Refined invariants and TQFT's from Homfly skein theory

Anna Beliakova *

Abstract: We work in the reduced $SU(N,K)$ modular category as constructed recently by Blanchet. We define spin type and cohomological refinements of the Turaev-Viro invariants of closed oriented 3-manifolds and give a formula relating them to Blanchet's invariants. Roberts' definition of the Turaev-Viro state sum is exploited. Furthermore, we construct refined Turaev-Viro and Reshetikhin-Turaev TQFT's and study the relationship between them.

Introduction

In [T] Turaev reduced the construction of quantum 3-manifold invariants and TQFT's to the construction of modular categories. A modular category is a monoidal category with additional structure (braiding, twist, duality, finite set of simple objects satisfying a domination property and a non-degeneracy axiom). A first example of the modular category was obtained from the representation theory of the quantum group $U_q(sl(2))$. Later an elementary approach, based on the Kauffman skein relations and leading to the same family of invariants, was developed by Lickorish in [L].

Yokota [Y] generalized his approach and constructed the $SU(N,K)$ modular category using Homfly skein theory. The underlying invariant $\tau^{SU(N)}$ coincides with the invariant of Turaev-Wenzl [TW] extracted from $U_q(sl(N))$ at level K. Recently Blanchet [Bl] defined the reduced $SU(N,K)$ modular category. His invariant τ can be considered as a generalization of $\tau^{PSU(N)}$ to the case when N and K are not coprime. For any closed oriented 3-manifold M holds (see [Bl])

$$\tau^{SU(N)}(M) \;=\; \tau^{U(1)}(M)\,\tau(M)$$

where $\tau^{U(1)}(M)$ is defined in [MOO]. Blanchet constructed cohomological and spin

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type refinements of $\tau(M)$ depending on the so-called spin^d structure on M with $d = \gcd(N, K)$. He showed that $\tau(M)$ splits into a sum of refined invariants.

In this article we work in the reduced $SU(N, K)$ modular category as constructed by Blanchet. We give a definition of the refinement $Z(M, s, h)$ of the Turaev-Viro state sum $Z(M)$ depending on the spin^d structure s on a closed 3-manifold M and the first $\mathbb{Z}/d\,\mathbb{Z}$-cohomology class h. We show that

$$Z(M) = \sum_{s,h} Z(M, s, h) \tag{1}$$

and prove the relation with Blanchet's invariants

$$Z(M, s, h) = \tau(M, s)\ \tau(-M, s + h). \tag{2}$$

Analogous formulas also hold for cohomological refinements. The definition of $Z(M, s, h)$ is given in terms of Roberts' chain-mail link. It turns out that (1) and (2) can be proved by minor modifications of Roberts' arguments. Nevertheless, we give a different proof of (2) which generalizes directly to the TQFT operators.

In the last section we construct spin topological quantum field theories (TQFT's) for type A modular categories. In the $SU(2, K)$ case these theories were studied in [BM] and [B]. The vector space asssociated to a surface with structure is defined as (a subspace of) a formal linear span of special colorings of some trivalent graph. In contrast to the unspun (or non-refined) theory, this vector space for a non-connected surface is not equal to the tensor product of spaces associated with connected components. We define operators corresponding to spin 3-cobordisms and prove the gluing property for them. Finally, we construct a weak spin TQFT which can be regarded as a 'zero graded part' of the spin TQFT. We show that the unspun theory is the sum of weak spin TQFT's.

Using the same approach, we extend Roberts' invariant $Z(M, s, h)$ to a refined Turaev-Viro TQFT. Here in oder to prove the gluing axiom we use an analog of (2) for spin 3-cobordisms.

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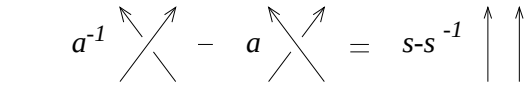
1 Definitions

Homfly skein theory. The manifolds throughout this paper are compact, smooth and oriented. By *links* we mean *isotopy classes of framed links*. A framing is a

trivialization of the normal bundle. This defines an orientation on the link. In all figures we use the blackboard framing convention.

Let M be a 3-manifold (possibly with a given set of framed oriented points on the boundary). We denote by $\mathcal{H}(M)$ the $\mathbb{C}$-vector space of formal sums of links in M (and framed arcs in M that meet ∂M in precisely the given set of points) modulo (isotopy keeping boundary points fixed and) the Homfly skein relation:



$$L \amalg \bigcirc = \frac{s^N - s^{-N}}{s - s^{-1}} L$$

with $a, s \in \mathbb{C}$. We call $\mathcal{H}(M)$ the skein of M.

For example, $\mathcal{H}(S^3) \cong \mathbb{C}$. The isomorphism sends any link L in S^3 to its Homfly polynomial $\langle L \rangle$.

Oriented embeddings induce natural maps between skeins. Let

$$\begin{array}{rccc} L_* : & \mathcal{H}(D^2 \times S^1)^{\otimes m} & \to & \mathcal{H}(S^3) \\ & x_1 \otimes ... \otimes x_m & \mapsto & \langle L(x_1, ..., x_m) \rangle. \end{array} \tag{3}$$

be the map induced by the embedding of m solid tori in S^3 with underlying m-component link L. We shall say that the components of L are cabled or colored with $x_1, ..., x_m$.

Specification of parameters. Let us fix a rank $N \in \mathbb{N}$ and a level $K \in \mathbb{N}$, such that $\gcd(N, K) = d$ is even, $N' = N/d$ and $K' = K/d$ are odd. Let s be a primitive $2(N + K)$ root of unity. We write $d = \alpha\beta$ with $\gcd(\alpha, 2K') = \gcd(\beta, N') = 1$ and choose the framing parameter a such that $(a^N s)^\alpha = 1$ and $(a^K s^{-1})^\beta = -1$.

Simple objects. Denote by $\lambda = (\lambda_1, ..., \lambda_p)$ the Young diagram with λ_i boxes in the i-th row. Set $|\lambda| = \sum_{i=1}^p \lambda_i$. In particular, let 1^N (resp. K) denote the diagram with one column (resp. one row) containing N (resp. K) cells.

The set of simple objects (colors) in the reduced $SU(N, K)$ modular category can be obtained from

$$\{(1^N)^{\otimes i} \otimes \lambda, \;\; 0 \leq i < \alpha, \; \lambda_1 \leq K, \; p < N\}$$

by identifying diagrams which differ by $K^{\otimes\beta}$. Recall that for any diagram λ with maximal $N-1$ rows and K columns $K\otimes\lambda = K+\lambda = (K,\lambda_1,...,\lambda_p)$. We denote by $\Gamma_{N,K}$ the resulting set of simple objects.

