Semiclassical Quantization of Classical Field Theories

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Abstract These lectures are an introduction to formal semiclassical quantization of classical field theory. First we develop the Hamiltonian formalism for classical field theories on space time with boundary. It does not have to be a cylinder as in the usual Hamiltonian framework. Then we outline formal semiclassical quantization in the finite dimensional case. Towards the end we give an example of such a quantization in the case of Abelian Chern-Simons theory.

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1 Introduction

The goal of these lectures is an introduction to the formal semiclassical quantization of classical gauge theories.

In high energy physics space time is traditionally treated as a flat Minkowski manifold without boundary. This is consistent with the fact the characteristic scale in high energy is so much smaller then any characteristic scale of the Universe.

As one of the main paradigms in quantum field theory, quantum fields are usually assigned to elementary particles. The corresponding classical field theories are described by relativistically invariant local action functionals. The locality of interactions between elementary particles is one of the key assumptions of a local quantum field theories and of the Standard Model itself.

The path integral formulation of quantum field theory makes it mathematically very similar to statistical mechanics. It also suggests that in order to understand the mathematical nature of local quantum field theory it is natural to extend this notion from Minkowski space time to a space time with boundary. It is definitely natural to do it for the corresponding classical field theories.

The concept of topological and conformal field theories on space time manifolds with boundary was advocated in [3, 28]. The renormalizability of local quantum field theory on a space time with boundary was studied earlier in [30]. Here we develop the gauge fixing approach for space time manifolds with boundary by adjusting the Faddeev-Popov (FP) framework to this setting. This gauge fixing approach is a particular case of the more general Batalin-Vilkovisky (BV) formalism for quantization of gauge theories. The classical Hamiltonian part of the BV quantization on space time manifolds with boundary, the BV-BFV formalism, is developed in [13]. In a subsequent publication we will extend it to the quantum level.

The goal of these notes is an overview of the FP framework in the context of space time manifolds with boundary. As a first step we present the Hamiltonian structure for such theories. We focus on the Hamiltonian formalism for first order theories. Other theories can be treated similarly, see for example [14] and references therein. In a subsequent publication we will connect this approach with the BV-BFV program.

In Sect. 2 we recall the concept of local quantum field theory as a functor from the category of space time cobordisms to the category of vector spaces. The Sect. 3 contains examples: the scalar field theory, Yang-Mills theory, Chern-Simons and BF theories. The concept of semiclassical quantization of first order quantum field theories is explained in Sect. 4 where we present a finite dimensional model for the gauge fixing for space time manifolds with or without boundary. In Sect. 5 we briefly discuss the example of Abelian Chern-Simons theory. The nonabelian case and the details of the gluing of partition functions for semiclassical Chern-Simons theories will be given elsewhere.

2 First Order Classical Field Theories

2.1 Space Time Categories

In order to define a classical field theory one has to specify a space time category, a space of fields for each space time and the action functional on the space of fields.

Two space time categories which are most important for these lectures are the category of smooth n-dimensional cobordisms and the category of smooth n-dimensional Riemannian manifolds.

The *d*-dimensional smooth category. Objects are smooth, compact, oriented $(d-1)$-dimensional manifolds with smooth d-dimensional collars. A morphism between Σ_1 and Σ_2 is a smooth d-dimensional compact oriented manifolds with $\partial M = \Sigma_1 \sqcup \overline{\Sigma_2}$ and the smooth structure on M agrees with smooth structure on collars near the boundary. The orientation on M should agree with the orientations of Σ_1 and be opposite to the one on Σ_2 in a natural way.

The composition consists of gluing two morphisms along the common boundary in such a way that collars with smooth structure on them fit.

In this and the subsequent examples of space time categories identity morphisms have to be adjoined formally. Note also that we are not taking the quotient of cobordisms by diffeomorphisms.

The *d*-dimensional Riemannian category. Objects are $(d-1)$-dimensional Riemannian manifolds with d-dimensional collars. Morphisms between two oriented $(d-1)$-dimensional Riemannian manifolds N_1 and N_2 are oriented d-dimensional Riemannian manifolds M with collars near the boundary, such that $\partial M = N_1 \sqcup \overline{N_2}$. The orientation on all three manifolds should naturally agree, and the metric on M agrees with the metric on N_1 and N_2 and on collar near the boundary. The composition is the gluing of such Riemannian cobordisms. For the details see [29].

This category is important for many reasons. One of them is that it is the underlying structure for statistical quantum field theories.

The *d*-dimensional pseudo-Riemannian category. The difference between this category and the Riemannian category is that morphisms are pseudo-Riemannian with the signature $(d-1, 1)$ while objects remain $(d-1)$-dimensional Riemannian. This is the most interesting category for particle physics.

Both objects and morphisms may have an extra structure such as a fiber bundle (or a sheaf) over it. In this case such structures for objects should agree with the structures for morphisms.

2.2 General Structure of First Order Theories

2.2.1 First Order Classical Field Theories

A first order classical field theory[1] is defined by the following data:

- A choice of space time category.
- A choice of the space of fields F_M for each space time manifold M. This comes together with the definition of the space of fields $F_{\partial M}$ for the boundary of the space time and the restriction mapping $\pi : F_M \to F_{\partial M}$.
- A choice of the action functional on the space F_M which is local and first order in derivatives of fields, i.e.

$$S_M(\phi) = \int_M L(d\phi, \phi)$$

  Here $L(d\phi, \phi)$ is linear in $d\phi$.

These data define:

- The space EL_M of solutions of the Euler-Lagrange equations.
- The 1-form $\alpha_{\partial M}$ on the space of boundary fields arising as the boundary term of the variation of the action [14].
- The Cauchy data subspace $C_{\partial M}$ of boundary values (at $\{0\} \times \partial M$) of solutions of the Euler-Lagrange equations in $[0, \epsilon) \times \partial M$.
- The subspace $L_M \subset C_{\partial M}$ of boundary values of solutions of the Euler-Lagrange equations in M, $L_M = \pi(EL_M)$.

When $C_{\partial M} \neq F_{\partial M}$ the Cauchy problem is overdetermined and therefore the action is degenerate. Typically it is degenerate because of the gauge symmetry.

A natural boundary condition for such system is given by a Lagrangian fibration[2] on the space of boundary fields such that the form $\alpha_{\partial M}$ vanishes at the fibers. The last conditions guarantees that solutions of Euler-Lagrange equations which are constrained to a leaf of such fibration are critical points of the action functional, i.e. not only the bulk term vanishes but also the boundary terms.

[1] It is not essential that we consider here only first order theories. Higher order theories where $L(d\phi, \phi)$ is not necessary a linear function in $d\phi$ can also be treated in a similar way, see for example [14] and references therein. In first order theories the space of boundary fields is the pull-back of fields in the bulk.

[2] In our examples, fibrations are actually fiber bundles. By abuse of terminology, terms "fibration" and "foliation" will be used interchangeably.

2.2.2 First Order Classical Field Theory as a Functor

First order classical field theory can be regarded as a functor from the category of space times to the category which we will call Euler-Lagrange category and will denote $\underline{EL}$. Here is an outline of this category:

An *object* of $\underline{EL}$ is a symplectic manifold F with a prequantum line bundle, i.e. a line bundle with a connection α_F, such that the symplectic form is the curvature of this connection. It should also have a Lagrangian foliation which is α_F-exact, i.e. the pull-back of α_F to each fiber vanishes.[3]

A *morphism* between F_1 and F_2 is a manifold F together with two surjective submersions $\pi_1 : F \to F_1$ and $\pi_2 : F \to F_2$, with a function S_F on F and with the subspace $EL \subset F$ such that $dS_F|_{EL}$ is the pull-back of $-\alpha_{F_1} + \alpha_{F_2}$ on $F_1 \times F_2$. The image of EL in $(F_1, -\omega_1) \times (F_2, \omega_2)$ is automatically an isotropic submanifold. Here $\omega_i = d\alpha_{F_i}$. We will focus on theories where these subspaces are Lagrangian.

The composition of morphisms (F, S_F) and $(F', S_{F'})$ is the fiber product of the morphism spaces F and F' over the intermediate object and $S_{F' \circ F} = S_F + S_{F'}$. This category is the $gh = 0$ part of the BV-BFV category from [13].

A first order classical field theory defines a functor from the space time category to the Euler-Lagrange category. An object N of the space time category is mapped to the space of fields F_N, a morphism M is mapped to (F_M, S_M), etc. Composition of morphisms is mapped to the fiber product of spaces of fields[4] and because of the assumption of locality of the action functional, it is additive with respect to the gluing.

This is just an outline of the Euler-Lagrange category and of the functor. For our purpose of constructing formal semiclassical quantization we will not need the precise details of this construction. But it is important to have this more general picture in mind.

2.3 Symmetries in First Order Classical Field Theories

The theory is relativistically invariant if the action is invariant with respect to geometric automorphisms of the space time. These are diffeomorphisms for the smooth category, isometries for the Riemannian category etc. In such theory the action is constructed using geometric operations such as de Rham differential and exterior multiplication of forms for smooth category. In Riemannian category in addition to these two operations we have Hodge star (or the metric).

If the space time category has an additional structure such as fiber bundle, the automorphisms of this additional structure give additional symmetries of the theory. In Yang-Mills, Chern-Simons and BF theories, gauge symmetry, or automorphisms

[3] Here we are assuming for simplicity of the exposition that the prequantum line bundle is trivial and thus we can identify the connection with its 1-form on F.

[4] We are not precise at this point. Rather, the value of the functor on a composition is *homotopic* (in the appropriate sense) to the fiber product.

of the corresponding principal G-bundle, are such a symmetry. A theory with such space time with the gauge invariant action is called gauge invariant. The Yang-Mills theory is gauge invariant, the Chern-Simons and the BF theories are gauge invariant only up to boundary terms.

There are more complicated symmetries when a distribution, not necessary integrable, is given on the space of fields and the action is annihilated by corresponding vector fields. Nonlinear Poisson σ-model is an example of such field theory [18].

3 Examples

3.1 First Order Lagrangian Mechanics

3.1.1 The Action and Boundary Conditions

In Lagrangian mechanics the main component which determines the dynamics is the Lagrangian function. This is a function on the tangent bundle to the configuration space $L(\xi, x)$ where $\xi \in T_x N$. In Newtonian mechanics the Lagrangian function is quadratic in velocity and the quadratic term is positive definite which turns N into a Riemannian manifold.

The most general form of first order Lagrangian is $L(\xi, x) =< \alpha(x), \xi > -H(x)$ where α is a 1-form on N and H is a function on N. The action of a first order Lagrangian mechanics is the following functional on parameterized paths $F_{[t_1,t_2]} = C^\infty([t_1, t_2], N)$

$$S_{[t_2,t_1]}[\gamma] = \int_{t_1}^{t_2} (\langle\alpha(\gamma(t)), \dot{\gamma}(t)\rangle - H(\gamma(t)))\, dt, \tag{1}$$

where γ is a parametrized path.

The Euler-Lagrange equations for this action are:

$$\omega(\dot{\gamma}(t)) - dH(\gamma(t)) = 0,$$

where $\omega = d\alpha$. Naturally, the first order Lagrangian system is called *non-degenerate*, if the form ω is non-degenerate. We will focus on non-degenerate theories here. Denote the space of solutions to Euler-Lagrange equations by $EL_{[t_1,t_2]}$.

Thus, a non-degenerate first order Lagrangian system defines an exact symplectic structure $\omega = d\alpha$ on a manifold N. The Euler-Lagrange equations for such system are equations for flow lines of the Hamiltonian on the symplectic manifold (N, ω) generated by the Hamiltonian H. It is clear that the action of a non-degenerate first order system is exactly the action for this Hamiltonian system.

The variation of the action on solutions of the Euler-Lagrange equations is given by the boundary terms:

$$\delta S_{[t_2,t_1]}[\gamma] = \langle \alpha(\gamma(t)), \delta\gamma(t)\rangle\,|_{t_1}^{t_2}\,.$$

If $\gamma(t_1)$ and $\gamma(t_2)$ are constrained to Lagrangian submanifolds in $L_{1,2} \subset N$ with $TL_{1,2} \subset \ker(\alpha)$, these terms vanish.

The restriction to boundary points gives the projection $\pi : F_{[t_1,t_2]} \to N \times N$. The image of the space of solutions of the Euler-Lagrange equations $L_{[t_1,t_2]} \subset N \times N$ for small $[t_1, t_2]$ is a Lagrangian submanifold with respect to the symplectic form $(d\alpha)_1 - (d\alpha)_2$ on $N \times N$.

Note that solutions of the Euler-Lagrange equation with boundary conditions in $L_1 \times L_2$ correspond to the intersections points $(L_1 \times L_2) \cap L_{[t_1,t_2]}$ which is generically a discrete set.

3.1.2 More on Boundary Conditions

The evolution of the system from time t_1 to t_2 and then to t_3 can be regarded as gluing of space times $[t_1, t_2] \times [t_2, t_3] \to [t_1, t_2] \cup [t_2, t_3] = [t_1, t_3]$. If we impose boundary conditions L_1, L_2, L_3 at times t_1, t_2, t_3 respectively there may be no continuous solutions of equations of motion for intervals $[t_1, t_2]$ and $[t_2, t_3]$ which would compose into a continuous solution for the interval $[t_1, t_3]$. This is why boundary conditions should come in families of Lagrangian submanifolds, so that by varying the boundary condition at t_2 we could choose L_2 in such a way that solutions for $[t_1, t_2]$ and $[t_2, t_3]$ would compose to a continuous solution.

This is why we will say that a boundary condition for a first order theory is a Lagrangian fibration on the space of boundary values of classical fields. In case of first order classical mechanics this is a Lagrangian fibration on N, boundary condition is a Lagrangian fibration of $(N, \omega) \times (N, -\omega)$. It is natural to choose boundary conditions independently for each connected component of the boundary of the space time. In case of classical mechanics this means a choice of Lagrangian fibration $p : N \to B$ for each endpoint of $[t_1, t_2]$. The form α should vanish on fibers of this fibration.

Remark 1 For semiclassical quantization we will need only classical solutions and infinitesimal neighborhood of classical solutions. This means that we need in this case a Lagrangian fibration on the space of boundary fields defined only locally, not necessary globally.

Let N be a configuration space (such as $\mathbb{R}^n$) and $T^*(N)$ be the corresponding phase space. Let γ be a parameterized path in $T^*(N)$ such that, writing $\gamma(t) = (p(t), q(t))$ (where p is momenta and q is position), we have $q(t_i) = q_i$ for two fixed points q_1, q_2. If γ_{cl} is a solution to the Euler-Lagrange equations, then

$$dS^{\gamma_{\mathrm{cl}}}_{t_1,t_2}(q_1, q_2) = \pi^*(p_1\, dq_1 - p_2\, dq_2) \tag{2}$$

where $p_1 = p(t_1)$, $p_2 = p(t_2)$ are determined by t_1, t_2, q_1, q_2. The function $S^{\gamma_{\mathrm{cl}}}_{t_1,t_2}$ is the Hamilton-Jacobi function.

3.2 Scalar Field Theory in an n-dimensional Space Time

The space time in this case is a smooth oriented compact Riemannian manifold M with dim $M = n$. The space of fields is

$$F_M = \Omega^0(M) \oplus \Omega^{n-1}(M). \tag{3}$$

where we write φ for an element of $\Omega^0(M)$ and p for an element of $\Omega^{n-1}(M)$. The action functional is

$$S_M(p,\varphi) = \int\limits_M p \wedge d\varphi - \frac{1}{2}\int\limits_M p \wedge *p - \int\limits_M V(\varphi)\, dx. \tag{4}$$

with $V \in C^\infty(\mathbb{R})$ a fixed potential; dx stands for the metric volume form.

The first term is topological and analogous to $\int_\gamma \alpha$ in (1). The second and third terms use the metric and together yield an analog of the integral of the Hamiltonian in (1).

