GRADED GEOMETRY AND GENERALIZED REDUCTION

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Abstract. We present general reduction procedures for Courant, Dirac and generalized complex structures, in particular when a group of symmetries is acting. We do so by taking the graded symplectic viewpoint on Courant algebroids and carrying out graded symplectic reduction, both in the coisotropic and hamiltonian settings. Specializing the latter to the exact case, we recover in a systematic way the reduction schemes of Bursztyn-Cavalcanti-Gualtieri.

Contents

1. Introduction

Courant algebroids, introduced in [LWX97], have drawn much attention in recent years due to their many connections with (higher) geometry and mathematical physics in such topics as Poisson–Lie theory, T-duality, sigma models, vertex algebras, shifted symplectic structures, see, e.g., [Bre07, CG10, PS20, Rog13, Roy07, Š15]. A key feature of Courant algebroids is that they provide a unifying framework for various geometric structures, especially by means of their Dirac structures. In their original setting, Dirac structures in the standard Courant algebroid [Cou90] were used to unify Poisson and presymplectic structures. In the same vein, Courant algebroids are central ingredients in generalized complex geometry [Gua11, Hit03], where a similar "unification" phenomenon allows complex and symplectic structures to be treated on equal footing.

A reduction procedure for Courant algebroids in the presence of symmetries was developed in [BCG07, BCG08] as the foundational step for the reduction of Dirac and generalized complex structures (for the latter see also [Hu09, LT06, SX08]). An important aspect of such Courant reduction is that symmetries were described by a new notion of "extended action" by "Courant algebras," rather than usual actions by Lie algebras or Lie groups. While this reduction scheme was motivated and guided by examples, the general procedure lacked a clear conceptual framework.

The goal of the present work is to give a broad and systematic approach to the reduction of Courant algebroids and related geometric structures, and our main tool is the viewpoint to these objects in terms of *graded symplectic geometry*. Specifically, our starting point is the correspondence between Courant algebroids and degree 2 symplectic $\mathbb{N}$-manifolds with a self-commuting degree 3 function [Roy02a, Šev05].

pseudo-euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$	symplectic degree 2 $\mathbb{N}$-manifold $(\mathcal{M}, \omega)$
Courant structure ρ, $[\![\cdot,\cdot]\!]$	$\Theta \in C(\mathcal{M})_3$, $\{\Theta, \Theta\} = 0$

Our strategy consists in expressing the graded analogues of usual reduction procedures in symplectic geometry in Courant-geometric terms via this correspondence. This involves, as a key step, adding new lines to the above dictionary so as to include, on the graded symplectic side, usual ingredients of symplectic reduction such as coisotropic submanifolds, hamiltonian actions and momentum maps. Their classical geometric counterparts lead to natural "coisotropic" and "hamiltonian" frameworks in which to carry out the reduction of Courant and related structures, and these are shown to include the constructions in [BCG07, BCG08, Zam08] as special cases.

Remark 1.1. During the long writing process of this paper, some related topics were explored in separate works. In the context of degree 1 $\mathbb{N}$-manifolds, the analogous perspective of graded symplectic reduction leads to general reduction procedures for Poisson manifolds described in [CZ09, CZ13]; in [Meh11], a more general framework for graded symplectic reduction is studied, based on actions by homotopy Poisson Lie groups; in the graded setting, coisotropic reduction relies on a version of the Frobenius theorem (on the integrability of involutive distributions) for $\mathbb{N}$-manifolds discussed in [BCM].

Statements of results and outline of the paper.

Geometrization of degree 2 $\mathbb{N}$-manifolds. We collect in §2 foundational results about degree 2 $\mathbb{N}$-manifolds, starting with their basic geometric description in terms of ordinary vector bundles. Any degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, with body M, is completely determined by a pair of vector bundles E_1 and $\widetilde{E}$ over M (whose duals codify functions of degrees 1 and 2) together with a vector-bundle map $\phi_E\colon \widetilde{E} \twoheadrightarrow \wedge^2 E_1$ (whose dual accounts for the multiplication of degree 1 functions). Such triples $(E_1, \widetilde{E}, \phi_E)$ can be naturally regarded as objects of a category, denoted by VB2, which is shown to be equivalent to the category 2Man of degree 2 $\mathbb{N}$-manifolds,

$$\text{VB2} \rightleftharpoons \text{2Man}, \tag{1.1}$$

see Proposition 2.4. We build on this equivalence to obtain classical geometric descriptions of submanifolds of degree 2 $\mathbb{N}$-manifolds (Proposition 2.9 and Theorem 2.13), as well as regular values of maps (Corollary 2.24) and their inverse images (Proposition 2.26).

In §3 we focus on *symplectic* degree 2 $\mathbb{N}$-manifolds, and derive their known correspondence with pseudo-euclidean vector bundles [Roy02a] (see Theorem 3.4) from the equivalence in (1.1): any vector bundle carrying a pseudo-riemannian metric naturally gives rise to an object in VB2 by means of its Atiyah sequence, and equipping a degree 2 $\mathbb{N}$-manifold with a symplectic structure amounts to identifying its corresponding object in VB2 with one of this type.

Coisotropic reduction. We consider coisotropic submanifolds of symplectic degree 2 $\mathbb{N}$-manifolds in §4, providing their geometric characterization in Theorem 4.5. For a pseudo-euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$ over M, coisotropic submanifolds of the corresponding symplectic degree 2 $\mathbb{N}$-manifold are identified with quadruples (N, K, F, ∇) consisting of a submanifold N supporting an isotropic subbundle $(K \to N) \subset (E \to M)$ and an involutive subbundle $F \subseteq TN$, along with a flat, metric F-connection ∇ on the quotient vector bundle $(K^\perp/K) \to N$. We refer to such quadruples (N, K, F, ∇) as *geometric coisotropic data* (Definition 4.6). As a special case, we recover the correspondence between graded lagrangian submanifolds and lagrangian subbundles $K \to N$ of E (Corollary 4.15) stated in [Šev05]. Geometric coisotropic data are the fundamental objects allowing constructions involving graded coisotropic submanifolds to be expressed in classical geometric terms.

In usual symplectic geometry, coisotropic submanifolds are often used to construct new symplectic manifolds via *reduction*: concretely, any coisotropic submanifold carries a "null" foliation (tangent to the kernel of the restriction of the symplectic form) whose leaf space, whenever smooth, is naturally equipped with a symplectic structure. A more fundamental fact underlying this construction is that the *basic* functions (i.e., leafwise constant) of a coisotropic submanifold always form a *Poisson* *algebra*. For coisotropic submanifolds of symplectic degree-2 $\mathbb{N}$-manifolds, Proposition 4.13 gives a geometric description of their sheaves of basic functions, together with their Poisson brackets, in terms of the data (N, K, F, ∇).

In § 5, we consider the reduction of a coisotropic submanifold $\mathcal{N}$ of a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, yielding (under suitable conditions) a new symplectic degree 2 $\mathbb{N}$-manifold $\underline{\mathcal{N}}$. The classical geometric counterpart of this construction is explained in Theorem 5.4. The set-up is that of a pseudo-euclidean vector bundle E with geometric coisotropic data (N, K, F, ∇) (corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively), with the additional requirements that F defines a *simple foliation* on N and that the F-connection ∇ on $K^\perp/K$ has *trivial holonomy* (which are the conditions for $\underline{\mathcal{N}}$ to exist). In this case, N has a smooth leaf space $\underline{N}$ and ∇ gives rise to a linear action on $K^\perp/K$ whose quotient is a a pseudo-euclidean vector bundle $E_{red} \to \underline{N}$ (see Lemma 5.1),

$$\begin{array}{ccc} \frac{K^\perp}{K} & \longrightarrow & E_{red} \\ \downarrow & & \downarrow \\ N & \longrightarrow & \underline{N}. \end{array}$$

It is shown in Theorem 5.4 that E_{red} is the pseudo-euclidean vector bundle corresponding to the reduced symplectic degree 2 $\mathbb{N}$-manifold $\underline{\mathcal{N}}$.

pseudo-euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$	symplectic degree 2 $\mathbb{N}$-manifold $(\mathcal{M}, \omega)$
(N, K, F, ∇) geometric coisotropic data	$\mathcal{N} \hookrightarrow \mathcal{M}$ coisotropic submanifold
$E_{red} \to \underline{N}$	coisotropic reduction $\underline{\mathcal{N}}$

Remark 1.2. The classical geometric description of graded coisotropic reduction can be extended to more general graded submanifolds with null distributions of "constant rank," but we restrict ourselves to the coisotropic setting for simplicity.

Building on graded coisotropic reduction, we obtain in § 6 a reduction procedure for Courant algebroids. As previously recalled, if E is a pseudo-euclidean vector bundle with corresponding symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, a Courant algebroid structure on E is equivalent to a *Courant function* on $\mathcal{M}$, i.e., a degree 3 function Θ satisfying $\{\Theta, \Theta\} = 0$. A function S on $\mathcal{M}$ is called *reducible* with respect to a coisotropic submanifold $\mathcal{N}$ if it satisfies $\{S, \mathcal{I}\} \subseteq \mathcal{I}$, where $\mathcal{I}$ is the vanishing ideal of $\mathcal{N}$ (i.e., if the hamiltonian vector field of S is tangent to $\mathcal{N}$); in the context of graded coisotropic reduction

$$\mathcal{N} \to \underline{\mathcal{N}},$$

reducibility ensures that S descends to a function S_{red} on $\underline{\mathcal{N}}$. Courant reduction is based on the fact that whenever a Courant function Θ on $\mathcal{M}$ is reducible, Θ_{red} is a Courant function of $\underline{\mathcal{N}}$, which in turn corresponds to a Courant algebroid structure on the pseudo-euclidean vector bundle $E_{red} \to \underline{N}$.

For a Courant algebroid structure on E with anchor ρ and bracket $[\![\cdot,\cdot]\!]$, we show in Theorem 6.5 that the reducibility of the corresponding Courant function with respect to a coisotropic submanifold defined by (N, K, F, ∇) amounts to the following conditions:

$$\rho(K^\perp) \subseteq TN, \quad \rho(K) \subseteq F, \quad [\rho(\Gamma^{flat}_{E,K^\perp}), \Gamma_{TM,F}] \subseteq \Gamma_{TM,F}, \quad [\![\Gamma^{flat}_{E,K^\perp}, \Gamma^{flat}_{E,K^\perp}]\!] \subseteq \Gamma^{flat}_{E,K^\perp}, \tag{1.2}$$

where $\Gamma^{flat}_{E,K^\perp}$ denotes the sheaf of sections of E whose restriction to N lie in $K^\perp$ and project to a ∇-flat section of $K^\perp/K$, and $\Gamma_{TM,F}$ is the sheaf of vector fields on M whose restriction to N lie in F. In the case of a graded lagrangian submanifold, reducibility amounts to K being a *Dirac structure* in E supported on N (see Def. 6.10).

The reduction of Courant algebroids corresponding to graded coisotropic reduction of Courant functions is presented in Theorem 6.11, and summarized below.

Theorem (Coisotropic Courant reduction). *Let $E \to M$ be a Courant algebroid equipped with geometric coisotropic data (N, K, F, ∇) such that F is simple and ∇ has trivial holonomy. If the reducibility conditions* (1.2) *hold, then the pseudo-euclidean vector bundle $E_{red} \to \underline{N}$ inherits a natural Courant algebroid structure.*

This result extends reductions in [LBM09, Zam08], see Example 6.13(ii) and Example 6.15.

In § 7, we use coisotropic Courant reduction as the basis for reduction schemes for *generalized complex* and *Dirac* structures:

- Generalized complex structures can be viewed as quadratic functions on symplectic degree 2 $\mathbb{N}$-manifolds suitably compatible with a Courant function, see [Gra06]; the geometric characterization of their reducibility (Lemma 7.1) leads to a reduction procedure for generalized complex structures with respect to geometric coisotropic data, see Theorem 7.2.
- Dirac structures are lagrangian submanifolds of symplectic degree 2 $\mathbb{N}$-manifolds for which the Courant function is reducible, see Cor. 6.9. Just as in usual symplectic geometry, graded lagrangian submanifolds can be reduced to coisotropic quotients upon a clean intersection condition (Theorem 7.6). By means of a geometric characterization of such clean intersections in the graded setting (Prop. 7.5) we obtain a reduction procedure for Dirac structures with respect to geometric coisotropic data in Theorem 7.7.

In the process of expressing reduction constructions from graded geometry in classical geometric terms, a recurrent issue is that of relating the Atiyah algebroid of a given (pseudo-euclidean) vector bundle A with the Atiyah algebroid of a quotient of A. Appendix A addresses this issue and presents the results needed in the paper.

Hamiltonian reduction. In § 8, we consider graded symplectic reduction in the hamiltonian setting; its formulation in classical terms leads to a hamiltonian version of Courant reduction that recovers constructions in [BCG07, BCG08].

In ordinary symplectic geometry, a hamiltonian action of a Lie algebra $\mathfrak{g}$ on a symplectic manifold M is given by a Lie algebra map $\mu^*\colon \mathfrak{g} \to C^\infty(M)$, known as a *comomentum map* (the dual map $\mu\colon M \to \mathfrak{g}^*$ is the *momentum map*). In the context of degree 2 $\mathbb{N}$-manifolds, we consider a graded Lie algebra $\widetilde{\mathfrak{g}}$ concentrated in degrees $-2, -1$ and 0,

$$\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g},$$

with a hamiltonian action on a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$ given by a comomentum map

$$\widetilde{\mu}^\sharp\colon \widetilde{\mathfrak{g}} \to C(\mathcal{M})[2], \tag{1.3}$$

see § 8.2. With this setup one may construct new symplectic degree 2 $\mathbb{N}$-manifolds by means of a graded version of Marsden–Weinstein reduction. For simplicity, we will focus on reduction at momentum level zero, which can be regarded as an instance of coisotropic reduction. Whenever $\mathcal{M}$ is equipped with a Courant function Θ, $C(\mathcal{M})[2]$ becomes a differential graded Lie algebra (DGLA) with differential $\{\Theta, \cdot\}$; in this case we also assume that $\widetilde{\mathfrak{g}}$ is a DGLA and that the comomentum map is a DGLA morphism. This ensures that Θ is reducible with respect to the coisotropic submanifold defined by the zero level of the momentum map, and hence defines a reduced Courant function Θ_{red} on the Marsden–Weinstein quotient $\mathcal{M}_{red}$.

For the classical geometric description of graded hamiltonian actions and reduction, we first notice that a DGLA $\widetilde{\mathfrak{g}}$ as above is equivalent to an ordinary Lie algebra $\mathfrak{g}$, together with $\mathfrak{g}$-modules $\mathfrak{a}$ and $\mathfrak{h}$, a symmetric bilinear map $\varpi \colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{h}$ and operators $\mathfrak{h} \stackrel{\delta}{\to} \mathfrak{a} \stackrel{\delta}{\to} \mathfrak{g}$ satisfying the conditions described in Propositions 8.1 and 8.2 (see (**B**) in § 8.4). The notion of a *hamiltonian action* on a Courant algebroid $E \to M$ in Definition 8.8 arises as the classical geometric counterpart of a comomentum map (1.3); it is given in terms of $(\mathfrak{g}, \mathfrak{a}, \mathfrak{h}, \varpi, \delta)$ and consists of a linear $\mathfrak{g}$-action on E and $\mathfrak{g}$-equivariant maps

$$\mu \colon M \to \mathfrak{h}^*, \qquad \varrho \colon \mathfrak{a} \to \Gamma(E)$$

with suitable compatibility conditions (see (**C**) in § 8.4). Reduction procedures for Courant, Dirac and GC structures in this hamiltonian setting are presented in Theorem 8.10, all based on graded Marsden–Weinstein reduction. (Since the zero level of the graded momentum map is coisotropic, these procedures rely on coisotropic Courant reduction.)

Hamiltonian action on Courant algebroid E	Hamiltonian action on $\mathcal{M}$, $\Theta \in C(\mathcal{M})_3$
$\mathfrak{g}$-action on E, $\mu \colon M \to \mathfrak{h}^*$, $\varrho \colon \mathfrak{a} \to \Gamma(E)$	comomentum map $\widetilde{\mathfrak{g}} \to C(\mathcal{M})[2]$
Courant reduction $E_{red} \to \mu^{-1}(0)/G$	Marsden–Weinstein reduction $\mathcal{M}_{red}$, Θ_{red}

We make contact with [BCG07, BCG08] by restricting our attention to hamiltonian actions of *exact* DGLAs on *exact* Courant algebroids. We show in Prop. 8.15 that exact DGLAs bijectively correspond to the (exact) Courant algebras introduced in [BCG07] (up to a minor technical difference with no noticeable effect). Further we prove in Prop. 8.23 (and Remark 8.24) that their hamiltonian actions on exact Courant algebroids are equivalent to the "extended actions with momentum maps," used in the reduction schemes in [BCG07, BCG08], of the corresponding Courant algebras.

Exact DGLA $\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}$	Exact Courant algebra $\mathfrak{a} \to \mathfrak{g}$
Hamiltonian $\widetilde{\mathfrak{g}}$-action on (exact) $E \to M$	Extended action of $\mathfrak{a} \to \mathfrak{g}$ on (exact) $E \to M$

This shows that our hamiltonian framework for Courant reduction, naturally derived from standard constructions in graded symplectic geometry, provides a a systematic approach to the reductions of Courant, Dirac and generalized complex structures, which recovers the results in [BCG07, BCG08] when specialized to the exact case.

Notation and conventions. For a $\mathbb{Z}$-graded vector space V, we define its degree shift $V[k]$ by $(V[k])_i = V_{k+i}$.

We will work in the category of smooth manifolds, usually denoted by M and N.

For a sheaf of rings $\mathcal{A}$ over M, we denote its evaluation on open subsets U by $\mathcal{A}(U)$, its restriction by $\mathcal{A}|_U$, and its stalk at x by $\mathcal{A}|_x$.

For a manifold M, we denote by C^∞_M its sheaf of algebras of (real-valued) smooth functions, and $C^\infty(M) = C^\infty_M(M)$. For a submanifold $N \subseteq M$, I_N denotes its sheaf of vanishing ideals.

For a smooth vector bundle $E \to M$, the sheaf of sections is denoted by Γ_E, and $\Gamma(E) = \Gamma_E(M)$. Vector bundle maps cover the identity map unless stated otherwise. Given a subbundle $K \to N$ of $E \to M$, we write $\Gamma_{E,K}$ for the subsheaf of sections of E whose restriction to N lie in K. Derivations of $E \to M$ are denoted by (X, D), with $D : \Gamma(E) \to \Gamma(E)$ and $X \in \mathfrak{X}(M) = \Gamma(TM)$ its symbol.

For a pseudo-euclidean vector bundle $E \to M$ (i.e., E is equipped with a fiberwise pseudo-Riemannian metric), we denote by $\mathbb{A}_E$ its Atiyah algebroid (whose sections are derivations of E that are compatible with the metric). We write $\Gamma^N_{\mathbb{A}_E}$ for the subsheaf of sections of $\mathbb{A}_E$ whose symbols are tangent to a submanifold N. For a subbundle $K \to N$ of $E \to M$, we denote by $\Gamma^{N,K}_{\mathbb{A}_E}$ the subsheaf of $\Gamma_{\mathbb{A}_E}$ whose sections (X, D) satisfy $X|_N \in TN$ and $D(\Gamma_{E,K}) \subseteq \Gamma_{E,K}$. When $N = M$, we simplify notation to $\Gamma^K_{\mathbb{A}_E}$ for the sheaf of derivations (X, D) with $D(\Gamma_K) \subseteq \Gamma_K$. We use a similar notation $\Gamma^{K,L}_{\mathbb{A}_E}$ when we have an additional subbundle $L \to M$ of $E \to M$, and impose the conditions $D(\Gamma_K) \subseteq \Gamma_K$ and $D(\Gamma_L) \subseteq \Gamma_L$.

We will make use of Einstein's convention for sums whenever there is no risk of confusion.

Acknowledgements: H.B. acknowledges the continual financial support of CAPES, CNPq and FAPERJ. A.S.C. acknowledges partial support of SNF Grant No. 200020-192080, of the Simons Collaboration on Global Categorical Symmetries, and of the COST Action 21109 (CaLISTA). A.S.C.'s research was (partly) supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation. R.M. acknowledges partial support from CNPq. M.Z. acknowledges partial support by Forschungskredit U. Zürich, by Ciencia 2007 (Portugal), SB2006-0141, MICINN RYC-2009-04065 and Severo Ochoa SEV-2011-0087 (Spain), CNPq-PVE grant 88881.030367/2013-01 (Brazil), Methusalem grants METH/15/026 and METH/21/03 - long term structural funding of the Flemish Government, the FWO research projects G083118N and G0B3523N, the FWO and FNRS under EOS projects G0H4518N and G0I2222N (Belgium). We are thankful to several institutions for hosting us during various stages of this project, including Erwin Schrödinger Institut, IMPA, KU Leuven and U. Zürich. We thank A. Cabrera, M. Cueca, F. del Carpio-Marek, P. H. Carvalho Silva for fruitful discussions related to this work.

