Localization-Delocalization Phenomena for Random Interfaces

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Abstract

We consider d-dimensional random surface models which for $d = 1$ are the standard (tied-down) random walks (considered as a random "string"). In higher dimensions, the one-dimensional (discrete) time parameter of the random walk is replaced by the d-dimensional lattice $\mathbb{Z}^d$, or a finite subset of it. The random surface is represented by real-valued random variables ϕ_i, where $i \in \mathbb{Z}^d$. A class of natural generalizations of the standard random walk are gradient models whose laws are (formally) expressed as

$$P\left(d\phi\right) = \frac{1}{Z} \exp\left[-\sum\nolimits_{|i-j|=1} V\left(\phi_i - \phi_j\right)\right] \prod_i d\phi_i,$$

$V : \mathbb{R} \to \mathbb{R}^+$, convex, and with some growth conditions.

Such surfaces have been introduced in theoretical physics as (simplified) models for random interfaces separating different phases. Of particular interest are localization-delocalization phenomena, for instance for a surface interacting with a wall by attracting or repulsive interactions, or both together. Another example are so-called heteropolymers which have a noise-induced interaction.

Recently, there had been developments of new probabilistic tools for such problems. Among them are:

- Random walk representations of Helffer-Sjöstrand type,
- Multiscale analysis,
- Connections with random trapping problems and large deviations.

We give a survey of some of these developments.

2000 Mathematics Subject Classification: 60.

1.   Introduction

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Gradient models are an important class of random interfaces and random surfaces. In the mathematical physics literature they are often called "effective interface models". The (discrete) random surface is described by random variables $(\phi_x)_{x \in V}$, where V is $\mathbb{Z}^d$ or a subset of it. The ϕ_x itself are either $\mathbb{Z}$-valued or $\mathbb{R}$-valued. We will mainly concentrate on the latter situation which is easier in some respects. If V is a finite subset of $\mathbb{Z}^d$, the law P_V of $\phi = (\phi_x)_{x \in V}$ is described via a Hamiltonian

$$H_V(\phi) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{x,y \in V} p(y - x) U(\phi_x - \phi_y) + \sum_{x \in V,\, y \notin V} p(y - x) U(\phi_x), \qquad (1)$$

where $U : \mathbb{R} \to \mathbb{R}^+$ is symmetric and convex, and $p : \mathbb{Z}^d \to [0, 1]$ is a symmetric probability distribution on $\mathbb{Z}^d$. The above choice of the Hamiltonian corresponds to 0 boundary conditions. Of course, one can consider more general ones, where the second summand is replaced by $\sum_{x \in V,\, y \notin V} p(y - x) U(\phi_x - \psi_y)$, ψ being a configuration outside V. We will be mainly interested in the nearest neighbor case $p(x) = 1/2d$, for $|x| = 1$, and $p(x) = 0$ otherwise, but more general conditions can also be considered. We always assume that the matrix $\mathcal{Q} = (q_{ij})$ given by

$$q_{ij} \stackrel{\text{def}}{=} \sum_x x_i x_j p(x) \qquad (2)$$

is positive definite, and that p has exponentially decaying tails. Furthermore, the random walk $(\eta_t)_{t \in \mathbb{N}}$ with transition probabilities p is assumed to be irreducible. The Hamiltonian defines a probability distribution on $\mathbb{R}^V$ by

$$P_V(d\phi) \stackrel{\text{def}}{=} \frac{1}{Z_V} \exp\left[-H_V(\phi)\right] \prod_{x \in V} d\phi_x, \qquad (3)$$

where $d\phi_x$ denotes the Lebesgue measure. Z_V is the norming constant

$$Z_V \stackrel{\text{def}}{=} \int_{\mathbb{R}^V} \exp\left[-H_V(\phi)\right] \prod_{x \in V} d\phi_x. \qquad (4)$$

In the one-dimensional case $d = 1$, P_V is the law of a tied down random walk: Let ξ_i, $i \geq 1$, be i.i.d. random variables with the density $\text{const} \times \mathrm{e}^{-U(x)}$. If $V = \{1, \ldots, n\}$, then P_V is the law of the sequence $\left(\sum_{j=1}^i \xi_j\right)_{1 \leq i \leq n}$, conditioned on $\sum_{j=1}^{n+1} \xi_j = 0$.

A special case is the *harmonic* one with $U(x) = x^2/2$. Then P_V is a Gaussian measure on $\mathbb{R}^V$ which is centered for 0-boundary conditions. We usually write P_V^{harm} in this case. The law is therefore given by its covariances

$$\gamma_V(x, y) \stackrel{\text{def}}{=} \int \phi_x \phi_y \, dP_V^{\text{harm}}.$$