The variation of the action is

$$\int\limits_M \delta p \wedge (d\varphi - *p) - (-1)^{n-1}\int\limits_M dp \wedge \delta\varphi + (-1)^{n-1}\int\limits_{\partial M} p\,\delta\varphi - \int\limits_M V'(\varphi)\,\delta\varphi\, dx. \tag{5}$$

The Euler-Lagrange equations are therefore

$$d\varphi - *p = 0, (-1)^{n-1} dp + V'(\varphi)\, dx = 0. \tag{6}$$

The first equation gives $p = (-1)^{n-1} * d\varphi$, and substituting this into the second equation gives

$$\Delta\varphi + V'(\varphi) = 0. \tag{7}$$

where $\Delta = *d * d$ is the Laplacian acting of functions.

Thus the space of all solutions of Euler-Lagrange equations is

$$EL_M = \{(p,\varphi) | p = (-1)^{n-1} * d\varphi, \quad \Delta\varphi + V'(\varphi) = 0\}$$

Remark 2 To recover the second-order Lagrangian compute the action at the critical point in p, i.e. substitute $p = (-1)^{n-1} * d\varphi$ into the action functional:

$$S_M((-1)^{n-1} * d\varphi, \varphi) = \int_M (-1)^{n-1} * d\varphi \wedge d\varphi - \frac{1}{2} \int * d\varphi \wedge ** d\varphi$$
$$- \int_M V(\varphi)\, dx = \frac{1}{2} \int_M d\varphi \wedge * d\varphi - \int_M V(\varphi)\, dx$$
$$= \int_M \left(\frac{1}{2} (d\varphi, d\varphi) - V(\varphi) \right) dx.$$

The boundary term in the variation gives the 1-form on boundary fields

$$\alpha_{\partial M} = \int_{\partial M} p\, \delta\varphi \in \Omega^1(F_{\partial M}). \tag{8}$$

Here δ is the de Rham differential on $\Omega^\bullet(F_{\partial M})$. The differential of this 1-form gives the symplectic form $\omega_{\partial M} = \delta\alpha_{\partial M}$ on $F_{\partial M}$.

Note that we can think of the space $F_{\partial M}$ of boundary fields as $T^*(\Omega^0(\partial M))$ in the following manner: if $\delta\varphi \in T_\varphi(\Omega^0(\partial M)) \cong \Omega^0(\partial M)$ is a tangent vector, then the value of the cotangent vector $A \in \Omega^{n-1}(\partial M)$ is

$$A(\delta\varphi) = \int_{\partial M} A \wedge \delta\varphi. \tag{9}$$

The symplectic form $\omega_{\partial M}$ is the natural symplectic form on $T^*\Omega^0(\partial M)$.

The image of the space EL_M of all solutions to the Euler-Lagrange equations with respect to the restriction map $\pi : F_M \to F_{\partial M}$ gives a subspace $L_M = \pi(EL_M) \subset F_{\partial M}$.

Proposition 1 *Suppose there is a unique solution*[5] *to* $\Delta\varphi + V'(\varphi) = 0$ *for any Dirichlet boundary condition* $\varphi|_{\partial M} = \eta$. *Then* $\pi(EL_M)$ *is a Lagrangian submanifold of* $F_{\partial M}$.

Indeed, in this case L_M is the graph of a map $\Omega^0(\partial M) \to F_{\partial M}$ given by $\eta \mapsto (p_\partial = \pi((-1)^{n-1} * d\varphi), \eta)$ where φ is the unique solution to the Dirichlet problem with boundary conditions η.

The space of boundary fields has a natural Lagrangian fibration $\pi_\partial : T^*(\Omega^0(\partial M)) \to \Omega^0(\partial M)$. This fibration corresponds to Dirichlet boundary conditions: we fix the value $\varphi|_{\partial M} = \eta$ and impose no conditions on $p|_{\partial M}$, i.e. we impose boundary condition $(p, \varphi)|_{\partial M} \in \pi_\partial^{-1}(\eta)$.

Another natural family of boundary conditions, Neumann boundary conditions, correspond to the Larganian fibration of $T^*(\Omega^0(\partial M)) \simeq \Omega^{n-1}(\partial M) \oplus \Omega^0(\partial M)$ where the base is $\Omega^{n-1}(\partial M)$. In the case we fix $*_\partial i^*(p) = \eta \in \Omega^0(\partial M)$. The

[5] It is unique if $-V(\varphi)$ is convex.

intersection of L_M and the fiber over η is the set of pairs $(*_\partial \eta, \xi) \in \Omega^{n-1}(\partial M) \oplus \Omega^0(\partial M)$ where $\xi = i^*(\phi)$ and ϕ is a solution to the Neumann problem

$$\Delta\phi + V'(\phi) = 0, \quad \partial_n \phi|_{\partial M} = \eta$$

where ∂_n is the normal derivative of ϕ at the boundary.

3.3 Classical Yang-Mills Theory

Space time is again a smooth compact oriented Riemannian manifold M. Let G be a compact semisimple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. We assume that it is a matrix group, i.e. we fix an embedding of G into Aut(V), and hence an embedding of $\mathfrak{g}$ into End(V) such that the Killing form on $\mathfrak{g}$ is $< a, b >= tr(ab)$. The space of fields in the first order Yang-Mills theory is

$$F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \tag{10}$$

where we think of $\Omega^1(M, \mathfrak{g})$ as the space of connections on a trivial G-bundle over M. If we use a nontrivial G-bundle over M then the first term should be replaced by the corresponding space of connections. We denote an element of F_M by an ordered pair (A, B), $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^{n-2}(M, \mathfrak{g})$. The action functional is

$$S_M(A, B) = \int_M \mathrm{tr}(B \wedge F(A)) - \frac{1}{2} \int_M \mathrm{tr}(B \wedge *B) \tag{11}$$

where $F(A) = dA + A \wedge A$ is the curvature of A as a connection.[6]

After integrating by part we can write the variation of the action as the sum of bulk and boundary parts:

$$\delta S_M(A, B) = \int_M \mathrm{tr}(\delta B \wedge (F(A) - *B) + \delta A \wedge d_A B) - \int_{\partial M} \mathrm{tr}(\delta A \wedge B) \tag{12}$$

The space EL_M of all solution to Euler-Lagrange equations is the space of pairs (A, B) which satisfy

$$B = *F(A), \quad d_A B = 0$$

[6] We will use notations $A \wedge B = \sum_{\{i\}\{j\}} A_{\{i\}} B_{\{j\}} dx^{\{i\}} \wedge dx^{\{j\}}$ for matrix-valued forms A and B. Here $\{i\}$ is a multiindex $\{i_1, \ldots, i_k\}$ and x^i are local coordinates on M. We will also write $[A \wedge B]$ for $\sum_{\{i\}\{j\}} [A_{\{i\}}, B_{\{j\}}] dx^{\{i\}} \wedge dx^{\{j\}}$.

3.3.1 Boundary structure

The boundary term of the variation defines the one-form on the space boundary fields $F_{\partial M} = \Omega^1(\partial M, \mathfrak{g}) \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$.

$$\alpha_{\partial M} = -\mathrm{tr} \int_{\partial M} \delta A \wedge B \in \Omega^1(F_{\partial M}). \tag{13}$$

Its differential defines the symplectic form $\omega_{\partial M} = \int_{\partial M} \mathrm{tr}(\delta A \wedge \delta B)$.

Note that, similarly to the scalar field theory, boundary fields can be regarded as $T^*\Omega^1(\partial M, \mathfrak{g})$ where we identify cotangent spaces with $\Omega^{n-2}(\partial M, \mathfrak{g})$, tangent spaces with $\Omega^1(\partial M, \mathfrak{g})$ with the natural pairing

$$\beta(\alpha) = \mathrm{tr} \int_{\partial M} \alpha \wedge \beta$$

The projection map $\pi : F_M \to F_{\partial M}$ which is the restriction (pull-back) of forms to the boundary defines the subspace $L_M = \pi(EL_M)$ of the space of boundary values of solutions to the Euler-Lagrange equations on M.

3.3.2 On Lagrange property of L_M

Let us show that this subspace is Lagrangian for Maxwell's electrodynamics, i.e. for the Abelian Yang-Mills with $G = \mathbb{R}$. In this case Euler-Lagrange equations are

$$B = *dA, \quad d * dA = 0$$

Fix Dirichlet boundary condition $i^*(A) = a$. Let A_0 be a solution to this equation satisfying Laurenz gauge condition $d^*A_0 = 0$. Such solution is a harmonic 1-form, $(dd^* + d^*d)A_0 = 0$ with boundary condition $i^*(A_0) = a$. If A_0' is another such form, then $A_0 - A_0'$ is a harmonic 1-form with boundary condition $i^*(A_0 - A_0') = 0$. The space of such forms is naturally isomorphic to $H^1(M, \partial M)$. Each of these solutions gives the same value for $B = *dA = *dA_0$ and therefore its boundary value $b = i^*(B)$ is uniquely determined by a. Therefore the projection of EL_M to the boundary is a graph of the map $a \to b$ and thus L_M is a Lagrangian submanifold.

The Dirichlet and Neumann boundary value problems for Yang-Mills theory were studied in [25].

Conjecture 1 *The submanifold L_M is Lagrangian for non-Abelian Yang-Mills theory.*

It is clear that this is true for small connections, when we can rely on perturbation theory starting from an Abelian connection. It is also easy to prove that L_M is isotropic.

3.3.3 The Cauchy subspace

Define the *Cauchy subspace*

$$C_{\partial M} = \pi_\epsilon(EL_{\partial M_\epsilon}) \tag{14}$$

where $\partial M_\epsilon = [0, \epsilon) \times \partial M$ and $\pi_\epsilon : F_{\partial M_\epsilon} \to F_{\partial M}$ is the restriction of fields to $\{0\} \times \partial M$. In other words $C_{\partial M}$ is the space of boundary values of solution to Euler-Lagrange equations in $\partial M_\epsilon = [0, \epsilon) \times \partial M$. It is easy to see that[7]

$$C_{\partial M} = \{(A, B) | d_A B = 0\}$$

We have natural inclusions

$$L_M \subset C_{\partial M} \subset F_{\partial M}$$

3.3.4 Gauge transformations

The automorphism group of the trivial principal G-bundle over M can be naturally identified with $C^\infty(M, G)$. Bundle automorphisms act on the space of Yang-Mills fields. Thinking of a connection A as an element $A \in \Omega^1(M, \mathfrak{g})$ we have the following formulae for the action of the bundle automorphism (gauge transformation) g on fields:

$$g : A \mapsto A^g = g^{-1} A g + g^{-1} dg, \quad B \mapsto B^g = g^{-1} B g. \tag{15}$$

Note that the curvature $F(A)$ is a 2-form and it transforms as $F(A^g) = g^{-1} F(A) g$. Also, if we have two connections A_1 and A_2, their difference is a 1-form and $A_1^g - A_2^g = g^{-1}(A_1 - A_2) g$.

The Yang-Mills functional is invariant under this symmetry:

$$S_M(A^g, B^g) = S_M(A, B) \tag{16}$$

which is just the consequence of the cyclic property of the trace.

The restriction to the boundary gives the projection map of gauge groups $\tilde{\pi} : G_M \to G_{\partial M}$ which is a group homomorphism. This map is surjective, so we obtain an exact sequence

$$0 \to \mathrm{Ker}(\tilde{\pi}) \to G_M \to G_{\partial M} \to 0 \tag{17}$$

[7] The subspace $C_{\partial M}$ also makes sense also in scalar field theory, where explicitly it consists of pairs $(p, \varphi) \in \Omega^{n-1}(\partial M) \oplus \Omega^0(\partial M)$ where p is the pullback of $p_0 = *d\varphi_0$ and φ is the boundary value of φ_0 which solves the Euler-Lagrange equation $\Delta\varphi_0 - V'(\varphi_0) = 0$. Since Cauchy problem has unique solution in a small neighborhood of the boundary, $C_{\partial M} = F_{\partial M}$ for the scalar field.

where $\mathrm{Ker}(\tilde{\pi})$ is the group of gauge transformations acting trivially at the boundary.

It is easy to check that boundary gauge transformations $G_{\partial M}$ preserve the symplectic form $\omega_{\partial M}$. The action of G_M induces an infinitesimal action of the Lie algebra $\mathfrak{g}_M = C^\infty(M, \mathfrak{g})$ of G_M by vector fields on F_M. For $\lambda \in \mathfrak{g}_M$ we denote by $(\delta_\lambda A, \delta_\lambda B)$ the tangent vector to F_M at the point (A, B) corresponding to the action of λ:

$$\delta_\lambda A = -[\lambda, A] + d\lambda = d_A\lambda, \delta_\lambda B = -[\lambda, B] \tag{18}$$

where the bracket is the pointwise commutator (we assume that $\mathfrak{g}$ is a matrix Lie algebra). Recall that the action of a Lie group on a symplectic manifold is Hamiltonian if vector fields describing the action of the Lie algebra $Lie(G)$ are Hamiltonian.

We have the following

Theorem 1 *The action of $G_{\partial M}$ on $F_{\partial M}$ is Hamiltonian.*

Indeed, let f be a function on $F_{\partial M}$ and let $\lambda \in \mathfrak{g}_{\partial M}$. Let $\delta_\lambda f$ denote the Lie derivative of the corresponding infinitesimal gauge transformation. Then

$$\delta_\lambda f(A, B) = \int_{\partial M} \mathrm{tr}\left(\frac{\delta f}{\delta A} \wedge d_A\lambda + \frac{\delta f}{\delta B} \wedge [\lambda, B]\right). \tag{19}$$

Let us show that this is the Poisson bracket $\{H_\lambda, f\}$ where

$$H_\lambda = \int_{\partial M} \mathrm{tr}(\lambda d_A B). \tag{20}$$

The Poisson bracket on functions on $F_{\partial M}$ is given by

$$\{f, g\} = \int_{\partial M} \mathrm{tr}\left(\frac{\delta f}{\delta A} \wedge \frac{\delta g}{\delta B} - \frac{\delta g}{\delta A} \wedge \frac{\delta f}{\delta B}\right). \tag{21}$$

We have

$$\frac{\delta H_\lambda}{\delta A} = \frac{\delta}{\delta A}\left(\int_{\partial M} \mathrm{tr}(\lambda\, dB + \lambda[A \wedge b])\right) = [\lambda, B] \tag{22}$$

and, using integration by parts:

$$\frac{\delta H_\lambda}{\delta B} = d_A B = dB + [A \wedge B]. \tag{23}$$

This proves the statement.

An important corollary of this fact is that the Hamiltonian action of G_M induces a moment map $\mu : F_{\partial M} \to \mathfrak{g}^*_{\partial M}$, and it is clear that

$$C_{\partial M} = \mu^{-1}(0)$$

This implies that $C_{\partial M} \subset F_{\partial M}$ *is a coisotropic submanifold.*

Remark 3 Let us show directly that $C_{\partial M} \subset F_{\partial M}$ is a coisotropic subspace of the symplectic space $F_{\partial M}$ when $\mathfrak{g} = \mathbb{R}$. We need to show that $C_{\partial M}^{\perp} \subset C_{\partial M}$ where $C^{\perp}$ is the symplectic orthogonal to C.

The subspace $C_{\partial M}^{\perp}$ consists of all $(\alpha, \beta) \in \Omega^1(\partial M) \oplus \Omega^{n-2}(\partial M)$ such that

$$\int_{\partial M} a \wedge \beta + \int_{\partial M} \alpha \wedge b = 0 \tag{24}$$

for all $(a, b) \in C_{\partial M} \subset \Omega^1(\partial M) \oplus \Omega^{n-2}(\partial M)$. This condition for all a gives that $\beta = 0$ and requiring this condition for all b gives that α is exact, so we have $C_{\partial M}^{\perp} = \Omega^1_{\mathrm{ex}}(\partial M) \subset C_{\partial M}$ as desired.