2. Degree 2 $\mathbb{N}$-manifolds

In this section we present foundational results about $\mathbb{N}$-manifolds in degree 2. We will consider $\mathbb{N}$-manifolds as in [BP13, Meh06] (see also [Roy02a, Šev05]). Some parallel results in the theory of supermanifolds can be found, e.g., in [CCF11, Lei80].

2.1. Degree 2 $\mathbb{N}$-manifolds and morphisms.

A *degree* 2 $\mathbb{N}$-*manifold* $\mathcal{M}$ of dimension $m_0|m_1|m_2$ is a pair $(M, C_\mathcal{M})$, where M (the *body* of $\mathcal{M}$) is a dimension m_0

manifold and $C_{\mathcal{M}}$ is a sheaf of graded-commutative algebras on M satisfying the following property: any $x \in M$ admits an open neighborhood $U \subseteq M$ such that

$$C_{\mathcal{M}}|_U \cong C^\infty_U \otimes \wedge\mathbb{R}^{m_1} \otimes \mathrm{S}\mathbb{R}^{m_2}, \tag{2.1}$$

where the elements of $\mathbb{R}^{m_q}$ are of degree q. In other words, there exist sections $\{x^i, e^\mu, p^I\}$ of $C_{\mathcal{M}}$ over U, where $\{x^i\}$ are usual coordinates on U, the elements $\{e^\mu\}$ and $\{p^I\}$ are respectively of degree 1 and 2, and every local section of $C_{\mathcal{M}}|_U$ can be expressed as a sum of functions that are smooth in x^i and polynomial in e^μ and p^I. We refer to the local generators

$$\{x^1, \dots, x^{m_0}, e^1, \dots, e^{m_1}, p^1, \dots, p^{m_2}\}$$

as *local coordinates* on $\mathcal{M}$, and we think of $C_{\mathcal{M}}$ as the sheaf of "smooth functions" on $\mathcal{M}$. We denote by $(C_{\mathcal{M}})_q$ the subsheaf of degree q elements of $C_{\mathcal{M}}$ and observe that $(C_{\mathcal{M}})_0$ is canonically isomorphic to C^∞_M. We use the notation $\dim(\mathcal{M}) = m_0|m_1|m_2$ and refer to $\mathrm{totdim}(\mathcal{M}) = m_0 + m_1 + m_2$ as the *total dimension*.

Remark 2.1. Degree-2 $\mathbb{N}$-manifolds with dimensions of special type $m_0|0|0$ are naturally identified with ordinary manifolds of dimension m_0. Those of dimension type $m_0|m_1|0$ are referred to as *degree 1* $\mathbb{N}$*-manifolds*.

If $\mathcal{M}$ and $\mathcal{N}$ are degree 2 $\mathbb{N}$-manifolds, a *morphism* $\Psi\colon \mathcal{N} \to \mathcal{M}$ is a pair $(\psi, \psi^\sharp)$, where $\psi\colon N \to M$ is a smooth map and

$$\psi^\sharp\colon C_{\mathcal{M}} \to \psi_* C_{\mathcal{N}}$$

is a morphism of sheaves over M; in particular, for each open subset $U \subseteq M$, $\psi^\sharp_U\colon C_{\mathcal{M}}(U) \to C_{\mathcal{N}}(\psi^{-1}(U))$ is a morphism of graded algebras. Degree 2 $\mathbb{N}$-manifolds and their morphisms define a category denoted by 2Man.

A standard argument (see, e.g., [Var04, Sec. 4.1]) shows that the restriction of $\psi^\sharp$ to degree zero functions agrees with the pullback map

$$\psi^*\colon C^\infty_M \to \psi_* C^\infty_N.$$

An *isomorphism* $\Psi\colon \mathcal{N} \to \mathcal{M}$ is a morphism that admits an inverse; equivalently, ψ is a diffeomorphism and $\psi^\sharp$ induces a bijection of stalks at each $x \in M$.

Let $U \subset \mathbb{R}^{m_0}$ be an open set, and denote by $\mathcal{U}$ the degree 2 $\mathbb{N}$-manifold with body U and structure sheaf $C^\infty_U \otimes \wedge\mathbb{R}^{m_1} \otimes \mathrm{S}\mathbb{R}^{m_2}$, described by coordinates $\{x^i, e^\mu, p^I\}$. If $\mathcal{N}$ is any degree 2 $\mathbb{N}$-manifold, then it is a simple matter to check that any morphism $\mathcal{N} \to \mathcal{U}$ is completely determined by the choice of a map $\psi : N \to U$ as well as elements $f^\mu, g^I \in C_{\mathcal{N}}(N)$, of degrees 1 and 2, respectively; indeed, the conditions

$$\psi^\sharp(x^i) = x^i \circ \psi, \;\; \psi^\sharp(e^\mu) = f^\mu, \;\; \psi^\sharp(p^I) = g^I,$$

uniquely determine a morphism of sheaves $\psi^\sharp : C_{\mathcal{U}} \to \psi_* C_{\mathcal{N}}$.

2.2. **Vector-bundle description.** As we explain in this section, degree 2 $\mathbb{N}$-manifolds can be completely described in terms of classical vector bundles. (A generalization of this description for $\mathbb{N}$-manifolds of higher degrees can be found in [BCM].)

Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold. The subsheaf $(C_{\mathcal{M}})_1$ of degree 1 functions is a locally free sheaf of C^∞_M-modules, so it can be identified with the sheaf of sections $\Gamma_{E_1^*}$ of some vector bundle $E_1 \to M$. In coordinates $\{x^i, e^\mu, p^I\}$, we may take $\{e^\mu\}$ as a local frame for E_1^*. Similarly, $(C_{\mathcal{M}})_2$ can be identified with $\Gamma_{\widetilde{E}^*}$ for some vector bundle $\widetilde{E} \to M$, and a local frame for $\widetilde{E}^*$ is given by $\{p^I, e^\mu e^\nu : \mu < \nu\}$. Since $(C_\mathcal{M})_1.(C_\mathcal{M})_1 \subseteq (C_\mathcal{M})_2$, there is a natural inclusion map

$$\iota_E \colon \wedge^2 E_1^* \hookrightarrow \widetilde{E}^*, \tag{2.2}$$

whose dual map $\iota_E^* \colon \widetilde{E} \twoheadrightarrow \wedge^2 E_1$ is surjective. It also follows that there is an identification

$$C_\mathcal{M} = \Gamma_{\wedge^\bullet E_1^*} \otimes \Gamma_{\mathrm{S}^\bullet \widetilde{E}^*}/I \tag{2.3}$$

where I is the sheaf of homogeneous ideals generated by $1 \otimes \iota_E(T) - T \otimes 1$, with T a local section of $\Gamma_{\wedge^2 E_1^*}$.

Let $\mathcal{N}$ be another degree 2 $\mathbb{N}$-manifold, with corresponding vector bundles

$$F_1 \to N, \quad \widetilde{F} \to N.$$

Lemma 2.2. *Given a smooth map $\psi \colon N \to M$, a morphism $\psi^\sharp \colon C_\mathcal{M} \to \psi_* C_\mathcal{N}$ is completely specified by either of the following:*

- (*a*) *A pair of morphisms of sheaves of C^∞_M-modules $\psi_i^\sharp \colon (C_\mathcal{M})_i \to \psi_*(C_\mathcal{N})_i$, $i = 1, 2$, such that $\psi_2^\sharp(ee') = \psi_1^\sharp(e)\psi_1^\sharp(e')$ for any local sections e, e' of $(C_\mathcal{M})_1$.*
- (*b*) *A pair of vector-bundle maps $\psi_1 \colon F_1 \to E_1$, $\widetilde{\psi} \colon \widetilde{F} \to \widetilde{E}$, covering ψ, such that $\wedge^2\psi_1 \circ \iota_F^* = \iota_E^* \circ \widetilde{\psi}$.*

Proof. Using the identifications $(C_\mathcal{M})_1 = \Gamma_{E_1^*}$ and $(C_\mathcal{M})_2 = \Gamma_{\widetilde{E}^*}$, we see that $\psi_1^\sharp$ and $\psi_2^\sharp$ in (a) naturally extend to morphisms $\Gamma_{\wedge E_1^*} \to \psi_* C_\mathcal{N}$ and $\Gamma_{\mathrm{S}\widetilde{E}^*} \to \psi_* C_\mathcal{N}$, leading to a morphism $\Gamma_{\wedge E_1^*} \otimes \Gamma_{\mathrm{S}\widetilde{E}^*} \to \psi_* C_\mathcal{N}$. The compatibility condition between $\psi_1^\sharp$ and $\psi_2^\sharp$ in (a) guarantees that this morphism descends to the quotient (2.3), and any morphism $\psi^\sharp : C_\mathcal{M} \to \psi_* C_\mathcal{N}$ is determined in this way.

To check that the data in (a) and (b) are equivalent, note that the morphisms $\psi_1^\sharp \colon \Gamma_{E_1^*} \to \psi_* \Gamma_{F_1^*}$ and $\psi_2^\sharp \colon \Gamma_{\widetilde{E}^*} \to \psi_* \Gamma_{\widetilde{F}^*}$ in (a) are equivalent (by adjointness) to morphisms (of sheaves of C^∞_N-modules)

$$\psi_1^\sharp \colon \psi^* \Gamma_{E_1^*} \to \Gamma_{F_1^*}, \quad \psi_2^\sharp \colon \psi^* \Gamma_{\widetilde{E}^*} \to \Gamma_{\widetilde{F}^*}.$$

These morphisms are defined by vector-bundle maps $\psi^* E_1^* \to F_1^*$ and $\psi^* \widetilde{E}^* \to \widetilde{F}^*$, and their duals are vector-bundle maps

$$\psi_1 \colon F_1 \to E_1, \quad \widetilde{\psi} \colon \widetilde{F} \to \widetilde{E}$$

covering $\psi \colon N \to M$. The compatibility of $\psi_1^\sharp$ and $\psi_2^\sharp$ in (a) translates into the compatibility in (b). □

Definition 2.3. We denote by VB2 the category whose objects are triples $(E_1, \widetilde{E}, \phi_E)$, where E_1 and $\widetilde{E}$ are vector bundles over a manifold M and $\phi_E \colon \widetilde{E} \twoheadrightarrow \wedge^2 E_1$ is a surjective bundle map covering the identity on M, and morphisms $(E_1, \widetilde{E}, \phi_E) \to (F_1, \widetilde{F}, \phi_F)$ are defined by pairs $(\psi_1, \widetilde{\psi})$ of vector bundle maps $\psi_1 \colon E_1 \to F_1$, $\widetilde{\psi} \colon \widetilde{E} \to \widetilde{F}$, covering a smooth map $\psi : M \to N$, such that

$$\wedge^2 \psi_1 \circ \phi_E = \phi_F \circ \widetilde{\psi}. \tag{2.4}$$

Given an object $(E_1, \widetilde{E}, \phi_E)$, we set $E_2 = \ker(\phi_E)$, so that we have a short exact sequence

$$0 \longrightarrow E_2 \longrightarrow \widetilde{E} \xrightarrow{\phi_E} \wedge^2 E_1 \longrightarrow 0. \tag{2.5}$$

We will use the injective map

$$\phi_E^* \colon \wedge^2 E_1^* \to \widetilde{E}^*$$

to view $\wedge^2 E_1^*$ as a subbundle of $\widetilde{E}^*$, so that $E_2^* = \widetilde{E}^* / \wedge^2 E_1^*$.

There is a functor from VB2 to the category of degree 2 $\mathbb{N}$-manifolds,

$$\mathcal{F} \colon \mathrm{VB2} \to \mathrm{2Man}, \tag{2.6}$$

defined on objects as in (2.3), for $\iota_E = \phi_E^*$, and on morphisms as in Lemma 2.2. In particular, for $\mathcal{M} = \mathcal{F}(E_1, \widetilde{E}, \phi_E)$,

$$(C_{\mathcal{M}})_1 = \Gamma_{E_1^*}, \quad \text{and} \quad (C_{\mathcal{M}})_2 = \Gamma_{\widetilde{E}^*}, \tag{2.7}$$

and hence local coordinates for $\mathcal{M}$ are equivalently given by local frames of E_1^* and E_2^* over some open subset of M.

By Lemma 2.2 and the discussion preceding it, one can verify the following result.

Proposition 2.4. *The functor* $\mathcal{F} \colon \mathrm{VB2} \to \mathrm{2Man}$ *is an equivalence of categories.*

An extension of this result for $\mathbb{N}$-manifolds of arbitrary degrees is presented in [BCM].

Remark 2.5. There is a related classical geometric description of degree 2 $\mathbb{N}$-manifolds in terms of special types of double vector bundles, see [dCM15, JL18, LB12].

We illustrate the above correspondence with simple examples.

Example 2.6. Any vector bundle $E \to M$ defines an object in VB2 with $E_1 = E$, $\widetilde{E} = \wedge^2 E_1$ and $\phi_E = \mathrm{Id}_{\wedge^2 E_1}$. In this way the category of vector bundles sits in VB2 as a (full) subcategory. The restriction of $\mathcal{F}$ to this subcategory recovers the well-known equivalence between vector bundles and degree 1 $\mathbb{N}$-manifolds. The image of $E \to M$ under this equivalence is denoted by $\mathcal{M} = E[1]$ and satisfies $C_{\mathcal{M}} = \Gamma_{\wedge E^*}$. ⋄

Example 2.7. A pair of vector bundles $E_1 \to M$ and $E_2 \to M$ defines an object in VB2 with $\widetilde{E} = E_2 \oplus \wedge^2 E_1$ and $\phi_E \colon \widetilde{E} \to \wedge^2 E_1$ the natural projection. A degree 2 $\mathbb{N}$-manifold corresponding to an object of this type via $\mathcal{F}$ is called *split*; in this case the sheaf $C_{\mathcal{M}}$ (see (2.3)) has a simpler description in terms of E_1 and E_2 as

$$C_{\mathcal{M}} = \Gamma_{\wedge E_1^*} \otimes \Gamma_{\mathrm{S} E_2^*}.$$

Given an arbitrary object $(E_1, \widetilde{E}, \phi_E)$, the fact that one can always choose a splitting of the exact sequence (2.5) implies that any degree 2 $\mathbb{N}$-manifold is isomorphic to one that is split, though this identification is generally noncanonical, see [BP13]. ⋄

Example 2.8. Any graded vector space $V = V_{-2} \oplus V_{-1} \oplus V_0$, concentrated in degrees -2, -1 and 0, may be regarded as a split degree 2 $\mathbb{N}$-manifold corresponding to the vector bundles

$$E_1 = (V_{-1} \times V_0) \to V_0, \qquad E_2 = (V_{-2} \times V_0) \to V_0.$$

⋄

2.3. **Submanifolds.** Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold of dimension $m_0|m_1|m_2$. A *submanifold* of $\mathcal{M}$ is defined by a degree 2 $\mathbb{N}$-manifold $\mathcal{N}$ of dimension $n_0|n_1|n_2$ with $N \subseteq M$, and a morphism $(\iota, \iota^\sharp) : \mathcal{N} \to \mathcal{M}$ such that $\iota : N \hookrightarrow M$ is the inclusion map and the following local condition holds: any $x \in N$ admits a neighborhood U in M with local coordinates $\{x^i, e^\mu, p^I\}$ with respect to which the map

$$\iota^\sharp|_U : C_{\mathcal{M}}|_U \to C_{\mathcal{N}}|_{U\cap N} \tag{2.8}$$

is given by

$$\iota^\sharp(x^i) = \iota^*(x^i) = 0, \quad 1 \le i \le r_0, \tag{2.9}$$

$$\iota^\sharp(e^\mu) = 0, \qquad 1 \le \mu \le r_1, \tag{2.10}$$

$$\iota^\sharp(p^I) = 0, \qquad 1 \le I \le r_2, \tag{2.11}$$

where $r_l = m_l - n_l$, $l = 0, 1, 2$, and in such a way that

$$\{\iota^\sharp(x^i), \iota^\sharp(e^\mu), \iota^\sharp(p^I)\}, \;\; r_0 < i \le m_0, \; r_1 < \mu \le m_1, \; r_2 < I \le m_2,$$

are local coordinates for $\mathcal{N}$ over $U \cap N$. Such local coordinates $\{x^i, e^\mu, p^I\}$ of $\mathcal{M}$ on U are called *adapted to* $\mathcal{N}$. In particular, note that $N \subseteq M$ is an embedded submanifold. We refer to $r_0|r_1|r_2$ as the *codimension* of $\mathcal{N}$.

Let $(\iota, j^\sharp) : \mathcal{N}' \to \mathcal{M}$ be another submanifold with $N' = N$. We consider the submanifolds $\mathcal{N}$ and $\mathcal{N}'$ to be *equivalent* if there is an isomorphism of sheaves $\psi^\sharp \colon C_{\mathcal{N}} \to C_{\mathcal{N}'}$ for which the induced isomorphism $\psi^\sharp \colon \iota_* C_{\mathcal{N}} \to \iota_* C_{\mathcal{N}'}$ satisfies $\psi^\sharp \circ \iota^\sharp = j^\sharp$. We will make no distinction between equivalent submanifolds, and we will keep the term *submanifold* to refer to an equivalence class.

There is a geometric description of submanifolds in light of Proposition 2.4. Let $(E_1, \widetilde{E}, \phi_E)$ be an object in VB2. By a *subobject* of $(E_1, \widetilde{E}, \phi_E)$ we mean an object $(F_1, \widetilde{F}, \phi_F)$ in VB2, where F_1 and $\widetilde{F}$ are vector bundles over N, equipped with a morphism into $(E_1, \widetilde{E}, \phi_E)$, defined by $j_1 \colon F_1 \to E_1$ and $j_2 \colon \widetilde{F} \to \widetilde{E}$, so that j_1 and j_2 are fiberwise injective and cover an embedding $\iota \colon N \to M$. We will naturally identify subobjects of $(E_1, \widetilde{E}, \phi_E)$ which are isomorphic through an isomorphism that commutes with their respective morphisms into $(E_1, \widetilde{E}, \phi_E)$. In this way, we can always assume that a subobject $(F_1, \widetilde{F}, \phi_F)$ is such that $F_1 \subseteq E_1|_N$, $\widetilde{F} \subseteq \widetilde{E}|_N$, and $\phi_E(\widetilde{F}) = \wedge^2 F_1 \subseteq \wedge^2 E_1$, so that $\phi_F = \phi_E|_{\widetilde{F}}$.

Suppose that $\mathcal{M}$ corresponds to $(E_1, \widetilde{E}, \phi_E)$ via the functor (2.6).

Proposition 2.9. *Submanifolds of $\mathcal{M}$ are equivalent to either of the following geometric data:*

- *Subobjects of $(E_1, \widetilde{E}, \phi_E)$ in* VB2*, i.e., pairs of vector subbundles $F_1 \subseteq E_1|_N$, $\widetilde{F} \subseteq \widetilde{E}|_N$, such that $\phi_E(\widetilde{F}) = \wedge^2 F_1 \subseteq \wedge^2 E_1$;*
- *Pairs of vector subbundles $K_1 \subseteq E_1^*$ and $\widetilde{K} \subseteq \widetilde{E}^*$ over a submanifold $N \subseteq M$ satisfying $\widetilde{K} \cap \wedge^2 E_1^*|_N = K_1 \wedge E_1^*|_N$.*

The explicit correspondence between the two sets of geometric data is given by $K_1 = \mathrm{Ann}(F_1)$ and $\widetilde{K} = \mathrm{Ann}(\widetilde{F})$.

Proof. It is a direct verification that, through the equivalence in Proposition 2.4, subobjects of $(E_1, \widetilde{E}, \phi_E)$ in VB2 are in one-to-one correspondence with submanifolds of $\mathcal{M}$.

Given a pair of vector subbundles $F_1 \subseteq E_1$ and $\widetilde{F} \subseteq \widetilde{E}$ over a submanifold $N \subseteq M$, let $K_1 = \mathrm{Ann}(F_1)$ and $\widetilde{K} = \mathrm{Ann}(\widetilde{F})$. Then the condition $\phi_E(\widetilde{F}) = \wedge^2 F_1$ holds if and only if $\widetilde{K} \cap \wedge^2 E_1^*|_N = K_1 \wedge E_1^*|_N$. □

In the special case of an object defined by a vector bundle $E \to M$ as in Example 2.6, the previous proposition recovers the correspondence between vector subbundles $F \to N$ of E and submanifolds $\mathcal{N} = F[1]$ of $\mathcal{M} = E[1]$. For a split degree 2 $\mathbb{N}$-manifold $\mathcal{M}$ corresponding to a pair of vector bundles E_1 and E_2 as in Example 2.7, a corresponding pair of vector subbundles defines a split submanifold of $\mathcal{M}$, but not every submanifold of $\mathcal{M}$ is split.

2.3.1. *Submanifolds and regular ideals.* Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold. Any submanifold $(\iota, \iota^\sharp) \colon \mathcal{N} \to \mathcal{M}$ gives rise to a subsheaf of homogeneous ideals

$$\mathcal{I} = \ker(\iota^\sharp) \subseteq C_{\mathcal{M}}, \tag{2.12}$$

called the *sheaf of vanishing ideals* of $\mathcal{N}$. Note that $\mathcal{I}_0 = \mathcal{I} \cap (C_{\mathcal{M}})_0 \subseteq C^\infty_M$ coincides with I_N, the vanishing ideal of $N \subseteq M$, the body of $\mathcal{N}$. The sheaf of functions on $\mathcal{N}$ is recovered from $\mathcal{I}$ via

$$C_{\mathcal{N}} = \iota^{-1}(C_{\mathcal{M}}/\mathcal{I}).$$

We will now characterize the sheaves of ideals in $C_{\mathcal{M}}$ that arise as vanishing ideals of submanifolds.