These covariances have a random walk representation: If V is a finite set then

$$\gamma_V(x, y) = \mathbb{E}_x \left(\sum_{s=0}^{\tau_V - 1} 1_y(\eta_s)\right), \qquad (5)$$

where $(\eta_s)_{s\geq 0}$ under $\mathbb{P}_x$ is a discrete time random walk on $\mathbb{Z}^d$ starting at x and with transition probabilities $\mathbb{P}_x(\eta_1 = y) = p(y - x)$. τ_V is the first exit time from V. As a consequence of this representation one sees that the thermodynamic limit

$$P^{\text{harm}}_\infty \overset{\text{def}}{=} \lim_{n\to\infty} P^{\text{harm}}_{V_n}, \ V_n \overset{\text{def}}{=} \{-n, -n+1, \ldots, n\}^d \tag{6}$$

exists for $d \geq 3$. P^{harm}_∞ is the centered Gaussian measure on $\mathbb{R}^{\mathbb{Z}^d}$ whose covariances are given by the Green's function of the random walk. It is important to notice that this random field has slowly decaying correlations:

$$\gamma_\infty(x, y) \approx \frac{\text{const}}{|x - y|^{d-2}}, \ |x - y| \to \infty.$$

For $d = 2$, the thermodynamic limit does not exist, and in fact

$$E^{\text{harm}}_{V_n}(\phi_0^2) \approx \text{const} \times \log n, \ n \to \infty.$$

For $d = 1$, the variance grows of course like n in the bulk. The harmonic surface is therefore localized for $d \geq 3$, but not for $d = 1, 2$.

Many of these properties carry over to non-harmonic cases with a convex and symmetric interaction function U in (1). Of particular importance is that there is a generalization of the representation (5), the Helffer-Sjöstrand representation, see [26]. The random walk (η_s) has to be replaced by a random walk in a dynamically changing random environment. Using this representation, many of the results for the harmonic case can be generalized to the case of a convex U, although often not in a quantitatively as precise form as in the harmonic case. For a probabilistic description of the Helffer-Sjöstrand representation, see [20].

The main topic of this paper are effects arising from interactions of the random surface (ϕ_x) with a "wall". The simplest case of such a wall is the configuration $\phi \equiv 0$. There are many type of interactions which had been considered in the literature, both in physics and in mathematics. The simplest one is a local attraction of the surface to this wall. It turns out that an arbitrary weak attraction localizes the random field in a strong sense, and in all dimensions. This will be discussed in a precise way in Section 2.. Interesting localization-delocalization phenomena may occur when mixed attractive and repulsive interactions are present, with phase transitions depending on the parameters regulating the strength of the interactions. Naturally, these phenomena are best understood for the one-dimensional case. A simple example is the following one, which is discussed in details in [25]: Let $\phi_0 = 0, \phi_1, \ldots, \phi_{2n-1}, \phi_{2n} = 0$ be a discrete time $\mathbb{Z}$-valued, and tied-down, simple random walk, i.e. P_n is simply the uniform distribution on all such paths which satisfy $|\phi_x - \phi_{x-1}| = 1$. Introducing an arbitrary pinning to the wall in the form

$$\hat{P}_{n,\beta}(\phi) = \frac{1}{\hat{Z}_{n,\beta}} \exp\left[\beta \sum_{x=1}^{2n-1} 1_0(\phi_x)\right] P_n(\phi), \ \beta > 0$$

strongly localizes the "random string", i.e. $\sup_{n,x} \hat{E}_{n,\beta}(\phi_x^2) < \infty$ holds for all $\beta > 0$. Furthermore, the correlations $\hat{E}_{n,\beta}(\phi_x \phi_y)$ are exponentially decaying in $|x - y|$, uniformly in n. These facts are easily checked.

On the other hand, if the string is confined to be on one side of the wall, the situation is completely different. Let $\Omega_{2n}^{+} \stackrel{\text{def}}{=} \{\phi : \phi_x \geq 0,\ 1 \leq x \leq 2n-1\}$, and $\hat{P}_{n,\beta}^{+}(\cdot) \stackrel{\text{def}}{=} \hat{P}_{n,\beta}\left(\cdot \mid \Omega_{2n}^{+}\right)$. Then there is a critical $\beta_c > 0$ such that the above localization property holds for $\beta > \beta_c$, but not for $\beta < \beta_c$, where the path measure converges, after Brownian rescaling, to the Brownian excursion. For a proof of this so called "wetting transition", see [25]. More precise information has been obtained recently in this one-dimensional situation in [28].

There are similar phase transitions for more complicated models. Some of them will be discussed in Section 3. and Section 4.. We begin in the next section by discussing the pinning effect alone mainly in the difficult two-dimensional case.

2.   Pinning of two-dimensional gradient fields

We consider now a gradient field (3), but we modify it by introducing an attractive local pinning to the wall $\{\phi \equiv 0\}$. This is often done by modifying the Hamiltonian in the following way: Let $\psi : \mathbb{R} \to \mathbb{R}^{-}$ be symmetric and with compact support. Then we put

$$H_{V,\psi}(\phi) = H_V(\phi) + \sum_{x \in V} \psi(\phi_x). \tag{7}$$

Evidently, the corresponding finite volume Gibbs measure favours surfaces which have the tendency to stick close to the wall. It should be emphasized that this is a much weaker attraction than in a so-called massive field, where one takes ψ to be convex, for instance $\psi(x) = x^2$. A formally slightly easier model can be obtained by not changing the Hamiltonian, but replacing the Lebesgue measure as the reference measure by a mixture of the Lebesgue measure and a Dirac measure at 0. This corresponds to the following probability measure on $\mathbb{R}^V$:

$$\hat{P}_{V,\varepsilon}(d\phi) \stackrel{\text{def}}{=} \frac{1}{\hat{Z}_{V,\varepsilon}} \exp\left[-H_V(\phi)\right] \prod_{x \in V} \left(d\phi_x + \varepsilon\delta_0(d\phi_x)\right),\ \varepsilon > 0. \tag{8}$$

