On the relation between two quantum group invariants of 3-cobordisms

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Abstract: We prove in the context of quantum groups at even roots of unity that a Turaev-Viro type invariant of a 3-dimensional cobordism M equals the tensor product of the Reshetikhin-Turaev invariants of M and M^*, where the latter denotes M with orientation reversed.

1 Introduction

According to [At] a 3-dimensional topological quantum field theory (TQFT) associates a finite dimensional vector space V_Σ to each compact closed oriented 2-dimensional surface Σ and a vector (partition function) $Z(M) \in V_\Sigma$ to each compact oriented 3-dimensional manifold M with boundary Σ, satisfying a certain set of axioms. Of particular relevance for the following discussion are the following: 1) V_{Σ^*} is the dual space of V_Σ for each surface Σ, where Σ^* denotes Σ with orientation reversed, 2) given an orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ between oriented surfaces, there exists an isomorphism $U(f) : V_\Sigma \to V_{\Sigma'}$ fulfilling $U(f_1 f_2) = U(f_1)U(f_2)$ for any pair of diffeomorphisms that can be composed, and 3) if M is obtained by gluing two 3-manifolds M_1 and M_2 along $\Sigma \in \partial M_1$ and $\Sigma^* \in \partial M_2$ then $Z(M)$ is obtained by contracting $Z(M_1) \otimes Z(M_2)$ with respect to V_Σ. In addition, the vectorspace associated to the empty surface is assumed to be the complex numbers, and if Σ is the disjoint union of two surfaces Σ_1 and

Σ_2 then $V_\Sigma = V_{\Sigma_1} \otimes V_{\Sigma_2}$. In particular, if M is a closed manifold $Z(M)$ is a complex number which is a topological invariant of M.

Alternatively, the gluing property 3) can be reformulated in terms of operators as follows. Viewing M_1 and M_2 as cobordisms with $\partial M_1 = \Sigma_1 \cup \Sigma$ and $\partial M_2 = \Sigma'^* \cup \Sigma_2$ we can correspondingly consider the state sums as operators $Z(M_1) : V_{\Sigma_1}{}^* \to V_\Sigma$ and $Z(M_2) : V_{\Sigma'} \to V_{\Sigma_2}$ by 1). Given an orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ and letting M denote the manifold obtained by gluing M_1 to M_2 along f, property 3) is equivalent to

$$Z(M) = Z(M_2)U(f)Z(M_1)\ . \tag{1.1}$$

Note that the symmetry of the gluing w.r.t. M_1 and M_2 requires that

$$U(f^*) = (U(f)^t)^{-1}, \tag{1.2}$$

where $f^* : \Sigma^* \to (\Sigma')^*$ denotes f with orientations on Σ and Σ' switched, and the superscript t indicates transposition. There now exists in the literature a variety of rigorous constructions of 3-dimensional TQFT's. In this note we shall consider the constructions by Reshetikhin-Turaev [RT] and the one by Turaev-Viro [TV] and their generalizations (see [T], [DJN], [KS], [BD]). These are all based on the algebraic structure of the representation theory of quantum groups with deformation parameter equal to a root of unity, and are known to be related to Chern-Simons theory with an arbitrary compact gauge group.

In [BD] we have proven that for closed manifolds the invariant $Z_{TV}(M)$ of the Turaev-Viro construction equals the modulus squared of the invariant $\tau(M)$ obtained by the Reshetikhin-Turaev construction for a general quantum group at simple even roots of unity (see also [Wa], [T] and [R]). The purpose of this paper is to extend this result to manifolds with boundary, i.e. we show that

$$Z_{TV}(M) = \tau(M) \otimes \tau(M^*)$$

for any 3-cobordism M. Here $Z_{TV}(M)$ and $\tau(M)$ denote the cobordism invariants defined in [BD] and [T], respectively. In section 2 we recall briefly the basic elements of the Turaev-Viro construction as developed in [BD] and refer the reader to that paper for fuller details. We then prove a basic lemma which yields certain isomorphisms from the state spaces of the theory onto certain explicitly realizable spaces. This result is used in Section 3 to obtain an equivalent TQFT for which the announced factorization property is then proven.

2 Turaev-Viro TQFT

In this section we briefly recall the formulation and basic properties of TQFT of the Turaev-Viro type (for more details see [BD]). The corresponding state sum will be denoted by $Z(M)$ (omiting the index TV in the following).

Originally, the Turaev-Viro invariant was defined for a compact connected closed oriented 3-manifold M as follows [TV]: Choose a triangulation of M and associate to each 1-simplex of the triangulation an index (or a colour) from a finite set $\mathcal{I}$ of so-called "admissible" representations of a quantum group. To each coloured tetrahedron one then associates a 6j-symbol, which is possible due to the invariance of 6j-symbols under the tetrahedral symmetry group. In addition, to each coloured link one attaches a factor ω_i^2, which equals the quantum dimension of the corresponding colour i, and to each vertex one attaches a factor ω^{-2}, where

$$\omega^2 = \sum_{i \in \mathcal{I}} \omega_i^4.$$

The invariant $Z(M)$ is then obtained as the sum over all colourings of the triangulation of the product of all factors associated to tetrahedra, links and vertices. It can be shown (using the Biedenharn-Elliott relations for 6j-symbols) that the resulting quantity is independent of the particular choice of triangulation.