Given a subsheaf of ideals $\mathcal{I} \subseteq C_{\mathcal{M}}$, we call it *regular* if

$$\mathcal{I}_0 := \mathcal{I} \cap (C_{\mathcal{M}})_0 = I_N \tag{2.13}$$

for a subset $N \subseteq M$ and the following local condition holds: There exist $r_j \in \{1, \dots, m_j\}$, $j = 0, 1, 2$, such that any $x \in N$ has a neighborhood $U \subset M$ with local coordinates $\{x^i, e^\mu, p^I\}$ satisfying the property that $\{x^1, \dots, x^{r_0}, e^1, \dots, e^{r_1}, p^1, \dots, p^{r_2}\}$ generates the sheaf of ideals $\mathcal{I}|_U$. Note that N is not uniquely specified by $\mathcal{I}$, since different subsets of M may have the same vanishing ideal. But the local condition implies that any $N \subseteq M$ satisfying (2.13) is an embedded submanifold.

Proposition 2.10. *For a submanifold $\mathcal{N}$ of $\mathcal{M}$, its sheaf of vanishing ideals is regular. Moreover, given a submanifold $N \subseteq M$, this correspondence establishes a bijection between submanifolds of $\mathcal{M}$ with body N and regular sheaves of ideals $\mathcal{I} \subseteq C_{\mathcal{M}}$ such that $\mathcal{I}_0 = I_N$.*

This result will follow from the geometric description of regular sheaves of ideals in the next lemma.

Suppose that $\mathcal{M}$ corresponds to the object $(E_1, \widetilde{E}, \phi_E)$ in VB2.

Lemma 2.11. *For a given submanifold $N \subseteq M$, regular sheaves of ideals $\mathcal{I} \subseteq C_{\mathcal{M}}$ with $\mathcal{I}_0 = I_N$ are equivalent to the following geometric data: vector subbundles $K_1 \subseteq E_1^*$ and $\widetilde{K} \subseteq \widetilde{E}^*$ over N satisfying $\widetilde{K} \cap \wedge^2 E_1^*|_N = K_1 \wedge E_1^*|_N$.*

The explicit correspondence is determined by the conditions

$$\mathcal{I}_0 = I_N, \quad \mathcal{I}_1 = \Gamma_{E_1^*, K_1}, \quad \text{and} \quad \mathcal{I}_2 = \Gamma_{\widetilde{E}^*, \widetilde{K}}, \tag{2.14}$$

where $\mathcal{I}_k = \mathcal{I} \cap (C_{\mathcal{M}})_k$, for $k = 0, 1, 2$. Recall from Section 1 that $\Gamma_{E_1^*, K_1}$ denotes the sheaf of sections of the vector bundle E_1^* which restrict to sections of K_1 over N, and similarly for $\Gamma_{\widetilde{E}^*, \widetilde{K}}$. Note also that a regular sheaf of ideals $\mathcal{I}$ coincides with the sheaf of ideals generated by $\mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2$ as a result of its local property.

Proof of Lemma 2.11. We first check that any regular sheaf of ideals $\mathcal{I}$ with $\mathcal{I}_0 = I_N$ is given as in (2.14). The local property of $\mathcal{I}$ can be rephrased as follows: any $x \in N$ has a neighborhood $U \subseteq M$ where $C_{\mathcal{M}}$ has local coordinates

$$\{x^1, \ldots, x^{m_0}, e^1, \ldots, e^{m_1}, p^1, \ldots, p^{m_2}\}$$

such that the sheaves of C^∞_M-modules $\mathcal{I}_0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ are generated by

$$\{x^1, \ldots, x^{r_0}\}, \tag{2.15}$$

$$\{e^1, \ldots, e^{r_1}, x^i e^\mu\}, \quad i = 1, \ldots, r_0, \tag{2.16}$$

$$\{p^1, \ldots, p^{r_2}, e^\mu e^\gamma, x^i p^I, x^i e^\mu e^\nu\}, \quad i = 1, \ldots, r_0, \quad \gamma = 1, \ldots, r_1, \tag{2.17}$$

respectively, where μ, ν, and I range through all indices.

The pullback sheaf $\iota^*(C(\mathcal{M})_1)$ with respect to the inclusion $\iota\colon N \hookrightarrow M$ is the sheaf of sections $\Gamma_{E_1^*|_N}$, and the local description (2.16) of $\mathcal{I}_1$ around points in N shows that $\iota^*(\mathcal{I}_1) = \Gamma_{K_1}$ for a vector subbundle $K_1 \subseteq E_1^*|_N$. Note that a section of E_1 over a small neighborhood $U \subseteq M$ of any $x \in N$ vanishes along $N \cap U$ if and only if it is a local section of $\mathcal{I}_1$, so $\mathcal{I}_1 = \Gamma_{E_1^*, K_1}$. By similar arguments, there is a vector subbundle $\widetilde{K} \subseteq \widetilde{E}^*|_N$ such that $\mathcal{I}_2 = \Gamma_{\widetilde{E}^*, \widetilde{K}}$. The description of $\mathcal{I}_2$ around points in N given by the generators (2.17) shows that $\widetilde{K} \cap \wedge^2 E^*|_N = K_1 \wedge E_1^*|_N$. Conversely, we can reverse the arguments to show that, given K_1 and $\widetilde{K}$ as in the statement, we obtain a corresponding regular sheaf of ideals by means of the conditions (2.14). $\square$

To conclude the proof of Proposition 2.10, note that if a submanifold $\mathcal{N}$ of $\mathcal{M}$ corresponds to a subobject $(F_1, \widetilde{F}, \phi_F)$, then its sheaf of vanishing ideals $\mathcal{I}$ corresponds, as in Lemma 2.11, to the subbundles $K_1 = \mathrm{Ann}(F_1)$ and $\widetilde{K} = \mathrm{Ann}(\widetilde{F})$. Hence Proposition 2.10 follows directly from the equivalences in Prop. 2.9.

2.3.2. Another geometric characterization of submanifolds.

We will make use of yet another geometric characterization of submanifolds based on a reformulation of the data in Lemma 2.11.

Lemma 2.12. *Let $K_1 \subseteq E_1^*$ and $\widetilde{K} \subseteq \widetilde{E}^*$ be vector subbundles over a submanifold $N \subseteq M$. Consider the following diagram of natural projections:*

$$\begin{array}{ccc} \widetilde{E}^*|_N & \xrightarrow{\quad\pi\quad} & E_2^*|_N \\ & {\scriptstyle \pi''}\searrow \qquad \nearrow{\scriptstyle \pi'} & \\ & \dfrac{\widetilde{E}^*|_N}{K_1 \wedge E_1^*|_N}. & \end{array} \tag{2.18}$$

The following are equivalent:

(a) $\widetilde{K} \cap \wedge^2 E_1^*|_N = K_1 \wedge E_1^*|_N$.

(b) *There is a vector subbundle $K_2 \subseteq E_2^*|_N$ and a vector-bundle map $\phi\colon K_2 \to \frac{\widetilde{E}^*|_N}{K_1 \wedge E_1^*|_N}$ satisfying $\pi' \circ \phi = Id$ such that $\widetilde{K} = (\pi'')^{-1}(\phi(K_2))$.*

Proof. Note that $\widetilde{K}$ is of the form $(\pi'')^{-1}(\phi(K_2))$ as in (*b*) if and only if

- $\pi(\widetilde{K})$ has constant rank (so we can set $K_2 = \pi(\widetilde{K})$),
- $\ker(\pi'') = K_1 \wedge E_1^*|_N \subseteq \widetilde{K}$, and

- $\pi'|_{\pi''(\widetilde{K})}$ is injective (so that it is an isomorphism onto K_2, and ϕ is its inverse).

The injectivity of $\pi'|_{\pi''(\widetilde{K})}$ amounts to the condition

$$\pi'(\pi''(k)) = \pi(k) = 0 \implies k \in \ker(\pi'') = K_1 \wedge E_1^*|_N,$$

for all $k \in \widetilde{K}$; i.e., $\ker(\pi) \cap \widetilde{K} = \wedge^2 E_1^*|_N \cap \widetilde{K} = K_1 \wedge E_1^*|_N$. It directly follows that (a) and (b) are equivalent. □

From Lemma 2.12 and the results in the previous subsection (see Prop. 2.10 and Lemma 2.11), we obtain a geometric characterization of submanifolds that will be useful in Section 4.

Theorem 2.13. *Submanifolds of $\mathcal{M}$ of codimension $r_0|r_1|r_2$ are equivalent to quadruples (N, K_1, K_2, ϕ), where*

- *$N \subseteq M$ is a submanifold of codimension r_0,*
- *$K_1 \subseteq E_1^*|_N$ and $K_2 \subseteq E_2^*|_N = \frac{\widetilde{E}^*|_N}{\wedge^2 E_1^*|_N}$ are vector subbundles of ranks r_1 and r_2, respectively, and*
- *$\phi\colon K_2 \to \dfrac{\widetilde{E}^*|_N}{K_1 \wedge E_1^*|_N}$ is a vector bundle map such that $\pi' \circ \phi = Id$, for π' defined in (2.18).*

We recall how to make the correspondence explicit. For a given quadruple (N, K_1, K_2, ϕ), we set

$$\widetilde{K} = (\pi'')^{-1}(\phi(K_2)),$$

as in Lemma 2.12. Let

$$F_1 = \mathrm{Ann}(K_1) \subseteq E_1|_N, \quad \widetilde{F} = \mathrm{Ann}(\widetilde{K}) \subseteq \widetilde{E}|_N, \quad \phi_F = \phi_E|_{\widetilde{F}}\colon \widetilde{F} \to \wedge^2 F_1.$$

Then the submanifold $\mathcal{N} \hookrightarrow \mathcal{M}$ corresponding to (N, K_1, K_2, ϕ) is the one defined by $(F_1, \widetilde{F}, \phi_F)$ via $(C_\mathcal{N})_1 = \Gamma_{F_1^*}$ and $(C_\mathcal{N})_2 = \Gamma_{\widetilde{F}^*}$. . From another perspective, the regular sheaf of vanishing ideals $\mathcal{I}$ representing the submanifold corresponding to (N, K_1, K_2, ϕ) is determined as in (2.14).

2.4. **Tangent bundle and differential calculus.** In this section we collect some basic results on the differential calculus on degree 2 $\mathbb{N}$-manifolds. Analogous results for supermanifolds can be found e.g. in [BC85, CCF11, Var04].

We will make use of Einstein's convention for sums whenever there is no risk of confusion.

2.4.1. *Vector fields and tangent vectors.*

Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold. A *degree q vector field* X on $\mathcal{M}$ is a degree q derivation of the sheaf $C_\mathcal{M}$; i.e., for each open subset $U \subseteq M$, X defines a linear map $C_\mathcal{M}(U) \to C_\mathcal{M}(U)$ such that for any $f, g \in C_\mathcal{M}(U)$, where f is homogeneous, $|X(f)| = |f| + q$ and

$$X(fg) = X(f)g + (-1)^{q|f|} f X(g).$$

Derivations form a sheaf of modules over $C_\mathcal{M}$, which we denote by $\mathcal{TM}$. We will also use the notation $\mathfrak{X}(\mathcal{M}) = \mathcal{TM}(M)$ Given two homogeneous vector fields X and Y, of degrees $|X|$ and $|Y|$, their Lie bracket is defined as the graded commutator

$$[X, Y] := XY - (-1)^{|X||Y|} YX,$$

so $\mathcal{TM}$ is also a sheaf of graded Lie algebras.

In local coordinates $\{x^i, e^\mu, p^I\}$ on $\mathcal{M}$, one has the associated vector fields

$$\frac{\partial}{\partial x^i}, \ \frac{\partial}{\partial e^\mu}, \ \frac{\partial}{\partial p^I}, \tag{2.19}$$

of degrees 0, -1 and -2, respectively; with respect to these coordinates, any degree q vector field X can be written in the form

$$X = a^i \frac{\partial}{\partial x^i} + b^\mu \frac{\partial}{\partial e^\mu} + c^I \frac{\partial}{\partial p^I},$$

where $|a^i| = q$, $|b^\mu| = q+1$, and $|c^I| = q+2$, showing that the sheaf of modules $\mathcal{TM}$ is locally free.

Example 2.14. For a degree 1 $\mathbb{N}$-manifold $\mathcal{M} = E[1]$ (see Example 2.6), so that $C_\mathcal{M} = \Gamma_{\wedge^\bullet E^*}$, degree 0 vector fields agree with *derivations* of $E \to M$ (or, equivalently, $E^* \to M$, by duality), i.e., pairs (X, D) where X is a (ordinary) vector field on M and $D\colon \Gamma(E) \to \Gamma(E)$ is $\mathbb{R}$-linear and satisfies $D(fe) = (\pounds_X f)e + fD(e)$, for $f \in C^\infty(M)$. Degree -1 vector fields are $C^\infty(M)$-linear maps $\Gamma(E^*) \to C^\infty(M)$, which can be seen as sections of E. See [ZZ13, Lemma 1.6] for more details. ⋄

Remark 2.15. For a degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, the geometric description of vector fields in degrees 0, -1 and -2 in terms of the corresponding object $(E_1, \widetilde{E}, \phi_E)$ in VB2 can be found in [BCM, §4.3] (see also [dCM15, § 3.5]). ⋄

Remark 2.16. A degree 1 vector field Q on $\mathcal{M}$ satisfying $[Q, Q] = 2Q^2 = 0$ is called a *Q-structure*, or *homological vector field*. For a degree 1 $\mathbb{N}$-manifold $\mathcal{M} = E[1]$ (see Example 2.6), a Q-structure is equivalent to a Lie algebroid structure on $E \to M$ [Vaĭ97] (Q is just the Lie algebroid differential on $\Gamma(\wedge^\bullet E^*)$). For degree 2 $\mathbb{N}$-manifolds, Q-structures (also known as *Lie 2-algebroids*) have been described in [dCM15, JL19, LB12]. (See [BLX20] for more on differential graded manifolds.) ⋄

Let $C_\mathcal{M}|_x$ be the stalk of $C_\mathcal{M}$ at a point $x \in M$. A homogeneous *tangent vector* to $\mathcal{M}$ at a point $x \in M$ is a linear map $X_x\colon C_\mathcal{M}|_x \to \mathbb{R}$ satisfying

$$X_x(fg|_x) = X_x(f|_x)g_0(x) + (-1)^{|X_x||f|} f_0(x) X_x(g|_x), \tag{2.20}$$

where $f_0(x), g_0(x)$ denote the evaluation of the degree zero components of $f|_x, g|_x \in C_\mathcal{M}|_x$ at x, respectively. Any $X \in \mathcal{TM}(U)$ defines a tangent vector X_x at each $x \in U$ by

$$X_x(f|_x) = (X(f))_0(x). \tag{2.21}$$

Whenever there is no risk of confusion, we may simplify the notation and write f instead of $f|_x$.

In terms of local coordinates $\{x^i, e^\mu, p^I\}$ around $x \in M$, the derivations (2.19) define tangent vectors at x, denoted by

$$\left(\frac{\partial}{\partial x^i}\right)_x, \ \left(\frac{\partial}{\partial e^\mu}\right)_x, \ \left(\frac{\partial}{\partial p^I}\right)_x, \tag{2.22}$$

of degrees 0, -1, and -2, respectively. The space of tangent vectors at $x \in M$ is a graded vector space over $\mathbb{R}$, denoted by $\mathcal{T}_x\mathcal{M}$, having (2.22) as a basis. In particular, note that if X is a homogeneous local section of $\mathcal{TM}$ and $X_x \neq 0$ at some x, then $|X| \in \{-2, -1, 0\}$.

Proposition 2.17. *Let $U \subseteq M$ be open and consider*

$$X_1, \dots, X_{k_0}, Y_1, \dots, Y_{k_1}, Z_1, \dots, Z_{k_2} \in \mathcal{TM}(U),$$

where $|X_j| = 0, |Y_\nu| = -1, |Z_J| = -2$. If $(X_1)_x, \dots, (Z_{k_2})_x \in \mathcal{T}_x\mathcal{M}$ are linearly independent for all $x \in U$, then $X_1, \dots, Z_{k_2}$ are linearly independent over $C_\mathcal{M}(U)$.

Note that the converse does not hold: the vector field $X = p\frac{\partial}{\partial p}$ of degree 0 is linearly independent over $C_\mathcal{M}$, but $X_x = 0$

Proof. To show that $X_1, \dots, Z_{k_2} \in \mathcal{TM}(U)$ are linearly independent, it suffices to consider U admitting coordinates $\{x^i, e^\mu, p^I\}$. Since each $(X_j)_x$ (resp. $(Y_\nu)_x$, resp. $(Z_J)_x$) is a linear combination of $(\partial/\partial x^i)_x$ (resp. $(\partial/\partial e^\mu)_x$, resp. $(\partial/\partial p^J)_x$), one may verify that the linear independence of $(X_1)_x, \dots, (Z_{k_2})_x$ is equivalent to the conditions

$$f^j(X_j)_x(x^i) = 0 \qquad \forall\ i = 1, \dots, m_0, \implies f^j = 0, \tag{2.23}$$

$$g^\nu(Y_\nu)_x(e^\mu) = 0 \qquad \forall\ \mu = 1, \dots, m_1, \implies g^\nu = 0, \tag{2.24}$$

$$h^J(Z_J)_x(p^I) = 0 \qquad \forall\ I = 1, \dots, m_2, \implies h^J = 0, \tag{2.25}$$

where $f^j, g^\nu, h^J \in \mathbb{R}$.

Consider the condition

$$a^j X_j + b^\nu Y_\nu + c^J Z_J = 0, \tag{2.26}$$

where $a^j, b^\nu, c^J \in C_\mathcal{M}(U)$. By applying the left-hand side of this equation on x^i, and since there are no functions of negative degrees, we see that

$$a^j(X_j(x^i)) = 0, \quad i = 1, \dots, m_0. \tag{2.27}$$

Since a^j is a polynomial in e^μ and p^I with coefficients in smooth functions of x^i, it suffices to assume that a^j is a monomial, i.e., of the form $f^j(x)e^{\mu_1} \dots e^{\mu_r} p^{I_1} \dots p^{I_s}$, for fixed $\mu_1 \leq \dots \leq \mu_r$ and $I_1 \leq \dots \leq I_s$. It follows from (2.27) that

$$(f^j X_j(x^i))(x) = f^j(x)(X_j)_x(x^i) = 0, \quad \forall x \in U,$$

and (2.23) implies that $f^j = 0$, so $a^j = 0$. Similarly, applying (2.26) on e^μ and using (2.24), one concludes that $b^\nu = 0$. Finally, applying (2.26) on p^I and using (2.25) one verifies that $c^J = 0$. □

We now give an alternative characterization of $\mathcal{T}_x\mathcal{M}$ for later use. For a fixed $x \in M$, let us consider the ideal

$$\mathcal{I}_{(x)} := \{f \in C_\mathcal{M}|_x\,,\ f_0(x) = 0\} \subseteq C_\mathcal{M}|_x, \tag{2.28}$$

and the graded vector space $\mathcal{I}_{(x)}/\mathcal{I}^2_{(x)}$. The derivation property (2.20) for an element $X_x \in \mathcal{T}_x\mathcal{M}$ implies that $X_x|_{\mathcal{I}^2_{(x)}} = 0$, so there is a natural degree-preserving map

$$\mathcal{T}_x\mathcal{M} \longrightarrow \left(\mathcal{I}_{(x)}/\mathcal{I}^2_{(x)}\right)^*. \tag{2.29}$$

Proposition 2.18. *The map* (2.29) *is an isomorphism of graded vector spaces.*

Proof. Denote by $[f]$ the class of $f \in \mathcal{I}_{(x)}$ in $\mathcal{I}_{(x)}/\mathcal{I}^2_{(x)}$. One directly checks that if $\alpha \in \mathcal{I}_{(x)}/\mathcal{I}^2_{(x)}$, then

$$C_\mathcal{M}|_x \to \mathbb{R}, \quad f \mapsto \alpha([f - f_0(x)]),$$

defines an element in $\mathcal{T}_x\mathcal{M}$, denoted by X_x^α. The map $(\mathcal{I}_{(x)}/\mathcal{I}_{(x)}^2)^* \to \mathcal{T}_x\mathcal{M}$, $\alpha \mapsto X_x^\alpha$, is the inverse of (2.29). □

2.4.2. Tangent vectors to submanifolds.

Let $(\iota, \iota^\sharp) : \mathcal{N} \to \mathcal{M}$ be a submanifold with corresponding sheaf of vanishing ideals $\mathcal{I}$ and codimension $r_0|r_1|r_2$. Let us consider the natural surjective map $r^\sharp \colon \mathcal{TM} \to \iota_*\iota^*\mathcal{TM}$, defined in local adapted coordinates $\{x^i, e^\mu, p^I\}$ by

$$a^i \frac{\partial}{\partial x^i} + b^\mu \frac{\partial}{\partial e^\mu} + c^I \frac{\partial}{\partial p^I} \mapsto \iota^\sharp(a^i)\frac{\partial}{\partial x^i} + \iota^\sharp(b^\mu)\frac{\partial}{\partial e^\mu} + \iota^\sharp(c^I)\frac{\partial}{\partial p^I}. \tag{2.30}$$

We denote the kernel of $r^\sharp$ by $\mathcal{I}.\mathcal{TM}$; in local adapted coordinates, it is given by the span over $\mathcal{I}$ of $\{\partial/\partial x^i, \partial/\partial e^\mu, \partial/\partial p^I\}$. So we have an exact sequence

$$0 \longrightarrow \mathcal{I}.\mathcal{TM} \longrightarrow \mathcal{TM} \xrightarrow{r^\sharp} \iota_*\iota^*\mathcal{TM} \longrightarrow 0. \tag{2.31}$$

In order to relate the sheaves $\mathcal{TN}$ and $\mathcal{TM}$, we consider the subsheaf of $C_\mathcal{M}$-modules $\mathcal{T}_\mathcal{I} \subseteq \mathcal{TM}$ given on open subsets $U \subseteq M$ by

$$\mathcal{T}_\mathcal{I}(U) := \{X \in \mathcal{TM}(U) \,|\, X(\mathcal{I}(U)) \subseteq \mathcal{I}(U)\}. \tag{2.32}$$

One may directly check that sections of $\mathcal{T}_\mathcal{I}$ are closed under the Lie bracket of vector fields,

$$[\mathcal{T}_\mathcal{I}, \mathcal{T}_\mathcal{I}] \subseteq \mathcal{T}_\mathcal{I}.$$

Note also that $\mathcal{I}.\mathcal{TM} \subseteq \mathcal{T}_\mathcal{I}$, so $r^\sharp$ restricts to an exact sequence

$$0 \longrightarrow \mathcal{I}.\mathcal{TM} \longrightarrow \mathcal{T}_\mathcal{I} \xrightarrow{r^\sharp} \iota_*\iota^*\mathcal{T}_\mathcal{I} \longrightarrow 0. \tag{2.33}$$

The following result is readily verified.