3.3.5 Reduction by gauge symmetry

The differential δS_M of the action functional is the sum of the bulk term defining the Euler-Lagrange equations and of the boundary term defining the 1-form $\alpha_{\partial M}$ on the space of boundary fields. The bulk term vanishes on solutions of the Euler-Lagrange equations, so we have

$$\delta S_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}) \tag{25}$$

where $\pi : F_M \to F_{\partial M}$ is the restriction to the boundary and $L_M = \pi(EL_M)$. This is analogous to the property of the Hamilton-Jacobi action in classical mechanics.

Because S_M is gauge invariant, it defines the functional on gauge classes of fields and thus, on gauge classes of solutions to Euler-Lagrange equations. Passing to gauge classes we now replace the chain of inclusions of gauge invariant subspaces $L_M \subset C_{\partial M} \subset F_{\partial M}$ with the chain of inclusions of corresponding gauge classes

$$L_M/G_{\partial M} \subset C_{\partial M}/G_{\partial M} \subset F_{\partial M}/G_{\partial M}. \tag{26}$$

The rightmost space is a Poisson manifold since the action of $G_{\partial M}$ is Hamiltonian. The middle space is the Hamiltonian reduction of $C_{\partial M}$ and is a symplectic leaf in the rightmost space. The leftmost space is still Lagrangian by the standard arguments from symplectic geometry.

3.3.6 Gauge invariant Lagrangian fibrations on the boundary

A natural Lagrangian fibration $p_\partial : \Omega^{n-2}(\partial M, \mathfrak{g}) \oplus \Omega^1(\partial M, \mathfrak{g}) \to \Omega^1(\partial M, \mathfrak{g})$ corresponds to the Dirichlet boundary conditions when we fix the pull-back of A to the boundary: $a = i^*(A)$. Such boundary conditions are compatible with the gauge action. Another example of the family of gauge invariant boundary conditions corresponds to Neumann boundary conditions and is given by the Lagrangian fibration $p_\partial : \Omega^{n-2}(\partial M, \mathfrak{g}) \oplus \Omega^1(\partial M, \mathfrak{g}) \to \Omega^{n-2}(\partial M, \mathfrak{g})$.

3.4 Classical Chern–Simons Theory

3.4.1 Classical theory with boundary

Spacetimes for classical Chern-Simons field theory are smooth, compact, oriented 3-manifolds. Let M be such manifold fields F_M on M are connections on the trivial G-bundle over M with G being compact, semisimple, connected, simply connected Lie group. We will identify the space of connections with the space of 1-forms $\Omega^1(M, \mathfrak{g})$. The action functional is

$$S(A) = \int_M \operatorname{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right) \tag{27}$$

where A is a connection.

The variation is

$$\delta S_M(A) = \int_M \operatorname{tr}(F(A) \wedge \delta A) + \frac{1}{2} \int_{\partial M} \operatorname{tr}(A \wedge \delta A) \tag{28}$$

so the space of solutions EL_M to the Euler-Lagrange equations is the space of flat connections:

$$EL_M = \{A | F(A) = 0\}$$

The boundary term defines the 1-form on boundary fields (connections on the trivial G-bundle over the boundary which we will identify with $\Omega^1(\partial M)$):

$$\alpha_{\partial M} = -\frac{1}{2} \int_{\partial M} \operatorname{tr}(A \wedge \delta A). \tag{29}$$

This 1-form on boundary fields defines the symplectic structure on the space of boundary fields:

$$\omega_{\partial M} = \delta\alpha_{\partial M} = -\frac{1}{2}\mathrm{tr}\int\limits_{\partial M} \delta A \wedge \delta A \tag{30}$$

3.4.2 Gauge symmetry and the boundary cocycle

The gauge group G_M is the group of bundle automorphisms of of the trivial principal G-bundle over M. It can be naturally be identified with the space of smooth maps $M \to G$ which transform connections as in (15) and we have:

$$S_M(A^g) = S_M(A) + \frac{1}{2}\mathrm{tr}\int\limits_{\partial M} (g^{-1}Ag \wedge g^{-1}\,dg) - \frac{1}{6}\mathrm{tr}\int\limits_{M} g^{-1}\,dg \wedge g^{-1}\,dg \wedge g^{-1}\,dg. \tag{31}$$

Assume the integrality of the Maurer-Cartan form on G:

$$\theta = -\frac{1}{6}\mathrm{tr}(dg\,g^{-1} \wedge dg\,g^{-1} \wedge dg\,g^{-1})$$

i.e. we assume that the normalization of the Killing form is chosen in such a way that $[\theta] \in H^3(M, \mathbb{Z})$. Then for a closed manifold M the expression

$$W_M(g) = -\frac{1}{6}\mathrm{tr}\int\limits_{M} dg\,g^{-1} \wedge dg\,g^{-1} \wedge dg\,g^{-1}$$

is an integer and therefore $S_M \mod \mathbb{Z}$ is gauge invariant (for details see for example [20]).

Proposition 2 *When the manifold M has a boundary, the functional $W_M(g) \mod \mathbb{Z}$ depends only on the restriction of g to ∂M.*

Indeed, let M' be another manifold with the boundary $\partial M'$ which differs from ∂M only by reversing the orientation, so that the result of the gluing $M \cup M'$ along the common boundary is smooth. Then

$$W_M(g) - W_{M'}(g') = -\frac{1}{6}\mathrm{tr}\int\limits_{M\cup M'}\int\limits_{M} d\tilde{g}\tilde{g}^{-1} \wedge d\tilde{g}\tilde{g}^{-1} \wedge d\tilde{g}\tilde{g}^{-1} \in \mathbb{Z}$$

Here $\tilde{g}$ is the result of gluing maps g and g' into a map $M \cup M' \to G$. Therefore, modulo integers, it does not depend on g and g'.

For a a connection on the trivial principal G-bundle over a 2-dimensional manifold Σ and for $g \in C^\infty(\Sigma, G)$ define

$$c_\Sigma(a, g) = \exp\left(2\pi i\left(\frac{1}{2}\int\limits_{\partial M} \mathrm{tr}(g^{-1}ag \wedge g^{-1}dg) + W_\Sigma(g)\right)\right)$$

Here we wrote $W_\Sigma(g)$ because $W_M(g) \mod \mathbb{Z}$ depends only on the value of g on ∂M.

The transformation property (31) of the Chern-Simons action implies that the functional

$$\exp(2\pi i S_M(A))$$

transforms as

$$\exp(2\pi i S_M(A^g)) = \exp(2\pi i S_M(A)) c_{\partial M}(i^*(A), i^*(g))$$

where i^* is the restriction to the boundary (pull-back). For further details on gauge aspects of Chern-Simons theory see [20, 21].

Now we can define the gauge invariant version of the Chern-Simons action. Consider the trivial circle bundle $\mathcal{L}_M = S^1 \times F_M$ with the natural projection $\mathcal{L}_M \to F_M$. Define the action of G_M on $\mathcal{L}_M$ as

$$g : (\lambda, A) \mapsto (\lambda c_{\partial M}(i^*(A), i^*(g)), A^g)$$

The functional $\exp(2\pi i S_M(A))$ is a G_M-invariant section of this bundle. The restriction of $\mathcal{L}_M$ to the boundary gives the trivial S^1-bundle over $F_{\partial M}$ with the $G_{\partial M}$-action

$$g : (\lambda, A) \mapsto (\lambda c_{\partial M}(A, g), A^g)$$

The 1-form $\alpha_{\partial M}$ is a $G_{\partial M}$-invariant connection of $\mathcal{L}_{\partial M}$. The curvature of this connection is the $G_{\partial M}$-invariant symplectic form ω_∂.

By definition of $\alpha_{\partial M}$ we have the Hamilton-Jacobi property of the action:

$$\delta S_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}). \tag{32}$$

3.4.3 Reduction

Now, when the gauge symmetry of the Chern-Simons theory is clarified, let us pass to gauge classes. The action of boundary gauge transformations on $F_{\partial M}$ is Hamiltonian with respect to the symplectic form (30). It is easy to check (and it is well known) that the vector field on $F_{\partial M}$ generating infinitesimal gauge transformation $A \to A + d_A\lambda$ is Hamiltonian with the generating function

$$H_\lambda(A) = \int_{\partial M} \mathrm{tr}(F(A)\lambda). \tag{33}$$

This induces the moment map $\mu : F_{\partial M} \to \mathfrak{g}^*_{\partial M}$ given by $\mu(A)(\lambda) = H_\lambda(A)$.

Let $C_{\partial M}$ be the space of Cauchy data, i.e. boundary values of connections which are flat in a small neighborhood of the boundary. It can be naturally identified with the space of flat G-connections on ∂M and thus, $C_{\partial M} = \mu^{-1}(0)$. Hence $C_{\partial M}$ is a coisotropic submanifold of $F_{\partial M}$. We have a chain of inclusions

$$L_M = \pi(EL_M) \subset C_{\partial M} \subset F_{\partial M} \tag{34}$$

where L_M is the space of flat connections on ∂M which extend to flat connections on M. Using Poincaré-Lefschetz duality for de Rham cohomology with coefficients in a local system, one can easily show that L_M is Lagrangian.

We have following inclusions of the spaces of gauge classes

$$L_M/G_{\partial M} \subset C_{\partial M}/G_{\partial M} \subset F_{\partial M}/G_{\partial M} \tag{35}$$

where the middle term is the Hamiltonian reduction $\mu^{-1}(0)/G_{\partial M} \cong \underline{C}_{\partial M}$, which is symplectic. The left term is Lagrangian, and the right term is Poisson. Note that the middle term is a finite dimensional symplectic leaf of the infinite dimensional Poisson manifold $F_{\partial M}/G_{\partial M}$.

The middle term $C_{\partial M}/G_{\partial M}$ is the moduli space $\mathcal{M}^G_{\partial M}$ of flat G-connections on ∂M. It is naturally isomorphic to the representation variety:

$$\mathcal{M}^G_{\partial M} \cong \mathrm{Hom}(\pi_1(\partial M), G)/G$$

where G acts on $\mathrm{Hom}(\pi_1(M), G)$ by conjugation. We will denote the symplectic structure on this space by $\underline{\omega}_{\partial M}$.

Similarly, we have $EL_M/G_M = \mathcal{M}^G_M \cong \mathrm{Hom}(\pi_1(M), G)/G$, which is the moduli space of flat G-connections on M. Unlike in Yang-Mills case, these spaces are finite-dimensional.

The image of the natural projection $\pi : \mathcal{M}^G_M \to \mathcal{M}^G_{\partial M}$ is the reduction of L_M which we will denote by $\underline{L}_M = L_M/G_M$.

Reduction of $\mathcal{L}_M$ and of $\mathcal{L}_{\partial M}$ gives line bundles $\underline{\mathcal{L}}_M = \mathcal{L}_M/G_M$ and $\underline{\mathcal{L}}_{\partial M} = \mathcal{L}_{\partial M}/G_{\partial M}$ over $\mathcal{M}^G_M$ and $\mathcal{M}^G_{\partial M}$ respectively. The 1-form $\alpha_{\partial M}$ which is also a $G_{\partial M}$-invariant connection on $\mathcal{L}_{\partial M}$ becomes a connection on $\underline{\mathcal{L}}_{\partial M}$ with the curvature $\underline{\omega}_{\partial M}$.

The Chern-Simons action yields a section cs of the pull-back of the line bundle $\mathcal{L}_{\partial M}$ over $\mathcal{M}^G_{\partial M}$. Because $\underline{L}_M$ is a Lagrangian submanifold, the symplectic form $\underline{\omega}_{\partial M}$ vanishes on it and the restriction of the connection $\underline{\alpha_{\partial M}}$ to $\underline{L}_M$ results in a flat connection over $\mathcal{L}_{\partial M}|_{\underline{L}_M}$. The section cs is horizontal with respect to the pull-back of the connection $\underline{\alpha_{\partial M}}$. It can be written as

$$(d - \pi^*(\underline{\alpha_{\partial M}}|_{L_M}))cs = 0. \tag{36}$$

This collection of data is the *reduced Hamiltonian structure* of the Chern-Simons theory.

3.4.4 Complex polarization

There are no natural non-singular Lagrangian fibrations on the space of connections on the boundary which are compatible with the gauge action. However, for formal semiclassical quantization we need such fibration only to exist locally near a preferred point in the space of connections. Now we will describe another structure on the space of boundary fields for the Chern-Simons theory which is used in geometric quantization [4].

Instead of looking for a real Lagrangian fibration, let us choose a complex polarization of $\Omega^1(M,\mathfrak{g})_{\mathbb{C}}$. Fixing a complex structure on the boundary, gives us the natural decomposition

$$\Omega^1(\partial M,\mathfrak{g})_{\mathbb{C}} = \Omega^{1,0}(\partial M,\mathfrak{g})_{\mathbb{C}} \oplus \Omega^{0,1}(\partial M,\mathfrak{g})_{\mathbb{C}}$$

and we can define boundary fibration as the natural projection to $\Omega^{1,0}(\partial M,\mathfrak{g})_{\mathbb{C}}$. Here elements of $\Omega^{1,0}(\partial M,\mathfrak{g})_{\mathbb{C}}$ are $\mathfrak{g}_{\mathbb{C}}$-valued forms which locally can be written as $a(z,\overline{z})\,dz$ and elements of $\Omega^{0,1}(\partial M,\mathfrak{g})_{\mathbb{C}}$ can be written as $b(z,\overline{z})\,d\overline{z}$. The decomposition above locally works as follows:

$$A = \mathcal{A} + \overline{\mathcal{A}}$$

where $\mathcal{A} = a(z,\overline{z})\,dz$.

In terms of this decomposition the symplectic form is

$$\omega = \int_{\partial M} \mathrm{tr}\; \delta\mathcal{A} \wedge \delta\overline{\mathcal{A}}$$

It is clear that subspaces $\mathcal{A} + \Omega^{0,1}(\partial M)$ are Lagrangian in the complexification of $\Omega(M,\mathfrak{g})$. Thus, we have a Lagrangian fibration $\Omega(M,\mathfrak{g})_{\mathbb{C}} \to \Omega^{0,1}(M,\mathfrak{g})_{\mathbb{C}}$. The action of the gauge group preserves the fibers.

However, the form $\alpha_{\partial M}$ does not vanish of these fibers. To make it vanish we should modify the action as

$$\tilde{S}_M = S_M + \frac{1}{2}\int_{\partial M} \mathrm{tr}\;(\mathcal{A}\wedge\overline{\mathcal{A}})$$

After this modification, the boundary term in the variation of the action gives the form

$$\tilde{\alpha}_{\partial M} = -\int_{\partial M} \mathrm{tr}\;(\overline{\mathcal{A}}\wedge\delta\mathcal{A})$$

This form vanishes on fibers. It is not gauge invariant as well as the modified action. The modified action transforms under gauge transformations as

$$\tilde{S}_M(A^g) = \tilde{S}_M(A) + \frac{1}{2}\mathrm{tr}\int_{\partial M} (g^{-1}\mathcal{A}g \wedge g^{-1}\overline{\partial}g) + W_M(g)$$

This gives the following cocycle on the boundary gauge group

$$\tilde{c}_\Sigma(A, g) = \exp(2\pi i(\frac{1}{2}\int_\Sigma \mathrm{tr}(g^{-1}\mathcal{A}g \wedge g^{-1}\overline{\partial}g) + W_\Sigma(g)))$$

This modification of the action and this complex polarization of the space of boundary fields is important for geometric quantization in Chern-Simons theory [4] and is important for understanding the relation between the Chern-Simons theory and the WZW theory, see for example [1, 16]. We will not expand this direction here, since we are interested in formal semiclassical quantization where real polarizations are needed.

3.5 BF-Theory

Space time M is smooth, oriented[8] and compact and is equipped with a trivial G-bundle where G is connected, simple or abelian compact Lie group. Fields are

$$F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \tag{37}$$

where $\Omega^1(M, \mathfrak{g})$ describes connections on the trivial G-bundle.

