

ON UNIFICATION OF COLORED ANNULAR \mathfrak{sl}_2 KNOT HOMOLOGY

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ABSTRACT. We show that the Khovanov and Cooper–Krushkal models for colored \mathfrak{sl}_2 homology are equivalent in the case of the unknot, when formulated in the quantum annular Bar-Natan category. Again for the unknot, these two theories are shown to be equivalent to a third colored homology theory, defined using the action of Jones–Wenzl projectors on the quantum annular homology of cables. The proof is given by conceptualizing the properties of all three models into a Chebyshev system and by proving its uniqueness. In addition, we show that the classes of the Cooper–Hogancamp projectors in the quantum horizontal trace coincide with those of the Cooper–Krushkal projectors on the passing through strands. As an application, we compute the full quantum Hochschild homology of Khovanov’s arc algebras. Finally, we state precise conjectures formalizing cabling operations and extending the above results to all knots.

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1. INTRODUCTION

At the beginning of our century knot theory was revolutionized by the Khovanov construction of a chain complex [Kho00] whose graded Euler characteristic is the Jones polynomial, and whose Poincaré polynomial is a new link invariant detecting the unknot [KM11]. Since then many modifications of this construction were introduced in the literature. The most relevant for us is the quantum annular Khovanov homology constructed by three authors of the present paper in [BPW19]. This is a triply graded homology theory for links in a thickened annulus \mathbb{A} equipped with a membrane. The differentials preserve the quantum and annular gradings and are given by annular cobordisms considered up to isotopies fixing the membrane, so that moving a cobordism S through the membrane contributes a factor $q^{\pm\chi(S)}$, where q is a quantum parameter and $\chi(S)$ is the Euler characteristic (see Figure 1). Such cobordisms form a category $\mathcal{BN}_q(\mathbb{A})$. Setting $q = 1$ we recover the usual annular Khovanov homology of Asaeda–Przytycki–Sikora [APS04].

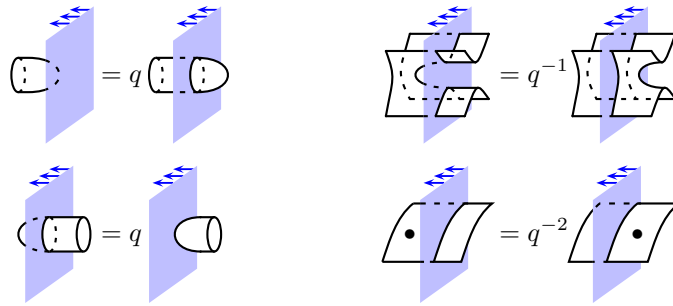


FIGURE 1. Membrane relations.

This paper aims to compare different approaches extending this construction to colored knots. Let us recall that given a framed oriented knot $K \subset S^3$, and an $(n + 1)$ -dimensional representation V_n of the quantum group $U_q(\mathfrak{sl}_2)$, the colored Jones polynomial $J(K, V_n)$ can be defined by inserting the Jones–Wenzl idempotent into a diagram representing n -cable of K and by resolving the crossings with the Kauffman bracket skein relations. Alternatively, the Reshetikhin–Turaev functor can be used to define $J(K, V_n)$.

In the categorified setting there are three different approaches to define colored annular Khovanov homology, or its quantization. To outline these constructions we will need some

preparation. Let us denote by \mathcal{TL} the Temperley–Lieb category, whose objects are natural numbers and whose morphisms from n to m are flat (n, m) -tangles (or crossingless matchings of $n + m$ points). The unknot considered as $(0, 0)$ morphism in \mathcal{TL} takes value $q + q^{-1}$. For generic q the Temperley–Lieb algebra $\mathrm{TL}_n := \mathcal{TL}(n, n)$ is semisimple equipped with a family of idempotents $\{p_\varepsilon\}$, $\varepsilon \in \{-1, 1\}^n$ projecting onto simple components. The most famous among them p_n corresponding to $\varepsilon = (1, \dots, 1)$ is called the Jones–Wenzl projector. There is a functor $\mathrm{HH}_0 : \mathcal{TL} \rightarrow S(\mathbb{A})$, to the skein algebra of the annulus, sending p_n to its braid closure $[p_n]$ satisfying the following Chebyshev recursion relation:

$$[p_n][p_1] = [p_{n+1}] + [p_{n-1}]$$

This relation also holds for classes $[V_n]$ in the Grothendieck ring of the representation category $\mathcal{Rep}(U_q(\mathfrak{sl}_2))$ or in the Hochschild homology of the latter, thus leading to an isomorphism between $\mathrm{HH}_0(\mathcal{TL})$ and $\mathrm{K}_0(\mathcal{Rep}(U_q(\mathfrak{sl}_2)))$.

The Bar-Natan 2-category \mathbf{BN} categorifies \mathcal{TL} by adding as 2-morphisms surfaces bounded by flat tangles. We will denote by \mathcal{BN}_n the category $\mathbf{BN}(n, n)$. There is also a natural categorification of the 0th Hochschild homology, called the *horizontal trace* of a 2-category. We will work with the q -twisted, or *quantum*, version of the horizontal trace developed in [BPW19], where it was shown that the quantum horizontal trace $\mathrm{hTr}_q(\mathbf{BN}^\oplus)$ is equivalent to $\mathcal{BN}_q(\mathbb{A})^\oplus$. Here the symbol \oplus indicates the additive closure.

Our first model for colored knot homology uses the Cooper–Krushkal categorification P_n of the Jones–Wenzl idempotent $p_n \in \mathrm{TL}_n$ living in an appropriate completion of the category of bounded complexes over the Bar-Natan category \mathbf{BN}_n . Let us denote by $[P_n]$ the class of P_n in $\mathrm{hTr}_q(\mathbf{BN}^\oplus)$. In the Cooper–Krushkal model the colored annular complex is constructed by composing $[P_n]$ with the quantum annular complex for the n cable of K .

The second approach is due to Khovanov and inspired by the following decomposition of $[V_n] \in \mathrm{K}_0(\mathcal{Rep}(U_q(\mathfrak{sl}_2)))$

$$(1.1) \quad [V_n] = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} [V_1^{\otimes(n-2k)}].$$

In [Kho05] Khovanov constructed a family of finite complexes \mathcal{V}_n over the Temperley–Lieb category \mathcal{TL} whose Euler characteristics satisfies (1.1) where the binomial coefficients are interpreted as grading shifts. Combining these complexes with those for the n cable of K we get the n -colored quantum annular complex.

The third construction makes use of a functor

$$S^1 \times (-) : \mathcal{TL} \rightarrow \mathcal{BN}_q(\mathbb{A})$$

that sends an object n to a collection of n essential circles in \mathbb{A} and a cap or cup to a band between neighboring circles. More generally, this functor sends a flat tangle T to the surface $S^1 \times T \subset \mathbb{A} \times I$. In [BPW19] it is shown that this functor induces an equivalence between \mathcal{TL}^\oplus and $\mathcal{BN}_q(\mathbb{A})^\oplus$. This allows us to define the colored complex for a knot K by composing $S^1 \times p_n \in \mathcal{BN}_q(\mathbb{A})^\oplus$ with the quantum annular complex of the n cable of K .

To conclude, all three models associate with a pair (K, V_n) a (bounded above) chain complex over the Karoubi envelope of \mathcal{TL}^\oplus . We denote the category of such chain complexes by $\mathrm{Ch}^-(\mathrm{Kar}(\mathcal{TL})^\oplus)$. Let us also fix a coefficient ring \mathbb{k} , such that q is a fixed invertible element of \mathbb{k} and $1 - q^d$ is invertible for all integers d in $(0, 2N]$ for a big enough N .

Our main result establishes an equivalence of these three models for the n -colored unknot.

Theorem A. *The quantum annular complexes for the n -colored unknot in all three models are homotopy equivalent, meaning that*

$$[P_n] \simeq \text{im } p_n \simeq \mathcal{V}_n \in \text{Ch}^-(\text{Kar}(\mathcal{TL})^\oplus).$$

where $\text{im } p_n$ denotes the idempotent p_n viewed as an object of $\text{Kar}(\mathcal{TL})$.

We remark that the above theorem fails in the setting of the usual (untwisted) horizontal trace, and in general the Cooper–Krushkal and Khovanov colored homologies are non-isomorphic.

To prove Theorem A we formalize properties of all three models by introducing a *Chebyshev system* defined for any monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ by the following data:

- a self-dual object V in \mathcal{C} ,
- a family of complexes $V^{(n)} \in \text{Ch}^-(\mathcal{C})$ with $V^{(0)} = \mathbb{1}$ and $V^{(1)} = V$,
- chain maps $\pi^{(n)}: V^{(n-1)} \otimes V \rightarrow V^{(n)}$ for $n \geq 1$,

such that $\pi^{(1)}: \mathbb{1} \otimes V \rightarrow V$ is the canonical isomorphism and there is a distinguished triangle

$$(1.2) \quad V^{(n-2)} \longrightarrow V^{(n-1)} \otimes V \xrightarrow{\pi^{(n)}} V^{(n)} \longrightarrow V^{(n-2)}[1]$$

in the homotopy category of bounded complexes over \mathcal{C} (see Definition 4.6 for more details). We prove that for a given V , a Chebyshev system in \mathcal{C} is unique up to homotopy and show that each of our three models defines a Chebyshev system in $\text{Ch}^-(\text{Kar}(\mathcal{TL})^\oplus)$, thus proving the claim.

1.1. Generalized projectors. Notice that as an abelian group, the Hochschild homology $\text{HH}_0(\text{TL}_n)$ is free on the classes $[p_n]$. To express a general endomorphism $x \in \text{TL}_n$ in this basis, we use a complete collection of primitive orthogonal idempotents $\{p_\varepsilon \in \text{TL}\}_{\varepsilon}$, labelled by certain length n sequences of ± 1 . In particular, for any $x \in \text{TL}_n$ we decompose

$$x = \sum_{\varepsilon} x p_{\varepsilon} \quad \text{or} \quad [x] = \left[\sum_{\varepsilon} p_{\varepsilon} x p_{\varepsilon} \right] \in \text{HH}_0(\mathcal{TL}).$$

Since each p_{ε} is a primitive idempotent, $p_{\varepsilon} \text{TL}_n p_{\varepsilon}$ is one-dimensional, hence $p_{\varepsilon} x p_{\varepsilon} = \text{Tr}_{\varepsilon}(x) p_{\varepsilon}$ for some scalars $\text{Tr}_{\varepsilon}(x)$ and

$$[x] = \sum_{\varepsilon} \text{Tr}_{\varepsilon}(x) [p_{\varepsilon}].$$

Moreover, $[p_{\varepsilon}]$ depends only on the integer $|\varepsilon|$ (the maximal number of coming through strands) and is characterized by $\text{im } p_{\varepsilon} \cong \text{im } p_{|\varepsilon|}$ or equivalently, by $[p_{\varepsilon}] = [p_{|\varepsilon|}]$ in $\text{HH}_0(\text{TL}_n)$, where $p_{|\varepsilon|}$ is the Jones–Wenzl projector. We then have

$$[x] = \sum_{\varepsilon} \text{Tr}_{\varepsilon}(x) [p_{|\varepsilon|}].$$

We conjecture that a categorified analogue of this statement holds as well.

Conjecture B. *For a given $X \in \text{Ch}^-(\mathcal{BN}_n)$, there exist complexes $\text{hTr}_{q,\varepsilon}(X) \in \text{Ch}^-(\mathbb{k}\text{-mod})$ such that*

$$[X] \simeq \bigoplus_{\varepsilon} \text{hTr}_{q,\varepsilon}(X) \otimes_{\mathbb{k}} [P_{|\varepsilon|}].$$

where $[P_{\varepsilon}]$ is the class of P_{ε} in the quantum horizontal trace and $\text{hTr}_{q,\varepsilon}(X)$ is characterized by

$$[X \star P_{\varepsilon}] = \text{hTr}_{q,\varepsilon}(X) \otimes [P_{\varepsilon}]$$

where P_ε is the categorified Temperley–Lieb algebra idempotent constructed by the second author and B. Cooper [CH15], and \star denotes the composition of 1-morphisms in \mathbf{BN} .

We will not pursue this conjecture in this paper, but we will prove the following special case.

Theorem C. *For each ε with $|\varepsilon| = k$ we have*

$$[P_\varepsilon] \simeq [P_k] \in \mathrm{Ch}^-(\mathcal{TL}^\oplus).$$

The proof of this theorem and our construction of P_ε is based on homological perturbation theory and a general form of the *combing hairs* lemma, that may be of independent interest for homotopy theorists.

As an application of Theorem C we show that the higher quantum Hochschild homology groups of the Khovanov arc algebra H^n vanishes and the rank of the zeroth homology is given by the n th Catalan number.

The results and tools developed in this paper form a part of our program aiming to investigate the behavior of Khovanov homology under cabling operations. Let us state two further conjectures that we plan to address in the sequel.

1.2. Colored knot homology as a functor. Given a knot K , a complex semisimple Lie algebra \mathfrak{g} , and a finite dimensional representation V of $U_q(\mathfrak{g})$, the Reshetikhin–Turaev construction produces a link invariant $P_{\mathfrak{g}}(K, V) \in \mathbb{Z}[q, q^{-1}]$. The aim of categorified Reshetikhin–Turaev invariants is to replace the Laurent polynomial $P_{\mathfrak{g}}(K, V)$ by a complex of vector spaces $C_{\mathfrak{g}}(K, V)$ whose graded Euler characteristic is $P_{\mathfrak{g}}(K, V)$. As is well known, in favorable situations $C_{\mathfrak{g}}(K, V)$ is functorial in K , with respect to framed oriented cobordisms.

It is also natural to wonder if such colored knot homology theories can be made functorial in V . That is to say, if we fix K , we should expect the assignment $V \mapsto C_{\mathfrak{g}}(K, V)$ to lift to a functor

$$\mathrm{Rep}(U_q(\mathfrak{g})) \rightarrow \mathrm{Ch}(\mathbb{k}\text{-mod}).$$

We plan to investigate the existence of such a functor in the case $\mathfrak{g} = \mathfrak{sl}_2$. In this case, $\mathrm{Rep}(U_q(\mathfrak{g}))$ is equivalent to the Temperley–Lieb category $\mathrm{Kar}(\mathcal{TL})^\oplus$.

In the sequel our aim will be to prove the following.

Conjecture D. *Each framed oriented knot K determines a dg functor*

$$C(K; -) : \mathrm{Ch}^-(\mathrm{Kar}(\mathcal{TL})^\oplus) \rightarrow \mathrm{Ch}^-(\mathbb{k}\text{-mod})$$

with the following properties:

- 1) $C(K; V_1^{\otimes n})$ is the quantum annular complex associated to the n -cable of K .
- 2) $C(K; V_n)$ is a quantum annular version of Khovanov’s colored \mathfrak{sl}_2 -homology.

1.3. The space of colors as Hochschild homology. Let $K \subset S^3$ be a framed oriented knot. Choose a presentation of K as the closure of a $(1, 1)$ -tangle diagram D (equipped with the blackboard framing), and consider the following construction. Suppose we are given an element of the Temperley–Lieb algebra $x \in \mathrm{TL}_n$. Let D^n denote the n -cable of D , and let $\langle D^n \rangle \in \mathrm{TL}_n$ denote the Kauffman bracket. We can multiply $\langle D^n \rangle x$ and then take the trace

$$P'(K, x) := \mathrm{Tr}(\langle D^n \rangle x) \in \mathbb{k}.$$

This assignment does not depend on the choice of $(1, 1)$ -tangle diagram D , and satisfies (by isotopy invariance and the nature of traces)

$$P'(K, xy) = P'(K, yx).$$

In fact this identity remains true when $xy \in \mathrm{TL}_n$ and $yx \in \mathrm{TL}_m$ for $m \neq n$. Thus, we can view the assignment $x \mapsto P'(K, x)$ as a linear map

$$\mathrm{HH}_0(\mathcal{TL}) \rightarrow \mathbb{k}.$$

Since TL_n is semisimple, its higher Hochschild homology groups vanish. Replacing the Hochschild homology by its full categorification, which is a dg version of the horizontal trace defined in [GHW22], we get the following conjecture.

Conjecture E. *Each complex $X \in \mathrm{Ch}^-(\mathcal{BN}_n)$ determines a colored complex*

$$C'(K, X) \in \mathrm{Ch}^-(\mathbb{k}\text{-mod}).$$

The assignment $X \mapsto C'(K, X)$ is functorial in X , in the sense that it factors through the dg quantum horizontal trace [GHW22]. Furthermore, if P_n denotes the Cooper–Krushkal categorified Jones–Wenzl idempotent, then $C'(K, P_n)$ is homotopy equivalent to the V_n -colored complex $C(K, V_n)$ from Conjecture D.

In particular, we expect that the Khovanov and Cooper–Krushkal models for the colored quantum annular homology coincide for all knots. The results of this paper and the previous one [Bel+19] will play a crucial role in the proof of these conjectures.

1.4. Organization. The paper is organized as follows. In Section 2, we provide the necessary background on categorical traces and shadows. We also introduce the main categories and bicategories used in the paper. In Section 3, we discuss homological perturbation theory for twisted complexes and prove a general version of the combing hairs lemma, needed for our proofs. In Section 4, we define the notion of a Chebyshev system in a triangulated category, and we prove a uniqueness result (Theorem 4.8). As examples of Chebyshev systems, we consider the three models from Theorem A. In the fourth section, we prove Theorem 4.20, which asserts that the Cooper–Krushkal model is indeed a Chebyshev system. To show this, we establish a new 2-periodic model for the Cooper–Krushkal projector (Theorem 5.1). The next section is devoted to the categorified idempotents P_ε . We first give a new proof of their existence and then establish Theorem C about the quantum horizontal trace of P_ε . Finally, we compute the quantum Hochschild homology of the Khovanov arc algebras.

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2. PRELIMINARIES

In this section, we will review the homological and categorical constructions that will be used in the later sections of the paper. We will also give a self-contained proof of the fact that for generic q , the Temperley–Lieb category is equivalent to the quantum horizontal trace of the Bar-Natan bicategory (Theorem 2.2). We will start by fixing some conventions.

2.1. Conventions. Throughout this paper, \mathbb{k} will be a commutative unital ring and $q \in \mathbb{k}^\times$ will be a fixed invertible element. We will assume that $1 - q^d$ is invertible for all integers $d \in (0, 2n]$ and n sufficiently large.

We will write \star for the horizontal composition of 1- and 2-morphisms in a bicategory. Similarly, we will use \circ to denote the vertical composition of 2-morphisms, as well as the ordinary composition of morphisms in a category. Moreover, the symbols \otimes and \sqcup will be used to denote the monoidal products in the Temperley–Lieb category and in the Bar-Natan bicategory, respectively. Finally, the superscript \oplus will refer to the additive closure. This superscript will be omitted in the later sections of the paper, where it is understood implicitly that categories are completed with respect to finite direct sums.

2.2. Complexes. If \mathcal{A} is a \mathbb{k} -linear category, we will let $\text{Ch}(\mathcal{A})$ denote category of complexes over \mathcal{A} . Objects of this category are complexes (with the cohomological convention for differentials)

$$\dots \xrightarrow{\delta^{k-1}} X^k \xrightarrow{\delta^k} X^{k+1} \xrightarrow{\delta^{k+1}} \dots$$

in which each $X^k \in \mathcal{A}$. Morphism spaces between objects X and Y in $\text{Ch}(\mathcal{A})$ are complexes

$$\text{Hom}_{\text{Ch}(\mathcal{A})}^k(X, Y) = \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(X^i, Y^{i+k})$$

equipped with the differential

$$f \mapsto \delta_Y \circ f - (-1)^{|f|} f \circ \delta_X =: [\delta, f]$$

where for $f \in \text{Hom}_{\text{Ch}(\mathcal{A})}^k(X, Y)$, $|f| = k$ is the (cohomological) degree of f . In this way, the category $\text{Ch}(\mathcal{A})$ is a *dg category*, i.e. a category in which hom spaces are complexes of \mathbb{k} -modules, and for which composition of morphisms satisfies the appropriate version of the Leibniz rule:

$$[\delta, f \circ g] = [\delta, f] \circ g + (-1)^{|f|} f \circ [\delta, g].$$

We use superscripts $-$, $+$, b to denote complexes which are bounded from the right, respectively left, respectively both right and left.

The category $\text{Ch}(\mathcal{A})$ is symmetric monoidal with the tensor product defined on objects by

$$(X \otimes Y)^k = \bigoplus_{i+j=k} X^i \otimes Y^j, \quad \delta_{X \otimes Y} = \delta_X \otimes \text{id}_Y + \text{id}_X \otimes \delta_Y$$

and on morphisms f, g by $(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y)$ with braidings

$$c_{X, Y} : X \otimes Y \longrightarrow Y \otimes X \quad \text{given by} \quad c_{X, Y}(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

For any ℓ , the translation \mathfrak{t}^ℓ shifts the complex X by ℓ steps to the right, i.e.

$$(\mathfrak{t}^\ell X)^k = X^{k-\ell} \quad \text{and} \quad \delta_{\mathfrak{t}^\ell X} = (-1)^\ell \delta_X.$$

A morphism $f \in \text{Hom}_{\text{Ch}(\mathcal{A})}(X, Y)$ is said to be *closed* if $[\delta, f] = 0$ and *null-homotopic* if $f = [\delta, h]$ for some $h \in \text{Hom}_{\text{Ch}(\mathcal{A})}(X, Y)$. In the later case h is called a null-homotopy for f . For $f, g \in \text{Hom}_{\text{Ch}(\mathcal{A})}(X, Y)$, we write $f \simeq g$ and say f and g are *homotopic* if $f - g$ is null-homotopic.