Lemma 2.19. *In local coordinates $\{x^i, e^\mu, p^i\}$ on $U \subseteq M$ adapted to $\mathcal{N}$, one has*

$$\mathcal{T}_\mathcal{I}|_U = \mathrm{span}_{C_\mathcal{M}|_U}\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial e^\mu}, \frac{\partial}{\partial p^I}\right\} + \mathcal{I}.\mathcal{TM}|_U, \tag{2.34}$$

for $i = r_0 + 1, \ldots, m_0$, $\mu = r_1 + 1, \ldots, m_1$, $I = r_2 + 1, \ldots, m_2$.

Proposition 2.20. *The following statements hold:*

(a) *There is a natural identification $\iota^*\mathcal{T}_\mathcal{I} = \mathcal{TN}$.*
(b) *For $U \subseteq M$ open, the map $r^\sharp \colon \mathcal{T}_\mathcal{I}(U) \to \mathcal{TN}(U \cap N)$ preserves Lie brackets and satisfies $r^\sharp(X)\iota^\sharp(f) = \iota^\sharp(Xf)$, for $X \in \mathcal{T}_\mathcal{I}(U)$ and $f \in C_\mathcal{M}(U)$.*
(c) *The map $\{X_x : X \in \mathcal{T}_\mathcal{I}(U)\} \to \mathcal{T}_x\mathcal{N}$, $X_x \mapsto (r^\sharp(X))_x$ is an isomorphism, where U is a small neighborhood of $x \in N$ in M.*

Proof. Using the local description of $\mathcal{T}_\mathcal{I}$ in adapted coordinates on $U \subseteq M$ in Lemma 2.19, we see that

$$\iota^*\mathcal{T}_\mathcal{I}|_V = r^\sharp(\mathcal{T}_\mathcal{I}|_U) = \mathrm{span}_{C_\mathcal{N}|_V}\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial e^\mu}, \frac{\partial}{\partial p^I}\right\},$$

where $i = r_0 + 1, \ldots, m_0$, $\mu = r_1 + 1, \ldots, m_1$, $I = r_2 + 1, \ldots, m_2$, and $V = U \cap N$. It follows that, once such adapted coordinates are chosen, there is a natural identification $\iota^*\mathcal{T}_\mathcal{I}|_V \cong \mathcal{TN}|_V$. Since this identification is clearly independent of the coordinates on $\mathcal{N}$ over V, it can be globalized, proving (a).

It suffices to verify (b) in local coordinates. By the Leibniz identity for the Lie bracket of vector fields, one checks that

$$[\mathcal{T}_{\mathcal{I}}, \mathcal{I}.\mathcal{T}\mathcal{M}] \subseteq \mathcal{I}.\mathcal{T}\mathcal{M}.$$

Then, using (2.34) and the observation that $\iota^\sharp : C_{\mathcal{M}}(U) \to C_{\mathcal{N}}(V)$ commutes with the derivations $\partial/\partial x^i$, $\partial/\partial e^\mu$, $\partial/\partial p^I$, for $i = r_0 + 1, \ldots, m_0$, $\mu = r_1 + 1, \ldots, m_1$, $I = r_2 + 1, \ldots, m_2$, one verifies both assertions in (b).

The isomorphism in (c) can be checked with the aid of local coordinates and the observation that, for any local section f of $C_{\mathcal{M}}$ around $x \in N$, we have $f_0(x) = \iota^\sharp(f)_0(x)$. □

2.4.3. *Tangent maps.* Any morphism $\Psi = (\psi, \psi^\sharp) : \mathcal{M} \to \mathcal{N}$ induces a *tangent map* $(d\Psi)_x : \mathcal{T}_x\mathcal{M} \to \mathcal{T}_{\psi(x)}\mathcal{N}$ by

$$(d\Psi)_x(X_x) \colon C_{\mathcal{N}}|_{\psi(x)} \to \mathbb{R}, \;\; (d\Psi)_x(X_x)(f) := X_x(\psi^\sharp f).$$

Let $\{x^j, e^\nu, p^J\}$ be local coordinates around x_0, and $\{\overline{x}^i, \overline{e}^\mu, \overline{p}^I\}$ be local coordinates around $\psi(x_0)$. With respect to these coordinates, the morphism Ψ can be written as

$$\psi^\sharp\overline{x}^i = \psi^i(x), \;\; \psi^\sharp\overline{e}^\mu = f^\mu_\nu(x)e^\nu, \;\; \psi^\sharp\overline{p}^I = g^I_{\mu\nu}(x)e^\mu e^\nu + h^I_J(x)p^J. \tag{2.35}$$

By considering the basis of $\mathcal{T}_{x_0}\mathcal{M}$ and $\mathcal{T}_{\psi(x_0)}\mathcal{N}$ relative to the choice of coordinates (as in (2.22)), one obtains the matrix expression

$$(d\Psi)_{x_0} = \begin{pmatrix} \left(\frac{\partial\psi^i}{\partial x^j}(x_0)\right) & 0 & 0 \\ 0 & \left(f^\mu_\nu(x_0)\right)^t & 0 \\ 0 & 0 & \left(h^I_J(x_0)\right)^t \end{pmatrix}. \tag{2.36}$$

Suppose that $\mathcal{M}$ and $\mathcal{N}$ correspond to $(E_1, \widetilde{E}, \phi_E)$ and $(F_1, \widetilde{F}, \phi_F)$, respectively, objects in VB2. Let

$$\psi_1 \colon E_1 \to F_1, \qquad \widetilde{\psi} \colon \widetilde{E} \to \widetilde{F}$$

be the vector bundle maps (covering $\psi \colon M \to N$) corresponding to Ψ. By the compatibility (2.4), $\widetilde{\psi}$ takes $E_2 = \ker(\phi_E)$ to $F_2 = \ker(\phi_F)$, and we denote the restricted map by

$$\psi_2 \colon E_2 \to F_2.$$

By viewing $\{e^\nu\}$ and $\{p^J\}$ as local frames for E_1^* and E_2^*, and $\{\overline{e}^\mu\}$ and $\{\overline{p}^I\}$ as local frames for F_1^* and F_2^*, respectively, we obtain the matrix expressions $(\psi_1)^*_{x_0} = (f^\mu_\nu(x_0))$ and $(\psi_2)^*_{x_0} = (h^I_J(x_0))$, i.e.,

$$(\psi_1)_{x_0} = (f^\mu_\nu(x_0))^t, \qquad (\psi_2)_{x_0} = (h^I_J(x_0))^t.$$

Comparing with (2.36), the next result readily follows.

Proposition 2.21. *For $x \in M$, the tangent map $(d\Psi)_x : \mathcal{T}_x\mathcal{M} \to \mathcal{T}_{\psi(x)}\mathcal{N}$ is injective (resp. surjective) if and only if the maps $(d\psi)_x \colon T_xM \to T_{\psi(x)}N$, $(\psi_1)_x \colon E_1|_x \to F_1|_{\psi(x)}$ and $(\psi_2)_x \colon E_2|_x \to F_2|_{\psi(x)}$ are injective (resp. surjective).*

By simple diagram chasing (see (2.38) below), one can verify that the injectivity of $(\psi_1)_x$ and $(\psi_2)_x$ implies that of $(\widetilde{\psi})_x$, and similarly for surjectivity.

2.4.4. Regular values. We now show how to use the previous proposition to directly derive versions of the inverse function theorem, and local normal form of submersions and immersions, for degree 2 $\mathbb{N}$-manifolds.

We call Ψ a *local isomorphism* around $x \in M$ if there are open neighborhoods of x in M and $\psi(x)$ in N, denoted by U and V, so that $\psi : U \to V$ is a diffeomorphism and $\psi^\sharp \colon C_{\mathcal{N}}|_V \to \psi_*(C_{\mathcal{M}}|_U)$ is an isomorphism of sheaves.

Corollary 2.22. *Let $\Psi = (\psi, \psi^\sharp) \colon \mathcal{M} \to \mathcal{N}$ be a morphism and $x_0 \in M$. Then $(d\Psi)_{x_0} \colon \mathcal{T}_{x_0}\mathcal{M} \to \mathcal{T}_{\psi(x_0)}\mathcal{N}$ is an isomorphism if and only if Ψ is a local isomorphism around x_0.*

Proof. By the equivalence in Prop. 2.4, the condition that Ψ is a local isomorphism around x_0 is equivalent to the existence of neighborhoods U of x_0 and V of $\psi(x_0)$ such that $\psi : U \to V$ is a diffeomorphism, and $\psi_1 \colon E_1|_U \to F_1|_V$ and $\psi_2 \colon E_2|_U \to F_2|_V$ are vector bundle isomorphisms. By the inverse function theorem for ψ, the existence of such neighborhoods is, in turn, equivalent to $(d\psi)_{x_0}$, $(\psi_1)_{x_0}$ and $(\psi_2)_{x_0}$ being isomorphisms, which is the same as $(d\Psi)_{x_0}$ being an isomorphism by Prop. 2.21. □

A morphism $\Psi = (\psi, \psi^\sharp) \colon \mathcal{M} \to \mathcal{N}$ is a *submersion at a point* $x_0 \in M$ if $(d\Psi)_{x_0} : \mathcal{T}_{x_0}\mathcal{M} \to \mathcal{T}_{\psi(x_0)}\mathcal{N}$ is onto, and it is a *submersion* if it is a submersion at every point. We will say that Ψ is a *surjective submersion* if it is a submersion and ψ is surjective.

Corollary 2.23. *Suppose that $\Psi = (\psi, \psi^\sharp) \colon \mathcal{M} \to \mathcal{N}$ is a submersion at $x_0 \in M$. Then there exist local coordinates $\{x^j, e^\nu, p^J\}$ around x_0, and $\{\overline{x}^i, \overline{e}^\mu, \overline{p}^I\}$ around $\psi(x_0)$, with respect to which Ψ has the form*

$$\psi^\sharp \overline{x}^i = \psi^i(x) = x^i, \;\; \psi^\sharp \overline{e}^\mu = e^\mu, \;\; \psi^\sharp \overline{p}^I = p^I. \tag{2.37}$$

Proof. By Prop. 2.21, the fact that Ψ is a submersion at x_0 is the same as the surjectivity of $(d\psi)_{x_0}$, $(\psi_1)_{x_0}$ and $(\psi_2)_{x_0}$. By the usual local normal form of submersions, there are neighborhoods of x_0 and $\psi(x_0)$, with coordinates $\{x^j\}$ and $\{\overline{x}^i\}$ satisfying the first condition in (2.37). Let us view the coordinates $\{e^\nu\}$ and $\{\overline{e}^\mu\}$ as local frames of E_1^* around x_0 and F_1^* around $\psi(x_0)$, respectively, and recall that the map $\psi^\sharp$ in degree 1 is identified with $\psi_1^* \colon \psi^* F_1 \to E_1$. We see that the local injectivity of ψ_1^* around x_0 (which follows from the surjectivity of $(\psi_1)_{x_0}$) ensures that, given any local frame $\{\overline{e}^\mu\}$, one can extend the set of sections $\{\psi_1^*\overline{e}^\mu\}$ to a local frame $\{e^\nu\}$ of E_1^*, and hence the second condition in (2.37) is satisfied. The same argument using the local injectivity of ψ_2^* shows the analogous result for degree 2 coordinates. □

One can obtain a local normal form for immersions with similar arguments.

We say that $c \in N$ is a *regular value* of $\Psi = (\psi, \psi^\sharp) : \mathcal{M} \to \mathcal{N}$ if $(d\Psi)_x$ is onto for all $x \in M$ with $\psi(x) = c$. The following is an immediate consequence of Prop. 2.21.

Corollary 2.24. *A point $c \in N$ is a regular value of $\Psi \colon \mathcal{M} \to \mathcal{N}$ if and only if*

(a) *c is a regular value for $\psi \colon M \to N$,*
(b) *the maps $\psi_i \colon E_i|_{\psi^{-1}(c)} \to F_i|_c$, $i = 1, 2$, are fiberwise surjective. (Equivalently, for each $x \in \psi^{-1}(c)$, the maps $(\psi_i)_x^* \colon F_i^*|_c \to E_i^*|_x$, $i = 1, 2$, are injective.)*

Recall that the second condition above implies that the map

$$\widetilde{\psi} \colon \widetilde{E}|_{\psi^{-1}(c)} \to \widetilde{F}|_c$$

is also fiberwise surjective.

Let c be a regular value of Ψ, and consider the submanifold $S = \psi^{-1}(c)$ and the subbundles $K_1 \subseteq E_1^*$ and $K' \subseteq \widetilde{E}^*$ over S, given by $K_1|_x = (\psi_1)_x^*(F_1^*|_c)$ and $K'|_x = (\widetilde{\psi})_x^*(\widetilde{F}^*|_c)$.

Lemma 2.25. *If c is a regular value of Ψ, then $K' \cap \wedge^2 E_1^*|_S = \wedge^2 K_1$.*

Proof. We have the following commutative diagram,

$$\begin{array}{ccccc} \wedge^2 F_1^* & \hookrightarrow & \widetilde{F}^* & \overset{\pi_F}{\twoheadrightarrow} & F_2^* \\ \downarrow{\scriptstyle \wedge^2\psi_1^*} & & \downarrow{\scriptstyle \widetilde{\psi}^*} & & \downarrow{\scriptstyle \psi_2^*} \\ \wedge^2 E_1^* & \hookrightarrow & \widetilde{E}^* & \overset{\pi_E}{\twoheadrightarrow} & E_2^* \end{array} \tag{2.38}$$

where we have omitted the base points x (on the bottom) and $c = \psi(x)$ (on the top). Recall that all the vertical maps are injective. We will check that $K' \cap \wedge^2 E_1^*|_S \subseteq \wedge^2 K_1$, since the opposite containment is clear (from the commutativity of the square on the left).

Let $k \in K' \cap \wedge^2 E_1^*$, which means that $k = \widetilde{\psi}^*(u)$ for $u \in \widetilde{F}^*$ and $\pi_E(k) = 0$. Note that $\pi_E(k) = \pi_E(\widetilde{\psi}^*(u)) = \psi_2^*(\pi_F(u)) = 0$, hence $\pi_F(u) = 0$, by the injectivity of ψ_2^*. So $u \in \wedge^2 F_1^*$, and therefore $k \in \wedge^2 K_1$.

□

It follows from the previous lemma that $K' \cap (K_1 \wedge E_1^*|_S) = \wedge^2 K_1$, which ensures that

$$\widetilde{K} := K' + K_1 \wedge E_1^*|_S \tag{2.39}$$

is a vector subbundle of $\widetilde{E}^*|_S$. It is also immediate that

$$\widetilde{K} \cap \wedge^2 E_1^*|_S = K_1 \wedge E_1^*|_S,$$

so by Prop. 2.9 we have

Proposition 2.26. *If c is a regular value of Ψ, then the subbundles $K_1 \subseteq E_1^*|_S$ and $\widetilde{K} \subseteq \widetilde{E}^*|_S$ define a submanifold of $\mathcal{M}$.*

We denote this submanifold by $\Psi^{-1}(c)$.

By (2.14), we see that the vanishing ideal of $\Psi^{-1}(c)$ coincides with the sheaf of ideals generated by $\psi^\sharp(\mathcal{I}_c)$, where $\mathcal{I}_c \subseteq C_\mathcal{N}$ is the subsheaf of vanishing ideals defined by $c \in N$,

$$\mathcal{I}_c(V) = \{f \in C_\mathcal{N}(V) \mid f_0(c) = 0 \text{ if } c \in V\}, \quad V \subseteq N \text{ open}.$$

2.5. **Distributions and the Frobenius theorem.** Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold of dimension $m_0|m_1|m_2$. A *distribution* of rank $d_0|d_1|d_2$ is a graded subsheaf of $C_\mathcal{M}$-modules $\mathcal{D}$ of $\mathcal{TM}$ satisfying the following local property: around any point in M there is a neighborhood U such that $\mathcal{D}|_U$ is generated by vector fields $X_1, \dots, X_{d_0}$ of degree 0, $Y_1, \dots, Y_{d_1}$ of degree -1, and $Z_1, \dots, Z_{d_2}$ of degree -2, and such that for each $x \in U$,

$$(X_1)_x, \dots, (X_{d_0})_x, (Y_1)_x, \dots, (Y_{d_1})_x, (Z_1)_x, \dots, (Z_{d_2})_x$$

are linearly independent elements of $\mathcal{T}_x\mathcal{M}$.

A distribution $\mathcal{D}$ is *involutive* if its spaces of local sections are closed under the Lie bracket on $\mathcal{TM}$. We have the following generalization of the Frobenius theorem.

Theorem 2.27. *Let $\mathcal{D}$ be an involutive distribution of rank $d_0|d_1|d_2$ on $\mathcal{M}$. Then any point in M has a neighborhood U with local coordinates $\{x^i, e^\mu, p^I\}$ such that $\mathcal{D}|_U$ is spanned by*

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{d_0}}, \frac{\partial}{\partial e^1}, \dots, \frac{\partial}{\partial e^{d_1}}, \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^{d_2}}.$$

This is proven in [BCM] (see, e.g., [CCF11, Var04] for the case of supermanifolds).

3. Symplectic degree 2 $\mathbb{N}$-manifolds

3.1. **Poisson brackets.** Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold of dimension $m_0|m_1|m_2$. A *Poisson structure* on $\mathcal{M}$ of degree q is an $\mathbb{R}$-bilinear operation

$$\{\cdot,\cdot\} : C_{\mathcal{M}} \times C_{\mathcal{M}} \to C_{\mathcal{M}}$$

such that, for each $U \subseteq M$ open and homogeneous elements $f \in (C_{\mathcal{M}}(U))_k$, $g \in (C_{\mathcal{M}}(U))_l$, $h \in (C_{\mathcal{M}}(U))_m$, the following holds:

(br1) $\{f, g\} \in (C_{\mathcal{M}}(U))_{k+l+q}$,
(br2) $\{f, g\} = -(-1)^{(k+q)(l+q)}\{g, f\}$,
(br3) $\{f, gh\} = \{f, g\}h + (-1)^{(k+q)l} g\{f, h\}$,
(br4) $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(k+q)(l+q)}\{g, \{f, h\}\}$

In particular, any global section $f \in C(\mathcal{M})_k$ defines a vector field on $\mathcal{M}$,

$$X_f := \{f, \cdot\},$$

of degree $k + q$, called the *hamiltonian vector field* of f. A simple consequence of (br4) is that, for $f \in C(\mathcal{M})_k$ and $g \in C(\mathcal{M})_l$, we have

$$X_{\{f,g\}} = [X_f, X_g]. \tag{3.1}$$

For each $x \in M$ and $f \in C_{\mathcal{M}}|_x$ of degree k, there is similarly an element $(X_f)_x \in \mathcal{T}_x\mathcal{M}$ of degree $k + q$ defined by

$$(X_f)_x(g) = \{f, g\}_0(x), \quad \text{for } g \in C_{\mathcal{M}}|_x,$$

where we keep the notation as in (2.21). It follows from (br2) and (br3) that if $f \in \mathcal{I}^2_{(x)}$ (see (2.28)), then $(X_f)_x = 0$. So there is an induced degree-preserving map

$$(\mathcal{T}_x\mathcal{M})^* \cong \mathcal{I}_{(x)}/\mathcal{I}^2_{(x)} \longrightarrow T_x\mathcal{M}[q],\ [f] \mapsto (X_f)_x. \tag{3.2}$$

Here "$[q]$" denotes the degree shift by q, as recalled in § 1. We say that a Poisson bracket $\{\cdot,\cdot\}$ is *non-degenerate* when the map (3.2) is an isomorphism for all $x \in M$.