The action functional of the BF theory is the topological term of Yang-Mills action:

$$S_M(A, B) = \int_M \mathrm{tr}(B \wedge F(A)). \tag{38}$$

For the variation of S_M we have:

$$\delta S_M = \mathrm{tr}\int_M \delta B \wedge F(A) + (-1)^{n-1}\mathrm{tr}\int_M d_A B \wedge \delta A + (-1)^{n-1}\mathrm{tr}\int_{\partial M} B \wedge \delta A. \tag{39}$$

The bulk term gives Euler-Lagrange equations:

$$EL_M = \{(A, B) : F(A) = 0, \quad d_A B = 0\}. \tag{40}$$

The boundary term gives a 1-form on the space of boundary fields $F_{\partial M} = \Omega^1(\partial M, \mathfrak{g}) \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$:

[8] The orientability assumption can be dropped, see [15].

$$\alpha_{\partial M} = \int_{\partial M} \mathrm{tr}(B \wedge \delta A). \tag{41}$$

The corresponding exact symplectic form is

$$\omega_{\partial M} = \delta\alpha_{\partial M} = \int_{\partial M} \mathrm{tr}(\delta B \wedge \delta A). \tag{42}$$

The space of Cauchy data is

$$C_{\partial M} = \{(A, B) | F_A = 0, \quad d_A B = 0\}$$

Boundary values of solutions of the Euler-Lagrange equations on M define the submanifold $L_M = \pi(EL_M) \subset F_{\partial M}$. This submanifold is Lagrangian. Thus we have the embedding:

$$L_M \subset C_{\partial M} \subset F_{\partial M}$$

where $F_{\partial M}$ is exact symplectic, $C_{\partial M}$ is co-isotropic, and L_M is Lagrangian.

3.5.1 Gauge symmetry and reduction

The space of bundle automorphisms G_M is the space of smooth maps $M \to G$. They act on $A \in \Omega^1(M, \mathfrak{g})$ by $A \mapsto g^{-1}Ag + g^{-1}dg$ and on $B \in \Omega^{n-2}(M, \mathfrak{g})$ by $B \mapsto g^{-1}Bg$. As in Yang-Mills theory the action is invariant with respect to these transformations.

In addition, it is almost invariant with respect to transformations $A \mapsto A$, $B \mapsto B + d_A\beta$ where $\beta \in \Omega^{n-3}(M, \mathfrak{g})$:

$$S_M(A, B + d_A\beta) = S_M(A, B) + \int_M \mathrm{tr}(d_A\beta \wedge F(A)). \tag{43}$$

After integration by parts in the second term we write it as

$$\int_M \mathrm{tr}(\beta \wedge d_A F(A)) + \int_{\partial M} \mathrm{tr}(\beta \wedge F(A)). \tag{44}$$

The bulk term here vanishes because of the Bianchi identity and the only additional contribution is a boundary term, thus:

$$S_M(A, B + d_A\beta) = S_M(A, B) + \mathrm{tr} \int\limits_{\partial M} (\beta \wedge F(A))$$

The additional gauge symmetry $B \mapsto B + d_A\beta$ gives us a larger gauge group

$$G_M^{BF} = G_M \times \Omega_M^{n-3}. \tag{45}$$

Its restriction to the boundary gives the boundary gauge group

$$G_{\partial M}^{BF} = G_{\partial M} \times \Omega_{\partial M}^{n-3}. \tag{46}$$

The action is invariant up to a boundary term. This means that the 1-form $\alpha_{\partial M}$ is not gauge invariant. Indeed, it is invariant with respect to G_M-transformations, but when $(A, B) \mapsto (A, B + d_A\beta)$ the forms $\alpha_{\partial M}$ transforms as

$$\alpha_{\partial M} \mapsto \alpha_{\partial M} + \int\limits_{\partial M} \mathrm{tr}\, d_A\beta \wedge \delta A$$

However, it is clear that the symplectic form $\omega_{\partial M} = \delta\alpha_{\partial M}$ is gauge invariant. Moreover, we have the following.

Theorem 2 *The action of $G_{\partial M}^{BF}$ is Hamiltonian.*

Indeed, if $\alpha \in \Omega^0(\partial M, \mathfrak{g})$ is an element of the Lie algebra of boundary gauge transformations and $\beta \in \Omega^{n-3}(\partial M, \mathfrak{g})$, then we can take

$$H_\alpha(A, B) = \int\limits_{\partial M} \mathrm{tr}(B \wedge d_A\alpha) \tag{47}$$

$$H_\beta(A, B) = \int\limits_{\partial M} \mathrm{tr}(A \wedge d_A\beta) \tag{48}$$

as Hamiltonians generating the action of corresponding infinite dimensional Lie algebra.

This defines a moment map $\mu : F_{\partial M} \to \Omega^0(\partial M, \mathfrak{g}) \oplus \Omega^{n-3}(\partial M, \mathfrak{g})$. It is clear that Cauchy submanifold is also $C_{\partial M} = \mu^{-1}(0)$. This proves that it is a co-isotropic submanifold.

Note also, that the restriction of $\alpha_{\partial M}$ to $C_{\partial M}$ is $G_{\partial M}^{BF}$-invariant. Indeed $\mathrm{tr} \int\limits_{\partial M} d_A\beta \wedge \delta A = -\mathrm{tr} \int\limits_{\partial M} \beta \wedge d_A\delta A$, and this expression vanishes when the form is pulled-back to the space of flat connections where $d_A\delta A = 0$. Therefore the Hamiltonian reduction of $F_{\partial M}$ which is $\underline{F}_{\partial M} = C_{\partial M}/G_{\partial M}^{BF}$ is an exact symplectic manifold.

It is easy to see that the reduced space of fields on the boundary $\underline{F}_{\partial M}$ can be naturally identified, as a symplectic manifold, with $T^*\mathcal{M}_{\partial M}^G$, the cotangent bundle to the moduli space of flat connections $\mathcal{M}^G_{\partial M} = \mathrm{Hom}(\pi_1(\partial M), G)/G$. The canonical 1-form on this cotangent bundle corresponds to the form $\alpha_{\partial M}$ restricted to $C_{\partial M}$. The Lagrangian subspace $L_M \subset F_{\partial M}$ is gauge invariant. It defines the Lagrangian submanifold

$$L_M/G^{BF}_{\partial M} \subset T^*\mathcal{M}^G_{\partial M}$$

The restriction of the action functional to EL_M is gauge invariant and defines the the function $\underline{S}_M$ on $EL_M/G^{BF}_{\partial M}$. The formula for the variation of the action gives the analog of the Hamilton-Jacobi formula:

$$d\underline{S}_M = \pi^*(\theta|_{L_M}) \tag{49}$$

where θ is the canonical 1-form on the cotangent bundle $T^*\mathcal{M}^G_{\partial M}$ restricted to $L_M/G^{BF}_{\partial M}$.

3.5.2 A gauge invariant Lagrangian fibration

One of the natural choices of boundary conditions is the Dirichlet boundary conditions. This is the Lagrangian fibration $\Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \to \Omega^1(M, \mathfrak{g})$. This fibration is gauge invariant. After the reduction it gives the standard Lagrangian fibration $T^*\mathcal{M}^G_{\partial M} \to \mathcal{M}^G_{\partial M}$.

4 Semiclassical Quantization of First Order Field Theories

In this section, after reminding briefly the general framework of local quantum field theory, we will concentrate on a finite-dimensional toy model for the path integral. In this model partition functions satisfy the gluing axiom by general properties of measure theoretic integrals (the Fubini theorem). One can also model the gauge symmetry in this setting, treated by a version of the Faddeev-Popov trick. We will see that the arising integrals can be evaluated, in the asymptotics $h \to 0$, by the stationary phase formula. The result of such an evaluation we call a “formal integral” (alluding to integration over a formal neighborhood of a critical point, as well as the fact that we forget the measure-theoretic definition of the integral we started with). We will obtain the gluing formula for such formal integrals which is satisfied a priori, since the construction comes from measure integrals. In the usual setting of local quantum field theory, partition functions are the path integrals where a measure theoretic definition is not accessible, while the “formal integral” can be defined as a formal power series in h where coefficients are the Feynman diagrams. In this setting the gluing formulae are not automatic and have to be proven, cf. e.g. [22].

4.1 The Framework of Local Quantum Field Theory

We will follow the framework of local quantum field theory which was outlined by Atiyah and Segal for topological and conformal field theories. In a nutshell it is a functor from an appropriate category of cobordisms to the category of vector spaces (or, more generally, to some category).

In this sense, a quantum field theory is the assignment of a vector space to the boundary $N = \partial M$ of a space time manifold M and a vector in this vector space to the manifold M:

$$N \mapsto H(N), \quad M \mapsto Z_M \in H(\partial M).$$

The identification of such assignments with linear maps is natural assuming that the vector space assigned to the boundary is the tensor product of vector spaces assigned to connected components of the boundary and that changing the orientation replaces the corresponding vector space by its dual.

The vector space assigned to the boundary is the space of boundary states. It may depend on the extra structure at the boundary. In this case it is a vector bundle over the space of admissible geometric data and Z_M is a section of this vector bundle. The vector Z_M is called the partition function or the amplitude.

These data should satisfy natural axioms, which can by summarized as follows:

1. The locality properties of boundary states:
$$H(\varnothing) = \mathbb{C}, \quad H(N_1 \sqcup N_2) = H(N_1) \otimes H(N_2),$$
2. The locality property of the partition function
$$Z_{M_1 \sqcup M_2} = Z_{M_1} \otimes Z_{M_2} \in H(\partial M_1) \otimes H(\partial M_2).$$
3. For each space N (an object of the space time category) there is a non-degenerate pairing
$$\langle ., . \rangle_N : H(\overline{N}) \otimes H(N) \to \mathbb{C}$$
such that $\langle ., . \rangle_{N_1 \sqcup N_2} = \langle ., . \rangle_{N_1} \otimes \langle ., . \rangle_{N_2}$.
4. The canonical orientation reversing isomorphism $\sigma : N \to \overline{N}$ induces a $\mathbb{C}$-antilinear mapping $\widehat{\sigma}_N : H(N) \to H(\overline{N})$ which agrees with locality of N and $\widehat{\sigma}_{\overline{N}} \widehat{\sigma}_N = id_N$. Together with the pairing $\langle ., . \rangle_N$ the orientation reversing mapping induces the Hilbert space structure on $H(N)$.
5. An orientation preserving isomorphism[9] $f : N_1 \to N_2$ induces a linear isomorphism

[9] By an isomorphism here we mean a mapping preserving the corresponding geometric structure.

$$T_f : H(N_1) \to H(N_2).$$

which is compatible with the pairing and $T_{f\sqcup g} = T_f \otimes T_g, \quad T_{f\circ g} = T_f T_g$ (possibly twisted by a cocycle of the group of automorphisms of the boundary).

6. The gluing axiom. This pairing should agree with partition functions in the following sense. Let $\partial M = N \sqcup \overline{N} \sqcup N'$, then

$$(\langle .,.\rangle \otimes id)Z_M = Z_{\tilde{M}} \in H(N') \tag{50}$$

where $\tilde{M}$ is the result of gluing of N with $\overline{N}$. The operation is known as the gluing axiom. For more details see [8].

7. The quantum field theory is (projectively) invariant with respect to transformations of the space time (diffeomorphisms, gauge transformations etc.) if for such transformation $f : M_1 \to M_2$,

$$T_{f_\partial} Z_{M_1} = c_{M_1}(f) Z_{M_2}$$

Here $c_M(f)$ is a cocycle $c_M(fg) = c_{gM}(f)c_M(g)$. When the theory is invariant, not only projectively invariant, $c_M(f) = 1$.

Remark 4 The gluing axiom in particular implies the functoriality of Z:

$$Z_{M_1 \circ M_2} = Z_{M_1} \circ Z_{M_2}\,.$$

Here $M_1 \circ M_2$ is the composition of cobordisms in the category of space time manifolds. In case of cylinders this is the semigroup property of propagators in the operator formulation of QFT.

Remark 5 This framework is very natural in models of statistical mechanics on cell complexes with open boundary conditions, also known as lattice models.

Remark 6 The main physical concept behind this framework is the locality of the interaction. Indeed, we can cut our space time manifold in small pieces and the resulting partition function Z_M in such framework is expected to be the composition of partition functions of small pieces. Thus, the theory is determined by its structure on ‘small’ space time manifolds, or at ‘short distances’. This is the concept of *locality*. To fully implement this concept one should consider the field theory on manifolds with corners where we can glue along parts of the boundary. In the case of topological theories, a particular realization of the concept of locality is the formalism of *extended/fully extended* topological quantum field theories of Baez-Dolan [7] and Lurie [23].

4.2 Path Integral and Its Finite Dimensional Model

4.2.1 Quantum Field Theory via Path Integrals

Given a first order classical field theory with boundary conditions given by Lagrangian fibrations, one can try to construct a quantum field theory by the path integral quantization. In this framework the space of boundary states $H(\partial M)$ is taken as the space of functionals on the base $B_{\partial M}$ of the Lagrangian fibration on boundary fields $F_{\partial M}$. The vector Z_M is the Feynman integral over the fields on the bulk with given boundary conditions

$$Z_M(b) = \int\limits_{f \in \pi^{-1} p_\partial^{-1}(b)} e^{\frac{i}{\hbar} S_M(f)} Df \tag{51}$$

where Df is the integration measure, $\pi : F_M \to F_{\partial M}$ is the restriction map and $p_\partial : F_{\partial M} \to B_{\partial M}$ is the boundary fibration.

The integral above is difficult to define when the space of fields is infinite dimensional. To clarify the functorial structure of this construction and to define the formal semiclassical path integral let us start with a model case when the space of fields is finite dimensional, when the integrals are defined and absolutely convergent. A "lattice approximation" of a continuous theory is a good example of such a finite dimensional model.

4.2.2 Finite Dimensional Classical Model

A finite dimensional model of a first order classical field theory on a space time manifold with boundary consists of the following data. Three finite dimensional manifolds F, F_∂, B_∂ should be complemented by the following structures.

- The manifold F_∂ is endowed with an exact symplectic form $\omega_\partial = d\alpha_\partial$.
- A surjective submersion $\pi : F \to F_\partial$.
- A function S on F, such that the submanifold $EL \subset F$, on which the form $dS - \pi^*(\alpha_\partial)$ vanishes, projects to a Lagrangian submanifold in F_∂.
- A Lagrangian fibration of F_∂ given by $p_\partial : F_\partial \to B_\partial$ such that α_∂ vanishes on fibers. We also assume that fibers are transversal to $L = \pi(EL) \subset F_\partial$.

We will say that this is a finite dimensional model of a *non-degenerate theory* if S has finitely many simple critical points on each fiber $\pi^{-1} p_\partial^{-1}(b)$.

The model is gauge invariant with the bulk gauge group G and the boundary gauge group G_∂ if the following holds.

- The group G acts on F, and G_∂ acts on F_∂.
- There is a group homomorphism $\tilde{\pi} : G \to G_\partial$ such that the restriction map satisfies $\pi(gx) = \tilde{\pi}(g)\pi(x)$.

- The function S is invariant under the G-action up to boundary terms:

$$S(gx) = S(x) + c_\partial(\pi(x), \tilde{\pi}(g))$$

  where $c_\partial(x, g)$ is a cocycle for G_∂ acting on F_∂:

$$c_\partial(x, gh) = c_\partial(hx, g) + c_\partial(x, h)$$

- The action of G_∂ is compatible with the fibration p_∂, i.e. it maps fibers to fibers. Assuming that the stabilizer subgroups $\mathrm{Stab}_b \subset G_\partial$ coincide for different fibers $\pi_\partial^{-1}(b)$, one can introduce a quotient group $\Gamma_\partial = G_\partial/\mathrm{Stab}_b$ acting on B_∂. One has then the quotient homomorphism $\tilde{p}_\partial : G_\partial \to \Gamma_\partial$. We require that the cocycle $c(g, x)$ is constant on fibers of p_∂, i.e. is a pullback of a cocycle $\tilde{c}$ of Γ_∂ acting on B_∂: $c(x, g) = \tilde{c}(p_\partial(x), \tilde{p}_\partial(g))$.

