

TOWARDS EQUIVARIANT YANG-MILLS THEORY

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ABSTRACT. We study four dimensional gauge theories in the context of an equivariant extension of the Batalin-Vilkovisky (BV) formalism. We discuss the embedding of BV Yang-Mills (YM) theory into a larger BV theory and their relation. Partial integration in the equivariant BV setting (BV push-forward map) is performed explicitly for the abelian case. As result, we obtain a non-local homological generalization of the Cartan calculus and a non-local extension of the abelian YM BV action which satisfies the equivariant master equation.

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1. INTRODUCTION

During the last thirty years equivariant localization played an instrumental role in the derivation of many exact results in quantum field

theory (QFT), in particular in the context of supersymmetric gauge theories. In QFT the equivariant localization is based on a clever combination of the supersymmetry transformations with BRST symmetry in such fashion that the resulting transformations are interpreted as the equivariant differential on the space of fields. In our previous work [5] we proposed a framework for the treatment of such theories within the equivariant extension of the Batalin-Vilkovisky (BV) formalism. In the present paper we continue to study this extension and discuss the equivariant partial BV integration (equivariant BV push-forward) map. Our main examples will be four-dimensional gauge theories.

We observe that the BV description of Yang-Mills (YM) theory given by Costello in [11] can be obtained restricting the BV action of Donaldson–Witten (DW) theory ([5], [7], [8], [12]) to a BV submanifold. Both theories can be recast in the AKSZ formulation, provided that, in the case of YM, one allows in the source a dga \mathcal{A} that is not locally freely generated. The BV push-forward map from DW theory to YM theory corrects the Costello’s BV YM action by terms that in the abelian case involve the zero modes of the ultraviolet (UV) fields but in the non abelian case are packed in a full perturbative expansion in the YM fields. In the equivariant case, we perform the explicit BV push-forward map in the case of abelian theory and obtain an effective action that is non-local and satisfies the equivariant master equation. From a geometrical point of view we can see this result as a non-local homological generalization of the Cartan calculus.

The paper is organized as follows: in Section 2 we recall basic notions within the BV formalism and the AKSZ construction of the solution of the classical master equation. We give the formal definition of the BV push-forward map and list its formal properties. In Section 3 we summarize the equivariant extension of the BV formalism and the corresponding equivariant version of the AKSZ construction. We define formally the equivariant analog of the BV push-forward map and discuss its properties. In Section 4 we define the 4d AKSZ theory to which we refer as the Donaldson–Witten (DW) theory. We discuss different formulations of this theory. In Section 5 we discuss the BV formulation of Yang–Mills (YM) theory and its relation to the DW theory. We briefly discuss the application of the BV push-forward map to the abelian DW theory. In Section 6 we introduce the equivariant extension of DW AKSZ theory. For the case of abelian DW theory we describe explicitly the equivariant BV push-forward map. The final result is a non-local extension of abelian YM theory which satisfies the equivariant master equation. This provides a non-local Cartan calculus on the dg algebra \mathcal{A} . We summarize the paper and discuss the open

questions in 7. Technical details regarding the Hodge decomposition and the relevant properties of the operators are collected in appendix A

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2. OVERVIEW OF BV FORMALISMS

2.1. **BV.** The BV formalism was introduced in [2, 3, 4] to study the (in)dependence of the partition function (or, more generally, expectation values of observables) from the gauge fixing. In this setting, the space of fields is extended to an odd symplectic manifold (for bookkeeping, it is convenient to have an additional \mathbb{Z} -grading with the symplectic form having degree -1 ; in this paper, we assume that the parity is the modulo two reduction of the integer degree). The action is extended to a BV action S (of degree 0) satisfying the quantum master equation

$$(1) \quad \frac{1}{2}\{S, S\} - i\hbar\Delta S = 0.$$

Here $\{, \}$ denotes the BV bracket, i.e., the odd Poisson bracket defined by the odd symplectic structure, and Δ is the BV Laplacian. The latter is a second-order differential operator which in Darboux coordinates $p_1, \dots, p_n, q^1, \dots, q^n$ (the parity of each p being opposite to that of the corresponding q) with respect to the standard Berezinian density $\mu_{\text{stand}} = d^n q d^n p$ takes the form $\Delta = \frac{\partial}{\partial p_i} \frac{\partial}{\partial q^i}$.

The first theorem of the BV formalism states that, if $f = \Delta g$ and \mathcal{L} is a Lagrangian submanifold on which the integral of g converges, then

$$\int_{\mathcal{L}} f \mu_{\text{stand}}^{\frac{1}{2}} = 0.$$

Here $\mu_{\text{stand}}^{\frac{1}{2}}$ denotes the half-density associated to μ_{stand} . Its restriction to a Lagrangian submanifold is canonically a density.

This fact implies the central theorem of the BV formalism which states that, for a family \mathcal{L}_t of Lagrangian submanifolds smoothly depending on t ,

$$I_t := \int_{\mathcal{L}_t} f \mu_{\text{stand}}^{\frac{1}{2}}$$

is constant, under the assumptions that $\Delta f = 0$ and that for each t the integral converges. In fact, one can show that

$$\frac{d}{dt} I_t = \int_{\mathcal{L}_t} (\Delta(\dot{\psi} f) + \dot{\psi} \Delta f) \mu_{\text{stand}}^{\frac{1}{2}} = \int_{\mathcal{L}_t} \dot{\psi} \Delta f \mu_{\text{stand}}^{\frac{1}{2}},$$

where we have used the previous theorem. Here $\dot{\psi}$ describes the variation of the gauge fixing. If $\Delta f = 0$, we get the result. For applications in physics, one considers $f = e^{\frac{i}{\hbar} S}$, and S is called the BV action.

The application in field theory then consists in replacing the ill-defined integral on the reference Lagrangian \mathcal{L}_0 , which naively describes the original divergent integral times a vanishing integral over the ghosts, with a well-defined integral over some nearby \mathcal{L}_t (gauge fixing).

The formalism can actually be extended globally to an odd symplectic manifold \mathcal{M} . In this case, the definition of the BV Laplacian requires the choice of a Berezinian density μ such that the resulting differential operator squares to zero. Alternatively, one can observe [13, 14] that there is a canonical Laplacian Δ acting on half-densities. For every half-density σ of the form $\Delta\tau$ and every Lagrangian submanifold \mathcal{L} on which τ is integrable, one has

$$(2) \quad \int_{\mathcal{L}} \sigma = 0.$$

Moreover, for every half-density σ satisfying $\Delta\sigma = 0$, one has that

$$I_t := \int_{\mathcal{L}_t} \sigma$$

is constant, under the assumption that for each t the integral converges. Typically, one chooses σ of the form $e^{\frac{i}{\hbar} S} \sigma_0$ with σ_0 a reference half-density satisfying $\Delta\sigma_0 = 0$. More generally, we have

$$(3) \quad \frac{d}{dt} I_t = \int_{\mathcal{L}_t} \dot{\psi} \Delta\sigma,$$

where $\dot{\psi}$ describes the variation of the gauge fixing.

In field theory, all this has to be regularized, as we are in an infinite-dimensional context, so neither the integral against $\mu_{\text{stand}}^{\frac{1}{2}}$ nor the BV Laplacian are defined. For the considerations in the present paper, it is enough to ignore the BV Laplacian, i.e., to assume that S satisfies the classical master equation

$$(4) \quad \{S, S\} = 0.$$

The integral is on the other hand understood in perturbation theory around a nondegenerate Gaussian (the choice of gauge fixing achieves this goal).

2.2. AKSZ. The AKSZ formalism [1] is a general procedure to construct solutions of the classical master equation (4) (one checks a posteriori, by restriction to the variables of degree 0 what physical theory it describes).

The construction, in d dimensions, requires an odd symplectic manifold Y , with exact symplectic form $\omega_Y = d\theta_Y$ of degree $d-1$, endowed with a solution S_Y (of degree d) of the associated classical master equation. Given a closed d -manifold Σ_d (our space-time), to these data one associates the odd symplectic manifold $\mathcal{M} = \text{Map}(T[1]\Sigma_d, Y)$ and a BV action S satisfying the classical master equation.