We have here assumed that the 6j-symbols are scalars, i.e. that the multiplicity of any representation $i \in \mathcal{I}$ in a tensor product of two representations in $\mathcal{I}$ is always 0 or 1, which e.g. is the case for $SU_q(2)$. For more general quantum groups the 6j-symbols are tensors. To be specific we associate to each oriented, coloured triangle t in $\Sigma = \partial M$ with oriented boundary links as indicated in Fig.1 (where the orientation of the plane is assumed to be counter clock-wise) the vector space V_{ij}^{k} of Clebsch-Gordan coefficients defined by

$$H_i \otimes H_j = \sum_{k \in \mathcal{I}} V_{ij}^{k} \otimes H_k,$$

where H_i denotes the vector space of the representation i.

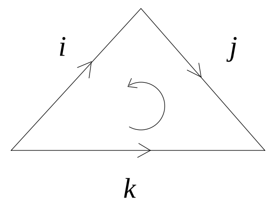


Fig.1 *An oriented* $\{i, j, k\}$*-coloured 2-simplex*

The canonically dual vector space $(V_{ij}^{k})^* = V_{k}^{ij}$ will be associated to the oppositely oriented triangle. For other configurations of arrows than that on Fig.1 the corresponding spaces are defined by requiring that reversing an arrow on a 1-simplex is equivalent to replacing its colour by the dual one (i.e. replacing the corresponding representation by its adjoint).

Moreover, the 6j-symbol associated to an oriented coloured tetrahedron with oriented links belongs to the tensor product of the vector spaces associated to the triangles in its boundary. Thus, we may define $Z(M)$ by replacing above the product of 6j-symbols by the corresponding tensor product and contracting with respect to the dual pairs of spaces associated to triangles (with some fixed orientation of links), and the result is again independent of the choice of triangulation as well as of the chosen orientation of links. In fact, this definition is easily extended to non-closed, oriented manifolds M by simply contracting only with respect to dual pairs of spaces associated to *interior* triangles of the triangulation. One then obtains a tensor $Z'(M)$ in the vector space $V'_{\partial M}$ defined as the direct sum over all colourings of the links in ∂M of the tensor product of the spaces associated to the triangles in ∂M. This space, of course, depends on the triangulation of ∂M. However, any two such triangulations may be connected by a triangulation of the cylinder $\partial M \times [0, 1]$ in the obvious sense, and $Z'(\partial M \times [0, 1])$ defines a *cylinder map* between the corresponding spaces. In particular, choosing the same triangulation at the two ends of the cylinder the map becomes a projection, and the supports of the projections so obtained may be canonically identified by the cylinder maps thus defining the vector space $V_{\partial M}$, and at the same time the partition functions $Z'(M)$ are also identified with a unique vector $Z(M) \in V_{\partial M}$ fulfilling the required properties.

Exploiting ideas of Turaev [Tu] an effective calculational tool was developed in [KS] by introducing coloured graphs $G_{\underline{x}}$ on the boundary of the manifold M and defining an associated state sum $Z(M, G_{\underline{x}})$ generalizing $Z(M)$. Here a coloured graph $G_{\underline{x}}$ is a closed 1-dimensional simplicial complex, whose 0-simplexes have order at most 3 and whose lines (i.e. maximal sequences of 1-simplexes joined by vertices of order 2 are oriented and coloured (by elements in $\mathcal{I}$), the collection of colours being indicated by $\underline{x}$. The graph is assumed to be embedded into ∂M such that over- and undercrossings are distinguished. The definition of $Z(M, G_{\underline{x}})$ proposed in [KS] has the following geometrical interpretation (see [BD]). One glues to the boundary Σ of M a certain pseudo-manifold P_G whose boundary consists partly of one copy of Σ^* (triangulated as Σ) and partly of a surface on which the dual graph of G determines a cell decomposition into triangles (corresponding to 3-vertices) and rectangles (corresponding to over- and undercrossings) and whose edges inherit a colouring from $\underline{x}$. The state sum $Z(M, G_{\underline{x}})$ then equals $Z(M_{G_{\underline{x}}})$, where $M_{G_{\underline{x}}}$ is the resulting pseudo-manifold with fixed colouring of boundary links given by $\underline{x}$. Actually, the construction requires a slight modification in case rectangles are present in the boundary (see [BD]). Suffice here to mention that $Z(M, G_{\underline{x}})$ in all cases belongs to the tensor product of the vector spaces associated to the triangles dual to the 3-vertices in $G_{\underline{x}}$ and is a homotopy invariant of the coloured graph $G_{\underline{x}}$ in Σ.

In case G is empty the pseudo-manifold P_G is the cone over Σ and the boundary of the resulting manifold degenerates to a point. On the other hand, if G is sufficiently "large" so that P_G is homeomorphic to the cylinder $\Sigma \times [0,1]$, then $M_{G_{\underline{x}}}$ is homeomorphic to M, and if G in addition has no over- or undercrossings it follows that $\oplus_{\underline{x}} Z(M, G_{\underline{x}})$ equals $Z'(M)$ with ∂M triangulated by the dual graph to G.

The gluing axiom described at the beginning of section 1 can now be reformulated in the language of graphs as follows. If M is obtained by gluing M_1 and M_2 along Σ we have

$$Z(M) = \frac{1}{\omega^2} \sum_{\underline{x}} \omega_{\underline{x}}^2 \, Z(M_1, G_{\underline{x}}^F) \, Z(M_2, G_{\underline{x}}) \tag{2.1}$$

for any canonical graph G without over- or undercrossings, and where F is the gluing homeomorphism and G^F denotes the image of G under F.