We denote by $\mathcal{K}(\mathcal{A})$ the *homotopy* category of $\text{Ch}(\mathcal{A})$ which has the same objects as $\text{Ch}(\mathcal{A})$ and whose morphisms are homotopy classes of degree zero closed morphisms in $\text{Ch}(\mathcal{A})$, i.e.

$$\text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) := \frac{\{f \in \text{Hom}_{\text{Ch}(\mathcal{A})}^0(X, Y) \mid [\delta, f] = 0\}}{\delta \left(\text{Hom}_{\text{Ch}(\mathcal{A})}^{-1}(X, Y) \right)}.$$

An isomorphism $f : X \rightarrow Y$ in $\mathcal{K}(\mathcal{A})$ is called a *homotopy equivalence* and in this case we write $X \simeq Y$. If $X \simeq 0$, we say that X is *contractible*.

If $f \in \text{Hom}_{\text{Ch}(\mathcal{A})}^1(X, Y)$ is a degree 1 closed morphism, the *cone* of f is the object $\text{Cone}(f) := X \oplus Y$ in $\text{Ch}(\mathcal{A})$ equipped with the differential $\begin{bmatrix} \delta_X & 0 \\ f & \delta_Y \end{bmatrix}$. If instead $f : X \rightarrow Y$ is a degree 0 closed morphism, then we replace f by a degree 1 closed morphism $\mathbb{t}^{-1}X \rightarrow Y$ and then apply the previous construction.

The construction of cones allows to translate statements about morphisms into statements about objects. For example, a closed map $f : X \rightarrow Y$ is a homotopy equivalence if and only if $\text{Cone}(f) \simeq 0$.

2.3. Gradings on categories and bicategories. We will say that a \mathbb{k} -linear is *pregraded* if its morphism sets are endowed with \mathbb{Z} -gradings which are additive under composition. A pregraded category is called *graded* if it comes with a “grading shift” automorphism \mathfrak{q} along with a natural degree 1 isomorphism $\mathfrak{s} : \mathbb{1} \rightarrow \mathfrak{q}$ from the identity functor to \mathfrak{q} .

A bicategory is called *\mathbb{k} -linear* if its 2-morphism sets are equipped with a \mathbb{k} -module structures such that both compositions are bilinear. Such a bicategory is *pregraded* if its morphism categories are pregraded in such a way that gradings are additive under horizontal (and vertical) composition. In the non-strict setting, it is also required that the associator and the unitors have degree 0.

Finally, a pregraded bicategory is called *graded* its morphism categories are graded in such a way that the grading shift functors and the natural degree 1 isomorphisms $\mathfrak{s} : \mathbb{1} \rightarrow \mathfrak{q}$ behave nicely under horizontal composition. By this, we mean that $(\mathfrak{q}f) \star g = \mathfrak{q}(f \star g) = f \star (\mathfrak{q}g)$ and correspondingly $\mathfrak{s} \star \mathbb{1} = \mathfrak{s} = \mathbb{1} \star \mathfrak{s}$ for composable 1-morphisms f and g .

Note that while the grading shift functors in a graded bicategory take a 1-morphism f to a “shifted” 1-morphism $\mathfrak{q}f$, they do not change the degrees of 2-morphisms. Indeed, $|\mathfrak{q}\alpha| = |\mathfrak{s} \circ \alpha \circ \mathfrak{s}^{-1}| = |\alpha|$, where we write $|\alpha| \in \mathbb{Z}$ for the degree of the 2-morphism α . In general, any homogeneous 2-morphism $\alpha : f \Rightarrow g$ in a graded bicategory corresponds to a homogenous 2-morphism $\mathfrak{s}^m \circ \alpha \circ \mathfrak{s}^{-n} : \mathfrak{q}^n f \Rightarrow \mathfrak{q}^m g$ of degree $|\alpha| + m - n$ for $m, n \in \mathbb{Z}$. In particular, the identity 2-morphism $\mathbb{1} : f \rightarrow f$ corresponds to $\mathfrak{s}_f : f \rightarrow \mathfrak{q}f$.

Given a bicategory which is only pregraded but not already graded, we can extend it to a graded bicategory with the same objects. The 1-morphisms in this graded bicategory are given by pairs of the form $\mathfrak{q}^n f := (f, n)$, where f is a 1-morphism from the original bicategory, and n is an integer, to be viewed as a formal grading shift. If $\mathfrak{q}^n f = (f, n)$ and $\mathfrak{q}^m g = (g, m)$ are two such pairs, then the 2-morphisms $\mathfrak{q}^n f \Rightarrow \mathfrak{q}^m g$ are given by copies of 2-morphisms $f \Rightarrow g$ from the original bicategory, but with degrees raised by $m - n$.

In a similar way, a category which is only pregraded but not already graded can be extended to a graded category with objects of the form $\mathfrak{q}^n x := (x, n)$.

If \mathbf{C} is a pregraded (possibly graded) bicategory, then we denote by \mathbf{C}_0 the bicategory which contains the same objects and 1-morphisms, but only the 2-morphisms of \mathbf{C} that have

degree 0. Likewise, for a pregraded (possibly graded) category \mathcal{C} , we denote by \mathcal{C}_0 the subcategory which has the same objects but only the morphisms of degree 0.

Note that if \mathbf{C} is graded, then every 2-morphism in \mathbf{C} can be obtained from a 2-morphism in \mathbf{C}_0 by pre- and post-composing with suitable powers of \mathfrak{s} . A similar remark applies to a graded category \mathcal{C} and its subcategory \mathcal{C}_0 .

2.4. Temperley–Lieb category. As before, let \mathbb{k} be a commutative ring and let $q \in \mathbb{k}^\times$ be a fixed invertible element. The *Temperley–Lieb category* is the \mathbb{k} -linear category \mathcal{TL} whose objects are nonnegative integers and whose morphisms sets are generated by isotopy classes of flat tangles modulo the relation

$$T \cup \bigcirc = (q + q^{-1})T$$

for any circular component \bigcirc . By a *flat tangle*, we here mean an unoriented compact 1-manifold $T \subset I^2$ with n bottom endpoints on $I \times \{0\}$ and m top endpoints on $I \times \{1\}$ for some $n, m \geq 0$.

The composition in \mathcal{TL} is given by vertical stacking of tangles, so that $T \circ T'$ denotes the result of placing the tangle T on top of the tangle T' and rescaling vertically. There is also a horizontal composition of tangles, given by placing tangles side-by-side, which induces a monoidal structure on \mathcal{TL} . For technical reasons, we will sometimes assume that the objects in \mathcal{TL} come with formal grading shifts, defined as in [BN05, Section 6].

The endomorphism algebra $\mathrm{TL}_n := \mathcal{TL}(n, n)$ is known as the *Temperley–Lieb algebra*. If the quantum integers $[k] := q^{k-1} + q^{k-3} + \dots + q^{1-k}$ are invertible for $1 < k \leq n$, then TL_n contains a distinguished idempotent p_n called the *Jones–Wenzl idempotent*. The idempotents p_n can be defined recursively by $p_0 := \mathrm{id}_0$, $p_1 := \mathrm{id}_1$, and

$$(2.1) \quad \begin{array}{c} \text{---} \\ | \\ \boxed{n} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{n-1} \\ | \\ \text{---} \end{array} - \frac{[n-1]}{[n]} \begin{array}{c} \text{---} \\ | \\ \boxed{n-1} \\ | \\ \text{---} \\ | \\ \boxed{n-1} \\ | \\ \text{---} \end{array}$$

for $n > 1$, where each box represents a Jones–Wenzl idempotent.

2.5. Khovanov homology and Bar-Natan’s bicategory. Given an oriented link diagram, we can resolve each of its crossings by applying the Kauffman bracket skein relation

$$\times = q^{(3\epsilon-1)/2} (\text{---} \text{---} \text{---} - q \text{---} \text{---})$$

where $\epsilon \in \{\pm 1\}$ is the sign of the crossing. Replacing circles in the resulting terms by factors of $q + q^{-1}$, we obtain the *Jones polynomial* of the link. The same algorithm associates with a tangle a morphism in \mathcal{TL} .

In [Kho00] Khovanov categorified this construction by adding morphisms between resolutions in form of saddle cobordisms. Arranging the crossing resolutions as the vertices of a hypercube and the saddle cobordisms as its edges, Khovanov assigned to a link diagram a commutative diagram in the category of 2-dimensional cobordisms. By applying a TQFT functor corresponding to the Frobenius algebra $\mathbb{k}[x]/x^2$, he then defined a chain complex whose homology is a link invariant, called *Khovanov homology*. The Jones polynomial is recovered as the graded Euler characteristic of Khovanov homology. This construction extends to tangles in surfaces [APS04] and is (projectively) functorial with respect to tangle cobordisms.

It was observed by Bar-Natan that most of Khovanov's construction can be performed formally on the level of a complex $[[T]]$ called the *formal Khovanov bracket* of $T \subset F$ [BN05]. Here F can be any smooth oriented surface with boundary and it is assumed that $\partial T = B$ for a set $B \subset \partial F$. The complex $[[T]]$ is constructed in the additive closure of the *Bar-Natan category* $\mathcal{BN}(F, B)$, the graded \mathbb{k} -linear category whose objects are flat tangles in F with boundary B and with grading shifts, and whose morphism sets are generated by compact surfaces in $F \times I$ decorated by dots. Such surfaces are required to have boundaries of the form $\partial S = (T \times \{0\}) \cup (B \times I) \cup (T' \times \{1\})$ and are considered up to isotopy and up to the following local relations:

- *sphere evaluations:*

$$(2.2) \quad \begin{array}{c} \text{blue sphere} \\ \hline \end{array} = 0 \quad \begin{array}{c} \text{blue sphere with dot} \\ \hline \end{array} = 1$$

- *neck cutting relation:*

$$(2.3) \quad \begin{array}{c} \text{blue neck} \\ \hline \end{array} = \begin{array}{c} \text{blue cup with dot} \\ \hline \end{array} + \begin{array}{c} \text{blue cup} \\ \hline \end{array}$$

- *dot reduction:*

$$(2.4) \quad \begin{array}{c} \text{blue parallelogram with two dots} \\ \hline \end{array} = 0$$

We define the (unshifted) degree of a cobordism $S \subset F \times I$ by

$$|S| := -\chi(S) + \frac{\#\{\text{corners in } S\}}{2} + 2\#\{\text{dots on } S\}$$

Note that the relations are degree preserving, and that the neck cutting relation evaluates a handle attached to a plane as a dot scaled by 2. Because of that, it is common to think of a dot as „half” of a handle, even when 2 is not an invertible scalar.

We will denote by $\mathcal{BN}(\mathbb{A}) := \mathcal{BN}(\mathbb{A}, \emptyset)$ the Bar-Natan category of the annulus. Moreover, we will denote by \mathbf{BN} the graded bicategory whose objects correspond to nonnegative integers, and whose morphism categories are given by the Bar-Natan categories $\mathcal{BN}(I^2, B_{n,m})$ where $B_{n,m} \subset \partial I^2$ consists of n points on the bottom boundary of the square I^2 and of m points on the top boundary. Note that the horizontal composition in \mathbf{BN} is induced by the composition of tangles, and that \mathbf{BN} has a monoidal structure corresponding to the monoidal structure on the category of flat tangles.

The formal Khovanov bracket is projectively functorial [BN05]. Indeed, there is a way of associating a chain map with each Reidemeister move and with any cobordism containing a unique critical point. One constructs a chain map for any smooth tangle cobordism by decomposing the cobordism into a sequence of the above elementary pieces and composing the associated maps; choosing a different decomposition may at most changes the map within its homotopy class, or changes the global sign of the map. This global sign can be fixed by using the explicit equivalence between $\mathcal{U}_q(\mathfrak{gl}_2)$ -foams and \mathbf{BN} constructed in [Bel+19]. Hence, there is a well-defined bifunctor

$$[[-]]: \mathbf{Tan} \longrightarrow \mathcal{K}(\mathbf{BN}^{\oplus}),$$

where \mathbf{Tan} denotes the bicategory whose 1-morphisms are oriented tangle diagrams in I^2 (possibly with crossings) and whose 2-morphisms are smooth tangle cobordisms in \mathbb{R}^4 , up to isotopy.

2.6. Quantum horizontal trace. There is a functorial construction, called the twisted horizontal trace, sending suitable tangles in $F \times I$ to links in a surface bundle over a circle with fiber F and monodromy $\sigma \in \text{Diff}(F)$. This construction works more generally for any bicategory \mathbf{C} equipped with an endofunctor $\Sigma: \mathbf{C} \rightarrow \mathbf{C}$. Let us recall its definition from [BPW19].

Definition 2.1. The Σ -twisted horizontal trace of \mathbf{C} is the category $\text{hTr}(\mathbf{C}, \Sigma)$, whose objects are 1-morphisms $f: \Sigma x \rightarrow x$ in \mathbf{C} and morphisms from $f \in \mathbf{C}(\Sigma x, x)$ to $g \in \mathbf{C}(\Sigma y, y)$ are equivalence classes $[p, \alpha]$ of squares

$$\begin{array}{ccc} \Sigma x & \xrightarrow{f} & x \\ \Sigma p \downarrow & \swarrow \alpha & \downarrow p \\ \Sigma y & \xrightarrow{g} & y \end{array}$$

where $\alpha: p \star f \Rightarrow g \star \Sigma p$ is a 2-morphism in \mathbf{C} , modulo the relation

$$(2.5) \quad \begin{array}{ccc} \Sigma x & \xrightarrow{f} & x \\ \Sigma p \left(\begin{array}{c} \swarrow \Sigma \tau \\ \leftarrow \end{array} \right) \swarrow \Sigma p' & \swarrow \alpha & \downarrow p \\ \Sigma y & \xrightarrow{g} & y \end{array} \sim \begin{array}{ccc} \Sigma x & \xrightarrow{f} & x \\ \Sigma p' \downarrow & \swarrow \alpha & \downarrow p \left(\begin{array}{c} \leftarrow \tau \\ \swarrow \end{array} \right) \\ \Sigma y & \xrightarrow{g} & y \end{array}$$

for 1-morphisms $p, p': x \rightarrow y$, and 2-morphisms $\alpha: p \star f \Rightarrow g \star \Sigma p'$ and $\tau: p' \Rightarrow p$. Here $[\mathbb{1}_x, \mathbb{1}_f]$ the identity on f and the composition

$$(2.6) \quad [q, \beta] \circ [p, \alpha] := [q \star p, (\beta \star \mathbb{1}_{\Sigma p}) \circ (\mathbb{1}_q \star \alpha)]$$

can be visualized as stacking squares one on top of the other:

$$(2.7) \quad \begin{array}{ccc} \Sigma y & \xrightarrow{g} & y \\ \Sigma q \downarrow & \swarrow \beta & \downarrow q \\ \Sigma z & \xrightarrow{h} & z \end{array} \circ \begin{array}{ccc} \Sigma x & \xrightarrow{f} & x \\ \Sigma p \downarrow & \swarrow \alpha & \downarrow p \\ \Sigma y & \xrightarrow{g} & y \end{array} := \begin{array}{ccc} \Sigma x & \xrightarrow{f} & x \\ \Sigma p \downarrow & \swarrow \alpha & \downarrow p \\ \Sigma y & \xrightarrow{g} & y \\ \Sigma q \downarrow & \swarrow \beta & \downarrow q \\ \Sigma z & \xrightarrow{h} & z \end{array}$$

Unitarity and associativity follows from (2.5) with an appropriate composition of associators and unitors as τ .

When \mathbf{C} is \mathbb{k} -linear and $\Sigma: \mathbf{C} \rightarrow \mathbf{C}$ is given by linear maps on 2-morphism sets, we modify the definition of the twisted horizontal trace by allowing formal \mathbb{k} -linear combinations of squares as morphisms. Such linear combinations are taken modulo relation (2.5) and modulo the relation $[p, a\alpha + \beta] = a[p, \alpha] + [p, \beta]$ for $a \in \mathbb{k}$. This definition ensures that $\text{hTr}(\mathbf{C}, \Sigma)$ is itself a \mathbb{k} -linear category.

If $\Sigma = \mathbb{1}$, we recover the horizontal trace from [Bel+17], and in this case we will omit Σ from the notation. For example,

$$\text{hTr}(\mathbf{BN}) \simeq \mathcal{BN}(\mathbb{A}),$$

where the equivalence sends a flat tangle $T \subset I^2$ to its annular closure $\widehat{T} \subset \mathbb{A}$.

In case \mathbf{C} is pregraded, we can define the *quantum endofunctor*, which acts as the identity on objects and 1-morphisms and which satisfies $\Sigma\alpha = q^{-|\alpha|}\alpha$ for any homogeneous 2-morphism α and for a fixed invertible element $q \in \mathbb{k}^\times$. The horizontal trace twisted by this endofunctor is called *quantum horizontal trace* and will be denoted by $\mathrm{hTr}_q(\mathbf{C})$.

We will denote the quantum horizontal trace of \mathbf{BN} by

$$\mathcal{BN}_q(\mathbb{A}) := \mathrm{hTr}_q(\mathbf{BN})$$

and call it the *quantum annular Bar-Natan category*. This category admits the following graphical description. Choose a radial arc $\mu \subset \mathbb{A}$ connecting the two components of $\partial\mathbb{A}$, and let $M \subset \mathbb{A} \times I$ be the *membrane* $M := \mu \times I$. The category $\mathcal{BN}_q(\mathbb{A})$ can be defined in the same way as $\mathcal{BN}(\mathbb{A})$, except that objects are required to be transverse to μ and cobordisms in $\mathbb{A} \times I$ are required to be transverse to M . Isotopic cobordisms are identified if the ambient isotopy fixes the membrane setwise. Otherwise, we rescale the target cobordism as shown in Figure 1, where the coorientation of the membrane is induced by the orientation of the core of the annulus.

There is a natural functor

$$S^1 \times (-) : \mathcal{TL} \longrightarrow \mathcal{BN}_q(\mathbb{A})$$

that sends an object n to a collection of n essential circles in \mathbb{A} and a cap or cup to a band between neighboring circles. More generally, this functor sends a flat tangle T to the surface $S^1 \times T \subset \mathbb{A} \times I$, which corresponds to the morphism $[T, \mathbb{1}_T]$ in the quantum horizontal trace.

The following result first appeared in [BPW19].

Theorem 2.2. *Suppose $1 - q^d$ is invertible for all $d \neq 0$. Then the functor $S^1 \times (-)$ induces an equivalence of categories*

$$\mathcal{TL}^\oplus \simeq \mathrm{hTr}_q(\mathbf{BN})^\oplus \cong \mathcal{BN}_q(\mathbb{A})^\oplus,$$

where objects in the involved categories are assumed to come with formal grading shifts.

A proof of this proposition will be given in Subsection 2.9. Note that the assumption on q implies that in $\mathcal{BN}_q(\mathbb{A})$, dots on essential components of a cobordism $S \subset \mathbb{A} \times I$ are equal to zero. Indeed, moving such a dot through the membrane, we see that $(1 - q^2)$ times the dot has to be equal to zero. This is a special case of the following result, in which $\alpha: p \Rightarrow p$ in a 2-endomorphism in a pregraded (and possibly graded) bicategory \mathbf{C} :

Lemma 2.3. *Suppose $1 - q^d$ is invertible in \mathbb{k} . If $|\alpha| = d$, then the square*

$$\begin{array}{ccc} x & \xrightarrow{\mathbb{1}_x} & x \\ p \downarrow & \alpha & \downarrow p \\ y & \xrightarrow{\mathbb{1}_y} & y \end{array}$$

is zero in $\mathrm{hTr}_q(\mathbf{C})$.

Proof. We have

$$\begin{array}{c} x \xrightarrow{\mathbb{1}_x} x \\ \downarrow p \quad \swarrow \alpha \quad \downarrow p \\ y \xrightarrow{\mathbb{1}_y} y \end{array} = \begin{array}{c} x \xrightarrow{\mathbb{1}_x} x \\ \downarrow p \quad \swarrow \alpha \quad \downarrow p \\ y \xrightarrow{\mathbb{1}_y} y \end{array} \sim q^d \cdot \begin{array}{c} x \xrightarrow{\mathbb{1}_x} x \\ \downarrow p \quad \swarrow \mathbb{1}_p \quad \downarrow p \\ y \xrightarrow{\mathbb{1}_y} y \end{array} = q^d \cdot \begin{array}{c} x \xrightarrow{\mathbb{1}_x} x \\ \downarrow p \quad \swarrow \alpha \quad \downarrow p \\ y \xrightarrow{\mathbb{1}_y} y \end{array}$$

which implies that the square from the lemma is annihilated by $1 - q^d$. The result follows because $1 - q^d$ is invertible in \mathbb{k} \square

In the graded setting, the lemma admits a generalization in which $\mathbb{1}_x$ and $\mathbb{1}_y$ are replaced by shifted copies $q^n \mathbb{1}_x$ and $q^m \mathbb{1}_y$. For the lemma to remain true, d must be the degree of α viewed as a 2-morphism between the *unshifted* 1-morphisms $p \star \mathbb{1}_x = p$ and $\mathbb{1}_y \star p = p$. To see this, consider a 2-morphism $\alpha: p \star (q^n \mathbb{1}_x) \Rightarrow (q^m \mathbb{1}_y) \star p$, and apply the original version of the lemma to the 2-morphism $\mathfrak{s}^{-m} \circ \alpha \circ \mathfrak{s}^n: p \star \mathbb{1}_x \Rightarrow \mathbb{1}_y \star p$.

In Subsection 2.9, we will reinterpret the lemma in terms of the vertical quantum trace.

Remark 2.4. An argument similar to the one used in the proof of the lemma shows that if the commutative ring \mathbb{k} is equipped with a nontrivial grading and if $a \in \mathbb{k}$ has degree $|a| = d$, then $(1 - q^d)a$ annihilates all morphisms in the quantum horizontal trace.