We present the construction explicitly for Y a finite-dimensional vector space and $\theta_Y = p_i dq^i$. In this case, \mathcal{M} has Darboux coordinates determined by the superfields \mathbf{P}_i s and \mathbf{Q}^i s, each of which is a sum of differential forms on Σ_d of all degrees: if a superfield corresponds to a target variable of degree n , then its k -form component is assigned degree $n-k$. The odd symplectic form on \mathcal{M} is then defined by

$$(5) \quad \omega = \int_{\Sigma_d} \delta \mathbf{P}_i \wedge \delta \mathbf{Q}^i,$$

where in each summand the integral selects the d -form. The BV action is defined as

$$(6) \quad S = \int_{\Sigma_d} (\mathbf{P}_i d\mathbf{Q}^i + S_Y(\mathbf{P}, \mathbf{Q})).$$

Remark 2.1. A typical gauge fixing in an AKSZ theory is obtained as follows. One picks a Riemannian metric g and defines the codifferential d^\dagger (see Appendix A). Then one defines \mathcal{L}_g as the Lagrangian submanifold obtained by requiring each differential form in the superfields to be in the image of d^\dagger .

2.3. BV push-forward. The central theorem of the BV formalism can be extended to partial integration (BV pushforward).¹

Namely, given a product $\mathcal{M} = \mathcal{M}_{IR} \times \mathcal{M}_{UV}$ of odd symplectic manifolds to which we refer as UV and IR respectively and with total BV Laplacian $\Delta = \Delta_{IR} + \Delta_{UV}$, one has, for every half-density σ on \mathcal{M} satisfying $\Delta\sigma = 0$ and for a family $\mathcal{L}_{UV,t}$ of Lagrangian submanifolds of \mathcal{M}_{UV} , that

$$\sigma_{IR}(t) := \int_{\mathcal{L}_{UV,t}} \sigma$$

defines a family of half-densities $\sigma_{IR}(t)$ on \mathcal{M}_{IR} satisfying $\Delta_{IR}\sigma_{IR} = 0$ for every t . Moreover, $\frac{d}{dt}\sigma_{IR}$ is Δ_{IR} -exact; in view of (2), this change of σ_{IR} is irrelevant in the BV formalism.

These results are obtained from the BV theorems we have listed above. In fact,

$$\Delta_{IR}\sigma_{IR}(t) = \int_{\mathcal{L}_{UV,t}} \Delta_{IR}\sigma = \int_{\mathcal{L}_{UV,t}} \Delta\sigma - \int_{\mathcal{L}_{UV,t}} \Delta_{UV}\sigma = - \int_{\mathcal{L}_{UV,t}} \Delta_{UV}\sigma = 0,$$

where we have used (2) on \mathcal{M}_{UV} . Similarly, by (3) on \mathcal{M}_{UV} ,

$$\frac{d}{dt}\sigma_{IR}(t) := \int_{\mathcal{L}_{UV,t}} \dot{\psi} \Delta_{UV}\sigma.$$

Therefore,

$$(7) \quad \frac{d}{dt}\sigma_{IR}(t) = \int_{\mathcal{L}_{UV,t}} \dot{\psi} \Delta_{UV}\sigma = \int_{\mathcal{L}_{UV,t}} \dot{\psi} \Delta\sigma - \Delta_{IR} \int_{\mathcal{L}_{UV,t}} \dot{\psi} \sigma,$$

which yields the result if $\Delta\sigma = 0$.

Typically, we have reference half-densities σ_0 on \mathcal{M} and $\sigma_{0,IR}$ on \mathcal{M}_{IR} , and $\sigma = e^{\frac{i}{\hbar}S}\sigma_0$. In this case, we write $\sigma_{IR} = e^{\frac{i}{\hbar}S_{\text{eff}}}\sigma_{0,IR}$ and call S_{eff} the effective action.

For applications in field theory (in particular, in this paper), the odd symplectic manifolds are affine spaces (with a given choice of global Darboux coordinates), and the reference half-densities are the standard ones. The effective action is computed in perturbation theory around nondegenerate Gaussians.

¹We follow the terminology of [9] and refer to its Section 2.2 for more details. To the best of our knowledge, the BV pushforward was first used in [10]. It was then realized by A. Losev that it can be used to define Wilsonian renormalization in a BV compatible way; this was first put to use, in the case of BF theories, in P. Mněv's thesis [15]. It was then extensively used by K. Costello [11] to study the renormalization of gauge theories.

3. OVERVIEW OF EQUIVARIANT BV FORMALISMS

3.1. **Equivariant BV.** In [5] an equivariant version of the BV formalism was introduced. The first observation was that, if the quantum master equation is violated,

$$(8) \quad \frac{1}{2}\{S, S\} - i\hbar\Delta S =: T \neq 0,$$

then one still has the central theorem for BV integrals and BV push-forwards if one restricts oneself to integrating on Lagrangian submanifolds on which T vanishes (we call them T -Lagrangian submanifolds); see Section 3.2.

In particular, the case of the extension of an AKSZ theory to the Cartan model for equivariant cohomology was considered. Namely, one starts with an infinitesimal action of a Lie algebra on the space-time Σ_d . We denote by $\{v_a\}$ the vector fields corresponding to some basis (e_a) and introduce degree-two variables u^a . One can then define the modified action

$$(9) \quad S_{eq} = S + u^a S_{\widehat{\iota}_{v_a}},$$

where S is as in (6), and

$$(10) \quad S_{\widehat{\iota}_{v_a}} = \int_{\Sigma_d} \mathbf{P}_i \iota_{v_a} \mathbf{Q}^i$$

with ι_{v_a} denoting the contraction operator and $\widehat{\iota}_{v_a}$ its extension to the space of fields. It then follows that

$$(11) \quad \frac{1}{2}\{S_{eq}, S_{eq}\} + u^a S_{\widehat{L}_{v_a}} = 0,$$

where

$$(12) \quad S_{\widehat{L}_{v_a}} = \int_{\Sigma_d} \mathbf{P}_i L_{v_a} \mathbf{Q}^i$$

with L_{v_a} denoting the Lie derivative with respect to v_a and \widehat{L}_{v_a} its extension to the space of fields. Assuming $\Delta S = 0$ and $\Delta S_{\widehat{\iota}_{v_a}} = 0$,² we get

$$(13) \quad T = -u^a S_{\widehat{L}_{v_a}}.$$

It follows that \mathcal{L}_g as in Remark 2.1 is a T -Lagrangian submanifold if the metric g is invariant.

²Both conditions are reasonable, as, formally, Δ on these functionals corresponds to taking the trace on $\Omega^\bullet(M)$ of operators of degree different from zero.

In this paper, we consider a variation on this theme. Namely, we consider a BV theory (specifically, abelian Yang–Mills in Costello’s BV formulation [11]) which is based on a generalized AKSZ construction. Since the source algebra admits Lie derivatives L_{v_a} and so their lift \widehat{L}_{v_a} to the space of fields, we look for an action which satisfies (11) with $S_{\widehat{L}_{v_a}}$ as in (12). However since there is not a standard Cartan calculus in the source and in particular no contraction vector field ι_{v_a} , we cannot use the solution as in (9). We will rather obtain a solution to (11) that has essentially the same properties as the one above; in particular, \mathcal{L}_g is a T -Lagrangian submanifold for every invariant metric g . On the other hand, we will obtain a generalized Cartan calculus that includes a new definition $\{\mathcal{S}_{\widehat{v}_a}, \}$ of the contraction operator together with higher operations, in a way that is compatible with the theory at hand.