Under a λ-colored line we understand $|\lambda|$ copies of it with the idempotent of the Hecke algebra corresponding to λ inserted (see [AM] for more details).

There exists an involution $i:\lambda\to\lambda^*$ on the set of colors, such that changing the orientation on the λ-colored link component is equivalent to replacing λ by λ^*. Note that $|\lambda| = -|\lambda^*| \bmod d$.

Definition of ω. Let $y_\lambda\in\mathcal{H}(D^2\times S^1)$ be the skein element obtained by cabling with λ a 0-framed circle $\{pt\}\times S^1$, $pt\in D^2-\partial D^2$. The image of y_λ under the map $\mathcal{H}(D^2\times S^1)\to\mathcal{H}(S^3)$ given by the standard embedding of the solid torus in S^3 is denoted by $\langle\lambda\rangle$.

For a cell c in λ with coordinates (i,j) we define its hook length $hl(c)$ and its content $cn(c)$ by formulas

$$hl(c) = \lambda_i + \check{\lambda}_j - i - j + 1, \quad cn(c) = j - i,$$

where $\check{\lambda}_j$ is the length of the j-th column of λ. Then (see [A])

$$\langle\lambda\rangle = \langle\lambda^*\rangle = \prod_{cells}\frac{[N+cn(c)]}{[hl(c)]} \quad \text{where} \quad [n] = \frac{s^n - s^{-n}}{s-s^{-1}}. \tag{4}$$

With this notation the element

$$\tilde{\omega} = \sum_{\lambda\in\Gamma_{N,K}}\langle\lambda\rangle\, y_\lambda \quad\in\quad \mathcal{H}(D^2\times S^1)$$

has the nice property that the Homfly polynomial of a link with an $\tilde{\omega}$-colored component is invariant under handleslides along this component. In addition, it is also independent of the orientation on this component.

We choose the normalization $\omega = \eta\tilde{\omega}$ with

$$\eta^{-2} = \langle\tilde{\omega}\rangle = (-1)^{\frac{N(N-1)}{2}}\frac{d(N+K)^{N-1}}{\prod_{j=1}^{N-1}(s^j - s^{-j})^{2(N-j)}}.$$

Then we have $\langle U_1(\omega)\rangle\langle U_{-1}(\omega)\rangle = 1$ where $\langle U_\epsilon(\omega)\rangle$ denotes the Homfly polynomial on the ϵ-framed unknot colored with ω.

Grading. The algebra $\mathcal{H}(D^2\times S^1)$ has a natural $\mathbb{Z}_d = \mathbb{Z}/d\,\mathbb{Z}$ grading (recall $d = \gcd(N,K)$) by taking the number of strands modulo d. According to this grading we decompose

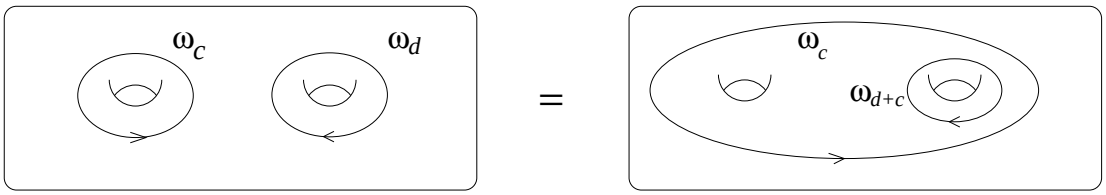
$$\omega = \omega_0 + \omega_1 + ... + \omega_{d-1}.$$

Modifying slightly the calculations in Lemma 2.4 [Bl] we get

$$\langle U_0(\omega)\rangle = d\langle U_0(\omega_i)\rangle = \eta^{-1}. \tag{5}$$

By Lemma 4.3 in [Bl] we have $\langle U_1(\omega)\rangle = \langle U_1(\omega_{d/2})\rangle$.

The graded handleslide property can be written as follows (see [Bl], Lemma 4.1)



By Proposition 1.5 in [Bl] we have

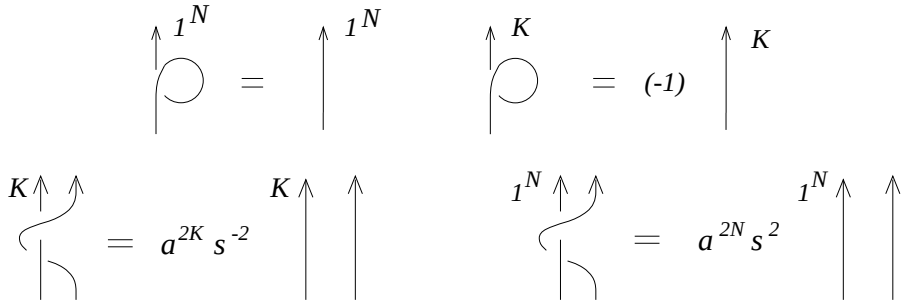
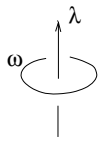


Figure 1: *Framing and twisting coefficients on* 1^N *and* K.

Killing property. If $\lambda \neq 0$, the following skein element



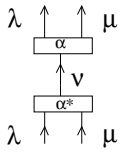
(6)

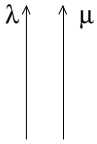
is zero. This is an analog of Lemma 2.5 in [Bl] in the reduced category.

Graded killing property. The skein element (6) with ω replaced by ω_i is zero if $\lambda \neq (1^N)^{\otimes k} \otimes K^{\otimes l}$ where $0 \leq k < \alpha$ and $0 \leq l < \beta$ (see Lemma 4.4 in [Bl]).

Fusion rules. We denote by $\mathcal{H}(D^3, a_1...a_n, b_1...b_m)$ the skein of a 3-ball with n outgoing and m incoming points on the boundary colored with $a_1, ..., a_n$ and $b_1, ..., b_m$ respectively. Then a natural pairing $\mathcal{H}(D^3, \lambda\mu, \nu) \times \mathcal{H}(D^3, \nu, \lambda\mu) \to \mathcal{H}(S^3)$ can be defined by gluing 3-balls together (identifying points of the same color).

With this notation, the domination property can be written as follows:

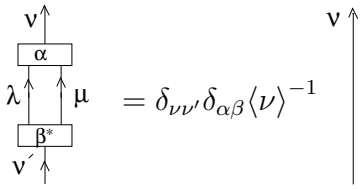
$$\sum_{\nu\in\Gamma_{N,K}}\sum_{\alpha}\langle\nu\rangle$$ 

$$=$$ 

, (7)

where α and α^* run over dual bases with respect to the pairing described above. In what follows we shall represent the elements of $\mathcal{H}(D^3, \lambda\mu, \nu)$ by colored 3-vertices for brevity. Let $N^\nu_{\lambda\mu}$ be the dimension of $\mathcal{H}(D^3, \lambda\mu, \nu)$. We say that a coloring (λ, μ, ν) of a 3-vertex is admissible if $N^\nu_{\lambda\mu} \neq 0$. We shall call $N^\nu_{\lambda\mu}$ the multiplicity of the colored 3-vertex.

