tilde{P}}$ is obtained by extending the functions in a constant way along $\mathfrak{k}^\circ$ to obtain functions $\hat{f}_1, \hat{f}_2$ on $\mathfrak{g}^*$, taking their Poisson bracket and restricting to $\tilde{P}$. Further (see Cor. 2.11

of [16]) the differential of $\{\hat{f}_1, \hat{f}_2\}$ at any point of $\tilde{P}$ kills $\mathfrak{k}^\circ$. So if the f_i are restrictions of linear functions on $\mathfrak{p}^\circ$ then the $\hat{f}_i$ will be linear functions on $\mathfrak{g}^*$ corresponding to elements of $\mathfrak{k}$, and $\{\hat{f}_1, \hat{f}_2\}$, which is a linear function on $\mathfrak{g}^*$, will also correspond to an element of $\mathfrak{k}$. We deduce that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ (and that the Poisson structure on $\tilde{P}$ induced from $\mathfrak{g}^*$ is the restriction of a linear Poisson structure on $\mathfrak{p}^\circ$).

To show $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ pick any $k \in \mathfrak{k}, p \in \mathfrak{p}$ and $y \in \tilde{P}$. Then $\langle [k, p], y \rangle = -\langle k, ad_p^*(y) \rangle = \langle k, \sharp d_y p \rangle = 0$, using Lemma 5.4 in the second equality, because $\sharp d_y p \subset \sharp N_y^* \tilde{P} = \mathfrak{k}^\circ$. This shows that $[k, p]$ annihilates $\tilde{P}$, hence it must annihilate its span $\mathfrak{p}^\circ$. $\qquad \square$

Remark 5.6. The text preceding Prop. 5.5 and the proposition itself give a way to start with a simple piece of data (a subspace of $\mathfrak{g}$) and, in favorable cases, obtain a decomposition $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ where $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. If $\mathfrak{g}$ admits a non-degenerate Ad-invariant bilinear form B, then the B-orthogonal $\mathfrak{p}$ of any subalgebra $\mathfrak{k}$ satisfies $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, because for any $k, k' \in \mathfrak{k}$ and $p \in \mathfrak{p}$ we have $B([k, p], k') = -B(p, [k, k']) = 0$. If B is positive-definite (such a B exists for example if the simply connected Lie group integrating $\mathfrak{g}$ is compact), then we clearly also have $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$. Hence for such Lie algebras one obtains the kind of decomposition of Prop. 5.5 in a much easier way.

A converse statement to Prop. 5.5 is given by

Proposition 5.7. *Assume that $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and there exists a point $y \in \mathfrak{p}^\circ$ at which none of the fundamental vector fields $\frac{d}{dt}|_0 Ad_{exp(tp)}^*(y)$ vanish, where p ranges over $\mathfrak{p} \setminus \{0\}$. Then there is an open subset $\tilde{P} \subset \mathfrak{p}^\circ$ which is cosymplectic and $\mathfrak{k}^\circ := \sharp N_x^* \tilde{P}$ is independent of $x \in \tilde{P}$. (Hence applying Prop. 5.5 it follows that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$).*

Proof. For all $x \in \mathfrak{p}^\circ$ we have $\sharp N_x^*(\mathfrak{p}^\circ) = ad_{\mathfrak{p}}^*(x) \subset \mathfrak{k}^\circ$, as can be seen using $\langle ad_p^*(x), \mathfrak{k} \rangle = \langle x, [p, \mathfrak{k}] \rangle = 0$ for all $p \in \mathfrak{p}$ (which holds because of $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$). The assumption on the coadjoint action at y means that the map $\mathfrak{p} \to \mathfrak{g}^*, p \mapsto ad_p^*(y)$ is injective; by continuity it is injective also on an open subset $\tilde{P} \subset \mathfrak{p}^\circ$, and by dimension counting we get $\sharp N_x^*(\mathfrak{p}^\circ) = \mathfrak{k}^\circ$ on $\tilde{P}$. $\qquad \square$

Now we display an example for Prop. 5.5

Example 5.8. Let $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the non-degenerate (indefinite) inner product $(A, B) = Tr(A \cdot B)$. Since it is Ad-invariant, the action of ad_X and ad_X^* on $\mathfrak{g}$ and $\mathfrak{g}^*$ are intertwined (up to sign).