2.7. Cyclicity isomorphism. For a bicategory with left and right duals, the twisted horizontal trace can be seen as the target category of a *universal twisted shadow* (see [BPW19]). In particular, this means that $\text{hTr}(\mathbf{C}, \Sigma)$ comes with the following extra structure: for each object $x \in \mathbf{C}$, there is a functor $[-]_x: \mathbf{C}(\Sigma x, x) \rightarrow \text{hTr}(\mathbf{C}, \Sigma)$ given by

$$[f]_x := f, \quad [\alpha]_x := [\mathbb{1}_x, \alpha].$$

Likewise, for each pair of 1-morphisms $\Sigma x \xrightarrow{f} y \xrightarrow{g} x$ in \mathbf{C} , there is a *cyclicity morphism* $\theta_{g,f}: [f \star \Sigma g]_y \rightarrow [g \star f]_x$ given by an identity 2-morphism (or by an associator in the non-strict setting):

$$\theta_{g,f} := \begin{array}{ccccc} \Sigma y & \xrightarrow{\Sigma g} & \Sigma x & \xrightarrow{f} & y \\ \Sigma g \downarrow & \swarrow \mathbb{1}_{\Sigma x} & \swarrow \mathbb{1}_y & \swarrow & \downarrow g \\ \Sigma x & \xrightarrow{f} & y & \xrightarrow{g} & x \end{array}$$

When \mathbf{C} has appropriate duals (which is the case for the bicategories considered in this paper), then the morphism $\theta_{g,f}$ turns out to be an isomorphism. Moreover, $\theta_{g,f}$ is natural in the sense that the following diagram commutes

$$\begin{array}{ccc} [f \star \Sigma g]_y & \xrightarrow{\theta_{g,f}} & [g \star f]_x \\ \downarrow [\alpha \star \Sigma \beta]_y & & \downarrow [\beta \star \alpha]_x \\ [f' \star \Sigma g']_y & \xrightarrow{\theta_{g',f'}} & [g' \star f']_x \end{array}$$

for 2-morphisms $\alpha: f \Rightarrow f'$ and $\beta: g \Rightarrow g'$.

When Σ is the quantum endofunctor of a pregraded bicategory \mathbf{C} , we drop the subscript $[f]_x$ referring to the object x and instead write $[f]$ for $[f]_x \in \mathrm{hTr}_q(\mathbf{C}) = \mathrm{hTr}(\mathbf{C}, \Sigma)$. In this case, the naturality property of θ can be stated as saying that

$$\begin{array}{ccc} [f \star g] & \xrightarrow{\theta_{g,f}} & [g \star f] \\ \downarrow [\alpha \star \beta] & & \downarrow q^{|\beta|} [\beta \star \alpha] \\ [f' \star g'] & \xrightarrow{\theta_{g',f'}} & [g' \star f'] \end{array}$$

commutes in $\mathrm{hTr}_q(\mathbf{C})$ for any 2-morphism $\alpha: f \Rightarrow f'$ and any homogeneous 2-morphism $\beta: g \Rightarrow g'$. More generally, we have a similar commutative diagram in which $[\alpha \star \beta]$ is replaced by $[\alpha_1 \star \beta_1] + \dots + [\alpha_n \star \beta_n]$ and $q^{|\beta|} [\beta \star \alpha]$ is replaced by

$$q^{|\beta_1|} [\beta_1 \star \alpha_1] + \dots + q^{|\beta_n|} [\beta_n \star \alpha_n]$$

for any 2-morphisms $\alpha_i: f \Rightarrow f'$ and homogeneous 2-morphisms $\beta_i: g \Rightarrow g'$.

The cyclicity morphism $\theta_{g,f}$ depends not just on the 1-morphism $g \star f$, but on the factorization of this 1-morphism into f and g . This point is important when working in a graded setting because a 1-morphism of the form $\mathbb{Q}^d(f \star g)$ can be factored as $(\mathbb{Q}^{d-n}f) \star (\mathbb{Q}^n g)$ for any integer $n \in \mathbb{Z}$. The corresponding cyclicity morphism in the quantum horizontal trace depends on the chosen n , but using relation (2.9) from the next subsection, one can see that

$$(2.8) \quad \theta_{\mathbb{Q}^n g, \mathbb{Q}^{d-n} f} = q^{n-m} \theta_{\mathbb{Q}^m g, \mathbb{Q}^{d-m} f}$$

for all $m, n \in \mathbb{Z}$. Thus, changing the chosen n only rescales the cyclicity morphism by an overall power of q .

We conclude this subsection by noting that the functor

$$[-]: \mathbf{C}(x, x) \longrightarrow \mathrm{hTr}_q(\mathbf{C})$$

is linear and thus induces a functor

$$[-]: \mathrm{Ch}(\mathbf{C}(x, x)) \longrightarrow \mathrm{Ch}(\mathrm{hTr}_q(\mathbf{C}))$$

between the corresponding categories of chain complexes and their associated homotopy categories. Applying this functor to the formal Bar-Natan bracket of an (n, n) tangle diagram, we get a chain complex in the quantum annular Bar-Natan category. For generic q , this complex can be viewed as a complex in $\mathcal{TL}^\oplus \subset \mathcal{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$ or, after forgetting the quantum group action, as a complex of \mathbb{k} -modules. As such, it generalizes the *annular Khovanov complex* defined in [APS].

2.8. Quantum horizontal trace of \mathbf{C}_0 . In this paper, we will usually work with the full Bar-Natan bicategory \mathbf{BN} . However, most of our results could also be formulated using the bicategory $\mathbf{BN}_0 \subset \mathbf{BN}$ in which 1-morphisms still have formal grading shifts, but 2-morphisms are required to have degree 0 (after taking into account these formal grading shifts). For our results to remain true for \mathbf{BN}_0 , we have to make a minor modification in our definition of the quantum horizontal trace.

Thus, let \mathbf{C} be a graded bicategory and let $\mathbf{C}_0 \subset \mathbf{C}$ be the bicategory which has the same objects and 1-morphisms but only 2-morphisms of degree 0. In the quantum horizontal trace

of \mathbf{C} , the relation

$$(2.9) \quad \begin{array}{ccc} x & \xrightarrow{f} & x \\ \mathfrak{q}p \downarrow & \swarrow \mathfrak{q}\alpha & \downarrow \mathfrak{q}p \\ y & \xrightarrow{g} & y \end{array} \sim \begin{array}{ccc} x & \xrightarrow{f} & x \\ p \downarrow & \swarrow \alpha & \downarrow p \\ y & \xrightarrow{g} & y \end{array}$$

is satisfied because

$$(2.10) \quad \begin{array}{ccc} x & \xrightarrow{f} & x \\ \mathfrak{q}p \left(\begin{array}{c} \xleftarrow{\Sigma \mathfrak{s}_p} \\ \downarrow p \end{array} \right) \swarrow \alpha \circ \mathfrak{s}_p^{-1} & & \downarrow \mathfrak{q}p \\ y & \xrightarrow{g} & y \end{array} \sim \begin{array}{ccc} x & \xrightarrow{f} & x \\ p \downarrow \swarrow \alpha \circ \mathfrak{s}_p^{-1} & & \mathfrak{q}p \left(\begin{array}{c} \xleftarrow{\mathfrak{s}_p} \\ \downarrow p \end{array} \right) \\ y & \xrightarrow{g} & y \end{array}$$

where Σ denotes the quantum endofunctor, and \mathfrak{s} is the natural degree 1 isomorphism between the identity functor and the grading shift functor \mathfrak{q} .

However, relation (2.9) is a priori not longer true in $\mathrm{hTr}_q(\mathbf{C}_0)$ because the 2-morphism \mathfrak{s}_p in (2.10) has nonzero degree.

Since (2.9) is important for our purposes, we therefore *redefine* $\mathrm{hTr}_q(\mathbf{C}_0)$ by imposing (2.9) as an extra relation, which is required to hold in addition to the usual horizontal trace relation. It is then easy to see that

$$(2.11) \quad \mathrm{hTr}_q(\mathbf{C}_0) \cong \mathrm{hTr}_q(\mathbf{C})_0$$

where $\mathrm{hTr}_q(\mathbf{C})_0$ denotes the subcategory of $\mathrm{hTr}_q(\mathbf{C})$ that contains only degree 0 morphisms, and where the degree of a morphism in $\mathrm{hTr}_q(\mathbf{C})$ is defined by $||[p, \alpha]|| := |\alpha|$.

The extra relation (2.9) (or the resulting isomorphism (2.11)) also ensures that equation (2.8) still holds in $\mathrm{hTr}_q(\mathbf{C}_0)$. A further consequence of (2.11) is that the naturality condition

$$q^{|\beta|}[\beta \star \alpha] = \theta_{g', f'} \circ [\alpha \star \beta] \circ \theta_{g, f}^{-1}$$

still holds in $\mathrm{hTr}_q(\mathbf{C}_0)$ whenever $|\alpha \star \beta| = 0$. This can also be seen directly as follows: suppose α has degree $|\alpha| =: n$ (so that $|\beta| = -n$), and let $\tilde{\alpha}: f \Rightarrow \mathfrak{q}^{-n} f'$ and $\tilde{\beta}: g \Rightarrow \mathfrak{q}^n g'$ be the 2-morphisms $\tilde{\alpha} := \mathfrak{s}^{-n} \circ \alpha$ and $\tilde{\beta} := \mathfrak{s}^n \circ \beta$, so that $\tilde{\alpha} \star \tilde{\beta} = \alpha \star \beta$ and $\tilde{\beta} \star \tilde{\alpha} = \beta \star \alpha$. Since $\tilde{\alpha}$ and $\tilde{\beta}$ have degree 0, we then have

$$[\tilde{\beta} \star \tilde{\alpha}] = \theta_{\mathfrak{q}^n g', \mathfrak{q}^{-n} f'} \circ [\tilde{\alpha} \star \tilde{\beta}] \circ \theta_{g, f}^{-1}$$

in $\mathrm{hTr}_q(\mathbf{C}_0)$, and since (2.8) still holds in $\mathrm{hTr}_q(\mathbf{C}_0)$, this implies the desired result.

Returning to the quantum horizontal trace of the full bicategory \mathbf{C} , we note that $\mathrm{hTr}_q(\mathbf{C})$ is pregraded via the definition $||[p, \alpha]|| := |\alpha|$ mentioned earlier. We can turn $\mathrm{hTr}_q(\mathbf{C})$ into a graded category by setting

$$(2.12) \quad \mathfrak{q}[f] := [\mathfrak{q}f], \quad \mathfrak{q}[p, \alpha] := [p, \mathfrak{q}\alpha], \quad \mathfrak{s}[f] := [\mathbb{1}, \mathfrak{s}f].$$

Note that the automorphism $\mathfrak{q}: \mathrm{hTr}_q(\mathbf{C}) \rightarrow \mathrm{hTr}_q(\mathbf{C})$ sends the subcategory $\mathrm{hTr}_q(\mathbf{C})_0 \subset \mathrm{hTr}_q(\mathbf{C})$ to itself, and that every morphism in $\mathrm{hTr}_q(\mathbf{C})$ can be obtained from a morphism in $\mathrm{hTr}_q(\mathbf{C})_0$ by pre- and post-composing with suitable powers of \mathfrak{s} .

2.9. Connection with the quantum vertical trace. Let \mathbf{C} be a pregraded bicategory and $q \in \mathbb{k}^\times$ be a fixed invertible element. The *quantum vertical trace* of \mathbf{C} is the category obtained from \mathbf{C} by replacing each morphism category $\mathbf{C}(x, y)$ by its quantum trace $\mathrm{Tr}_q(\mathbf{C}(x, y))$ (see [BPW19]). Equivalently:

Definition 2.5. The *quantum vertical trace* of \mathbf{C} is the \mathbb{k} -linear category $\mathrm{vTr}_q(\mathbf{C})$ with the same objects as \mathbf{C} , and whose morphism are given by formal \mathbb{k} -linear combinations of 2-endomorphisms in \mathbf{C} , subject to the relation $\alpha \circ \beta \sim q^{|\alpha|} \beta \circ \alpha$ for 2-morphisms $\alpha: p \Rightarrow p'$ and $\beta: p' \Rightarrow p$. Composition in $\mathrm{vTr}_q(\mathbf{C})$ is induced by the horizontal composition in \mathbf{C} .

If \mathbf{C} is graded, then we modify the definition of $\mathrm{vTr}_q(\mathbf{C}_0)$ by also imposing the relation $q\alpha \sim q\alpha$ for all 2-endomorphisms α in \mathbf{C}_0 . This is to ensure that

$$(2.13) \quad \mathrm{vTr}_q(\mathbf{C}_0) \cong \mathrm{vTr}_q(\mathbf{C})_0.$$

In general, the quantum vertical trace admits a fully faithful embedding $\mathrm{vTr}_q(\mathbf{C}) \rightarrow \mathrm{hTr}_q(\mathbf{C})$ given by sending each object x to its identity 1-morphism $\mathbb{1}_x$ and each $\alpha: p \Rightarrow p$ to the square from Lemma 2.3. This embedding can be used to reestablish the result from Lemma 2.3. Indeed, in $\mathrm{vTr}_q(\mathbf{C})$, we have

$$\alpha \sim \alpha \circ \mathbb{1} \sim q^d(\mathbb{1} \circ \alpha) \sim q^d \alpha$$

for any homogeneous 2-endomorphism α of degree $|\alpha| = d$, and thus α is annihilated by $1 - q^d$. When $1 - q^d$ is invertible, this implies the lemma. Moreover, if $1 - q^d$ is invertible for all $d \neq 0$, then it follows that

$$(2.14) \quad \mathrm{vTr}_q(\mathbf{C}) = \mathrm{vTr}_q(\mathbf{C})_0$$

because in this case there are no nonzero morphisms of nonzero degree. In particular, the quantum vertical trace is usually only pregraded, with all morphisms supported in degree 0.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Ignoring grading shifts, every object in $\mathcal{BN}_q(\mathbb{A})$ can be viewed as a closed 1-manifold C embedded in the annular \mathbb{A} . By applying an ambient isotopy to C , we can move trivial components in C away from the arc $\mu \subset \mathbb{A}$ and straighten essential components so that they become concentric circles. The resulting 1-manifold $C' \subset \mathbb{A}$ is a closure of an identity tangle with some trivial circles. Moreover, the isotopy from C to C' can be realized as a composition of cyclicity isomorphisms in $\mathrm{hTr}_q(\mathbf{BN})$ and isotopies in I^2 .

In $\mathrm{hTr}_q(\mathbf{BN})^\oplus$, we also have a “delooping isomorphism” $\bigcirc \cong q\emptyset \oplus q^{-1}\emptyset$ given by the matrix $[S_\bullet \ S]^\dagger$ where S is a cap cobordism and S_\bullet denotes S with a dot. Applying this isomorphism to each trivial circle in C' , we can reduce C' to a direct sum of shifted copies of a closure of an identity tangle.

We conclude that $\mathrm{hTr}_q(\mathbf{BN})^\oplus$ is obtained from image of $\mathrm{vTr}_q(\mathbf{BN}) \rightarrow \mathrm{hTr}_q(\mathbf{BN})$ by allowing grading shifts and direct sums. Since $1 - q^d$ is invertible for all $d \neq 0$, we also have $\mathrm{vTr}_q(\mathbf{BN}) = \mathrm{vTr}_q(\mathbf{BN})_0 \cong \mathrm{vTr}_q(\mathbf{BN}_0)$ by (2.14) and (2.13), where the latter isomorphism uses that we have imposed the additional relation $q\alpha \sim q\alpha$ in the definition of $\mathrm{vTr}_q(\mathbf{BN}_0)$.

It remains to show that $\mathrm{vTr}_q(\mathbf{BN}_0) \simeq \mathcal{TL}$. This was essentially shown by Bar-Natan in [BN05, Prop. 10.10], but we will prove the result using a slightly different argument. For this, let

$$\varphi: \mathcal{TL} \longrightarrow \mathrm{vTr}_q(\mathbf{BN}_0)$$

denote the functor which sends each object of \mathcal{TL} to the corresponding object of \mathbf{BN}_0 , and each flat tangle T to the class of its identity 2-endomorphism in $v\mathrm{Tr}_q(\mathbf{BN}_0)$.

To see that φ is compatible with isotopy of tangles and with the relation $\bigcirc = q + q^{-1}$, recall from [BPW19] that the assignment $T \mapsto \mathbb{1}_T$ induces a well-defined map from the split Grothendieck group of $\mathbf{BN}_0(n, m)^\oplus$ to the quantum trace

$$\mathrm{Tr}_q(\mathbf{BN}_0(n, m)^\oplus) = \mathrm{Tr}_q(\mathbf{BN}_0(n, m)^\oplus) = v\mathrm{Tr}_q(\mathbf{BN}_0)(n, m).$$

Since isotopic tangles are isomorphic as objects of $\mathbf{BN}_0(n, m)$, it follows that their identity 2-morphisms are equal in the vertical trace. Likewise, the “delooping isomorphism” implies $\mathbb{1}_\bigcirc \sim q\mathbb{1}_\emptyset + q^{-1}\mathbb{1}_\emptyset \sim q\mathbb{1}_\emptyset + q^{-1}\mathbb{1}_\emptyset$ in $v\mathrm{Tr}_q(\mathbf{BN}_0)$, which corresponds to $\bigcirc = q + q^{-1}$.

It is clear that φ is bijective on objects, so the theorem will follow if we can prove that φ is full and faithful. For this, we can focus on a single morphism category $\mathbf{BN}_0(n, m)$. Using the “delooping isomorphism” and the invariance of the quantum trace under graded equivalences and under additive closures, we can further restrict our attention to the full subcategory $\mathcal{T} \subset \mathbf{BN}_0(n, m)$ generated by tangles without closed components. That is, we only need to consider 2-morphisms of the form $\alpha: \mathbb{q}^i T \Rightarrow \mathbb{q}^j T'$ where T and T' are flat tangles without closed components.

Let $S: T \Rightarrow T'$ be a dotted cobordism representing such an α . We first claim that the unshifted degree of S must satisfy $|S| \geq 0$, and $|S| = 0$ if and only if S is isotopic to a scalar multiple of an identity cobordism. To see this, note that the surface $S \subset I^3$ can be obtained from a cobordism $S' \subset I^3$ with only disk components by adding tubes, dots, and closed components. Using that undotted spheres are equal to zero and that T and T' have no closed components, it is easy to see that $|S| \geq |S'| \geq 0$. The second part of the claim follows because closed components evaluate to scalars.

We conclude that any 2-endomorphism $\alpha: \mathbb{q}^i T \Rightarrow \mathbb{q}^j T$ in $\mathcal{T} \subset \mathbf{BN}_0(n, m)$ must be a scalar multiple of an identity 2-morphism, which proves that φ is full.

To see that φ is faithful, consider a relation of the form $\alpha \circ \beta \sim q^{|\alpha|} \beta \circ \alpha$ for 2-morphisms $\alpha: \mathbb{q}^i T \Rightarrow \mathbb{q}^j T'$ and $\beta: \mathbb{q}^j T' \Rightarrow \mathbb{q}^i T$ in \mathcal{T} . We can drop the factor of $q^{|\alpha|}$ because $|\alpha| = 0$, and we can further assume that the 2-morphisms α and β are represented by dotted cobordisms $S: T \Rightarrow T'$ and $S': T' \Rightarrow T$. Since $|S| + |S'| = |S \circ S'| = |\alpha \circ \beta| = 0$ and since $|S|$ and $|S'|$ are nonnegative, it follows that $|S| = |S'| = 0$. However, this implies that S and S' are isotopic to scalar multiples of identity cobordisms, and that $i = j$.

In summary, the relation $\alpha \circ \beta \sim q^{|\alpha|} \beta \circ \alpha$ only has the effect of identifying the identity 2-morphisms $\mathbb{q}^i \mathbb{1}_{T'}$ and $\mathbb{q}^i \mathbb{1}_T$ of isotopic tangles in \mathcal{T} . By applying Khovanov’s universal TQFT, one can further see that $c\mathbb{1}_T$ is nonzero in \mathcal{T} for any $c \neq 0$ (for this to make sense, regard $\mathbb{1}_T \subset I^3$ as a cobordism from the empty 1-manifold to $\partial\mathbb{1}_T$).

We conclude that φ is faithful, which completes the proof of the theorem. \square

3. HOMOLOGICAL TOOLKIT

In this section we discuss the dg notions of twisting, splicing, and homological perturbation. The tools introduced here will be indispensable when discussing models for the Cooper–Krushkal projector.

We discuss the notion of twisting, and we do so first in an abstract setting. So let \mathcal{A} be a \mathbb{k} -linear category, and consider the dg category $\mathrm{Ch}(\mathcal{A})$ of complexes.

If (X, δ_X) is a complex over \mathcal{A} , then any complex of the form $(X, \delta_X + \alpha)$ will be referred to as a *twist* of X , written $\mathrm{tw}_\alpha(X)$. If X is equipped with a direct sum decomposition $X =$

$\bigoplus_{i \in I} X_i$ (in which the differential on X is the direct sum of differentials on X_i) then α can be expressed in terms of its components $\alpha_{ij} \in \text{Hom}_{\text{Ch}(\mathcal{A})}^1(X_j, X_i)$, and the condition that $\delta_X + \alpha$ be a differential on X can be expressed componentwise as

$$(\delta_{X_i} \circ \alpha_{ij} + \alpha_{ij} \circ \delta_{X_j}) + \sum_k \alpha_{ik} \circ \alpha_{kj} = 0.$$

for all $i, j \in I$. The twist $\text{tw}_\alpha(X)$ is *one-sided* if there is a direct sum decomposition $X \cong \bigoplus_{i \in I} X_i$ where the indexing set I is equipped with a partial order with respect to which the matrix representing α is strictly lower triangular (i.e., $\alpha_{ij} = 0$ unless $i > j$). Expressions of the form $\text{tw}_\alpha(\bigoplus_i X_i)$ are also called *one-sided twisted complexes*.