3.2. Equivariant BV push-forward. Suppose S satisfies (8), i.e.,

$$\Delta f = \left(\frac{i}{\hbar}\right)^2 T f$$

with $f = e^{\frac{i}{\hbar}S}$. Consider the half-density $\sigma = f\sigma_0$, with σ_0 a reference Δ -closed half-density. We then have

$$\Delta\sigma = \left(\frac{i}{\hbar}\right)^2 T\sigma.$$

We now perform the BV pushforward with the extra assumption that $T = T_{IR} + T_{UV}$ with T_{IR} and T_{UV} functions on \mathcal{M}_{IR} and \mathcal{M}_{UV} respectively. The half-density $\sigma_{IR}(t)$ on \mathcal{M}_{IR} defined by the pushforward along a T_{UV} -Lagrangian $\mathcal{L}_{UV,t}$ on \mathcal{M}_{UV} then satisfies

$$\Delta_{IR}\sigma_{IR}(t) = \int_{\mathcal{L}_{UV,t}} \Delta\sigma = \left(\frac{i}{\hbar}\right)^2 \int_{\mathcal{L}_{UV,t}} (T_{IR} + T_{UV})\sigma = \left(\frac{i}{\hbar}\right)^2 T_{IR} \int_{\mathcal{L}_{UV,t}} \sigma,$$

i.e.,

$$\Delta_{IR}\sigma_{IR}(t) = \left(\frac{i}{\hbar}\right)^2 T_{IR}\sigma_{IR}(t),$$

which shows that the relation is well-behaved under BV-pushforward (with the above assumptions).

As for deformations of the T_{UV} -Lagrangian submanifold, a direct computation using (7) simply yields

$$\frac{d}{dt}\sigma_{IR}(t) = -\left(\frac{i}{\hbar}\right)^2 T_{IR} \int_{\mathcal{L}_{UV,t}} \dot{\psi}\sigma - \Delta_{IR} \int_{\mathcal{L}_{UV,t}} \dot{\psi}\sigma.$$

In view of (2), the Δ_{IR} -exact term is irrelevant in the BV formalism. As the other term is proportional to T_{IR} , it is also irrelevant, since we are going to consider only T_{IR} -Lagrangian submanifolds for the further gauge fixing.

If we now apply this to the equivariant BV case where T is defined by (13) and \mathcal{M}_{IR} and \mathcal{M}_{UV} are based on AKSZ superfields, we see that the assumptions are satisfied and that the effective theory is precisely as described at the end of Section 3.1. Namely, for the effective theory we still have

$$S_{\widehat{L}_{v_a}}^{\text{eff}} = \int_{\Sigma_d} \mathbf{P}_i L_{v_a} \mathbf{Q}^i$$

where now the \mathbf{P} 's and \mathbf{Q} 's are the superfields on \mathcal{M}_{IR} . On the other hand, $S_{\widehat{L}_{v_a}}^{\text{eff}}$ will in general have a very complicated, nonlocal expression.

4. BV FORMALISM FOR DW

In this section we describe the AKSZ construction [1] for a four-dimensional field theory to which we refer as the Donaldson-Witten (DW) theory since upon the appropriate gauge fixing it can be reduced to the standard cohomological field theory described by Witten in [16]. This DW BV theory has been discussed previously in [5, 7, 8, 12]

Let \mathfrak{g} be a finite dimensional quadratic Lie algebra with the Lie bracket $[\cdot, \cdot]$ and with the invariant metric $\langle \cdot, \cdot \rangle$. Consider the graded vector space

$$(14) \quad \mathfrak{g}[1] \oplus \mathfrak{g}[2] ,$$

which is equipped with the symplectic structure of degree 3

$$(15) \quad \omega = \langle \delta c, \delta \phi \rangle ,$$

where c is the coordinate of degree 1 and ϕ is the coordinate of degree 2. We can introduce the Hamiltonian function of degree 4

$$(16) \quad \Theta = \frac{1}{2} \langle \phi, \phi \rangle + \frac{1}{2} \langle \phi, [c, c] \rangle ,$$

which generates the following odd vector field

$$(17) \quad \mathbf{q}c = \phi + \frac{1}{2} [c, c] ,$$

$$(18) \quad \mathbf{q}\phi = [c, \phi] ,$$

such that $\mathbf{q}^2 = 0$. Following the standard AKSZ construction [1] we can define the space of maps

$$(19) \quad \mathcal{M}_{DW} = \{ T[1]\Sigma_4 \longrightarrow \mathfrak{g}[1] \oplus \mathfrak{g}[2] \} ,$$

where Σ_4 is a 4-manifold. Denoting by \mathbf{C} and $\mathbf{\Phi}$ the superfields of degree 1 and 2, respectively, the BV symplectic form is defined as

$$(20) \quad \omega_{DW} = \int_{T[1]\Sigma_4} d^4x d^4\theta \delta\mathbf{C}^a \wedge \delta\mathbf{\Phi}^b \eta_{ab}$$

and the AKSZ action as

$$(21) \quad S_{DW} = \int_{T[1]\Sigma_4} d^4x d^4\theta \left(\langle \mathbf{\Phi}, d\mathbf{C} \rangle + \frac{1}{2} \langle \mathbf{\Phi}, \mathbf{\Phi} \rangle + \frac{1}{2} \langle \mathbf{\Phi}, [\mathbf{C}, \mathbf{C}] \rangle \right) .$$

The BV transformations are

$$(22) \quad \delta\mathbf{C} = d\mathbf{C} + \mathbf{\Phi} + \frac{1}{2}[\mathbf{C}, \mathbf{C}] ,$$

$$(23) \quad \delta\mathbf{\Phi} = d\mathbf{\Phi} + [\mathbf{C}, \mathbf{\Phi}] .$$

We adopt the following convention for the expansion of superfields in terms of differential forms

$$(24) \quad \mathbf{C} = c_0 + c_1 + c_2 + c_3 + c_4 ,$$

$$(25) \quad \mathbf{\Phi} = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 .$$

The BV symplectic structure can then be written as

$$(26) \quad \omega_{DW} = \int_{\Sigma_4} (\delta c_0 \wedge \delta \phi_4 + \delta c_1 \wedge \delta \phi_3 + \delta c_2 \wedge \delta \phi_2 + \delta c_3 \wedge \delta \phi_1 + \delta c_4 \wedge \delta \phi_0) ,$$

and the action (21) reads

$$(27) \quad \begin{aligned} S_{DW} = & \int \langle \phi_0, d_{c_1} c_3 \rangle + \langle \phi_1, d_{c_1} c_2 \rangle + \langle \phi_2, F(c_1) \rangle + \langle \phi_3, d_{c_1} c_0 \rangle \\ & + \langle \phi_0, \phi_4 \rangle + \langle \phi_1, \phi_3 \rangle + \frac{1}{2} \langle \phi_2, \phi_2 \rangle + \langle \phi_0, [c_0, c_4] \rangle \\ & + \frac{1}{2} \langle \phi_0, [c_2, c_2] \rangle + \langle \phi_1, [c_0, c_3] \rangle + \langle \phi_2, [c_0, c_2] \rangle + \frac{1}{2} \langle \phi_4, [c_0, c_0] \rangle \end{aligned}$$

where $d_{c_1} = d + [c_1, \]$ and $F(c_1) = dc_1 + \frac{1}{2}[c_1, c_1]$. Restricting to the physical (*i.e.* degree 0) fields

$$(28) \quad S_{DW}|_{cl} = \int \langle \phi_2, F(c_1) \rangle + \frac{1}{2} \langle \phi_2, \phi_2 \rangle ,$$

where upon the integration of ϕ_2 we obtain the standard topological term on which the original treatment of the DW theory is based.

The 4D AKSZ action (21) can also be encoded in the following space

$$(29) \quad T[1]\Sigma_4 \oplus \mathbb{R}[-1] \longrightarrow \mathfrak{g}[1] ,$$

where we denote θ 's as odd coordinates on $T[1]\Sigma_4$ and ξ as an odd coordinate on $\mathbb{R}[-1]$. The new superfield is written in terms of old as follows

$$(30) \quad \mathbf{A} = \xi \Phi + \mathbf{C}$$

The symplectic structure is

$$(31) \quad \omega_{DW} = \int_{T[1]\Sigma_4 \oplus \mathbb{R}[-1]} d^4 x d^4 \theta d\xi \delta \mathbf{A}^a \wedge \delta \mathbf{A}^b \eta_{ab}$$

and the AKSZ action becomes

$$(32) \quad S_{DW} = \int_{T[1]\Sigma_4 \oplus \mathbb{R}[-1]} d^4 x d^4 \theta d\xi \left(\langle \mathbf{A}, (d + \frac{\partial}{\partial \xi}) \mathbf{A} \rangle + \mathbf{A} \mathbf{A} \mathbf{A} \right)$$

