Integrality of quantum 3–manifold invariants and rational surgery formula

Anna Beliakova and Thang T. Q. Lê

Abstract

We prove that the Witten–Reshetikhin–Turaev (WRT) $SO(3)$ invariant of an arbitrary 3–manifold M is always an algebraic integer. Moreover, we give a rational surgery formula for the unified invariant dominating WRT $SO(3)$ invariants of rational homology 3–spheres at roots of unity of order co–prime with the torsion. As an application, we compute the unified invariant for Seifert fibered spaces and for Dehn surgeries on twist knots. We show that this invariant separates Seifert fibered integral homology spaces and can be used to detect the unknot.

Introduction

The Witten–Reshetikhin–Turaev (WRT) invariant was first introduced by Witten using physics heuristic ideas, and then mathematically rigorously by Reshetikhin and Turaev [20]. The invariant, depending on a root ξ of unity, was first defined for the Lie group $SU(2)$, and was later generalized to other Lie groups. The $SO(3)$ version of the invariant was introduced by Kirby and Melvin [8]. For this $SO(3)$ version the quantum parameter ξ must be a root of unity of *odd* order. One important result in quantum topology, first proved by H. Murakami for rational homology spheres [18] and then generalized by Masbaum and Roberts [16], is that the WRT $SO(3)$ invariant (also known as quantum $SO(3)$ invariant) $\tau_M(\xi)$ of an arbitrary 3–manifold M is an algebraic integer, when the order of the root of unity ξ is an *odd prime*. The first integrality result for all roots of unity, but for the restricted set of 3–manifolds (integral homology 3–spheres) was obtained by Habiro in [7]. Recently, the second author proved [11] that if the order of ξ is co–prime with the cardinality of the torsion of $H_1(M,\mathbb{Z})$, then the $SO(3)$ quantum invariant $\tau_M(\xi) \in \mathbb{Z}[\xi]$. In this paper we remove all the restrictions on the order of ξ.

Theorem 1. *For every closed 3–manifold and every root ξ of unity of odd order, the quantum $SO(3)$ invariant $\tau_M(\xi)$ belongs to $\mathbb{Z}[\xi]$.*

The integrality has many important applications, among them is the construction of an integral topological quantum field theory and representations of mapping class groups over $\mathbb{Z}$ by Gilmer and Masbaum (see e.g. [6]). The integrality is also a key property required for the categorification of quantum 3–manifold invariants [10].

Our proof of integrality is inspired by Habiro's work. In [7], Habiro constructed an invariant of integral homology 3–spheres with values in the universal ring

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q;q)_n)$$

2000 Mathematics Subject Classification 57N10 (primary), 57M25 (secondary)
Keywords: 3–manifolds, unified quantum invariants, cyclotomic completion ring, q–hypergeometric series
The first author wishes to express her gratitude to Christian Krattenthaler for valuable suggestions.

where $(z;q)_n := (1-z)(1-qz)\dots(1-q^{n-1}z)$ and for any $f \in \mathbb{Z}[q]$, (f) denotes the ideal generated by f. Habiro's invariant specializes at a root ξ of unity to $\tau_M(\xi)$.

In [11], the second author generalized Habiro's theory to rational homology 3–spheres. For a rational homology sphere M with $|H_1(M,\mathbb{Z})| = a$, he constructed an invariant I_M which dominates the $SO(3)$ invariants of M at roots of unity of order co–prime to a. Habiro's universal ring was modified by inverting a and cyclotomic polynomials of order not co–prime to a. Applications of this theory are the new integrality properties of quantum invariants, new results about Ohtsuki series and a better understanding of the relation between LMO invariant, Ohtsuki series and quantum invariants.

In this paper we give a rational surgery formula for the unified invariant I_M defined in [11] and refine the ring that contains the values of I_M. Let us summarize our main results.

Let $t := q^{1/a}$ and $\mathcal{R}_{a,k} \subset \mathbb{Q}(t)$ be the subring generated over $\mathbb{Z}[t^{\pm 1}]$ by $\frac{(t;t)_k}{(q;q)_k}$. Note that

$$\mathcal{R}_{a,1} \subset \mathcal{R}_{a,2} \subset \dots \subset \mathcal{R}_a,$$

where $\mathcal{R}_a = \cup_{k=1}^{\infty} \mathcal{R}_{a,k}$. The analog of the Habiro ring constructed in [11] can be defined as[1]

$$\widehat{\mathcal{R}}_a := \lim_{\overleftarrow{n}} \mathcal{R}_a/((q;q)_n).$$

Let U_a be the set of all complex roots of unity with orders *odd* and co–prime with a. For every $\xi \in U_a$ and every $f \in \widehat{\mathcal{R}}_a$ one can define an evaluation $\mathrm{ev}_\xi(f) \in \mathbb{C}$, replacing q by ξ, see section 1.2. It was shown in [11] that if $|H_1(M,\mathbb{Z})| = a$, then $I_M \in \widehat{\mathcal{R}}_a$ and $\mathrm{ev}_\xi(I_M)$, after a simple normalization, is the $SO(3)$ quantum invariants of M at $q = \xi$.

It will be shown that for $f \in \widehat{\mathcal{R}}_a$, $\mathrm{ev}_\xi(f) \in \mathbb{Z}[\frac{\xi}{a}]$, and in general, one cannot avoid the denominator. We will single out a subring Γ_a of $\widehat{\mathcal{R}}_a$ such that $\mathrm{ev}_\xi(\Gamma_a) = \mathbb{Z}[\xi]$.

Let

$$x_n := \frac{(q^{n+1};q)_{n+1}}{1-q} = \frac{(1-q^{n+1})(1-q^{n+2})\dots(1-q^{2n+1})}{1-q} \in \mathbb{Z}[q].$$

Then $(q;q)_n$ divides x_n, which, in turns, divides $(q;q)_{2n+1}$. Hence the ideals (x_n) and $(q;q)_n$ are cofinal in $\mathcal{R}_a$, and we have $\widehat{\mathcal{R}}_a := \lim_{\overleftarrow{n}} \mathcal{R}_a/(x_n)$. Every element $f \in \widehat{\mathcal{R}}_a = \lim_{\overleftarrow{n}} \mathcal{R}_a/(x_n)$ can be represented as an infinite series of the form

$$f = \sum_{n=0}^{\infty} f_n(t) x_n, \quad \text{where} \quad f_n(t) \in \mathcal{R}_a. \tag{1}$$

Let Γ_a be the set of all elements f of $\widehat{\mathcal{R}}_a$ that have a presentation (1) such that $f_n(t) \in \mathcal{R}_{a,2n+1}$. It is easy to see that Γ_a is a subring of $\widehat{\mathcal{R}}_a$. The following shows that Γ_a is strictly smaller than $\hat{\mathcal{R}}_a$: it enjoys stronger integrality.

Proposition 1. *Suppose $f \in \Gamma_a$ and $\xi \in U_a$, i.e. ξ is a root of unity whose order is odd and co-prime with a. Then $\mathrm{ev}_\xi(f) \in \mathbb{Z}[\xi]$. On the other hand, $\mathrm{ev}_\xi(\hat{\mathcal{R}}_a) = \mathbb{Z}[\frac{\xi}{a}]$.*

Now we can formulate our next result.

Theorem 2. *Let M be a rational homology 3–sphere with $|H_1(M,\mathbb{Z})| = a$. Then $I_M \in \Gamma_a$.*

In particular, Theorem 2 and Proposition 1 give a new proof of the integrality of $SO(3)$ quantum invariant of rational homology 3–sphere M with $|H_1(M,\mathbb{Z})| = a$ at a root of unity $\xi \in U_a$, a result proved in [11].

[1] We are grateful to the referee for pointing out that $\widehat{\mathcal{R}}_a$ coincides with the ring $\hat{\Lambda}_a$ in [11].

Further, we compute the unified invariant for Seifert fibered spaces and for Dehn surgeries on twist knots.

Theorem 3. *The unified invariant separates Seifert fibered integral homology spheres.*

Theorem 3 follows also from [5], where the computation were done for the LMO invariant combined with the sl_2 weight system, i.e. for the Ohtsuki series. By the result of Habiro in [7], the Ohtsuki series is just the Taylor expansion of I_M at $q = 1$, which determines I_M.

For a knot K, let $M(K,a)$ denotes the 3–manifold obtained by surgery on the knot K with framing a. In general, there are different K, K' such that $M(K,a) = M(K',a)$ for some a.

Theorem 4. *Suppose that for infinitely many $a \in \mathbb{Z}$, the Ohtsuki series of $M(K,a)$ and $M(K',a)$ coincide, i.e. $I_{M(K,a)} = I_{M(K',a)}$. Then K and K' have the same colored Jones polynomial.*

In particular, using the recent deep result of Andersen [1], that the colored Jones polynomial detects the unknot, we see that (under the assumption of the theorem), if K is the unknot, then so is K'.

1. Quantum invariants

Recall that $q = t^a$. We will use the following notations:

$$\{n\} = q^{n/2} - q^{-n/2}, \quad \{n\}! = \prod_{i=1}^{n} \{i\}, \quad [n] = \frac{\{n\}}{\{1\}}, \quad \left[\begin{array}{c} n \\ k \end{array} \right] = \frac{\{n\}!}{\{k\}!\{n-k\}!}.$$

1.1 The colored Jones polynomial

Suppose L is framed, oriented link in S^3 with m ordered components. For every positive integer n there is a unique irreducible sl_2–module V_n of dimension n. For positive integers $n_1, \dots, n_m$ one can define the quantum invariant $J_L(n_1, \dots, n_m) := J_L(V_{n_1}, \dots, V_{n_m})$ known as the colored Jones polynomial of L (see e.g. [20]). Let us recall here a few well–known formulas. For the unknot U with 0 framing one has

$$J_U(n) = [n] = \{n\}/\{1\}. \tag{2}$$

If L' is obtained from L by increasing the framing of the i–th component by 1, then

$$J_{L'}(n_1, \dots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \dots, n_m). \tag{3}$$

In general, $J_L(n_1, \dots, n_m) \in \mathbb{Z}[q^{\pm 1/4}]$. However, there is a number $b \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ such that $J_L(n_1, \dots, n_m) \in q^b \mathbb{Z}[q^{\pm 1}]$.