Now take $\mathfrak{h} = \{ \left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix} \right) : b, c, d \in \mathbb{R} \}$, which is not a subalgebra. Its annihilator is identified with the line C spanned by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$. Since C is one-dimensional and the Poisson structure on $\mathfrak{g}^*$ linear it is clear that $\sharp N_x^* C$ is independent of $x \in C \setminus \{0\}$ and $C \setminus \{0\}$ is pre-Poisson. Using Lemma 5.4 we compute $\sharp N_x^* C = \{ \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix} \right) : b, c \in \mathbb{R} \}$, so as complement R to $T_x C + \sharp N_x^* C$ we can take the line spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Then

$\mathfrak{p}^\circ := R \oplus C$ is given by the diagonal matrices, and $\mathfrak{p} \subset \mathfrak{g}$ is given by matrices with only zeros on the diagonal. For any $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \mathfrak{p}^\circ$ we compute $\sharp N^*_{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)}\mathfrak{p}^\circ$ using Lemma 5.4 and obtain the set of matrices with only zeros on the diagonal if $a \neq d$ and $\{0\}$ otherwise. So the open set $\tilde{P}$ on which $\mathfrak{p}^\circ$ is cosymplectic is a plane with a line removed, and $\mathfrak{k}^\circ := \sharp N^*_{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)}\tilde{P}$ is independent of the footpoint $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \tilde{P}$. $\mathfrak{k} \subset \mathfrak{g}$ coincides hence with the set of diagonal matrices. As predicted by Lemma 5.5 $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$; one can check easily that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ too.

Since $\mathfrak{k}$ is abelian, the linear Poisson structure induced on $\tilde{P}$ is the zero Poisson structure. This can be seen also looking at the explicit Poisson structure on $\mathfrak{g}^*$, which with respect to the coordinates given by the basis $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $c = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $d = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathfrak{g}^*$ is

$$-b\partial_a \wedge \partial_b + c\partial_a \wedge \partial_c + (d-a)\partial_b \wedge \partial_c - b\partial_b \wedge \partial_d + c\partial_c \wedge \partial_d.$$

Indeed at a point $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ of $\mathfrak{p}^\circ$ the bivector field reduces to $(d-a)\partial_b \wedge \partial_c$. Finally remark that if we had chosen R to be spanned by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ instead we would have obtained as $\sharp N^*_{\left(\begin{smallmatrix} a & b \\ 0 & b \end{smallmatrix}\right)}\mathfrak{p}^\circ$ the span of $\left(\begin{smallmatrix} -b & b \\ a-b & b \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & b-a \\ 0 & 0 \end{smallmatrix}\right)$, which obviously is not constant on any open subset of $\mathfrak{p}^\circ$

6   Subgroupoids associated to pre-Poisson submanifolds

In Section 2 we defined Lie algebroids and recalled that for every Poisson manifold P there is an associated Lie algebroid, namely the cotangent bundle T^*P.

In analogy to the fact that finite dimensional Lie algebras integrate to Lie groups (uniquely if required to be simply connected), Lie algebroids - when integrable - integrate to objects called *Lie groupoids*. Recall that a Lie groupoid over P is given by a manifold Γ endowed with surjective submersions $\mathbf{s},\mathbf{t}$ (called source and target) to the base manifold P, a smooth associative multiplication defined on elements $g, h \in \Gamma$ satisfying $\mathbf{s}(g) = \mathbf{t}(h)$, an embedding of P into Γ as the spaces of "identities" and a smooth inversion map $\Gamma \to \Gamma$; see for example [14] for the precise definition. The total space of the Lie algebroid associated to the Lie groupoid Γ is $ker(\mathbf{t}_*|_P) \subset T\Gamma|_P$, with a bracket on sections defined using left invariant vector fields on Γ and $\mathbf{s}_*|_P$ as anchor. A Lie algebroid A is said to be integrable if there exists a Lie groupoid whose associated Lie algebroid is isomorphic to A; in this case there is a unique (up to isomorphism) integrating Lie groupoid with simply connected source fibers.