We will be interested in Poisson brackets of degree $q = -2$.

Proposition 3.1. *Let $\mathcal{M}$ be equipped with a Poisson bracket $\{\cdot,\cdot\}$ of degree $q = -2$, and let $\{x^i, e^\mu, p^I\}$ be local coordinates on $\mathcal{M}$ around $x \in M$. Then the map* (3.2) *is an isomorphism if and only if $m_0 = m_2$ and*

$$\det(\{p^I, x^j\}(x)) \neq 0, \ \ \det(\{e^\mu, e^\nu\}(x)) \neq 0. \tag{3.3}$$

Proof. Relative to the bases $[x^i], [e^\mu], [p^I]$ of $\mathcal{I}_{(x)}/\mathcal{I}^2_{(x)}$ and (2.22) of $\mathcal{T}_x\mathcal{M}$, with $i = 1, \dots, m_0$, $\mu = 1, \dots, m_1$, $I = 1, \dots, m_2$, the linear map (3.2) is given by the matrix

$$\begin{pmatrix} \{x^i, x^j\}_0(x) & \{e^\mu, x^j\}_0(x) & \{p^I, x^j\}_0(x) \\ \{x^i, e^\nu\}_0(x) & \{e^\mu, e^\nu\}_0(x) & \{p^I, e^\nu\}_0(x) \\ \{x^i, p^J\}_0(x) & \{e^\mu, p^J\}_0(x) & \{p^I, p^J\}_0(x) \end{pmatrix}, \tag{3.4}$$

which, for $q = -2$, takes the form

$$\begin{pmatrix} 0 & 0 & \{p^I, x^j\}(x) \\ 0 & \{e^\mu, e^\nu\}(x) & 0 \\ \{x^i, p^J\}(x) & 0 & 0 \end{pmatrix}.$$

This matrix is invertible if and only if $m_0 = m_2$ and (3.3) holds. □

Definition 3.2. We will refer to a degree 2 $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a non-degenerate Poisson bracket of degree -2 as a *symplectic* degree 2 $\mathbb{N}$-manifold.

Corollary 3.3. *Let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold and $\{x^i, e^\mu, p^I\}$ be local coordinates. Then the local vector fields $\{x^i, \cdot\}$, $\{e^\mu, \cdot\}$, $\{p^I, \cdot\}$ are linearly independent.*

Proof. The nondegeneracy conditions in (3.3) imply that, for every $x \in M$ in the domain of the local coordinates, the tangent vectors $(X_I)_x := \{p^I, \cdot\}(x)$, $(Y_\mu)_x := \{e^\mu, \cdot\}(x)$, $(Z_i)_x := \{x^i, \cdot\}(x)$ are linearly independent in $\mathcal{T}_x\mathcal{M}$ (see (2.23), (2.24), (2.25)). The result now follows from Proposition 2.17. □

3.2. **Equivalence with pseudo-euclidean vector bundles.** Let $E \to M$ be a vector bundle. A *pseudo-euclidean* structure on E is a fiberwise bilinear pairing $\langle \cdot, \cdot \rangle$ which is symmetric and non-degenerate. As observed in [Roy02a, Šev05], pseudo-euclidean vector bundles are equivalent to symplectic degree 2 $\mathbb{N}$-manifolds:

Theorem 3.4 ([Roy02a],Thm. 3.3)**.** *There is a natural one-to-one correspondence between pseudo-euclidean vector bundles and symplectic degree 2 $\mathbb{N}$-manifolds (up to isomorphisms).*

This result can be deduced from the equivalence in Proposition 2.4; indeed, as we will now see, any pseudo-euclidean vector bundle naturally gives rise to an object in VB2, and such objects, in turn, are in correspondence with symplectic degree 2 $\mathbb{N}$-manifolds via the functor (2.6).

Recall that any pseudo-euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$ over M has an associated *Atiyah Lie algebroid*

$$\mathbb{A}_E \to M,$$

see, e.g., [CdSW99]. As a vector bundle, $\mathbb{A}_E$ is characterized by the property that its sheaf of sections $\Gamma_{\mathbb{A}_E}$ agrees with the sheaf of *derivations* (also called *covariant differential operators*) of E that preserve the pseudo-euclidean metric. I.e., for each open subset $U \subseteq M$, elements in $\Gamma_{\mathbb{A}_E}(U) = \Gamma(\mathbb{A}_E|_U)$ are pairs (X, D), where $X \in \mathfrak{X}(U)$ and $D\colon \Gamma(E|_U) \to \Gamma(E|_U)$ is an $\mathbb{R}$-linear endomorphism satisfying

$$D(fe) = (\pounds_X f)e + fD(e), \tag{3.5}$$

$$\pounds_X \langle e, e' \rangle = \langle D(e), e' \rangle + \langle e, D(e') \rangle, \tag{3.6}$$

for $f \in C^\infty(U)$, $e, e' \in \Gamma(E|_U)$. The Lie bracket on $\Gamma_{\mathbb{A}_E}$ is given by commutators and the anchor σ of $\mathbb{A}_E$ (also called the *symbol map*) is the projection $(X, D) \mapsto X$. The Lie algebroid $\mathbb{A}_E$ is transitive and fits into the exact sequence (known as the *Atiyah sequence*)

$$0 \longrightarrow \mathfrak{so}(E) \cong \wedge^2 E^* \longrightarrow \mathbb{A}_E \xrightarrow{\sigma} TM \longrightarrow 0, \tag{3.7}$$

where $\mathfrak{so}(E) \subseteq \mathrm{End}(E)$ is the space of skew-symmetric endomorphisms

$$T : E \to E, \;\; \langle Te, e' \rangle = -\langle e, Te' \rangle,$$

that is naturally identified with $\wedge^2 E^*$.

For a pseudo-euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$, the object $(E_1, \widetilde{E}, \phi_E)$ in VB2 canonically associated to it is defined by

$$E_1 = E, \qquad \widetilde{E} = \mathbb{A}_E^*, \qquad \phi_E \colon \widetilde{E} \twoheadrightarrow \wedge^2 E_1, \tag{3.8}$$

where ϕ_E is the map dual to the inclusion $\wedge^2 E^* \to \mathbb{A}_E$ in (3.7). In this way pseudo-euclidean vector bundles are identified with special objects in VB2. The next result allows us to characterize the corresponding degree 2 $\mathbb{N}$-manifolds.

Lemma 3.5. *Let $\mathcal{M}$ be a degree 2 $\mathbb{N}$-manifold corresponding to an object $(E_1, \widetilde{E}, \phi_E)$ in* VB2. *A symplectic structure on $\mathcal{M}$ is equivalent to the following additional data on $(E_1, \widetilde{E}, \phi_E)$:*

- *a pseudo-euclidean metric on E_1^* (or, equivalently, on E_1),*
- *a vector-bundle isomorphism $\widetilde{E}^* \cong \mathbb{A}_{E_1}$ commuting with the inclusions $\wedge^2 E_1^* \to \widetilde{E}^*$ and $\wedge^2 E_1^* \to \mathbb{A}_{E_1}$.*

Remark 3.6. A general Poisson bracket of degree -2 (not necessarily symplectic) on $\mathcal{M}$ gives rise to a Lie-algebroid structure on $\widetilde{E}^*$ and a Lie-algebroid morphism $\widetilde{E}^* \to \mathbb{A}_{E_1}$ commuting with the inclusions of $\wedge^2 E_1^*$, see [dCM15]. The nondegeneracy condition in the symplectic case makes this morphism into an isomorphism, leading to the simplified formulation of Lemma 3.5.

Proof. Let $\mathcal{M}$ be equipped with a symplectic structure given by the Poisson bracket

$$\{\cdot, \cdot\} \colon (C_{\mathcal{M}})_k \times (C_{\mathcal{M}})_l \to (C_{\mathcal{M}})_{k+l-2}.$$

We will obtain the data described in the lemma by restricting this bracket to functions of degrees 0, 1 and 2.

The restriction to degree 1 functions,

$$\{\cdot, \cdot\} \colon \Gamma_{E_1^*} \times \Gamma_{E_1^*} \to C^\infty_M,$$

is symmetric and C^∞_M-linear by properties (br2) and (br3), and non-degenerate by the second condition in (3.3). So it defines a pseudo-euclidean structure on E_1^*, or equivalently on E_1, through the identification of E_1 with E_1^* via the pseudo-euclidean metric.

The Poisson brackets of functions in degrees 2 and 0 define a map

$$\Gamma(\widetilde{E}^*) \to \mathfrak{X}(M), \;\; \widetilde{e} \mapsto (\{\widetilde{e}, \cdot\} : C^\infty(M) \to C^\infty(M)),$$

which is $C^\infty(M)$-linear, i.e., it comes from a vector-bundle map

$$\rho \colon \widetilde{E}^* \to TM.$$

The Poisson brackets of functions in degrees 2 and 1 now lead to a map $\Gamma(\widetilde{E}^*) \to \Gamma(\mathbb{A}_{E_1^*})$, $\widetilde{e} \mapsto (X, D)$, where $X = \{\widetilde{e}, \cdot\} \colon C^\infty(M) \to C^\infty(M)$ and

$$D = \{\widetilde{e}, \cdot\} : \Gamma(E_1^*) \to \Gamma(E_1^*). \tag{3.9}$$

It is a direct verification that this map comes from a vector-bundle map $\widetilde{E}^* \to \mathbb{A}_{E_1^*}$. By means of the identification $E_1 \cong E_1^*$, we alternatively obtain a map

$$\widetilde{E}^* \to \mathbb{A}_{E_1}. \tag{3.10}$$

It follows from condition (br3) that this map commutes with the inclusions $\wedge^2 E_1^* \to \widetilde{E}^*$ and $\wedge^2 E_1^* \to \mathbb{A}_{E_1}$. Finally, the first nondegeneracy condition in (3.3) implies the exactness of the sequence

$$0 \longrightarrow \wedge^2 E^* \xrightarrow{\phi_E^*} \widetilde{E}^* \xrightarrow{\rho} TM \longrightarrow 0. \tag{3.11}$$

The fact that the morphism (3.10) commutes with the inclusions and projections of the exact sequences (3.11) and (3.7) implies that it is an isomorphism. This shows that a symplectic structure on $\mathcal{M}$ gives rise to the data in the lemma.

For the converse, we reverse the arguments and use the additional data on $(E_1, \widetilde{E}, \phi_E)$ to define the nontrivial Poisson-bracket relations involving functions on $\mathcal{M}$ of degrees 0, 1 and 2. (Note that the Poisson bracket on degree 2 functions is identified with the Lie bracket on $\Gamma_{\mathbb{A}_{E_1}}$.) Since $C_{\mathcal{M}}$ is locally generated by such functions, one can verify (using (br3)) that there is a unique Poisson bracket on $C_{\mathcal{M}}$ of degree -2 satisfying these relations, which is moreover non-degenerate. □

We can now complete the proof of the theorem (c.f. [Roy02a]).

Proof of Theorem 3.4. Given a pseudo-euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$, the object in VB2 canonically associated with it, see (3.8), naturally carries the additional data in Lemma 3.5. It follows that the corresponding degree 2 $\mathbb{N}$ manifold carries a natural symplectic structure.

On the other hand, consider a degree 2 $\mathbb{N}$ manifold $\mathcal{M}$ corresponding to an object $(E_1, \widetilde{E}, \phi_E)$ in VB2. By Lemma 3.5, if $\mathcal{M}$ is equipped with a symplectic structure, then $(E_1, \widetilde{E}, \phi_E)$ is isomorphic in VB2 to the object associated with a pseudo-euclidean structure on E_1. The latter has a corresponding symplectic degree 2 $\mathbb{N}$ manifold $\mathcal{M}'$. Since this isomorphism in VB2 preserves all the additional data in Lemma 3.5 – which encode the Poisson-bracket relations of functions in degrees 0, 1, and 2 – it follows that the corresponding isomorphism $\mathcal{M} \cong \mathcal{M}'$ preserves symplectic structures. □

In particular, for a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$ corresponding to a pseudo-euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$, we have identifications

$$(C_{\mathcal{M}})_1 = \Gamma_E, \qquad (C_{\mathcal{M}})_2 = \Gamma_{\mathbb{A}_E}. \tag{3.12}$$

Example 3.7. Given a vector bundle $A \to M$, one can define a pseudo-euclidean vector bundle $E := A \oplus A^*$, with metric

$$\langle (a_1, \xi_1), (a_2, \xi_2) \rangle = \xi_2(a_1) + \xi_1(a_2).$$

The corresponding degree 2 symplectic $\mathbb{N}$-manifold is denoted by $T^*[2]A[1]$ and plays the role of the cotangent bundle of $A[1]$ (as in Example 2.6) in the context of degree

2 $\mathbb{N}$-manifolds. The projection $T^*[2]A[1] \to A[1]$ is defined by the natural inclusion

$$\kappa^\sharp \colon C_{A[1]} \to C_{T^*[2]A[1]},$$

given by the identity map in degree 0 and

$$(C_{A[1]})_1 = \Gamma_{A^*} \hookrightarrow \Gamma_{0\oplus A^*} \subseteq \Gamma_{A\oplus A^*} = (C_{T^*[2]A[1]})_1.$$

For graded cotangent bundles in higher degrees, see [Cue21]. ◇

4. Coisotropic submanifolds

For a symplectic manifold M, a submanifold $N \hookrightarrow M$ is called *coisotropic* if its sheaf of vanishing ideals is closed under the Poisson bracket,

$$\{I_N, I_N\} \subseteq I_N,$$

or, equivalently, if $TN^\omega \subseteq TN$, where TN^ω is the symplectic orthogonal to TN. A coisotropic submanifold N carries a foliation with leaves tangent to TN^ω, and a (local) function on N is called *basic* if it is constant along the leaves. An important property of N is that the sheaf of basic functions $(C^\infty_N)_{bas}$ inherits a natural Poisson bracket given by

$$\{f, g\}_{bas} = \{\widetilde{f}, \widetilde{g}\}|_N,$$

where f, g are (local) basic functions and $\widetilde{f}$, $\widetilde{g}$ are arbitrary (local) extensions to M. In this section we will present versions of these results for degree 2 symplectic $\mathbb{N}$-manifolds.

Remark 4.1. The previous discussion, as well as the results in this section, can be extended to the broader class of submanifolds $\iota\colon N \hookrightarrow M$ for which $TN \cap TN^\omega$, which coincides with the kernel distribution of $\iota^*\omega$, has constant rank. We will restrict ourselves to coisotropic submanifolds for simplicity.

4.1. Two viewpoints.

Let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold, with Poisson bracket $\{\cdot,\cdot\}$. If $(\iota, \iota^\sharp) : \mathcal{N} \to \mathcal{M}$ is a submanifold with sheaf of vanishing ideals $\mathcal{I} \subseteq C_{\mathcal{M}}$, we denote by $\mathfrak{N}_{\mathcal{I}} \subseteq C_{\mathcal{M}}$ its sheaf of *Lie normalizers*, given for each open subset $U \subseteq M$ by

$$\mathfrak{N}_{\mathcal{I}}(U) := \{f \in C_{\mathcal{M}}(U) \,|\, \{f, \mathcal{I}(U)\} \subseteq \mathcal{I}(U)\}. \tag{4.1}$$

Note that local sections of $\mathfrak{N}_{\mathcal{I}}$ are closed under the Poisson bracket, so $\mathfrak{N}_{\mathcal{I}}$ is a sheaf of Poisson algebras.

We say that $\mathcal{N}$ is *coisotropic* if $\mathcal{I} \subseteq \mathfrak{N}_{\mathcal{I}}$,; i.e., if

$$\{\mathcal{I}, \mathcal{I}\} \subseteq \mathcal{I}. \tag{4.2}$$

One can also approach coisotropic submanifolds through the more geometric condition "$\mathcal{T}\mathcal{N}^\omega \subseteq \mathcal{T}\mathcal{N}$". To make sense of the symplectic orthogonal distribution $\mathcal{T}\mathcal{N}^\omega$, we follow the classical viewpoint that regards it as the distribution locally generated by hamiltonian vector fields corresponding to functions in $\mathcal{I}$.

Let $\mathcal{T}_{\mathcal{I}^\omega} \subseteq \mathcal{T}\mathcal{M}$ be the subsheaf of $C_{\mathcal{M}}$-modules given, for each open subset $U \subseteq M$, by vector fields $X \in \mathcal{T}\mathcal{M}(U)$ with the property that any $x \in U$ admits a neighborhood where X can be written as a combination of hamiltonian vector fields $\{f, \cdot\}$, with f a section of $\mathcal{I}$ in that neighborhood. Consider the sheaf of $C_{\mathcal{N}}$-modules given by

$$\mathcal{T}\mathcal{N}^\omega := \iota^*\mathcal{T}_{\mathcal{I}^\omega} \subseteq \iota^*\mathcal{T}\mathcal{M}. \tag{4.3}$$

Let $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}\mathcal{M}$ be the subsheaf defined in (2.32).

Lemma 4.2. *The submanifold $\mathcal{N}$ is coisotropic if and only if $\mathcal{T}_{\mathcal{I}^\omega} \subseteq \mathcal{T}_{\mathcal{I}}$.*

Proof. Suppose that $\mathcal{T}_{\mathcal{I}^\omega} \subseteq \mathcal{T}_{\mathcal{I}}$, and let $f \in \mathcal{I}(U)$, $U \subseteq M$ open. Then $\{f, \cdot\} \in \mathcal{T}_{\mathcal{I}^\omega}(U) \subseteq \mathcal{T}_{\mathcal{I}}(U)$, which implies that $\{f, \mathcal{I}(U)\} \subseteq \mathcal{I}(U)$. Hence $\mathcal{I}(U) \subseteq \mathfrak{N}_{\mathcal{I}}(U)$ for all open U, i.e., $\mathcal{N}$ is coisotropic.

Conversely, let $X \in \mathcal{T}_{\mathcal{I}^\omega}(U)$. To show that $X \in \mathcal{T}_{\mathcal{I}}(U)$, it suffices to verify that any $x \in U$ admits an open neighborhood W in U for which $X|_W \in \mathcal{T}_{\mathcal{I}}(W)$. By the definition of $\mathcal{T}_{\mathcal{I}^\omega}$, one can always find such W where

$$X|_W = a^l\{f_l, \cdot\},$$

for $f_l \in \mathcal{I}(W)$ and $a^l \in C^\infty(W)$; then the condition $\{\mathcal{I}, \mathcal{I}\} \subseteq \mathcal{I}$ implies that $X|_W \in \mathcal{T}_{\mathcal{I}}(W)$. □

Lemma 4.3. *Let $\{x^i, e^\mu, p^I\}$ be local coordinates on an open subset $U \subseteq M$ adapted to $\mathcal{N}$. Then*

$$\mathcal{T}_{\mathcal{I}^\omega}(U) = \mathrm{span}_{C_{\mathcal{M}}(U)}\{\{x^i, \cdot\}, \{e^\mu, \cdot\}, \{p^I, \cdot\}\},$$

for $i = 1, \dots, r_0$, $\mu = 1, \dots, r_1$, $I = 1, \dots, r_2$, where $r_0|r_1|r_2 = \mathrm{codim}(\mathcal{N})$.

Proof. Any point in U admits a neighborhood W such that

$$\mathcal{T}_{\mathcal{I}^\omega}(W) = \mathrm{span}_{C_{\mathcal{M}}(W)}\{\{x^i, \cdot\}, \{e^\mu, \cdot\}, \{p^I, \cdot\}\},$$

for $1 \le i \le r_0$, $1 \le \mu \le r_1$, $1 \le I \le r_2$, since such x^i, e^μ, p^I are generators of $\mathcal{I}(U)$. Let $\{W_k\}$ be an open cover of U such that $\mathcal{T}_{\mathcal{I}^\omega}(W_k)$ is given as above. Let $X \in \mathcal{T}_{\mathcal{I}^\omega}(U)$. Then, for each k, $X|_{W_k} = a^i_k\{x^i, \cdot\} + b^\mu_k\{e^\mu, \cdot\} + c^I_k\{p^I, \cdot\}$ for unique sections a^i_k, b^μ_k, c^I_k in $C_{\mathcal{M}}(W_k)$, see Corollary 3.3. It also follows that, on overlaps $W_l \cap W_k$, we have that $a^i_l = a^i_k$, $b^\mu_l = b^\mu_k$ and $c^I_l = c^I_k$. So by the gluing property of sheaves we can find a^i, b^μ, c^I such that $X = a^i\{x^i, \cdot\} + b^\mu\{e^\mu, \cdot\} + c^I\{p^I, \cdot\}$. □

Proposition 4.4. *A submanifold $\mathcal{N} \to \mathcal{M}$ is coisotropic if and only if*

$$\mathcal{T}\mathcal{N}^\omega \subseteq \mathcal{T}\mathcal{N}.$$

In this case, $\mathcal{T}\mathcal{N}^\omega$ is an involutive distribution of $\mathcal{T}\mathcal{N}$ with rank $r_0|r_1|r_2 = \mathrm{codim}(\mathcal{N})$.