and it reduces to (21). It is a Chern-Simons action with source the dg-manifold $(T[1]\Sigma_4 \times \mathbb{R}[-1], D + \partial/\partial \xi)$ and degree -3 integral $\int = \int d^4 x d^4 \theta d\xi$; the dga of global functions is then $(\Omega(\Sigma_4)[\xi], D + \frac{\partial}{\partial \xi})$. The underlying complex can be described as

$$(33) \quad \begin{array}{ccccccccc} & & \Omega^0(\Sigma_4) & \rightarrow & \Omega^1(\Sigma_4) & \rightarrow & \Omega^2(\Sigma_4) & \rightarrow & \Omega^3(\Sigma_4) & \rightarrow & \Omega^4(\Sigma_4) \\ & \nearrow & & & \nearrow & & \nearrow & & \nearrow & & \nearrow \\ \Omega^0(\Sigma_4) & \rightarrow & \Omega^1(\Sigma_4) & \rightarrow & \Omega^2(\Sigma_4) & \rightarrow & \Omega^3(\Sigma_4) & \rightarrow & \Omega^4(\Sigma_4) & & \end{array}$$

where the horizontal arrows denote de Rham differential d and the diagonal arrows denote $\partial/\partial \xi$. The field components are defined by expanding the superfield \mathbf{A} along the following basis of local coordinates (as a $\Omega^0(\Sigma_4)$ -module)

$$\begin{aligned} & \xi, \text{ degree } -1 \\ & 1, \theta^i \xi, \text{ degree } 0 \\ & \theta^i, \theta^i \theta^j \xi, \text{ degree } 1 \\ & \theta^i \theta^j, \theta^i \theta^j \theta^k \xi, \text{ degree } 2 \\ & \theta^i \theta^j \theta^k, \theta^1 \theta^2 \theta^3 \theta^4 \xi, \text{ degree } 3 \\ & \theta^1 \theta^2 \theta^3 \theta^4, \text{ degree } 4. \end{aligned}$$

5. BV FORMALISM FOR YM

Let us introduce a metric on Σ_4 and let us decompose the two form components in self-dual and anti-self-dual forms as

$$(34) \quad \phi_2 = \phi_2^+ + \phi_2^-, \quad c_2 = c_2^+ + c_2^-.$$

If we denote with

$$(35) \quad \mathcal{M}_{IR} = \{c_2^- = c_3 = c_4 = \phi_0 = \phi_1 = \phi_2^- = 0\}$$

and

$$(36) \quad \mathcal{M}_{UV} = \{c_0 = c_1 = c_2^+ = \phi_2^+ = \phi_3 = \phi_4 = 0\}$$

we see that \mathcal{M}_{DW} is decomposed in the direct product of symplectic manifolds

$$(37) \quad \mathcal{M}_{DW} = \mathcal{M}_{IR} \times \mathcal{M}_{UV} ,$$

where the symplectic forms read as

$$(38) \quad \omega_{IR} = \int_{\Sigma_4} (\delta c_0 \wedge \delta \phi_4 + \delta c_1 \wedge \delta \phi_3 + \delta c_2^+ \wedge \delta \phi_2^+) ,$$

and

$$(39) \quad \omega_{UV} = \int_{\Sigma_4} (\delta c_2^- \wedge \delta \phi_2^- + \delta c_3 \wedge \delta \phi_1 + \delta c_4 \wedge \delta \phi_0) .$$

As in Sections 2 and 3 we are using the subscripts IR and UV that stand for *infrared* and *ultraviolet* respectively.

Restricting S_{DW} to \mathcal{M}_{IR} (i.e., setting to zero ϕ_2^- , c_2^- , ϕ_1 , c_3 , ϕ_0 and c_4 in (27)) we get

$$(40) \quad \begin{aligned} S_{IR} \equiv S_{DW}|_{\mathcal{M}_{IR}} &= \int \langle \phi_2^+, F(c_1) \rangle + \langle \phi_3, d_{c_1} c_0 \rangle + \frac{1}{2} \langle \phi_2^+, \phi_2^+ \rangle \\ &+ \langle \phi_2^+, [c_0, c_2^+] \rangle + \frac{1}{2} \langle \phi_4, [c_0, c_0] \rangle . \end{aligned}$$

It is crucial to observe that we obtain the same formula if we restrict S_{DW} to $\mathcal{M}_{IR} \times \mathcal{L}_{UV}$, where \mathcal{L}_{UV} is the lagrangian submanifold of \mathcal{M}_{UV} defined by $\phi_0 = \phi_1 = \phi_2^- = 0$ (i.e. keeping c_2^- , c_3 and c_4 different from zero). It is easy to see that this implies that

$$(41) \quad \{S_{IR}, S_{IR}\} = 0 .$$

By restricting (40) further to the physical (i.e. degree 0) fields we get

$$S_{IR}|_{cl} = \int_{\Sigma_4} \langle \phi_2^+, F(c_1) \rangle + \frac{1}{2} \langle \phi_2^+, \phi_2^+ \rangle \equiv S_{YM}^{fo}$$

that is the first order formulation of the Yang-Mills action. Upon the integration of ϕ_2^+ we obtain the standard Yang-Mills action (up to topological term). This proves that S_{IR} is a BV extension of the Yang-Mills action; we are going to prove that it coincides with Costello's solution in [11]. A similar result can be found also in [8].

5.1. Costello's formulation of Yang-Mills. Let us consider now the Chern-Simons formulation of S_{DW} written in (32). The restriction to \mathcal{M}_{IR} imposes constraints to the superfield (30) so that S_{IR} is not anymore in the AKSZ form; let us see how it can still be put in this form.

Let us consider $\mathcal{A}_0 = \{\omega + (\nu_2^+ + \nu_3 + \nu_4)\xi\}$ that is a dg-subalgebra of $(\Omega(\Sigma_4)[\xi], d + \frac{\partial}{\partial \xi})$ together with its dg-ideal $\mathcal{I}_0 = \{\omega_2^- + \omega_3 + \omega_4\}$ (recall that $\nu_2^+ \wedge \omega_2^- = 0$). The quotient dga

$$(42) \quad (\mathcal{A} = \mathcal{A}_0/\mathcal{I}_0, d + \frac{\partial}{\partial \xi})$$

encodes the following complex

$$(43) \quad \begin{array}{ccccccc} \Omega^0(\Sigma_4) & \longrightarrow & \Omega^1(\Sigma_4) & \longrightarrow & \Omega^{2+}(\Sigma_4) & & \\ & & & & \nearrow & & \\ & & \Omega^{2+}(\Sigma_4) & \longrightarrow & \Omega^3(\Sigma_4) & \longrightarrow & \Omega^4(\Sigma_4) \end{array}$$

where the horizontal map is the de Rham differential (composed with the projection to self dual forms in degree 1 and to 0 in degree 2 of the upper line) and the diagonal one is the identity. Analogously, the algebra structure is the wedge product composed with the same projections when it involves forms of the upper line. By inspection we check that in (40) there is no difference if we assume that forms are multiplied according to this exotic ring structure instead of just with the wedge product. We then showed that also (40) is an AKSZ solution of the master equation with target $\mathfrak{g}[1]$ and with source the (spec of the) dg-algebra $(\mathcal{A}, d + \frac{\partial}{\partial \xi})$ endowed with the degree -3 integral $\int : \mathcal{A} \rightarrow \mathbb{R}$ defined as

$$\int (\omega + \nu \xi) = \int_{\Sigma_4} \nu .$$

A local basis of \mathcal{A} (as a $\Omega^0(\Sigma_4)$ -module) is

$$(44) \quad \begin{aligned} & 1, \text{ degree } 0 \\ & \theta^i, \pi^+(\theta^i \theta^j) \xi, \text{ degree } 1 \\ & \pi^+(\theta^i \theta^j), \theta^i \theta^j \theta^k \xi, \text{ degree } 2 \\ & \theta^1 \theta^2 \theta^3 \theta^4 \xi, \text{ degree } 3 \end{aligned}$$

where π^+ is the projection on self-dual two forms. We can summarize by saying that

$$(45) \quad S_{IR} = \int \langle \mathbf{A}, (d + \frac{\partial}{\partial \xi}) \mathbf{A} \rangle + \mathbf{A} \mathbf{A} \mathbf{A} ,$$

where now \mathbf{A} is expanded along the local basis (44) and the product is taken in \mathcal{A} . This CS solution of the master equation for Yang-Mills was first introduced in [11]. In the following sections we will refer to $(\mathcal{A}, d + \frac{\partial}{\partial \xi})$ as the Costello dg-algebra.