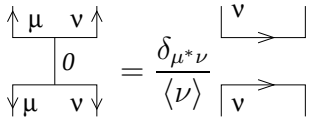
We choose a normalization of 3-vertices, so that the following equation hold (see [BD]):



$= \delta_{\nu\nu'}\delta_{\alpha\beta}\langle\nu\rangle^{-1}$

. (8)

After closing the ends in (8) and applying (7) we get $\langle\lambda\rangle\langle\mu\rangle = \sum_\nu N^\nu_{\lambda\mu}\langle\nu\rangle$. As a consequence, we have the following rule for deleting and/or introducing of a 0-colored line:



(9)

1.1 Spin^d Structures

All homology and cohomology groups throughout this article will have $\mathbb{Z}_d$ coefficients where d is an even integer.

An oriented manifold has spin structure if the rotational group SO as the structural group of its stable tangent bundle can be replaced by its 2-fold covering group $Spin$ (see [LM, p.80]). The notion of a spin^d structure is a natural generalization of this construction which corresponds to the lifting of the structural group SO to its d-fold covering group $Spin^d = (Spin \times \mathbb{Z}_d)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ acts by $(-1, d/2)$ on $Spin \times \mathbb{Z}_d$). Such structures always exist on oriented n-manifolds with $n \leq 3$ due to the vanishing of the second Stiefel-Whitney class.

Definition 1 *Let N be an n-manifold (possibly with boundary), where $n = 2, 3$. Let FN be the space of oriented orthonormal 3-frames on N (= the principle stable*

tangent bundle). A spin^d structure on N is a cohomology class $s \in H^1(FN)$ whose restriction to each fibre is non-trivial.

We denote by $\mathrm{Spin}^d(N)$ the set of spin^d structures on N. Using Künneth formula one can show that the following sequence

$$0 \to H^1(N) \to H^1(FN) \to H^1(SO(3)) \to 0$$

is exact. Therefore, $\mathrm{Spin}^d(N)$ is affinely isomorphic to $H^1(N)$ and consists of $s \in H^1(FN)$, which are equal to $d/2$ on homologically trivial 0-framed curves in N.

If a closed 3-manifold $M = S^3(L)$ is obtained by surgery on S^3 along an m-component link L, $\mathrm{Spin}^d(M)$ is in bijection with the solutions $c = (c_1, ..., c_m) \in (\mathbb{Z}_d)^m$ of the following system of equations

$$\sum_{j=1}^{m} L_{ij}\, c_j = d/2\; L_{ii} \mod d, \quad 1 \le i \le m \tag{10}$$

where $\{L_{ij}\}_{1 \le i,j \le m}$ denotes the linking matrix of L with framing on the diagonal.

Let M be a 3-manifold with parametrized boundary, i.e. its boundary components are supplied with diffeomorphisms to the standard surface. Then we glue (along the parametrization) to each $\Sigma \in \partial M$ the standard handlebody. The result is a closed 3-manifold $\tilde{M}$. Deformation retracts of the handlebodies glued to M can be viewed as a 3-valent graph G in $\tilde{M}$ (see Figure 3). Let $A = \{a_1, ..., a_p\}$ be the set of circles of G, where $p = 1 - \chi(\partial M)/2$, $\chi(\Sigma)$ being the Euler characteristic of Σ. Let $\tilde{M} = S^3(L)$ be obtained by surgery on an m-component link L. Denote by $\{\tilde{L}_{ij}\}$ the linking matrix of $L \cup A$. Then $\mathrm{Spin}^d(M)$ is in bijection with the solutions $\tilde{c} = (c_1, ..., c_m, z_1, ..., z_p)$ of the following equations

$$\sum_{j} \tilde{L}_{ij} \tilde{c}_j = d/2\; L_{ii} \mod d, \quad 1 \le i \le m. \tag{11}$$

The proof in the case of spin structures can be found in [B]. The generalization is straightforward.

2 Spin state sum invariants

Definition. Let M be a closed, connected 3-manifold. Choose a handle decomposition of M with d_0, a, b, d_3 handles of indices 0,1,2 and 3 respectively. Let H be a handlebody given by the union of 0- and 1-handles. Denote by $m = \{m_1, ..., m_a\}$

and $\varepsilon = \{\varepsilon_1, ..., \varepsilon_b\}$ the meridian curves of 1-handles and the attaching curves of 2-handles on ∂H respectively. We choose an orientation on all these curves and extract the normal vector from the orientation of ∂H. Let $j(m)$ and $j(\varepsilon)$ be the images of m and ε under an orientation preserving embedding $j : H \hookrightarrow S^3$. Then $R = j(m)_+ \cup j(\varepsilon)_-$ is the Roberts chain-mail link. Here $+$ (resp. $-$) means the push-off in the direction of the outgoing (resp. incoming) normal to ∂H.

Let $s \in \mathrm{Spin}^d(M)$. Let s_0 be the unique spin^d structure on S^3. Then $x = s|_H - s_0|_{j(H)}$ assigns $\mathbb{Z}_d$- numbers $\{x_1, ..., x_a\}$ to 1-handles. Here we assume that the cores of 1-handles are 0-framed and oriented in such a way that they have the linking number one with the corresponding meridians. Choose a 2-cycle $y = \sum_i y_i \varepsilon_i$ representing a second homology class of M. Let $h = D(y) \in H^1(M)$ be its Poincare dual class. We define

$$Z(M, s, h) = (d\eta)^{d_3+d_0-2} \langle R(\omega_{x_1}, ..., \omega_{x_a}, \omega_{y_1}, ..., \omega_{y_b}) \rangle.$$

Let $-M$ be M with the reversed orientation. By definition we have that $Z(M, s, h) = Z(-M, s, h)$.

Theorem 2 *$Z(M, s, h)$ is an invariant of (M, s, h).*

Proof: We need to show that $Z(M, s, h)$ does not depend on the orientation of R, embedding j, the handle decomposition and the representatives for x and y.

Let $\tilde{R}$ be R with the orientation on the first component reversed. After changing the orientation, the numbers $\{-x_1, x_2, ..., x_a\}$ will be assigned to 1-handles. Applying the involution to the set of colors we have

$$\langle R(\omega_{x_1}, ...) \rangle = \langle \tilde{R}(\omega_{-x_1}, ...) \rangle.$$

Other cases can be treated analogously.