1.2 Evaluation map and Gauss sum

Throughout this paper let ξ be a primitive root of unity of *odd* order r. We first define, for each ξ, the evaluation map ev_ξ, which replaces q by ξ. Suppose $f \in \mathbb{Q}[q^{\pm 1/h}]$, where h is co–prime with r, the order of ξ. There exists an integer b, unique modulo r, such that $(\xi^b)^h = \xi$. Then we define

$$\mathrm{ev}_\xi f := f|_{q^{1/h} = \xi^b}.$$

Suppose $t := q^{1/a}$ and $\mathbb{N}_a$ is the set of all positive integers coprime to a. Denote by $\Phi_s(t)$ the s–th cyclotomic polynomial. Recall that

$$1 - t^n = \prod_{s|n} \Phi_s(t).$$

It follows that $\frac{(t;t)_k}{(q;q)_k}$ is the inverse of the product of many $\Phi_s(t)$ with $s \notin \mathbb{N}_a$. Recall that U_a is the set of all complex roots of unity with orders odd and coprime with a. When $\xi \in U_a$ and $s \notin \mathbb{N}_a$, $\Phi_s(\xi) \neq 0$. Thus one can evaluate $\mathrm{ev}_\xi\left(\frac{(t;t)_k}{(q;q)_k}\right)$. The definition also extends to $\mathrm{ev}_\xi : \widehat{\mathcal{R}}_a \to \mathbb{C}$, since $\mathrm{ev}_\xi((q;q)_n) = 0$ if $n \geqslant r$.

Suppose $f(q; n_1, \dots, n_m)$ is a function of variables q and integers $n_1, \dots, n_m$. Let

$$\sum_{n_i}^{\xi} f := \sum_{n_i} \mathrm{ev}_\xi(f),$$

where in the sum all the n_i run the set of *odd* numbers between 0 and $2r$. A variation $\gamma_d(\xi)$ of the Gauss sum is defined by

$$\gamma_d(\xi) := \sum_{n}^{\xi} q^{d\frac{n^2-1}{4}}.$$

It is known that, for odd r, $|\gamma_d(\xi)| = \sqrt{r}$, and hence is never 0.

Let

$$F_L(\xi) := \sum_{n_i}^{\xi} J_L(n_1, \dots, n_m) \prod_{i=1}^{m} [n_i].$$

The following result is well–known (compare [11]).

Lemma 1.1. *For the unknot $U^\pm$ with framing ± 1, one has $F_{U^\pm}(\xi) \neq 0$. Moreover,*

$$F_{U^\pm}(\xi) = \mp 2\gamma_{\pm 1}(\xi)\, \mathrm{ev}_\xi\left(\frac{q^{\mp 1/2}}{\{1\}}\right). \tag{4}$$

1.3 Definition of $SO(3)$ invariant of 3–manifolds

All 3–manifolds in this paper are supposed to be closed and oriented. Every link in a 3–manifold is framed, oriented, and has components ordered.

Suppose M is an oriented 3–manifold obtained from S^3 by surgery along a framed, oriented link L. (Note that M does not depend on the orientation of L). Let σ_+ (respectively, σ_-) be the number of positive (resp. negative) eigenvalues of the linking matrix of L. Suppose ξ is a root of unity of odd order r. Then the quantum $SO(3)$ invariant is defined by

$$\tau_M(\xi) = \tau_M^{SO(3)}(\xi) := \frac{F_L(\xi)}{(F_{U^+}(\xi))^{\sigma_+}\,(F_{U^-}(\xi))^{\sigma_-}}.$$

For connected sum, one has $\tau_{M\#N}(\xi) = \tau_M(\xi)\tau_N(\xi)$.

1.4 Laplace transform

In [4], we together with Blanchet developed the Laplace transform method to compute $\tau_M(\xi)$. Here we generalize this method to the case where r is not co–prime with torsion.

Suppose r is an odd number, and d is positive integer. Let

$$c := (r, d), \quad d_1 := d/c, \quad r_1 := r/c.$$

Let $\mathcal{L}_{d;n} : \mathbb{Z}[q^{\pm n}, q^{\pm 1}] \to \mathbb{Z}[q^{\pm 1/d}]$ be the $\mathbb{Z}[q^{\pm 1}]$–linear operator, called the Laplace transform, defined by

$$\mathcal{L}_{d;n}(q^{na}) := \begin{cases} 0 & \text{if } c \not| a; \\ q^{-a^2/d} & \text{if } a = ca_1, \end{cases} \tag{5}$$

LEMMA 1.2. *Suppose $f \in \mathbb{Z}[q^{\pm n}, q^{\pm 1}]$. Then*

$$\sum_n^{\xi} q^{d\frac{n^2-1}{4}} f = \gamma_d(\xi)\, \mathrm{ev}_\xi(\mathcal{L}_{d;n}(f)).$$

Proof. It's enough to consider the case when $f = q^{na}$, with a an integer. This case is proven by Lemma 1.3 in the next subsection. □

The point is that $\mathcal{L}_{d;n}(f)$, unlike the left hand side $\sum_n^{\xi} q^{d\frac{n^2-1}{4}} f$, does not depend on ξ, and will help us to define a "universal invariant". Note that Lemma 1.2 with $d = \pm 1$ and $f = [n]^2$ implies Lemma 1.1.

1.5 Reduction from r to r_1

Let O_r be the set of all odd integers between 0 and $2r$. This set O_r can be partitioned into r_1 subsets $O_{r;s}$ with $s \in O_{r_1}$, where $O_{r;s}$ is the set of all $n \in O_r$ which are equal to s modulo r_1. In other words, $O_{r;s} := \{s + 2jr_1, j = 0, 1, \dots, c-1\}$. The point is, the value of $\xi^{d\frac{n^2-1}{4}}$ remains the same for all n in the same set $O_{r;s}$. Let $\zeta = \xi^c$, then the order of ζ is r_1.

LEMMA 1.3. *One has*

$$\gamma_d(\xi) = c\gamma_{d_1}(\zeta). \tag{6}$$

$$\sum_n^{\xi} q^{d\frac{n^2-1}{4}} q^{an} = \begin{cases} 0 & \textit{if } c \not| a; \\ \gamma_d(\xi)\zeta^{-a_1^2 d_1^*} & \textit{if } a = ca_1, \end{cases} \tag{7}$$

where d_1 is an integer satisfying $d_1 d_1^* \equiv 1 \pmod{r_1}$.

Proof. One has

$$\sum_n^{\xi} q^{d\frac{n^2-1}{4}} q^{an} = \sum_{n \in O_r} \xi^{d\frac{n^2-1}{4}} \xi^{an} = \sum_{s \in O_{r_1}} \sum_{n \in O_{r;s}} \xi^{d\frac{n^2-1}{4}} \xi^{an}.$$

Using the fact that $\xi^{d\frac{n^2-1}{4}}$ remains the same for all n in the same set $O_{r;s}$, we get

$$\sum_n^{\xi} q^{d\frac{n^2-1}{4}} q^{an} = \sum_{s \in O_{r_1}} \xi^{d\frac{s^2-1}{4}} \sum_{n \in O_{r;s}} \xi^{an} \tag{8}$$

$$= \sum_{s \in O_{r_1}} \xi^{d\frac{s^2-1}{4}} \xi^{sa} \left(\sum_{j=0}^{c-1} \xi^{2ar_1 j} \right) \tag{9}$$

Note that (6) follows from (8) with $a = 0$.

$$\sum_{j=0}^{c-1} \xi^{2ar_1 j} = \sum_{j=0}^{c-1} (\xi^{2ar_1})^j. \tag{10}$$

If $c \not| a$, then $(\xi^{2ar_1}) \neq 1$, but a root of unity of order dividing c, hence the right hand side of (10) is 0. It follows that the right hand side of (9) is also 0, or $\sum_n^\xi q^{d\frac{n^2-1}{4}} q^{an} = 0$.

If $c|a$, then the right hand side of (10) is c. Hence from (9) we have

$$\begin{aligned}
\sum_n^\xi q^{d\frac{n^2-1}{4}} q^{an} &= c \sum_{s\in O_{r_1}} \xi^{d\frac{s^2-1}{4}} \xi^{sa} \\
&= c \sum_{s\in O_{r_1}} \zeta^{d_1\frac{s^2-1}{4}} \zeta^{sa_1} = c \sum_n^\zeta q^{d_1\frac{n^2-1}{4}} q^{a_1 n} \\
&= c\gamma_{d_1}(\zeta)\zeta^{-a_1^2 d_1^*}
\end{aligned}$$

The last equality follows by the standard square completion argument. Using (6) we get the result. □

1.6 Habiro's cyclotomic expansion of the colored Jones polynomial

In [7], Habiro defined a new basis P'_k, $k = 0, 1, 2, \ldots$, for the Grothendieck ring of finite–dimensional sl_2–modules, where

$$P'_k := \frac{1}{\{k\}!} \prod_{i=1}^{k} (V_2 - q^{(2i-1)/2} - q^{-(2i-1)/2}).$$

For any link L, one has

$$J_L(n_1, \ldots, n_m) = \sum_{0\leqslant k_i \leqslant n_i - 1} J_L(P'_{k_1}, \ldots, P'_{k_m}) \prod_{i=1}^{m} \begin{bmatrix} n_i + k_i \\ 2k_i + 1 \end{bmatrix} \{k_i\}! \tag{11}$$

Since there is a denominator in the definition of P'_k, one might expect that $J_L(P'_{k_1}, \ldots, P'_{k_m})$ also has non–trivial denominator. A difficult and important integrality result of Habiro [7] is

THEOREM 5. *[7, Thm. 3.3] If L is algebraically split and zero framed link in S^3, then*

$$J_L(P'_{k_1}, \ldots, P'_{k_m}) \in \frac{\{2k+1\}!}{\{k\}!\{1\}} \mathbb{Z}[q^{\pm 1/2}] = \begin{bmatrix} 2k+1 \\ k \end{bmatrix} (q^2)_k \, \mathbb{Z}[q^{\pm 1/2}],$$

where $k = \max\{k_1, \ldots, k_m\}$.