This measure can be obtained from measures defined by the Hamiltonian (7) via an appropriate limiting procedure. The nice feature of (8) is that $\hat{P}_{V,\varepsilon}$ can trivially be expanded into a mixture of "free" measures: We just have to expand out the product:

$$\hat{P}_{V,\varepsilon}(d\phi) = \sum_{A \subset V} \varepsilon^{|V \setminus A|} \frac{Z_A}{\hat{Z}_{V,\varepsilon}} \frac{1}{Z_A} \exp\left[-H_V(\phi)\right] \prod_{x \in A} d\phi_x \prod_{x \in V \setminus A} \delta_0(d\phi_x) \tag{9}$$

$$= \sum_{A \subset V} \varepsilon^{|V \setminus A|} \frac{Z_A}{\hat{Z}_{V,\varepsilon}} P_A(d\phi),$$

where P_A is the measure defined by (3), extended by 0 outside A. Remark that

$$\nu_{V,\varepsilon}(A) \stackrel{\text{def}}{=} \varepsilon^{|V \setminus A|} \frac{Z_A}{\hat{Z}_{V,\varepsilon}}$$

defines a probability distribution on the set of subsets of V. Therefore, we have represented $\hat{P}_{V,\varepsilon}$ as a mixture of free measures P_A. It should be remarked that similar but technically more involved expansions are possible also in the case of the Hamiltonian (7). The case of $\psi(x) = -a1_{[-b,b]}(x)$ is discussed in [10]. Probably, more general cases could be handled with the help of the Brydges-Fröhlich-Spencer random walk representation (see [15]), but the results presented here have not been derived in this more general case. For the sake of simplicity, we stick here to the δ-pinning case (8).

The above representation easily leads to a representation of the covariances of the pinned field. This is particularly simple in the harmonic case $U(x) = x^2/2$, where one gets

$$\int \phi_x \phi_y \hat{P}_{V,\varepsilon}^{\mathrm{harm}}(d\phi) = \sum_{A \subset V} \nu_{V,\varepsilon}(A) \, \mathbb{E}_x \left(\sum_{s=0}^{\tau_A - 1} 1_y(\eta_s) \right).$$

The problem is therefore reduced to a problem of a random walk among random traps: The distribution $\nu_{V,\varepsilon}$ defines a random trapping configuration, let's denote it by $\mathcal{A}$, i.e. $P_{\mathrm{trap}}(\mathcal{A} = V^c \cup (V \backslash A)) \stackrel{\mathrm{def}}{=} \nu_{V,\varepsilon}(A)$, and the covariances of our pinned measure are given in terms of the discrete Green's function among these random traps which are killing the random walk when it enters one of these traps. A difficult point is a precise analysis of the distribution of $\mathcal{A}$, and a crucial step is a comparison with Bernoulli measures. The two-dimensional case is the most difficult one. In three and more dimensions, a comparison of the distribution of $\mathcal{A}$ with a Bernoulli measure is quite easy.

It turns out that the pinning localizes the field in a strong sense. First of all, the variance of the variables stay bounded as $V \uparrow \mathbb{Z}^d$. Secondly, there is exponential decay of the covariances, uniformly in V. Results of this type have a long history. For $d \geq 3$, and for the harmonic case with pinning of the type (7), the localization has been obtained in [15]. In [24], boundedness of the absolute first moment has been proved for $d = 2$, but no exponential decay of the correlations. The first proof of exponential decay of correlations in the two-dimensional case has been obtained in [7] for the harmonic case. One drawback of the method used there was that it uses reflection positivity, which holds only under restrictive assumptions on p. Also, periodic boundary conditions are required, and so the results are not directly valid for the 0-boundary case. A satisfactory approach had then been obtained in [21] and [27]. The quantitatively precise results presented here are from [10], where the critical exponents for the depinning transition $\varepsilon \to 0$ have been derived, including the correct log-corrections for $d = 2$.

We define the mass $m_\varepsilon(x)$, $x \in S^{d-1}$, by

$$m_\varepsilon(x) \stackrel{\mathrm{def}}{=} - \lim_{k \to \infty} \frac{1}{k} \log \lim_{V \uparrow \mathbb{Z}^d} \hat{E}_{V,\varepsilon}\left(\phi_0 \phi_{[kx]}\right).$$

The most precise results we have are for the harmonic case:

Theorem 1   *a) If $d = 2$, then for small enough ε :*

$$\left| \lim_{V \uparrow \mathbb{Z}^d} \hat{E}_{V,\varepsilon}^{\mathrm{harm}}\left(\phi_0^2\right) - \frac{|\log \varepsilon|}{2\pi \sqrt{\det \mathcal{Q}}} \right| \leq \mathrm{const} \times \log |\log \varepsilon|$$

b) *If $d = 2$, then for all $x \in S^{d-1}$ and small enough ε :*

$$\text{const} \times \frac{\sqrt{\varepsilon}}{|\log \varepsilon|^{3/4}} \leq m_\varepsilon^{\text{harm}}(x) \leq \text{const} \times \frac{\sqrt{\varepsilon}}{|\log \varepsilon|^{3/4}}.$$

c) *If $d \geq 3$, then for all $x \in S^{d-1}$ and small enough ε :*

$$\text{const} \times \sqrt{\varepsilon} \leq m_\varepsilon^{\text{harm}}(x) \leq \text{const} \times \sqrt{\varepsilon}.$$

The constants depend on the dimension d and p only.