The state sums $Z(M, G_{\underline{x}})$ satisfy a number of simple relations under certain elementary changes of the graph $G_{\underline{x}}$, which together with (2.1) can be used to show that the dimension of V_{Σ_g}, where Σ_g is a connected surface of genus $g \geq 1$, is given by the square of the Verlinde formula:

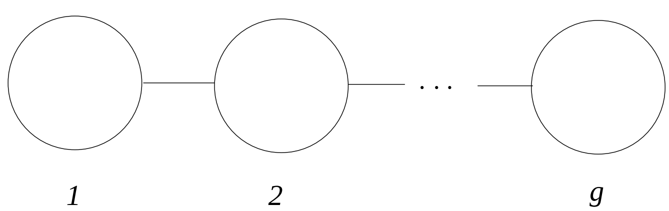
$$dim V_{\Sigma_g} = tr\ id_{V_{\Sigma_g}} = tr Z(\Sigma_g \times I) = Z(\Sigma_g \times S^1) = (tr \vec{N}^{2(g-1)})^2 \tag{2.2}$$

where $\vec{N}^2 = \sum_a (N^a)^2$ and (N^a) is the multiplicity matrix given by

$$(N^a)_{bc} = N_{bc}^a = dim V_{bc}^a \tag{2.3}$$

for $a, b, c \in \mathcal{I}$.

It is even possible to realize the space V_{Σ_g} explicitly as follows. Consider a handle body M_g of genus g in R^3 with $\partial M_g = \Sigma_g$ and introduce two copies c^L and c^R of the graph depicted below such that they are deformation retracts of Σ in M_g and such that they are disjoint (and not linked).



(2.4)

Clearly c^L and c^R then possess tubular neighborhoods that are disjoint and diffeomorphic to M_g and whose boundaries are homotopic to Σ_g in M_g. Removing two such tubular neighborhoods from M_g yields a manifold $\tilde{M}_g$ with three boundary components Σ_g, $(\Sigma_g^L)^*$ and $(\Sigma_g^R)^*$ all of genus g. We now choose the coordinates so that the cores c^L and c^R lie in the xy-planes and their z-components are equal to 1 and -1 respectively. We will call the part of Σ_g^L (resp. Σ_g^R) where $z > 1$ (resp. $z > -1$) the upper side and the other part where $z < 1$ (resp. $z < -1$) the back side of Σ_g^L (resp. Σ_g^R).

Next, we embed a copy G^L of the graph (2.4) on the upper side of Σ_g^L in such a way that the graph is homotopic to the core c^L. Analogously, we embed the second copy G^R of the graph (2.4) on the back side of Σ_g^R.

Finally, we make G^L lefthanded and G^R righthanded, i.e. we introduce meridians on each of the tubes corresponding to the lines of c^L, resp. c^R, which undercross, resp. overcross, the lines of G^L on Σ_g^L, resp. G^R on Σ_g^R. We then define

$$K_{\underline{e},\underline{f}} = \sum_{\underline{x},\underline{y}} \prod_{i=1}^{3g-3} \frac{\omega_{x_i}^2}{\omega^2} \frac{\omega_{y_i}^2}{\omega^2} \; Z(\tilde{M}_g, G_{\underline{e}}^L \cup m_{\underline{x}}^L \cup G_{\underline{f}}^R \cup m_{\underline{y}}^R \cup G^g) \tag{2.5}$$

where $\underline{e}$, resp. $\underline{f}$, is a colouring of G^L, resp. G^R, the product is over meridians and the sum is over colourings $\underline{x}$ and $\underline{y}$ of the meridians m^L and m^R, on Σ_g^L and Σ_g^R, respectively, and G^g is some canonical graph on Σ_g without over- or undercrossings.

We denote by V_g^L, resp. V_g^R, the vector space associated to G^L, resp. G^R, regarded as embedded into Σ_g^L, resp. Σ_g^R, i.e.

$$V_g^L = \oplus_{\underline{e}} V_g^L(\underline{e}), \tag{2.6}$$

where $V_g^L(\underline{e})$ is the tensor product of vector spaces associated to the coloured 3-vertices of G^L taking into account the orientation of Σ_g^L and similarly for G^R. Then

$$dim V_g^L = dim V_g^R = tr(\vec{N}^2)^{(g-1)} \tag{2.7}$$

by a simple counting, and hence

$$dim(V_g^L \otimes V_g^R) = dim V_{\Sigma_g}. \tag{2.8}$$

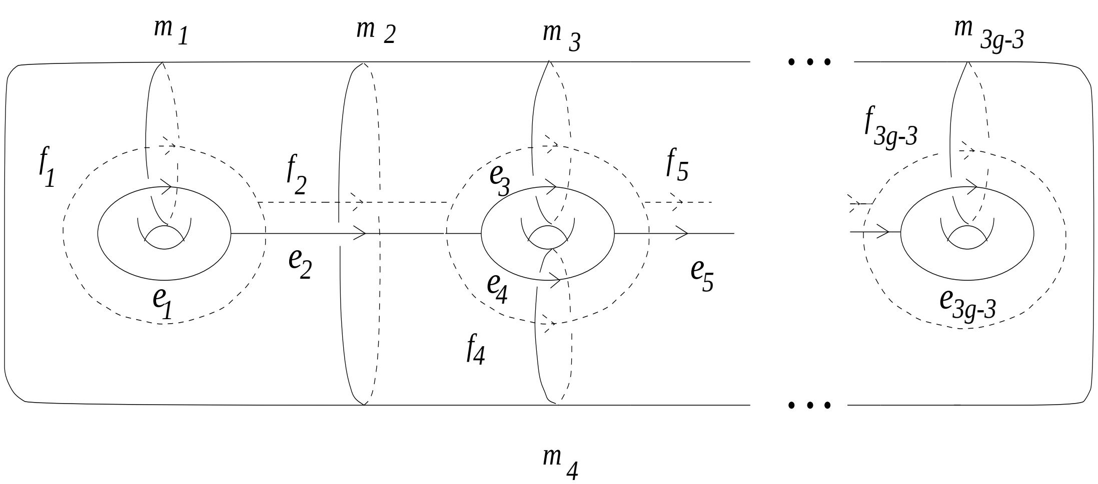
Moreover, with the chosen orientation convention we have (see [BD]) $K_{\underline{e},\underline{f}} \in V_g^L(\underline{e})^* \otimes V_g^R(\underline{f})^* \otimes V_{\Sigma_g}$ and hence (2.5) defines an operator