Thus, $J_L(P'_{k_1}, \ldots, P'_{k_m})$ is not only integral, but also divisible by $(q)_k$.

Suppose L is an algebraically split link with 0–framing on each component. Then we have

$$\mathrm{ev}_\xi(J_L(n_1, \ldots, n_m)) = \mathrm{ev}_\xi \left(\sum_{k_1,\ldots,k_m=0}^{(r-3)/2} J_L(P'_{k_1}, \ldots, P'_{k_m}) \prod_{i=1}^{m} \begin{bmatrix} n_i + k_i \\ 2k_i + 1 \end{bmatrix} \{k_i\}! \right)$$

2. Integrality of quantum invariants for all roots of unity

Throughout this section we assume that $c = (d, r) > 1$, $r/c = r_1$, $d/c = d_1$ and $d_1 d_1^* = 1 \mod r_1$, where r is the order of ξ and d is the order of the torsion part of $H_1(M, \mathbb{Z})$.

2.1 Quantum invariants of links with diagonal linking matrix

The following proposition plays a key role in the proof of integrality.

Proposition 2.1. *For $k \leqslant (r-3)/2$, we have*

$$\frac{1}{\gamma_{\pm 1}(\xi)} \sum_n^{\xi} q^{d\frac{n^2-1}{4}} \left[\begin{array}{c} n+k \\ 2k+1 \end{array} \right] \{k\}!\{n\} \in \mathbb{Z}[\xi]. \tag{12}$$

Proof of Theorem 1 (diagonal case) Suppose M is obtained from S^3 by surgery along an algebraically split m–component link L with integral framings $d_1, d_2, \ldots, d_m$. Inserting into the definition of $\tau_M(\xi)$ (see Section 1.3) the formulas (4) and (11) and using Lemma 1.2, we see that Proposition 2.1 and Theorem 5 imply integrality if $d_i \neq 0$ for all i. If some of d_i are zero, then by same argument as in Section 3.4.2 of [11] we have

$$\sum_n^{\xi} \left[\begin{array}{c} n+k \\ 2k+1 \end{array} \right] \{k\}!\{n\}, = 2\mathrm{ev}_\xi \left(q^{(k+1)(k+2)/4} \, (q^{k+2}; q)_{r-k-2} \right).$$

The result follows now from the fact $\gamma_d(\xi)/\gamma_1(\xi) \in \mathbb{Z}[\xi]$. □

2.1.1 *Technical results* This subsection is devoted to the proof of Proposition 2.1.

Lemma 2.2. (a) *Suppose $x \in \mathbb{Q}(\xi)$ such that $x^2 \in \mathbb{Z}[\xi]$, then $x \in \mathbb{Z}[\xi]$.*

(b) *Suppose $x, y \in \mathbb{Z}[\xi]$ such that x^2 is divisible by y^2, then x is divisible by y.*

Proof. (a) Suppose $a = x^2$, then $a \in \mathbb{Z}[\xi]$ and x is a solution of $x^2 - a = 0$, hence x is integral over $\mathbb{Z}[\xi]$, which is integrally closed. It follows that $x \in \mathbb{Z}[\xi]$.

(b) We have that $(x/y)^2 = x^2/y^2$ is in $\mathbb{Z}[\xi]$, hence by part (a), $x/y \in \mathbb{Z}[\xi]$. □

Recall that

$$(q^l; q)_m = \prod_{j=l}^{l+m-1} (1 - q^j).$$

Let $\widetilde{(q^l;q)}_m$ be the product on the right hand side, only with j *not divisible by c*. Also let $\widehat{(q^l;q)}_m$ be the complement, i.e. $\widehat{(q^l;q)}_m := (q^l;q)/\widetilde{(q^l;q)}_m$. Using $(\xi;\xi)_{r-1} = r, \quad (\xi^c;\xi^c)_{r_1-1} = r_1$, we see that

$$\widetilde{(\xi;\xi)}_{r-1} = c, \tag{13}$$

where $(a;b)_m := (1-a)(1-ab)\ldots(1-ab^{m-1})$. Note that $1-\xi^j$ is invertible in $\mathbb{Z}[\xi]$ iff $(j,r) = 1$. Let

$$z := \widetilde{(\xi;\xi)}_{(r-1)/2}, \quad \text{and} \quad z' := \widetilde{(\xi^{(r+1)/2};\xi)}_{(r-1)/2}. \tag{14}$$

Then zz' is the left hand side of (13), hence $zz' = c$. We use the notation $x \sim y$ if the ratio x/y is a unit in $\mathbb{Z}[\xi]$. Note that $z \sim z'$. This is because $1-\xi^k \sim 1-\xi^{r-k}$. Thus we have

$$z^2 \sim c \tag{15}$$

Lemma 2.3. *$\gamma_d(\xi)/\gamma_1(\xi)$ is divisible by z.*

Proof. Using Lemma 2.2(b) and (15), one needs only to show that $(\gamma_d(\xi))^2/(\gamma_1(\xi))^2$ is divisible by c. The values of $\gamma_b(\xi)$ are well–known when b is co–prime with r, the order of ξ. In particular, $\gamma_b(\xi) \sim \gamma_1(\xi)$, see [13].

Recall that $\zeta = \xi^c$ has order r_1. Since d_1 and r_1 are co–prime, we have

$$\gamma_{d_1}(\zeta) \sim \gamma_1(\zeta).$$

Using Lemma 1.3, we have
$$\frac{(\gamma_d(\xi))^2}{(\gamma_1(\xi))^2} = c^2\frac{(\gamma_{d_1}(\zeta))^2}{(\gamma_1(\xi))^2} \sim c^2\frac{(\gamma_1(\zeta))^2}{(\gamma_1(\xi))^2} \tag{16}$$
Using explicit formula for $\gamma_1(\xi) = \sum_{0\leqslant j<r}\xi^{j^2+j}$ (given e.g. by Thm. 2.2 of [13]), we have that
$$(\gamma_1(\xi))^2 = \pm r\xi^{-2^*} = cr_1\xi^{-2^*}, \quad (\gamma_1(\zeta))^2 = \pm r_1\zeta^{-2^*}$$
where 2^* is the inverse of 2. Plugging this in (16), we get the result. □

For $k, b \in \mathbb{Z}$ we define
$$Y_c(k,b) := (-1)^k \sum_{n=-\lfloor k/c\rfloor}^{\lfloor (k+1)/c\rfloor} (-1)^n \begin{bmatrix} 2k+1 \\ k+nc \end{bmatrix} q^{cbn^2} \tag{17}$$

LEMMA 2.4. *Suppose $d_1d_1^* \equiv 1 \pmod{r_1}$, where $r = cr_1$ is the order of ξ, then*
$$\sum_n^{\xi} q^{d\frac{n^2-1}{4}} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}\{k\}!\{n\} = -2\gamma_d(\xi)\,\mathrm{ev}_\xi\left(\frac{Y_c(k,-d_1^*)\{k\}!}{\{2k+1\}!}\right).$$

Proof. By Lemma 1.2 we have to compute $\mathcal{L}_{d;n}(\{n\}\,\{n+k\}!/\{n-k-1\}!)$. Since $\mathcal{L}_{d;n}$ is invariant under $n \to -n$, one has
$$\mathcal{L}_{d;n}(\{n\}\,\{n+k\}!/\{n-k-1\}!) = -2\mathcal{L}_{d;n}(q^{-nk}\,(q^{n-k};q)_{2k+1}). \tag{18}$$
By the q–binomial formula we have
$$q^{-nk}(q^{n-k};q)_{2k+1} = \sum_{j=0}^{2k+1}(-1)^j \begin{bmatrix} 2k+1 \\ j \end{bmatrix} q^{n(j-k)}. \tag{19}$$
Using the definition of $\mathcal{L}_{d;n}$ we get
$$\mathrm{ev}_\xi(\mathcal{L}_{d;n}(\{n\}\,\{n+k\}!/\{n-k-1\}!)) = -2\mathrm{ev}_\xi(Y_c(k,-d_1^*)).$$
Multiplying by $\{k\}!/\{2k+1\}!$, we get the result. □

THEOREM 6. *For $b \in \mathbb{Z}$ and $k \leqslant (r-3)/2$, $\frac{\gamma_d(\xi)}{\gamma_1(\xi)}\mathrm{ev}_\xi(Y_c(k,b))$ is divisible by $\mathrm{ev}_\xi\left(\frac{\{2k+1\}!}{\{k\}!}\right)$.*

Here we modify the proof of Theorem 7 in [11].

Proof. The case $b = 0$ is trivial. Let us assume $b \neq 0$. Separating the case $n = 0$ and combining positive and negative n, we have
$$Y_c(k,b) = (-1)^k\begin{bmatrix} 2k+1 \\ k \end{bmatrix} + (-1)^k\sum_{n=1}^{\lfloor k/c\rfloor}(-1)^n q^{cbn^2}\left(\begin{bmatrix} 2k+1 \\ k+nc \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-nc \end{bmatrix}\right).$$
Using
$$\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix} = \frac{\{k+1\}}{\{2k+2\}}\begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix}(q^{cn/2}+q^{-cn/2})$$
and $\begin{bmatrix} 2k+2 \\ k+1 \end{bmatrix} = \begin{bmatrix} 2k+1 \\ k \end{bmatrix}\frac{\{2k+2\}}{\{k+1\}}$ we get
$$Y_c(k,b) = (-1)^k\begin{bmatrix} 2k+1 \\ k \end{bmatrix} S_N \tag{20}$$

where $N = k+1$ and

$$S_N = 1 + \sum_{n=1}^{\infty} \frac{q^{Ncn}(q^{-N};q)_{cn}}{(q^{N+1};q)_{cn}} \left(1+q^{cn}\right) q^{cbn^2}.$$

For z defined by (14), we show the divisibility of $\mathrm{ev}_\xi(S_N)z$ by $(\xi;\xi)_N$ in Section 2.1.2. This implies the result, since $z|\frac{\gamma_d(\xi)}{\gamma_1(\xi)}$ by Lemma 2.3 and

$$\begin{bmatrix} 2k+1 \\ k \end{bmatrix} \{k+1\}! = \frac{\{2k+1\}!}{\{k\}!}.$$

□

Proof of Proposition 2.1 Combining Lemma 2.4 with Theorem 6 we get Proposition 2.1. □