Two embeddings of H in S^3 may be related by unknotting of 1-handles and reframing (twisting of 1-handles across their meridian discs).

An unknotting move can be realized by sliding all ε-curves in a 1-handle over a meridian of the other. This does not change the grading on the meridian, because the boundary of y is zero and therefore the number of ε-strands in each 1-handle is 0 modulo d.

Independence of the reframing move can be shown as follows: Add to R an $\omega_{d/2}$-colored ± 1-framed unknot (unlinked with R), slide all ε-curves in the ith 1-handle over it, twisting them. By the same argument as before the grading of the unknot remains unchanged. Finally, slide the unknot over the meridian of this 1-handle and

remove it. This changes the grading of the meridian by $d/2$, but the coefficient x_i is also changed by $d/2$ after reframing.

Two handle decompositions of M can be related by births or deaths of 0-1-, 1-2- and 2-3-handle pairs and handleslides of 1-1- or 2-2-pairs. The handleslides do not affect the invariant. Births of 0-1- or 2-3-handle pairs add to R a 0-framed unknot which can be slid over other 'parallel' components and deleted just like in [R]. The 1-2-handle pair adds to R a (0,0)-framed Hopf link colored by (ω_0, ω_0) or a $(\pm 1, 0)$-framed $(\omega_0, \omega_{d/2})$-colored one. In both cases the corresponding skein elements are equal to one by the lemma below.

Representatives of (x or) y differ by changing all labels in the (co-)boundary of some (0- or) 3-handles. This can be realized by adding a 0-framed ω_i-colored unknot, sliding it over all (m-curves or) ε-curves in the (co-)boundary of these handles and removing it. □

Remark. By desregarding grading in the proof we can see that

$$Z(M) = \eta^{d_0+d_3-2}\langle R(\omega, ..., \omega)\rangle$$

is an invariant of M.

Lemma 3 *Let $H_{\epsilon,0}$ be the $(\epsilon, 0)$-framed Hopf link with $\epsilon = 0, \pm 1$. Then for $i, j \in \mathbb{Z}_d$ we have*

$$\langle H_{\epsilon,0}(\omega_i, \omega_j)\rangle = \begin{cases} 1, & \text{if} \quad \epsilon = 0, i = 0,\ j = 0; \\ & \text{or} \quad \epsilon = \pm 1, i = 0,\ j = d/2; \\ 0, & \text{otherwise}\,. \end{cases} \tag{12}$$

Proof: Graded killing property implies that the ϵ-framed component of the Hopf link should be 0-graded. Using the identities on Figure 1 we can write

$$\langle H_{\epsilon,0}(\omega_0, \omega_j)\rangle = \frac{1}{d}\sum_{k=1}^{\alpha-1}(a^N s)^{2kj}\sum_{l=1}^{\beta-1}(-1)^{\epsilon l}(a^K s^{-1})^{2lj}$$

which is non-zero only in the two cases mentioned above. □

Theorem 4 *For a closed connected 3-manifold M, the Turaev-Viro invariant $Z(M)$ decomposes as a sum of the refined invariants:*

$$Z(M) = \sum_{s,h} Z(M, s, h).$$

Proof: The identification of $Z(M)$ with the Turaev-Viro invariant in the reduced $SU(N,K)$ modular category can be made analogously to Theorem 3.6 in [R1] (see also [BP]). The main difference is that $6j$-symbols are not numbers, but the elements of the tensor product of four vector spaces. In the definition of the Turaev-Viro state sum a contraction over these spaces is added (see [T] or [BD] for more details).

We will show the decomposition formula in the special case when the handle decomposition of M is a Heegaard splitting and H is embedded standardly in S^3. For any grading of ε-curves which does not correspond to homology classes, R contains a meridian curve linked with ε-strands whose total grading is not 0 modulo d. This is zero by the killing property. If the grading of m-curves does not correspond to spin^d structures, there exists a homologically trivial 1-cycle in M, such that the sum over gradings of 1-handles representing it is not 0 mod d. After handleslides (if necessary) we represent this cycle by an ε-curve. Now the invariant vanish by Lemma 4.2 in [Bl]. □

3 Relation with Blanchet's invariants

In [Bl] the refined Reshetikhin-Turaev invariants for the reduced $SU(N,K)$ modular category were defined in the following way: Let $M = S^3(L)$ be given by surgery on L. Let c be the solution of the modulo d characteristic equations (10) corresponding to $s \in \text{Spin}^d(M)$. Then

$$\tau(M,s) = \Delta^{-\sigma(L)} \langle L(\omega_{c_1}, ..., \omega_{c_m}) \rangle \tag{13}$$

is Blanchet's invariant of (M,s). Here $\Delta = \langle U_1(\omega_{d/2}) \rangle$ and $\sigma(L)$ is the signature of the linking matrix. This invariant is multiplicative with respect to connected sums and normalized at 1 for S^3. We denote by $\bar{L}$ the mirror of L. Then

$$\tau(-M,s) = \Delta^{\sigma(L)} \langle \bar{L}(\omega_{c_1}, ..., \omega_{c_m}) \rangle.$$

Theorem 5 *For a closed connected 3-manifold M,*

$$Z(M,s,h) = \tau(M, s+h)\ \tau(-M,s) = \tau(M,s)\ \tau(-M, s+h). \tag{14}$$

Proof: Once again, we take Heegaard splitting as handle decomposition and we choose the standard embedding of H in S^3. Denote by Σ the boundary of H with the standard homology basis $\{m_i, l_i\}_{1 \le i \le g}$. Write $M = H \underset{\phi}{\cup} -H$ where $\phi : \Sigma \to -\Sigma$

is a gluing diffeomorphism. Note that $\varepsilon_i = \phi^{-1}(m_i)$. Then $R = m_+ \cup \varepsilon_-$ with grading $\{x_1, ..., x_g, y_1, ..., y_g\}$. For any link L in a 3-manifold N (possibly with boundary) we denote by $N(L)$ the result of surgery on N along L.