$$K_{\underline{e},\underline{f}} : V_g^L(\underline{e}) \otimes V_g^R(\underline{f}) \to V_{\Sigma_g}$$

in an obvious way. We intend to show that the direct sum over $\underline{e}, \underline{f}$ of these operators yields an isomorphism between $V_g^L \otimes V_g^R$ and V_{Σ_g}. This was proven for the case $g = 1$ in [BD]. In the general case it is a consequence of Lemma 1 below in which, however, we have found it convenient first to rewrite $K_{\underline{e},\underline{f}}$, up to a factor $\omega^{2(-g+1)}$, as

$$K_{\underline{e},\underline{f}} = \sum_{\underline{x}} \prod_{i=1}^{3g-3} \frac{\omega_{x_i}^2}{\omega^2} \, Z(M'_g, G_{\underline{e},\underline{f}} \cup m_{\underline{x}} \cup G^g), \tag{2.9}$$

where M'_g is the manifold with boundary components Σ_g and $\Sigma'^{\,*}_g$ obtained by removing one tubular neighborhood instead of two as above and where $G_{\underline{e},\underline{f}}$ is the coloured graph on Σ'_g indicated on the figure below together with a system m of meridians (of which there are $3g-3$ for $g \geq 1$, and 1 for $g = 1$), and G^g is as above.



(2.10)

The equivalence of (2.5) and (2.9) follows by merging Σ_g^L and Σ_g^R as in the proof of Lemma 4.4 *ii)* in [BD]; see also the proof of Lemma 1 below, where the same technique is used. We shall henceforth take (2.9) as the definition of $K_{\underline{e},\underline{f}}$.

We now introduce an operator

$$L_{\underline{e},\underline{f}} : V_{\Sigma_g} \to V_g^L(\underline{e}) \otimes V_g^R(\underline{f}) \ \subseteq V_g^L \otimes V_g^R$$

as a mirror image of $K_{\underline{e},\underline{f}}$ w.r.t. a plane parallel to the z-axis and not intersecting the handlebody M_g. More precisely,

$$L_{\underline{e},\underline{f}} = \sum_{\underline{x}} \prod_{i=1}^{3g-3} \frac{\omega_{x_i}^2}{\omega^2}\; Z(M_g'', \bar{G}_{\underline{e},\underline{f}} \cup m_{\underline{x}} \cup \bar{G}^g), \tag{2.11}$$

where M_g'' is the mirror image of M_g' and $\partial M_g'' = \Sigma_g^* \cup \Sigma_g''$. The graphs $\bar{G}_{\underline{e},\underline{f}} \in \Sigma_g''^*$ and $\bar{G}_g \in \Sigma_g^*$ are the mirror images of $G_{\underline{e},\underline{f}} \in \Sigma_g'$ and $G_g \in \Sigma_g$ respectively.

Gluing $(M_g', G_{\underline{e},\underline{f}} \cup m_{\underline{x}})$ and $(M_g'', \bar{G}_{\underline{e}',\underline{f}'} \cup m_{\underline{y}})$ along Σ_g we obtain $(N_g, G_{\underline{e},\underline{f}} \cup m_{\underline{x}}, \tilde{G}_{\underline{e}',\underline{f}'} \cup m_{\underline{y}})$ where N_g is diffeomorphic to $\Sigma_g \times [0,1]$ with boundary $\Sigma_g'' \cup {\Sigma_g'}^*$. The graph $\tilde{G}_{\underline{e}',\underline{f}'} \cup m_{\underline{y}} \in \Sigma_g''$ can be obtained from the standard graph $G_{\underline{e},\underline{f}} \cup m_{\underline{x}} \in \Sigma_g'$ depicted in (2.10) by changing the colourings $\underline{e} \to \underline{e}'$, $\underline{f} \to \underline{f}'$, $\underline{x} \to \underline{y}$ and replacing all overcrossings by undercrossings and vice versa.

Eq. (2.1) implies that

$$L_{\underline{e}',\underline{f}'} K_{\underline{e},\underline{f}} = \sum_{\underline{x},\underline{y}} \prod_i \frac{\omega_{x_i}^2}{\omega^2} \frac{\omega_{y_i}^2}{\omega^2}\; Z(N_g, G_{\underline{e},\underline{f}} \cup m_{\underline{x}} \cup \tilde{G}_{\underline{e}',\underline{f}'} \cup m_{\underline{y}})\,. \tag{2.12}$$

We are now in position to state the announced lemma.