The distribution $\mathcal{T}\mathcal{N}^\omega \subseteq \mathcal{T}\mathcal{N}$ is called the *null distribution* of the coisotropic submanifold $\mathcal{N}$.

Proof. By Lemma 4.2, the first assertion amounts to checking that

$$\mathcal{T}_{\mathcal{I}^\omega} \subseteq \mathcal{T}_{\mathcal{I}} \iff \iota^*\mathcal{T}_{\mathcal{I}^\omega} \subseteq \iota^*\mathcal{T}_{\mathcal{I}}.$$

It suffices to verify that this holds locally. Let us consider local coordinates of $\mathcal{M}$ over U adapted to $\mathcal{N}$, and let $V = U \cap N$. Then $\iota^*\mathcal{T}_{\mathcal{I}^\omega}|_V$ is the image of $\mathcal{T}_{\mathcal{I}^\omega}|_U$ under the map $r^\sharp$ in (2.30), so the "$\implies$" direction is clear. For the reverse direction, consider $X \in \mathcal{T}_{\mathcal{I}^\omega}(U)$. If its image under (2.30) lies in $\iota^*\mathcal{T}_{\mathcal{I}}(V)$, it means that there is $Y \in \mathcal{T}_{\mathcal{I}}(U)$ such that $X - Y \in \mathcal{I}.\mathcal{T}\mathcal{M}(U)$, see (2.33). Since $\mathcal{I}.\mathcal{T}\mathcal{M}(U) \subseteq \mathcal{T}_{\mathcal{I}}(U)$, it follows that $X \in \mathcal{T}_{\mathcal{I}}(U)$.

For the second part of the proposition, note that Lemma 4.3 implies that

$$\mathcal{T}\mathcal{N}^\omega|_V = \mathrm{span}_{C_{\mathcal{N}}|_V}\{X_I, Y_\mu, Z_i\},$$

for $i = 1,\dots,r_0$, $\mu = 1,\dots,r_1$, $I = 1,\dots,r_2$, where X_I, Y_μ and Z_i are the images of the local sections $\{p^I,\cdot\}$, $\{e^\mu,\cdot\}$ and $\{x^i,\cdot\}$ of $\mathcal{T}_{\mathcal{I}}|_U$ under $r^\sharp$ (2.30), respectively. We have already observed that $\{p^I,\cdot\}(x), \{e^\mu,\cdot\}(x)$ and $\{x^i,\cdot\}(x)$ are linearly independent in $\mathcal{T}_x\mathcal{M}$ for all $x \in U$ (see the proof of Corollary 3.3), so $(X_I)_x, (Y_\mu)_x, (Z_i)_x \in \mathcal{T}_x\mathcal{N}$ are linearly independent for all $x \in V$ as a consequence of the isomorphism in Proposition 2.20(c). Hence $\mathcal{T}\mathcal{N}^\omega \subseteq \mathcal{T}\mathcal{N}$ is a distribution.

It follows from (3.1) and the coisotropicity condition (4.2) that the Lie bracket of any two of $\{x^i,\cdot\}$, $\{e^\mu,\cdot\}$ and $\{p^I,\cdot\}$, $i = 1,\dots,r_0$, $\mu = 1,\dots,r_1$, $I = 1,\dots,r_2$, lies again in $\mathcal{T}_{\mathcal{I}^\omega}|_U$. The involutivity of $\mathcal{T}\mathcal{N}^\omega$ follows from the fact that $r^\sharp$ preserves Lie brackets, see Proposition 2.20(b). □

4.2. **Geometric description of coisotropic submanifolds.** We now present a description of coisotropic submanifolds in classical geometric terms which builds on Theorem 2.13, see Theorem 4.5 below. We begin with some technical observations.

4.2.1. *Preliminaries.* Let $(E, \langle\cdot,\cdot\rangle)$ be a pseudo-euclidean vector bundle over M, and let $\iota\colon N \hookrightarrow M$ be a submanifold. The restriction $\iota^*E = E|_N$ is a pseudo-euclidean vector bundle over N. Let $K \subset E|_N$ be an isotropic subbundle, so

$$E_{quot} := K^\perp/K \tag{4.4}$$

is also a pseudo-euclidean vector bundle over N. We now collect some results relating the Atiyah algebroids of E, $E|_N$ and E_{quot}, following Appendix A.

Let $\Gamma^N_{\mathbb{A}_E}$ be the subsheaf of $\Gamma_{\mathbb{A}_E}$ given by metric-preserving derivations whose symbols are tangent to N. I.e., for each open subset $U \subseteq M$,

$$\Gamma^N_{\mathbb{A}_E}(U) = \{(X,D) \in \Gamma_{\mathbb{A}_E}(U) \,|\, X|_x \in T_xN,\ \forall x \in U \cap N\}.$$

Then there is a natural restriction map

$$\Gamma^N_{\mathbb{A}_E} \twoheadrightarrow \iota_*\Gamma_{\mathbb{A}_{E|_N}} \tag{4.5}$$

that takes $(X,D) \in \Gamma^N_{\mathbb{A}_E}(U)$ to $(X_N, D_N) \in \Gamma_{\mathbb{A}_{E|_N}}(U \cap N)$, where $X_N := X|_{U\cap N}$, and D_N is defined by

$$D_N(e|_{U\cap N}) := D(e)|_{U\cap N}.$$

It is clear that (4.5) is onto. If we now consider the subsheaf $\Gamma^{N,K}_{\mathbb{A}_E} \subseteq \Gamma^N_{\mathbb{A}_E}$ defined by

$$\Gamma^{N,K}_{\mathbb{A}_E}(U) = \{(X,D) \in \Gamma^N_{\mathbb{A}_E}(U) \,|\, D(\Gamma_{E,K}(U)) \subseteq \Gamma_{E,K}(U)\}, \tag{4.6}$$

then the map (4.5) restricts to a map

$$\Gamma^{N,K}_{\mathbb{A}_E} \twoheadrightarrow \iota_*\Gamma^K_{\mathbb{A}_{E|_N}}, \tag{4.7}$$

where $\Gamma^K_{\mathbb{A}_{E|_N}}$ is the subsheaf $\Gamma_{\mathbb{A}_{E|_N}}$ given by metric-preserving derivations of $E|_N$ that preserve Γ_K.

Since sections of $\Gamma^K_{\mathbb{A}_{E|_N}}$ also preserve $\Gamma_{K^\perp}$ (by compatibility with the pseudo-euclidean metric), we have a natural map

$$\Gamma^K_{\mathbb{A}_{E|_N}} \to \Gamma_{\mathbb{A}_{E_{quot}}}, \tag{4.8}$$

defined on sections over any open $V \subseteq N$ by $(X,D) \mapsto (X,[D])$, where $[D]\colon \Gamma_{E_{quot}}(V) \to \Gamma_{E_{quot}}(V)$ is given by

$$[D]([e]) := [D(e)],$$

with $e \mapsto [e]$ denoting the projection map $\Gamma_{K^\perp}(V) \to \Gamma_{E_{quot}}(V)$. The fact that (4.8) is onto is shown in Prop. A.2, and we have an exact sequence (see (A.2))

$$0 \to \Gamma_{K\wedge E|_N} \to \Gamma^K_{\mathbb{A}_{E|_N}} \to \Gamma_{\mathbb{A}_{E_{quot}}} \to 0. \tag{4.9}$$

4.2.2. Geometric characterization. We now proceed to the geometric description of coisotropic submanifolds of a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$. As explained in § 3.2, $\mathcal{M}$ corresponds to a pseudo-euclidean vector bundle E, so that, in terms of the associated object in VB2, we have identifications

$$E = E_1 \cong E_1^*, \quad \widetilde{E}^* = \mathbb{A}_E, \quad E_2^* = TM,$$

where $E_2 = \ker(\phi_E)$ and $E_1 \cong E_1^*$ via the pseudo-euclidean metric. The description of an arbitrary submanifold $\mathcal{N}$ of $\mathcal{M}$ in Theorem 2.13 amounts to the following geometric data: a submanifold $N \subseteq M$, subbundles $K_1 \subseteq E|_N$ and $K_2 \subseteq TM|_N$, as well as a bundle map

$$\phi\colon K_2 \to \frac{\mathbb{A}_E|_N}{K_1 \wedge E|_N} \tag{4.10}$$

such that $\pi' \circ \phi = Id$. Here the map π' is obtained by factoring the anchor (symbol) map $\sigma\colon \mathbb{A}_E \to TM$, whose kernel is $\wedge^2 E$ (see (4.9)), as the composition

$$\mathbb{A}_E|_N \xrightarrow{\pi''} \frac{\mathbb{A}_E|_N}{K_1 \wedge E|_N} \xrightarrow{\pi'} TM|_N = \frac{\mathbb{A}_E|_N}{\wedge^2 E|_N}, \tag{4.11}$$

c.f. (2.18). Recall (see (2.14) and Thm. 2.13) that the sheaf of vanishing ideals $\mathcal{I}$ corresponding to $\mathcal{N}$ is generated by its homogeneous components in degrees 0, 1 and 2,

$$\mathcal{I}_0 = I_N, \;\; \mathcal{I}_1 = \Gamma_{E,K_1}, \;\; \mathcal{I}_2 = \Gamma_{\mathbb{A}_E,\widetilde{K}}, \tag{4.12}$$

where

$$\widetilde{K} := (\pi'')^{-1}(\phi(K_2)) \subseteq \mathbb{A}_E|_N. \tag{4.13}$$

Theorem 4.5. *Coisotropic submanifolds $\mathcal{N}$ of codimension $r_0|r_1|r_2$ are equivalent to quadruples (N, K, F, ∇), where*

- *$\iota\colon N \hookrightarrow M$ is a submanifold of codimension r_0,*
- *K is an isotropic subbundle of $E|_N$ of rank r_1,*
- *$F \subseteq TN$ is a regular, integrable distribution of rank r_2,*
- *∇ is a flat, metric partial F-connection on the vector bundle $E_{quot} = K^\perp/K \to N$.*

Recall that a metric partial F-connection on the pseudo-Euclidean vector bundle E_{quot} can be defined as a vector bundle map

$$\nabla\colon F \to \mathbb{A}_{E_{quot}} \tag{4.14}$$

such that $\sigma \circ \nabla = \mathrm{Id}_F$; flatness corresponds to this map being a Lie algebroid morphism (i.e., preserving Lie brackets on spaces of sections).

Definition 4.6. Given a pseudo-euclidean vector bundle $E \to M$, we will refer to quadruples (N, K, F, ∇) as in Thm. 4.5 as *geometric coisotropic data.*

As the proof of the theorem will show, see below, the equivalence can be made explicit as follows. For geometric coisotropic data (N, K, F, ∇), the corresponding coisotropic submanifold $\mathcal{N}$ has body N and sheaf of vanishing ideals $\mathcal{I}$ such that

$$\mathcal{I}_0 = I_N, \qquad \mathcal{I}_1 = \Gamma_{E,K}, \tag{4.15}$$

and with $\mathcal{I}_2$ given as follows: for each open subset $U \subseteq M$,

$$\mathcal{I}_2(U) = \{(X, D) \in \Gamma^{N,K}_{\mathbb{A}_E}(U) \mid X_N \in \Gamma_F(U \cap N),\, [D_N] = \nabla_{X_N}\}, \tag{4.16}$$

where $(X_N, D_N) \in \Gamma_{\mathbb{A}_{E|_N}}(U \cap N)$ is the restriction of (X, D) to N (as in (4.7)) and $(X_N, [D_N]) \in \Gamma_{\mathbb{A}_{E_{quot}}}(U \cap N)$ its image under (4.8).

Remark 4.7. The condition $[D_N] = \nabla_{X_N}$ in (4.16) can be equivalently written as $[D_N](\Gamma^{flat}_{E_{quot}}) = 0$, where $\Gamma^{flat}_{E_{quot}}$ is the sheaf of ∇-flat sections of E_{quot} (since the difference $[D_N] - \nabla_{X_N}$ is C^∞_N-linear and vanishes on ∇-flat sections, which locally generate $\Gamma_{E_{quot}}$).

Conversely, given a coisotropic submanifold $\mathcal{N}$ with sheaf of vanishing ideals $\mathcal{I}$, the corresponding data (N, K, F, ∇) are such that N is the body of $\mathcal{N}$, $\Gamma_K = \iota^*\mathcal{I}_1$, and F and ∇ are defined by

$$\Gamma_F = \sigma(\iota^*\mathcal{I}_2), \qquad \nabla_Y[e|_N] = [D(e)|_N], \tag{4.17}$$

for all sections e of $K^\perp$, where Y is a section of F and D is such that (X, D) is a section of $\mathcal{I}_2 \subseteq \Gamma_{\mathbb{A}_E}$ with $X|_N = Y$. (Here $\sigma : \Gamma_{\mathbb{A}_{E|_N}} \to TN$ is the symbol map.)

Proof of Thm. 4.5. Let $\mathcal{N}$ be a coisotropic submanifold. Let (N, K_1, K_2, ϕ) be the geometric data associated with the submanifold $\mathcal{N}$ as in Theorem 2.13 (the precise correspondence was recalled just before Theorem 4.5). We will check that this quadruple is equivalent to the one in the theorem. Indeed, the condition for $\mathcal{N}$ being coisotropic is equivalent to

(a) $\{\mathcal{I}_1, \mathcal{I}_1\} \subseteq \mathcal{I}_0$,
(b) $\{\mathcal{I}_2, \mathcal{I}_0\} \subseteq \mathcal{I}_0$,
(c) $\{\mathcal{I}_2, \mathcal{I}_1\} \subseteq \mathcal{I}_1$,
(d) $\{\mathcal{I}_2, \mathcal{I}_2\} \subseteq \mathcal{I}_2$.

Since $\mathcal{I}_0 = I_N$ and $\mathcal{I}_1 = \Gamma_{E,K_1}$, (a) means that if $e_1, e_2 \in \Gamma_{E,K_1}(U)$, then

$$0 = \{e_1, e_2\}|_{U\cap N} = \langle e_1|_{U\cap N}, e_2|_{U\cap N}\rangle;$$

i.e., $K_1 \subset E|_N$ is isotropic with respect to $\langle\cdot,\cdot\rangle$. We set $K := K_1$.

If $(X, D) \in \mathcal{I}_2(U) \subseteq \Gamma_{\mathbb{A}_E}(U)$ then (b) says that $\mathcal{L}_X(I_N(U)) \subseteq I_N(U)$, while (c) says that $D(\Gamma_{E,K}(U)) \subseteq \Gamma_{E,K}(U)$. I.e., (b) and (c) are equivalent to

$$\mathcal{I}_2 \subseteq \Gamma^{N,K}_{\mathbb{A}_E}, \tag{4.18}$$

see (4.6). Since $\mathcal{I}_2 = \Gamma_{\mathbb{A}_E, \widetilde{K}}$, we see that

$$\Gamma_{\widetilde{K}} = \iota^*\mathcal{I}_2 \subseteq \iota^*\Gamma^{N,K}_{\mathbb{A}_E} = \Gamma^K_{\mathbb{A}_{E|_N}}. \tag{4.19}$$

In particular, $\widetilde{K} \subseteq \mathbb{A}_{E|_N} \subseteq \mathbb{A}_E|_N$. Since $K_2 = \widetilde{K}/\wedge^2 E|_N$, it follows from the Atiyah sequence for $E|_N$,

$$0 \to \wedge^2 E|_N \to \mathbb{A}_{E|_N} \to TN \to 0,$$

that $K_2 \subseteq TN \subseteq TM|_N$. Let $F := K_2$.

By the exact sequence (4.9), we have an identification

$$\Gamma_{\mathbb{A}_{E_{quot}}} = \pi''(\Gamma^K_{\mathbb{A}_{E|_N}}) \subseteq \frac{\Gamma_{\mathbb{A}_E|_N}}{\Gamma_{K\wedge E|_N}}.$$

We claim that the image of the map ϕ in (4.10) lies in $\mathbb{A}_{E_{quot}} \subseteq \frac{\mathbb{A}_E|_N}{K\wedge E|_N}$. Indeed, by (4.13), we know that $\phi(F) = \pi''(\widetilde{K})$, and $\pi''(\widetilde{K}) \subseteq \mathbb{A}_{E_{quot}}$ since $\Gamma_{\widetilde{K}} \subseteq \Gamma^K_{\mathbb{A}_{E|_N}}$, see (4.19). So ϕ is equivalent to a C^∞_M-linear map $\Gamma_F \to \Gamma_{\mathbb{A}_{E_{quot}}}$, and this is precisely a metric partial F-connection. Let us denote it by ∇, so we write $\phi(Y) = \nabla_Y$ for $Y \in F$.

We can now describe $\mathcal{I}_2 = \Gamma_{\mathbb{A}_E,\widetilde{K}}$ as follows. For any open $U \subseteq M$, an element $(X, D) \in \Gamma^{N,K}_{\mathbb{A}_E}(U)$ is in $\mathcal{I}_2(U)$ if and only if its restriction $(X_N, D_N) \in \Gamma^K_{\mathbb{A}_{E|_N}}(V)$, where $V = U \cap N$, lies in $\Gamma_{\widetilde{K}}(V)$, which, by (4.13) and (4.11), is equivalent to the condition

$$\pi''(X_N, D_N) = (X_N, [D_N]) = \phi(X_N) = \nabla_{X_N},$$

from where the description of $\mathcal{I}_2$ in (4.16) follows.

Finally, condition (d) is equivalent to

$$[\Gamma_F, \Gamma_F] \subseteq \Gamma_F, \quad \text{and } \nabla_{[X_N,Y_N]} = \nabla_{X_N}\nabla_{Y_N} - \nabla_{Y_N}\nabla_{X_N},$$

for all $X_N, Y_N \in \Gamma_F(V)$, $V \subseteq N$ open; i.e., F is an integrable distribution and ∇ is flat. □

We will use the following general remark to obtain natural examples of geometric coisotropic data by means of equivariant vector bundles.

Remark 4.8 (Equivariant bundles and partial connections). For a Lie group G, the infinitesimal counterpart of a G-action on a vector bundle $A \to N$ by vector-bundle automorphisms is a representation of the action Lie algebroid $\mathfrak{g} \ltimes N$ on A. Assuming that the G-action on N is (locally) free, and letting $F \subseteq TN$ be the distribution tangent to the G-orbits, we have an identification of F with $\mathfrak{g} \ltimes N$ as Lie algebroids. So a representation of $\mathfrak{g} \ltimes N$ on A is equivalent to a flat, partial F-connection ∇ on A.

Example 4.9 (Coisotropic data from invariant subbundles). Let $E \to M$ be a G-equivariant vector bundle with an invariant pseudo-euclidean structure. Let $K \to N$ be a G-invariant isotropic subbundle of $E \to M$. When the G-action on N is (locally) free, we obtain geometric coisotropic data (N, K, F, ∇) as follows. The G-action on E also preserves $K^\perp \to N$, hence $E_{quot} = K^\perp/K$ is G-equivariant, and its induced pseudo-euclidean structure is G-invariant. We then set F to be the orbit distribution on N, and ∇ is the flat, metric partial F-connection corresponding to G-action on E_{quot}, as in Remark 4.8. ◇

4.3. **Basic functions.** Let $\mathcal{N}$ be a coisotropic submanifold of a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, with $\dim(\mathcal{N}) = n_0|n_1|n_2$ and $\mathrm{codim}(\mathcal{N}) = r_0|r_1|r_2$, and vanishing ideal $\mathcal{I}$. Consider the subsheaf of algebras

$$(C_{\mathcal{N}})_{bas} := \iota^{-1}(\mathfrak{N}_{\mathcal{I}}/\mathcal{I}) \subseteq C_{\mathcal{N}}. \tag{4.20}$$

More concretely, considering local coordinates on $U \subseteq M$ adapted to $\mathcal{N}$ and the map $\iota^\sharp \colon C_{\mathcal{M}}(U) \to C_{\mathcal{N}}(V)$, where $V = U \cap N$, we see that

$$(C_{\mathcal{N}})_{bas}(V) = \iota^\sharp(\mathfrak{N}_{\mathcal{I}}(U)). \tag{4.21}$$

We will refer to $(C_{\mathcal{N}})_{bas}$ as the sheaf of *basic functions* on $\mathcal{N}$ relative to the null distribution, as indicated by property (b) of the next result.

Proposition 4.10. *The following holds:*
(a) $(C_{\mathcal{N}})_{bas}$ *is a sheaf of Poisson algebras.*
(b) *For* $V \subseteq N$ *open,*

$$(C_{\mathcal{N}})_{bas}(V) = \{f \in C_{\mathcal{N}}(V) \,|\, X(f) = 0 \; \forall\, X \in \mathcal{T}\mathcal{N}^{\omega}(V)\}.$$

Proof. By the local characterization of $(C_{\mathcal{N}})_{bas}$ in (4.21), we may write any section f in $(C_{\mathcal{N}})_{bas}(V)$ as $\iota^{\sharp}(\widetilde{f})$, for $\widetilde{f} \in \mathfrak{N}_{\mathcal{I}}(U)$. Using the fact that $\{\mathfrak{N}_{\mathcal{I}}, \mathcal{I}\} \subseteq \mathcal{I}$, we see that

$$\{f, g\}_{bas} := \iota^{\sharp}\{\widetilde{f}, \widetilde{g}\} \tag{4.22}$$

is a well-defined Poisson bracket (of degree -2) on $(C_{\mathcal{N}})_{bas}(V)$. Since $\iota^{\sharp}$ is a map of sheaves $C_{\mathcal{M}} \to \iota_* C_{\mathcal{N}}$, it is a simple matter to verify that this Poisson bracket globalizes to the sheaf $(C_{\mathcal{N}})_{bas}$.