The cotangent bundle T^*P of a Poisson manifold P carries more data then just a Lie algebroid structure; when it is integrable, the corresponding Lie groupoid Γ is actually a *symplectic groupoid* [12], i.e. [14] there is a symplectic form Ω on Γ such that the graph of the multiplication is a lagrangian submanifold of $(\Gamma \times \Gamma \times \Gamma, \Omega \times \Omega \times (-\Omega))$. Ω is uniquely determined (up to symplectic groupoid automorphism) by the requirement that $\mathbf{t} \colon \Gamma \to P$ be a Poisson map; a canonical Lie algebroid isomorphism

between T^*P and $ker(\mathbf{t}_*|_P)$ is then given by mapping du (for u a function on P) to the hamiltonian vector field $-X_{\mathbf{s}^*u}$. For example, if P carries the zero Poisson structure, then the symplectic groupoid is T^*P with the canonical symplectic structure and fiberwise addition as multiplication. We will describe in Example 6.2 below the symplectic groupoid of the dual of a Lie algebra.

In this Section we want to investigate how a pre-Poisson submanifold C of a Poisson manifold (P, Π) gives rise to subgroupoids of the source simply connected symplectic groupoid Γ (assuming that T^*P is an integrable Lie algebroid). By Prop. 3.6 of [6] $N^*C \cap \sharp^{-1}TC$ is a Lie subalgebroid of T^*P. When $\sharp N^*C$ has constant rank there is another Lie subalgebroid associated to C, namely $\sharp^{-1}TC = (\sharp N^*C)^\circ$. We want to describe the subgroupoids[3] of Γ integrating $N^*C \cap \sharp^{-1}TC$ and $\sharp^{-1}TC$.

Proposition 6.1. *[Prop. 7.2 of [6]] Let C be a pre-Poisson submanifold of (P, Π). Then the subgroupoid of (Γ, Ω) integrating $N^*C \cap \sharp^{-1}TC$ is an isotropic subgroupoid.*

We exemplify Prop. 6.1 considering the dual of a Lie algebra $\mathfrak{g}$ as a Poisson manifold, as in Section 5. The symplectic groupoid of $\mathfrak{g}^*$ (see Ex. 3.1 of [14]) is T^*G with its canonical symplectic form, where G is the simply connected Lie group integrating $\mathfrak{g}$. To describe the groupoid structure we identify T^*G with $\mathfrak{g}^* \times G$ by (the cotangent lift of) right translation. Then the target map $\mathfrak{g}^* \times G \to \mathfrak{g}^*$ is $\mathbf{t}(\xi, g) = \xi$ and the source map is $\mathbf{s}(\xi, g) = Ad^*_{g^{-1}}\xi$, and the multiplication is $(\xi, g_1) \cdot (Ad^*_{g^{-1}}\xi, g_2) = (\xi, g_1g_2)$.

Example 6.2. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$. By Prop. 5.1 we know that $C := \mathfrak{h}^\circ + \lambda$ is a pre-Poisson submanifold of $\mathfrak{g}^*$. We claim here that the subgroupoid of $\mathfrak{g}^* \times G$ integrating the Lie subalgebroid $N^*C \cap \sharp^{-1}TC$ is $C \times D$, where the subgroup $D \subset G$ is the connected component of the identity of $\{g \in H : (Ad^*_g\lambda)|_{\mathfrak{h}} = \lambda|_{\mathfrak{h}}\}$. By Prop. 6.1 we know that it is an isotropic subgroupoid.

To prove our claim, we first make the Lie subalgebroid more explicit: for all $x \in C$ using Remark 5.2 we have

$$N^*_xC \cap \sharp^{-1}T_xC = (\mathfrak{h}^\circ + ad^*_{\mathfrak{h}}(\lambda))^\circ = \mathfrak{h} \cap \{v \in \mathfrak{g} : (ad^*_v\lambda)|_{\mathfrak{h}} = 0\} =: \mathfrak{d},$$

so that the Lie subalgebroid $N^*C \cap \sharp^{-1}TC \subset T^*\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}$ is just the product $C \times \mathfrak{d}$. The canonical Lie algebroid isomorphism $T^*P \cong ker(\mathbf{t}_*|_P), du \mapsto -X_{\mathbf{s}^*u}$ is just the identity on $\mathfrak{g}^* \times \mathfrak{g}$, as can be checked using the explicit formula for the symplectic form on the groupoid $\mathfrak{g}^* \times G$ given in Ex. 3.1 of [14]. Now notice that the Lie subalgebra $\mathfrak{d}$ integrates to the connected subgroup D defined above. Using the definition of D one checks that $\mathbf{t}$ and $\mathbf{s}$ map $C \times D$ into C, and the fact that D is a subgroup allows us to check that $C \times D$ is actually a Lie subgroupoid of $\mathfrak{g}^* \times G$, proving our claim.