Lemma 1 *The operator $L_{\underline{e}',\underline{f}'} K_{\underline{e},\underline{f}} : V_g^L(\underline{e}) \otimes V_g^R(\underline{f}) \to V_g^L(\underline{e}') \otimes V_g^R(\underline{f}')$ satisfies*

$$\omega^{2g-2} \omega_{\underline{e}} \omega_{\underline{f}} \omega_{\underline{e}'} \omega_{\underline{f}'} L_{\underline{e}',\underline{f}'} K_{\underline{e},\underline{f}} = \delta_{\underline{e},\underline{e}'} \delta_{\underline{f},\underline{f}'}\; \mathbb{1}_{\,V_g^L(\underline{e}) \otimes V_g^R(\underline{f})}, \tag{2.13}$$

where we have introduced the notation $\omega_{\underline{e}} = \prod_{i=1}^{3g-3} \omega_{e_i}$ and $\delta_{\underline{e},\underline{e}'} = \prod_{i=1}^{3g-3} \delta_{e_i,e_i'}$.

Proof: The idea of the argument is the following. By introducing tubes between Σ_g' and Σ_g'' we step by step lift the lines of $G_{\underline{e}\underline{f}}^g \in \Sigma_g'$ on Σ_g'' and cut the handles traversed by these lines. Applying the technique developed in [BD] and [KS] we will arrive on (2.13).

Due to Lemma 3.3 in [BD] introduction of a tube with an a-coloured meridian (which is *not* normalized by ω^{-2}) does not change the state sum. Pictorially this looks as follows:

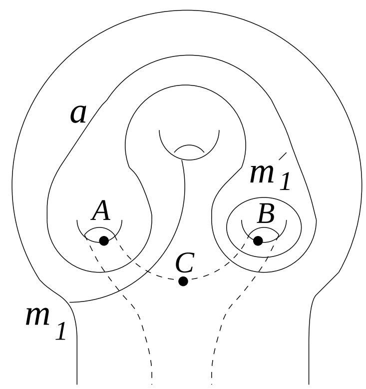


Fig.2 *A part of the manifold N_g where the boundary component Σ_g' of the tube is connected to Σ_g'' by a tube with an a-coloured meridian on it*

where we do not draw the $\underline{e}$-, $\underline{f}$- and $\underline{e}'$-, $\underline{f}'$-coloured lines. Applying Lemma 4.2 *ii)* in [BD] (or the Wigner-Eckart type relation (A.15) in [KS]) to the meridians m_1, m_1' and a we can change the graph so that the handle $(ABC)\times I$ will be traversed by a single line only. According to Remark 3.6 in [BD] the colour of this line can be set to zero and the handle cut. This yields a manifold N_g' as depicted on Fig.3.

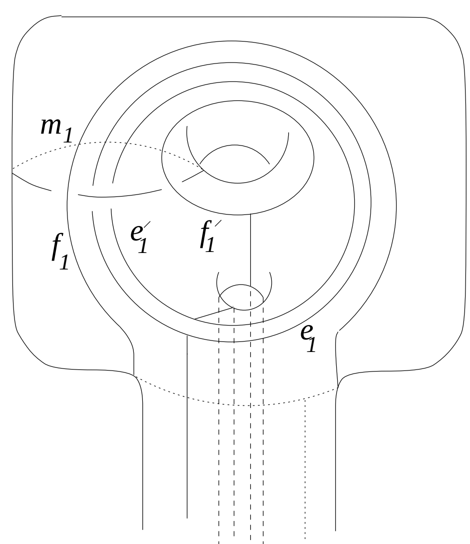


Fig.3 *A part of the manifold N_g' with associated graph on it*

Using lemma 4.2 *ii)* in [BD] once more (see also example 5.8 *iii)* in [KS]) one can cut the handle traversed by e_1'- , e_1- , f_1'- and f_1-coloured lines. After that the state sum of the resulting $(g-1)$-cylinder becomes multiplied by $\omega_{e_1}^{-2}\omega_{f_1}^{-2}\delta_{e_1' e_1}\delta_{f_1' f_1}$.

Continuing this procedure analogously we obtain the desired result:

$$L_{\underline{e}'\,\underline{f}'}K_{\underline{e}\,\underline{f}} = \omega^{-2g+2}\ \delta_{\underline{e}\,\underline{e}'}\delta_{\underline{f}\,\underline{f}'}\ (\omega_{\underline{e}}^2\ \omega_{\underline{f}}^2)^{-1}\ \mathbb{1}_{\ V_g^L(\underline{e})\otimes V_g^R(\underline{f})}\ .$$

□

Defining the operators $K : V_g^L \otimes V_g^R \to V_{\Sigma_g}$ and $L : V_{\Sigma_g} \to V_g^L \otimes V_g^R$ by

$$K = \omega^{g-1} \oplus_{\underline{e},\underline{f}}\ \omega_{\underline{e}}\omega_{\underline{f}}K_{\underline{e},\underline{f}}\ ,\ \ L = \omega^{g-1} \oplus_{\underline{e},\underline{f}}\ \omega_{\underline{e}}\omega_{\underline{f}}L_{\underline{e},\underline{f}}$$

it follows from (2.13) that $LK = \mathbb{1}_{\ V_g^L\otimes V_g^R}$ and hence by (2.8) K and L are isomorphisms and

$$L = K^{-1}\ . \tag{2.14}$$

Although we shall strictly speaking not use them in the following let us introduce the left- and righthanded counterparts $K_{\underline{e}}^L$ and $K_{\underline{f}}^R$ of $K_{\underline{e},\underline{f}}$ by replacing in eq. (2.9) the graph $G_{\underline{e},\underline{f}}$ by its left- and righthanded parts $G_{\underline{e}}^L$ and $G_{\underline{f}}^R$, respectively, and similarly $L_{\underline{e}}^L$ and $L_{\underline{f}}^R$ by replacing $\bar{G}_{\underline{e},\underline{f}}$ in eq. (2.11) by $\bar{G}_{\underline{e}}^L$ and $\bar{G}_{\underline{f}}^R$, respectively. The proof of Lemma 1 then yields