Our first aim is to see that $S^3(R) = M\# - M$. We proceed as follows. Let us cut S^3 with R inside along Σ. We get

$$S^3(R) = (S^3 - H)(m_+) \cup H(\varepsilon_-).$$

Once again, cut out from H a cylinder containing ε_-. Then

$$S^3(R) = (S^3 - H)(m_+) \cup (\Sigma \times I)(\varepsilon_-) \cup H. \tag{15}$$

Observe that surgery along m_+ on the handlebody $S^3 - H$ interchanges the contractible and non-contractible cycles in the homology basis of its boundary, i.e

$$\begin{aligned} S^3(R) &= -H \underset{id}{\cup} (\Sigma \times I)(\varepsilon_-) \underset{id}{\cup} H = \\ &= -H \underset{\phi}{\cup} (\Sigma \times I)(m_-) \underset{\phi^{-1}}{\cup} H. \end{aligned} \tag{16}$$

Here we have used that $\phi(\varepsilon_i) = m_i$. Taking into account that $(\Sigma \times I)(m_-)$ can be mapped to $H\# - H$ by a diffeomorphism which is the identity on the boundary, we get

$$S^3(R) = (-H \underset{\phi}{\cup} H)\#(-H \underset{\phi^{-1}}{\cup} H) = -M\#M.$$

It remains to find out to which spin^d structure on $-M\#M$ corresponds the grading of R. According to the definition, the spin^d structure on $S^3(R)$ does not extend over meridians of not 0-graded components of R. In (16) the structure does not extend over 1-handles of H and $-H$ with $x_i \neq 0$. This spin^d structure is equal to $s_0 + \sum x_i l_i$ and coincides with s. We have the additional obstruction on $\Sigma \times I$ given by meridians of curves $m_i = \phi(\varepsilon_i)$ with $y_i \neq 0$ pushed slightly into interior. After surgery, they become homologous to l_i on $-H$ and add the Poincare dual class of y to the spin^d structure on M. For the second equality in (14) we use the independence of $Z(M, s, h)$ of the orientation of M. □

Cohomological refinements

We need to change the specification of parameters in the Homfly polynomial. The spin case, considered above, is here excluded.

For a given rank N and level K choose s be a primitive root of unity of order $2(N + K)$ if $N + K$ is even and of order $N + K$ if $N + K$ is odd. As before,

$d = \gcd(N, K)$. If $N + K$ is even, we suppose that $N' = N/d$ is odd. Then $d = \alpha\beta$ with $\gcd(\alpha, 2K') = \gcd(\beta, N') = 1$ and we can find a satisfying

$$(a^K s^{-1})^\beta = (-1)^{N+K+1}, \qquad (a^N s)^\alpha = 1.$$

The main difference to the previous case is that the twist on the K-colored line is trivial and therefore $\langle U_1(\omega_0)\rangle = \langle U_1(\omega)\rangle = \Delta$.

Let $M = S^3(L)$ and $h \in H^1(M)$. Denote by $c = (c_1, ..., c_m)$ the element in the kernel of the linking matrix (modulo d) corresponding to h. Then

$$\tau(M, h) = \Delta^{-\sigma(L)} \langle L(\omega_{c_1}, ..., \omega_{c_m})\rangle$$

is Blanchet's invariant. Analogously to the spin case, we can define $Z(M, x, h)$ for any $x \in H^1(M)$ and show its invariance. The principal modifications are that the reframing is performed with an ω_0-colored unknot and that a birth of a 1-2-handle pair introduces an (ω_0, ω_0)-colored Hopf link with at least one 0-framed component.

Analogously we get

$$Z(M) = \sum_{x,h} Z(M, x, h) \quad \text{and} \quad Z(M, x, h) = \tau(M, x)\ \tau(-M, x + h).$$

4 Spin topological quantum field theories

A TQFT is a functor from the category of 3-cobordisms to the category of finite-dimensional vector spaces. It associates to any closed surface Σ a vector space V_Σ and to any 3-cobordism M with $\partial M = -\partial_- M \cup \partial_+ M$ an operator $Z(M) : V_{\partial_- M} \to V_{\partial_+ M}$. Crucial is the functorial behavior with respect to the composition of cobordisms (gluing property). Two well-known examples of such a construction are the Reshetikhin-Turaev (RT) and Turaev-Viro (TV) TQFT's (see [T]).

A spin TQFT is a TQFT based on the category $\mathcal{S}$ of spin 3-cobordisms. To define $\mathcal{S}$ we need a homotopy-theoretical definition of the notion of a spin^d structure.

Definition 6 *Let π be the fibration $BSpin^d \to BSO$. Let N be an n-dimensional manifold, possibly with boundary. A w_2-structure on N is a map $f : N \to BSpin^d$, such that $\pi \circ f$ classifies the stable tangent bundle of N. A spin^d structure on N is a homotopy class of w_2-structures.*
Let us fix a w_2-structure on a subset $A \subset N$. A relative spin^d structure on N is a homotopy class (relative to A) of w_2-structures on N extending the one given on A.

The category of spin cobordisms. Let Σ be a closed surface. Let us mark a point in each connected component of Σ and denote by P the resulting set of points. We choose a w_2-structure on P. Let σ be the relative spin^d structure on Σ extending the one given on P. The set of such structures is affinely isomorphic to $H^1(\Sigma, P) \cong H^1(\Sigma)$ by the obstruction theory (see [Sp, p.434]).

The triple (Σ, P, σ) is an object of $\mathcal{S}$. A morphism from (Σ, P, σ) to (Σ', P', σ') is a 3-cobordism M with $\partial M = -\Sigma \amalg \Sigma'$ supplied with a relative spin^d structure extending the one on $P \cup P'$, such that its restriction to the boundary is equal to $\sigma \amalg \sigma'$. The set of such structures on M is affinely isomorphic to $H^1(M, \partial M)$ (use the exact sequence $0 \to H^1(M, \partial M) \to H^1(M, P \cup P') \to H^1(\partial M, P \cup P') \to ...$). Here we identify $H^1(M, \partial M)$ with its image in $H^1(M, P \cup P')$.

Let us assume that the boundary of M is parametrized. Then we can extend the parametrization diffeomorphism to the map $M \to \partial M$, which composed with $\sigma \amalg \sigma'$ defines the relative spin^d structure $\tilde{\sigma}$ on M. Any other relative spin^d structure on M (with the given restriction to the boundary) is of the form $\tilde{\sigma} + \dot{s}$ for some $\dot{s} \in H^1(M, \partial M)$.

Spin RT TQFT. Let $(\Sigma, P, \sigma) \in \mathrm{Ob}(\mathcal{S})$ consist of n connected components. Let $\phi : \Sigma \to \Sigma^{st}$ be the parametrization diffeomorphism respecting the order of components and $\Sigma^{st} = \Sigma_{g_1} \cup ... \cup \Sigma_{g_n}$. Let us construct a framed graph $\hat{G}^{\Sigma}$ by taking the graph $\hat{G}^{g_1} \cup ... \cup \hat{G}^{g_n}$ (see Figure 2) and by connecting its 1-vertices with a fixed trivalent graph F_n.