We will denote one-sided twisted complexes of the form $\text{tw}_\alpha(\bigoplus_{i \in I} X_i)$ diagrammatically, as in the following examples:

$$(3.1a) \quad \text{tw}_{\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}}(X \oplus Y) =: \left(X \xrightarrow{\alpha} Y \right)$$

$$(3.1b) \quad \text{tw}_{\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{bmatrix}}(X \oplus Y \oplus Z) =: \left(\begin{array}{ccc} & \xrightarrow{\gamma} & \\ X & \xrightarrow{\alpha} & Y \xrightarrow{\beta} & Z \end{array} \right)$$

In particular the mapping cone of a degree zero chain map $f : X \rightarrow Y$ will be expressed as

$$\text{Cone}(f) = \left(\mathbb{t}^{-1}X \xrightarrow{f} Y \right).$$

Here, as always, we abuse notation by regarding the degree zero morphism $f : X \rightarrow Y$ also as a degree 1 morphism $\mathbb{t}^{-1}X \rightarrow Y$.

3.1. Homological perturbation. If $X \simeq Y$ then a given twist $\text{tw}_\alpha(X)$ may or may not be homotopy equivalent to a corresponding twist of Y . For instance if X is any nonzero but contractible complex and $Y = 0$, then there certainly exist twists of X which are not contractible (and therefore not homotopy equivalent to any twist of Y).

Homological perturbation concerns the problem twisting a homotopy equivalence $X \simeq Y$ to a homotopy equivalence $\text{tw}_\alpha(X) \simeq \text{tw}_\beta(Y)$, provided that certain conditions are met.

Lemma 3.1. *Let I be a partially ordered set satisfying the following ascending chain condition: for each $i \in I$ there are finitely many $j \in I$ with $j \geq i$. Suppose we have a one-sided twist $\text{tw}_\alpha(\bigoplus_i X_i)$, and complexes $Y_i \simeq X_i$ for all i . Then there exists a one-sided twist $\text{tw}_\beta(\bigoplus_{i \in I} Y_i)$ such that $\text{tw}_\beta(\bigoplus_{i \in I} Y_i) \simeq \text{tw}_\alpha(\bigoplus_i X_i)$.*

In more details, if $f_i \in \text{Hom}^0(X_i, Y_i)$ and $g_i \in \text{Hom}^0(Y_i, X_i)$ are inverse homotopy equivalences with $h_i \in \text{End}^{-1}(X_i)$ satisfying $[\delta_{X_i}, h_i] = \text{Id}_{X_i} - g_i \circ f_i$, then the twist β is given in terms of components by

$$\begin{aligned} \beta_{kl} &= f_k \circ \alpha_{kl} \circ g_l - \sum_{k > i_1 > l} f_k \circ \alpha_{k, i_1} \circ h_{i_1} \circ \alpha_{i_1, l} \circ g_l \\ &+ \sum_{k > i_1 > i_2 > l} f_k \circ \alpha_{k, i_1} \circ h_{i_1} \circ \alpha_{i_1, i_2} \circ h_{i_2} \circ \alpha_{i_2, l} \circ g_l + \cdots \end{aligned}$$

if $k > l$ and $\beta_{kl} = 0$ otherwise, and the homotopy equivalence $F : \text{tw}_\alpha(X) \rightarrow \text{tw}_\beta(Y)$ is given in terms of components by $F_{kk} = f_k$ and

$$\begin{aligned} F_{kl} &= -f_k \circ \alpha_{k,l} \circ h_l + \sum_{k > i_1 > l} f_k \circ \alpha_{k,i_1} \circ h_{i_1} \circ \alpha_{i_1,l} \circ h_l \\ &\quad - \sum_{k > i_1 > i_2 > l} f_k \circ \alpha_{k,i_1} \circ h_{i_1} \circ \alpha_{i_1,i_2} \circ h_{i_2} \circ \alpha_{i_2,l} \circ h_l + \cdots \end{aligned}$$

if $k > l$, and $F_{kl} = 0$ otherwise.

Proof. See [Mar01, Theorems 2 and 3, and Subs. 8.6]. \square

Remark 3.2. In particular, if indices $k < l \in I$ are “adjacent”, meaning there are no indices $i \in I$ with $k < i < l$, then the component β_{kl} equals $g_k \circ \alpha_{kl} \circ f_l$.

Remark 3.3. The assumption that I satisfies the ascending chain condition can be weakened as follows. For each $k \in \mathbb{Z}$, let $I_k \subset I$ be the subset consisting of those $i \in I$ for which the chain group $(X_i)^k$ is nonzero. The statement of Lemma 3.1 is true under the assumption that I_k satisfies the ascending chain condition for all k . We may refer to this as the *local ascending chain condition relative to X* .

Remark 3.4. As a special case of Lemma 3.1, if each complex X_i is contractible then any one-sided twist $\text{tw}_\alpha(\bigoplus_{i \in I} X_i)$ is also contractible, provided that the partially ordered set I satisfies the ascending chain condition.

Remark 3.5. There is a version of Lemma 3.1 in which the ascending chain condition is replaced by the *descending chain condition* and the direct sum is replaced by direct product.

3.1.1. *Splicing.* If we have twisted complexes (mapping cones)

$$X_1 = \left(A \xrightarrow{\alpha} E \right), \quad X_2 = \left(\mathbb{t}^{-1}E \xrightarrow{\beta} B \right),$$

then the result of *splicing* X_1 and X_2 is the twisted complex

$$Z = \left(A \xrightarrow{\beta \circ \alpha} B \right).$$

The spliced complex Z is homotopy equivalent to the mapping cone:

$$Z \simeq \left(X_1 \xrightarrow{-\pi} X_2 \right)$$

where π is the projection-followed-by-inclusion of the E summand; this has homological degree 1, given that E appears in X_2 with shift \mathbb{t}^{-1} .

The operation of splicing can be performed for “longer” one-sided twisted complexes. For instance the result of splicing together twisted complexes

$$X_1 = \left(\begin{array}{ccc} & \alpha' & \\ & \curvearrowright & \\ A & \xrightarrow{\delta_{BA}} & B \xrightarrow{\alpha} E \end{array} \right)$$

and

$$X_2 = \left(\begin{array}{ccc} & \xrightarrow{\beta'} & \\ \mathfrak{t}^{-1}E & \xrightarrow{\beta} C & \xrightarrow{\delta_{DC}} D \\ & & \end{array} \right),$$

is the twisted complex

$$(3.2) \quad Z = \left(\begin{array}{cccc} & & \xrightarrow{\delta_{DA}} & \\ A & \xrightarrow{\delta_{BA}} B & \xrightarrow{\delta_{CB}} C & \xrightarrow{\delta_{DC}} D \\ & & & \end{array} \right),$$

where

$$\delta_{CB} = \beta \circ \alpha, \quad \delta_{CA} = \beta \circ \alpha', \quad \delta_{DB} := \beta' \circ \alpha, \quad \delta_{DA} = \beta' \circ \alpha'.$$

We may iterate the operation of splicing, even countably infinitely many times. For instance, suppose we have a collection of twisted complexes of the form

$$X_i = \left(\begin{array}{ccc} & \xrightarrow{\gamma_i} & \\ \mathfrak{t}^{-1}E_{i+1} & \xrightarrow{\alpha_i} C_i & \xrightarrow{\beta_i} E_i \\ & & \end{array} \right),$$

where C_i and E_i are some complexes ($i = 0, 1, 2, \dots$). Let $\pi_i \in \text{Hom}^1(X_i, X_{i+1})$ be the degree 1 chain map which projects onto then includes the E_{i+1} term. Then $\pi_{i+1} \circ \pi_i = 0$ and we have a complex of the form

$$Y = \left(X_0 \xrightarrow{-\pi_0} X_1 \xrightarrow{-\pi_1} X_2 \xrightarrow{-\pi_2} \dots \right).$$

Strictly speaking we can express this complex as $Y = \text{tw}_\alpha(\bigoplus_{i \geq 0} X_i)$ where the twist α is given in terms of components by $-\pi_i$. One can also express Y as the total complex of a bicomplex.

We will call Y the *pre-spliced* complex constructed from the complexes X_i . The following lemma shows that Y is homotopy equivalent to a *spliced* complex.

Lemma 3.6. *Retain notation as in the paragraph above. Then Y is homotopy equivalent to a twisted complex of the form*

$$Y \simeq \left(\begin{array}{ccc} & & \\ & & \\ & & \\ \dots & \longrightarrow C_1 & \longrightarrow C_0 & \longrightarrow E_0 \\ & & & \end{array} \right),$$

in which we have not drawn all of the (countably many) components of the differential, but all of them point rightward.

Proof. We can write Y as a twisted complex of the form

$$\left(\begin{array}{c} \begin{array}{ccccc} & & & \xrightarrow{\gamma_0} & \\ & & & \searrow^{\beta_0} & \\ & & & & E_0 \\ & & & \nearrow^{\alpha_0} & \\ \mathbb{t}^{-1}E_1 & \xrightarrow{\alpha_0} & C_0 & \xrightarrow{\beta_0} & \\ & & & \searrow^{-\text{id}} & \\ & & & & E_1 \\ & & & \nearrow^{\gamma_1} & \\ \mathbb{t}^{-1}E_2 & \xrightarrow{\alpha_1} & C_1 & \xrightarrow{\beta_1} & \\ & & & \searrow^{-\text{id}} & \\ & & & & E_2 \\ \dots & \xrightarrow{\beta_2} & & & \end{array} \end{array} \right)$$

This can be rearranged into a twisted complex of the form

$$\left(\begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \dots & \xrightarrow{\quad} & C_2 & \xrightarrow{\quad} & (\mathbb{t}^{-1}E_2 \rightarrow E_2) & \xrightarrow{\quad} & C_1 & \xrightarrow{\quad} & (\mathbb{t}^{-1}E_1 \rightarrow E_1) & \xrightarrow{\quad} & C_0 & \xrightarrow{\quad} & E_0 \end{array} \right)$$

Apply Lemma 3.1 to cancel the contractible terms $(\mathbb{t}^{-1}E_i \xrightarrow{-\text{id}} E_i)$. Even though there are infinitely many such terms, this cancelation is valid since the differential respects a partial order (satisfying the ascending chain condition) on the terms. \square

3.2. Op-well-ordered twisted complexes.

Definition 3.7. A partially ordered set I is *op-well-ordered* if every non-empty subset $S \subset I$ has a unique maximum element.

Note that if I is op-well-ordered, then the partial order is a total order: every pair of elements i, j is comparable since $\{i, j\}$ has a maximum. Moreover, every op-well-ordered set satisfies the *ascending chain condition*: any increasing sequence $i_0 \leq i_1 \leq \dots$ eventually stabilizes to the maximum element of $\{i_0, i_1, \dots\}$.

Example 3.8. Suppose $r: E \rightarrow B$ is a map of sets so that the base B is op-well-ordered and the fibers $r^{-1}(b)$ are op-well-ordered for all b . Then E has an op-well-ordering given by $e \leq e'$ if $r(e) < r(e')$ in B or $r(e) = r(e') =: b$ and $e \leq e'$ in $r^{-1}(b)$.

Corollary 3.9. If I and J are op-well-ordered then so is $I \times J$ with the lexicographic ordering $(i, j) \leq (i', j')$ if $i < i'$ or $i = i'$ and $j \leq j'$. \square

Op-well-ordered sets arise as sets which index iterated one-sided twisted complexes.

Definition 3.10. Suppose \mathcal{A} is an additive category. An *op-well-ordered twisted complex* in $\text{Ch}^-(\mathcal{A})$ is a one-sided twisted complex $\text{tw}_\delta(\bigoplus_{i \in I} \mathbb{t}^{a_i} X_i)$ where I is op-well-ordered, and the shifts a_i are bounded above.

If \mathcal{X} is a collection of complexes in $\text{Ch}^-(\mathcal{A})$, we let $\langle \mathcal{X} \rangle$ denote full subcategory of $\text{Ch}^-(\mathcal{A})$ consisting of op-well ordered twisted complexes whose terms are finite direct sums of shifts of direct summands of objects in \mathcal{X} .

Definition 3.11. If \mathcal{X} is a collection of complexes in $\text{Ch}^-(\mathcal{A})$, then we let $\overline{\mathcal{X}}$ denote the closure of \mathcal{X} with respect to homotopy equivalences.

Lemma 3.12. Let $\mathcal{X}, \mathcal{Y} \subset \text{Ch}^-(\mathcal{A})$ be some collection of complexes such that each $N \in \mathcal{X}$ is homotopy equivalent to a complex in $\langle \mathcal{Y} \rangle$. Then $\langle \mathcal{X} \rangle \subset \overline{\langle \mathcal{Y} \rangle}$.

Proof. Suppose $X = \text{tw}_\delta(\bigoplus_{i \in I} X_i)$ with $X_i \in \mathcal{X}$ and I op-well-ordered, and suppose each X_i is homotopy equivalent to $\text{tw}_{\varepsilon(i)}(\bigoplus_{j \in J_i} Y_{i,j})$ with J_i op-well-ordered for all $i \in I$, and $Y_{i,j} \in \mathcal{Y}$. Let E denote the set of pairs (i, j) with $j \in J_i$, and let $r: E \rightarrow I$ be the projection map $(i, j) \mapsto i$. Then E has the structure of an op-well-ordered set via $(i, j) \leq (i', j')$ if $i < i'$ or $i = i'$ and $j < j'$.

Since I satisfies the ascending chain condition, the homological perturbation lemma applies, so

$$\text{tw}_\delta \left(\bigoplus_{i \in I} X_i \right) \simeq \text{tw}_{\delta'} \left(\bigoplus_{i \in I} \text{tw}_{\varepsilon_i} \left(\bigoplus_{j \in J_i} Y_{i,j} \right) \right) \cong \text{tw}_{\delta' + \varepsilon} \left(\bigoplus_{(i,j) \in E} Y_{i,j} \right).$$

where δ' strictly increases the index i and $\varepsilon = \sum_i \varepsilon(i)$ is a sum of terms which fix the index i but strictly increase j . Thus, the twist $\delta' + \varepsilon$ is strictly increasing with respect to the partial order on E , and this last complex is in $\langle \mathcal{Y} \rangle$. \square

Corollary 3.13. The category $\langle \mathcal{X} \rangle \subset \text{Ch}^-(\mathcal{A})$ is closed under taking op-well-ordered twisted complexes. \square

Remark 3.14. This corollary would be false if we had only allowed $\mathbb{Z}_{\leq 0}$ -indexed one-sided twisted complexes in the definition of $\langle \mathcal{X} \rangle$.

3.3. Combing hairs. We use the phrase ‘‘combing hairs’’ to refer to the construction of isomorphisms of the form $\text{tw}_\delta(\bigoplus_{i \in I} X_i) \cong \text{tw}_{\delta'}(\bigoplus_{i \in I'} X_i)$ in which I is an op-well-ordered-set and $I' = I$ with a coarser partial order \leq' . Being coarser means that the condition $i \leq' j$ is more restrictive than $i \leq j$. Here are some instances where we wish to apply such ideas (we will expand on these momentarily):

- 1) Suppose \mathcal{X} is a collection of complexes such that $\text{Hom}(N, M)$ has no cohomology in positive degrees for all $N, M \in \mathcal{X}$, and $X = \text{tw}_\delta(\bigoplus_{i \in I} \mathfrak{t}^{a_i} N_i)$ is an op-well-ordered twisted complex with $N_i \in \mathcal{X}$. Let $I' = I$ with the partial order given by $i <' j$ if $i < j$ and $a_i < a_j$. Combing hairs in such situations amounts to the elimination of components of the twist $\mathfrak{t}^{a_i} N_i \rightarrow \mathfrak{t}^{a_j} N_j$ which have positive cohomological degree when regarded as elements of $\text{Hom}(N_i, N_j)$.
- 2) Suppose $(\Omega, \trianglelefteq)$ is a partially ordered set and $\mathcal{X}_\omega \subset \text{Ch}^-(\mathcal{A})$ is a collection of complexes indexed by $\omega \in \Omega$ such that if $N \in \mathcal{X}_\omega$ and $M \in \mathcal{X}_\nu$ then $\text{Hom}(N, M) \simeq 0$ unless $\omega \trianglelefteq \nu$. Suppose $X = \text{tw}_\delta(\bigoplus_{i \in I} X_i)$ is an op-well-ordered twisted complex with $X_i \in \mathcal{X}_{\omega(i)}$. Let $I' = I$ with the partial order given by $i \leq' j$ if $i \leq j$ and $\omega(i) \trianglelefteq \omega(j)$. Combing hairs in such situations amounts to the elimination of components of the twist $X_i \rightarrow X_j$ in which $\omega(j)$ is lower or incomparable to $\omega(i)$.

Our main lemma will combine both of the above notions. For technical reasons we will need some boundedness and compactness assumptions on our complexes in \mathcal{X}_ω . To this end, fix an additive category \mathcal{A} , suppose (Ω, \leq) is a partially ordered set, and for each $\omega \in \Omega$ suppose we have a collection of complexes \mathcal{X}_ω such that:

- 1) Each complex in \mathcal{X}_ω is supported in non-positive cohomological degrees.
- 2) If $N \in \mathcal{X}_\omega$ and $M \in \mathcal{X}_\nu$, then the chain objects N^k (zero unless $k \leq 0$) satisfy

$$\mathrm{Hom}(N^k, M) \simeq 0$$

if $\omega \not\leq \nu$ or $\omega = \nu$ and $k < 0$.

Lemma 3.15. *Retain the setup from the previous paragraph. Suppose I is an op-well-ordered set and we have a mapping $I \rightarrow \Omega \times \mathbb{Z}$, $(i \mapsto (\omega(i), a(i)))$ and a twisted complex $X = \mathrm{tw}_\delta(\bigoplus_{i \in I} \mathfrak{t}^{a(i)} X_i)$ with $X_i \in \mathcal{X}_{\omega(i)}$. Assume that the mapping $a: I \rightarrow \mathbb{Z}$ has finite preimages and $a^{-1}(k) = \emptyset$ for $k \gg 0$. Let $I' = I$ with the partial order $i <' j$ if $i < j$ and $(\omega(i), a(i)) < (\omega(j), a(j))$ in the lexicographic order ($\omega(i) < \omega(j)$ or $\omega(i) = \omega(j)$ and $a(i) < a(j)$). Then $X \cong \mathrm{tw}_{\delta'}(\bigoplus_{i \in I'} \mathfrak{t}^{a(i)} X_i)$ where the twist δ' respects the partial order $<'$.*

Proof. Let's introduce some terminology and notation for the proof. We denote by 0 the unique maximum element of I . We may assume that I has a unique minimum element $-\infty \in I$ (if not, then we may replace I by $\{-\infty\} \sqcup I$ and set $X_{-\infty} = 0$). Recall, a subset $K \subset I$ is *convex* if $k \leq i \leq k'$ and $k, k' \in K$ implies $i \in K$. Given $i \leq j$ in I we have the interval $[i, j] := \{k \in I \mid i \leq k \leq j\}$, and $(i, j] = [i, j] \setminus \{i\}$ (and so on). Note that intervals are convex.

If $K \subset I$ is convex then $X_K := \mathrm{tw}_{\delta_K}(\bigoplus_{i \in K} \mathfrak{t}^{a(i)} X_i)$ is a well-defined one-sided twisted complex, where the twist δ_K is obtained from δ by restriction. A *combing* of X_K is a pair (Y_K, φ_K) where Y_K is a one-sided twisted complex of the form in the statement (with respect to the coarse partial order) and φ_K is an isomorphism $X_K \xrightarrow{\cong} Y_K$ which preserves the fine partial order on K .

We wish to show that $X = X_{[-\infty, 0]}$ can be combed. Our plan is to show that each $X_{[i, 0]}$ can be combed, by transfinite induction. In the base case, we observe that $(X_{\{0\}}, \mathrm{id})$ is a combing of $X_{\{0\}}$, trivially.

Now, fix i and assume by induction that we have constructed combings $(Y_{[j, 0]}, \varphi_{[j, 0]})$ of $X_{[j, 0]}$, for all $j > i$, and assume that these combings are compatible in the sense that if $j < j'$ then $\varphi_{[j', 0]}$ is obtained from $\varphi_{[j, 0]}$ by restriction. Then we have a combing $(Y_{(i, 0]}, \varphi_{(i, 0]})$ of $X_{(i, 0]}$ by taking the union of combings $(Y_{[j, 0]}, \varphi_{[j, 0]})$ for $j > i$.

To complete the proof we must construct a combing of $X_{[i, 0]}$. Note that

$$(3.3) \quad X_{[i, 0]} \cong (\mathfrak{t}^{a(i)} X_i \xrightarrow{\partial} X_{(i, 0]}) \cong (\mathfrak{t}^{a(i)} X_i \xrightarrow{\partial'} Y_{(i, 0]}),$$

where the second isomorphism is $\begin{bmatrix} \mathrm{id} & 0 \\ 0 & \varphi_{(i, 0]} \end{bmatrix}$. Let us decompose $(i, 0]$ into subsets $J_1 \sqcup J_2 \sqcup J_3$ as follows. Let J_3 consist of those $j > i$ with $\omega(i) < \omega(j)$ or $\omega(i) = \omega(j)$ and $a(i) < a(j)$. Let J_2 consist of those $j > i$ with $\omega(i) = \omega(j)$ and $a(i) \geq a(j)$. Let J_1 consist of those $j > i$ with $\omega(i) \not\leq \omega(j)$.

Note that each J_l is convex with respect to the coarse partial order so the combed $Y_{(i, 0]}$ can be written as

$$Y_{(i, 0]} = (Y_{J_1} \rightarrow Y_{J_2} \rightarrow Y_{J_3})$$

(with a long arrow connecting Y_{J_1} to Y_{J_3}) where Y_{J_i} contains those direct summands $\mathbb{t}^{a(j)} X_j$ with $j \in J_i$.