2.1.2 *Andrews's identity* Let α_n, β_n be a Bailey pair as defined in Section 3.4 of [2], with $a=1$. Then for any numbers $b_i, c_i, i = 1, \ldots, k$ and positive integer N we have the identity (3.43) of [2]:

$$\sum_{n\geqslant 0} (-1)^n \alpha_n q^{-\binom{n}{2}+kn+Nn} \frac{(q^{-N})_n}{(q^{N+1})_n} \prod_{i=1}^{k} \frac{(b_i)_n}{b_i^n} \frac{(c_i)_n}{c_i^n} \frac{1}{(\frac{q}{b_i})_n (\frac{q}{c_i})_n} =$$

$$\frac{(q)_N \, (\frac{q}{b_k c_k})_N}{(\frac{q}{b_k})_N \, (\frac{q}{c_k})_N} \sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 0} \beta_{n_1} \frac{q^{n_k}(q^{-N})_{n_k} (b_k)_{n_k} (c_k)_{n_k}}{(q^{-N} b_k c_k)_{n_k}} \prod_{i=1}^{k-1} \frac{q^{n_i} \frac{(b_i)_{n_i}}{b_i^{n_i}} \frac{(c_i)_{n_i}}{c_i^{n_i}} (\frac{q}{b_i c_i})_{n_{i+1}-n_i}}{(q)_{n_{i+1}-n_i} (\frac{q}{b_i})_{n_{i+1}} (\frac{q}{c_i})_{n_{i+1}}}. \tag{21}$$

A special Bailey pair is given by (see section 3.5 of [2]):

$$\alpha_0 = 1, \quad \alpha_n = (-1)^n q^{n(n-1)/2}(1+q^n) \quad \text{for } n \geqslant 1.$$
$$\beta_0 = 1, \quad \beta_n = 0 \quad \text{for } n \geqslant 1.$$

Using the decomposition

$$(q^x;q)_{nc} = (q^x;q^c)_n (q^{x+1};q^c)_n \ldots (q^{x+c-1};q^c)_n$$

for $x=-N$ and $x = N+1$, we can identify S_N with the LHS of (21) where the parameters are chosen as follows. Let $s = (c+1)/2$ and $k = b+s$. Suppose $N = mc+t$ with $0 \leqslant t \leqslant c-1$. We consider the limit $b_i, c_i \to \infty$ for $i = s+1, \ldots, k$. We put $b_s = q^{t-N}$ and $c_s = q^{Nc+c}$. For $j = 1, 2, \ldots, s-1$, among the integers $\{0, 1, 2, \ldots, c-1\}$ there is exactly one u_j and v_j, such that $u_j = j+t \pmod{c}$ and $v_j = -j+t \pmod{c}$. We choose $b_j = q^{u_j - N}$ and $c_j = q^{v_j - N}$ for $j = 1, 2, \ldots, s-1$. The base q in the identity should be replaced by q^c. Therefore, in the rest of this section

$$(q^a)_m := (q^a;q^c)_m \,.$$

The RHS of the identity gives us the following expression for S_N.

$$S_N = \sum_{N \geqslant n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_2 \geqslant 0} \hat{F}(n_k, \ldots, n_2) \tilde{F}(n_k, \ldots, n_2) \tag{22}$$

where

$$\hat{F}(n_k, \ldots, n_2) \sim \frac{(q^c)_N (q^{-Nc})_{n_k} (q^{Nc+c})_{n_s} (q^{-mc})_{n_s} (q^{mc-Nc})_{n_{s+1}-n_s}}{(q^{-Nc})_{n_{s+1}} (q^{c+mc})_{n_{s+1}} \prod_{i=1}^{k-1} (q^c)_{n_{i+1}-n_i}} \prod_{j=1}^{s-1} (q^{c+2N-v_j-u_j})_{n_{j+1}-n_j} \,;$$

$$\tilde{F}(n_k,\dots,n_2) \sim \prod_j^{s-1} \frac{(q^{u_j-N})_{n_j}(q^{v_j-N})_{n_j}}{(q^{c+N-u_j})_{n_{j+1}}(q^{c+N-v_j})_{n_{j+1}}}\,.$$

Here $x \sim y$ means x/y is a unit in $\mathbb{Z}[q^{\pm 1}]$. Note that $c-2N-v_j-u_j$, which is equal to $2N-2t\pm c$ or $2N-2t$, is always a multiple of c.

Observe that $\hat{F}(n_k,\dots,n_2)\neq 0$ iff the following inequalities hold

$$n_k \leqslant N\,, \qquad n_s \leqslant \lfloor N/c \rfloor = m \tag{23}$$

(otherwise $(q^{-Nc})_{n_k}$ or $(q^{-mc})_{n_s}$ is zero);

$$n_{s+1}-n_s \leqslant N-m \tag{24}$$

(otherwise $(q^{mc-Nc})_{n_{s+1}-n_s}$ is zero).

Let us assume that q is a primitive r–th root of unity, then we have in addition

$$N \leqslant r/c\,, \qquad Nc+cn_s < r \tag{25}$$

(otherwise $(q^c)_N$ or $(q^{Nc+c})_{n_s}$ is zero). Note that if $\hat{F}(n_k,\dots,n_2)\neq 0$ then it is also well–defined.

Lemma 2.5. *Suppose q is a primitive r–th root of unity, then $z\tilde{F}(n_k,\dots,n_2)$ is divisible by $\widetilde{(q;q)}_N$.*

Proof. It suffice to show that z is divisible by $\widetilde{(q;q)}_N D$, where D is the denominator of $\tilde{F}(n_k,\dots,n_2)$. Since $n_2\leqslant n_3\leqslant\cdots\leqslant n_s$, we have

$$D \mid (q^{1+N})_{n_s}(q^{2+N})_{n_s}\dots(q^{c+N})_{n_s} = \widetilde{(q^{1+N};q)}_{cn_s}$$

and so $\widetilde{(q;q)}_N D$ divides $\widetilde{(q;q)}_N\widetilde{(q^{1+N};q)}_{cn_s}=\widetilde{(q;q)}_{N+cn_s}$, but $N+cn_s\leqslant(r-1)/2$. Indeed,

$$2N+2cn_s \leqslant 3N+cn_s\leqslant Nc+cn_s<r$$

by (23), (25). Hence,

$$\widetilde{(q;q)}_{N+cn_s} \;\mid\; \widetilde{(q;q)}_{(r-1)/2}=z\,.$$

$\square$

Lemma 2.6. *For a primitive r–th root of unity q, $\hat{F}(n_k,\dots,n_2)$ is divisible by $\widehat{(q;q)}_N = (q^c;q^c)_m$.*

Proof. Using for integer $a\geqslant b>0$ the formula

$$(q^{-ac})_b \sim \frac{(q^c)_a}{(q^c)_{a-b}},$$

we have

$$\frac{(q^{-Nc})_{n_k}}{(q^{-Nc})_{n_{s+1}}}\frac{\prod_{j=1}^{s-1}(q^{c+2N-v_j-u_j})_{n_{j+1}-n_j}}{\prod_{i=1}^{k-1}(q^c)_{n_{i+1}-n_i}} \sim \frac{(q^c)_{N-n_{s+1}}}{(q^c)_{N-n_k}}\frac{\prod_{j=1}^{s-1}(q^{c+2N-v_j-u_j})_{n_{j+1}-n_j}}{\prod_{i=1}^{k-1}(q^c)_{n_{i+1}-n_i}}.$$

The latter, using the fact that $(q^c)_a$ divides $(q^{c+2N-v_j-u_j})_a$, is divisible by $\frac{1}{(q^c)_{n_{s+1}-n_s}}$. Thus $\hat{F}(n_k,\dots,n_2)/(q^c)_m$ is divisible by

$$\frac{(q^c)_{N-m}}{(q^c)_{n_{s+1}-n_s}}\;\frac{(q^c)_{N+n_s}}{(q^c)_{m+n_{s+1}}(q^c)_{N-m-n_{s+1}+n_s}}\;(q^{-mc})_{n_s}\,.$$

Note that in the first factor the denominator divides the numerator due to (24), and in the second factor because of the binomial integrality. $\square$

2.2 Diagonalization of the linking matrix

We say that a closed 3–manifold is of diagonal type if it can be obtained by integral surgery along an algebraically split link.

PROPOSITION 2.7. *Suppose M is a closed 3–manifold. There exist lens spaces $M_1, \dots, M_k$ of the form $L(2^l, a)$ such that the connected sum of $(M\#M)$ and these lens spaces is of diagonal type.*

We modify the proof of a similar result in [11].

2.2.1 *Linking pairing* Recall that *linking pairing* on a finite Abelian group G is a non–singular symmetric bilinear map from $G \times G$ to $\mathbb{Q}/\mathbb{Z}$. Two linking pairings ν, ν' on respectively G, G' are isomorphic if there is an isomorphism between G and G' carrying ν to ν'. With the obvious block sum, the set of equivalence classes of linking pairings is a semigroup.

One type of linking pairing is given by non–singular square symmetric matrices with integer entries: any such $n\times n$ matrix A gives rise to a linking pairing $\phi(A)$ on $G = \mathbb{Z}^n/A\mathbb{Z}^n$ defined by $\phi(A)(v, v') = v^t A^{-1} v' \in \mathbb{Q} \mod \mathbb{Z}$, where $v, v' \in \mathbb{Z}^n$. If there is a *diagonal* matrix A such that a linking pairing ν is isomorphic to $\phi(A)$, then we say that ν is *of diagonal type*.

Another type of pairing is the pairing $\phi_{b,a}$, with a, b non–zero co–prime integers, defined on the cyclic group $\mathbb{Z}/b$ by $\phi_{b,a}(x, y) = axy/b \mod \mathbb{Z}$. It is clear that $\phi_{b,\pm1}$ is also of the former type, namely, $\phi_{b,\pm1} = \phi(\pm b)$, where $(\pm b)$ is considered as the 1×1 matrix with entry $\pm b$.