We will say that the theory with gauge invariance is non-degenerate if critical points of S form finitely many G-orbits and if the corresponding points on $F(b)/G$ are simple (i.e. isolated) on each fiber $F(b)$ of $p_\partial\pi$.

4.2.3 Finite Dimensional Quantum Model

To define quantum theory assume that F and B_∂ are defined together with measures dx and db respectively. Assume also that there is a measure $\frac{dx}{db}$ on each fiber $F(b) = \pi^{-1}p_\partial^{-1}(b)$ such that $dx = \frac{dx}{db}db$.

Define the vector space H_∂ together with the Hilbert space structure on it as follows:

$$H_\partial = L^2(B_\partial)$$

When the function S is only projectively invariant with respect to the gauge group, the space of boundary states is the space of L^2-sections of the corresponding line bundle.

Remark 7 It is better to consider the space of half-forms on B_∂ which are square integrable but we will not do it here. For details see for example [9].

The partition function Z_F is defined as an element of H_∂ given by the integral over the fiber $F(b)$:

$$Z_F(b) = \int_{F(b)} \exp(\frac{i}{h}S(x))\frac{dx}{db} \tag{52}$$

When there is a gauge group the partition function transforms as

$$Z_F(\gamma b) = Z_F(b) \exp(\frac{i}{h} c_\partial(b, \gamma))$$

In such a finite dimensional model the gluing property follows from Fubini's theorem allowing to change the order of integration. Suppose we have two spaces F_1 and F_2 fibered over B_∂ and two functions S_1 and S_2 defined on F_1 and F_2 respectively such that integrals $Z_{F_1}(b)$ and $Z_{F_2}(b)$ converge absolutely for generic b. For example, we can assume that all spaces F, F_∂ and B_∂ are compact. Then changing the order of integration we have

$$\int_{B_\partial} Z_{F_1}(b) Z_{F_2}(b)\, db = Z_{F_1 \times_{B_\partial} F_2} \tag{53}$$

where

$$Z_{F_1 \times_{B_\partial} F_2} = \int_{F_1 \times_{B_\partial} F_2} \exp(\frac{i}{h}(S_1(x_1) + S_2(x_2))) \frac{dx_1}{db} \frac{dx_2}{db}\, db$$

Here $F_1 \times_{B_\partial} F_2 = \{(x, x') \in F_1 \times F_2 | \pi_1(x) = \pi_2(x')\}$ is the fiber product of F_1 and F_2 over B_∂. The measure $\frac{dx}{db}\frac{dx'}{db} db$ is induced by measures on $F_1(b)$, $F_2(b)$ and on B_∂.

Remark 8 The quantization is not functorial. We need to make a choice of measure of integration.

Remark 9 We will not discuss here quantum statistical mechanics where instead of oscillatory integrals we have integrals of probabilistic type representing Boltzmann measure. Wiener integral is among the examples of such integrals.

Remark 10 When the gauge group is non-trivial, the important subgroup in the total gauge group is the bulk gauge group, i.e. the symmetry of the integrand in the formula for $Z_F(b)$. If Γ_∂ is the gauge group acting on the base of the boundary Lagrangian fibration, then the bulk gauge group G^B is the kernel in the exact sequence of groups $1 \to G^B \to G \to \Gamma_\partial \to 1$.

An example of such construction is the discrete time quantum mechanics which is described in Appendix A.

4.2.4 The Semiclassical Limit, Non-degenerate Case

The asymptotical expansion of the integral (52) can be computed by the method of stationary phase (see for example [19, 26] and references therein).

Here we assume that the function S has finitely many simple critical points on the fiber $F(b)$ for each $b \in B_\partial$. Denote the set of such critical points by $C(b)$.

Using the stationary phase approximation we obtain the following expression for the asymptotical expansion of the partition function as $h \to 0$:

$$Z(b) \simeq \sum_{c \in C(b)} Z_c \tag{54}$$

where Z_c is the contribution to the asymptotical expansion from the critical point c. To describe Z_c let us choose local coordinates x^i on $F(b)$ near c, then

$$Z_c = (2\pi h)^{\frac{N}{2}} \frac{1}{\sqrt{|\det(B_c)|}} e^{\frac{iS(c)}{h} + \frac{i\pi}{4}\operatorname{sign}(B_c)} (v(c) + \sum_{\Gamma} \frac{(ih)^{-\chi(\Gamma)} F_c(\Gamma)}{|\operatorname{Aut}(\Gamma)|}) \tag{55}$$

Here $N = \dim F(b)$ and $(B_c)_{ij} = \frac{\partial^2 S(c)}{\partial x^i \partial x^j}$, $v(x)$ is the volume density in local coordinates $\{x^i\}_{i=1}^N$ on $F(b)$, $\frac{dx}{db} = v(x)\, dx^1, \ldots, dx^N$, $\chi(\Gamma)$ is the Euler characteristic of the graph Γ, $|\operatorname{Aut}(\Gamma)|$ is the number of automorphisms of the graph and the summation is taken over finite graphs where each vertex has valency at least 3. The weight of a graph $F_c(\Gamma)$ is given by the "state sum" which is described in the Appendix B. Note that this formula by the construction is invariant with respect to changes of local coordinates. This is particularly clear at the level of determinants. Indeed, let J be the Jacobian of the coordinate change $x^i \mapsto f^i(x)$. Then $v \mapsto v|\det(J)|$ and $|\det(B_c)| \mapsto |\det(B_c)| \det(J)^2$ and the Jacobians cancel. For higher level contributions, see [22].

4.2.5 Gluing Formal Semiclassical Partition Functions in the Non-degenerate Case

The image $L = \pi(EL)$, according to our assumptions is transversal to generic fibers of $p_\partial : F_\partial \to B_\partial$. By varying the classical background c we can span the subspace $T_{\pi(c)}L \subset T_{\pi(c)}F_\partial$ which is, according to the assumption of transversality, isomorphic to $T_{p_\partial \pi(c)}B_\partial$.

We will call the partition function Z_c the *formal semiclassical partition function* on the classical background c. We will also say that it is given by the formal integral of $\exp(\frac{iS}{h})$ over the formal neighborhood of c:

$$Z_c = \int_{T_c F(b)}^{formal} \exp(\frac{iS}{h}) \frac{dx}{db}$$

with $b = p_\partial \pi(c)$. The formal integral on the right hand side here is defined to be the right hand side of (55).

Passing to the limit $h \to 0$ in (53) we obtain the gluing formula for formal semiclassical partition functions (under the assumption of non-degeneracy of critical points):

$$\int\limits_{T_{b_0}B_\partial}^{formal} Z_{c_1(b)} Z_{c_2(b)} db = Z_c \tag{56}$$

Here c is a simple critical point of S on $F_1 \times_{B_\partial} F_2$, $b_0 = p_\partial \pi_1 \pi(c) = p_\partial \pi_2 \pi'(c)$ where $\pi : F_1 \times_{B_\partial} F_2 \to F_1$ and $\pi' : F_1 \times_{B_\partial} F_2 \to F_2$ are natural projections and $c_1(b)$ and $c_2(b)$ are critical points of S_1 and S_2 on fibers $F_1(b)$ and $F_2(b)$ respectively which are formal deformations of $c_1(b_0) = \pi_1(c)$ and of $c_2(b_0) = \pi_2(c)$. The left hand side of (56) stands for the stationary phase evaluation of the integral (note that the integrand has the appropriate asymptotics at $h \to 0$). In [22] this formula was used to prove that formal semiclassical propagator satisfies the composition property.

4.3 Gauge Fixing

4.3.1 Gauge Fixing in the Integral

Here we will outline a version of the Faddeev-Popov trick for gauge fixing in the finite dimensional model in the presence of boundary. We assume that the action function S, the choice of boundary conditions, and group action on F satisfy all properties described in Sect. 4.2.2.

The goal here is to calculate the asymptotics of the partition function

$$Z_F(b) = \int\limits_{F(b)} e^{\frac{i}{h}S(x)} \frac{dx}{db} \tag{57}$$

when $h \to 0$. Here, as in the previous section $F(b) = \pi^{-1} p_\partial^{-1}(b)$ but now a Lie group G acts on F and the function S and the integration measure dx are G-invariant. As in Sect. 4.2.2 we assume that there is an exact sequence $1 \to G^B \to G \to \Gamma_\partial \to 1$, where Γ_∂ acts on B_∂ in such a way that db is Γ_∂-invariant and the subgroup G^B acts fiberwise so that the measure $\frac{dx}{db}$ is G^B-invariant. We will denote the Lie algebra of G^B by $\mathfrak{g}^B$.

Assume that the function S has finitely many isolated G^B-orbits of critical points on $F(b)$ and that the measure of integration is supported on a neighborhood of these points.[10] We denote by $v(x)$ the density of the measure in local coordinates, $\frac{dx}{db} = v(x)\, dx^1, \ldots, dx^N$ with $\{x^i\}$ the local coordinates on F_b.

[10] In the asymptotics $h \to 0$, one can replace any invariant G^B-invariant measure by one with this property, since we are working with oscillatory integrals.

Let $O \subset F(b)$ be a critical G^B-orbit of the action S. Denote $U_O \subset F(b)$ the connected component[11] of the support of the density v containing O. The integral (57) is a sum of contributions of individual critical orbits:

$$\int_{F(b)} e^{\frac{i}{h}S(x)} \frac{dx}{db} = \sum_O \int_{U_O} e^{\frac{i}{h}S(x)} \frac{dx}{db}$$

For a fixed critical orbit O, let $\varphi : U_O \to \mathfrak{g}^B$ be some function with zero a regular value. Denote $\Lambda_\varphi = \varphi^{-1}(0) \subset U_O$ – the "gauge-fixing surface". Assume that Λ_φ intersects O transversally. Note that we do not assume that Λ_φ is a section of the G^B-action (i.e. of the projection $U_O \to U_O/G^B$).

Let c be one of the intersection points of the orbit O with Λ_φ. Denote $V_{O,c} \subset U_O$ the connected component of c in the intersection $U_O \cap \Lambda_\varphi$ and let $U_{O,c} \subset U_O$ be an open tubular neighborhood of $V_{O,c}$ in U_O (thin enough not to contain zeroes of φ lying outside $V_{O,c}$). Using Faddeev-Popov construction, the contribution of U_O to the integral (57) can be written as follows:

$$\int_{U_O} e^{\frac{i}{h}S(x)} \frac{dx}{db} = |G^B| \int_{U_{O,c}} e^{\frac{i}{h}S(x)} \det(L_\varphi(x))\delta(\varphi(x)) \frac{dx}{db} \tag{58}$$

We have a natural isomorphism $U_O \simeq V_{O,c} \times G^B$ given by the action of G^B on points of $V_{O,c}$, hence $V_{O,c} \simeq U_O/G^B$ and therefore the integral on the right hand side of (58) can be thought of as an integral supported on the quotient U_O/G^B. To describe $L_\varphi(x)$ choose a basis e_a in the Lie algebra $\mathfrak{g}^B$. The action of e_a on F_b is given by the vector field $\sum_i l_a^i(x)\partial_i$. Matrix elements of $L_\varphi(x)$ are $\sum_i l_a^i(x)\partial_i\varphi^b(x)$. The factor $|G^B|$ in (58) stands for the volume of the group G^B (with respect to the Haar measure compatible with the basis $\{e_a\}$ in $\mathfrak{g}^B$).

It is convenient to write (58) as a Grassmann integral:

$$\frac{|G^B|}{(2\pi i)^{\dim G^B}} \int_{\mathcal{F}_c(b)} \exp \frac{i}{h}\left(S(x) + \sum_a \lambda_a \varphi^a(x) + \sum_a \bar{\mathsf{c}}_a L_\varphi(x)^a_b \mathsf{c}^b \right) \frac{dx}{db}\, d\lambda\, d\bar{\mathsf{c}}\, d\mathsf{c} \tag{59}$$

where $\mathcal{F}_c(b) = U_{O,c} \oplus \mathfrak{g}^B_{odd} \oplus (\mathfrak{g}^B_{odd})^* \oplus (\mathfrak{g}^B_{even})^*$ and $\bar{\mathsf{c}}$ and c are odd variables. See for example [19] for details on Grassman integration. The asymptotical stationary phase expansion of (58) as $h \to 0$ can be understood[12] as a formal integral over the (formal) neighborhood of c in the supermanifold $\mathcal{F}_c(b)$. The functions $S(x)$, $\varphi^a(x)$, $L_\varphi(x)^a_b$ should be understood as the Taylor expansions in parameter $\frac{x-c}{\sqrt{h}}$, just as in the previous section. The result is the asymptotical expression given by Feynman diagrams where two types of edges correspond to the even and odd Gaussian terms in the

[11] In the case when the group G^B is disconnected, we define U_O to be the union of connected components of O_k in $\mathrm{supp}(v)$, where O_k are the connected components of O.

[12] The logic is that the formal integral is *defined* to be stationary phase asymptotics of (58).

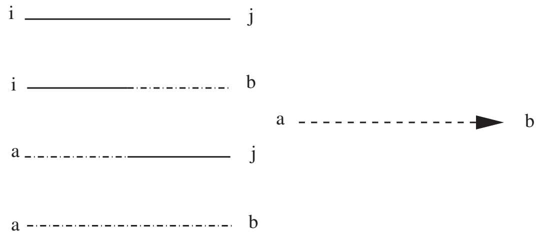

Fig. 1 Bosonic (*left*) and fermionic (*right*) edges for Feynman diagrams in (60) with states at their endpoints

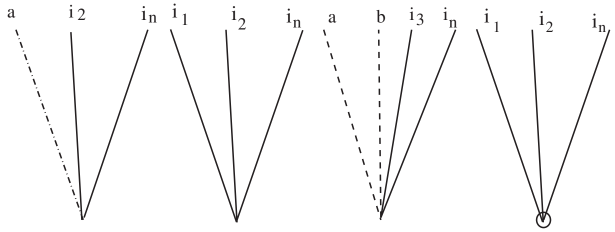

Fig. 2 Vertices for Feynman diagrams in (60) with states on their stars

integral :

$$Z_c = \int_{T_c\mathcal{F}(b)}^{formal} e^{\frac{i}{h}S(x)}\frac{dx}{db} = |G^B|(2\pi h)^{\frac{\dim F(b)-\dim G^B}{2}}$$
$$\times \frac{1}{\sqrt{|\det(B(c))|}}\det(L_\varphi(c))\cdot\exp\left(\frac{i}{h}S(c)+\frac{i\pi}{4}\operatorname{sign}(B(c))\right)$$
$$\times\left(v(c)+\sum_{\Gamma\neq\varnothing}\frac{(ih)^{-\chi(\Gamma)}(-1)^{c(D(\Gamma))}F_c(D(\Gamma))}{|\operatorname{Aut}(\Gamma)|}\right), \tag{60}$$

Here $D(\Gamma)$ is the planar projection of Γ, a Feynman diagram. Feynman diagrams in this formula have bosonic edges and fermionic oriented edges, $c(D(\Gamma))$ is the number of crossings of fermionic edges.[13] The structure of Feynman diagrams is the same as in (55). The propagators corresponding to Bose and Fermi edges are shown in Fig. 1. The weights of vertices are shown on Fig. 2.