We claim that $\text{Hom}(X_i, Y_{J_1}) \simeq 0$. To prove this claim, consider the chain objects $(X_i)^k$ of X_i . We have $\text{Hom}((X_i)^k, X_j) \simeq 0$ for all $j \in J_1$ since $\omega(i) \not\leq \omega(j)$. Since $(X_i)^k$ is an object of \mathcal{A} , it is compact when regarded as an object of $\text{Ch}^-(\mathcal{A})$. Thus

$$\text{Hom}((X_i)^k, \bigoplus_{j \in J_1} X_j) = \bigoplus_{j \in J_1} \text{Hom}((X_i)^k, X_j) \simeq 0.$$

Now, homological perturbation tells us $\text{Hom}((X_i)^k, Y_{J_1}) \simeq 0$; this applies since the indexing set J_1 satisfies the ascending chain condition. Finally, $\text{Hom}(X_i, Y_{J_1})$ is a one-sided twist (indexed by $\mathbb{Z}_{\geq 0}$ with the opposite partial order) of $\prod_{k \geq 0} \text{Hom}((X_i)^{-k}, Y_{J_1})$, which is contractible by another application of homological perturbation.

Next, we claim that $\text{Hom}(\mathbb{t}^{a(i)} X_i, Y_{J_2})$ has no homology in degree 1. Indeed, for each $k < 0$ the chain group $(X_i)^k$ satisfies $\text{Hom}((X_i)^k, X_j) \simeq 0$ by hypothesis. It follows that

$$\text{Hom}(\mathbb{t}^{a(i)} X_i, \mathbb{t}^{a(j)} X_j) \cong \mathbb{t}^{a(j)-a(i)} \text{Hom}(X_i, X_j) \simeq \mathbb{t}^{a(j)-a(i)} \text{Hom}((X_i)^0, X_j),$$

which is supported in cohomological degrees ≤ 0 since $a(i) \leq a(j)$ from the way J_2 is defined. Two applications of homological perturbation tell us that $\text{Hom}(\mathbb{t}^{a(i)} X_i, Y_{J_2})$ is homotopy equivalent to a complex supported in cohomological degrees ≤ 0 . In particular, there is no homology in degree 1.

Combining the above two paragraphs, we obtain that the inclusion $Y_{J_3} \hookrightarrow Y_{(i,0]}$ induces an isomorphism $H_1(\text{Hom}(\mathbb{t}^{a(i)} X_i, Y_{J_3})) \rightarrow H_1(\text{Hom}(\mathbb{t}^{a(i)} X_i, Y_{(i,0]}))$. It follows that the connecting differential ∂' in (3.3) is homotopic to an element $\partial'' = \partial' + d(h)$ which has components $X_i \rightarrow X_j$ only if $j \in J_3$ (i.e. j is greater than i in the coarse partial order). Therefore $\varphi_{[i,0]} := \begin{bmatrix} \text{id} & 0 \\ h & \text{id} \end{bmatrix} \circ \begin{bmatrix} \text{id} & 0 \\ 0 & \varphi_{(i,0]} \end{bmatrix}$ gives an isomorphism

$$X_{[i,0]} = (\mathbb{t}^{a(i)} X_i \xrightarrow{\partial} X_{[i,0]}) \rightarrow (\mathbb{t}^{a(i)} X_i \xrightarrow{\partial''} Y_{(i,0]}) =: Y_{[i,0]}.$$

This defines the desired combing of $X_{[i,0]}$ and completes the inductive proof that X can be combed. \square

4. CHEBYSHEV SYSTEMS

Definition 4.1. Let R be a ring. We say that a family of elements $v_k \in R$ ($k \in \mathbb{Z}_{\geq 0}$) satisfies the Chebyshev II recursion if $v_0 = 1$ and $v_{n-1}v_1 = v_n + v_{n-2}$ for $n \geq 2$.

Remark 4.2. In such a family, the element v_1 determines all of the elements v_2, v_3, \dots

The Chebyshev II recursion is intimately tied with the representation theory of quantum \mathfrak{sl}_2 , as illustrated in the following examples.

Example 4.3. If $R = \mathbb{Z}[q, q^{-1}]$ then $v_n := [n] := q^{n-1} + q^{n-3} + \dots + q^{1-n}$ satisfies the Chebyshev II recursion.

Example 4.4. If $R = K_0(\mathcal{TL})$ then the classes of simple objects $v_n := [V_n]$ satisfy the Chebyshev II recursion because of the Clebsch-Gordan rule:

$$V_{n-1} \otimes V_1 \cong V_n \oplus V_{n-2}.$$

We now turn our attention to a categorical analogue of the Chebyshev II recursion. So we let $(\mathcal{C}, \otimes, I)$ be a monoidal category. Naively, we might say that a collection of objects $V_n \in (n \in \mathbb{Z}_{\geq 0})$ satisfies the categorified Chebyshev II recursion if $V_0 \cong \mathbb{1}$ and $V_{n-1} \otimes V \cong V_n \oplus V_{n-2}$ for $n \geq 2$. However this is much too naive; it is much better to have knowledge of the maps involved in such an isomorphism, and include them as part of the structure. This is accomplished in Definition 4.6 below.

Definition 4.5. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *self-dual object* (or *self-biadjoint*) in V is a triple $(V, \text{ev}, \text{coev})$ where $V \in \mathcal{C}$ is an object and $\text{coev}: \mathbb{1} \leftrightarrow V \otimes V: \text{ev}$ are maps satisfying

$$(\text{id}_V \otimes \text{ev}) \circ (\text{coev} \otimes \text{id}_V) = \text{id}_V = (\text{ev} \otimes \text{id}_V) \circ (\text{id}_V \otimes \text{coev}).$$

Definition 4.6. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. A *Chebyshev system* in \mathcal{C} is the data of:

- a self-dual object $(V, \text{ev}, \text{coev})$ in \mathcal{C} ,
- a family of complexes $V^{(n)} \in \text{Ch}^-(\mathcal{C})$ with $V^{(0)} = \mathbb{1}$ and $V^{(1)} = V$,
- chain maps $\pi^{(n)}: V^{(n-1)} \otimes V \rightarrow V^{(n)}$ for $n \geq 1$,

such that $\pi^{(1)}: \mathbb{1} \otimes V \rightarrow V$ is the canonical isomorphism and there is a distinguished triangle

$$(4.1) \quad V^{(n-2)} \xrightarrow{\iota^{(n-2)}} V^{(n-1)} \otimes V \xrightarrow{\pi^{(n)}} V^{(n)} \rightarrow V^{(n-2)}[1]$$

in the homotopy category $\mathcal{K}^-(\mathcal{C})$, where $\iota^{(n-2)}$ is defined by commutativity of the triangle

$$(4.2) \quad \begin{array}{ccc} V^{(n-2)} & \xrightarrow{\text{id} \otimes \text{coev}} & V^{(n-2)} \otimes V \otimes V \\ & \searrow \iota^{(n-2)} & \downarrow \pi^{(n-1)} \otimes \text{id} \\ & & V^{(n-1)} \otimes V \end{array}$$

Remark 4.7. Our interest in Chebyshev systems stems from their expected relation to colored \mathfrak{sl}_2 link homology. Specifically we regard Chebyshev systems as models for colored \mathfrak{sl}_2 link homology of the unknot. Under an appropriate cabling functor, one hopes to obtain models for colored \mathfrak{sl}_2 link homology of other knots as well.

The following result shows Chebyshev systems are determined up to homotopy equivalence by $V^{(1)}$.

Theorem 4.8. Let $(V^{(n)}, \pi^{(n)})$ and $(V'^{(n)}, \pi'^{(n)})$ be two Chebyshev systems in a category \mathcal{C} , with $V^{(1)} \cong V'^{(1)}$ as self-dual objects. Then there are homotopy equivalences $\theta^{(n)}: V^{(n)} \xrightarrow{\simeq} V'^{(n)}$ that form isomorphisms of distinguished triangles

$$(4.3) \quad \begin{array}{ccccccc} V^{(n-2)} & \xrightarrow{\iota^{(n-2)}} & V^{(n-1)} \otimes V & \xrightarrow{\pi^{(n)}} & V^{(n)} & \longrightarrow & V^{(n-2)}[1] \\ \downarrow \theta^{(n-2)} & & \downarrow \theta^{(n-1)} \otimes \text{id} & & \downarrow \theta^{(n)} & & \downarrow \theta^{(n-2)}[1] \\ V'^{(n-2)} & \xrightarrow{\iota'^{(n-2)}} & V'^{(n-1)} \otimes V & \xrightarrow{\pi'^{(n)}} & V'^{(n)} & \longrightarrow & V'^{(n-2)}[1] \end{array}$$

Proof. We prove the thesis by induction on n . Since $V^{(0)} = V'^{(0)} = \mathbb{1}$, we take $\theta^{(0)} = \text{id}_{\mathbb{1}}$. Set $V := V^{(1)}$ and $V' := V'^{(1)}$. We denote by coev and coev' the coevaluation morphisms for V

and V' . Let $\theta^{(1)}: V \rightarrow V'$ be an isomorphism of self-dual objects, so in particular

$$(\theta^{(1)} \otimes \theta^{(1)}) \circ \text{coev} = \text{coev}'.$$

Thus, for $n = 2$ the left square in (4.3) commutes (in this case, $\iota^{(0)} = \text{coev}$ and $\iota'^{(0)} = \text{coev}'$). It follows (by properties of distinguished triangles) that there is a map $\theta^{(2)}$ which completes $\theta^{(0)}$ and $\theta^{(1)} \otimes \theta^{(1)}$ to a morphism of triangles. By the five lemma for triangulated categories, $\theta^{(2)}$ is a homotopy equivalence.

Now, let $n \geq 3$ be given and assume by induction that $\theta^{(k)}$ is constructed for $k < n$. Note that if there exists $\theta^{(n)}$ which fits into a morphism of distinguished triangles (4.3), then $\theta^{(n)}$ is automatically a homotopy equivalence, again by the five lemma. To prove the existence of $\theta^{(n)}$ we have to show that the left square of (4.3) commutes up to homotopy. To see this, consider the following diagram:

$$\begin{array}{ccccc} V^{(n-2)} & \xrightarrow{\text{id} \otimes \text{coev}} & V^{(n-2)} \otimes V \otimes V & \xrightarrow{\pi^{(n-1)} \otimes \text{id}} & V^{(n-1)} \otimes V \\ \downarrow \theta^{(n-2)} & & \downarrow \theta^{(n-2)} \otimes \theta^{(1)} \otimes \theta^{(1)} & & \downarrow \theta^{(n-1)} \otimes \theta^{(1)} \\ V'^{(n-2)} & \xrightarrow{\text{id} \otimes \text{coev}'} & V'^{(n-2)} \otimes V' \otimes V' & \xrightarrow{\pi'^{(n-1)} \otimes \text{id}} & V'^{(n-1)} \otimes V' \end{array}$$

The left square commutes since $\theta^{(1)}$ is a morphism of self-dual objects, and the right square commutes by commutativity of the middle square in (4.3) with n lowered by 1. The composition of the horizontal arrows are $\iota^{(n-2)}$ and $\iota'^{(n-2)}$. This proves that the left square in (4.3) commutes, which gives us existence of $\theta^{(n)}$. \square

Remark 4.9. There is a notion of a dual Chebyshev system, obtained from the one in Definition 4.6 by reversing the direction of all arrows and by replacing Ch^- with Ch^+ . Note that if $\{V^{(n)}, \pi^{(n)}\}$ is a Chebyshev system in a rigid category then $\{(V^{(n)})^\vee, (\pi^{(n)})^\vee\}$ is a dual Chebyshev system, where $(-)^{\vee}$ denotes the dualization functor. Explicitly, $(-)^{\vee}: \text{Ch}^-(\mathcal{C}) \rightarrow \text{Ch}^+(\mathcal{C})$ negates the homological grading and transforms differentials into their duals.

In what follows we discuss several Chebyshev systems, in \mathcal{TL} and related categories. By Theorem 4.8 these systems are all equivalent in $\text{Ch}^-(\text{Kar}(\mathcal{TL}))$.

4.1. The Jones–Wenzl system. In this section we take $\mathcal{C} := \text{Kar}(\mathcal{TL})$, the idempotent completion of the Temperley–Lieb category \mathcal{TL} . Recall that \mathcal{TL} contains the Jones–Wenzl projectors p_n , each representing a projection of $V_1^{\otimes n}$ onto its simple $(n+1)$ -dimensional quotient V_n . We shall abuse notation and denote by V also the object $1 \in \mathcal{TL}$. Because the Jones–Wenzl projectors satisfy the identity

$$p_n \circ (p_{n-1} \otimes \text{id}_V) = p_n = (p_{n-1} \otimes \text{id}_V) \circ p_n,$$

$$\begin{array}{ccc}
 \text{im } p_n & = & \left(\begin{array}{ccc} & 0 & \longrightarrow \text{im } p_n \\ \uparrow & & \uparrow \\ 0 & & \rho^{(n)} \end{array} \right) \\
 \varphi^{(n)} \downarrow \uparrow \bar{\varphi}^{(n)} & & \downarrow \uparrow \pi^{(n)} \\
 \text{Cone}(\iota^{(n-2)}) & = & \left(\begin{array}{ccc} \text{im } p_{n-2} & \xrightarrow{d=\iota^{(n-2)}} & \text{im } p_{n-1} \otimes V \\ \longleftarrow h=\kappa^{(n-2)} & & \end{array} \right)
 \end{array}$$

 FIGURE 2. The homotopy equivalences between $\text{im } p_{n+2}$ and $\text{Cone}(\iota^{(n)})$.

we have well-defined morphisms in $\text{Kar}(\mathcal{TL})$

$$\text{im } p_{n-1} \otimes V \xrightleftharpoons[\rho^{(n)}]{\pi^{(n)}} \text{im } p_n$$

$$\text{im } p_{n-2} \xrightleftharpoons[\kappa^{(n-2)}]{\iota^{(n-2)}} \text{im } p_{n-1} \otimes V$$

Theorem 4.10. *The data $\{\text{im } p_n, \pi^{(n)}\}_{n \geq 0}$ is a Chebyshev system. in $\text{Kar}(\mathcal{TL})$.*

Proof. We must construct a homotopy equivalence $\varphi^{(n)}: \text{im } p_n \rightarrow \text{Cone}(\iota^{(n-2)})$, such that $\varphi^{(n)} \circ \pi^{(n)}$ is homotopic to the canonical inclusion $\text{im } p_{n-1} \otimes V \rightarrow \text{Cone}(\iota^{(n-2)})$. For that consider Figure 2. Clearly, the composition $\bar{\varphi}^{(n)} \circ \varphi^{(n)}$ is the identity on $\text{im } p_{n+2}$. We compute

$$\begin{aligned}
 dh &= \frac{[n-1]}{[n]} \left(\begin{array}{c} [n-1] \\ \vdots \\ [n-1] \end{array} \right) = \left(\begin{array}{c} [n-1] \\ \vdots \\ [n-1] \end{array} \right) - \left(\begin{array}{c} [n] \\ \vdots \\ [n] \end{array} \right) = \text{id}_{\text{im } p_{n-1} \otimes V} - \rho^{(n)} \circ \pi^{(n)}, \\
 hd &= \frac{[n-1]}{[n]} \left(\begin{array}{c} [n-1] \\ \vdots \\ [n-1] \end{array} \right) = \frac{[n-1]}{[n]} \left(\left(\begin{array}{c} [n-2] \\ \vdots \\ [n-2] \end{array} \right) - \frac{[n-2]}{[n-1]} \left(\begin{array}{c} [n-2] \\ \vdots \\ [n-2] \end{array} \right) \right) \\
 &= \frac{[n-1][2] - [n-2]}{[n]} \left(\begin{array}{c} [n-2] \\ \vdots \\ [n-2] \end{array} \right) = \left(\begin{array}{c} [n-2] \\ \vdots \\ [n-2] \end{array} \right) = \text{id}_{\text{im } p_{n-2}}.
 \end{aligned}$$

Hence, h is a homotopy between $\varphi^{(n)} \circ \bar{\varphi}^{(n)}$ and the identity on the cone complex. Note that the above computation shows also that h is a homotopy between $\varphi^{(n)} \circ \pi^{(n)}$ and the canonical inclusion of $\text{im } p_{n-1} \otimes V$ into the cone complex. \square

4.2. The Khovanov system. In this section we take $\mathcal{C} := \mathcal{K}^b(\mathcal{TL})$, the bounded homotopy category of complexes over the Temperley–Lieb category. Inside \mathcal{C} we shall construct a Chebyshev system which is suggested by Khovanov’s categorification of the colored Jones polynomial.

For that let us review the complex \mathcal{V}_n assigned to the unknot colored by the simple $(n+1)$ -dimensional representation V_n , which we shall interpret as an object of $\text{Ch}^-(\mathcal{TL})$.

Recall that

$$[V_n] = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} [V_1^{\otimes(n-2k)}],$$

in the Grothendieck group of the representation category of quantum \mathfrak{sl}_2 . Following Khovanov, we will construct a family of complexes $\mathcal{V}_n \in \text{Ch}^-(\mathcal{TL})$ whose Euler characteristics satisfy the same relation as above.

The binomial coefficient $\binom{n-k}{k}$ counts the number of ways of selecting k disjoint pairs of neighbored dots from n dots placed on a horizontal line. We call such choices k -pairings and we denote by $I_{k,n}$ the set of all such k -pairings. Each $s \in I_{k,n}$ represents an object $\mathcal{V}_s \in \mathcal{TL}$ that consists of $n - 2k$ dots.

Let Γ_n be a graph, the vertices of which are k -pairings of n dots for all $k \geq 0$. Two vertices of Γ_n are connected by a directed edge if $s = s' \cup \{p\}$ for a single pair p . We decorate a vertex s with the object \mathcal{V}_s and an edge $s \rightarrow s'$ with the morphism

$$\cup_{s',s} := \begin{array}{c} | \quad | \quad | \quad \cup \quad | \quad | \\ \hline 1 \quad \dots \quad i \end{array},$$

where i counts unpaired dots in s that lie to the left of the pairing that does not belong to s' . An example for $n = 4$ is shown in Figure 3. Note that Γ_n is a commuting diagram in \mathcal{TL} . In

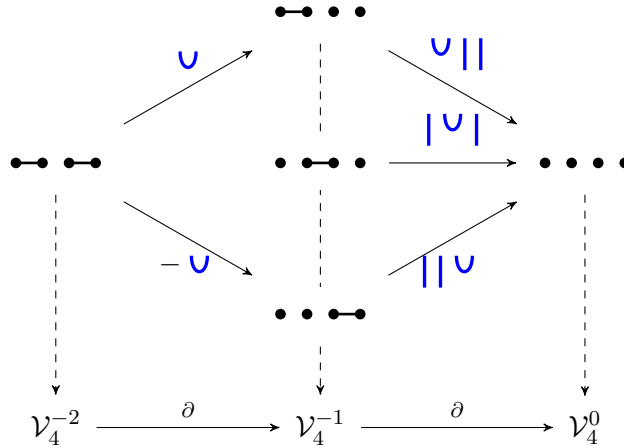


FIGURE 3. The graph Γ_4 and the complex \mathcal{V}_4 .

order to make all squares anticommute, scale $\cup_{s',s}$ by $(-1)^{(s,s')}$, where (s, s') is the number of pairs in s that lie to the right of the unique pair in $s \setminus s'$. The complex \mathcal{V}_n is the result of collapsing the graph by taking direct sums of k -pairings with the same k :

$$\mathcal{V}_n^{-k} := \bigoplus_{s \in I_{k,n}} \mathcal{V}_s$$

and the differential $\delta^{-k}: \mathcal{V}_n^{-k} \rightarrow \mathcal{V}_n^{-k+1}$ is given by the matrix with entries

$$\delta_{s',s}^{-k} := \begin{cases} (-1)^{(s,s') \cup_{s',s}} & \text{if there is an edge } s \rightarrow s', \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We may identify $\mathcal{V}_n \otimes V$ with the subcomplex of \mathcal{V}_{n+1} spanned by those k -pairings in which the right most dot does not belong to a pair.

Theorem 4.11. *The data $\{\mathcal{V}_n, \pi^{(n)}\}_{n \geq 0}$ is a Chebyshev system in $\mathcal{K}^b(\mathcal{TL})$, where $\pi^{(n)}: \mathcal{V}_{n-1} \otimes V \rightarrow \mathcal{V}_n$ is the inclusion of a subcomplex.*

Proof. The graph Γ_n has two types of vertices, corresponding to two types of pairings: those in which the rightmost dot belongs to a pair, and those for which it does not. Vertices of the second type generate the subcomplex $\mathcal{V}_{n-1} \otimes V$, whereas vertices of the first type generate a quotient complex that is isomorphic to $\mathfrak{t}^{-1}\mathcal{V}_{n-2}$. We can thus write \mathcal{V}_n as a mapping cone

$$\mathcal{V}_n = \text{Cone}(\mathcal{V}_{n-2} \rightarrow \mathcal{V}_{n-1} \otimes V),$$

where the chain map in the cone is induced by all morphisms $\cup_{s',s}$ that correspond to edges in Γ_n connecting vertices of the two different types—each is given by the diagram with a cup on its right edge. Explicitly, this chain map can be written as $\text{id}_{\mathcal{V}_{n-2}} \otimes \text{coev}_V$ (with codomain restricted to $\mathcal{V}_{n-1} \otimes V$) and it coincides with $(\pi^{(n-1)} \otimes \text{id}_V) \circ (\text{id}_{\mathcal{V}_{n-2}} \otimes \text{coev}_V)$. This shows that \mathcal{V}_{n-2} , \mathcal{V}_n , and $\mathcal{V}_{n-1} \otimes V$ are related by a distinguished triangle as in Definition 4.6. \square

4.3. The Cooper–Krushkal system. One of the main goals of this paper is to show how the Cooper–Krushkal categorified Jones–Wenzl idempotents yield a Chebyshev system.

Definition 4.12. Recall that the *width* of a flat tangle T is the minimal number of points on a horizontal section of T . In other words, the width of an (m, n) -tangle T is the minimal k such that T can be written $T = T' \circ T''$ with T' an (m, k) -tangle and T'' a (k, n) -tangle.