The extension of the AKSZ construction to the case when the source dg algebra is not locally freely generated (and so is not the algebra of functions on a dg manifold) has been discussed in [6].

5.2. BV-pushforward. The decomposition (37) defines also the (trivial) fibration $\mathcal{M}_{DW} \rightarrow \mathcal{M}_{IR}$ where we interpret

$$(46) \quad IR \equiv \{c_0, c_1, c_2^+, \phi_2^+, \phi_3, \phi_4\} \in \mathcal{M}_{IR}$$

as *infrared fields* and

$$(47) \quad UV \equiv \{c_2^-, c_3, c_4, \phi_0, \phi_1, \phi_2^-\} \in \mathcal{M}_{UV}$$

as *ultraviolet fields*. We are then in a position to define the BV pushforward of the topological DW theory to the sector of the physical YM fields.

Let us decompose the full BV action (27) as

$$S_{DW} = S_{IR} + S_{UV} + S_{mixed}$$

where

$$(48) \quad S_{UV} = \int \langle \phi_0, dc_3 \rangle + \langle \phi_1, dc_2^- \rangle + \frac{1}{2} \langle \phi_2^-, \phi_2^- \rangle + \frac{1}{2} \langle \phi_0, [c_2^-, c_2^-] \rangle$$

and

$$(49) \quad \begin{aligned} S_{mixed} = & \int \langle \phi_0, [c_1, c_3] \rangle + \langle \phi_1, dc_1 c_2^+ \rangle + \langle \phi_1, [c_1, c_2^-] \rangle + \langle \phi_2^-, F(c_1) \rangle \\ & + \langle \phi_0, \phi_4 \rangle + \langle \phi_1, \phi_3 \rangle + \langle \phi_0, [c_0, c_4] \rangle + \langle \phi_1, [c_0, c_3] \rangle \\ & + \langle \phi_2^-, [c_0, c_2^-] \rangle + \frac{1}{2} \langle \phi_0, [c_2^+, c_2^+] \rangle . \end{aligned}$$

In order to perform the vertical integration it is necessary to separate the UV fields in cohomology and fluctuations, by using the Hodges decomposition of forms. Let us define

$$(50) \quad c = \tilde{c} + h \quad \phi = \tilde{\phi} + \sigma$$

where $h \equiv \{h_2^-, h_3, h_4\} = P(c)$, $\sigma \equiv \{\sigma_0, \sigma_1, \sigma_2^-\} = P(\phi)$ are harmonic forms (P denotes the projection to harmonic forms) and $\tilde{c}, \tilde{\phi}$ denote the fluctuations. We call σ, h the *zero modes*; they are colored in red because, like the IR fields, they are not integrated out and must be treated as external fields. The zero modes symplectic form reads

$$\int (\delta \sigma_0 \wedge \delta h_4 + \delta \sigma_1 \wedge \delta h_3 + \delta \sigma_2^- \wedge \delta h_2^-) .$$

The gauge fixing is now imposed on the UV-fluctuations by choosing a Lagrangian submanifold \mathcal{L}_{UV} defined by

$$(51) \quad \tilde{c}_4 = \tilde{\phi}_2^- = 0, \quad \tilde{c}_3, \tilde{\phi}_1 \in \text{Im}d^\dagger,$$

where d^\dagger denotes the codifferential. The effective action is now defined by

$$e^{\frac{i}{\hbar}S_{\text{eff}}} = \int_{\mathcal{L}_{UV}} e^{\frac{i}{\hbar}S_{DW}}.$$

Let us analyze the abelian case first. The explicit computation is postponed to the next section where the more general equivariant case will be analyzed. We anticipate here that in the non equivariant abelian case the only corrections to (40) come from the UV zero modes contributions, *i.e.*

$$(52) \quad S_{\text{eff}} = S_{IR} + \int \langle \sigma_0, \phi_4 \rangle + \langle \sigma_1, \phi_3 \rangle + \frac{1}{2} \langle \sigma_2^-, \sigma_2^- \rangle$$

where the last terms come from the evaluation of (48) and (49) on zero modes. Since all remaining fields are red, we suppressed the coloring. The abelian YM-BV differential is corrected to

$$(53) \quad \begin{aligned} \delta c_0 &= \sigma_0, & \delta c_1 &= dc_0 + \sigma_1, & \delta c_2^+ &= \pi^+ dc_1 + \phi_2^+ \\ \delta \phi_2^+ &= 0, & \delta \phi_3 &= d\phi_2^+, & \delta \phi_4 &= d\phi_3 \end{aligned}$$

The UV zero modes are not integrated out so far and so they survive as dynamical variables. In particular we have

$$(54) \quad \begin{aligned} \delta \sigma_0 &= \delta \sigma_1 = \delta \sigma_2^- = 0 \\ \delta h_2^- &= \sigma_2^-, & \delta h_3 &= P\phi_3, & \delta h_4 &= P\phi_4. \end{aligned}$$

By construction δ satisfies

$$\delta^2 = 0.$$

Remark 5.1. It is easy to check that the effective action (52) is the CS action (45) with the replacement $\mathbf{A} \rightarrow \mathbf{A} + h + \xi\sigma$, where $h = h_2^- + h_3 + h_4$ and $\sigma = \sigma_0 + \sigma_1 + \sigma_2^-$ are the superfields encoding the UV zero modes. The homological vector field $d + \frac{\partial}{\partial \xi}$ is encoded in the following complex that is an enlarged version of (43)

$$(55) \quad \begin{array}{ccccccc} & & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{\pi^+ d \oplus 0} & \Omega^{2+} \oplus H^{2-} & \xrightarrow{P} & H^3 & \xrightarrow{P} & H^4 \\ & \nearrow i & & & \nearrow i & & \nearrow id & & & & \\ H^0 & & & & H^1 & & \Omega^{2+} \oplus H^{2-} & \xrightarrow{d \oplus 0} & \Omega^3 & \xrightarrow{d} & \Omega^4 \end{array}$$

where i denotes the embedding of de Rham cohomology into differential forms as harmonic forms.

Remark 5.2. The calculation of S_{eff} for non-abelian theory involves all orders in the perturbation theory and will be discussed in a future paper.

6. EQUIVARIANT BV PUSHFORWARD MAP

In the previous section we have constructed 4d DW theory as the standard AKSZ theory with the source being $T[1]\Sigma_4$ and the target $\mathfrak{g}[1] \oplus \mathfrak{g}[2]$. Let us consider now a vector field $v \in \Gamma(T\Sigma_4)$ (possibly a complete one whose flux defines a group action) and let us consider the equivariant extension of the ASKZ action as described in Section 3.

Borrowing the notation from the previous section, the equivariant AKSZ action is defined on the same space of fields \mathcal{M}_{DW} as

$$(56) \quad \begin{aligned} S_{eq-DW} &= S_{DW} + S_{\widehat{\iota}_v} \\ &= \int_{T[1]\Sigma_4} d^4x d^4\theta \left(\langle \Phi, d_v \mathbf{C} \rangle + \frac{1}{2} \langle \Phi, \Phi \rangle + \frac{1}{2} \langle \Phi, [\mathbf{C}, \mathbf{C}] \rangle \right), \end{aligned}$$

where $d_v = d + \iota_v$ is the (Cartan) equivariant differential and

$$(57) \quad S_{\widehat{\iota}_v} = \int_{T[1]\Sigma_4} d^4x d^4\theta \langle \Phi, \iota_v \mathbf{C} \rangle$$

is the hamiltonian defining the lift $\widehat{\iota}_v$ of the contraction vector field ι_v to \mathcal{M}_{DW} . For the clarity of forthcoming formulas here we suppress the formal variable u and assume that we have just one v , in the final result is easy to restore u^a and v_a . This action satisfies the equivariant master equation

$$(58) \quad \frac{1}{2} \{S_{eq-DW}, S_{eq-DW}\} = -S_{\widehat{L}_v},$$

where

$$(59) \quad S_{\widehat{L}_v} = \int_{T[1]\Sigma_4} d^4x d^4\theta \langle \Phi, L_v \mathbf{C} \rangle$$

is the hamiltonian defining the lift \widehat{L}_v of L_v to \mathcal{M}_{DW} . In components S_{eq-DW} is given by

$$(60) \quad S_{eq-DW} = S_{DW} + \int \langle \phi_1, \iota_v c_4 \rangle + \langle \phi_2, \iota_v c_3 \rangle + \langle \phi_3, \iota_v c_2 \rangle + \langle \phi_4, \iota_v c_1 \rangle$$

where the component expression of S_{DW} is given in (27).