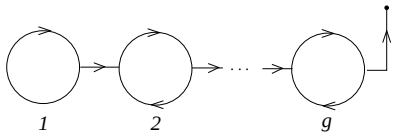


Figure 2 *The graph* $\hat{G}^g$

As before, we denote by $\{m_i\}$ the 0-framed meridians of the standardly embedded surface Σ^{st}. Let z_i be the result of the evaluation of the cohomology class corresponding to σ on the homology class of $\phi(m_i)$.

Under a *special coloring* e of $\hat{G}^{\Sigma}$ we understand an admissible coloring of $\hat{G}^{\Sigma}$, such that the grading of colors on the ith circle is equal to z_i[†], and the color of the ith line of F_n is $(1^N)^{\otimes k_i} \otimes K^{l_i}$ with $0 \le k_i < \alpha$ and $0 \le l_i < \beta$. We denote by $\hat{G}^{\Sigma}_e$ the e-colored graph. We set $\langle e \rangle = \prod_{e_i \in e} \langle e_i \rangle$ if $\mathrm{card}(e) > 1$ and $\langle e \rangle = 1$ otherwise.

[†]Note that the grading is well-defined on the circles of $\hat{G}$, because all lines connecting two circles are 0-graded.

Let us choose the numbering of the lines of F_n, so that the line containing kth 1-vertex becomes the number k. Then for an ordered set $u = \{0, u_2, ..., u_n, 0..., 0\}$ of $2n-3$ elements we define $u_e = (a^N s)^{2\sum k_i u_i} (a^K s^{-1})^{2\sum l_i u_i}$.

Let $(M, \dot{s})$ be a spin 3-cobordism from $(\partial_- M, P_-, \sigma_-)$ to $(\partial_+ M, P_+, \sigma_+)$. The boundary of M is parametrized and $\dot{s} \in H^1(M, \partial M)$. We assume that $\partial_- M$ (resp. $\partial_+ M$) has n_- (resp. n_+) connected components. We connect the marked points of $\partial_- M$ (resp. $\partial_+ M$) by the trivalent graph F_{n_-} (resp. a mirror image of F_{n_+}) in M. Let us glue (along the parametrization) to each connected component of $\partial_- M$ of genus g a tubular neighborhood of the graph $\hat{G}^g$, containing the graph itself inside. The 1-vertex of $\hat{G}^g$ is glued to the marked point. Analogously, to each connected component of $\partial_+ M$ of genus g we glue a tubular neighborhood of a mirror image of $\hat{G}^g$ (with respect to the plane orthogonal to that of the picture) containing the graph itself inside. The result is a closed 3-manifold $\tilde{M}$ with two closed 3-valent graphs $\hat{G}^+$ and $\hat{G}^-$ inside.

We denote by s the spin^d structure on M given by the homotopy class of $\tilde{\sigma} + \dot{s}$. Let u_i, $2 \leq i \leq n_+$, (resp. u'_i, $2 \leq i \leq n_-$) be the number associated by $\dot{s}$ to the cycle in $M/\partial M$ obtained by identifying the first and ith marked points of F_{n_+} (resp. F_{n_-}).

Let $\tilde{M} = S^3(L)$. Let $\tilde{c} = (c_1, ..., z_p)$ be the solution of (11) corresponding to s. Choose a special coloring e (resp. e') of $\hat{G}^+$ (resp. $\hat{G}^-$). Their grading on the circles is determined by $\{z_i\}$. We define

$$\tau_{ee'}(M, \dot{s}) = \Delta^{-\sigma(L)} \eta^{-\chi(\partial_+ M)/2} \sqrt{\langle e \rangle \langle e' \rangle}\; u_e u'_{e'} \langle L(\omega_{c_1}, ..., \omega_{c_m}) \cup \hat{G}^+_e \cup \hat{G}^-_{e'} \rangle$$

and interpret it as an (e, e')-coordinate of the operator $\tau(M, \dot{s})$ from the vector space spanned by the special colorings of $\hat{G}^-$ to the vector space spanned by the special colorings of $\hat{G}^+$.

The operator $\tau(M, \dot{s})$ is an invariant of the spin 3-cobordism $(M, \dot{s})$ with parametrized boundary. This is because, it is an isotopy invariant of the graphs $\hat{G}^+$ and $\hat{G}^-$ and it does not change under refined Kirby moves in $\tilde{M}$.

We set $\hat{G} = \hat{G}^+ \cup \hat{G}^-$. We call $L \cup \hat{G}$ the graph representing M, because M can be reconstructed from it (see [T, p.172]).

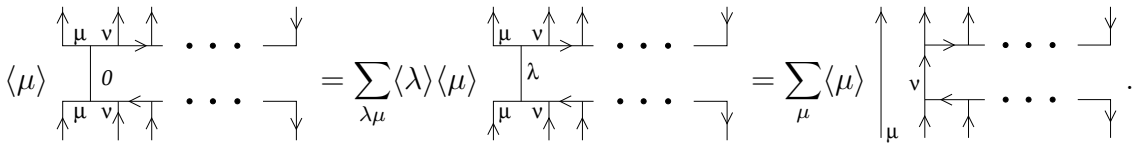
Theorem 7 *(Gluing property with anomaly) If the spin^d 3-cobordism $(M, \dot{s})$ is obtained from $(M_1, \dot{s}_1)$ and $(M_2, \dot{s}_2)$ by gluing along a diffeomorphism $f : \partial_+ M_1 \to \partial_- M_2$ which preserves the relative spin^d structures and commutes with parametrizations, then*

$$\tau_{ee''}(M, \dot{s}) = k \sum_{e'} \tau_{ee'}(M_2, \dot{s}_2)\; \tau_{e'e''}(M_1, \dot{s}_1)\,, \tag{17}$$

where $k = \Delta^{\sigma(L)-\sigma(L_1)-\sigma(L_2)}$ *is an anomaly factor and* L, L_1 *and* L_2 *are the surgery links of* $\tilde{M}$, $\tilde{M}_1$ *and* $\tilde{M}_2$ *respectively.*

Remark. To avoid the anomaly, we should supply cobordisms with so-called p_1-structures or 2-framings (see [BM] for more details).