To prove (b), it is enough to consider the case where $U \subseteq M$ has local coordinates $\{x^i, e^{\mu}, p^I\}$ adapted to $\mathcal{N}$ and $V = U \cap N$. Let $f = \iota^{\sharp}(\widetilde{f}) \in C_{\mathcal{N}}(V)$, for $\widetilde{f} \in C_{\mathcal{M}}(U)$. As a result of the second assertion in Prop. 2.20(b), the condition that $X(f) = 0$ for all $X \in \mathcal{T}\mathcal{N}^{\omega}(V)$ is equivalent to

$$\{x^i, \widetilde{f}\} \in \mathcal{I}(U), \quad \{e^{\mu}, \widetilde{f}\} \in \mathcal{I}(U), \quad \{p^I, \widetilde{f}\} \in \mathcal{I}(U),$$

for $1 \leq i \leq r_0$, $1 \leq \mu \leq r_1$, $1 \leq I \leq r_2$, which means that $\widetilde{f} \in \mathfrak{N}_{\mathcal{I}}(U)$, i.e., $f \in (C_{\mathcal{N}})_{bas}(V)$. □

Corollary 4.11. *Around any point in* N*, there are local coordinates*

$$\{x^1, \ldots, x^{n_0}, e^1, \ldots, e^{n_1}, p^1, \ldots, p^{n_2}\} \tag{4.23}$$

on $\mathcal{N}$ *with respect to which sections of* $(C_{\mathcal{N}})_{bas}$ *are the sections of* $C_{\mathcal{N}}$ *which depend only on* $x^{j>r_0}$*,* $e^{\nu>r_1}$*, and* $p^{J>r_2}$*; in particular,* $(C_{\mathcal{N}})_{bas}$ *is locally generated by* $((C_{\mathcal{N}})_{bas})_0$*,* $((C_{\mathcal{N}})_{bas})_1$*, and* $((C_{\mathcal{N}})_{bas})_2$*.*

Proof. By Frobenius' theorem (Theorem 2.27), there are local coordinates (4.23) with respect to which the distribution $\mathcal{T}\mathcal{N}^{\omega}$ is spanned by $\partial/\partial x^j$, $\partial/\partial e^{\nu}$, $\partial/\partial p^J$, $j = 1, \ldots, r_0$, $\nu = 1, \ldots, r_1$, $J = 1, \ldots, r_2$. The result now follows from Prop. 4.10(b). □

We will now give a description of the sheaf of basic functions in terms of the geometric data (N, K, F, ∇) corresponding to $\mathcal{N}$, as in Thm. 4.5. Consider the sheaves I_N, $\Gamma_{TM,F}$, $\Gamma_{E,K^{\perp}}$, $\Gamma^{N,K}_{\mathbb{A}_E}$, and $\Gamma^{flat}_{E,K^{\perp}}$, given by

$$\Gamma^{flat}_{E,K^{\perp}}(U) := \{e \in \Gamma_{E,K^{\perp}}(U) \,|\, \nabla[e|_{U\cap N}] = 0\}, \tag{4.24}$$

for $U \subseteq M$ open.

We start with a description of the sheaf of Lie normalizers $\mathfrak{N}_{\mathcal{I}}$.

Lemma 4.12. *For* $U \subseteq M$ *open,* $\mathfrak{N}_{\mathcal{I}}$ *satisfies*

$$(\mathfrak{N}_{\mathcal{I}})_0(U) = \{f \in C^{\infty}_M(U) \,|\, X(f) \in I_N(U) \;\; \forall\, X \in \Gamma_{TM,F}(U)\},$$
$$(\mathfrak{N}_{\mathcal{I}})_1(U) = \Gamma^{flat}_{E,K^{\perp}}(U),$$
$$(\mathfrak{N}_{\mathcal{I}})_2(U) = \{(X, D) \in \Gamma^{N,K}_{\mathbb{A}_E}(U) \,|\, [X, \Gamma_{TM,F}(U)] \subseteq \Gamma_{TM,F}(U),$$
$$D(\Gamma^{flat}_{E,K^{\perp}}(U)) \subseteq \Gamma^{flat}_{E,K^{\perp}}(U)\}.$$

Proof. Recall from Theorem 4.5 that $\mathcal{I}_0 = I_N$, $\mathcal{I}_1 = \Gamma_{E,K}$, and $\mathcal{I}_2$ is given by (4.16). To show that a local section g of $C_{\mathcal{M}}$ belongs to $\mathfrak{N}_{\mathcal{I}}$, it suffices to check that $\{g, \mathcal{I}_i\} \subseteq \mathcal{I}$, for $i = 0, 1, 2$. Note that, as a consequence of the surjectivity of (4.7) and (4.8), the map

$$\mathcal{I}_2 \to \Gamma_F, \tag{4.25}$$

given on sections over $U \subseteq M$ by $(X, D) \mapsto X_N = X|_{U\cap N}$, is onto.

If $f \in C^\infty_M(U)$ and $(X, D) \in \mathcal{I}_2(U)$, then $\{f, (X, D)\} \in \mathcal{I}(U)$ means that $X(f) \in I_N(U)$. But, by the surjectivity of (4.25), $X \in \Gamma_{TM}(U)$ is the symbol of an element in $\mathcal{I}_2(U)$ if and only if $X \in \Gamma_{TM,F}(U)$. This provides the desired description of $(\mathfrak{N}_{\mathcal{I}})_0$.

If $e \in \Gamma_E(U)$, then $\{e, \mathcal{I}_1(U)\} \subseteq \mathcal{I}(U)$ is equivalent to $\langle e, e'\rangle = I_N(U)$ for all $e' \in \Gamma_{E,K}(U)$, i.e., $e \in \Gamma_{E,K^\perp}(U)$. The condition $\{e, \mathcal{I}_2(U)\} \subseteq \mathcal{I}(U)$ amounts to $D(e) \in \Gamma_{E,K}(U)$ whenever $(X, D) \in \mathcal{I}_2(U)$. This is equivalent to $[D_N]([e|_{U\cap N}]) = \nabla_{X_N}([e|_{U\cap N}]) = 0$, which in turn means that $e \in \Gamma^{flat}_{E,K^\perp}(U)$, since (4.25) is onto.

Let $(X, D) \in \Gamma_{\mathbb{A}_E}(U)$. The conditions $\{(X, D), \mathcal{I}_i(U)\} \subseteq \mathcal{I}(U)$, $i = 0, 1$, simply say that $(X, D) \in \Gamma^{N,K}_{\mathbb{A}_E}(U)$. Condition $\{(X, D), \mathcal{I}_2(U)\} \subseteq \mathcal{I}(U)$ is equivalent, using the surjectivity of (4.25), to

$$[X, \Gamma_{TM,F}(U)] \subseteq \Gamma_{TM,F}(U), \quad [[D_N], \nabla_Y] = \nabla_{[X_N,Y]}, \tag{4.26}$$

for all $Y \in \Gamma_F(V)$, where $V = U \cap N$. . Now note that the second condition in (4.26) is equivalent to $[D_N] : \Gamma_{E_{quot}}(U) \to \Gamma_{E_{quot}}(U)$ preserving ∇-flat sections, which is the same as $D(\Gamma^{flat}_{E,K^\perp}(U)) \subseteq \Gamma^{flat}_{E,K^\perp}(U)$. □

We will now use the previous result to obtain a geometric description of $(C_{\mathcal{N}})_{bas}$. For that, let us consider the sheaf of basic functions on N with respect to $F \subseteq TN$, denoted by $(C^\infty_N)_{bas} \subseteq C^\infty_N$; i.e., for each open subset $V \subseteq N$,

$$(C^\infty_N)_{bas}(V) = \{f \in C^\infty_N(V) \mid X(f) = 0 \ \forall X \in \Gamma_F(V)\}. \tag{4.27}$$

Recall that $\Gamma^{flat}_{E_{quot}}$ denotes the sheaf of ∇-flat sections of E_{quot}, which is naturally a sheaf of $(C^\infty_N)_{bas}$-modules.

We also need to consider the vector bundle

$$\mathbb{A}^\nabla_{E_{quot}} := \mathbb{A}_{E_{quot}}/\nabla(F) \tag{4.28}$$

over N, where the vector subbundle $\nabla(F) \subseteq \mathbb{A}_{E_{quot}}$ is the image of (4.14). For a section (Y, Δ) of $\mathbb{A}_{E_{quot}}$, we denote its class in $\mathbb{A}^\nabla_{E_{quot}}$ by $\overline{(Y, \Delta)}$.

As explained in § A.2, the flat F-connection ∇ induces a flat F-connection $\overline{\nabla}$ on $\mathbb{A}^\nabla_{E_{quot}}$ by

$$\overline{\nabla}_Z(\overline{(Y, \Delta)}) = \overline{([Z, Y], [\nabla_Z, \Delta])},$$

where Z is a section of F, in such a way that a section $\overline{(Y, \Delta)}$ is flat if and only if

$$[Y, \Gamma_F] \subseteq \Gamma_F, \qquad \Delta(\Gamma^{flat}_{E_{quot}}) \subseteq \Gamma^{flat}_{E_{quot}}.$$

Proposition 4.13. *For a coisotropic submanifold $\mathcal{N}$ corresponding to (N, K, F, ∇), the sheaf $(C_{\mathcal{N}})_{bas}$ is such that*

$$((C_{\mathcal{N}})_{bas})_0 = (C^\infty_N)_{bas}, \quad ((C_{\mathcal{N}})_{bas})_1 = \Gamma^{flat}_{E_{quot}}, \quad ((C_{\mathcal{N}})_{bas})_2 = \Gamma^{flat}_{\mathbb{A}^\nabla_{E_{quot}}}.$$

Proof. Using Lemma 4.12 as well as 4.15 and (4.16) (see also Remark 4.7), we directly obtain the equalities in degrees 0 and 1, and a description of basic functions in degree 2 as follows: for $V \subseteq N$ open, $((C_{\mathcal{N}})_{bas})_2(V)$ is given by operators $(Y, D) \in \Gamma^{K}_{\mathbb{A}_{E|_N}}(V)$ satisfying

$$[Y, \Gamma_F(V)] \subseteq \Gamma_F(V), \quad [D](\Gamma^{flat}_{E_{quot}}(V)) \subseteq \Gamma^{flat}_{E_{quot}}(V),$$

modulo those such that $Y \in \Gamma_F(V)$ and $[D](\Gamma^{flat}_{E_{quot}}(V)) = 0$ (or, equivalently, $[D] = \nabla_Y$). We denote the equivalence class of (Y, D) by $\overline{(Y, D)}$. We then have a natural map

$$((C_{\mathcal{N}})_{bas})_2 \to \Gamma^{flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}, \quad \overline{(Y, D)} \mapsto \overline{(Y, [D])}, \tag{4.29}$$

which is readily seen to be injective. Its surjectivity follows from the surjectivity of (4.8), so we have an isomorphism. □

The Poisson bracket relations in $(C_{\mathcal{N}})_{bas}$ involving functions of degree 0, 1 and 2 (c.f. Prop. 4.10, part (a)) are computed to be as follows: for $f \in (C^\infty_N)_{bas}(V)$, $e, e' \in \Gamma^{flat}_{E_{quot}}(V)$, and $\overline{(Y, \Delta)}, \overline{(Y', \Delta')} \in \Gamma^{flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$, we have

$$\begin{aligned}
&\{e, e'\} = \langle e, e' \rangle,\\
&\{\overline{(Y, \Delta)}, f\} = \pounds_Y(f),\\
&\{\overline{(Y, \Delta)}, e\} = \Delta(e),\\
&\{\overline{(Y, \Delta)}, \overline{(Y', \Delta')}\} = \overline{([Y, Y'], [\Delta, \Delta'])}.
\end{aligned} \tag{4.30}$$

4.4. **Lagrangian submanifolds.** A submanifold $\mathcal{N}$ of a degree 2 symplectic $\mathbb{N}$-manifold $\mathcal{M}$ is called *lagrangian* if

$$\mathcal{T}\mathcal{N}^\omega = \mathcal{T}\mathcal{N}. \tag{4.31}$$

Theorem 4.14. *The following are equivalent:*

(a) *$\mathcal{N}$ is lagrangian;*
(b) *$\mathcal{N}$ is coisotropic and $(C_{\mathcal{N}})_{bas} = \{f \in C^\infty_N \mid f$ is locally constant $\}$;*
(c) *$\mathcal{N}$ is coisotropic, and its corresponding quadruple (N, K, F, ∇) satifies $K = K^\perp$ and $F = TN$;*
(d) *$\mathcal{N}$ is coisotropic and totdim$(N) = m_0 + \frac{m_1}{2} = \frac{1}{2}$totdim$(\mathcal{M})$.*

We obtain the following known geometric description of lagrangian submanifolds (see [Šev05]) as a direct consequence of part (c) of the theorem.

Let $E \to M$ be the pseudo-euclidean vector bundle corresponding to $\mathcal{M}$.

Corollary 4.15. *Lagrangian submanifolds of $\mathcal{M}$ are equivalent to lagrangian subbundles $K \to N$ of $E \to M$. Explicitly, the lagrangian submanifold corresponding to $K \to N$ has body N and sheaf of vanishing ideals determined by $\mathcal{I}_0 = I_N$, $\mathcal{I}_1 = \Gamma_{E,K}$ and $\mathcal{I}_2 = \Gamma^{N,K}_{\mathbb{A}_E}$.*

Proof of Theorem 4.14. The equivalence between (a) and (b) is a consequence of Prop. 4.10(b) and the fact, verified by a direct computation, that, for any degree 2 $\mathbb{N}$-manifold $\mathcal{N}$ and $f \in C_{\mathcal{N}}(V)$ (with V an open subset in the body), the condition $X(f) = 0$ for all $X \in \mathcal{T}\mathcal{N}(V)$ holds if and only if $f \in C^\infty(V)$ and is locally constant.

From Prop. 4.13 we see that $F = TN$ and $K = K^\perp$ if and only if $((C_\mathcal{N})_{bas})_0 \subseteq C^\infty_N$ is the space of locally constant functions and $((C_\mathcal{N})_{bas})_1$ is zero. So (b) implies (c). If we assume that (c) holds, then Lemma 4.12 shows that $(\mathfrak{N}_\mathcal{I})_2 = \Gamma^{N,K}_{\mathbb{A}_E} = \mathcal{I}_2$. It follows that $((C_\mathcal{N})_{bas})_2$ is zero, and (b) holds since $(C_\mathcal{N})_{bas}$ is locally generated in degrees 0, 1 and 2, see Cor. 4.11.

If $n_0|n_1|n_2 = \dim(\mathcal{N})$, then (c) implies that $n_1 = m_1 - \operatorname{rank}(K) = m_1/2$ and $n_2 = m_2 - n_0 = m_0 - n_0$. So $\operatorname{totdim}(\mathcal{N}) = m_0 + m_1/2 = \operatorname{totdim}(\mathcal{M})/2$, which is the condition in (d) . Conversely, note that

$$n_0 + n_1 + n_2 = n_0 + m_1 - \operatorname{rank}(K) + m_0 - \operatorname{rank}(F),$$

and suppose that this agrees with $m_0 + m_1/2$. Since $\operatorname{rank}(K) \leq m_1/2$ (as K is isotropic), we have

$$n_0 + m_1/2 + m_0 - \operatorname{rank}(F) \leq n_0 + m_1 - \operatorname{rank}(K) + m_0 - \operatorname{rank}(F) = m_0 + m_1/2,$$

hence $\operatorname{rank}(F) \geq n_0$. But since $F \subseteq TN$, we must have $F = TN$. Hence $\operatorname{totdim}(\mathcal{N}) = m_1 - \operatorname{rank}(K) + m_0 = m_1/2 + m_0$. This implies that $\operatorname{rank}(K) = m_1/2$, so $K = K^\perp$. Hence (d) implies (c). □

Example 4.16. For a vector bundle $A \to M$, consider the graded cotangent bundle $T^*[2]A[1]$ as in Example 3.7; recall that it corresponds to the pseudo-euclidean vector bundle $E = A \oplus A^*$. Given a vector subbundle $B \to N$ of A, we obtain a lagrangian subbundle $K = B \oplus \operatorname{Ann}(B) \subseteq E$. The corresponding lagrangian submanifold of $T^*[2]A[1]$ is denoted by $\nu^*[2]B[1]$ and plays the role of the conormal bundle of the submanifold $B[1] \subseteq A[1]$. ◇

5. Reduction of coisotropic submanifolds

The classical setting of coisotropic reduction is that of a symplectic manifold (M, ω) and a coisotropic submanifold $\iota\colon N \to M$ whose null foliation is simple, i.e., its leaf space is a smooth manifold $\underline{N}$ such that the quotient map $p\colon N \to \underline{N}$ is a surjective submersion. Then $\underline{N}$ acquires a unique symplectic structure ω_{red} such that $p^*\omega_{red} = \iota^*\omega$. Equivalently, recalling that $C^\infty(N)_{bas}$ is a Poisson algebra, ω_{red} is characterized (in terms of its Poisson bracket) by the fact that the identification

$$p^*\colon C^\infty(\underline{N}) \to C^\infty(N)_{bas}$$

is a Poisson isomorphism.

In this section we present coisotropic reduction for symplectic degree 2 $\mathbb{N}$-manifolds. We start by recalling general facts about quotients of vector bundles along surjective submersions from [Mac05, § 2.1].

5.1. **Quotients of vector bundles along submersions.** Any surjective submersion $p : N \to \underline{N}$ gives rise to a *submersion groupoid* $\mathcal{R}_p := N \times_{\underline{N}} N$, given by the equivalence relation that sets $x, y \in N$ as equivalent if $p(x) = p(y)$. The pullback $p^*\underline{A}$ of any vector bundle $\underline{A} \to \underline{N}$ carries a natural representation of $\mathcal{R}_p$. Conversely, as proven in [Mac05, Thm. 2.1.2], if a vector bundle $A \to N$ is equipped with a representation of $\mathcal{R}_p$, then there exists a vector bundle $\underline{A} \to \underline{N}$, unique up to isomorphism, such that $A \cong p^*\underline{A}$ through a vector-bundle isomorphism that preserves the representations of $\mathcal{R}_p$.

Let $F \subseteq TN$ be an integrable distribution on a manifold N. We will refer to F as *simple* if its underlying foliation is simple, in the sense that there is a manifold $\underline{N}$

and a surjective submersion $p : N \to \underline{N}$ with connected fibres so that $\ker(dp)_x = F_x$ for all $x \in N$ (so $\underline{N}$ is identified with the leaf space of the foliation). The pull-back map $C^\infty_{\underline{N}} \to p_* C^\infty_N$, defined for each local section f of $C^\infty_{\underline{N}}$ by $p^* f = f \circ p$, provides an identification

$$C^\infty_{\underline{N}} \cong p_*(C^\infty_N)_{bas}, \tag{5.1}$$

where $(C^\infty_N)_{bas} \subseteq C^\infty_N$ is as in (4.27).

Let F be a simple distribution on N with surjective submersion $p\colon N \to \underline{N}$, and let $A \to N$ be a vector bundle. A representation of $\mathcal{R}_p$ on A gives rise, through differentiation, to a representation of $F \to N$ (viewed as a Lie algebroid) on A, which is equivalent to a flat, partial F-connection ∇ on A. It is not always possible to integrate such ∇ to a representation of $\mathcal{R}_p$; when this is the case, we say that ∇ has *trivial holonomy*. The first part of the following lemma is a consequence of [Mac05, Thm. 2.1.2] mentioned above.

Lemma 5.1. *Let F be a simple distribution on N with surjective submersion $p\colon N \to \underline{N}$, and let $A \to N$ be a vector bundle equipped with a flat, partial F-connection ∇. Then ∇ has trivial holonomy if and only if there is a vector bundle $\underline{A} \to \underline{N}$ (unique, up to isomorphism) fitting into the pull-back diagram*

$$\begin{array}{ccc} A & \longrightarrow & \underline{A} \\ \downarrow & & \downarrow \\ N & \xrightarrow[p]{} & \underline{N} \end{array}$$

and such that the natural pull-back map $\Gamma_{\underline{A}} \to p_* \Gamma_A$ is an isomorphism onto $p_* \Gamma_A^{flat}$. Moreover, if A is pseudo-euclidean and ∇ is metric, then $\underline{A}$ inherits a pseudo-euclidean structure for which the pull-back map is an isometry.

Regarding the second assertion in the lemma, note that if A has a pseudo-euclidean structure $\langle \cdot, \cdot \rangle$ and ∇ is metric, then

$$\left\langle \Gamma_A^{flat}, \Gamma_A^{flat} \right\rangle \subseteq (C^\infty_N)_{bas}. \tag{5.2}$$

By means of the identifications $C^\infty_{\underline{N}} \cong p_*(C^\infty_N)_{bas}$ and

$$\Gamma_{\underline{A}} \cong p_* \Gamma_A^{flat}, \tag{5.3}$$

we see that $\langle \cdot, \cdot \rangle$ induces a pseudo-euclidean structure on $\underline{A}$.