Let us choose now a metric on Σ_4 invariant under L_v and consider the splitting $\mathcal{M}_{DW} = \mathcal{M}_{IR} \times \mathcal{M}_{UV}$ defined in (37). The restriction S_{eq-DW} to \mathcal{M}_{IR} does not satisfy the equivariant master equation (58) (since the lift $\widehat{\iota}_v$ to \mathcal{M}_{DW} of the contraction operator $\iota_v \in \text{Der}(\Omega(\Sigma_4)[\xi])$ does not restrict to \mathcal{M}_{IR}). In other terms, the invariance of the metric guarantees that \mathcal{M}_{IR} inherits from \mathcal{M}_{DW} the \widehat{L}_v action but not the Cartan complex for equivariant cohomology.

Remark 6.1. We saw in Section 5.1 that the infrared action S_{IR} , which is a BV extension of Yang-Mills, is an AKSZ action by itself with source the dg algebra \mathcal{A} defined in (42). It is natural then to look for the definition of the Cartan complex for equivariant cohomology on \mathcal{A} and then lifting it to \mathcal{M}_{IR} . It is easy to check that while the Lie derivative $L_v \in \text{Der}(\Omega(\Sigma)[\xi])$ descends to $\text{Der}(\mathcal{A})$, the same is not true for $\iota_v \in \text{Der}(\Omega(\Sigma)[\xi])$. So there is not a natural notion for the contraction operator.

We are going instead to compute the BV push-forward map for S_{eq-DW} to the space \mathcal{M}_{IR} ; by construction the effective action will satisfy the equivariant master equation.

The general non-abelian theory would require the full perturbation theory. At the same time in the abelian case the calculation can be carried out exactly still being non-trivial. Thus from now on we switch to the abelian equivariant DW theory.

Borrowing the notations from the previous section we can split the abelian S_{eq-DW} as follows

$$(61) \quad S_{eq-DW} = S_{IR} + S_{UV} + S_{mixed}$$

with $\{c_0, c_1, c_2^+, \phi_2^+, \phi_3, \phi_4\} \in \mathcal{M}_{IR}$

$$(62) \quad S_{IR} = \int \left(\phi_2^+ dc_1 + \phi_3 dc_0 + \frac{1}{2}(\phi_2^+)^2 + \phi_3 \iota_v c_2^+ + \phi_4 \iota_v c_1 \right)$$

and $\{c_2^-, c_3, c_4, \phi_0, \phi_1, \phi_2^-\} \in \mathcal{M}_{UV}$

$$(63) \quad S_{UV} = \int \left(\phi_0 dc_3 + \phi_1 dc_2^- + \frac{1}{2}(\phi_2^-)^2 + \phi_1 \iota_v c_4 + \phi_2^- \iota_v c_3 \right)$$

and

$$(64) \quad S_{mixed} = \int (\phi_1 dc_2^+ + \phi_2^- dc_1 + \phi_0 \phi_4 + \phi_1 \phi_3 + \phi_3 \iota_v c_2^- + \phi_2^+ \iota_v c_3)$$

We stress that we use the L_v -invariant metric to define all fields.

We decompose the UV fields as in (50) by using the Hodge decomposition with respect to the invariant metric. We then define the lagrangian submanifold \mathcal{L}_{UV} as follows

$$(65) \quad \tilde{c}_4 = \tilde{\phi}_2^- = 0, \quad \tilde{c}_3, \tilde{\phi}_1 \in \text{Im}d^\dagger,$$

and by construction \mathcal{L}_{UV} is invariant under the action of L_v . Before imposing the conditions $\tilde{c}_3, \tilde{\phi}_1 \in \text{Im}d^\dagger$ explicitly let us rewrite a bit S_{IR} , S_{UV} and S_{mixed} assuming just that $\tilde{c}_4 = \tilde{\phi}_2^- = 0$. Assuming the decomposition (50) S_{UV} is written as

$$(66) \quad S_{UV} = \int \left(\tilde{\phi}_0 d\tilde{c}_3 + \tilde{\phi}_1 d\tilde{c}_2^- + \frac{1}{2}(\sigma_2^-)^2 + \tilde{\phi}_1 \iota_v h_4 + \sigma_2^- \iota_v \tilde{c}_3 + \sigma_1 \iota_v h_4 + \sigma_2^- \iota_v h_3 \right)$$

and S_{mixed} as follows

$$(67) \quad S_{mixed} = \int \left(\tilde{\phi}_1 d\mathbf{c}_2^+ + \tilde{\phi}_0 \phi_4 + \sigma_0 \phi_4 + \tilde{\phi}_1 \phi_3 + \sigma_1 \phi_3 + \phi_3 \iota_v \tilde{c}_2^- + \phi_3 \iota_v \sigma_2^- + \phi_2^+ \iota_v \tilde{c}_3 + \phi_2^+ \iota_v h_3 \right).$$

We are going to integrate out only the blue fields $\{\tilde{\phi}_0, \tilde{c}_3, \tilde{\phi}_1, \tilde{c}_2^-\}$ and the red fields (IR fields and UV zero modes) are treated as external. Therefore it is useful to identify S_0 as containing only quadratic terms in the blue fields

$$(68) \quad S_0 = \int \left(\tilde{\phi}_0 d\tilde{c}_3 + \tilde{\phi}_1 d\tilde{c}_2^- \right) = \int \left(\frac{1}{2} \tilde{\phi}_0 d\tilde{c}_3 + \frac{1}{2} \tilde{c}_3 d\tilde{\phi}_0 + \frac{1}{2} \tilde{\phi}_1 d\tilde{c}_2^- + \frac{1}{2} \tilde{c}_2^- (\pi^- d) \tilde{\phi}_1 \right),$$

where π^\pm are the projectors on Ω^{2^\pm} , S_1 as containing the linear terms in the blue fields

$$(69) \quad S_1 = \int \left(\tilde{\phi}_0 \phi_4 - \tilde{c}_3 (\iota_v \phi_2^+ + \iota_v \sigma_2^-) + \tilde{\phi}_1 (\phi_3 + d\mathbf{c}_2^+ + \iota_v h_4) + \tilde{c}_2^- \iota_v \phi_3 \right)$$

and new \tilde{S}_{IR} containing only red fields (including UV zero modes)

$$(70) \quad \tilde{S}_{IR} = \int \left(\phi_2^+ d\mathbf{c}_1 + \phi_3 d\mathbf{c}_0 + \frac{1}{2}(\phi_2^+)^2 + \phi_3 \iota_v \mathbf{c}_2^+ + \phi_4 \iota_v \mathbf{c}_1 + \frac{1}{2}(\sigma_2^-)^2 + \sigma_1 \iota_v h_4 + \sigma_2^- \iota_v h_3 + \sigma_0 \phi_4 + \sigma_1 \phi_3 + \phi_3 \iota_v \sigma_2^- + \phi_2^+ \iota_v h_3 \right),$$

so that $S_{eq-DW} = S_0 + S_1 + \tilde{S}_{IR}$. It is convenient to write the quadratic form appearing in S_0 as

$$(71) \quad \frac{1}{2} \begin{pmatrix} \tilde{\phi}_0 & \tilde{c}_3 & \tilde{\phi}_1 & \tilde{c}_2^- \end{pmatrix} \begin{pmatrix} 0 & d & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & (\pi^- d) & 0 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_0 \\ \tilde{c}_3 \\ \tilde{\phi}_1 \\ \tilde{c}_2^- \end{pmatrix}$$

where the wedge product is assumed. Analogously the integrand in S_1 is written as

$$(72) \quad (\tilde{\phi}_0 \quad \tilde{c}_3 \quad \tilde{\phi}_1 \quad \tilde{c}_2^-) (1 - P) \begin{pmatrix} \phi_4 \\ -\iota_v \phi_2^+ - \iota_v \sigma_2^- \\ \phi_3 + d c_2^+ + \iota_v h_4 \\ \pi^- \iota_v \phi_3 \end{pmatrix}$$

where the wedge product is assumed and $(1 - P)$ is the project outside of harmonic forms (cohomology). The claim is that the quadratic form in (71) is non degenerate once the gauge fixing conditions (65) are taken into account so that the UV integration is well defined. This will be discussed in the next subsection.