Proof: Let us suppose that $\partial_+ M_1$ has n connected components. We put the graph representing M_2 on top of the graph representing M_1 and introduce a 0-colored line connecting $F_{n_\pm}$-lines of these graphs. Then we get



In the second equality we have used the fact that for $\lambda \neq 0$ the Homfly polynomial of the colored graph is zero. The sum over all kinds of (λ, μ, ν)-vertices is assumed. In (17) the sum over all ν of the form $(1^N)^{\otimes k} \otimes K^l$ is taken with $(a^N s)^{2k(u_1+u_1')}(a^K s^{-1})^{2l(u_1+u_1')}$ as coefficients, where u_i (resp. u_i') is assigned to the i-th F_{n_+}-line by $\dot{s}_1$ (resp. to the ith F_{n_-}-line by $\dot{s}_2$) and $u_1 = u_1' = 0$ by construction. This is equivalent to introducing a small ω_0-colored circle around the ν-colored line and allowing ν to run over $\Gamma_{N,K}$. Continuing this procedure we will replace the figure drawn above by n vertical strands, where the ith strand $(2 \leq i \leq n)$ is linked with a small $\omega_{u_i+u_i'}$-colored circle. After that, the sum over all colors of the remaining lines should be taken. Applying fusion rules again, we get a graph representing (M, s) (compare [T,p.177]). □

In $\mathcal{S}$ the identity morphism on (Σ, P, σ) is given by the cylinder $(\Sigma \times I, \dot{\sigma})$, where $\dot{\sigma}$ is the trivial extension of σ. We define $V(\Sigma, \sigma)$ to be the image of the projector $\tau(\Sigma \times I, \dot{\sigma})$ associated to the cylinder.

The operator $\tau(M, \dot{s}) : V(\partial_- M, \sigma_-) \to V(\partial_+ M, \sigma_+)$ defines the spin RT TQFT.

Remark. In the spin TQFT the vector space associated with a non-connected surface with structure is not equal to the tensor product of vector spaces associated with connected components. Therefore, the operators strongly depend on the cobordism structure of a given 3-manifold. For example, the operators $\tau(\Sigma \times I, \dot{s}) : V(\Sigma, \sigma) \to V(\Sigma, \sigma)$ are equal for all extensions $\dot{s}$ of σ. But the operators $\tau(\Sigma \times I, \dot{s}) : V(\emptyset) \to V((-\Sigma, \sigma) \amalg (\Sigma, \sigma))$ distinguish $\dot{s}$.

Weak spin RT TQFT. Let us replace $\mathcal{S}$ with a weaker category, where the objects are surfaces with spin^d structure and any 3-cobordism M from $(\partial_- M, \sigma_-)$ to

$(\partial_+ M, \sigma_+)$ is supplied with $s \in \mathrm{Spin}^d(M)$, such that $s|_{\partial_\pm M} = \sigma_\pm$. Strictly speaking, it is not a category, because the spin^d structure on the composition of such cobordisms along (Σ, σ) is uniquely defined only *if* Σ *is connected*. In order to get a category we should allow cobordisms with a 'superposition' (or collection) of spin structures.

To define the invariant $\tau(M, s)$ we only need to replace $\hat{G}^g$ with G^g, depicted below, in the previous construction.

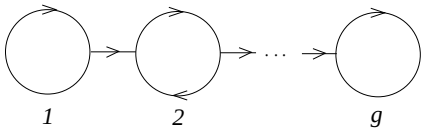


Figure 3 *The graph* G^g.

We denote by $G = G^+ \cup G^-$ the resulting graph in $\tilde{M}$. The set of special colorings of G is a subset of the special colorings of $\hat{G}$ consisting of colorings which are zero on $F_{n_+} \cup F_{n_-}$. The resulting TQFT we shall call a weak spin RT TQFT.

In this TQFT, the vector space associated to the disjoint union of surfaces is equal to the tensor product of spaces assigned to each of them. But we have a weak form of the gluing property (compare Theorem 4 in [B]).

Theorem 8 *If the 3-cobordism* (M, s) *is obtained from* (M_1, s_1) *and* (M_2, s_2) *by gluing along a diffeomorphism* $f : \partial_+ M_1 \to \partial_- M_2$ *which preserves spin*^{d} *structures and commutes with parametrizations, then*

$$\sum_s \tau_{ee''}(M, s) = k \sum_{e'} \tau_{ee'}(M_2, s_2)\, \tau_{e'e''}(M_1, s_1)\,, \tag{18}$$

where the sum is taken over all s *such that* $s|_{M_i} = s_i$, $i = 1, 2$.

Remark. The weak spin TQFT is the 'zero graded part' of the spin TQFT constructed in [BM]. The grading given by Theorem 11.2 in [BM] corresponds here to the orthogonal decomposition of $V(\Sigma, \sigma)$ into subspaces generated by colorings fixed on $F_{n_+} \cup F_{n_-}$.

For a 3-cobordism M, we define the vector $\tau(M) \in V(\partial M)$ by its coordinates

$$\tau_{ee'}(M) = \Delta^{-\sigma(L)} \eta^{-\chi(\partial_+ M)/2} \sqrt{\langle e \rangle \langle e' \rangle}\, \langle G_e^+ \cup L(\omega, ..., \omega) \cup G_{e'}^- \rangle$$

in the basis of $V(\partial M)$ given by admissible colorings of G. The pair $(\tau(M), V(\partial M))$ defines the unspun RT TQFT. Note that the number of admissible colorings of G (given by Verlinde formula) coincides with the dimension of $V(\partial M)$ (see [L1]).

Analogous to Lemma 4.2 in [Bl], we can prove the 'transfer theorem', which identifies the unspun theory with the sum of weak spin TQFT's:

$$\tau(M) = \sum_{s \in \mathrm{Spin}^d(M)} \tau(M, s)$$

Refined TV TQFT. Let us define a new cobordism category, where an object is $(\Sigma, P, \sigma, \tilde{h})$ with $\tilde{h} \in H^1(\Sigma, P)$ and the structure on 3-cobordisms extends the one given on the boundary.

Let $(M, \dot{s}, \dot{h})$ be such a 3-cobordism with parametrized boundary. Here $\dot{s} \in H^1(M, \partial M)$ determines the extension of the relative spin^d structure σ on ∂M and likewise, $\dot{h} \in H^1(M, \partial M)$ defines an extension of $\tilde{h} \in H^1(\partial M, P)$ to M.

We construct $(\tilde{M}, \hat{G})$ as in the spin RT TQFT. Choose a handle decomposition of $\tilde{M}$ in such a way that $\hat{G} \subset H$. Here H is as before the union of 0- and 1-handles.

The chain-mail graph for $(\tilde{M}, \hat{G})$ is the image under the embedding $j : H \hookrightarrow S^3$ of the graph consisting of

- a copy $\hat{G}^1$ of the graph $\hat{G}$ in the interior of H;
- attaching curves of 2-handles pushed slightly into H;
- a copy $\hat{G}^2$ of $\hat{G}$ on ∂H;
- meridian curves of 1-handles pushed slightly into $S^3 - H$.