The weight of the fermionic edge on Fig. 1 is $(L_\varphi(c)^{-1})_{ab}$. Weights of the bosonic edges from Fig. 1 correspond to matrix elements of $B(c)^{-1}$ where

[13] The sign rule is equivalent to the usual $(-1)^{\#\text{fermionic loops}}$ which is used in physics literature.

$$B(c) = \begin{pmatrix} \frac{\partial^2 S(c)}{\partial x^i \partial x^j} & \frac{\partial \varphi^a(c)}{\partial x^i} \\ \frac{\partial \varphi^b(c)}{\partial x^j} & 0 \end{pmatrix}$$

The weights of vertices with states on their stars from Fig. 2 are (from left to right):

$$\frac{\partial^{n-1}\varphi^a(c)}{\partial x^{i_2}, \ldots, \partial x^{i_n}}, \quad \frac{\partial^n S(c)}{\partial x^{i_1}, \ldots, \partial x^{i_n}}, \quad \frac{\partial^{n-2} L_\varphi(c)^a_b}{\partial x^{i_3}, \ldots, \partial x^{i_n}}, \quad \frac{\partial^n v(c)}{\partial x^{i_1}, \ldots, \partial x^{i_n}}$$

The last vertex should appear exactly once in each diagram.

This formula, by construction, does not depend on the choice of local coordinates. It is easy to see this explicitly at the level of determinants. Indeed, when we change local coordinates, we have

$$B(c) \mapsto \begin{pmatrix} J^T & 0 \\ 0 & 1 \end{pmatrix} B(c) \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}, \quad v \mapsto |\det(J)| v$$

where J is the Jacbian of the coordinate transformation. It is clear that the ratio $v/|\det(B(c))|$ is invariant with respect to such transformations.

Note that because we defined the formal integral (60) as the contribution to the asymptotical expansion of the integral (58) from the critical orbit of S passing through c, the coefficients in (60) do not depend on the choice of gauge constraint φ and

$$Z_c = Z_{[c]}$$

where $[c] = O$ is the orbit of G^B passing through c.

4.3.2 Gluing Formal Integrals for Gauge Theories

Assume that as in Sect. 4.2.3 we have two spaces F_1 and F_2 fibered over B_∂ and two functions S_1 and S_2 defined on F_1 and F_2 respectively such that the integrals $Z_{F_1}(b)$ and $Z_{F_2}(b)$ converge absolutely for generic b. For example, we can assume that spaces F_1, F_2 and B_∂ are compact. Denote by F the fiber product $F_1 \times_{B_\partial} F_2$ and set $N_i = \dim F_i$, $N_\partial = \dim B_\partial$. Let Lie groups G_1, G_2 and Γ_∂ act as $G_i : F_i \to F_i$ and $\Gamma_\partial : B_\partial \to B_\partial$ and assume that functions S_i are G_i-invariant and Γ_∂ appears in exact sequences:

$$1 \to G_1^B \to G_1 \to \Gamma_\partial \to 1, \quad 1 \to G_2^B \to G_2 \to \Gamma_\partial \to 1$$

where kernels G_1^B and G_2^B are bulk gauge groups for F_1 and F_2.

Changing the order of integration we obtain (53). As $h \to 0$ the gluing identity (53) becomes the identity between formal integrals just as in the non-degenerate case

$$\int\limits_{T_{b_0} B_\partial}^{formal} Z_{[c_1(b)]} Z_{[c_2(b)]} db = Z_{[c]}$$

which should be regarded as the contribution of the critical point c to Z_F written as an iterated integral.[14] After a gauge fixing in the integral over b we arrive to the following formula for the left side:

$$\begin{aligned} Z_{[c]} = |G_1^B||G_2^B||\Gamma_\partial|(2\pi h)^{\frac{N-n}{2}} & \frac{\det(L_{\varphi_1}(c_1))\ \det(L_{\varphi_2}(c_2))\ \det(L_{\varphi_\partial}(c_\partial))}{\sqrt{|\det(B_1(c_1))|\ |\det(B_2(c_2))|\ |\det(B_\partial(c_\partial))|}} \\ & \times \exp\left(\frac{i}{h}(S_1(c_1)+S_2(c_2)) + \frac{i\pi}{4}(\operatorname{sign}(B_1(c_1)) + \operatorname{sign}(B_2(c_2)) + \operatorname{sign}(B_\partial(c_\partial)))\right) \\ & \times \left(v_1(c_1)v_2(c_2)v_\partial(c_\partial) + \sum_{\Gamma\neq\varnothing} \text{composite Feynman diagrams} \right), \end{aligned} \tag{61}$$

Here $N = N_1 + N_2 - N_\partial = \dim F$ and $n = n_1 + n_2 - n_\partial$ were $n_i = \dim G_i$ and $n_\partial = \dim \Gamma_\partial$. Composite Feynman diagrams consist of Feynman diagrams for F_1, Feynman diagrams for F_2 and Feynman diagrams connecting them which come from formal integration over boundary fields in the formal neighborhood of b_0. Factors $v_1(c_1)$, $v_2(c_2)$, $v_\partial(c_\partial)$ are densities of corresponding measures in local coordinates which we used in (61).

Comparing this expression with (60) besides the obvious identity $S(c) = S(c_1) + S(c_2)$ we obtain identities

$$\begin{aligned} & \frac{\det(L_{\varphi_1}(c_1))\ \det(L_{\varphi_2}(c_2))\ \det(L_{\varphi_\partial}(c_\partial))}{\sqrt{|\det(B_1(c_1))|\ |\det(B_2(c_2))|\ |\det(B_\partial(c_\partial))|}} \\ & \quad \cdot \exp\left(\frac{i\pi}{4}(\operatorname{sign}(B_1(c_1)) + \operatorname{sign}(B_2(c_2)) + \operatorname{sign}(B_\partial(c_\partial)))\right) \\ & \quad = \frac{\det(L_\varphi(c))}{\sqrt{|\det(B(c))|}} \exp\left(\frac{i\pi}{4}\operatorname{sign}(B(c))\right) \end{aligned} \tag{62}$$

In addition to this, in each order h^m with $m > 0$ we will have the following identity: the sum of all composite Feynman diagrams of order m for F_1, F_2, B_∂ equals the sum of all Feynman diagrams of order m for F.

5 Abelian Chern-Simons Theory

In TQFT's there are no ultraviolet divergencies but there is a gauge symmetry to deal with. Perhaps the simplest non-trivial example of TQFT is the Abelian Chern-Simons theory with the Lie group $\mathbb{R}$. Fields in such theory are connections on the

[14] Recall that db is a Γ_∂-invariant measure on B_∂ such that $\frac{dx}{db}db$ is a G-invariant measure on F.

trivial $\mathbb{R}$-bundle over a compact, smooth, oriented 3-dimensional manifold M. We will identify fields with 1-forms on M. The action is

$$S(A) = \frac{1}{2} \int_M A \wedge dA$$

Solutions of the Euler-Lagrange equations are closed 1-forms on M. The variation of this action induces the exact symplectic form on $\Omega^1(\partial M)$ (see Sect. 3.4).

5.1 The Classical Action and Boundary Conditions

A choice of metric on M induces a metric on ∂M and the Hodge decomposition:

$$\Omega(\partial M) = d\Omega(\partial M) \oplus H(\partial M) \oplus d^* \Omega(\partial M)$$

The Lagrangian subspace of boundary values of solutions to Euler-Lagrange equations is

$$L_M = H^1_M(\partial M) \oplus d\Omega^0(\partial M)$$

where $H_M(\partial M)$ is the space of harmonic representatives of cohomology classes on the boundary coming from cohomology classes $H^1(M)$ of the bulk by pull-back with respect to inclusion of the boundary.

Choose a decomposition of $H(\partial M)$ into a direct sum of two Lagrangian subspaces:

$$H(\partial M) = H_+(\partial M) \oplus H_-(\partial M)$$

This induces a decomposition of forms $\Omega(\partial M) = \Omega_+(\partial M) \oplus \Omega_-(\partial M)$ where

$$\Omega_+(\partial M) = H_+(\partial M) \oplus d\Omega(\partial M), \quad \Omega_-(\partial M) = H_-(\partial M) \oplus d^* \Omega(\partial M)$$

Choose the boundary Lagrangian fibration as

$$p_\partial : \Omega(\partial M) \to B(\partial M) = \Omega_+(\partial M)$$

with fibers

$$p_\partial^{-1}(b) = b + \Omega_-(\partial M) \simeq H_-(\partial M) \oplus d^* \Omega(\partial M).$$

This fibration is not $\alpha_{\partial M}$-exact, i.e. the restriction of $\alpha_{\partial M}$ to fibers is zero. Let us modify the action, by adding a boundary term such that the form $\alpha_{\partial M}$ will vanish on fibers of p. Define the new action as

$$\widetilde{S}(A) = S(A) + \frac{1}{2}\int\limits_{\partial M} A_+ \wedge A_-$$

where $A_\pm$ are $\Omega_\pm$-components of $i^*(A)$.

The new form on boundary connections is

$$\widetilde{\alpha}_{\partial M}(a) = \alpha_{\partial M}(a) + \frac{1}{2}\delta\int\limits_{\partial M} a_+ \wedge a_- = -\int\limits_{\partial M} a_- \wedge \delta a_+$$

and it vanishes on the fibers of p_∂ because on each fiber $\delta a_+ = 0$.

Note that the modified action is gauge invariant. Indeed, on components $A_\pm$ gauge transformations act as $A_+ \mapsto A_+ + d\theta$ and $A_- \mapsto A_-$, i.e. gauge transformations act trivially on fibers.

5.2 Formal Semiclassical Partition Function

5.2.1 More on Boundary Conditions

For this choice of Lagrangian fibration the bulk gauge group G^B is $\Omega^0(M, \partial M)$. The boundary gauge group acts trivially on fibers. Indeed, the boundary gauge group $\Omega^0(\partial M)$ acts naturally on the base $B(\partial M) = H^1(\partial M)_+ \oplus d\Omega^0(\partial M), \alpha \mapsto \alpha + d\lambda$. It acts on the base shifting the fibers: $p(\beta + d\lambda) = p(\beta) + d\lambda$.

According to the general scheme outlined in Sect. 4.3, in order to define the formal semiclassical partition function we have to fix a background flat connection a and "integrate" over the fluctuations $\sqrt{h}\alpha$ with boundary condition $i^*(\alpha)_+ = 0$. We have

$$\widetilde{S}(a + \alpha) = \tilde{S}(\alpha) + \frac{1}{2}\int\limits_{\partial M} a_+ \wedge a_-$$

Note that $da = 0$ which means that a restricted to the boundary is a closed form which we can write as $i^*(a) = [a]_+ + [a]_- + d\theta$ where $[a]_\pm \in H_\pm(\partial M)$. Therefore, for the action we have:

$$\widetilde{S}(a + \alpha) = \widetilde{S}(\alpha) + \frac{1}{2} < [a]_+, [a]_- >_{\partial M}$$

where $\langle ., .\rangle$ is the symplectic pairing in $H(\partial M)$.

For semiclassical quantization we should choose the gauge fixing submanifold $\Lambda \subset \Omega(M)$, such that $(T_a F_M)_+ = T_a EL \oplus T_a \Lambda$. Here $(T_a F_M)_+$ is the space of 1-forms (α-fields) with boundary condition $i^*(\alpha)_+ = 0$. As it is shown in Appendix D the action functional restricted to fields with boundary values in an isotropic subspace $I \subset \Omega^1(\partial M)$ is non-degenerate on

$$T_a\Lambda_I = d^*\Omega^2_N(M, I^\perp) \cap \Omega^1_D(M, I)$$

For our choice of boundary conditions $I = \Omega^1_-(\partial M)$.

5.2.2 Closed Space Time

First, assume the space time has no boundary. Then the formal semiclassical partition function is defined as the product of determinants which arise from gauge fixing and from the Gaussian integration as in (60). In the case of Abelian Chern-Simons the gauge condition is $d^*A = 0$ and the action of the gauge Lie algebra $\Omega^0(M)$ on the space of fields $\Omega^1(M)$ is given by the map $d : \Omega^0(M) \to \Omega^1(M)$ (here we identified $\Omega^1(M)$ with its tangent space at any point). Thus, the FP action (59) in our case is

$$S(A, \overline{c}, c, \lambda) = \frac{1}{2}\int\limits_M A \wedge dA + \int\limits_M \overline{c}\ \ \Delta c\ \ d^3x + \int\limits_M \lambda\ \ d^*A\ \ d^3x$$

where $\overline{c}$, c are ghost fermion fields, and λ is the Lagrange multiplier for the constraint $d^*A = 0$.

By definition the corresponding Gaussian integral is

$$Z_a = C\frac{|\det'(\Delta_0)|}{\sqrt{|\det'(\widehat{*d})|}}\exp(\frac{i\pi}{4}(2\mathrm{sign}(\Delta_0) + \mathrm{sign}(\widehat{*d})))$$

Here $\det'$ is a regularized determinant and $\mathrm{sign}(A)$ is the signature of the differential operator A. The constant depends of the choice of regularization. The usual choice is the ζ-regularization. The signature is up to a normalization the eta invariant [31]. The operator $\widehat{*d}$ acts on $\Omega^1(M) \oplus \Omega^0(M)$ as

$$\begin{pmatrix} *d & d \\ d^* & 0 \end{pmatrix} \tag{63}$$

Its square is the direct sum of Laplacians:

$$\widehat{*d}^2 = \begin{pmatrix} d^*d + dd^* & 0 \\ 0 & d^*d \end{pmatrix}$$

Thus the regularized determinant of $\widehat{*d}$ is the product of determinants acting on 1-forms and on 0-forms:

$$|\det'(\widehat{*d})|^2 = |\det'(\Delta_1)||\det'(\Delta_0)|$$

This gives the following formula for the determinant contribution to the partition function:

$$\frac{|\det'(\Delta_0)|}{\sqrt{|\det'(\widehat{*d})|}} = \frac{|\det'(\Delta_0)|^{\frac{3}{4}}}{|\det'(\Delta_1)|^{\frac{1}{4}}} \tag{64}$$

Taking into account that $*\Omega^i(M) = \Omega^{3-i}(M)$ we can write this as

$$T^{1/2} = |\det'(\Delta_1)|^{\frac{1}{4}} |\det'(\Delta_2)|^{\frac{2}{4}} |\det'(\Delta_3)|^{\frac{3}{4}}$$

where T is the Ray-Singer torsion. This gives well-known formula for the absolute value of the partition function of the Abelian Chern-Simons theory on a closed manifold.

$$|Z| = CT^{1/2} \tag{65}$$

We will not discuss here the η-invariant part.

Remark 11 The operator $\widehat{*d}$ is easy to identify with $L_- = *d + d*$, acting on $\Omega^1(M) \oplus \Omega^3(M)$ from [31]. Indeed, using Hodge star we can identify $\Omega^0(M)$ and $\Omega^3(M)$. After this the operators are related as

$$L_- = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \widehat{*d} \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}^{-1}$$

Remark 12 There is one more formula in the literature for gauge fixing. Assume that a Lie group G has an invariant inner product, the space of fields F is a Riemannian manifold and G acts by isometries on F. In this case there is a natural gauge fixing which leads to the following formula for an integral of a G-invariant function [27]:

$$\int_F h(x)dx = |G| \int_{F/G} h(x)(\det'(\tau_x^* \tau_x))^{\frac{1}{2}} [dx]$$

Here we assume that the G-action does not have stabilizers. The linear mapping $\tau_x : \mathfrak{g} \to T_x F$ is given by the G-action, the Hermitian conjugate is taken with respect to the metric structure on F and on G, dx is the Riemannian volume on F and $[dx]$ is the Riemannian volume on F/G with respect to the natural Riemannian structure on the quotient space.

For the Abelian Chern-Simons a choice of metric on the space time induces metrics on $G = \Omega^0(M)$ and on $F = \Omega^1(M)$. The gauge group G acts on F by isometries and $\tau_x = d$, the de Rham differential. This gives another expression for the absolute value of the partition function

$$|Z| = C \frac{|\det'(\Delta_0)|^{\frac{1}{2}}}{|\det'(*d)|^{\frac{1}{2}}} \tag{66}$$

Here $*d : \Lambda \to \Lambda$, and $\Lambda = d^*\Omega^2(M)$ is the submanifold on which the action functional is non-degenerate. It is clear that this formula coincides with (65).