PROPOSITION 2.8. *Suppose ν is a linking pairing on a finite group G. There are pairs of integers $(b_j, a_j), j = 1, \dots, s$ with b_j a power of 2 and a_j either -1 or 3, such that the block sum of $\nu \oplus \nu$ and all the ϕ_{b_j,a_j} is of diagonal type.*

Proof. The following pairings, in 3 groups, generates the semigroup of linking pairings, see [14, 21]:

Group 1: $\phi(\pm p^k)$, where p is a prime, and $k > 0$.

Group 2: $\phi_{b,a}$ with $b = p^k$ as in group 1, and a is a non–quadratic residue modulo p if p is odd, or $a = \pm3$ if $p = 2$.

Group 3: E_0^k on the group $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ with $k \geqslant 1$ and E_1^k on the group $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ with $k \geqslant 2$.

For explicit formulas of E_0^k and E_1^k, see [14]. We will use only a few relations between these generators, taken from [14, 21].

Any pairing ϕ in group 1 is already diagonal by definition, hence $\phi\oplus\phi$ is also diagonal.

A pairing $\phi = \phi_{b,a}$ in group 2 might not be diagonal, but its double $\phi \oplus \phi$ is always so: Suppose b is odd, then one of the relations is $\phi_{b,a} \oplus \phi_{b,a} = \phi(b) \oplus \phi(b)$, which is diagonal type. Suppose b is even, then $b = 2^k$, $a = \pm3$, and one of the relations says $\phi_{b,\pm3} \oplus \phi_{b,\pm3} = \phi(\mp b) \oplus \phi(\mp b)$.

Thus $\nu \oplus \nu$ is the sum of a diagonal linking pairing and generators of group 3.

Some of the relations concerning group 3 generators are

$$E_0^k \oplus \phi_{2^k,-1} = \phi(2^k) \oplus \phi(-2^k) \oplus \phi(-2^k)$$
$$E_1^k \oplus \phi_{2^k,3} = \phi(2^k) \oplus \phi(2^k) \oplus \phi(2^k).$$

Thus by adding to $\nu \oplus \nu$ pairings of the forms $\phi_{2^k,a}$ with $a=-1$ or $a=3$, we get a new linking pairing which is diagonal. $\square$

2.2.2 *Proof of Proposition 2.7* Every closed 3–manifold M defines a linking pairing, which is the linking pairing on the torsion of $H_1(M,\mathbb{Z})$. Connected sum of 3–manifolds corresponds to block sum of linking pairings.

First suppose M is a rational homology 3–sphere, i.e. M is obtained from S^3 by surgery along a framed oriented link L, with non–degenerate linking matrix A. Then the linking pairing on $H_1(M,\mathbb{Z})$ is exactly $\phi(A)$. Also, the lens space $L(b,a)$ has linking pairing $\phi_{b,a}$. Proposition 2.7 follows now from Proposition 2.8 and the well–known fact that if the linking pairing on $H_1(M,\mathbb{Z})$ is of diagonal type, then M is of diagonal type, see [19, 11].

The case when M has higher first Betti number reduces to the case of rational homology 3–spheres just as in [11].

2.3 Proof of Theorem 1 (general case)

LEMMA 2.9. *Suppose $(a,r)=1$, and $M=L(a,b)$, then lens space. Then $\tau_M(\xi)\in\mathbb{Z}[\xi]$ and moreover, $\tau_M(\xi)$ is invertible in $\mathbb{Z}[\xi]$.*

Proof. This follows from the explicit formula for the $SO(3)$ invariant of a lens space given below (26). Note that if $a^*a=1 \pmod r$, then

$$\frac{1-\xi}{1-\xi^{a^*}}=\frac{1-\xi^{aa^*}}{1-\xi^{a^*}}\,.$$

$\square$

Proof of Theorem 1 (general case) Choose the lens space $M_1,\dots,M_k$ as in Proposition 2.7. Since $N := M\#M\#M_1\#\dots\#M_k$ is of diagonal type, its $SO(3)$ invariant is in $\mathbb{Z}[\xi]$. Note that the orders of the first homology of $M_1,\dots,M_k$ are powers of 2, and hence co–prime with r. Lemma 2.9 shows that the $SO(3)$ invariant of $M\#M$ is in $\mathbb{Z}[\xi]$, and by Lemma 2.2, the $SO(3)$ invariant of M is in $\mathbb{Z}[\xi]$ too. $\square$

3. Rational surgery formula

3.1 Hopf chain

Let a,b be co–prime integers with $b>0$. It is well known that rational surgery with parameter a/b over a link component can be achieved by shackling that component with a framed Hopf chain and then performing integral surgery, in which the framings $m_{1,2,\dots,n}$ are related to a/b via a continued fraction expansion:

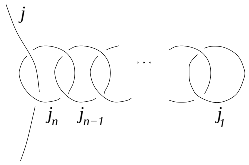
$$\frac{a}{b}=-\cfrac{1}{m_n-\cfrac{1}{m_{n-1}-\dots\cfrac{1}{m_2-\cfrac{1}{m_1}}}}$$

Let $D:=(F_{U^+}(\xi))^{\sigma_+^H}\,(F_{U^-}(\xi))^{\sigma_-^H}$ where $\sigma_\pm^H$ is the number of the (positive/negative) eigenvalues of the linking matrix for the Hopf chain. Let $\left(\frac{d}{r}\right)$ be the Jacobi symbol and $s(b,a)$ the Dedekind sum. Recall that

$$s(b,a) := \sum_{i=1}^{|a|-1} \left(\left(\frac{i}{a}\right)\right)\left(\left(\frac{ib}{a}\right)\right), \quad \text{where } ((x)) := x - \lfloor x \rfloor - 1/2.$$

Lemma 3.1. *For odd r with $(b,r)=1$, we have*

$$\frac{\mathrm{ev}_\xi\left([j]\right)}{D} \sum_{j_1,\dots,j_n}^{\xi} \prod_{i=1}^{n} q^{m_i \frac{j_i^2-1}{4}} [j_i]$$



$$= \mathrm{ev}_\xi\left(\left(\frac{b}{r}\right) q^{3s(a,b)} \left[\frac{j}{b}\right] q^{\frac{a(j^2-1)}{4b}}\right)$$

Proof. The colored Jones polynomial of the (j_1,j_2)–colored Hopf link is $[j_1 j_2]$. Thus we have to compute

$$\sum_{j_1,\dots,j_n}^{\xi} q^{\sum_i m_i \frac{j_i^2-1}{4}} (q^{\frac{j_1}{2}} - q^{-\frac{j_1}{2}})(q^{\frac{j_1 j_2}{2}} - q^{-\frac{j_1 j_2}{2}}) \dots (q^{\frac{j_n j}{2}} - q^{-\frac{j_n j}{2}})$$

The result is given by Lemma 4.12 in [13], where $A = \xi^{1/4}$ has the same order as ξ, because r is odd. Moreover, p and q in [13] are related to our parameters as follows: $a = -q$ and $b = p$. Computations analogous to Lemmas 4.15–4.21 in [13] imply the result. □

If $(r,a)=1$, the $SO(3)$ invariant of the lens space $L(a,b)$, which is obtained by surgery along the unknot with rational framing a/b, can be easily computed.

$$\tau_{L(a,b)}(\xi) = \left(\frac{a}{r}\right) \mathrm{ev}_\xi \left(q^{-3s(b,a)} \frac{q^{1/2a} - q^{-1/2a}}{q^{1/2} - q^{-1/2}} \right). \tag{26}$$

Here we used the Dedekind reciprocity law (see e.g. [9]), where $\mathrm{sn}(d)$ is the sign of d,

$$12\left(s(a,b) + s(b,a)\right) = \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} - 3\,\mathrm{sn}(ab)\,, \tag{27}$$

multiplicativity of the Jacobi symbols $\left(\frac{ab}{r}\right) = \left(\frac{a}{r}\right)\left(\frac{b}{r}\right)$ and

$$\frac{\gamma_d(\xi)}{\gamma_{\mathrm{sn}(d)}(\xi)} = \left(\frac{|d|}{r}\right) \mathrm{ev}_\xi(q^{(\mathrm{sn}(d)-d)/4}). \tag{28}$$

which holds for any nonzero integer d. Note that $\tau_{L(a,b)}(\xi)$ is invertible in $\mathbb{Z}[\xi]$.

3.2 Laplace transform

Laplace transform method, developed in [4], allows us to construct unified invariant by computing the Laplace transform of $\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}[n]$, and by proving its divisibility by $\frac{\{2k+1\}!}{\{k\}!}$. Let us show how this strategy works for rational framings.

Suppose that one component of L has rational framing a/b. Then by Lemma 3.1 we have to compute

$$\mathcal{L}_{a/b;n}\left(\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}\{k\}!\left\{\frac{n}{b}\right\}\right) = \frac{\{k\}!}{\{2k+1\}!}\mathcal{L}_{a/b;n}\left(\frac{\{n/b\}\,\{n+k\}!}{\{n-k-1\}!}\right).$$

Let $Y_k(q,n,b) := \{n/b\}\,\{n+k\}!/\{n-k-1\}!$. One can easily see that $Y_k(q,n,b) = Y_k(q,-n,b)$ and $Y_k(q,n,b) = Y_k(q^{-1},n,b)$. This implies for $H_k(q,a/b) := \mathcal{L}_{-a/b;n}(Y_k(q,n,b))$ that

$$H_k(q,a/b) = H_k(q^{-1},-a/b)\,.$$

Therefore, it is sufficient to compute $H_k(q,a/b)$ for $a>0$.

3.3 Divisibility of the Laplace transform images

Proposition 3.2. *For $a, b \in \mathbb{N}$, $(a, r) = 1$, $(b, r) = 1$ and $k \leqslant (r-3)/2$, we have*

$$\sum_n^{\xi} q^{\frac{a(1-n^2)}{4b}} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix} \{k\}! \left\{ \frac{n}{b} \right\} = 2\, q^{\frac{(b-1)^2}{4ab}} \gamma_{-a/b}(\xi)\, \mathrm{ev}_{\xi}(F_k(q,a,b))\,.$$

where $F_k(q,a,b) \in q^{\frac{(3k+2)(k+1)}{4}} \mathcal{R}_{a,2k+1}$.