If $X \in \text{Ch}^-(\mathbf{BN}_{m,n})$ then we say that X has *width $\leq k$ up to homotopy* if X is homotopy equivalent to a complex $Y \in \text{Ch}^-(\mathbf{BN}_{m,n})$ whose chain groups consist of sums of tangles with width $\leq k$.

Hereafter fix a natural number n and the following tangles

$$\mathbb{1}_n = \begin{array}{c} | \quad | \quad | \quad | \quad | \\ 1 \quad \dots \quad n \end{array} \quad \cup_i = \begin{array}{c} 1 \dots i \dots n \\ | \quad | \quad \cup \quad | \quad | \end{array} \quad \cap_i = \begin{array}{c} | \quad | \quad \cap \quad | \quad | \\ 1 \dots i \dots n \end{array} \quad B_i = \begin{array}{c} | \quad | \quad \cup \quad | \quad | \\ | \quad | \quad \cap \quad | \quad | \\ 1 \dots i \dots n \end{array}$$

for $i = 1, \dots, n-1$. We say that a complex $X \in \text{Ch}^-(\mathbf{BN}_{n,n})$ *kills turnbacks* if $\cap_i \star X \simeq 0 \simeq X \star \cup_i$ for all $1 \leq i \leq n-1$.

Theorem 4.13 ([CK12]). *There exists a complex $P_n \in \text{Ch}^-(\mathbf{BN}_{n,n})$ and a chain map $\eta: \mathbb{1}_n \rightarrow P_n$ satisfying the following properties:*

(P1) P_n kills turnbacks.

(P2) $\text{Cone}(\eta)$ has width $< n$ up to homotopy.

These properties determine the pair (P_n, η) uniquely up to homotopy.

Remark 4.14. In fact, it can be shown [Hog12, Cor. 5.13] that for any $X \in \text{Ch}^-(\mathbf{BN})$ we have $X \star \cup_i \simeq 0$ for all $1 \leq i \leq n-1$ if and only if $\cap_i \star X \simeq 0$ for all $1 \leq i \leq n-1$. Thus, to show P_n kills turnbacks it suffices to show that P_n kills turnbacks from the right ($P_n \star \cup_i \simeq 0$ for $1 \leq i \leq n-1$) or from the left ($\cap_i \star P_n \simeq 0$ for $1 \leq i \leq n-1$).

Definition 4.15. A Cooper–Krushkal projector on n strands is any pair (P_n, η) satisfying axioms (P1) and (P2) of Theorem 4.13.

Proposition 4.16. If $X \in \text{Ch}^-(\mathbf{BN}_{m,n})$ and $Y \in \text{Ch}^-(\mathbf{BN}_{n,m})$ have width $< n$ up to homotopy then

$$X \star P_n \simeq 0 \simeq P_n \star Y.$$

Proof. This follows immediately from the definitions and from homological perturbation theory. \square

Remark 4.17. In practice, axiom (P2) is typically realized in the following way. Suppose P_n can be expressed as a chain complex of the form

$$\cdots \rightarrow P_n^{-3} \rightarrow P_n^{-2} \rightarrow P_n^{-1} \rightarrow \mathbb{1}_n,$$

where each P_n^k for $k < 0$ is spanned by tangles of width smaller than n , and let $\eta: \mathbb{1}_n \rightarrow P_n$ be the inclusion of the degree zero chain object. Then

$$\text{Cone}(\eta) \simeq \cdots \rightarrow P_n^{-3} \rightarrow P_n^{-2} \rightarrow P_n^{-1} \rightarrow 0$$

and axiom (P2) is satisfied.

The axioms imply that P_n absorbs P_{n-1} in the following sense.

Lemma 4.18. Let (P_{n-1}, η_{n-1}) be a Cooper–Krushkal projector on $n - 1$ strands. Then the map

$$P_n \cong \mathbb{1}_n \star P_n \xrightarrow{(\eta_{n-1} \sqcup \text{id}_{\mathbb{1}_1}) \star \text{id}_{P_n}} (P_{n-1} \sqcup \mathbb{1}_1) \star P_n$$

is a homotopy equivalence.

Proof. Recall that a chain map f is a homotopy equivalence if and only if $\text{Cone}(f)$ is contractible. The cone of the map in the statement is isomorphic to $(\text{Cone}(\eta_{n-1}) \sqcup \mathbb{1}_1) \star P_n$ which is contractible since $\text{Cone}(\eta_{n-1})$ has width $< n$ up to homotopy and P_n kills turnbacks. See [CK12, Prop. 3.3] for details. \square

Proposition 4.19. There is a unique map $\nu: P_{n-1} \sqcup \mathbb{1}_1 \rightarrow P_n$ up to homotopy with the property that $\nu \circ \eta_{n-1} \simeq \eta_n$.

Proof. For existence we may define ν to be the composition of chain maps

$$P_{n-1} \sqcup \mathbb{1}_1 \cong (P_{n-1} \sqcup \mathbb{1}_1) \star \mathbb{1}_n \xrightarrow{\text{id} \star \eta_n} (P_{n-1} \sqcup \mathbb{1}_1) \star P_n \simeq P_n$$

(we leave it as an exercise to show $\nu \circ \eta_{n-1} \simeq \eta_n$). Uniqueness and other properties of ν follow from general facts about categorical idempotents [Hog17, Thm. 1.6]. \square

Theorem 4.20. Given $X \in \text{Ch}^-(\mathbf{BN}_{n,n})$, let $[X] \in \text{Ch}^-(\text{hTr}_q(\mathbf{BN}))$ denote the quantum horizontal trace of X . There is a Chebyshev system $\{[P_n], \pi^{(n)}\}_{n \geq 0}$, where

$$\pi^{(n)} = [\nu]: [P_{n-1} \sqcup \mathbb{1}_1] \longrightarrow [P_n]$$

and ν is as in Proposition 4.19.

In this theorem, it is assumed that $1 - q^d$ is invertible for all integers $d > 0$. We will prove Theorem 4.20 in the next section by using the following lemma:

Lemma 4.21. Let $f: P_n \rightarrow P_n$ be an endomorphism of quantum degree d , and suppose $1 - q^d$ is invertible. Then the induced endomorphism $[f]: [P_n] \rightarrow [P_n]$ is null-homotopic.

Proof. Let

$$\theta: [P_n \star P_n] \longrightarrow [P_n \star P_n]$$

be the chain isomorphism whose components are given by sums of cyclicity isomorphisms $\theta_{P_n^j, P_n^i}: [P_n^i \star P_n^j] \rightarrow [P_n^j \star P_n^i]$. Then the naturality of the cyclicity isomorphisms implies

$$[\text{id} \star f] = \theta \circ [f \star \text{id}] \circ \theta^{-1} \simeq \theta \circ [\text{id} \star f] \circ \theta^{-1} = q^d [f \star \text{id}] \simeq q^d [\text{id} \star f]$$

where we have used that $\text{id} \star f \simeq f \star \text{id}$ for any endomorphism $f: P_n \rightarrow P_n$ (this is a general fact about unital categorical idempotents, see [BD14, Lemma 3.18]). Hence $(1 - q^d)[\text{id} \star f] \simeq 0$, and so $[\text{id} \star f] \simeq 0$. This also implies $[f] \simeq 0$ because $(\eta_m \star \text{id}) \circ f = (\text{id} \star f) \circ (\eta_m \star \text{id})$ and $\eta_m \star \text{id}$ is a homotopy equivalence. \square

5. THE CATEGORICAL IDEMPOTENT MODEL

The purpose of this section is to prove Theorem 4.20. To this, we must carefully choose our preferred model for P_n . We begin with some setup in §5.1. Then in §5.2 we establish our model for P_n which manifests a sort of 2-periodicity similar to that of P_2 .

5.1. Setup. It will be beneficial to set up some notation which will be used throughout. Firstly, let

denote degree \mathfrak{q} chain maps induced by saddle cobordisms. By abuse we will also regard s, s^* as morphisms

(via the homotopy equivalence $(P_{n-1} \sqcup \mathbb{1}_1)^{\star 2} \simeq P_{n-1} \sqcup \mathbb{1}_1$). Note that since

is a sum of dots on the “cup” and “cap”.

Recall also that in [Hog14] distinguished endomorphisms $u_1, \dots, u_n \in \text{End}_{\text{Ch}^-(\mathbf{BN}_n)}(P_n)$ were constructed. The endomorphism u_k generates the \mathbb{k} -module of endomorphisms of P_n of degree $\mathfrak{t}^{2-2k} \mathfrak{q}^{2k}$ modulo homotopy, and is uniquely characterized up to homotopy and invertible scalar by this fact.

Finally, let

denote the degree $\mathbb{t}^{4-2n}\mathbb{q}^{2n-2}$ morphisms given by the action of applying u_{n-1} on the top or bottom factors of P_{n-1} , respectively.

5.2. The 2-periodic model for P_n . Our proof of Theorem 4.20 uses the following description of P_n .

Theorem 5.1. *There is a model for P_n of the form*

$$(5.1) \quad \left(\dots \xrightarrow{u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}}} \mathbb{t}^{-3}\mathfrak{a}_3 \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \xrightarrow{s^* \circ s} \mathbb{t}^{-2}\mathfrak{a}_2 \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \xrightarrow{u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}}} \mathbb{t}^{-1}\mathfrak{a}_1 \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \xrightarrow{s} \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \right)$$

(where we are omitting longer arrows which point to the right) in which the shifts satisfy:

$$\mathfrak{a}_1 := \mathbb{q} \quad \mathfrak{a}_{2k} = \mathbb{t}^{4-2n}\mathbb{q}^{2n-2}\mathfrak{a}_{2k-1} \quad \mathfrak{a}_{2k+1} = \mathbb{q}^2\mathfrak{a}_{2k}$$

for all $k \geq 1$. Moreover, with respect to this expression, the endomorphism u_n acts as

$$(5.2) \quad \begin{array}{ccccccc} \dots & \longrightarrow & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \longrightarrow & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \xrightarrow{s} & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow s \\ \dots & \longrightarrow & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \longrightarrow & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \longrightarrow & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \longrightarrow & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \xrightarrow{s} & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \end{array}$$

together with arrows pointing strictly to the right.

Remark 5.2. The expression (5.1) categorifies the right-hand side of the recursion (2.1) for the Jones–Wenzl idempotent.

This theorem will be a consequence of the following.

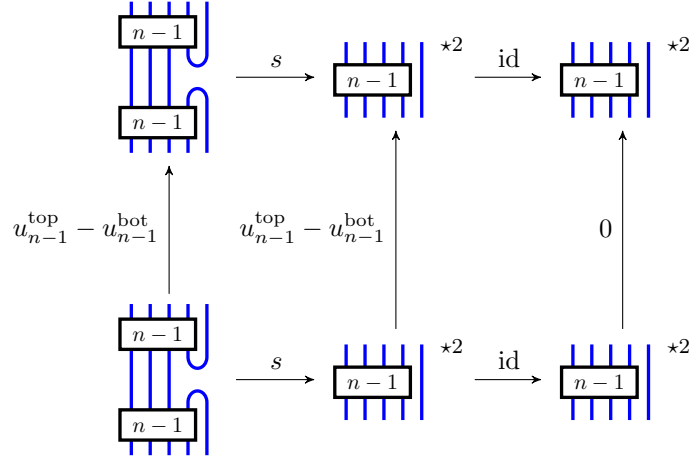
Lemma 5.3. *There is a complex of the form*

$$(5.3) \quad Q_n = \left(\begin{array}{ccccccc} & & & \gamma & & & \\ & & & \curvearrowright & & & \\ \mathbb{t}^{-3}\mathfrak{a}_3 \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \xrightarrow{s^*} & \mathbb{t}^{-2}\mathfrak{a}_2 \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \xrightarrow{u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}}} & \mathbb{t}^{-1}\mathfrak{a}_1 \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} & \xrightarrow{s} & \begin{array}{c} \boxed{n-1} \\ \vdots \\ \boxed{n-1} \end{array} \\ & & \searrow k & \nearrow h & & & \\ & & & & & & \end{array} \right)$$

for the appropriate shifts \mathfrak{a}_i . Specifically $\mathfrak{a}_1 = \mathbb{q}$, $\mathfrak{a}_2 = \mathbb{t}^{4-2n}\mathbb{q}^{2n-1}$, $\mathfrak{a}_3 = \mathbb{t}^{4-2n}\mathbb{q}^{2n}$. This complex kills turnbacks.

Compare (5.3) with the expression (4.10) in [Hog18].

Proof. The existence of h and k will follow from the vanishing of $s \circ (u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}})$ and $(u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}}) \circ s^*$ up to homotopy. We prove only the first of these, since the second is similar. Consider the diagram



The first square obviously commutes, and the second square commutes since $f \star \text{id} \simeq \text{id} \star f$ as endomorphisms of $P_{n-1} \star P_{n-1}$ for any endomorphism f of P_{n-1} (this is a general fact about unital categorical idempotents, see [BD14, Lemma 3.18]). This implies that $s \circ (u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}}) \simeq 0$, which gives us the existence of h . A similar argument gives us the existence of k . Finally, the obstruction to constructing γ is an endomorphism of $P_{n-1} \sqcup \mathbb{1}_1$ of degree $\mathfrak{t}^{3-2n} \mathfrak{q}^{2n}$. Any such obstruction vanishes up to homotopy by an argument identical to that in [Hog18, Lemma 4.34]. This completes the construction of Q_n . \square

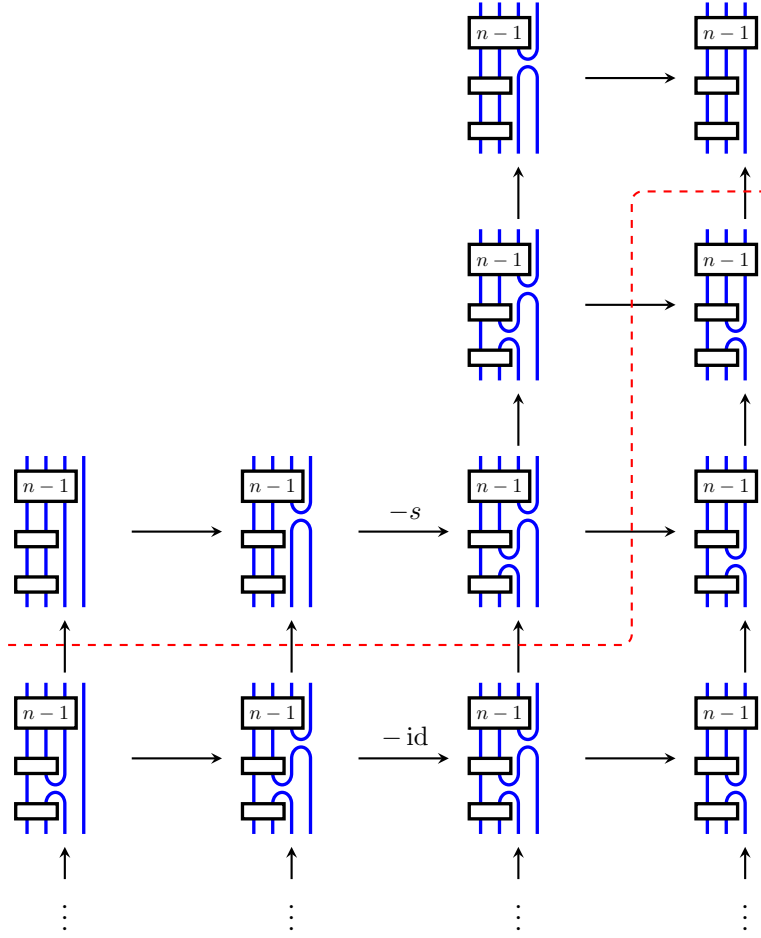
Using this Q_n we construct our 2-periodic model for P_n .

Proof of Theorem 5.1. It is clear that splicing together infinitely many copies of Q_n will result in a complex of the form (5.1). In more details, Q_n comes equipped with an endomorphism ∂ of degree $\mathfrak{t}^{1-2n} \mathfrak{q}^{2n}$ which projects onto the left-most term of (5.3) and includes this as the right-most term. The pre-spliced complex is given by

$$(5.4) \quad \left(Q_n \longrightarrow \mathfrak{t}^{2-2n} \mathfrak{q}^{2n} Q_n \longrightarrow \mathfrak{t}^{4-4n} \mathfrak{q}^{4n} Q_n \longrightarrow \dots \right)$$

which obviously carries an endomorphism U' of degree $\mathfrak{t}^{2-2n} \mathfrak{q}^{2n}$ which just shifts the above expression one unit to the right. By construction, $\text{Cone}(U') \simeq Q_n$. After cancelling contractible terms we obtain the honest spliced complex (5.1) with a chain endomorphism U of the form (5.2) (this is an easy exercise in Gaussian elimination, but also note that the homotopy equivalence $\text{Cone}(U) \simeq \text{Cone}(U') \simeq Q_n$ puts serious restrictions on U). Let us denote this spliced complex by P'_n . Once we show that $P'_n \simeq P_n$, the fact that U is a model for $\pm u_n$ will follow for degree reasons.

It is clear that P'_n satisfies axiom (P2) from Theorem 4.13, so to complete the proof we must show that P'_n kills turnbacks, say, from the right. The one-sided twisted complex (5.4) satisfies a local ascending chain condition because Q_n is bounded from above, so by homological

FIGURE 4. The complex Q_n after expanding the lower projector.

perturbation, P'_n kills turnbacks if and only if Q_n does. We assume by induction that we have already established $P'_{n-1} \simeq P_{n-1}$.

Consider the expression (5.3) for Q_n . This complex annihilates B_k for $1 \leq k \leq n-2$ because $P_{n-1} \sqcup \mathbb{1}_1$ does. So we must show that B_{n-1} annihilates Q_n . For this it will be useful to find a “six-term” expression for Q_n . We expand the lower projector P_{n-1} on the right in (5.3), according to the recursion (5.1) for P_{n-1} , obtaining the expression for Q_n shown in Figure 4.

In this figure, we have omitted longer arrows including arrows pointing strictly to the top right. The arrows labeled $-s$ and $-id$ come from the expansion of $-u_{n-1}^{bot}$, while the arrows corresponding to u_{n-1}^{top} would point to the top right.

For the purpose of this proof, the arrows coming from the maps h and γ in Q_n are irrelevant, but it will be important to make sure that all arrows in the expansion of the map k point (non-strictly) to the top right. This can be achieved by combing hairs, or directly as follows. From the proof of Lemma 5.3, we know that $k = s^* \circ k'$ where s^* is a saddle cobordism and k' is a null-homotopy for the endomorphism $u := u_{n-1}^{top} - u_{n-1}^{bot}$ of $P_{n-1}^{*2} \sqcup \mathbb{1}_1$. If we now expand the lower projector in $P_{n-1}^{*2} \sqcup \mathbb{1}_1$, we obtain the one-sided twisted complex shown in the

leftmost column of Figure 4. In this complex, all terms beyond the first one are contractible. Using homological perturbation theory, we thus get a homotopy equivalence, denoted F , from this twisted complex to its leading term. The inverse homotopy equivalence, denoted G , is given by inclusion, and there is a homotopy H between the identity map of $P_{n-1}^{*2} \sqcup \mathbb{1}_1$ and GF such that no arrows in H point downwards. In the definition of k , we can now replace k' by a pair of homotopies $u \simeq GFu \simeq 0$ given by Hu and GFk' . This yields a new map k , whose components point (non-strictly) to the top right.

With this in mind, we can now cancel the contractible terms in Figure 4 that lie below the dashed line. After absorbing $(P_{n-1} \sqcup \mathbb{1}_1) \star (P_{n-2} \sqcup \mathbb{1}_2) \simeq P_{n-1} \sqcup \mathbb{1}_1$, this yields a homotopy equivalence

$$Q_n \simeq \left(\begin{array}{cccccc} \begin{array}{|c|} \hline n-1 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|} \hline n-1 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|} \hline n-1 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|} \hline n-1 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|} \hline n-1 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|} \hline n-1 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|} \hline n-1 \\ \hline \end{array} \\ \begin{array}{|c|} \hline \square \\ \hline \end{array} & & \begin{array}{|c|} \hline \square \\ \hline \end{array} & & \begin{array}{|c|} \hline \square \\ \hline \end{array} & & \begin{array}{|c|} \hline \square \\ \hline \end{array} & & \begin{array}{|c|} \hline \square \\ \hline \end{array} & & \begin{array}{|c|} \hline \square \\ \hline \end{array} & & \begin{array}{|c|} \hline \square \\ \hline \end{array} \end{array} \right)$$

in which all length one maps are saddle maps except for the map in the middle which is $u_{n-2}^{\text{top}} - u_{n-2}^{\text{bot}}$ (here we use that the components of k point to the top right). The left half and right half each are killed by $-\star B_{n-1}$ (see for example [CK12, Subs. 7.1.1], so Q_n is killed by $-\star B_{n-1}$ as well. This completes the proof. \square

5.3. The quantum trace of P_n . In this section, we use the complex $[Q_n]$ to show that $[P_n]$ is homotopy equivalent to $V_n := \text{im } p_n$. In particular, this will lead to a proof of Theorem 4.20.

Lemma 5.4. *We have*

$$[Q_n] \simeq \left(\begin{array}{c} \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] \xrightarrow{c} \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] \xrightarrow{[u_{n-1}]} \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] \xrightarrow{a} \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] \end{array} \right)$$

where we are omitting longer arrows, and where a and c are both induced by cyclicity isomorphisms and saddles.