$$\omega^{2g-2}\ \omega_{\underline{e}}\ \omega_{\underline{e}'}\ L_{\underline{e}}^L K_{\underline{e}'}^L = \delta_{\underline{e},\underline{e}'}\mathbb{1}_{\ V_g^L(\underline{e})}$$

and

$$\omega^{2g-2}\ \omega_{\underline{f}}\ \omega_{\underline{f}'}\ L_{\underline{f}}^R K_{\underline{f}'}^R = \delta_{\underline{f},\underline{f}'}\mathbb{1}_{\ V_g^R(\underline{f})}$$

and consequently

$$L^L K^L = \mathbb{1}_{\ V_g^L}\ ,\ \ L^R K^R = \mathbb{1}_{\ V_g^R}\ ,$$

where $K^L : V_g^L \to V_{\Sigma_g}$ and $L^L : V_{\Sigma_g} \to V_g^L$ are defined by

$$K^L = \omega^{g-1} \oplus_{\underline{e}}\ \omega_{\underline{e}}K_{\underline{e}}^L\ ,\ \ L^L = \omega^{g-1} \oplus_{\underline{e}}\ \omega_{\underline{e}}L_{\underline{e}}^L, \tag{2.15}$$

and similarly for $K^R : V_g^R \to V_{\Sigma_g}$ and $L^R : V_{\Sigma_g} \to V_g^R$.

3 Factorization of state sums

For each genus $g \geq 0$ we fix once and for all manifolds M'_g and M''_g as defined in Section 2 with $\partial M'_g = \Sigma_g \cup \Sigma_g'^{\,*}$ and $\partial M''_g = \Sigma_g^* \cup \Sigma''_g$, where Σ_g, Σ'_g and Σ''_g are fixed oriented surfaces of genus g and where fixed graphs $G^g_{\underline{e},\underline{f}}$ and $\bar{G}^g_{\underline{e},\underline{f}}$ are embedded in Σ'_g and Σ''^*_g respectively, together with the associated sets of meridians. We have here made the dependence of the graphs and meridians on the genus explicit, and will do so likewise for the associated operators $K_{\underline{e},\underline{f}}, L_{\underline{e},\underline{f}}$ etc.

By a parametrized surface of genus g we mean a pair (Σ, ϕ), where Σ is a compact, connected, oriented surface of genus g and $\phi : \Sigma \to \Sigma_g$ is a diffeomorphism. We call ϕ a parametrization of Σ and set

$$\tilde{V}_\Sigma(\phi) = V_g^L \otimes V_g^R \,.$$

Let us consider a 3-dimensional cobordism M whose boundary $\partial M = \Sigma_1^* \cup \Sigma_2$ consists of two compact, connected, oriented surfaces of genus g_1 and g_2, respectively, which are parametrized by ϕ_1 and ϕ_2. An operator $\tilde{Z}(M) : \tilde{V}_{\Sigma_1}(\phi_1) \to \tilde{V}_{\Sigma_2}(\phi_2)$ can be defined as follows:

$$\tilde{Z}(M) = L(\phi_2) Z(M) K(\phi_1)\,,$$

where

$$K(\phi_1) = U(\phi_1) K^{g_1}, \qquad L(\phi_2) = L^{g_2} U(\phi_2)$$

and $U(\phi) : V_\Sigma \to V_{\Sigma_g}$ satisfying (1.2).

More generally, given a compact, oriented cobordism M with boundary components $\Sigma^{1\;*}_{g_1}, ..., \Sigma^{m\;*}_{g_m}, \Sigma^{m+1}_{g_{m+1}}, ..., \Sigma^n_{g_n}$ and parametrization ϕ_i of $\Sigma^i_{g_i}$ we set

$$\tilde{Z}(M) = L(\phi_{m+1}, ..., \phi_n) Z(M) K(\phi_1, ..., \phi_n) \tag{3.1}$$

where

$$K(\phi_1, ..., \phi_k) = \otimes_{i=1}^k K(\phi_i)$$

and $L(\phi_1, ..., \phi_k)$ is defined analogously.

Equivalently, (3.1) can be expressed as follows. Let $\bar{M}$ denote the manifold obtained by gluing M'_{g_i} onto M along ϕ_i for $1 < i < m$, and gluing

M''_{g_i} onto M along ϕ_i in case $i > m$. Then, clearly, $\bar{M}$ is diffeomorphic to M and has boundary components $(\Sigma'_{g_1})^*, ..., (\Sigma'_{g_m})^*$, $\Sigma''_{g_{m+1}}, ..., \Sigma''_{g_n}$ with embedded graphs $G^{g_1}_{\underline{e}^1,\underline{f}^1}, ..., G^{g_m}_{\underline{e}^m,\underline{f}^m}, \bar{G}^{g_{m+1}}_{\underline{e}^{m+1},\underline{f}^{m+1}}, ..., \bar{G}^{g_n}_{\underline{e}^n,\underline{f}^n}$, respectively. With the notation $\tilde{e} = (\underline{e}^1, ..., \underline{e}^n)$ and