5.2.3 Space Time with Boundary

Now let us consider the case when ∂M is non-empty. In this case the bulk gauge group G^B is $\Omega_D(M, \{0\})$ which we will denote just $\Omega_D(M)$. The space of fluctuations is $\Omega^1_D(M, \Omega_-(\partial M))$. The bilinear from in the Faddeev-Popov action is

$$\frac{1}{2}\int_M \alpha \wedge d\alpha + \int_M \lambda\, d^*\alpha\, d^3x - i\int_M \bar{c}\, \Delta c\, d^3x$$

The even part of this form is symmetric if we impose the boundary condition $i^*(\lambda) = 0$. Similarly to the case of closed space time we can define the partition function as

$$Z_{a,M} = C|\det{}'(\widehat{*d})|^{-1/2}|\det{}'(\Delta_0^{D,\{0\}})|\exp(\frac{i\pi}{4}(2\,\mathrm{sign}(\Delta_0) + \mathrm{sign}(\widehat{*d})))$$
$$\exp(\frac{i}{h} < [a]_+, [a]_- >_{\partial M}) \tag{67}$$

Here $\Delta_0^{D,\{0\}}$ is the Laplace operator action on $\Omega_D(M, \{0\})$ and $[a]_\pm$ are the $\pm$ components of the cohomology class of the boundary value $i^*(a)$ of a. The operator $\widehat{*d}$ acts on $\Omega^1_D(M, \Omega_-(\partial M)) \oplus \Omega^0_D(M, \{0\})$ and is given by (63). This ratio of determinants is expected to give a version of the Ray-Singer torsion for appropriate boundary conditions. The signature contributions are expected to be the η-invariant with the appropriate boundary conditions. For the usual choices of boundary conditions, such as tangent, absolute, or APS boundary conditions at least some of these relations are known, for more general boundary conditions it is a work in progress.

5.2.4 Gluing

According to the finite dimensional gluing formula we expect a similar gluing formula for the partition function. A consequence of this formula is the multiplicativity of the version of the Ray-Singer torsion with boundary conditions described above. To illustrate this, let us take a closer look at the exponential part of (67).

Recall that $L_M \subset \Omega^1(\partial M)$ is the space of closed 1-forms which are boundary values of closed 1-forms on M. To fix boundary conditions we fixed the decomposition $\Omega^1(\partial M) = \Omega^1(\partial M)_+ \oplus \Omega^1(\partial M)_-$ (see above).

Let β be a tangent vector to L_M at the point $i^*(a) \in L_M$. We have natural identifications

$$T_{i^*(a)}\Omega^1(\partial M)_- = H^1(\partial M)_- \oplus d^*\Omega^2(\partial M),\;\; T_{i^*(a)}\Omega^1(\partial M)_+ = H^1(\partial M)_+ \oplus d\Omega^0(\partial M)$$

Denote by $\beta_\pm$ the components of β in $T_{i^*(a)}L_\pm$ respectively. Since $d\beta = 0$ we have $\beta_+ = [\beta]_+ + d\theta$, and $\beta_- = [\beta]_-$, where $[\beta]_\pm$ are components of the cohomology $[\beta]_\pm$ in $H^1(\partial M)_\pm$. If the reduced tangent spaces $[T_{i^*(a)}L_M] = H^1_\pm(\partial M)$ and $[T_{i^*(a)}\Omega^1(\partial M)_\pm] = H^1_M(\partial M)$ are transversal, which is what we assume here, projections to $[T_{i^*(a)}\Omega^1(\partial M)_\pm]$ give linear isomorphisms $A^{(\pm)}_M : H^1_M(\partial M) \to H^1_\pm(\partial M)$. This defines the linear isomorphism

$$B_M = A^{(-)}_M (A^{(+)}_M)^{-1} : H^1(\partial M)_+ \to H^1(\partial M)_-$$

acting as $B_M([\beta]_+) = [\beta]_-$ for each $[\beta] \in H^1_M(\partial M)$. This is the analog of the Dirichlet-to-Neumann operator.

Now considering small variations around a have

$$\begin{aligned} Z_{[a+\sqrt{h}\beta]} = Z_{[a]} \exp(\frac{i}{\sqrt{h}}(&< [i^*(a)]_+, B_M([i^*(\beta)]_+) >_{\partial M} \\ &+ < [i^*(\beta)]_+, [i^*(a)]_- >_{\partial M}) + i < [i^*(\beta)]_+, B_M([i^*(\beta)]_+) >_{\partial M}) \end{aligned} \tag{68}$$

The gluing formula for this semiclassical partition function at the level of exponents gives the gluing formula for Hamilton-Jacobi actions. At the level of pre-exponents it also gives the gluing formula for torsions and for the η-invariant for appropriate boundary conditions. Changing boundary conditions results in a boundary contribution to the partition function and to the gluing identity. One should also expect the gluing formula for correlation functions. The details of these statements require longer discussion and substantial analysis and will be done elsewhere.

There are many papers on Abelian Chern-Simons theory. The appearance of torsions and η-invariants in the semiclassical asymptotics of the path integral for the Chern-Simons action was first pointed out in [31]. For a geometric approach to compact Abelian Chern-Simons theory and a discussion of gauge fixing and the appearance of torsions in the semiclassical analysis see [24]. For the geometric quantization approach to the Chern-Simons theory with compact Abelian Lie groups see [2].

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Appendix

A Discrete Time Quantum Mechanics

An example of a finite dimensional version of a classical field theory is a discrete time approximation to the Hamiltonian classical mechanics of a free particle on $\mathbb{R}$. We denote coordinates on this space (p, q) where p represents the momentum and q represents the coordinate of the system.

In this case the space time is an ordered collection of n points which represent the discrete time interval. If we enumerate these points $\{1, \ldots, n\}$ the points $1, n$ represent the boundary of the space time. The space of fields is $\mathbb{R}^{n-1} \times \mathbb{R}^n$ with coordinates p_i where $i = 1, \ldots, n-1$ represents the "time interval" between points i and $i+1$ and q_i where $i = 1, \ldots, n$. The coordinates p_1, p_{n-1}, q_1, q_n are boundary fields.[15] The action is

$$S = \sum_{i=1}^{n-1} p_i(q_{i+1} - q_i) - \sum_{i=1}^{n-1} \frac{p_i^2}{2}$$

We have

$$dS = \sum_{i=1}^{n-2}(q_{i+1} - q_i - p_i)\, dp_i + \sum_{i=2}^{n-1}(p_{i-1} - p_i)\, dq_i + p_{n-1}dq_n - p_1 dq_1$$

From here we derive the Euler-Lagrange equations

$$q_{i+1} - q_i = p_i, \quad i = 1, \ldots n-1,$$

$$p_{i-1} - p_i = 0, \quad i = 2, \ldots, n-1$$

and the boundary 1-form

$$\alpha = p_{n-1}dq_n - p_1 dq_1$$

This gives the symplectic structure on the space of boundary fields with

$$\omega_\partial = dp_{n-1} \wedge dq_n - dp_1 \wedge dq_1$$

The boundary values of solutions of the Euler-Lagrange equations define the subspace

$$L = \pi(EL) = \{(p_1, q_1, p_{n-1}, q_n) | p_1 = p_{n-1}, q_n = q_1 + (n-1)p_1\}$$

---

[15] In other words the space time is a 1-dimensional cell complex. Fields assign coordinate function q_i to the vertex i and p_i to the edge $[i, i+1]$.

It is clear that this a Lagrangian subspace.

B Feynman Diagrams

Let Γ be a graph with vertices of valency ≥ 3 with one special vertex which may also have valency 0, 1, 2. We define the weight $F_c(\Gamma)$ as follows.

A state on Γ is a map from the set of half-edges of Γ to the set $1, \ldots, n$, for an example see Fig. 3. The weight of Γ is defined as

$$F_c(\Gamma) = \sum_{states} \left(\frac{\partial^l v}{\partial x^{j_1}, \ldots, \partial x^{j_l}}(c) \prod_{vertices} \frac{\partial^k S}{\partial x^{i_1}, \ldots, \partial x^{i_k}}(c) \prod_{edges} (B_c^{-1})_{ij} \right)$$

Here the sum is taken over all states on Γ, and $i_1, \ldots, i_k$ are states on the half-edges incident to a vertex. The first factor is the weight of the special vertex where v is the density of the integration measure in local coordinates $\frac{dx}{db} = v(x)dx^1, \ldots, dx^N$. The pair (i, j) is the pair of states at the half-edges comprising an edge. Note that weights of vertices and the matrix B_c are symmetric. This makes the definition meaningful.

C Gauge Fixing in Maxwell's Electromagnetism

In the special case of electromagnetism ($G = \mathbb{R}, \mathfrak{g} = \mathbb{R}$), the space of fields is $F_M = \Omega^1(M) \oplus \Omega^{n-2}(M)$ and similarly for the boundary. If M has no boundary, the gauge group $G_M = \Omega^0(M)$ acts on fields as follows: $A \mapsto A + d\alpha$, $B \mapsto B$. We can construct a global section of the corresponding quotient using Hodge decomposition: we know that

$$\Omega^\bullet(M) \cong \Omega^\bullet_{\text{exact}}(M) \oplus H^\bullet(M) \oplus \Omega^\bullet_{\text{coexact}}(M) \tag{69}$$

where the middle term consists of harmonic forms. In particular,

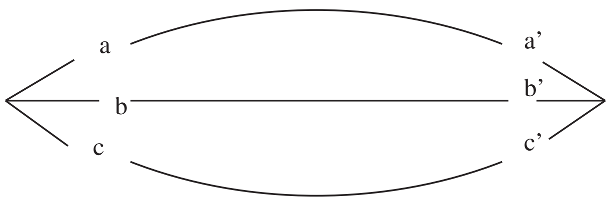

Fig. 3 The "theta" diagram

$$\Omega^1(M) = d\Omega^0(M) \oplus H^1(M) \oplus d^*\Omega^2(M) \tag{70}$$

where the last two terms give a global section. In physics, choosing a global section is called gauge fixing, and this particular choice of gauge is called the Lorentz gauge, where $d^*A = 0$.

D Hodge Decomposition for Riemannian Manifolds With Boundary

D.1 Hodge Decomposition With Dirichlet and Neumann Boundary Conditions

Let M be a smooth oriented Riemannian manifold with boundary ∂M. Recall some basic facts about the Hodge decomposition of differential forms on M. Choose local coordinates near the boundary in which the metric has the product structure with t being the coordinate in the normal direction. Near the boundary any smooth form can be written as

$$\omega = \omega_{tan} + \omega_{norm} \wedge dt$$

where ω_{tan} is the tangent component of ω near the boundary and ω_{norm} is the normal component.

We will denote by $\Omega_D(M)$ the space of forms satisfying the Dirichlet boundary conditions $\iota^*(\omega) = 0$ where ι^* is the pull-back of the form ω to the boundary. This condition can be also written as $\omega_{tan} = 0$.

We will denote by $\Omega_N(M)$ the space of forms satisfying the Neumann boundary conditions $\iota^*(*\omega) = 0$. Here $* : \Omega^i(M) \to \Omega^{n-i}(M)$ is the Hodge star operation, recall that $*^2 = (-1)^{i(n-i)}\mathrm{id}$ on $\Omega^i(M)$. Because $\omega_{norm} = *'\iota^*(*\omega)$ the Neumann boundary condition can be written as $\omega_{norm} = 0$.

Denote by $d^* = (-1)^i *^{-1} d*$ the formal adjoint of d, and by $\Delta = dd^* + d^*d$ the Laplacian on M. Denote by $\Omega_{cl}(M)$ closed forms on M, $\Omega_{ex}(M)$ exact forms on, $\Omega_{cocl}(M)$ the space of coclosed forms, i.e. closed with respect to d^* and by $\Omega_{coex}(M)$ the space of coexact forms.

Define subspaces:

$$\Omega_{cl,cocl}(M) = \Omega_{cl}(M) \cap \Omega_{cocl}(M), \quad \Omega_{cl,coex}(M) = \Omega_{cl}(M) \cap \Omega_{coex}(M)$$

and similarly $\Omega_{ex,cocl}(M)$, $\Omega_{cl,cocl,N}(M)$ and $\Omega_{cl,cocl,D}(M)$.

Theorem 3 (1) *The space of forms decomposes as*

$$\Omega(M) = d^*\Omega_N(M) \oplus \Omega_{cl,cocl}(M) \oplus d\Omega_D(M)$$

(2) *The space of closed, coclosed forms decomposes as*

$$\Omega_{cl,cocl}(M) = \Omega_{cl,cocl,N}(M) \oplus \Omega_{ex,cocl}(M)$$

$$\Omega_{cl,cocl}(M) = \Omega_{cl,cocl,D}(M) \oplus \Omega_{cl,coex}(M)$$

We will only outline the proof of this theorem. For more details and references on the Hodge decomposition for manifolds with boundary and Dirichlet and Neumann boundary conditions see [17]. Riemannian structure on M induces the scalar product on forms

$$(\omega, \omega') = \int_M \omega \wedge *\omega' \tag{71}$$

For two forms of the same degree we have $\omega(x) \wedge *\omega'(x) = \langle \omega(x), \omega'(x) \rangle \, dx$ where dx is the Riemannian volume form and $\langle ., . \rangle$ is the scalar product on $\wedge^k T_x^* M$ induced by the metric. This is why (71) is positive definite.

Lemma 1 *With respect to the scalar product* (71)

$$(d\Omega_D(M))^\perp = \Omega_{cocl}$$

Proof By the Stokes theorem for any form $\theta \in \Omega_D^{i-1}(M)$ we have

$$(\omega, d\theta) = \int_M \omega \wedge *d\theta = (-1)^{(i+1)(n-i)} (\int_{\partial M} \iota^*(*\omega) \wedge \iota^*(\theta) + \int_M d * \omega \wedge \theta)$$

The boundary integral is zero because $\theta \in \Omega_D(M)$. Thus $(\omega, d\theta) = 0$ for all θ if and only if $d * \omega = 0$ which is equivalent to $\omega \in \Omega_{cocl}(M)$.

Corollary 1 *Because $d\Omega_D(M) \subset \Omega_{cl}(M)$, we have $\Omega_{cl}(M) = \Omega_{cl}(M) \cap (d\Omega_D(M))^\perp \oplus d\Omega_D(M)$. i.e.*

$$\Omega_{cl}(M) = \Omega_{cl,cocl}(M) \oplus d\Omega_D(M)$$

Here we are sketchy on the analytical side of the story. If $U \subset V$ is a subspace in an inner product space, in the infinite dimensional setting more analysis might be required to prove that $V = U \oplus U^\perp$. Here and below we just assume that this does not create problems. Similarly to Lemma 1 we obtain

$$(d^*\Omega_N(M))^\perp = \Omega_{cl}(M)$$

This completes the sketch of the proof of the first part. The proof of the second part is similar.

Note that the spaces in the second part of the theorem are harmonic forms representing cohomology classes:

$$\Omega_{cl,cocl,N}(M) = H(M), \quad \Omega_{cl,cocl,D}(M) = H(M, \partial M)$$

D.2 More General Boundary Conditions

D.2.1 General setup

Assume that M is a smooth compact Riemannian manifold, possibly with non-empty boundary ∂M. Let $\pi : \Omega^i(M) \to \Omega^i(\partial M)$, $i = 0, \dots, n-1$ be the restriction map (the pull-back of a form to the boundary) and $\pi(\Omega^n(M)) = 0$.