A similar formula in the case $b = 1$ was obtained in [11]. Proposition 3.2 implies that

$$F_k(q,a,1) = \frac{\{k\}!}{\{2k+1\}!} Y(k,a) \tag{29}$$

where $Y(k,a)$ was defined in [11] as follows.

$$Y(k,a) := \sum_{j=0}^{2k+1} (-1)^j \begin{bmatrix} 2k+1 \\ j \end{bmatrix} q^{(j-k)^2/a} \tag{30}$$

The proof of Proposition 3.2 is given in Appendix. In the rest of the section we define $F_k(q,a,b)$. We put

$$C_{k,a,b} = (-1)^k q^{\frac{(5k+2)(k+1)}{4}} t^{\frac{k(k+1)}{2}(2b-3)} \frac{(t;t)_{2k+1}}{(q;q)_{2k+1}}$$

Let w be a primitive root of unity of order a. Let t be the a–th primitive root of q, i.e. $t^a = q$. We use the following notation $(q^x)_y = (q^x;q)_y$, $(t^x)_y = (t^x;t)_y$ and $(w^{\pm i} t^y)_x = (w^i t^y;t)_x (w^{-i} t^y;t)_x$.

Case a is odd. For odd a we define $c = (a-1)/2$, $l = c+b-1$, $x_i = \sum_{j=1}^{c-1+i} m_j$.

$$\frac{F_k(q,a,b)}{C_{k,a,b}} := \sum_{m_1,\dots,m_l \geqslant 0, x_b \leqslant k} (-1)^{m_1} t^{-\frac{m_1(m_1+1)}{2} + \sum_{i=1}^b x_i(x_i-1)} \frac{(t^{4k+2})^{m_{c-1}+2m_{c-2}+\dots+(c-1)m_1}}{(t^{2k})^{m_l+2m_{l-1}+\dots+lm_1}}$$
$$\frac{(q^{-k})_{k-m_1} (t^{2k+2})_{m_1} (t^{2k+2})_{m_2} \dots (t^{2k+2})_{m_c}}{(t)_{k-x_b} (t)_{m_2} (t)_{m_3} \dots (t)_{m_l}} \frac{(w^{\pm 2} t^{-2k-1})_{m_1} (w^{\pm 3} t^{-2k-1})_{m_1+m_2} \dots (w^{\pm c} t^{-2k-1})_{x_0}}{(w^{\pm 2} t^{m_1+1})_{m_2} (w^{\pm 3} t^{m_1+1})_{m_2+m_3} \dots (w^{\pm c} t^{m_1+1})_{x_1-m_1}} \tag{31}$$

Case a is even. For even a we define $c = a/2-1$, $l = c+b$, $x_i = \sum_{j=1}^{c+i} m_j$.

$$\frac{F_k(q,a,b)}{C_{k,a,b}} := \sum_{m_1,\dots,m_l \geqslant 0, x_b \leqslant k} (-1)^{m_1} t^{-\frac{m_1(m_1+1)}{2} + \sum_{i=1}^b x_i(x_i-1)} \frac{(t^{4k+2})^{m_{c-1}+2m_{c-2}+\dots+(c-1)m_1}}{(t^{2k})^{m_l+2m_{l-1}+\dots+lm_1}}$$
$$\frac{(t^{3k+1})^{x_{-1}} (-w^c t^{4k+2})^{m_c} \;\; (q^{-k})_{k-m_1} (t^{2k+2})_{m_1} \dots (t^{2k+2})_{m_{c-1}} (-w^{-c} t^{k+1})_{m_c} (-w^c t^{2k+2})_{m_{c+1}}}{(t)_{k-x_b} (t)_{m_2} (t)_{m_3} \dots (t)_{m_l}}$$
$$\frac{(w^{\pm 2} t^{-2k-1})_{m_1} (w^{\pm 3} t^{-2k-1})_{m_1+m_2} \dots (w^c t^{-2k-1})_{x_{-1}} (w^{-c} t^{-2k-1})_{x_0} (-t^{-2k-1})_{x_0}}{(w^{\pm 2} t^{m_1+1})_{m_2} (w^{\pm 3} t^{m_1+1})_{m_2+m_3} \dots (w^c t^{m_1+1})_{x_0-m_1} (-t^{-k+x_{-1}})_{m_c} (w^{-c} t^{m_1+1})_{x_1-m_1} (-t^{m_1+1})_{x_1-m_1}} \tag{32}$$

Example.

$$F_k(q,1,b) := q^{-\frac{(3k-2)(k+1)}{4}} q^{kb(k+1)} \sum_{m_1,m_2,\dots,m_b \geqslant 0,\, \sum m_i = k} q^{\sum_{i=1}^{b-1}(x_i^2 - (2k+1)x_i)} \frac{(q)_k}{\prod_{i=1}^b (q)_{m_i}} \tag{33}$$

where $x_p = \sum_{i=1}^p m_i$.

Note that (33) coincides up to units with the formula for the coefficient $c'_{k,b}$ in the decomposition of ω^b computed in [15, (46)]. (The same coefficient (up to units) appears also in the cyclotomic expansion of the Jones polynomial of twist knots (35)). This is because surgery on the $(-1/b)$–framed component can be achieved by replacing this component by b (-1)–framed copies. Indeed, changing the variables in (33) as follows: $s_1 = k - m_1$, $s_2 = k - m_1 - m_2, ..., s_{b-1} = k - m_1 - \cdots - m_{b-1}$, we get

$$F_k(q,1,b) = q^{\frac{(k+2)(k+1)}{4}} \sum_{k \geqslant s_1 \geqslant s_2 \geqslant \cdots \geqslant s_{b-1} \geqslant 0} q^{s_1^2 + s_2^2 + \cdots + s_{b-1}^2 + s_1 + \cdots + s_{b-1}} \frac{(q)_k}{\prod_{i=1}^{b-1} (q)_{s_i - s_{i+1}}} .$$

4. Universal invariant

In this section we assume that $(r, a) = 1$, where r is the order of the root of unity ξ and $a = |H_1(M, \mathbb{Z})|$.

Let $M = L(a, b)$ be a lens space with $a > 0$. Then the universal invariant I_M was defined in [11] as follows.

$$I_M := q^{3s(1,a) - 3s(b,a)} \frac{1 - q^{-1/a}}{1 - q^{-1}}.$$

Note that $3(s(1, a) - s(b, a)) \in \mathbb{Z}$ and I_M is invertible in Λ_a.

For an arbitrary rational homology sphere M with $a = |H_1(M, \mathbb{Z})|$, it was shown in [11] that there are lens spaces $M_1, \ldots, M_l$ such that $M' = (\#_{i=1}^l M_i)\#M$ can be obtained by surgery on an algebraically split link and I_{M_j} are invertible in Λ_a. Then we can define

$$I_M = I_{M'} (\prod_{i=1}^{l} I_{M_i})^{-1}.$$

It remains to define I_M when M is given by surgery along an algebraically split link L. Assume L has m components with nonzero rational framings $\frac{a_1}{b_1}, \ldots \frac{a_m}{b_m}$. Then we have $|H_1(M, \mathbb{Z})| = a$ for $a = \prod_i a_i$. Let L_0 be L with all framings switched to zero.

THEOREM 7. *For M as above, the unified invariant is given by the following formula.*

$$I_M = q^{(a-1)/4} \sum_{k_i=0}^{\infty} J_{L_0}(P'_{k_1}, \ldots, P'_{k_m}) \prod_{i=1}^{m} \operatorname{sn}(a_i) q^{\frac{1}{2a_i} - 3s(b_i, a_i)} F_{k_i}(q^{-\operatorname{sn}(a_i)}, |a_i|, b_i) \qquad (34)$$

Moreover,

$$\left(\frac{a}{r}\right) \tau_M(\xi) = \operatorname{ev}_\xi(q^{(1-a)/4} I_M)$$

Proof. Note first that, if $b_i = 1$ for all i, our formula coincides with (21) in [11]. It follows from (29) and

$$q^{\frac{3\operatorname{sn}(a_i) - a_i}{4}} q^{3s(1,a_i)} = q^{\frac{1}{2a_i}} .$$

Here we used that $3s(1, a) = \frac{1}{2a} + \frac{a - 3\operatorname{sn}(a)}{4}$ by the reciprocity law (27).

Let us collect the coefficients in the definition of τ_M. From Lemmas 1.1, 3.1, Proposition 3.2 and (28) we have

$$q^{3s(a_i,b_i) - \frac{(b_i-1)^2}{4a_i b_i} + \frac{3\operatorname{sn}(a_i)}{4} - \frac{a_i}{4b_i}} = q^{-3s(b_i,a_i)} q^{\frac{1}{2a_i}}$$

The Corollary 0.3 (d) in [11] allows to drop the conditions $(b_i, r) = 1$, because I_M is determined by its values at any infinite sequence of roots of prime power order from U_a. □

4.1 Proof of Theorem 2

The statement holds trivially if $M = L(a, b)$. Indeed, we have $m = 1$, $f_k = 0$ for $k > 0$, and $f_0 = q^{3s(1,a)-3s(b,a)}(1-t)/(1-q) \in \Gamma_a$.

The general case follows from (34), Proposition 3.2 and Theorem 5. Note that multiplication of I_M by the inverse of $I_{L(a,b)}$ multiply all f_{k_i} by an element of $\mathbb{Z}[t^{\pm 1}]$. Moreover, I_M does not contain fractional powers of $q^{1/a}$ (compare Proof of Lemma 4.2 in [11]). □

4.2 On the ring $\widehat{\mathcal{R}}_a$

Here we present the proof (of the referee) that $\widehat{\mathcal{R}}_a$ coincides with the ring $\hat{\Lambda}_a$ of [11]. The ring $\hat{\Lambda}_a$ in [11] is obtained from $\mathcal{R}_a$ by first inverting a, and then completing using $(q;q)_n$:

$$\hat{\Lambda}_a := \varprojlim_n \mathcal{R}_a[1/a]/((q;q)_n).$$

To prove $\hat{\Lambda}_a = \widehat{\mathcal{R}}_a$ one needs only to show that a is invertible in $\widehat{\mathcal{R}}_a$.