The proof of the results depends on a comparison of the laws of the trapping configurations with Bernoulli measures. This is particularly delicate in $d = 2$. The following result is the key comparison of the distribution of traps with Bernoulli measures. We formulate it only in the harmonic case. Somewhat weaker results are proved in [10] also for the anharmonic situation.

Theorem 2 *Let $\mathcal{A}_{\varepsilon,V}$ be a random subset of V with $P\left(\mathcal{A}_{\varepsilon,V} = V \backslash A\right) = \nu_{V,\varepsilon}(A)$. Assume $d = 2$, and $U(x) = x^2/2$.*

a) *Let $\alpha > 0$. There exists $\varepsilon_0 > 0$ and $C(\alpha) > 0$ such that for $\varepsilon \leq \varepsilon_0$, any finite set $V \subset \mathbb{Z}^d$, and any $B \subset V$ with $\text{dist}(B, V^c) > \varepsilon^{-\alpha}$, one has the estimate*

$$P\left(\mathcal{A}_{\varepsilon,V} \cap B = \emptyset\right) \geq \left(1 - C(\alpha) \frac{\varepsilon}{\sqrt{|\log \varepsilon|}}\right)^{|B|}.$$

b) *There exist $C > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, any finite set $V \subset \mathbb{Z}^d$, and all $B \subset V$, one has*

$$P\left(\mathcal{A}_{\varepsilon,V} \cap B = \emptyset\right) \leq \left(1 - C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}}\right)^{|B|}.$$

The case of dimension $d \geq 3$ is simpler and somewhat better estimates can be obtained. With the help of the above theorem and the random walk representation (5), a comparison can be made, relating the quantities in Theorem 1 to random trapping problems for Bernoulli traps. For instance, when investigating the variance, we get

$$\hat{E}_{V,\varepsilon}^{\text{harm}}\left(\phi_0^2\right) = E_{\text{traps}} \mathbb{E}_0 \left(\sum_{t=0}^{\tau-1} 1_0\left(\eta_t\right)\right) = E_{\text{traps}} \sum_{t=0}^{\infty} p_t(0)\, \mathbb{P}_{0,0}^{(t)}\left(\mathcal{A} \cap \eta_{[0,t]} = \emptyset\right),$$

where $\mathcal{A}$ is the random set of points with traps, as introduced above, τ is the first entrance time into this trapping set and $\eta_{[0,t]}$ is the set of points visited by the random walk between time 0 and t. $\mathbb{P}_{0,0}^{(t)}$ refers to a random walk bridge from 0 to 0 in time t, and $p_t(x)$, $x \in \mathbb{Z}^d$ are the transition probabilities of the random walk. With the help of Theorem 2, the right hand side can be estimated in terms of a

Bernoulli trapping problem. If the traps are Bernoulli on $\mathbb{Z}^d$, with probability ρ that a trap is present at a given site, then (in the $V \uparrow \mathbb{Z}^d$ limit)

$$E_{\text{traps}} \sum_{t=0}^{\infty} p_t(0) \, \mathbb{P}_{0,0}^{(t)} \left(\mathcal{A} \cap \eta_{[0,t]} = \emptyset \right) = \sum_{t=0}^{\infty} p_t(0) \, \mathbb{E}_{0,0}^{(t)} \exp \left[\left| \eta_{[0,t]} \right| \log(1-\rho) \right].$$

There are classical results about the right hand side, due to Donsker and Varadhan [23], Sznitman [30], and most recently in [1] investigating such questions. The classical Donsker-Varadhan result is not sharp enough to prove the results of Theorem 1, but a modification of the arguments in [1] is exactly what is needed. The following result is a discrete but somewhat weaker version of one of the main results in [1].

Proposition 3 *Assume $d = 2$. There exists a function $\mathbb{R}^+ \ni a \rightarrow r(a) \in \mathbb{R}^+$, satisfying $\lim_{a \to 0} r(a) = \infty$, such that*

$$\mathbb{P}_{0,0}^{(t)} \left(\left| \eta_{[0,t]} \right| \leq a \frac{t}{\log t} \right) \leq t^{-r(a)}$$

for large enough t.

(In [1], a variational formula for $r(a)$ is given, in the continuous Wiener sausage case.) This proposition and Theorem 2 lead to the appropriate variance estimates in Theorem 1 a).

For the anharmonic case, the results are less precise, but we still get the correct leading order dependence of the variance on the pinning parameter ε. Assume that there is a $C > 0$ such that

$$1/C \leq U''(x) \leq C, \ \forall x.$$

Under this condition we have the following result:

Theorem 4 *Assume $d = 2$. There exists a constant D, depending on p, such that*

$$\frac{1}{D} \left| \log \varepsilon \right| \leq \sup_V \hat{E}_{V,\varepsilon} \left(\phi_0^2 \right) \leq D \left| \log \varepsilon \right|.$$

The upper bound is in [21] and [27], and the lower bound is in [10].