The convention for the framing is the same as before. Denote by $A = \{a_1, ..., a_p\}$ the set of circles of $\hat{G}$ and by B the set of its meridians. Let u be the union of the sets u and u' used in the spin RT TQFT. Analogously, $t = \{0, t_2, ..., t_{n_+}, 0, ..., 0\} \cup \{0, t'_2, ..., t'_{n_-}, 0, ..., 0\}$, where t_i (resp. t'_i) is the number associated by $\dot{h}$ to the cycle in $M/\partial M$ obtained by identifying the first and ith marked points of F_{n_+} (resp. F_{n_-}). We denote by $h \in H^1(M)$ the cohomology class determined by $\dot{h}$ and the cohomology class on the boundary.

Then $s|_H - s_0|_{j(H)}$ assigns the numbers $\{x_1, ..., x_a\}$ to the 1-handles and the numbers $\{w_1, ..., w_p\}$ to the elements of B. Choose $y = \sum_i y_i \varepsilon_i$ representing $D(h) \in H_2(M, \partial M)$. Then $\partial y = \sum_i v_i a_i$. Choose a special coloring f (resp. e) of $\hat{G}$, such that the grading of the colors on its ith circle is equal to $-w_i$ (resp. $w_i - v_i$).

We set

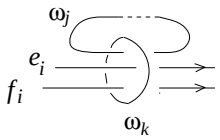
$$Z_{ef}(M, \dot{s}, \dot{h}) = (d\eta)^{d_0+d_3-2} \eta^{-\chi(\partial M)/2} \sqrt{\langle e \rangle \langle f \rangle} u_f u_e t_e \langle \hat{G}^2_f \cup R(\omega_{x_1}, ..., \omega_{y_b}) \cup \hat{G}^1_e \rangle. \quad (19)$$

We interpret $Z_{ef}(M, \dot{s}, \dot{h})$ as an (e, f)-coordinate of the vector $Z(M, \dot{s}, \dot{h})$ in the space spanned by special colorings of the graph $\hat{G} \cup \hat{G}$.

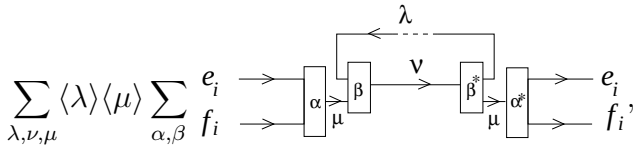
Theorem 9 *$Z(M,\dot{s},\dot{h})$ is an invariant of the 3-cobordism $(M,\dot{s},\dot{h})$ with parametrized boundary.*

Proof: By definition, $Z(M,\dot{s},\dot{h})$ is an isotopy invariant of $\hat{G}^1 \cup \hat{G}^2$. The rest of the proof is analogous to the proof of Theorem 1. Note that the number of lines in each 1-handle is 0 mod d. Therefore the unknotting und reframing moves can be performed analogously.

Only the births and deaths of 1-2-handle pairs require modifications. It can happen that a birth of such a pair introduces a 0-framed ω_k-colored ($k = 0, d/2$) unknot linked with 3-strands, as depicted below:



We use the fusion rules to replace these strands by one. Then applying the graded killing property we get



where we sum over all ν of the form $(1^N)^{\otimes k} \otimes K^l$ and all λ such that $|\lambda| = j$ mod d. Note that $\langle\nu\rangle = 1$. Let us apply (8) to the μ-colored line. After that, the sum $\sum_{\beta,\lambda}\langle\lambda\rangle = \sum_\lambda N^\nu_{\lambda\mu}\langle\lambda\rangle = \langle\mu\rangle$ factorizes and using (7) we can delete the 1-2-handle pair. □

Theorem 10 *For a 3-cobordism $(M,\dot{s},\dot{h})$,*

$$Z(M,\dot{s},\dot{h}) = \tau(-M,\dot{s}) \otimes \tau(M,\dot{s}+\dot{h})$$

where $\dot{s}+\dot{h} \in H^1(M,\partial M)$ is the extension of $\sigma + \tilde{h}$ to M.

The proof is analogous to the proof of Theorem 5. The difference is that the handlebodies in (15) contain a copy of $\hat{G}$.

Theorems 10 and 7 provide the gluing property (without anomaly) for the invariant $Z(M,\dot{s},\dot{h})$. This completes the construction of the refined TV TQFT.

TV TQFT. Consider a 3-cobordism M with parametrized boundary $\partial M = -\partial_- M \cup \partial_+ M$. We construct $(\tilde{M}, G)$ as in the weak spin TQFT. The admissible colorings of $G \cup G$ provide a basis of the vector space $V_{\partial M}$ associated with ∂M.

Then the vector $Z(M) \in V_{\partial M}$ with coordinates

$$Z_{ef}(M) = \eta^{d_0+d_3-2}\eta^{-\chi(\partial_+ M)}\sqrt{\langle e\rangle\langle f\rangle}\langle G_f^2 \cup R(\omega, ..., \omega) \cup G_e^1\rangle \tag{20}$$

is an invariant of M (by fogetting about the grading in the proof of Theorem 7). In fact, $Z(M)$ is equal to the invariant $\tilde{Z}(M)$ defined in [BD1]. This identifies the pair $(Z(M), V_{\partial M})$ with the Turaev-Viro TQFT.

We recall that $\tilde{Z}(M)$ is defined as the Turaev-Viro state sum operator of M with fixed triangulation of the boundary (given by two copies of the dual graph to G^g for each connected component of ∂M of genus g). The equality of $Z(M)$ and $\tilde{Z}(M)$ can be shown (in the spirit of Theorem 3.9 in [R1]) as follows: Choose the dual triangulation of $\tilde{M}$ as handle decomposition. Using fusion rules and the killing property for 1-handle curves, we can split the graph $G^2 \cup R \cup G^1$ into parts sitting in 0-handles. This associates 6j-symbols to 0-handles with no 3-vertices of the graph inside and products of 6j-symbols to the others. The definition of $\tilde{Z}(M)$ can then be reconstructed term-by-term. (The details will be omitted.)

The operator associated with a 3-cobordism (M, s, h) by the weak refined TV TQFT is denoted by $Z(M, s, h)$.

Corollary 11 *For a 3-cobordism (M, s, h),*

$$Z(M, s, h) = \tau(-M, s) \otimes \tau(M, s + h).$$

Corollary 12 *The Turaev-Viro operator invariant of a 3-cobordism M splits into a sum of weak refined invariants, i.e. $Z(M) = \sum_{s,h} Z(M, s, h)$.*

Finally, we note that an explicit calculation of a Homfly polynomial of a colored graph requires the knowledge of 6j-symbols which have apparently not yet been determined for $N > 2$.

Mathematisches Institut, Universität Bern, Sidlerstr. 5, CH-3012 Bern
e-mail: beliak@math-stat.unibe.ch

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