Suppose p is a prime factor of a. For any integer $l \in \mathbb{N}_a$ we have $\Phi_{pl}(t) \in (p, \Phi_l(t))$ in $\mathbb{Z}[t^{\pm 1}]$. Since $pl \notin \mathbb{N}_a$, Φ_{pl} is invertible in $\mathcal{R}_a$. Therefore $(p, \Phi_l(t)) = 1$ in $\mathcal{R}_a$, or p is invertible in $\mathcal{R}_a/(\Phi_l)$. Since

$$t^m - 1 = \prod_{l|m} \Phi_l(t),$$

it follows that if $m \in \mathbb{N}_a$, then p is invertible in $\mathcal{R}_a/((t^m-1)^j)$ for every $j \geqslant 1$. Hence p is invertible in the completion of $\mathcal{R}_a$ with respect to the directed system of ideals $\{(t^m - 1)^j\mathcal{R}_a\}_{j\geqslant 1, m\in\mathbb{N}_a}$. Note that $\{(q;q)_n\mathcal{R}_a\}_{n\geqslant 1}$ and $\{(t^m-1)^j\mathcal{R}_a\}_{j\geqslant 1, m\in\mathbb{N}_a}$ are cofinal, hence they define the same completion. This completes the proof.

4.3 Proof of Proposition 1

First note that if $f = 1/a$, then $f \in \widehat{\mathcal{R}}_a$, and $\mathrm{ev}_\xi(f) = 1/a$. It follows that $\mathrm{ev}_\xi \widehat{\mathcal{R}}_a = \mathbb{Z}[\xi/a]$.

Assume the order of $\xi \in U_a$ is r. Suppose $f \in \Gamma_a$ has a presentation given by formula (1). Since $\mathrm{ev}_\xi(x_n) = 0$ if $2n + 1 \geqslant r$, we have

$$\mathrm{ev}_\xi(f) = \sum_{n=0}^{(r-3)/2} \mathrm{ev}_\xi(f_n(t))\, \mathrm{ev}_\xi(x_n).$$

Since $x_n \in \mathbb{Z}[q^{\pm 1}]$, $\mathrm{ev}_\xi(x_n) \in \mathbb{Z}[\xi]$. We will show that $\mathrm{ev}_\xi(f_n(t)) \in \mathbb{Z}[\xi]$ for $n \leqslant (r-3)/2$.

Note that $f_n(t) \in \mathcal{R}_{a,k} = \mathbb{Z}[t^{\pm 1}]\left[\frac{(t;t)_k}{(q;q)_k}\right]$, with $k = r - 1$. Since $k < r$, $\mathrm{ev}_\xi((q;q)_k) \neq 0$. Hence, with an integer a^* such that $aa^* \equiv 1 \pmod{r}$, we have

$$\mathrm{ev}_\xi\left(\frac{(t;t)_k}{(q;q)_k}\right) = \prod_{j=1}^{k} \frac{1 - \xi^{a^* j}}{1 - \xi^j} \in \mathbb{Z}[\xi].$$

It follows that $\mathrm{ev}_\xi(f_n(t)) \in \mathbb{Z}[\xi]$ for every $f_n(t) \in \mathcal{R}_{a,k}$. This completes the proof of Proposition 1. □

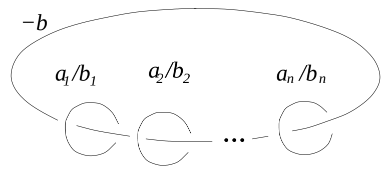
5. Applications

In this section we compute the universal invariant I_M for Seifert fibered spaces and for a/b surgeries on twist knots.

5.1 Seifert fibered spaces with a spherical base

Let $M = L(b; a_1/b_1, \ldots, a_n/b_n)$ be the Seifert fibered space with base space S^2 and with n exceptional fibers with orbit invariants (a_i, b_i) $(a_i > 0, 0 \leqslant b_i \leqslant a_i,\ (a_i, b_i) = 1)$, and with bundle invariant $b \in \mathbb{Z}$.

It is well–known that M is a rational homology sphere if $e := b + \sum b_i/a_i \neq 0$ and $|H_1(M, \mathbb{Z})| = |e| \prod_i a_i$. Furthermore, M can be obtained by surgery on the following (rationally framed) link.



THEOREM 8. *Let $M = L(b; a_1/b_1, \ldots, a_n/b_n)$ as above. Assume $e \neq 0$, and $|H_1(M, \mathbb{Z})| = d$.*

$$I_M = q^{\frac{d-1}{4}} \frac{q^{(e-3\,\mathrm{sn}(e))/4} q^{-3\sum_i s(b_i,a_i)}}{\{1\}} \mathcal{L}_{-e;j} \left(\frac{\prod_{i=1}^n \left\{ \frac{j}{a_i} \right\}}{\{j\}^{n-2}} \right)$$

Proof. The linking matrix of the surgery link has n positive eigenvalues and the sign of the last eigenvalue is equal to $-\,\mathrm{sn}(e) = -\,\mathrm{sn}(b)$. Let us color the rationally framed components of the surgery link by j_i, $i = 1, \ldots, n$ and the $-b$ framed component by j.

The main ingredient of the proof is the following computation. Using Lemmas 1.1, 3.1 we have

$$\left(\frac{b_i}{r}\right) \frac{q^{3s(a_i,b_i)}}{F_{U^+(\xi)}} \sum^{\xi} q^{\frac{a_i(j_i^2-1)}{4b_i}} \begin{bmatrix} j_i \\ b_i \end{bmatrix} [j j_i] = \left(\frac{a_i}{r}\right) q^{-3s(b_i,a_i)} q^{-\frac{b_i(j^2-1)}{4a_i}} \begin{bmatrix} j \\ a_i \end{bmatrix}$$

Applying finally the Laplace transform $\mathcal{L}_{-e;j}$ and collecting the factors we get the result. $\square$

5.2 Proof of Theorem 3

Note that $M = L(b; a_1/b_1, \ldots, a_n/b_n)$ is an integral homology sphere if $e^{-1} = \pm \prod_i a_i$. M is uniquely determined by the pairwise co–prime integers a_i. (Knowing a_i's and e, one can compute b_i's and b using the Chinese remainder theorem).

Suppose for simplicity that $e > 0$. Rewriting

$$\frac{1}{\{j\}^{n-2}} = (-1)^{n-2} q^{(n-2)/2} \sum_{k=0} c_k q^k$$

with $c_k \in \mathbb{Z}$, we see that the image of the Laplace transform is the sum of the following terms:

$$(-1)^{n-2} c_k\, q^{\frac{\prod_i a_i}{4}(\pm 1/a_1 \pm 1/a_2 \cdots \pm 1/a_n + 2k + n - 2)^2}$$

The leading term in I_M for $k \to \infty$ behaves asymptotically like

$$q^{k^2 \prod_i a_i + k(n-2)\prod_i a_i + k \sum_i a_1 \ldots \hat{a}_i \ldots a_n}$$

where $\hat{a}_i$ means delete a_i. This allows to determine the a_i's. In the case $e<0$, we have the same asymptotic after replacing q by q^{-1}. $\square$

5.3 Dehn surgeries on twist knots

Let K_p be the twist knot with p twists. Masbaum [15] calculated the P'_n colored Jones polynomial of this knot. For $p>0$ we have

$$J_{K_p}(P'_n) = q^{n(n+3)/2} \sum_{i_1,i_2,\dots,i_p\geqslant 0, \sum_j i_j = p} q^{\sum_i (s_i^2+s_i)} \frac{(q)_n}{\prod_{j=1}^p (q)_{i_j}} \tag{35}$$

where $s_k = \sum_{j=1}^k i_j$. The formula for the negative p can be obtained from the given one by sending $p\to -p$, $q\to q^{-1}$, forgetting the factor $q^{n(n+3)/2}$ and multiplying the result by $(-1)^n$.

Corollary 5.1. *Let $M_{a/b}$ is obtained by (a/b) surgery in S^3 on the twist knot K_p. Then*

$$I_{M_{a/b}} := q^{(a-1)/4}\,\mathrm{sn}(a) q^{-3s(b,a)+\frac{1}{2a}} \sum_{n=0}^{\infty} J_{K_p}(P'_n) F_n(q^{-\mathrm{sn}(a)}, |a|, b) \tag{36}$$

5.4 Proof of Theorem 4

Assume K and K' are 0–framed. We expand the function $Q_K(N) := J_K(N)[N]$ around $q=1$ into power series. Suppose $q=e^h$, then we have

$$Q_K(N)|_{q=e^h} = \sum_{2j\leqslant n+2} c_{j,n}(K) N^j h^n \,.$$

It is known that $c_{j,n}$ is zero if j is odd. Applying Laplace transform, we have to replace N^{2j} by $\frac{(-2)^j(2j-1)!!}{a^j h^j}$ (see [12]). Therefore, the following expression coincides (up to some standard factor) with the Ohtsuki series

$$\sum_{2j\leqslant n+2} c_{2j,n}(K)(-2)^j(2j-1)!!\; a^{-j}h^{n-j}\,.$$

From the fact that the Ohtsuki series for $M(K,a)$ and $M(K',a)$ coincide, we derive

$$\sum_{2j\leqslant n+2} \big(c_{2j,n}(K) - c_{2j,n}(K')\big)(-2)^j(2j-1)!!\; a^{-j}h^{n-j} = 0\,.$$

Because the last system of equations should hold for infinitely many $a\in\mathbb{Z}$, we have $c_{2j,n}(K) = c_{2j,n}(K')$ and $J_K(N) = J_{K'}(N)$ for any $N\in\mathbb{N}$. $\square$