3. Entropic repulsion and the wetting transition

In view of the example of Fisher discussed shortly in Section 1. it is natural to ask similar question for higher-dimensional interfaces. The first task is to investigate the effect of a wall on the random surface without the presence of a pinning effect. There are different ways to take the presence of a wall into account. We have mainly worked with a "hard wall", i.e. where the measure is simply conditioned on the event $\Omega_V^+ \overset{\text{def}}{=} \{ \phi : \phi_x \geq 0 \ \forall x \in V \}$. There are other possibilities, for instance by

introducing a "soft wall". This means that the Hamiltonian (1) is changed by adding $\sum_{x \in V} f(\phi_x)$, where $f : \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to -\infty} f(x) = \infty$. We will only work with a hard wall here, and consider the conditional law for the random field $P\left(\cdot | \Omega_V^+\right)$.

What is the effect of the presence of the wall on the surface? The crucial point is that the surface has local fluctuations, which push the interface away from the wall. On the other hand, the long-range correlations give the surface a certain global stiffness. In order to understand what is going on, consider first the case where there are no such long-range correlations, in the extreme case, where the ϕ_x are just i.i.d. random variables. In that case, evidently nothing interesting happens: The variables are individually conditioned to stay positive. In particular, $E\left(\phi_x | \Omega_V^+\right)$ stays bounded for $V \uparrow \mathbb{Z}^d$. This picture remains the same for fields with rapidly decaying correlations. However, gradient fields behave differently, and so do interfaces in more realistic statistical physics models. As the surface has some global stiffness, the energetically best way for the surface to leave some room for the local fluctuations is to move away from the wall in some global sense. This effect is called "entropic repulsion" and is well known in the physics literature.

The first mathematically rigorous treatment of entropic repulsion appeared in the paper by Bricmont, Fröhlich and El Mellouki [14]. In a series of papers [4], [17], [18], and [6], sharp quantitative results have been derived, the most accurate ones for the harmonic case.

In most of these and related questions, the two-dimensional case is the most difficult but also the most interesting one. In fact, interfaces in the "real world" are mostly two-dimensional.

We first present the results for $d \geq 3$. For gradient non-Gaussian models, some results in the same spirit have been obtained in [18], but they are not as precise as the ones obtained in the Gaussian model. The case where one starts with the field P_∞ (which exists for $d \geq 3$) is somewhat easier than the field on the finite box V_n with zero boundary condition. In the latter case, there are some boundary effects complicating the situation without changing it substantially. This is investigated in [17]. Despite the fact that we consider P_∞, we consider the wall only on a finite box, i.e., we consider $P_\infty\left(\cdot | \Omega_{V_n}^+\right)$, and we are interested in what happens as $n \to \infty$. We usually write Ω_n^+ for $\Omega_{V_n}^+$. Our first task is to get information about $P_\infty\left(\Omega_n^+\right)$. The following results are proved only for the case of nearest neighbor interactions, i.e. when $p(x) = 1/2d$ for $|x| = 1$.

Theorem 5 *Let $d \geq 3$. Then*

a)
$$P_\infty^{\mathrm{harm}}\left(\Omega_n^+\right) = \exp\left[-2\Gamma(0)\mathrm{cap}\left(V\right)n^{d-2}\log n\left(1 + o(1)\right)\right],$$

where $V = [-1, 1]^d$, $\mathrm{cap}(A)$ denotes the Newtonian capacity of A

$$\mathrm{cap}(A) \stackrel{\mathrm{def}}{=} \inf\left\{\|\nabla f\|^2 : f \geq 1_A\right\},$$

and $\Gamma(0) = \gamma_\infty(0, 0)$ is the variance of ϕ_0 under P_∞^{harm}.

b)

$$E_\infty^{\text{harm}}\left(\phi_0|\Omega_n^+\right) = 2\sqrt{\Gamma(0)\log n}(1 + o(1)).$$

c)

$$\mathcal{L}_{P_\infty^{\text{harm}}(\cdot|\Omega_n^+)}\left(\left(\phi_x - E_\infty(\phi_x|\Omega_n^+)\right)_{x\in\mathbb{Z}^d}\right) \to P_\infty^{\text{harm}} \text{ weakly,}$$

as $n \to \infty$, where $\mathcal{L}_{P_\infty^{\text{harm}}(\cdot|\Omega_n^+)}$ denotes the law of the field under the conditioned measure.

Part b) gives the exact rate at which the random surface escapes to infinity, while part c) states that the effect of the entropic repulsion essentially is only this shifting: after subtraction of the shift by the expectation, the surface looks as it does without the wall. However, there is some subtlety in this picture. From the theorem in particular part c), one might conclude that $\lim_{n\to\infty} P_\infty^{\text{harm}}\theta_{2\sqrt{\Gamma(0)\log n}}^{-1}(\Omega_n^+) = 1$, where $\theta_a : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is the shift mapping $\theta_a\left((\phi_x)_{x\in\mathbb{Z}^d}\right) = (\phi_x + a)_{x\in\mathbb{Z}^d}$. But this is not the case. In fact $P_\infty^{\text{harm}}\theta_{2\sqrt{\Gamma(0)\log n}}^{-1}(\Omega_n^+)$ converges rapidly to 0. As part c) states only the weak convergence, this is no contradiction. Parts a) and b) of Theorem 5 had been proved in [4], part c) in [18].