6.1. Integration of UV sector. The canonical way to restrict fields to \mathcal{L}_{UV} in (65) is by adding a trivial sector of auxiliary fields. Indeed, let us add $(\varphi, \varphi^\vee, \eta, \eta^\vee)$ with $\deg \varphi = -2$, $\deg \varphi^\vee = 1$, $\deg \eta = -1$ and $\deg \eta^\vee = 0$, with φ being (even) zero form (φ^\vee is top form) and η being (odd) zero form (η^\vee top form). The BV symplectic form in UV sector is now

$$(73) \quad \tilde{\omega}_{UV} = \int \left(\delta \tilde{\phi}_2^- \wedge \delta \tilde{c}_2^- + \delta \tilde{\phi}_1 \wedge \delta \tilde{c}_3 + \delta \tilde{\phi}_0 \wedge \delta \tilde{c}_4 + \delta \varphi \wedge \delta \varphi^\vee + \delta \eta \wedge \delta \eta^\vee \right)$$

and the BV action is extended to

$$(74) \quad S_{eq-DW} + \int (\varphi^\vee \eta + \eta^\vee L_v \varphi) .$$

The new fields contain zero modes that must be decoupled from fluctuations, as we did for the UV fields. It is easy to see that zero modes decouple and so they can be ignored in the following. The extra term in the action just modifies the quadratic term (68) in UV fields

$$(75) \quad S'_0 = S_0 + \int (\varphi^\vee \eta + \eta^\vee L_v \varphi)$$

The following gauge fixing fermion (with L_v invariant metric)

$$(76) \quad \Psi = \int \tilde{\varphi} (d \star \tilde{\phi}_1)$$

defines the following \mathcal{L}_{UV}

$$(77) \quad \tilde{c}_4 = 0, \quad \tilde{\phi}_2^- = 0, \quad \tilde{\eta}^\vee = 0, \quad \tilde{c}_3 = \star d \tilde{\varphi}, \quad \tilde{\varphi}^\vee = d \star \tilde{\phi}_1 .$$

Evaluating the quadratic part (75) of the action we obtain

$$(78) \quad S'_0|_{\mathcal{L}_{UV}} = \int \left(\tilde{\phi}_0 d \star d \tilde{\varphi} + \tilde{\phi}_1 d \tilde{c}_2^- + d \star \tilde{\phi}_1 \tilde{\eta} \right) .$$

Introducing the inner product $(\ , \)$ as in (91) we can rewrite it in the following matrix form

$$(79) \quad S'_0|_{\mathcal{L}_{UV}} = \frac{1}{2}(x, Dx)$$

where $x = (\tilde{\phi}_0, \tilde{\varphi}, \tilde{\phi}_1, \tilde{c}_2^-, \tilde{\eta}) \in V \equiv (1 - P)(\Omega^0 \oplus \Omega^0 \oplus \Omega^1 \oplus \Omega^{2-} \oplus \Omega^0)$ and $D : V \rightarrow V$ is defined as

$$(80) \quad D = \begin{pmatrix} 0 & -\Delta & 0 & 0 & 0 \\ -\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d^\dagger & d \\ 0 & 0 & -\pi^- d & 0 & 0 \\ 0 & 0 & d^\dagger & 0 & 0 \end{pmatrix}$$

The differential operator D is by construction self-adjoint and invertible, its inverse being as follows

$$(81) \quad D^{-1} = \begin{pmatrix} 0 & -\Delta^{-1} & 0 & 0 & 0 \\ -\Delta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\Delta^{-1}d^\dagger & \Delta^{-1}d \\ 0 & 0 & -2\pi^-\Delta^{-1}d & 0 & 0 \\ 0 & 0 & \Delta^{-1}d^\dagger & 0 & 0 \end{pmatrix}.$$

In order to check it, one has to use the properties (102) and the explicit definition of Δ on zero forms.

Next the restriction of the linear term S_1 in (69) to \mathcal{L}_{UV} gives

$$(82) \quad S_1|_{\mathcal{L}_{UV}} = \int \left(\tilde{\phi}_0 \phi_4 - \star d \tilde{\varphi} (\iota_v \phi_2^+ + \iota_v \sigma_2^-) + \tilde{\phi}_1 (\phi_3 + d c_2^+ + \iota_v h_4) + \tilde{c}_2^- \iota_v \phi_3 \right),$$

that can be written as as

$$S_1|_{\mathcal{L}_{UV}} = (x, a)$$

with $a \in V$ being

$$a = (1 - P) (\star \phi_4, -d^\dagger \iota_v (\phi_2^+ + \sigma_2^-), -\star \phi_3 + d^\dagger c_2^+ - \star \iota_v h_4, -\pi^- \iota_v \phi_3, 0).$$

Combining these terms together we have the following structure

$$\begin{aligned} (S'_0 + S_1)|_{\mathcal{L}_{UV}} &= \frac{1}{2}(x, Dx) + (x, a) \\ &= \frac{1}{2}(x + D^{-1}a, D(x + D^{-1}a)) - \frac{1}{2}(a, D^{-1}a) \end{aligned}$$

so that, after integrating out the fields in UV sector, we obtain

$$S_{\text{eff}} = \tilde{S}_{IR} - \frac{1}{2}(a, D^{-1}a).$$

with \tilde{S}_{IR} defined in (70). Upon integration we also obtain the field independent determinants which however depend on the metric. In our consideration we ignore those determinants, but for physical discussions (e.g., like metric dependence etc) they do play an important role. Since we are left with red fields, IR fields and UV zero modes, from now on we stop the colour marking of the fields. After some computations one gets the following effective BV action for the IR fields

$$(83) \quad S_{\text{eq-eff}} = \int \left(\phi_2^+ dc_1 + \phi_3 dc_0 + \frac{1}{2}(\phi_2^+)^2 + \sigma_0 \phi_4 + \sigma_1 \phi_3 + \frac{1}{2}(\sigma_2^-)^2 \right. \\ \left. \phi_3 \iota_v c_2^+ + \phi_4 \iota_v c_1 + \sigma_1 \iota_v h_4 + \sigma_2^- \iota_v h_3 + \phi_3 \iota_v h_2^- + \phi_2^+ \iota_v h_3 \right. \\ \left. - \phi_4 K \iota_v (\phi_2^+ + \sigma_2^-) - 2(\phi_3 + \iota_v h_4) K \pi^- \iota_v \phi_3 - 2c_2^+ dK \pi^- \iota_v \phi_3 \right) .$$

By construction it satisfies the equivariant master equation (58) with

$$S_{\widehat{L}_v a}^{\text{eff}} = \int \left(\phi_4 L_v c_0 + \phi_3 L_v c_1 + \phi_2^+ L_v c_2^+ + \sigma_0 L_v h_4 + \sigma_1 L_v h_3 + \sigma_2^- L_v h_2^- \right)$$

By putting $v = 0$ one recognizes the non equivariant effective action in (52) which coincides, up to zero modes, with the BV extension of abelian Yang-Mills theory. We can say then that (83) is an equivariant extension of the abelian Yang-Mills BV theory.