Appendix

The main technical ingredient we use in the proof of Proposition 3.2 is the Andrews's generalization of Watson's identity ([3, Theorem 4, p.199]):

$$ {}_{2p+4}\phi_{2p+3}\left[\begin{matrix} \alpha, t\sqrt{\alpha}, -t\sqrt{\alpha}, b_1, c_1, \dots, b_p, c_p, t^{-N} \\ \sqrt{\alpha}, -\sqrt{\alpha}, \alpha t/b_1, \alpha t/c_1, \dots, \alpha t/b_p, \alpha t/c_p, \alpha t^{N+1} \end{matrix}\; ; t, \frac{\alpha^p t^{p+N}}{b_1c_1\dots b_pc_p}\right] = \frac{(\alpha t)_N(\alpha t/b_pc_p)_N}{(\alpha t/b_p)_N(\alpha t/c_p)_N}$$
$$\sum_{m_1,\dots,m_{p-1}\geqslant 0} \frac{(b_p)_{\sum_i m_i}(c_p)_{\sum_i m_i}(t^{-N})_{\sum_i m_i}}{(b_pc_pt^{-N}/\alpha)_{\sum_i m_i}} \prod_{i=1}^{p-1} \frac{t^{m_i}(\alpha t)^{(p-i-1)m_i}(\alpha t/b_ic_i)_{m_i}(b_i)_{\sum_{j<i}m_j}(c_i)_{\sum_{j<i}m_j}}{(t)_{m_i}(\alpha t/b_i)_{\sum_{j\leqslant i}m_j}(\alpha t/c_i)_{\sum_{j\leqslant i}m_j}(b_ic_i)^{\sum_{j<i}m_j}} \tag{37}$$

where

$$
{}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; t, z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(t)_n (b_1)_n \dots (b_s)_n} \left[(-1)^n t^{\binom{n}{2}} \right]^{1+s-r} z^n \tag{38}
$$

are the basic q–hypergeometric series and $(a)_n := (a;t)_n$.

Proof of Proposition 3.2

We have to compute $\mathcal{L}_{-a/b;n}(\{n/b\}\,\{n+k\}!/\{n-k-1\}!)$. Note

$$
\{n/b\}\,\{n+k\}!/\{n-k-1\}! = q^{-n/2-nk-n/(2b)}(1-q^{n/b})(q^{n-k})_{2k+1}\,.
$$

Using the q–binomial theorem and (5) (with $c=1$) we get

$$
q^{\frac{(2bk+b+1)^2}{4ba}} \sum_{j=0}^{\infty} \frac{(q^{-2k-1})_j}{(q)_j} q^{\frac{b}{a}j^2+(1-2\frac{b}{a})kj+(1-\frac{(b+1)}{a})j}(1-q^{\frac{2j-2k-1}{a}})
$$

We put $t := q^{1/a}$ and choose a primitive a–th root of unity w. Then using

$$
(q^x;q)_j = (t^x;t)_j (wt^x;t)_j (w^2t^x;t)_j \dots (w^{a-1}t^x;t)_j
$$

we can rewrite the previous sum as follows.

$$
\sum_{j=0}^{2k+1} \frac{(t^{-2k-1})_j (wt^{-2k-1})_j \dots (w^{a-1}t^{-2k-1})_j}{(t)_j (wt)_j \dots (w^{a-1}t)_j} t^{bj^2+(a-2b)kj+(a-b-1)j}(1-t^{2j-2k-1}) \tag{39}
$$

The main point is that (39) is equal to $(1-t^{-2k-1})$ times the LHS of the generalized Watson identity (37) with the specialization of parameters described below. We consider the limit $\alpha \to t^{-2k-1}$. We set $p = \max\{b, a+b-2\}$, $b_i, c_i \to \infty$ for $i = a-1, ..., p-1$; $b_p \to t^{-k}$, $c_p \to \infty$ and $N \to \infty$.

Case a is odd. We put $c = (a-1)/2$; $b_i = w^i t^{-2k-1}$, $c_i = w^{-i}t^{-2k-1}$ for $i = 1, ..., c$; $b_i, c_i \to t^{-k}$ for $i = c+1, ..., a-2$.

Case a is even. We put $c = a/2 - 1$. Let $p = a+b-2$, $b_i = w^i t^{-2k-1}$, $c_i = w^{-i}t^{-2k-1}$ for $i = 1, ..., c-1$; $b_c = w^c t^{-2k-1}$, $c_c = -t^{-k}$; $b_{c+1} = -t^{-2k-1}$, $c_{c+1} = w^{-c}t^{-2k-1}$; $b_i, c_i \to t^{-k}$ for $i = c+2, ..., a-2$.

To simplify (37) we use the following limits.

$$
\lim_{c\to\infty} \frac{(c)_n}{c^n} = (-1)^n t^{n(n-1)/2} \qquad \lim_{c\to\infty} \left(\frac{t}{c}\right)_n = 1
$$

$$
\lim_{c_1,c_2\to\infty} \frac{(c_1)_n (c_2)_n}{(t^{-N}c_1c_2)_n} = (-1)^n t^{n(n-1)/2} t^{Nn} \qquad \lim_{\alpha\to t^{-2k-1}} \frac{(\alpha t)_\infty}{(\sqrt{\alpha t})_\infty} = 2(t^{-2k})_k
$$

Finally, the formulas below allow us to separate the factor $\frac{(t)_{2k+1}}{(q)_{2k+1}}$.

$$
\frac{\{2k+1\}!}{\{k\}!} = q^{-\frac{(3k+2)(k+1)}{4}}(-1)^{k+1}(q^{k+1})_{k+1}
$$

$$
(q)_j = (-1)^{(k-j)} q^{\frac{(j-k)(k+j+1)}{2}} \frac{(q)_k}{(q^{-k})_{k-j}}
$$

$$
(t^{-k})_j = (-1)^j t^{-kj+\frac{j(j-1)}{2}} \frac{(t)_k}{(t)_{k-j}}
$$

The next lemma implies the result. □

Lemma 5.2. $\frac{F_k(q,a,b)}{C_{k,a,b}} \in \mathbb{Z}[t^{\pm 1}]$.

Proof. First note that $F_k(q,a,b)$ does not depend on w, because in the LHS of the identity w does not occur.

Suppose a is odd. Let $z := x_1 - m_1 = m_2 + m_3 + \cdots + m_c$. Let us complete $(w^c t^{m_1+1})_z$ to $(q^{m_1+1})_z$ by multiplying the numerator and the denominator of (31) with

$$(t^{m_1+1})_z (w^{\pm} t^{m_1+1})_z (w^{\pm 2} t^{m_1+m_2})_{z-m_2} \dots (w^{\pm(c-1)} t^{m_1+z-m_c})_{m_c}\,.$$

Now up to units the denominator of (31) is equal to $(q^{m_1+1})_{x_1-m_1}(t)_{k-x_b}(t)_{m_2}(t)_{m_3}\dots(t)_{m_l}$ which divides the numerator. The even case is similar. □

References

1 J. Andersen, *The coloured Jones polynomials is an unknot detector*, to appear

2 G. Andrews, *q-series: their development and applications in analysis, number theory, combinatorics, physics, and computer algebra*, regional conference series in mathematics, 66 Amer. Math. Soc. 1985

3 G. Andrews, *Problems and prospects for basic hypergeometric functions*, in Theory and application of special functions, ed. R. Askey (1975) 191–224

4 A. Beliakova, C. Blanchet, T. T. Q. Le, *Unified quantum invariants and their refinements for homology 3–spheres with 2–torsion*, preprint `math.QA/0509394`.

5 D. Bar–Natan, R. Lawrence, *A rational surgery formula for the LMO invariant*, Israel J. Mathem. **140** (2004) 29–60

6 P. Gilmer, G. Masbaum, *Integral lattices in TQFT*, math.GT/0411029

7 K. Habiro, *On the quantum $\mathfrak{sl}_2$ invariants of knots and integral homology spheres*, Geom. Topol. Monogr. **4** (2002) 55–68; *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, math.GT/0605314

8 R. Kirby and P. Melvin, *The* 3*-manifold invariants of Witten and Reshetikhin-Turaev for* sl$(2, C)$, Invent. Math. **105** (1991), 473–545.

9 R. Kirby, P. Melvin, *Dedekind sums, μ-invariants and the signature cocycle*, Math. Ann. **299** (1994) 231–267

10 M. Khovanov, *Hopfological algebra and categorification at a root of unity: the first steps*, math.QA/0509083

11 T.T.Q. Lê, *Strong integrality of quantum invariants of 3–manifolds*, arXiv:math.GT/0512433, to appear in Trans. AMS.

12 T. T. Q. Lê, *Quantum invariants of 3–manifolds: integrality, splitting, and perturbative expansion*, Topology Appl. **127** (2003) 125–152.

13 B.-H. Li, T.-J. Li, *Generalized Gaussian sums: Chern-Simons-Witten-Jones invariants of lens-spaces*, J. Knot Theory Ramifications **5** (1996) 183–224.

14 A. Kawauchi and S. Kojima, *Algebraic classification of linking pairings on* 3*-manifolds*, Math. Ann. **253** (1980) 29–42

15 G. Masbaum, *Skein–theoretical derivation of some formulas of Habiro*, Algeb. Geom. Topology **3** (2003) 537–556

16 G. Masbaum and J. Roberts, *A simple proof of integrality of quantum invariants at prime roots of unity*, Math. Proc. Camb. Phil. Soc. **121** (1997) 443–454

17 J. M. Montesinos, *Classical tessellations and three–manifolds*, Springer 1985

18 H. Murakami, *Quantum $SO(3)$-invariants dominate the $SU(2)$-invariant of Casson and Walker*, Math. Proc. Camb. Phil. Soc. **117** (1995) 237–249

19 T. Ohtsuki, *A polynomial invariant of rational homology* 3*-spheres*, Invent. Math. **123** (1996), 241–257.

20 V. Turaev, *Quantum invariants of knots and 3–manifolds*, de Gruyter Studies in Math. 18, 1994

21 C. T. C. Wall, *Quadratic forms on finite groups, and related topics*, Topology **2** (1963) 281–298

Anna Beliakova anna@math.unizh.ch
Institut für Mathematik, Universität Zurich, Winterthurerstrasse 190, 8057 Zürich, Switzerland

Thang T. Q. Lê letu@math.gatech.edu
Department of Mathematics , Georgia Institute of Technology , Atlanta, GA 30332-0160, USA