We come now to the two-dimensional case which is considerably more delicate than the higher dimensional one. We again consider only the harmonic case. We write P_n for P_{V_n}. If the lattice is two-dimensional, a thermodynamic limit of the measures P_n does not exist as the variance blows up. $P_n^{\text{harm}}(\Omega_n^+)$ is of order $\exp[-cn]$, as has been shown in [17]. As remarked above, this is mainly a boundary effect and is not really relevant for the phenomenon of the entropic repulsion. To copy somehow the procedure of the case $d \geq 3$, we consider a subset $D \subset V = [-1,1]^2$ which has a nice boundary and a positive distance from the boundary of V. To be specific, just think of taking $D \overset{\text{def}}{=} \lambda V$ for some $\lambda < 1$. Then let $D_n \overset{\text{def}}{=} nD\cap\mathbb{Z}^2$ and $\Omega_{D_n}^+ \overset{\text{def}}{=} \{\phi_x \geq 0,\, x \in D_n\}$. In contrast to $P_n(\Omega_n^+)$, $P_n\left(\Omega_{D_n}^+\right)$ decays much slower, but still faster than any polynomial rate. In [6] we proved the following result:

Theorem 6 *Assume $d = 2$ and let $g \overset{\text{def}}{=} 1/2\pi$.*

a)

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} \log P_n^{\text{harm}}(\Omega_{D_n}^+) = -2g\text{cap}_V(D),$$

where $\text{cap}_V(D)$ is the relative capacity of D with respect to V:

$$\text{cap}_V(D) \overset{\text{def}}{=} \inf\left\{\|\nabla f\|_2^2 : f \in H_0^1(V),\, f \geq 1 \text{ on } D\right\}.$$

Here, $H_0^1(V)$ is the Sobolev space of (weakly) differentiable functions f with square integrable gradient and $f|_{\partial V} = 0$.

b) For any $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{x\in D_n} P_n^{\text{harm}}\left(|\phi_x - 2\sqrt{g}\log n| \geq \varepsilon\log n|\Omega_{D_n}^+\right) = 0.$$

This corresponds to parts a) and b) of Theorem 5. Part c) does not make sense here as P_∞^{harm} does not exist. Remark that under the unconditional law P_n^{harm}, $|\phi_x|$ is typically of order $\sqrt{\log n}$ in the bulk.

Roughly speaking, the delicacy in the two-dimensional case is coming from the fact that the relevant "spikes" responsible for the repulsion are thicker than in the higher dimensional case, where essentially just very local spikes are responsible for the effect. This makes necessary to apply a multiscale analysis separating the scales of the spikes.

It is well-known that the two-dimensional harmonic field has much similarity with a hierarchical field defined in the following way: We call a sequence $\alpha = \alpha_1\alpha_2\ldots\alpha_m$, $\alpha_i \in \{0,1\}$ a *binary string*. $\ell(\alpha) = m$ is the length. $\emptyset$ is the empty string of length 0. We write T for the set of all such strings of finite length, and $T_m \subset T$ for the set of strings of length m. If $\alpha \in T_m$, $0 \leq k \leq m$, we write $[\alpha]_k$ for the truncation at level k :

$$[\alpha_1\alpha_2\ldots\alpha_m]_k \stackrel{\mathrm{def}}{=} \alpha_1\alpha_2\ldots\alpha_k.$$

If α, $\beta \in T_m$ we define the hierarchical distance

$$d_H(\alpha,\beta) \stackrel{\mathrm{def}}{=} m - \max\{k \leq m : [\alpha]_k = [\beta]_k\}.$$

We consider the following family $(X_\alpha)_{\alpha \in T_m}$ of centered Gaussian random variables by

$$\mathrm{cov}\,(X_\alpha, X_\beta) = \gamma\,(m - d_H(\alpha,\beta))\,, \tag{10}$$

with a parameter $\gamma > 0$. We argue now that there is much similarity between the two dimension harmonic field $(\phi_x)_{x \in D_n}$ and the field $(X_\alpha)_{\alpha \in T_m}$. To see this, we first match the number of variables, i.e. put $2^m = |D_n|$. As $|D_n|$ is of order n^2, this just means that $m \sim 2\log n/\log 2$. Then we should also match the variances, i.e. take $\gamma = g/2\log 2$. For the free field (ϕ_x), it is known that $\mathrm{cov}(\phi_x, \phi_y)$ behaves like $g\,(\log n)\,/\log|x - y|$, if x, y are not too close to the boundary. This follows from the random walk representation. Comparing this with (10), we see that for any number $s \in (0, g)$

$$\#\,\{y \in D_n : \mathrm{cov}\,(\phi_x, \phi_y) \leq s\log n\} \sim \#\,\{\beta \in T_m : \mathrm{cov}\,(X_\alpha, X_\beta) \leq s\log n\} \tag{11}$$

in first order, for any $x \in D_n$, $\alpha \in T_m$. Therefore, the two fields have roughly the same covariance structure. The hierarchical field is much simpler and is very well investigated (see e.g. [2], [12], [22]), and the entropic repulsion is much easier to discuss than for the harmonic field. The approach to prove Theorem 6 consist in introducing a hierarchical structure in the (ϕ_x)-field with the help of successive conditionings on a hierarchy of scales, and then adapt the methods from the purely hierarchical case.