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[3]Here, for any Lie subalgebroid A of T^*P integrating to a source simply connected Lie groupoid H, we take "subgroupoid" to mean the (usually just immersed) image of the (usually not injective) morphism $H \to \Gamma$ induced from the inclusion $A \to T^*P$.

Now we consider $\sharp^{-1}TC$ and assume that it has constant rank, or equivalently that the characteristic distribution $TC \cap \sharp N^*C$ have constant rank[4]. Then $\sharp^{-1}TC$ is a Lie subalgebroid of T^*P, and quoting part of Prop. 7.2 of [6]:

Proposition 6.3. *The subgroupoid of Γ integrating $\sharp^{-1}TC$ is $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$, and it is a presymplectic submanifold of (Γ, Ω).*

Remark 6.4. In this case the foliation integrating the characteristic distribution of $\mathbf{s}^{-1}(C) \cap \mathbf{t}^{-1}(C)$ (i.e. the kernel of the pullback of Ω) is given by the orbits of the action by right and left multiplication of the source-connected isotropic subgroupoid integrating $N^*C \cap \sharp^{-1}TC$.

Example 6.5. Let C be a submanifold of $\mathfrak{g}^*$ such that $T_xC \cap T_x\mathcal{O} = \{0\}$ at every point x where C intersects a coadjoint orbit $\mathcal{O}$. Then C is pre-Poisson iff $\sharp^{-1}TC$ has constant rank, which in this case just means that the coadjoint orbits that C intersects all have the same dimension. By the above proposition the source connected subgroupoid of $\mathfrak{g}^* \times G$ integrating $\sharp^{-1}TC$ is $\{(x,g) : x \in C, Ad^*_{g^{-1}}(x) = x\}$, a bundle of groups integrating a bundle of isotropy Lie algebras of the coadjoint action. We also have the following alternative description for this bundle of Lie algebras, which sometimes is more convenient for computations: $\sharp^{-1}T_xC = (\sharp N_x^*C)^\circ$ can be described as $N_x^*\mathcal{O}$, for $\mathcal{O}$ the coadjoint orbit through x.

If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$, we know that $C := \mathfrak{h}^\circ + \lambda$ is a pre-Poisson submanifold of $\mathfrak{g}^*$, but generally $\sharp^{-1}TC$ does not have constant rank. A case where it does have constant rank is the following. As in Example 5.3 consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and the pre-Poisson submanifold $C := \{(0, t, t+1) : t \in \mathbb{R}\}$. As remarked there C intersects transversely the symplectic leaves of $\mathfrak{g}^*$, which are the level sets of the Casimir function $\nu_1^2 + \nu_2^2 - \nu_3^2$. At $x = (0, t, t+1)$ we have $N_x^*\mathcal{O} = \mathbb{R}(t d\nu_2 - (t+1)d\nu_3)$, which in terms of the basis $e_1 = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $e_2 = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $e_3 = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of $\mathfrak{sl}(2, \mathbb{R})$ used in Example 5.3 is $\mathbb{R}\left(\begin{smallmatrix} t & -(t+1) \\ t+1 & -t \end{smallmatrix}\right)$. As seen above, integrating these Lie algebras to subgroups of G (the simply connected Lie group integrating $\mathfrak{sl}(2, \mathbb{R})$) we obtain the source connected subgroupoid of $\mathfrak{g}^* \times G$ integrating $\sharp^{-1}TC$.

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[4]Indeed more generally we have the following for any submanifold C of P: if any two of $\sharp^{-1}TC$, $\sharp N^*C + TC$ or $TC \cap \sharp N^*C$ have constant rank, then the remaining one also has constant rank. This follows trivially from $rk(\sharp N^*C + TC) = rk(\sharp N^*C) + \dim C - rk(TC \cap \sharp N^*C)$.

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