$$\omega_{\tilde{e}} = \prod_{i=1}^{n} \omega_{\underline{e_i}}$$

we then have

$$\tilde{Z}(M) = \oplus_{\tilde{e},\tilde{f}} \tilde{Z}_{\tilde{e},\tilde{f}}(M), \tag{3.2}$$

where the coloured state sum $\tilde{Z}_{\tilde{e},\tilde{f}}(M)$ is defined by

$$\tilde{Z}_{\tilde{e},\tilde{f}}(M) = \omega^{g_1+...+g_n-n} \omega_{\tilde{e}}\ \omega_{\tilde{f}}\ \sum_{\tilde{x}} \prod_{i,j} \frac{\omega^2_{x^j_i}}{\omega^2} Z(\bar{M}, \mathcal{G}_{\tilde{e},\tilde{f}} \cup \mathcal{M}_{\tilde{x}}) \tag{3.3}$$

where

$$\mathcal{G}_{\tilde{e},\tilde{f}} = G^{g_1}_{\underline{e}^1,\underline{f}^1} \cup ... \cup \bar{G}^{g_n}_{\underline{e}^n,\underline{f}^n}$$

and

$$\mathcal{M}_{\tilde{x}} = m^1_{\underline{x}^1} \cup ... \cup m^n_{\underline{x}^n}\,.$$

Finally, we define an isomorphism $\tilde{U}(f) : \tilde{V}_\Sigma(\phi) \to \tilde{V}_{\Sigma'}(\phi')$ by

$$\tilde{U}(f) = L(\phi')U(f)K(\phi), \tag{3.4}$$

for any orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$ between parametrized surfaces (Σ, ϕ) and (Σ', ϕ') of genus g. This definition is extended in an obvious way to orientation preserving diffeomorphisms between arbitrary compact, oriented surfaces in terms of tensor products.

The objects $\tilde{V}, \tilde{U}, \tilde{Z}$ define a TQFT on compact, oriented 3-manifolds with parametrized boundary. This can be easily verified using the definition of these objects and eq. (2.14). The TQFT based on $\tilde{V}, \tilde{U}$ and $\tilde{Z}$ is equivalent to the theory defined in the previous section. The equivalence is given by the K and L-operators (see [T] or [DJ]).

We are now ready to state and prove the main result of this paper.

Theorem 2 *Let M be a compact, oriented 3-manifold. For any colouring $(\tilde{e}, \tilde{f})$ as defined above we have*

$$\tilde{Z}_{\tilde{e},\tilde{f}}(M) \;=\; \tau_{\tilde{e}}(M) \otimes \tau_{\tilde{f}}(M^*) \tag{3.5}$$

where the invariant $\tau_{\tilde{e}}$ is given by eq. (3.8) below and coincides with the invariant introduced in [T] up to normalization.

Proof:

As remarked earlier, we can replace each tube in $\bar{M}$ defined above with graph $G^{g_i}_{\underline{e}^i,\underline{f}^i} \cup m^i_{\underline{x}^i}$ by two tubes with graphs $(G^{g_i}_{\underline{e}^i})^L \cup (m^i_{\underline{x}^i})^L$ and $(G^{g_i}_{\underline{f}^i})^R \cup (m^i_{\underline{y}^i})^R$, respectively, at the cost of a factor $\omega^{2(g_i-1)}$. Let us assume we have done so for each $i = 1, ..., n$ and denote the resulting manifold also by $\bar{M}$. As is well known, the closed manifold obtained from $\bar{M}$ by filling all $2n$ tubes has a representation by surgery on S^3 along a set of links $l_1, ..., l_N$ which, of course, may be assumed not to intersect the filled tubes. Using Lemma 1 for the case $g = 1$ as in the proof of Theorem 5.2 in [BD] one obtains

$$\begin{aligned}
&Z(\bar{M}, G^{g_1}_{\underline{e}^1,\underline{f}^1} \cup m^1_{\underline{x}^1} \cup ... \cup \bar{G}^{g_n}_{\underline{e}^n,\underline{f}^n} \cup m^n_{\underline{x}^n}) = \\
= \quad &\omega^{2(g_1+...+g_n-n-N)} \sum_{\tilde{a},\tilde{z},\tilde{b},\tilde{z}'} \omega^2_{\tilde{a}}\, \omega^2_{\tilde{b}}\, \frac{\omega^2_{\tilde{z}}}{\omega^{2N}}\, \frac{\omega^2_{\tilde{z}'}}{\omega^{2N}} \\
&Z(\tilde{S}^3, \mathcal{L}^L_{\tilde{a}} \cup (\mathcal{M}'_{\tilde{z}})^L \cup \mathcal{L}^R_{\tilde{b}} \cup (\mathcal{M}'_{\tilde{z}'})^R \cup \mathcal{G}^L_{\tilde{e}} \cup \mathcal{M}^L_{\tilde{x}} \cup \mathcal{G}^R_{\tilde{f}} \cup \mathcal{M}^R_{\tilde{y}})
\end{aligned} \tag{3.6}$$

where we have introduced the shorthand notation

$$\mathcal{G}^L_{\tilde{e}} = (G^{g_1}_{\underline{e}^1})^L \cup ... \cup (\bar{G}^{g_n}_{\underline{e}^n})^L$$

and similarly for the righthanded part and the meridians. Furthermore, $\tilde{S}^3$ denotes the manifold obtained from $\bar{M}$ by removing two disjoint tubular neighborhoods T^L_i and T^R_i for each $i = 1, ..., N$. We define T^L_i and T^R_i by splitting a tubular neighborhood of l_i into two nearby ones as was done previously for the graphs $G^{g_1}, ..., G^{g_n}$. Finally, $\mathcal{L}^L = {L_1}^L \cup ... \cup {L_N}^L$ (together with associated meridians $\mathcal{M}^L = m^L_1 \cup ... \cup m^L_N$) is a collection of lefthanded graphs on the boundary components $\partial T^L_1, ..., \partial T^L_N$ of $\tilde{S}^3$, where the graphs are determined by the surgery prescription, and similarly for $\mathcal{L}^{\mathcal{R}}$ and $\mathcal{M}^{\mathcal{R}}$.