We come now back to the question of a wetting transition, as discussed in the one-dimensional case by Michel Fisher [25]. One is interested in the behavior of $\hat{P}_{V,\varepsilon}\left(\cdot \mid \Omega_V^+\right)$ for large V, where $\hat{P}_{V,\varepsilon}$ is the pinned measure introduced in (8). Unfortunately, we are not able to describe this path measure. The simplest way to

discuss the wetting transition is in terms of free energy considerations. For this we expand $\hat{P}_{V,\varepsilon}\left(\Omega_V^+\right)$ (see (9)):

$$\hat{P}_{V,\varepsilon}\left(\Omega_V^+\right) = \sum_{A \subset V} \varepsilon^{|V \setminus A|} \frac{Z_A}{\hat{Z}_{V,\varepsilon}} P_A\left(\Omega_V^+\right).$$

It is plausible, that pinning "wins" over entropic repulsion, if this sum is much larger than the contribution to the sum coming from subsets A having essentially no pinning sites, i.e. $A \approx V$. It is therefore natural to consider the quantity

$$p_+\left(\varepsilon\right) \overset{\text{def}}{=} \lim_{V \uparrow \mathbb{Z}^d} \frac{1}{|V|} \log \frac{\hat{Z}_{V,\varepsilon}\hat{P}_{V,\varepsilon}\left(\Omega_V^+\right)}{Z_V P_V\left(\Omega_V^+\right)} = \lim_{V \uparrow \mathbb{Z}^d} \frac{1}{|V|} \log \frac{\hat{Z}_{V,\varepsilon}\hat{P}_{V,\varepsilon}\left(\Omega_V^+\right)}{Z_V}.$$

The limit is easily seen to exist. It is also not difficult to see that $p_+\left(\varepsilon\right) > 0$ for large enough $\varepsilon > 0$, and in any dimension (see [9]). Similar to the discrete random walk case in [25], the Gaussian model has a wetting transition, too, for $d = 1$: There exists an $\varepsilon_{\text{crit}} > 0$, such that $p_+\left(\varepsilon\right) = 0$ for $\varepsilon < \varepsilon_{\text{crit}}$. This is easy to see for $d = 1$. For the harmonic model, there is remarkably no such transition for $d \geq 3$, but for $d = 2$ there is a wetting transition.

Theorem 7 [5] *For $d \geq 3$, $p_+^{\text{harm}}\left(\varepsilon\right) > 0$ for all $\varepsilon > 0$.*

Theorem 8 [16] *For $d = 2$, there exists $\varepsilon_{\text{crit}}^{\text{harm}} > 0$, such that $p_+^{\text{harm}}\left(\varepsilon\right) = 0$ for $\varepsilon < \varepsilon_{\text{crit}}^{\text{harm}}$.*

Remarkably, too, Caputo and Velenik have proved that such a wetting transition exists for $d \geq 3$ for some non-harmonic models, e.g. for $U\left(x\right) = |x|$.

There are many open questions concerning this wetting transition, which is very poorly understood (mathematically). For instance, the methods discussed in Section 2. do not apply, and we are not able to prove that in the pinning dominated region $p_+\left(\varepsilon\right) > 0$, the measure is pathwise localized, i.e. that

$$\sup_V \sup_{x \in V} \hat{P}_{V,\varepsilon}\left(\phi_x^2 \mid \Omega_V^+\right) < \infty,$$

which certainly should be expected. To discuss the nature of the transition (first order or second order?) is probably even much more delicate.

4. Localization-delocalization transitions for one-dimensional copolymers

We stick here to the standard simple random walk case where P_n simply is the uniform distribution on the set of paths $\phi_0 = 0, \phi_1, \ldots, \phi_n \in \mathbb{Z}$, satisfying $|\phi_i - \phi_{i-1}| = 1$, $1 \leq i \leq n$. There is not much difference when considering more general random walks, or the tied-down situation, but most of the published results are for the simple random walk. An interesting case of a mixed attractive-repulsive interaction is given in the following way. Regard the above random walk as a (very

simplified) model of a polymer chain imbedded in two liquids, say water and oil. The water is at the bottom, say at points $(i, j) \in \mathbb{N} \times \mathbb{Z}$, $j \leq 0$, and the oil above at $j > 0$. The polymer chain is attached with one end at the interface between the two liquids, and interacts with them in the following way: To each "node" (i, ϕ_i) of the polymer chain, we attach a value $\sigma_i \in \mathbb{R}$ which is < 0 if the node is water-repellent, and > 0 if it is oil-repellent. The overall effect is described by the Hamiltonian

$$H_{n,\sigma}(\phi) \overset{\text{def}}{=} \sum_{i=1}^{n} \sigma_i \operatorname{sign}(\phi_i),$$

where we put $\operatorname{sign}(0) \overset{\text{def}}{=} 0$. With this Hamiltonian, we define the σ-dependant path measure

$$P_{n,\beta,\sigma}(\phi) \overset{\text{def}}{=} \frac{1}{Z_{n,\beta,\sigma}} \exp\left[-\beta H_{n,\sigma}(\phi)\right],$$