Proof. We first observe that we have the following diagram which commutes up to homotopy:

$$\begin{array}{ccccccc} \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] & \xrightarrow[\theta]{\cong} & \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] & \xrightarrow{\text{id}} & \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] & \xrightarrow{\cong} & \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] \\ \downarrow [u_{n-1}^{\text{top}} - u_{n-1}^{\text{bot}}] & & \downarrow [u_{n-1}^{\text{bot}} - q^{2n-2}u_{n-1}^{\text{top}}] & & \downarrow [u_{n-1}^{\text{top}} - q^{2n-2}u_{n-1}^{\text{top}}] & & \downarrow [(1 - q^{2n-2})u_{n-1}] \\ \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] & \xrightarrow[\theta]{\cong} & \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] & \xrightarrow{\text{id}} & \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] & \xrightarrow{\cong} & \left[\begin{array}{|c|} \hline n-1 \\ \hline \end{array} \right] \end{array}$$

The composition of horizontal maps gives a homotopy equivalence

$$\left[\begin{array}{c} \boxed{n-1} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \simeq \left[\begin{array}{c} \boxed{n-1} \end{array} \right]$$

Applying this homotopy equivalence to the two middle terms in $[Q_n]$, we get

$$[Q_n] \simeq \left(\left[\begin{array}{c} \boxed{n-1} \end{array} \right] \xrightarrow{c} \left[\begin{array}{c} \boxed{n-1} \end{array} \right] \xrightarrow{f} \left[\begin{array}{c} \boxed{n-1} \end{array} \right] \xrightarrow{a} \left[\begin{array}{c} \boxed{n-1} \end{array} \right] \right)$$

where $f = (1 - q^{2n-2})[u_{n-1}]$. The lemma now follows because $1 - q^{2n-2}$ is assumed to be invertible. \square

We are now ready to prove:

Lemma 5.5. $[P_n] \simeq V_n$

Proof. By Theorem 5.1, we have an expansion

$$\left[\begin{array}{c} \boxed{n-1} \end{array} \right] \simeq \left(\cdots \longrightarrow \left[\begin{array}{c} \boxed{n-1} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \longrightarrow \left[\begin{array}{c} \boxed{n-1} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \longrightarrow \left[\begin{array}{c} \boxed{n-1} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \xrightarrow{s} \left[\begin{array}{c} \boxed{n-1} \end{array} \right] \right)$$

in which the right-hand side is homotopic to

$$(5.5) \quad \cdots \longrightarrow \left[\begin{array}{c} \boxed{n-2} \end{array} \right] \longrightarrow \left[\begin{array}{c} \boxed{n-2} \end{array} \right] \longrightarrow \left[\begin{array}{c} \boxed{n-2} \end{array} \right] \xrightarrow{\text{id}} \left[\begin{array}{c} \boxed{n-2} \end{array} \right] \oplus \left[\begin{array}{c} \boxed{n-2} \end{array} \right]$$

We now replace the middle terms in the complex from Lemma 5.4 by the above expansion. This yields the expression for $[Q_n]$ shown in Figure 5.

As indicated by the dashed lines, the complex from Figure 5 can be viewed as a one-sided twisted complex of the form

$$C \begin{array}{c} \curvearrowright \\ \longrightarrow \\ \longrightarrow \end{array} B \longrightarrow A$$

where B and the bottom part of C are contractible. After collapsing these contractible pieces, this twisted complex becomes

$$(5.6) \quad \left[\begin{array}{c} \boxed{n-1} \end{array} \right] \longrightarrow \left[\begin{array}{c} \boxed{n-2} \end{array} \right] \longrightarrow \left[\begin{array}{c} \boxed{n-2} \end{array} \right] \xrightarrow{a'} \left[\begin{array}{c} \boxed{n-1} \end{array} \right]$$

We now use induction on n to replace the remaining copies of $[P_{n-1} \sqcup \mathbb{1}_1]$ and $[P_{n-2}]$ by $V_{n-1} \otimes V_1$ and V_{n-2} . Taking into account the shifts of the cohomological grading (but not of the quantum grading), this yields

$$(5.7) \quad [Q_n] \simeq \left(\mathbb{t}^{1-2n} V_{n-1} \otimes V_1 \longrightarrow \mathbb{t}^{2-2n} V_{n-2} \quad \mathbb{t}^{-1} V_{n-2} \longrightarrow V_{n-1} \otimes V_1 \right)$$

On the other hand, the proof of Theorem 5.1 shows that

$$(5.8) \quad [Q_n] \simeq \left([\mathfrak{t}^{1-2n} P_n] \xrightarrow{[u_n]} [P_n] \right) \simeq \left([\mathfrak{t}^{1-2n} P_n] \xrightarrow{0} [P_n] \right) = [\mathfrak{t}^{1-2n} P_n] \oplus [P_n]$$

where the second homotopy equivalence follows from Lemma 4.21 because u_n has nonzero quantum degree.

Taking into account grading shifts, it is now easy to see that (5.8) can only hold if both arrows in (5.7) are nonzero and $[P_n] \simeq V_n$. Indeed, we have $V_{n-1} \otimes V_1 \cong V_n \oplus V_{n-2}$, and the two middle terms in (5.7) cannot be connected by an arrow for grading reasons (here we can assume $n > 2$ because the $n = 2$ case is straightforward). \square

Remark 5.6. The map a in Lemma 5.4 is induced by the composition $s \circ \theta^{-1}$ where s is a saddle and θ is a chain isomorphism given by cyclicity isomorphisms. From this, one can easily see that the map a' in (5.6) and in Figure 5 is induced by an annulus $[P_{n-2}] \rightarrow [P_{n-2} \sqcup \mathbb{1}_2]$. On the other hand, the leftmost map in (5.6) is more difficult to describe because it potentially depends on longer arrows connecting the leftmost term in Figure 5 to the rightmost column of C . As a consequence, it is not clear whether this map is nonzero, although this follows indirectly from the proof of Lemma 5.5.

Although Theorem 4.20 follows implicitly from Lemma 5.5 and its proof, we will now give a concrete proof of this theorem by showing explicitly that $[P_n]$ deformation retracts to the complex A from Figure 5. Note, however, that a part of this proof relies on Lemma 5.5.

Proof of Theorem 4.20. By using an analogue of Lemma 5.4, we can write the trace of P_n as

$$[P_n] \simeq \left(\dots \xrightarrow{[u_{n-1}]} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \xrightarrow{b} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \xrightarrow{[u_{n-1}]} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \xrightarrow{a} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \right)$$

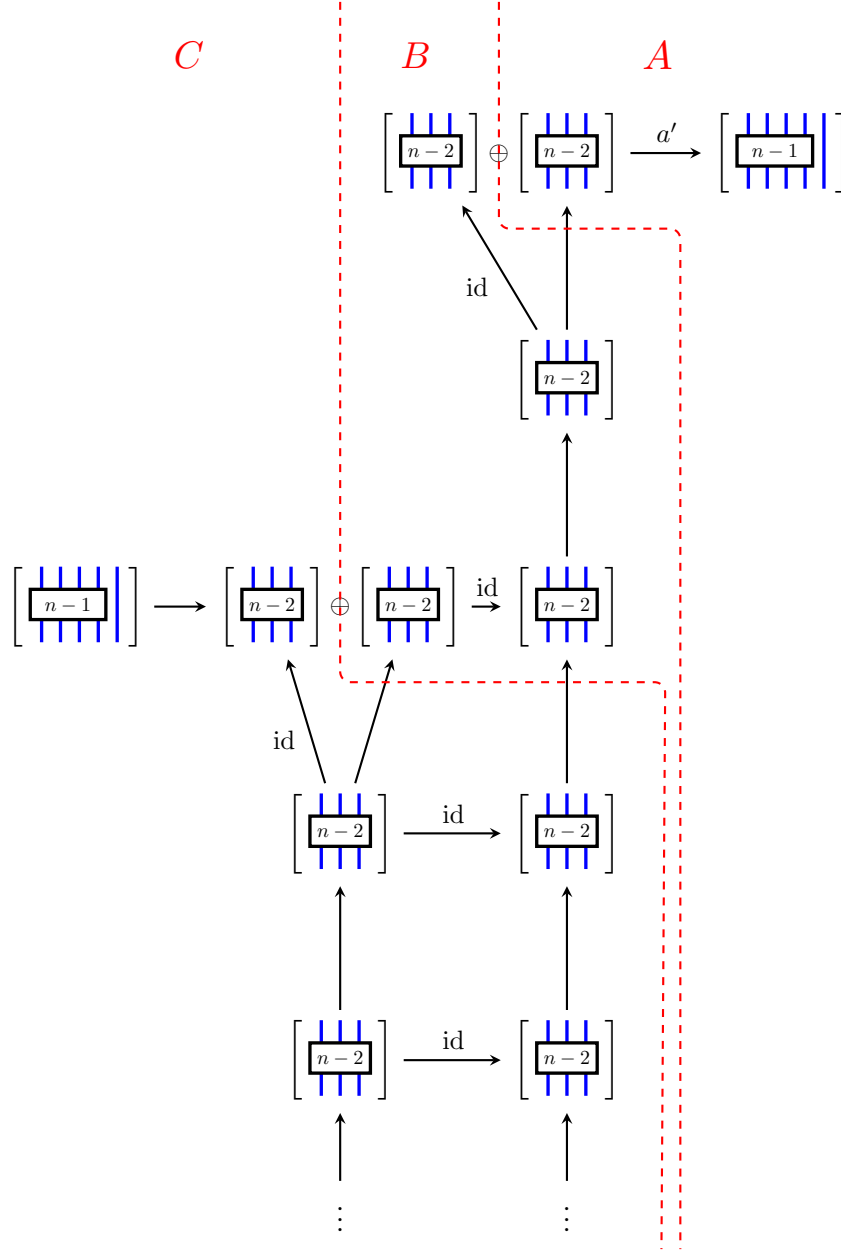
where b is given by a dot placed on the looped strand. In this expansion of $[P_n]$, we now replace each term (beyond the first) by the expansion from (5.5). This yields the expression for $[P_n]$ shown in Figure 6.

Note that the complexes A and B in Figure 6 are identical with the ones from Figure 5. While the complex B is obviously contractible, it is not clear whether D is contractible because there could be longer arrows connecting the leftmost term in D to its rightmost column.

However, a simple grading argument shows that A becomes a direct summand of $[P_n]$ if one replaces $[P_{n-2}]$ and $[P_{n-1} \sqcup \mathbb{1}_1]$ by V_{n-2} and $V_{n-1} \otimes V_1$. Because the resulting complex $V_{n-2} \rightarrow V_{n-1} \otimes V_1$ necessarily contains V_n as a direct summand, and because $[P_n] \simeq V_n$ by Lemma 5.5, it then follows that the complex $\dots \rightarrow D \rightarrow B \rightarrow D \rightarrow B$ in the expansion of $[P_n]$ must be contractible. Hence $[P_n]$ deformation retracts to the complex

$$A = \left(\left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{n-2} \end{array} \right] \xrightarrow{a'} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{n-1} \end{array} \right] \right)$$

where a' is as described in Remark 5.6. The theorem now follows from the definition of a model. \square

FIGURE 5. Expansion of $[Q_n]$ without longer arrows.

6. QUANTUM TRACES OF GENERALIZED PROJECTORS

In [CH15], B. Cooper and the second author introduced a family of complexes $P_\varepsilon \in \text{Ch}^-(\mathcal{BN}_n)$ which categorify the indecomposable idempotents in \mathcal{TL} . In this section, we give a new and streamlined proof of the existence of these complexes. We then show that the class of P_ε in the quantum horizontal trace depends only on the through-degree of P_ε (Theorem C). As in the previous sections, we need to assume that $1 - q^d$ is invertible for $d > 0$.

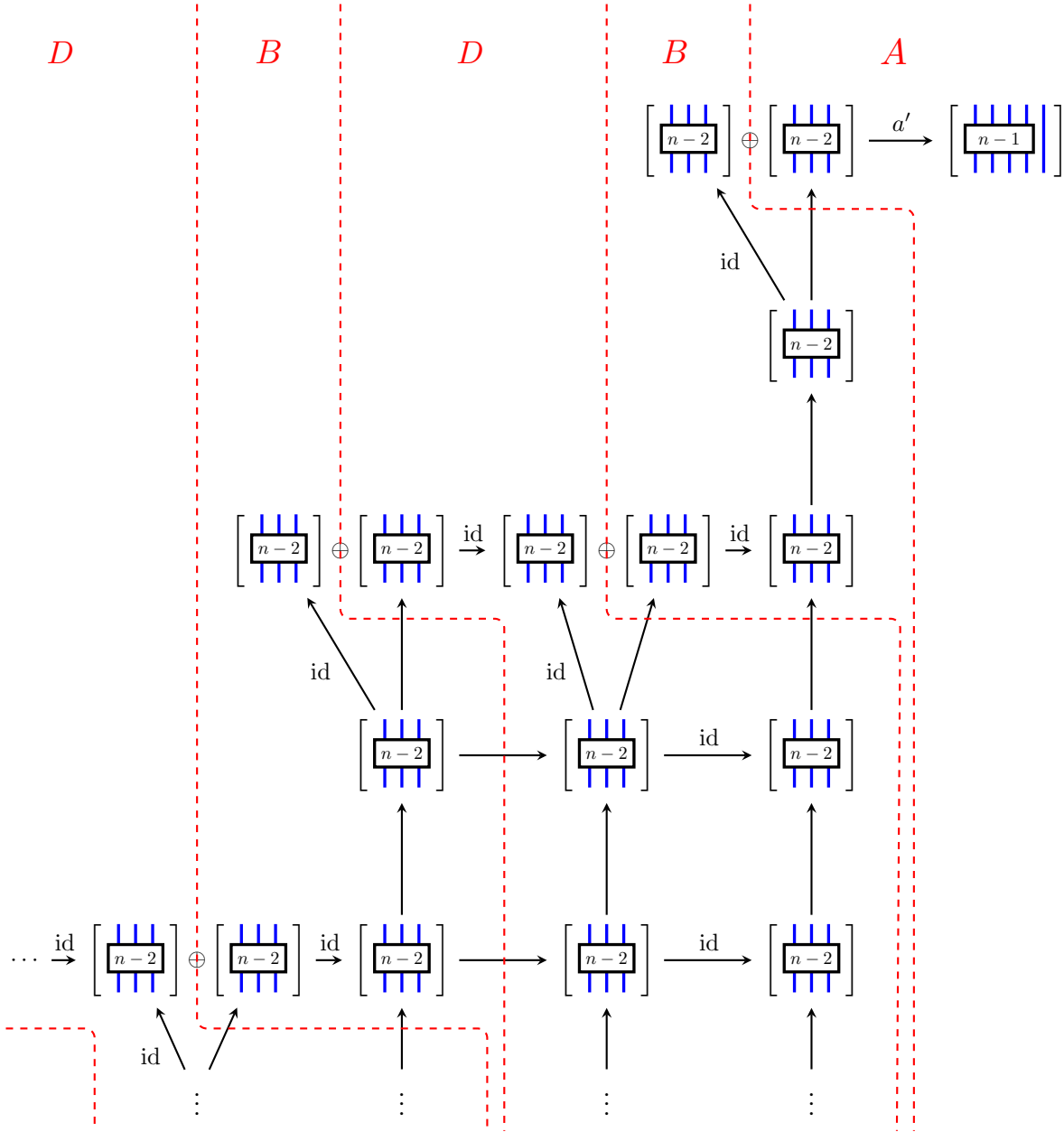


FIGURE 6. Expansion of $[P_n]$ without longer arrows.

6.1. Idempotents in \mathcal{TL} . Recall the Karoubi envelope of a \mathbb{k} -linear category \mathcal{C} is the category $\text{Kar}(\mathcal{C})$ whose objects are formal expressions $\text{im } e$, where e is an idempotent endomorphism in \mathcal{C} . If e, e' are idempotent endomorphisms in \mathcal{C} , then any morphism in \mathcal{C} of the form $e \circ f \circ e'$ is regarded as a morphism $\text{im } e \leftarrow \text{im } e'$ in $\text{Kar}(\mathcal{C})$.

Recall the Temperley–Lieb category \mathcal{TL} . This category has identity maps id_n indexed by $n \in \mathbb{Z}_{\geq 0}$, and the space of morphisms $\text{im } \text{id}_n \leftarrow \text{im } \text{id}_m$ is the Temperley–Lieb space of (n, m) tangles, denoted $\mathcal{TL}_{n,m}$. We also denote $\mathcal{TL}_n := \mathcal{TL}_{n,n}$.

We also have the Jones–Wenzl idempotents $p_n = \text{id}_n \circ p_n \circ \text{id}_n$.

Proposition 6.1. *There is a monoidal functor from $\text{Kar}(\mathcal{TL})$ to the category of finite dimensional $U_q(\mathfrak{sl}_2)$ representations sending*

$$\text{im id}_n \mapsto V_1^{\otimes n}, \quad \text{im } p_n \mapsto V_n,$$

where V_n denotes the $n + 1$ dimensional simple representation of $U_q(\mathfrak{sl}_2)$. This functor is an equivalence of categories.

As is well known

$$(6.1) \quad V_n \otimes V_1 \cong V_{n+1} \oplus V_{n-1}$$

for $n \geq 1$, and

$$V_0 \otimes V_1 \cong V_1.$$

Since the summands on the right-hand side above are distinct simples there is a unique pair of complementary idempotent endomorphisms of $V_n \otimes V_1$ which project onto summands isomorphic to $V_{n\pm 1}$.

Iterating (6.1) allows us to obtain a preferred collection of idempotent endomorphisms which project $V_1^{\otimes n}$ onto its simple direct summands. The goal of this section is to arrive at such idempotent decompositions inside \mathcal{TL} , by means which are amenable to categorification. We begin by introducing the central idempotents $p_{n,k} \in \mathcal{TL}_n$, which under the functor to $U_q(\mathfrak{sl}_2)$ representations become the projections of $V^{\otimes n}$ onto its V_k -isotypic summand.

Proposition 6.2. *There exist a unique family of elements $p_{n,k} \in \mathcal{TL}_n$ indexed by integers $0 \leq k \leq n$ with $k \equiv n \pmod{2}$ such that*

- 1) $\text{id}_n = \sum_k p_{n,k}$.
- 2) $p_{n,k}$ is a sum of elements factoring through $p_k \in \mathcal{TL}_k$.

Proof. The proof of existence is by induction on n . In the base cases we have $p_{0,0} = \text{id}_0$ and $p_{1,1} = \text{id}_1$. So assume that $p_{n,k}$ have been constructed as in the statement. By hypothesis, $p_{n,k}$ can be written as

$$p_{n,k} = \sum_i x_{n,k,i} p_k y_{n,k,i}$$

for some elements $x_{n,k,i} \in \mathcal{TL}_{n,k}$ and $y_{n,k,i} \in \mathcal{TL}_{k,n}$. Adding a strand yields

$$p_{n,k} \sqcup \text{id}_1 = \sum_k (x_{n,k,i} \sqcup \text{id}_1)(p_k \sqcup \text{id}_1)(y_{n,k,i} \sqcup \text{id}_1).$$

The recursion satisfied by the Jones–Wenzl idempotents tells us that $p_k \sqcup \text{id}_1$ is a sum of p_{k+1} plus terms which factor through p_{k-1} . It follows that $p_{n,k} \sqcup \text{id}_1$ can be written as a sum of terms factoring through p_{k+1} or p_{k-1} , hence by induction that id_{n+1} can be written as a sum of terms factoring through p_l for various l , which completes the inductive proof on the existence of $\{p_{n,k}\}_{n,k}$.

We make some observations. If $f \in \mathcal{TL}_{n,m}$ factors through p_k , then $l \neq k$ implies $f p_{n,l} = 0$, hence

$$f = f \sum_l p_{n,l} = f p_{n,k}.$$

A similar argument shows that $f = p_{m,k} f$. Now, suppose that $f \in \mathcal{TL}_n$ is a sum of terms factoring through p_k , and $\text{id}_n - f$ is a sum of terms factoring through p_l for various $l \neq k$. Then $p_{n,k} f = f$ and $p_{n,k}(\text{id}_n - f) = 0$. In other words,

$$f = p_{n,k} f = p_{n,k}.$$

This proves the uniqueness of $p_{n,k}$. \square

Additional properties of the $p_{n,k}$ are recorded below.

Corollary 6.3. *The $p_{n,k}$ are a complete collection of mutually orthogonal idempotents in \mathcal{TL}_n , with $p_{n,n} = p_n$.* \square

Corollary 6.4. *The $p_{n,k}$ are central, in the sense that $p_{n,k}f = fp_{m,k}$ for all $f \in \mathcal{TL}_{n,m}$ (interpreted as zero if $n < k$ or $m < k$).*

Proof. If f factors through p_k for some k then $f = p_{n,k}f = fp_{m,k}$. Any $f \in \mathcal{TL}_{n,m}$ is a sum of terms factoring through some p_k , which completes the proof. \square

Corollary 6.5. *We have $(p_{n-1,k} \sqcup \text{id}_1)p_{n,l} = 0$ unless $l = k + 1$ or $l = k - 1$.*

Proof. We have seen that $p_{n-1,k} \sqcup \text{id}_1$ is a sum of terms factoring through p_{k+1} or p_{k-1} ; the corollary follows. \square

We remark that the $p_{n,k}$ are not primitive idempotents. To construct a complete collection of primitive idempotents in \mathcal{TL}_n we use the following.