Next we recall from [BD] (see also [KS]) that two tubes with left- and righthanded lines, respectively, have trivial braiding, i.e. they may be deformed through each other. Using this and the fact that $\tilde{S}^3$ is a 3-sphere with a collection of $2(n+N)$ tubes removed, together with the factorization property of $Z(M,G)$ w.r.t. connected sums (see Lemma 3.2 in [BD]), we obtain by substituting (3.6) into (3.3) that

$$\tilde{Z}_{\tilde{e},\tilde{f}}(M) = \omega^{2(g_1+\dots g_n-n-N+1)} \sum_{\tilde{a},\tilde{b}} \mathcal{Z}(S^3, \mathcal{L}^L_{\tilde{a}} \cup \mathcal{G}^L_{\tilde{e}}) \otimes \mathcal{Z}(S^3, \mathcal{L}^R_{\tilde{b}} \cup \mathcal{G}^R_{\tilde{f}}), \quad (3.7)$$

where we have introduced

$$\begin{aligned} \mathcal{Z}(S^3, \mathcal{L}^L_{\tilde{a}} \cup \mathcal{G}^L_{\tilde{e}}) \;&=\; \omega^{g_m+\dots+g_n-(n-m)} \omega_{\tilde{e}} \omega^2_{\tilde{a}} \sum_{\tilde{z},\tilde{x}} \frac{\omega^2_{\tilde{z}}}{\omega^{2N}} \prod_{i,j} \frac{\omega^2_{x^j_i}}{\omega^2} \\ & Z((\tilde{S}^3)^L, \mathcal{L}^L_{\tilde{a}} \cup (\mathcal{M}'_{\tilde{z}})^L \cup \mathcal{G}^L_{\tilde{e}} \cup \mathcal{M}^L_{\tilde{x}}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{Z}(S^3, \mathcal{L}^R_{\tilde{b}} \cup \mathcal{G}^R_{\tilde{f}}) \;&=\; \omega^{g_1+\dots+g_m-m} \omega_{\tilde{f}} \omega^2_{\tilde{b}} \sum_{\tilde{z}',\tilde{y}} \frac{\omega^2_{\tilde{z}'}}{\omega^{2N}} \prod_{i,j} \frac{\omega^2_{y^j_i}}{\omega^2} \\ & Z((\tilde{S}^3)^R, \mathcal{L}^R_{\tilde{b}} \cup (\mathcal{M}'_{\tilde{z}'})^R \cup \mathcal{G}^R_{\tilde{f}} \cup \mathcal{M}^R_{\tilde{y}}) \end{aligned}$$

where $(\tilde{S}^3)^L$ is defined in analogy with $\tilde{S}^3$ except that only tubes with lefthanded graphs or links are removed from S^3 and $(\tilde{S}^3)^R$ is defined similarly.

Finally, setting

$$\Delta_L = \sum_{c \in \mathcal{I}} q_c^2 \omega_c^4,$$

we define

$$\tau_{\tilde{e}}(M) = \omega^{g_1+\dots+g_n-n-N+1} (\Delta_L \omega^{-1})^{\sigma(\mathcal{L})} \sum_{\tilde{a}} \mathcal{Z}((\tilde{S}^3)^L, \mathcal{L}^L_{\tilde{a}} \cup \mathcal{G}^L_{\tilde{e}}), \quad (3.8)$$

where $\sigma(\mathcal{L})$ is the signature of a certain 4-manifold whose boundary is $\bar{M}$ with tubes filled in. Similarly, the righthanded counterpart $\tau^R_{\tilde{f}}$ is defined with Δ_R given by the same formula as Δ_L except that q_c should be replaced by q_c^{-1}. Then

$$\Delta_L \Delta_R = \omega^2$$

(see [T]) and hence (3.7) can be rewritten as

$$Z_{\tilde{e},\tilde{f}}(M) = \tau_{\tilde{e}}(M) \otimes \tau_{\tilde{f}}^{R}(M).$$

By arguments identical to those in [BD] one shows that

$$\tau_{\tilde{f}}^{R}(M) = \tau_{\tilde{f}}(M^{*})$$

thus proving (3.5). Likewise the argument that $\tau_{\tilde{e}}(M)$ equals the ribbon graph invariant introduced in [T] follows as in [BD] by projecting the tubes in $(\tilde{S}^3)^L$ with graphs and links onto a plane. $\square$

4 Concluding remarks

The proof of Theorem 2 can be extended in a straightforward manner to the case where punctures are introduced on the boundary components of M. We shall, however, not elaborate on that case here (see also [T]).

It should be mentioned that the equivalence of the TQFT defined in section 2 and the one defined in terms of $\tilde{V}, \tilde{U}, \tilde{Z}$ follows from the equality of the corresponding state sums of closed manifolds, shown in [BD] and [T], once it is known that the two theories are non-degenerate (see e.g. [T]). The method of this paper gives the equivalence explicitly and at the same time prepares the ground for the proof of (3.5).

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