where $\beta > 0$ is a parameter governing the strength of the interaction. We assume that the σ_i change sign either in a periodic way or randomly. There may be two competing effects. The polymer chain may try to follow the preferences described by the σ's as closely as possible in which case the path evidently would have to stay close to the oil-water-interface and gets localized. On the other hand, this strategy may be entropically too costly, in particular if there is no balance between oil-repellence and water-repellence. We will always assume that

$$h \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \sigma_j,$$

exists, and we assume it to be ≥ 0. (The case $h \leq 0$ can be treated symmetrically). It turns out that typically, there is a non-trivial curve in the (β, h)-plane which separates the localized from the delocalized region. This phase separation line is quite model dependent, but the behavior near $(0, 0)$ appears to be much more universal but it is completely different depending whether the σ_i are random or periodic.

    The first rigorous results in this direction had been obtained by Sinai [29] who proved the following result in the balanced case (i.e. $h = 0$). Let $\mathbb{P}$ be the symmetric Bernoulli-measure on $\{-1, 1\}^{\mathbb{N}}$.

Theorem 9 *Let $\beta > 0$. There exist constants C and $\rho(\beta) > 0$, and for $\mathbb{P}$-almost all $\sigma = (\sigma_i)_{i \geq 1}$, there exists a sequence $(R_n(\sigma))_{n \in \mathbb{N}}$ of natural numbers such that*

$$P_{n,\beta,\sigma}(|\phi_n| \geq r) \leq C \exp\left[-\rho(\beta) r\right]$$

for $r \geq R_n(\sigma)$. The sequence (R_n) is stochastically bounded, i.e.

$$\lim_{m \to \infty} \sup_{n} \mathbb{P}(R_n \geq m) = 0.$$

    In a paper with Frank den Hollander [8] we proved that there is a localization-delocalization transition in the random non-balanced case. This transition is discussed in this paper in terms of the free energy. To describe the results, let

$\sigma_i = \pm 1 + h$ with probabilities $1/2$, and independently, $h \geq 0$. One strategy of the path could be just to stay on the negative side all the time, i.e. $\phi_i < 0$ for all $i \leq n$. This leads to a trivial lower bound of the free energy

$$f(\beta, h) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log Z_{n,\beta,\sigma}$$

which is easily seen to exist, and is non-random:

$$Z_{n,\beta,\sigma} \geq E_n \left(\exp\left[-\beta H_{n,\sigma}(\phi)\right] 1_{\{\phi_i < 0, \, \forall i \leq n\}} \right)$$
$$= \exp\left[\beta \sum_{i=1}^{n} \sigma_i \right] P(\phi_i < 0, \, \forall i \leq n).$$

From this we get

$$f(\beta, h) \geq \beta h.$$

It is quite plausible that localization dominates in the case where there is a strict inequality, and that delocalization holds if $f(\beta, h) = \beta h$.

Theorem 10 *There exists a positive, continuous, and increasing function $\beta \to h^*(\beta)$ such that*

$$f(\beta, h) > \beta h \text{ for } 0 \leq h < h^*(\beta), \tag{12}$$
$$f(\beta, h) = \beta h \text{ for } h > h^*(\beta). \tag{13}$$

The function $\beta \to h^*(\beta)$ has a positive tangent at $\beta = 0$.

The phase separating function h^* is certainly very much model dependent, but we expect that the tangent at 0 is model independent, and would be the same for any random law of the σ-sequence which has variance 1 and a expectation h, and has exponentially decaying tails, but this is not proved in [8]. In physics literature, there are non-rigorous arguments claiming that the tangent is 1, but we neither have been able to prove or disprove it, yet. We prove that the tangent at 0 can be described in terms of a phase separation line for a continuous model, where the random walk is replaced by a Brownian motion, and the random environment σ is replaced by (biased) white noise. In this case, the phase separation line is a straight line, and we prove that this line is the tangent at 0 of our model. It should be remarked that the $(\beta, h) \approx (0, 0)$ situation, cannot be handled by simple perturbation techniques.

A natural question is if in the localized region $f(\beta, h) > \beta h$ the path measure is really localized in the sense described in the paper of Sinai. This is indeed the case and has been proved by Biskup and den Hollander [3]. One might also wonder if in the localized region $f(\beta, h) = \beta h$ or at least in the interior of it, the path measure is really delocalized, which should mean, that it converges, after Brownian rescaling, to the limit of a random walk conditioned to stay negative, which is the negative of the Brownian meander. This seems to be a rather difficult question and has not been answered, yet.

The positive tangent is essentially tied to the randomness of the sequence. For the periodic case, the situation is different, as has recently been proved in [11]:

Theorem 11 *Let $\sigma_i = \omega_i + h$, where $\omega_i \in \{-1, 1\}$ is periodic, i.e. such that there exists T with $\omega_{i+2T} = \omega_i$ for all i, and $\sum_{i=1}^{2T} \omega_i = 0$. Then there is a function h^* such that (12) and (13) hold. In this case*

$$C = \lim_{\beta \to 0} \frac{h^*(\beta)}{\beta^3}$$

exists and is positive.

In this paper an expression for C in terms of a variational problem is derived, where the exact nature of the periodic sequence enters.

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