Definition 6.6. A sequence $\varepsilon \in \{1, -1\}^n$ will be called *admissible* if $\varepsilon_1 + \dots + \varepsilon_i \geq 0$ for all $1 \leq i \leq n$. For each admissible $\varepsilon \in \{1, -1\}^n$ define an element $p_\varepsilon \in \mathcal{TL}_n$ by the formula

$$p_\varepsilon = \prod_{i=1}^n (p_{i, \varepsilon_1 + \dots + \varepsilon_i} \sqcup \text{id}_{n-i})$$

Proposition 6.7. *The set $\{p_\varepsilon \mid \varepsilon \in \{1, -1\}^n \text{ is admissible}\}$ is a complete collection of orthogonal idempotents in \mathcal{TL}_n . Moreover, $\text{im } p_\varepsilon \cong \text{im } p_{|\varepsilon|}$ in $\text{Kar}(\mathcal{TL})$; in particular the idempotents p_ε are primitive.*

Proof. We multiply together our idempotent decompositions of id_n :

$$\text{id}_n = \prod_{m=1}^n \left(\sum_k p_{m,k} \sqcup \text{id}_{n-m} \right) = \sum_{k_1, \dots, k_n} p'_{1,k_1} p'_{2,k_2} \cdots p'_{n,k_n},$$

where we are abbreviating by writing $p'_{m,k} := p_{m,k} \sqcup \mathbb{1}_{n-m}$, and the sum on the right is indexed by sequences (k_1, \dots, k_n) with $0 \leq k_m \leq m$ and $k_m \equiv m \pmod{2}$. Observe that all of the factors $p'_{m,k}$ commute with one another, hence the $p'_{1,k_1} p'_{2,k_2} \cdots p'_{n,k_n}$ are orthogonal idempotents. Corollary 6.5 says that $p'_{1,k_1} p'_{2,k_2} \cdots p'_{n,k_n}$ is zero unless $k_m = k_{m-1} \pm 1$ for all $2 \leq m \leq n$. We obtain an idempotent decomposition

$$\text{id}_n = \sum_{\varepsilon} p_\varepsilon$$

indexed by admissible sequences $\varepsilon \in \{1, -1\}^n$ as claimed.

To see that $\text{im } p_\varepsilon \cong \text{im } p_{|\varepsilon|}$ in $\text{Kar}(\mathcal{TL})$, we argue by induction on n . The statement is trivial for $n = 0$ or $n = 1$, since in these cases $p_\varepsilon = p_n = \text{id}_n$. Assume by induction that $\text{im } p_\varepsilon \cong \text{im } p_k$, where $k = |\varepsilon|$. The decomposition $p_\varepsilon \sqcup \text{id}_1 = p_{(\varepsilon,1)} + p_{(\varepsilon,-1)}$ yields an isomorphism

$$\text{im}(p_\varepsilon) \otimes \text{im}(\text{id}_1) \cong \text{im}(p_{(\varepsilon,1)}) \oplus \text{im}(p_{(\varepsilon,-1)}).$$

On the other hand,

$$\text{im}(p_\varepsilon) \otimes \text{im}(\text{id}_1) \cong \text{im}(p_k) \otimes \text{im}(\text{id}_1) \cong \text{im}(p_{k+1}) \oplus \text{im}(p_{k-1})$$

Since $p_{(\varepsilon, \pm 1)}$ factors through $p_{k \pm 1}$, the only possibility is that $\text{im}(p_{(\varepsilon, \pm 1)}) \cong \text{im}(p_{k \pm 1})$. This completes the proof. \square

6.2. The categorified central idempotents. In this section we describe the categorical analogues of the idempotent decompositions from §6.1. As discussed in §4.3, the Cooper–Krushkal projectors $P_n \in \text{Ch}^-(\mathbf{BN}_n)$ are categorical analogues of the Jones–Wenzl idempotents $p_n \in \mathcal{TL}_n$.

Definition 6.8. Given integers $0 \leq k \leq n$ with $k \equiv n \pmod{2}$, we let $\mathcal{X}_{n,k}$ denote the collection of complexes which are direct sums of q -shifts of complexes of the form $T \star P_k \star T'$, where T, T' are objects of $\mathbf{BN}_{n,k}$ and $\mathbf{BN}_{k,n}$, respectively. We say that $X \in \text{Ch}^-(\mathbf{BN}_n)$ *factors through* P_k if $X \simeq X'$ with $X' \in \langle \mathcal{X}_{n,k} \rangle$.

Note that we do not allow cohomological shifts in the definition, so each complex in $\mathcal{X}_{n,k}$ is supported in cohomological degrees ≤ 0 .

Lemma 6.9. *If $X \in \text{Ch}^-(\mathbf{BN}_n)$ factors through P_k then $X \sqcup \mathbb{1}_1 \in \text{Ch}^-(\mathbf{BN}_{n+1})$ is homotopy equivalent to a complex factoring through P_{k+1} and P_{k-1} .*

Proof. The recursion satisfied by Cooper–Krushkal projectors ensures that if $X = T \star P_k \star T'$ then $X \sqcup \mathbb{1}_1 = (T \sqcup \mathbb{1}_1) \star (P_k \sqcup \mathbb{1}_1) \star (T' \sqcup \mathbb{1}_1)$ is homotopy equivalent to a complex in $\langle \mathcal{X}_{n+1,k+1} \cup \mathcal{X}_{n+1,k-1} \rangle$. Then Lemma 3.12 completes the proof. \square

Lemma 6.10. *Suppose we have $X \in \mathcal{X}_{n,k}$ and $Y \in \mathcal{X}_{n,l}$. We have:*

- 1) *if $k < l$ then each chain object X^m satisfies $\text{Hom}(X^m, Y) \simeq 0$.*
- 2) *if $k = l$ and $m < 0$, then $\text{Hom}(X^m, Y) \simeq 0$.*

Proof. We use the duality isomorphism $\text{Hom}(X^m, Y) \cong \text{Hom}(\mathbb{1}_n, (X^m)^\vee \star Y)$ together with the fact that the through degree of X^m is $\leq k$, with equality if and only if $m = 0$. Since Y factors through P_l , $X^m \star Y$ is contractible if the through degree of X^m is strictly less than l , which will occur if $k < l$ or $k = l$ and $m < 0$. \square

Theorem 6.11. *There exist a unique family of complexes $P_{n,k} \in \text{Ch}^-(\mathbf{BN}_n)$ such that*

- 1) $\mathbb{1}_n \simeq (P_{n,n} \rightarrow P_{n,n-2} \rightarrow \cdots \rightarrow P_{n,0} \text{ or } 1)$.
- 2) $P_{n,k}$ *factors through* P_k .

Proof. Fix n , and let $\Omega = \{k \mid 0 \leq k \leq n, k \equiv n \pmod{2}\}$ with order $k \trianglelefteq l$ if $k > l$ (note the reversal of order). Iterating Lemma 6.9 we see that $\mathbb{1}_n$ is homotopy equivalent to a complex $E \in \langle \mathcal{X}_{n,k} \mid k \in \Omega \rangle$. The rest of the construction is achieved by combing hairs.

To be precise, note that the collections of complexes $\mathcal{X}_{n,k}$ (indexed by $k \in \Omega$) satisfy the hypotheses of Lemma 3.15 by Lemma 6.10. Combing hairs results in a complex $E' \simeq \mathbb{1}_n$ which can be written as $E = \text{tw}_{\delta'}(\bigoplus_{i \in I'} \uparrow^{a(i)} X_i)$ in which I is a partially ordered set, $X_i \in \mathcal{X}_{\omega(i)}$, such $i \leq j$ implies $\omega(i) \triangleleft \omega(j)$ or $\omega(i) = \omega(j)$ and $a(i) < a(j)$. We may collect terms according to the through degree k and the cohomological shift a , obtaining

$$\mathbb{1}_n \simeq \text{tw}_{\delta+\varepsilon} \left(\bigoplus_{k \in \Omega} \bigoplus_{a \leq 0} E'_{k,a} \right)$$

where the twist δ maps $E'_{k,a}$ to $E'_{l,b}$ only if $k \triangleleft l$ and the twist ε maps $E'_{k,a}$ to $E'_{l,b}$ only if $k = l$ and $a \leq b$. Then the complexes $P_{n,k} = \text{tw}_{\varepsilon}(\bigoplus_{a \leq 0} E'_{k,a})$ are as in the statement. \square

Proposition 6.12. *Given $X \in \text{Ch}^-(\mathbf{BN}_{n,m})$ we have $P_{n,k} \star X \simeq X \star P_{m,k}$, naturally in X up to homotopy.*

Proof. Define complexes

$$P_{n,\leq k} = (P_{n,k} \rightarrow P_{n,k-2} \rightarrow \cdots \rightarrow P_{n,0 \text{ or } 1}),$$

$$P_{n,\geq k} = (P_{n,n} \rightarrow P_{n,n-2} \rightarrow \cdots \rightarrow P_{n,k}).$$

Then $P_{n,k} \simeq P_{n,\geq k} \star P_{n,\leq k}$. So it suffices to show that $P_{n,\geq k}$ and $P_{n,\leq k}$ are central, as in the statement. The centrality of these complexes follows from [Hog17, Theorem 4.24]. \square

The following is straightforward.

Lemma 6.13. *Suppose $X \in \text{Ch}^-(\mathbf{BN}_n)$ factors through P_k . Then $P_{n,l} \star X \simeq 0$ if $l \neq k$, and $P_{n,k} \star X \simeq X$. With $l \neq k$, then $P_{n,k} \star X \simeq 0$.*

Proof. Suppose $X, Y \in \text{Ch}^-(\mathbf{BN}_n)$ are complexes such that X factors through P_k and Y factors through P_l . Then $X \star Y \simeq 0$. In particular, $P_{n,l} \star X \simeq 0$. Using the idempotent decomposition of $\mathbb{1}_n$ into central idempotents, we obtain:

$$X \simeq \text{tw}_\delta \left(\bigoplus_k P_{n,k} \right) \star X \simeq \text{tw}_\delta \left(\bigoplus_k P_{n,k} \star X \right) \simeq P_{n,k} \star X,$$

where in the last step we contracted the contractible complexes $P_{n,l} \star X$ ($l \neq k$) using homological perturbation. \square

6.3. The primitive categorified idempotents. The following defines the categorical analogues of the idempotent elements $p_\varepsilon \in \mathcal{TL}_n$ from Definition 6.6.

Definition 6.14. For each admissible $\varepsilon \in \{1, -1\}$ define $P_\varepsilon \in \text{Ch}^-(\mathbf{BN}_n)$ by the formula

$$P_\varepsilon = \star_{i=1}^n (P_{i,\varepsilon_1+\cdots+\varepsilon_i} \sqcup \mathbb{1}_{n-i}).$$

Proposition 6.15. *We have $P_\varepsilon \star P_\nu \simeq 0$ if $\varepsilon \neq \nu$, and $\mathbb{1}_n \simeq \text{tw}_\delta(\bigoplus_\varepsilon P_\varepsilon)$ in which the twist δ strictly decreases the partial order $(\varepsilon_1, \dots, \varepsilon_n)T \geq (\nu_1, \dots, \nu_n)$ if $\varepsilon_1 + \cdots + \varepsilon_i \geq \nu_1 + \cdots + \nu_i$ for all $i = 1, \dots, n$.*

Proof. This is analogous to Proposition 6.7, and we leave many details to the reader. We tensor together our idempotent decompositions of $\mathbb{1}_n = \mathbb{1}_m \sqcup \mathbb{1}_{n-m}$ for $1 \leq m \leq n$:

$$\text{id}_n \simeq \star_{m=1}^n \left(\sum_k P_{m,k} \sqcup \text{id}_{n-m} \right) = \bigoplus_{k_1, \dots, k_n} P'_{1,k_1} P'_{2,k_2} \cdots P'_{n,k_n},$$

with a twist that decreases one or more indices k_i , where we are abbreviating by writing $P'_{m,k} := P_{m,k} \sqcup \mathbb{1}_{n-m}$, and the sum on the right is indexed by sequences (k_1, \dots, k_n) with $0 \leq k_m \leq m$ and $k_m \equiv m \pmod{2}$. Observe that all of the factors $P'_{m,k}$ commute with one another up to homotopy, hence the $P'_{1,k_1} P'_{2,k_2} \cdots P'_{n,k_n}$ are orthogonal idempotents up to homotopy. Lemma 6.13 and Lemma 6.9 tell us that $P'_{1,k_1} P'_{2,k_2} \cdots P'_{n,k_n}$ is contractible unless $k_m = k_{m-1} \pm 1$ for all $2 \leq m \leq n$. Homological perturbation gives an idempotent decomposition

$$\mathbb{1}_n \simeq \text{tw}_\delta \left(\bigoplus_\varepsilon P_\varepsilon \right)$$

in which the twist decreases one or more indices k_i . \square

6.4. The quantum annular trace of categorified idempotents.

Theorem C. *For every admissible $\varepsilon \in \{1, -1\}^n$ we have $[P_\varepsilon] \simeq V_{|\varepsilon|}$ in $\text{Ch}^-(\text{Kar}(\mathcal{BN}_q(\mathbb{A})))$.*

Proof. Induction on n . In the base case $n = 0$ there is nothing to prove. Assume by induction that $[P_\varepsilon] \simeq V_k$, where $k = |\varepsilon|$. Then $[P_\varepsilon \sqcup \mathbb{1}_1] \simeq V_{k+1} \oplus V_{k-1}$. But also $[P_\varepsilon \sqcup \mathbb{1}_1] \simeq ([P_{(\varepsilon,1)}] \rightarrow [P_{(\varepsilon,-1)}])$, by construction. Computations in the trace tell us that $P_{(\varepsilon,\pm 1)}$ is homotopy equivalent to a complex built from copies of $[P_{k\pm 1}] \simeq V_{k\pm 1}$. The only possibility is $[P_{(\varepsilon,\pm 1)}] \simeq V_{k\pm 1}$. \square

Corollary 6.16. *We have $[P_{n,k}] \simeq V_k^{\oplus C_{n,k}} \cong \text{im}(p_{n,k})$ where $C_{n,k}$ is the number of admissible $\varepsilon \in \{1, -1\}^n$ with $|\varepsilon| = k$. Explicitly, $C_{n,k} = \frac{k+1}{m+1} \binom{n}{m}$ where $m = (n+k)/2$.*

Proof. It follows from Proposition 6.15 that $P_{n,k}$ is homotopy equivalent to a one-sided twisted complex built from idempotents P_ε with $|\varepsilon| = k$. Because of Theorem C, this implies that $[P_{n,k}]$ is a one-sided twisted complex built from copies of V_k , and since all of these copies are in cohomological degree zero, there can be no differentials between them.

The explicit formula for $C_{n,k}$ can be obtained from the hook length formula by identifying admissible sequences $\varepsilon \in \{1, -1\}^n$ with $|\varepsilon| = k$ with standard Young tableaux of shape $\lambda = (m, n-m)$. Specifically, an admissible sequence $\varepsilon \in \{1, -1\}^n$ corresponds to a standard two-row Young tableau whose first row contains the indices i with $\varepsilon_i = 1$, and whose second row contains the indices i with $\varepsilon_i = -1$. \square

7. QUANTUM HOCHSCHILD HOMOLOGY OF H^n

As an application of the results from the previous section, we will now compute the full quantum Hochschild homology of Khovanov's arc ring H^n [Kho02]. We will first recall the definition of H^n .

Let B^n denote the set of all isotopy classes of flat tangles without closed components, and with $2n$ upper endpoints and no lower endpoints. Given $a, b \in B^n$, let a^\vee denote the reflection of a along a horizontal line, and $a^\vee b := a^\vee \star b$ denote the vertical composition of a^\vee and b . Note that $a^\vee b$ is a disjoint collection of closed components in the plane \mathbb{R}^2 .

By applying Khovanov's functor from [Kho00, Subs. 2.3] (for $c = 0$), we can assign to each pair of elements $a, b \in B^n$ a graded \mathbb{k} -module

$${}_a H_b^n := \mathbb{Q}^n \mathcal{F}(a^\vee b).$$

Since \mathcal{F} is naturally isomorphic to $\text{Hom}_{\text{BN}_{0,0}}(\emptyset, -)$ [BN05, Subs 9.1], this \mathbb{k} -module can also be viewed as the morphism set

$$\mathbb{Q}^n \text{Hom}_{\text{BN}_{0,0}}(\emptyset, a^\vee b) \cong \text{Hom}_{\text{BN}_{2n,2n}}(a, b).$$

As a graded \mathbb{k} -module, Khovanov's arc algebra H^n can now be defined as the direct sum

$$H^n := \bigoplus_{a,b} {}_a H_b^n$$

for $a, b \in B^n$. The algebra multiplication is induced by cobordism maps

$${}_a H_b^n \otimes_{\mathbb{k}} {}_b H_c^n \longrightarrow {}_a H_c^n,$$

or equivalently, by the composition in $\text{BN}_{2n,2n}$.

The algebra H^n comes with a distinguished collection of orthogonal idempotents $(a) \in H^n$, one for each $a \in B^n$, which are given by labeling each circle in $a^\vee \star a$ by the generator 1 of Khovanov's Frobenius algebra $\mathbb{k}[x]/(x^2)$. Correspondingly, there are projective left modules

$$H_a^n := H^n(a) = \bigoplus_b H_a^n$$

and projective right modules

$${}_a H^n := (a)H^n = \bigoplus_b {}_a H_b^n.$$

Khovanov's construction further gives rise to a functor

$$\mathcal{F}_n: \mathbf{BN}_{2n,2n} \longrightarrow H^n\text{-Bimod}$$

which sends a flat $(2n, 2n)$ tangle T to a sweet graded H^n -bimodule $\mathcal{F}_n(T)$, in the sense of [Kho02, Def. 1]. As a \mathbb{k} -module, this bimodule decomposes as

$$\mathcal{F}_n(T) := \bigoplus_{a,b} {}_a \mathcal{F}(T)_b$$

for $a, b \in B^n$, where

$${}_a \mathcal{F}(T)_b := \mathbb{Q}^n \mathcal{F}(a^\vee \star T \star b).$$

Moreover, Khovanov proved that the functors \mathcal{F}_n extend to a bifunctor

$$\mathbf{F}: \mathbf{BN}_{\text{even}} \longrightarrow \mathbf{Bimod},$$

where \mathbf{Bimod} denotes the bicategory of sweet graded bimodules, and $\mathbf{BN}_{\text{even}} \subset \mathbf{BN}$ denotes the bicategory whose objects are given by even integers $2n \geq 0$, and whose 1-morphism categories are identical with the ones in \mathbf{BN} .

Before proceeding, let us also recall that the 0 th quantum Hochschild homology of a graded bimodule M over a graded algebra $A = \bigoplus_d A_d$ is given by the quantum space of coinvariants

$$\text{coInv}_q(M) := M/[A, M]_q,$$

where

$$[A, M]_q := \text{Span}_{\mathbb{k}}\{am - q^d ma \mid a \in A_d, m \in M\}.$$

More generally, the full quantum Hochschild homology of M can be computed as the homology of the complex $\text{coInv}_q(\mathcal{P})$ where $\mathcal{P} \rightarrow M$ is a projective resolution of the graded bimodule M .

It was observed in [BPW19] that the assignment $M \mapsto \text{coInv}_q(M)$ extends to a quantum shadow on the bicategory \mathbf{Bimod} . Since this bicategory has left and right duals, Theorem 3.5 from [BPW19] implies that this quantum shadow factors through a functor

$$\overline{\text{coInv}}_q: \text{hTr}_q(\mathbf{Bimod}) \longrightarrow \mathbb{k}\text{-mod}.$$

We can now prove the following theorem, which holds under the assumption that $1 - q^d$ is invertible for all integers d in $(0, 4n]$.

Theorem 7.1. *The higher quantum Hochschild homology of H^n vanishes and the 0th quantum Hochschild homology of H^n is isomorphic to $\mathbb{k}^{\oplus C_{2n,0}}$ where $C_{2n,0} = |B^n|$ is the n th Catalan number.*

Proof. By Proposition 6.15, we have a homotopy equivalence

$$\mathbb{1}_{2n} \simeq \left(P_{2n,2n} \longrightarrow P_{2n,2n-2} \longrightarrow \dots \longrightarrow P_{2n,0} \right)$$

and applying \mathcal{F}_n gives

$$(7.1) \quad H^n \simeq \left(\mathcal{P}_{2n,2n} \longrightarrow \mathcal{P}_{2n,2n-2} \longrightarrow \dots \longrightarrow \mathcal{P}_{2n,0} \right)$$

where $\mathcal{P}_{2n,k} := \mathcal{F}_n(P_{2n,k})$.

Now observe that, as a complex of \mathbb{k} -modules, $\mathcal{P}_{2n,k}$ is built from complexes of the form

$${}_a\mathcal{F}_n(T \star P_k \star T')_b = \mathcal{F}(a^\vee \star T \star P_k \star T' \star b)$$

for $a, b \in B^n$. In particular, $\mathcal{P}_{2n,k}$ is acyclic for $k > 0$ because P_k kills turnbacks.

On the other hand, $\mathcal{P}_{2n,0}$ is built from flat tangles with through-degree 0, and after removing closed components, every such tangle can be written as $a \star (b^\vee)$ for $a, b \in B^n$. Since closed components in a flat tangle T don't affect the bimodule multiplication on $\mathcal{F}_n(T)$, this shows that $\mathcal{P}_{2n,0}$ is built from projective bimodules of the form $\mathcal{F}_n(a \star (b^\vee)) = H_a^n \otimes_{\mathbb{k}} {}_b H^n$ for $a, b \in B^n$.

In summary, we see that the inclusion of the subcomplex $\mathcal{P}_{2n,0}$ into the right-hand side of (7.1) defines a quasi-isomorphism $\mathcal{P}_{2n,0} \rightarrow H^n$, and hence $\mathcal{P}_{2n,0}$ is a projective resolution of H^n . By the remarks preceding this proof, we can therefore compute the quantum Hochschild homology of H^n from the complex

$$\mathrm{coInv}_q(\mathcal{P}_{2n,0}) = \overline{\mathrm{coInv}_q}([\mathcal{P}_{2n,0}]) = \overline{\mathrm{coInv}_q \circ \mathbf{F}}([\mathcal{P}_{2n,0}])$$

The theorem now follows from Corollary 6.16 for $k = 0$. □

Remark 7.2. An analogous result for the Chen–Khovanov algebras A^n was shown in [BPW19, Prop. 6.6]. Unlike Theorem 7.1, the latter result holds without any restrictions on q .

Remark 7.3. Explicitly, the 0th quantum Hochschild homology of H^n is spanned by the idempotents (a) for $a \in B^n$. Indeed, these idempotents remain linearly independent in $\mathrm{coInv}_q(H^n)$ because they are mutually orthogonal and span the degree 0 part of the positively graded algebra H^n .

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