We can read off the equivariant BV transformations for IR fields

$$(84) \quad \begin{aligned} \delta_{\text{eq}} c_0 &= \iota_v c_1 + \sigma_0 - K \iota_v \phi_2^+ - K \iota_v \sigma_2^- \\ \delta_{\text{eq}} c_1 &= dc_0 + \sigma_1 + \iota_v h_2^- + \iota_v c_2^+ - 2K(\pi^- \iota_v \phi_3) \\ &\quad - 2\iota_v(\pi^- K \phi_3) + 2\iota_v(\pi^- dK c_2^+) - 2\iota_v \pi^- K \iota_v h_4 \\ \delta_{\text{eq}} c_2^+ &= \pi^+ \left(dc_1 + \phi_2^+ + \iota_v h_3 - \iota_v(K \phi_4) \right) \\ \delta_{\text{eq}} \phi_2^+ &= \pi^+ \left(\iota_v \phi_3 - 2dK(\pi^- \iota_v \phi_3) \right) \\ \delta_{\text{eq}} \phi_3 &= d\phi_2^+ + \iota_v \phi_4 \\ \delta_{\text{eq}} \phi_4 &= d\phi_3 \end{aligned}$$

and for UV zero modes

$$(85) \quad \begin{aligned} \delta_{\text{eq}} \sigma_0 &= -2P \iota_v K \pi^- \iota_v \phi_3 + P \iota_v \sigma_1 \\ \delta_{\text{eq}} \sigma_1 &= P \iota_v \phi_2^+ + P \iota_v \sigma_2^- \\ \delta_{\text{eq}} \sigma_2^- &= P \pi^- \iota_v \phi_3 \\ \delta_{\text{eq}} h_2^- &= \pi^- P \iota_v (-K \phi_4 + h_3) + \sigma_2^- \\ \delta_{\text{eq}} h_3 &= P(\phi_3 + \iota_v h_4) \\ \delta_{\text{eq}} h_4 &= P \phi_4 \end{aligned}$$

The reader, with a straightforward but tedious calculation, can check that δ_{eq} satisfies

$$(86) \quad \delta_{eq}^2 = -\hat{L}_v .$$

Remark 6.2. The invariant metric defining the UV gauge fixing is arbitrary. In the computation of (84) we implicitly assumed that it is in the same conformal class of the metric defining the IR space (or equivalently the YM theory). There is then a family of equivariant extensions parametrized by invariant metrics that are conformally equivalent to the YM metric. From the general discussion of subsection 3.2 it follows that they give equivalent extensions of abelian YM.

There are terms in δ_{eq} that are quadratic in the vector field v . Restoring the formal parameter u of degree 2, one can expand δ_{eq} in powers of u and get

$$(87) \quad \delta_{eq} = \delta + u\delta_{-1} + u^2\delta_{-3} .$$

From (86) they satisfy the following relations

$$(88) \quad \begin{aligned} \delta^2 &= 0 , \\ \delta\delta_{-1} + \delta_{-1}\delta &= L_v , \\ \delta\delta_{-3} + \delta_{-3}\delta + \delta_{-1}^2 &= 0 , \\ \delta_{-1}\delta_{-3} + \delta_{-3}\delta_{-1} &= 0 , \\ \delta_{-3}^2 &= 0 \end{aligned}$$

The first two lines are the standard Cartan relations that make us to interpret δ_{-1} as the contraction operator; the third lines states that δ_{-1} squares to zero only up to $\text{ad}(\delta)$. We can think (88) as a higher analog of the Cartan calculus.

Remark 6.3. Analogously to Remark 5.1 we can read from (84) and (85) the equivariant differential on the enlarged Costello complex (55). We can organize it as

$$(89) \quad d_u^{\text{eff}} = d_{+1} + ud_{-1} + u^2d_{-3} ,$$

where d_{+1} is the degree 1-differential described in (55). The operation d_{-1} lowers the degree by 1 and can be summarized by the following diagram

$$\begin{array}{ccccccc} \Omega^0 & \xleftarrow{\iota_v} & \Omega^1 & \xleftarrow{\iota_v(1+2\pi^-dK)\oplus\iota_v} & \Omega^{2+} \oplus H^{2-} & \xleftarrow{\pi^+\iota_v\oplus\pi^-P\iota_v} & H^3 & \xleftarrow{P\iota_v} & H^4 \\ & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & \\ & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & \\ & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & \\ H^0 & \xleftarrow{P\iota_v} & H^1 & \xleftarrow{P\iota_v\oplus P\iota_v} & \Omega^{2+} \oplus H^{2-} & \xleftarrow{\pi^+(\iota_v-2dK\pi^-\iota_v)\oplus\pi^-P\iota_v} & \Omega^3 & \xleftarrow{\iota_v} & \Omega^4 \end{array}$$

in the same fashion as the equivariant considerations help calculate the ordinary integrals of the closed differential forms. But this requires further study.

Another interesting result of this work is the homological generalization of Cartan calculus with the explicit constructions in terms of non-local operators. This structure requires further study and better understanding both at the level of the formal properties and at the level of concrete examples. It would be interesting to see if any of the standard considerations for the equivariant cohomology and equivariant localization can be generalized to this new setting.

APPENDIX A. HODGE DECOMPOSITION AND RELEVANT OPERATORS

In this appendix we collect the relevant properties of the Hodge decomposition and the related differential operators. All formulas are written for a 4d manifold Σ_4 . Let d be de Rham operator. We pick up the metric and introduce the inner product on differential forms

$$(91) \quad (\alpha, \beta) = \int \alpha \wedge \star \beta ,$$

where \star is the Hodge star operator with the property

$$(92) \quad \star^2 \omega_r = (-1)^r \omega_r ,$$

where $\omega_r \in \Omega^r(\Sigma_4)$. We define the adjoint operator d^\dagger for the de Rham differential

$$(93) \quad (\alpha, d\beta) = (d^\dagger \alpha, \beta)$$

with the explicit definition given by

$$(94) \quad d^\dagger = - \star d \star .$$

Next we define the Laplace operator

$$(95) \quad \Delta = dd^\dagger + d^\dagger d ,$$

which is a self-adjoint operator $\Delta^\dagger = \Delta$. The Hodge decomposition for Ω^r corresponds to the decomposition into three orthogonal spaces

$$(96) \quad \Omega^r = (\text{Im } d) \oplus (\text{Im } d^\dagger) \oplus (\text{Harm}^r) ,$$

where $(\text{Harm}^r) = (\ker \Delta)$ stands for harmonic forms which can be identified with the de Rham cohomology. We denote by P the projector on (Harm^r) and by $(1 - P)$ the projector to $(\text{Im } d) \oplus (\text{Im } d^\dagger)$. Away from (Harm^r) the Laplace operator is invertible, thus whenever we write Δ^{-1} we assume that it is defined only on the $(1 - P)$ space. The operator

Δ^{-1} is self-adjoint ($\Delta^{-1\dagger} = \Delta^{-1}$) and we have the following obvious properties

$$[d, \Delta] = 0, \quad [d^\dagger, \Delta] = 0, \quad [d, \Delta^{-1}] = 0, \quad [d^\dagger, \Delta^{-1}] = 0,$$

where the relations with Δ^{-1} understood only for the $(1 - P)$ space. It is convenient to define the homotopy operator

$$(97) \quad K = \int_0^\infty dt e^{-t\Delta} d^\dagger,$$

which is defined everywhere and satisfies the property

$$(98) \quad Kd + dK = 1 - P.$$

If understood correctly K can be expressed as follows

$$(99) \quad K = \Delta^{-1} d^\dagger.$$

On two forms we have the following projectors

$$(100) \quad \pi^\pm = \frac{1}{2}(1 \pm \star),$$

which splits $\Omega^2 = \Omega^{2+} \oplus \Omega^{2-}$ into self-dual and anti-self-dual two forms correspondingly. We have the following useful properties

$$(101) \quad \pi^\pm \Delta = \Delta \pi^\pm, \quad \pi^\pm \Delta^{-1} = \Delta^{-1} \pi^\pm$$

and

$$(102) \quad \Delta \pi^\pm = 2\pi^\pm dd^\dagger \pi^\pm.$$

Assume that Σ_4 admits the action of a vector field v (typically coming from some group action). On the differential forms v acts via the Lie derivative

$$(103) \quad L_v = d\iota_v + \iota_v d,$$

where ι_v is contraction of v with a differential form. We pick an invariant metric g such that $\mathcal{L}_v g = 0$. In this case we have the following properties

$$(104) \quad [L_v, d] = 0, \quad [L_v, d^\dagger] = 0, \quad [L_v, \Delta] = 0.$$

and thus L_v preserves the subspaces in the corresponding Hodge decomposition. Moreover we have the following property

$$(105) \quad L_v^\dagger = -L_v.$$

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