The Riemannian structure on M induces the metric on ∂M. Denote by $*$ the Hodge star for M, and by $*_\partial$ the Hodge star for the boundary $*_\partial : \Omega^i(\partial M) \to \Omega^{n-1-i}(\partial M)$. Define the map $\widetilde{\pi} : \Omega(M) \to \Omega(\partial M)$, $i = 1, \dots, n$ as the composition $\widetilde{\pi}(\alpha) = *_\partial \pi(*\alpha)$. Note that $\tilde{\pi}(\Omega^0(M)) = 0$.

Denote by $\Omega_D(M, L)$ and $\Omega_N(M, L)$ the following subspaces:

$$\Omega_D(M, L) = \pi^{-1}(L), \quad \Omega_N(M, L) = \widetilde{\pi}^{-1}(L)$$

where $L \subset \Omega(\partial M)$ is a subspace.

Denote by $L^\perp$ the orthogonal complement to L with respect to the Hodge inner product on the boundary. The following is clear:

Lemma 2

$$(*L^{(i)})^\perp = *(L^{(i)})^\perp, *(L^\perp) = L^{sort}$$

Here L^{sort} is the space which is symplectic orthogonal to L.

Proposition 3 $(d^* \Omega_N(M, L))^\perp = \Omega_D(M, L^\perp)_{cl}$

Proof Let ω be an i-form on M such that

$$\int_M \omega \wedge d * \alpha = 0$$

for any α. Applying Stocks theorem we obtain

$$\int_M \omega \wedge d * \alpha = (-1)^i \int_{\partial M} \pi(\omega) \wedge *_\partial \tilde{\pi}(\alpha) + (-1)^{i+1} \int_M d\omega \wedge *\alpha$$

The boundary integral is zero for any α if and only if $\pi(\omega) \in L^\perp$ and the bulk integral is zero for any α if and only if $d\omega = 0$.

As a corollary of this we have the orthogonal decomposition

$$\Omega(M) = \Omega_D(M, L^\perp)_{cl} \oplus d^* \Omega_N(M, L)$$

Similarly, for each subspace $L \subset \Omega(\partial M)$ we have the decomposition

$$\Omega(M) = \Omega_N(M, L^\perp)_{cocl} \oplus d\Omega_D(M, L)$$

Now, assume that we have two subspaces $L, L_1 \subset \Omega(\partial M)$ such that

$$d_\partial(L_1^\perp) \subset L^\perp, \tag{72}$$

Note that this implies $d_\partial^* L \subset L_1$. Indeed, fix $\alpha \in L$, then (72) implies that for any $\beta \in L_1^\perp$ we have

$$\int_{\partial M} \alpha \wedge *d_\partial \beta = 0$$

This is possible if and only if

$$\int_{\partial M} *d_\partial * \alpha \wedge *\beta = 0$$

Thus, $d_\partial^* \alpha \in L_1$. Here we assumed that $(L_1^\perp)^\perp = L_1$.

Because $\pi d = d_\partial \pi$ and $\tilde{\pi} d^* = d_\partial^* \tilde{\pi}$ we also have

$$d\Omega_D(M, L_1^\perp) \subset \Omega_D(M, L^\perp)_{cl}, \quad d^*\Omega_N(M, L) \subset \Omega_N(M, L_1)_{cocl}$$

Theorem 4 *Under assumption* (72) *we have*

$$\Omega(M) = d^*\Omega_N(M, L) \oplus \Omega_D(M, L^\perp)_{cl} \cap \Omega_N(M, L_1)_{cocl} \oplus d\Omega_D(M, L_1^\perp) \tag{73}$$

Indeed, if $V, W \subset \Omega$ are liner subspaces in the scalar product space Ω such that $W \subset V^\perp$ and $V \subset W^\perp$ then $\Omega = V \oplus V^\perp = W \oplus W^\perp$ and

$$\Omega = V \oplus W^\perp \cap V^\perp \oplus W$$

We will call the identity (73) the Hodge decomposition with boundary conditions. The following is clear:

Theorem 5 *The decomposition* (73) *agrees with the Hodge star operation if and only if*

$$*L_1^\perp = L$$

Remark 13 In the particular case $L = \{0\}$ and $L_1^\perp = \{0\}$ we obtain the decomposition from the previous section:

$$\Omega(M) = d^*\Omega_N(M) \oplus \Omega_{cl,cocl}(M) \oplus d\Omega_D(M)$$

Lemma 3 *If $L \subset \Omega(\partial M)$ is an isotropic subspace then $*L \subset \Omega(\partial M)$ is also an isotropic subspace.*

Indeed, if L is isotropic then for any $\alpha, \beta \in L$ we have $\int\limits_{\partial M} \alpha \wedge *\beta = 0$, but

$$\int\limits_{\partial M} *\alpha \wedge *^2\beta = \pm \int\limits_{\partial M} \alpha \wedge *\beta$$

therefore $*L$ is also isotropic.

Remark 14 We have

$$*\Omega_N(M) = \Omega_D(M), \quad *H(M) = H(M, \partial M)$$

In the second formula $H(M)$ is the space of closed-coclosed forms with Neumann boundary conditions and $H(M, \partial M)$ is the space of closed-coclosed forms with Dirichlet boundary conditions. They are naturally isomorphic to corresponding cohomology spaces. Note that as a consequence of the first identity we have $*d^*\Omega_N(M) = d\Omega_D(M)$. We also have more general identity

$$*\Omega_N(M, L) = \Omega_D(M, *_\partial L)$$

and consequently $*\Omega_D(M, L) = \Omega_N(M, *_\partial L)$.

Let π and $\tilde{\pi}$ be maps defined at the beginning of this section. Because π commutes with de Rham differential and $\tilde{\pi}$ commutes with its Hodge dual, we have the following proposition

Proposition 4 *Let $H_M(\partial M)$ be the space of harmonic forms on ∂M extendable to closed forms on M, then*

$$\pi(\Omega_{cl}(M)) = H_M(\partial M) \oplus d\Omega(\partial M), \quad \tilde{\pi}(\Omega_{cocl}(M)) = H_M(\partial M)^\perp \oplus d^*\Omega(\partial M)$$

Here is an outline of the proof. Indeed, let $\theta \in \Omega_{cl}(M)$ and $\sigma \in \Omega_{cocl}(M)$. Then

$$\int\limits_{\partial M} \pi(\theta) \wedge *_\partial \tilde{\pi}(\sigma) = \int\limits_{\partial M} \pi(\theta) \wedge \pi(*\sigma) = \int\limits_{M} d(\theta \wedge *\sigma)$$

The last expression is zero because by the assumption θ and $*\sigma$ are closed. The proposition follows now from the Hodge decomposition for forms on the boundary and from $\pi(\Omega_{cl}(M)) \subset \Omega_{cl}(\partial M)$, $\tilde{\pi}(\Omega_{cocl}(M)) \subset \Omega_{cocl}(\partial M)$.

D.2.2 dim $M = 3$

Let us look in details at the 3-dimensional case. In order to have the Hodge decomposition with boundary conditions we required

$$dL_1^{\perp} \subset L^{\perp}$$

If we want it to be *invariant with respect to the Hodge star* we should also have $*L_1^{\perp} = L$. Together these two conditions imply that L should satisfy $d * L \subset L^{\perp}$ or

$$\int_{\partial M} d * \alpha \wedge *\beta = 0$$

for any $\alpha, \beta \in L$. This condition is equivalent to

$$\int_{\partial M} d^* \alpha \wedge \beta = 0$$

for any $\alpha \in L^{(1)}$ and any $\beta \in L^{(2)}$.

Note that if $L^{(2)} = \{0\}$ we have no conditions on the subspace $L^{(1)}$. In this case for any choice of $L^{(0)}$ and $L^{(1)}$ the $*$-invariant Hodge decomposition is:

$$\Omega^0(M) = d^* \Omega_N^1(M, L^{(0)}) \oplus \Omega_D^0(M, L^{(0)\perp})_{cl}$$

$$\Omega^1(M) = d^* \Omega_N^2(M, L^{(1)}) \oplus \Omega_D^1(M, L^{(1)\perp})_{cl} \cap \Omega_N^1(M, L^{(0)})_{cocl} \oplus d\Omega_D^0(M)$$

Here we used $\Omega_N^i(M, L_1) = \Omega_N^i(M, L_1^{(i-1)}) = \Omega_N^i(M, (*L^{(3-i)})^{\perp})$. The condition $L^{(2)} = \{0\}$ implies that $\Omega_N^1(M, (*L^{(2)})^{\perp}) = \Omega^1(M)$. We also used $\Omega_D^0(M, L_1^{\perp}) = \Omega^0(M, *L^{(2)}) = \Omega_D^0(M)$.

The decomposition of 2- and 3-forms is the result of application of Hodge star to these formulae.

D.2.3 The gauge-fixing subspace

Consider the bilinear form

$$B(\alpha, \beta) = \int_M \beta \wedge d\alpha \tag{74}$$

on the space $\Omega^\bullet(M)$.

Let $I \subset \Omega^\bullet(\partial M)$ be an isotropic subspace.

Proposition 5 *The form B is symmetric on the space $\Omega_D(M, I)$.*

Indeed

$$\int_M (\beta \wedge d\alpha) = (-1)^{|\beta|+1} \int_{\partial M} \pi(\beta) \wedge \pi(\alpha) + \int_M d\beta \wedge \alpha = (-1)^{(|\alpha|+1)(|\beta|+1)} B(\alpha, \beta)$$

The boundary term vanishes because boundary values of α and β are in an isotropic subspace I.

Proposition 6 *Let $I \subset \Omega(\partial M)$ be an isotropic subspace, then B is nondegenerate on $d^*\Omega_N(M, I^\perp) \cap \Omega_D(M, I)$.*

Proof If I is isotropic, $\beta \in \Omega_D(M, I)$ and $B(\beta, \alpha) = 0$ for any $\alpha \in \Omega_D(M, I)$, we have:

$$B(\beta, \alpha) = B(\alpha, \beta) = \int_M \alpha \wedge d\beta$$

and therefore $d\beta = 0$. Therefore $\Omega_D(M, I)_{cl}$ is the kernel of the form B on $\Omega_D(M, I)$. But we have the decomposition

$$\Omega(M) = \Omega_D(M, I)_{cl} \oplus d^*\Omega_N(M, I^\perp)$$

This implies

$$\Omega_D(M, I) = \Omega_D(M, I)_{cl} \oplus d^*\Omega_N(M, I^\perp) \cap \Omega_D(M, I)$$

This proves the statement.

In particular, the restriction of the bilinear form B is nondegenerate on $\Lambda_I = d^*\Omega^2_N(M, I^{(1)\perp}) \cap \Omega^1_D(M, I^{(1)})$. For the space of all 1-forms with boundary values in $I^{(1)}$ we have:

$$\Omega^1_D(M, I^{(1)}) = \Omega^1_D(M, I^{(1)})_{cl} \oplus d^*\Omega^2_N(M, I^{(1)\perp}) \cap \Omega^1_D(M, I^{(1)})$$

The first part is the space of solutions to the Euler-Lagrange equations with boundary values in $I^{(1)}$.

References

1. A. Alekseev, Y. Barmaz, P. Mnev, Chern-Simons Theory with Wilson lines and boundary in the BV-BFV formalism, arXiv:1212.6256
2. J.E. Andersen, Deformation quantization and geometric quantization of abelian moduli spaces. Comm. Math. Phys. **255**, 727–745 (2005)
3. M. Atiyah, On framings of 3-manifolds. Topology **29**(1), 1–7 (1990)
4. S. Axelrod, S.Della Pietra, E. Witten, Geometric quantization of Chern Simons gauge theory. J. Diff. Geom **33**, 787–902 (1991)
5. S. Axelrod, I.M. Singer, *Chern-Simons perturbation theory. I*, Perspectives in mathematical physics, pp. 17–49. in *Conference on Proceedings of Lecture Notes Mathematical Physics, III* (Int. Press, Cambridge, 1994)
6. S. Axelrod, I.M. Singer, Chern-Simons perturbation theory. II. J. Diff. Geom. **39**(1), 173–213 (1994)

7. J. Baez, J. Dolan, Higher-dimensional algebra and topological quantum field theory. J. Math. Phys. **36**(11), 6073–6105 (1995)
8. B. Bakalov, A. Kirillov, *Lectures on Tensor Categories and Modular Functors*, vol. 21, University Lecture Series (American Mathematical Society, Providence, 2001)
9. S. Bates, A. Weinstein, *Lectures on the Geometry of Quantization*, Berkeley Mathematics Lecture Notes series (AMS, Providence, 1997)
10. R. Bott, A.S. Cattaneo, Integral invariants of 3-manifolds. J. Diff. Geom. **48**(1), 91–133 (1998)
11. R. Bott, A.S. Cattaneo, Integral invariants of 3-manifolds. II. J. Diff. Geom. **53**(1), 1–13 (1999)
12. A.S. Cattaneo, P. Mnev, Remarks on Chern-Simons invariants. Comm. Math. Phys. **293**(3), 803–836 (2010)
13. A.S. Cattaneo, P. Mnev, N. Reshetikhin, Classical BV theories on manifolds with boundary. Comm. Math. Phys. **332**(2), 535–603 (2014)
14. A.S. Cattaneo, P. Mnev, N. Reshetikhin, Classical and quantum Lagrangian field theories with boundary, PoS(CORFU2011)044, arXiv:1207.0239 [math-ph]
15. A.S. Cattaneo, C.A. Rossi, Higher-dimensional BF theories in the Batalin-Vilkovisky formalism: the BV action and generalized Wilson loops, arXiv:math/0010172
16. S. Elitzur, G. Moore, A. Schwimmer, N. Seiberg, Remarks on the canonical quantization of the Chern-Simons-Witten theory. Nucl. Phys. **B326**, 108 (1989)
17. S. Cappell, D. DeTurck, H. Gluck, E. Miller, Cohomology of harmonic forms on Riemannian manifolds with boundary, arXiv:math/0508372 (math.DG)
18. A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula. Comm. Math. Phys. **212**(3), 591–611 (2000)
19. P. Deligne et al., *Quantum Fields and Strings: A Course for Mathematicians*, vol. 1 and 2 (American Mathematical Society, Providence, 1999)
20. D. Freed, Classical Chern-Simons theory. I. Adv. Math. **113**(2), 237–303 (1995)
21. B. Himpel, Lie groups and Chern-Simons theory, Lecture Notes, Centre for Quantum Geometry of Moduli Spaces, Aarhus University
22. T. Johnson-Freyd, Feynman-diagrammatic description of the asymptotics of the time evolution operator in quantum mechanics. Lett. Math. Phys. **94**(2), 123–149 (2010)
23. J. Lurie, On the classification of topological field theories. Curr. Dev. Math. **2009**, 129–280 (2008)
24. M. Manoliu, Abelian Chern-Simons Theory, arXiv:dg-ga/961000
25. A. Marini, Dirichlet and Neumann boundary value problems for Yang-Mills connections. Comm. Pure App. Math., XLV (1992), pp. 1015–1050
26. N. Reshetikhin, Lectures on quantization of gauge systems, in Proceedings of the Summer School "New paths towards quantum gravity", Holbaek, Denmark. ed. by B. Booss-Bavnbek, G. Esposito, M. Lesch. Lecture Notes in Physics, vol. 807 (Springer, Berlin, 2010), pp. 125–190
27. A. Schwarz, Instantons and fermions in the field of instanton. Comm. Math. Phys. **64**, 233–268 (1979)
28. G. Segal, The Definition of Conformal Field Theory, in *Topology, Geometry and Quantum Field Theory*, London Mathematical Society Lecture Note Series, vol. 308 (Cambridge University Press, 2004), pp. 421–577
29. S. Stoltz, P. Teichner, Supersymmetric Euclidean field theories and generalized cohomology, a survey, preprint, 2008
30. K. Symanzik, Schrödinger representation and Casimir effect in renormalizable quantum field theory, Nuclear Phys. *B 190* no. 1, FS 3 (1981) 1–44
31. E. Witten, Quantum field theory and the Jones polynomial. Comm. Math. Phys. **121**(3), 351–399 (1989)