Definition 5.2. We refer to $\underline{A}$ in the previous lemma as the *quotient* of $A \to N$ with respect to F and ∇.

The following special case will be useful in Section 8.

Example 5.3 (Quotients of vector bundles by group actions). Let G be a Lie group, and let $A \to N$ be a G-equivariant vector bundle such that the G-action on N is free and proper. Then the quotient map $p\colon N \to N/G$ is a surjective submersion, and there is a well-defined vector bundle

$$A/G \to N/G$$

with the property that $p^*(A/G) = A$ and $\Gamma(A/G)$ is identified with the space of G-invariant sections of A. In this setting, the G-action on A is equivalent to a representation of the action groupoid $G \ltimes N$ on A, and $G \ltimes N$ is naturally identified with the submersion groupoid $\mathcal{R}_p = N \times_{N/G} N$.

Let $F \subseteq TN$ be the distribution tangent to the G-orbits. As recalled in Remark 4.8, the infinitesimal $\mathfrak{g}$-action on A is defined by a representation of $\mathfrak{g} \ltimes N = F$ on A, which is the same as a flat, partial F-connection ∇. When G is connected, N/G is the leaf space of F, and A/G is the quotient of A with respect to F and ∇, as in Def. 5.2. ◇

Let us recall from § A.2 how the Atiyah algebroids $\mathbb{A}_A$ and $\mathbb{A}_{\underline{A}}$ are related. The vector bundle

$$\mathbb{A}_A^\nabla = \mathbb{A}_A/\nabla(F)$$

carries a flat, partial F-connection $\overline{\nabla}$ induced from ∇, see (A.4), with flat sections described in Lemma A.6. Then $\mathbb{A}_{\underline{A}}$ is the quotient of $\mathbb{A}_A^\nabla$ with respect to F and $\overline{\nabla}$,

$$\begin{array}{ccc} \mathbb{A}_A^\nabla & \longrightarrow & \mathbb{A}_{\underline{A}} \\ \downarrow & & \downarrow \\ N & \xrightarrow[p]{} & \underline{N}, \end{array}$$

and the identification

$$\Gamma_{\mathbb{A}_{\underline{A}}} \cong p_* \Gamma^{flat}_{\mathbb{A}_A^\nabla} \tag{5.4}$$

is given as follows (see Prop. A.7): a section $\overline{(Y, \Delta)}$ of $\Gamma^{flat}_{\mathbb{A}_A^\nabla}(V)$ corresponds to a section $(\underline{Y}, \underline{\Delta})$ of $\Gamma_{\mathbb{A}_{\underline{A}}}(\underline{V})$, for $\underline{V} \subseteq \underline{N}$ open and $V = p^{-1}(\underline{V})$, if and only if $\underline{Y} = p_*(Y)$ and the restriction of Δ to flat sections, $\Delta\colon \Gamma_A^{flat}(V) \to \Gamma_A^{flat}(V)$, corresponds to $\underline{\Delta}\colon \Gamma_{\underline{A}}(\underline{V}) \to \Gamma_{\underline{A}}(\underline{V})$ upon the identification $\Gamma_{\underline{A}} \cong p_* \Gamma_A^{flat}$.

5.2. **Coisotropic reduction.** Let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold corresponding to the pseudo-euclidean vector bundle $E \to M$. Let $\mathcal{N}$ be a coisotropic submanifold of $\mathcal{M}$ corresponding to the geometric data (N, K, F, ∇), as in Theorem 4.5. Consider $(C_{\mathcal{N}})_{bas}$, as in (4.20) and Prop. 4.10.

Theorem 5.4. *The following are equivalent:*

(a) *F is simple and ∇ has trivial holonomy;*
(b) *there is a symplectic degree 2 $\mathbb{N}$-manifold $\underline{\mathcal{N}}$ and a surjective submersion $(p, p^\sharp) : \mathcal{N} \to \underline{\mathcal{N}}$ such that $p^\sharp\colon C_{\underline{\mathcal{N}}} \to p_*(C_{\mathcal{N}})_{bas}$ is an isomorphism of sheaves of Poisson algebras.*

In this equivalence, $\underline{N}$ (the body of $\underline{\mathcal{N}}$) is the leaf space of F, and the pseudo-euclidean vector bundle corresponding to $\underline{\mathcal{N}}$ is the quotient of $E_{quot} = K^\perp/K$ with respect to F and ∇.

Note that the properties of $\underline{\mathcal{N}}$ in (b) define it uniquely, up to symplectomorphisms.

Definition 5.5. We will refer to $\underline{\mathcal{N}}$ as the *reduction* of $\mathcal{N}$.

Proof of Thm. 5.4. Assume that (a) holds. Let $\underline{N}$ be the leaf space of F, with quotient map $p\colon N \to \underline{N}$, and let $E_{red} \to \underline{N}$ be the quotient of $E_{quot} = K^\perp/K \to N$ with respect to F and ∇, as in Lemma 5.1. Since ∇ is metric, E_{red} inherits a pseudo-euclidean structure.

Let now $\underline{\mathcal{N}}$ be the symplectic degree 2 $\mathbb{N}$-manifold corresponding to the pseudo-euclidean vector bundle E_{red} (as in Theorem 3.4), determined by

$$(C_{\underline{\mathcal{N}}})_0 = C^\infty_{\underline{N}},\ (C_{\underline{\mathcal{N}}})_1 = \Gamma_{E_{red}},\ (C_{\underline{\mathcal{N}}})_2 = \Gamma_{\mathbb{A}_{E_{red}}}. \tag{5.5}$$

We will define a surjective submersion $(p, p^\sharp) : \mathcal{N} \to \underline{\mathcal{N}}$. For that, it suffices to determine $p^\sharp$ in degrees 1 and 2, see Lemma 2.2, part (a). We set

$$p^\sharp_1 \colon \Gamma_{E_{red}} \xrightarrow{\sim} p_*\Gamma^{flat}_{E_{quot}} = p_*((C_{\mathcal{N}})_{bas})_1 \subseteq p_*(C_{\mathcal{N}})_1, \tag{5.6}$$

$$p^\sharp_2 \colon \Gamma_{\mathbb{A}_{E_{red}}} \xrightarrow{\sim} p_*\Gamma^{flat}_{\mathbb{A}^{\overline{\nabla}}_{E_{quot}}} = p_*((C_{\mathcal{N}})_{bas})_2 \subseteq p_*(C_{\mathcal{N}})_2, \tag{5.7}$$

to be the identifications in (5.3) and (5.4). In order for $p^\sharp_1$ and $p^\sharp_2$ to determine a map $p^\sharp \colon C_{\underline{\mathcal{N}}} \to p_*C_{\mathcal{N}}$, we still need to check that they are compatible in the sense of Lemma 2.2, part (a), i.e.,

$$p^\sharp_1(e).p^\sharp_1(e') = p^\sharp_2(e.e'). \tag{5.8}$$

Consider the natural inclusion

$$\wedge^2\Gamma^{flat}_{E_{quot}} \to \Gamma^{flat}_{\mathbb{A}^{\overline{\nabla}}_{E_{quot}}}, \quad e_1 \wedge e_2 \mapsto (0, e_1 \wedge e_2),$$

that corresponds to the map $p_*((C_{\mathcal{N}})_{bas})_1.p_*((C_{\mathcal{N}})_{bas})_1 \hookrightarrow p_*((C_{\mathcal{N}})_{bas})_2$ (note that the image of $\wedge^2\Gamma^{flat}_{E_{quot}}$ coincides with the flat sections of $\wedge^2 E_{quot} \subseteq \mathbb{A}^{\overline{\nabla}}_{E_{quot}}$ with respect to the connection $\overline{\nabla}$). To verify (5.8) we recall that $\overline{(Y, \Delta)} = p^\sharp_2((\underline{Y}, \underline{\Delta}))$ satisfies $\Delta \circ p^\sharp_1 = p^\sharp_1 \circ \underline{\Delta}$. It follows that, for local sections e_1, e_2 and e of E_{red},

$$\begin{aligned} p^\sharp_2(e_1 \wedge e_2)(p^\sharp_1(e)) &= p^\sharp_1((e_1 \wedge e_2)(e)) = p^\sharp_1(\langle e_1, e\rangle e_2 - \langle e_2, e\rangle e_1) \\ &= p^*\langle e_1, e\rangle p^\sharp_1(e_1) - p^*\langle e_2, e\rangle p^\sharp_1(e_1) \\ &= (p^\sharp_1(e_1) \wedge p^\sharp_1(e_2))(p^\sharp_1(e)), \end{aligned}$$

where we used that, since ∇ is metric, we have

$$\left\langle p^\sharp_1(e), p^\sharp_1(e')\right\rangle = p^*\langle e, e'\rangle. \tag{5.9}$$

Therefore (5.8) holds, and (5.6) and (5.7) uniquely determine a morphism $(p, p^\sharp)$: $\mathcal{N} \to \underline{\mathcal{N}}$.

Since p^* defines an isomorphism

$$p^* \colon C^\infty_{\underline{N}} \xrightarrow{\sim} p_*(C^\infty_N)_{bas} = p_*((C_{\mathcal{N}})_{bas})_0, \tag{5.10}$$

and $p^\sharp_1$ and $p^\sharp_2$ are isomorphisms onto $p_*((C_{\mathcal{N}})_{bas})_1$ and $p_*((C_{\mathcal{N}})_{bas})_2$, respectively, we conclude that $p^\sharp \colon C_{\underline{\mathcal{N}}} \to p_*(C_{\mathcal{N}})_{bas}$ is an isomorphism, see Cor. 4.11. It is also a consequence of Cor. 4.11 that there are local coordinates in $\mathcal{N}$ and $\underline{\mathcal{N}}$ with respect to which $(p, p^\sharp)$ is in the normal form of Cor. 2.23, so it is a surjective submersion.

Comparing with (4.30), we see that the fact that $p^\sharp$ preserves Poisson brackets is equivalent to (5.9), the identities

$$p^*(\pounds_{\underline{Y}} f) = \pounds_Y(p^*f), \qquad \Delta(p^\sharp_1 e) = p^\sharp_1(\underline{\Delta}(e)),$$

when $\overline{(Y, \Delta)} = p^\sharp_2(\underline{Y}, \underline{\Delta})$, and the straightforward fact that $p^\sharp_2$ preserves commutators. This completes the proof that (a) implies (b).

For the converse, suppose that (b) holds, and let $E_{red} \to \underline{N}$ be the pseudo-euclidean vector bundle corresponding to $\underline{\mathcal{N}}$. Then (5.10) and (5.6) hold, and (5.10) implies that the fibres of p are the leaves of F, so F is simple. On the other hand, (5.6) induces a bundle map $E_{quot} \to E_{red}$ covering $p : N \to \underline{N}$ which defines an identification $E_{quot} \cong p^* E_{red}$. The fact that ∇ has trivial holonomy then follows from Lemma 5.1. □

Example 5.6. Consider a G-equivariant, pseudo-euclidean vector bundle $E \to M$. Let $K \to N$ be a G-invariant isotropic subbundle, and suppose that the G-action on N is free and proper. Consider the associated coisotropic geometric data (N, K, F, ∇), as in Example 4.9. In this case the foliation F is simple and ∇ has trivial holonomy, following Example 5.3; when G is connected, coisotropic reduction yields the pseudo-euclidean vector bundle $E_{red} = E_{quot}/G$, for $E_{quot} = K^\perp/K$. ⋄

6. Coisotropic reduction of Courant algebroids

6.1. **Courant algebroids.** Let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold, and let $(E, \langle\cdot,\cdot\rangle)$ be the corresponding pseudo-euclidean vector bundle, as in Thm. 3.4. Suppose that $\mathcal{M}$ is equipped with a global section of $C_{\mathcal{M}}$ of degree 3, $\Theta \in C(\mathcal{M})_3$, satisfying

$$\{\Theta, \Theta\} = 0. \tag{6.1}$$

We will refer to such Θ as a *Courant function*. The terminology is motivated by the fact that, in terms of the pseudo-euclidean vector bundle E, a Courant function Θ is equivalent to a *Courant algebroid* structure on E [LWX97], see, e.g., [Roy02a, Thm. 4.5]. In other words, Θ is equivalent to a bundle map $\rho\colon E \to TM$, referred to as the *anchor*, and a bilinear bracket $[\![\cdot,\cdot]\!]\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying

- (C1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
- (C2) $[\![e_1, f e_2]\!] = f[\![e_1, e_2]\!] + (\pounds_{\rho(e_1)} f) e_2$,
- (C3) $\pounds_{\rho(e_1)} \langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,
- (C4) $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]$,
- (C5) $\langle e_1, [\![e_2, e_2]\!]\rangle = \frac{1}{2}\pounds_{\rho(e_1)}\langle e_2, e_2\rangle$,

for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$. Explicitly, the relationship between Θ and ρ and $[\![\cdot,\cdot]\!]$ is given by the following derived-bracket formulas:

$$\pounds_{\rho(e)} f = \{\{\Theta, e\}, f\}, \quad [\![e_1, e_2]\!] = \{\{\Theta, e_1\}, e_2\}, \tag{6.2}$$

for $e, e_1, e_2 \in \Gamma(E) = C(\mathcal{M})_1$ and $f \in C^\infty(M)$.

Remark 6.1. Conditions (C2) and (C3) say that, for each $e \in \Gamma(E)$,

$$\mathrm{ad}_e\colon = [\![e, \cdot]\!]\colon \Gamma(E) \to \Gamma(E)$$

defines a metric-preserving derivation of E, i.e., $(\rho(e), [\![e,\cdot]\!]) \in \Gamma(\mathbb{A}_E)$ (cf. (3.5) and (3.6)). By (C1), ad_e is also a derivation for the bracket, so it defines an infinitesimal automorphism of the Courant algebroid.

Example 6.2 (Doubles). Any Lie algebroid gives rise to a Courant algebroid, referred to as its *double*, as follows.

Let $A \to M$ be a vector bundle, and consider the degree 2 symplectic $\mathbb{N}$-manifold $\mathcal{M} = T^*[2]A[1]$ corresponding to pseudo-euclidean vector bundle $A \oplus A^*$ (see Example 3.7). To see how a Lie algebroid structure on A defines a Courant function on

$\mathcal{M}$, we will use a natural injective map of $C(A[1])$-modules,

$$\mathfrak{X}(A[1]) \to C(\mathcal{M})[2], \tag{6.3}$$

given by (see Example 2.14):

$$(\mathfrak{X}(A[1]))_{-1} = \Gamma(A) \to \Gamma(A \oplus 0) \subset \Gamma(A \oplus A^*) = C(\mathcal{M})_1,$$
$$(\mathfrak{X}(A[1]))_0 = \mathrm{Der}(A) \to \Gamma(\mathbb{A}_{A \oplus A^*}) = C(\mathcal{M})_2,$$

where the second map takes a derivation (X, D) of A to $(X, D \oplus D^*)$. This map is a morphism of graded Lie algebras (with respect to the Poisson bracket on $C(\mathcal{M})$). In analogy with the classical case, we refer to functions in the image of (6.3) as "fiberwise linear" on $T^*[2]A[1]$.

Now suppose that $A \to M$ is a Lie algebroid, and denote by $Q = d_A$ its Lie algebroid differential, regarded as a degree 1 vector field on $A[1]$ (see Remark 2.16). By means of (6.3), Q defines a (fiberwise linear) degree 3 function on $\mathcal{M}$, which is a Courant function (since $[Q, Q] = 0$).

Denoting by ρ_A and $[\cdot,\cdot]_A$ the Lie-algebroid anchor and bracket, the corresponding Courant algebroid structure on the pseudo-euclidean vector bundle $E = A \oplus A^*$ has anchor $\rho(a, \xi) = \rho_A(a)$ and bracket

$$[\![(a_1, \xi_1), (a_2, \xi_2)]\!] = ([a_1, a_1], \pounds_{a_1}\xi_2 - i_{a_2} d_A \xi_1),$$

where $\pounds_a = d_A i_a + i_a d_A$. These Courant algebroids are special cases of "doubles" of Lie (quasi-)bialgebroids [LWX97, Roy02b]. For instance, given a d_A-closed element $\chi \in \Gamma(\wedge^3 A^*)$, one obtains a more general Courant bracket on $A \oplus A^*$ given by

$$[\![(a_1, \xi_1), (a_2, \xi_2)]\!] = ([a_1, a_1], \pounds_{a_1}\xi_2 - i_{a_2} d_A \xi_1 + i_{a_2} i_{a_1}\chi).$$

◇

Example 6.3 (Exact Courant algebroids). For a manifold M, by considering the tangent Lie algebroid $A = TM \to M$ in the previous example, one obtains the *standard Courant algebroid* structure on $E = TM \oplus T^*M$, with anchor ρ the natural projection onto TM and

$$[\![(X, \alpha), (Y, \beta)]\!] = ([X, Y], \pounds_X \beta - i_Y d\alpha).$$

The symplectic degree 2 $\mathbb{N}$-manifold corresponding to $TM \oplus T^*M$ is denoted by $T^*[2]T[1]M$, and the Courant function Θ in this case corresponds to the de Rham differential on M via (6.3). Any set of local coordinates $\{x^i\}$, $i = 1, \dots, n$, on M induces local frames on TM and T^*M, which give rise to local coordinates

$$\{x^1, \dots, x^n, v^1, \dots, v^n, \xi_1, \dots, \xi_n, p_1, \dots, p_n\}$$

on $T^*[2]T[1]M$, of respective degrees $0, 1, 1$ and 2, satisfying

$$\{p_j, x^i\} = \delta_{ij}, \quad \{v^i, \xi_j\} = \delta_{ij},$$

while the other brackets vanish. In these coordinates, $\Theta = v^i p_i$. More generally, for a closed $\chi \in \Omega^3(M)$, locally written as $\frac{1}{6}\chi_{ijk} dx^i dx^j dx^k$, we have that $\Theta_\chi = v^i p_i + \frac{1}{6}\chi_{ijk} v^i v^j v^k$ is a Courant function that defines the χ-twisted Courant bracket on $TM \oplus T^*M$ [ŠW01]; these types of Courant algebroids are known as *exact*.

We recall that χ-twisted Courant brackets admit interesting symmetries called gauge transformations [ŠW01] (also known as B-field transforms [Gua11]). Any 2-form $B \in \Omega^2(M)$ gives rise to a vector-bundle automorphism

$$TM \oplus T^*M \to TM \oplus T^*M, \qquad (X, \alpha) \mapsto (X, \alpha + i_X B) \tag{6.4}$$

that preserves the χ-twisted Courant structure if and only if $dB = 0$. From a graded-geometric viewpoint, as explained in [Roy02b, §4], B is viewed as a degree 2 function on $T^*[2]T[1]M$ (since $\Omega^2(M) \subseteq \Gamma(\wedge^2 E) \subseteq \Gamma(\mathbb{A}_E)$), so it defines a degree 0 hamiltonian vector field $\{B, \cdot\}$ whose time 1 flow corresponds to the pseudo-euclidean automorphism (6.4); the condition $\{B, \Theta_\chi\} = 0$ (saying that the hamiltonian flow of B preserves the Courant function Θ_χ) is equivalent to $dB = 0$. ◇

6.2. **Reducible Courant functions.** Let $\mathcal{N}$ be a coisotropic submanifold of $\mathcal{M}$, and suppose that $\underline{\mathcal{N}}$ is the reduction of $\mathcal{N}$ (in the sense of Def. 5.5). Any function S on $\mathcal{M}$ that is a section of the subsheaf $\mathfrak{N}_\mathcal{I} \subseteq C_\mathcal{M}$ defines a section of $(C_\mathcal{N})_{bas}$ (see (4.1) and (4.21)), and hence a function S_{red} of the reduction $\underline{\mathcal{N}}$ (via the identification in Theorem 5.4, part (b)).

Definition 6.4. We refer to functions on $\mathcal{M}$ that are sections of $\mathfrak{N}_\mathcal{I}$ as *reducible.*

Let Θ be a Courant function on $\mathcal{M}$, with corresponding Courant algebroid defined as in (6.2). If Θ is reducible, then Θ_{red} is a Courant function on $\underline{\mathcal{N}}$, which is in turn equivalent to a Courant algebroid over $\underline{N}$. Our next goal is to express the reducibility condition for a Courant function in geometric terms, so as to obtain a geometric reduction procedure for Courant algebroids.

Let $\mathcal{N}$ be a coisotropic submanifold of $\mathcal{M}$ with sheaf of vanishing ideals $\mathcal{I}$ and corresponding geometric data given by (N, K, F, ∇) (as in Thm. 4.5). Recall the notation $\Gamma^{flat}_{E,K^\perp}$ from (4.24).

Theorem 6.5. *A Courant function $\Theta \in C(\mathcal{M})_3$ is reducible (i.e., $\{\Theta, \mathcal{I}\} \subseteq \mathcal{I}$) if and only if the following conditions hold:*

- (R1) $\rho(K^\perp) \subseteq TN$,
- (R2) $\rho(K) \subseteq F$,
- (R3) $[\rho(\Gamma^{flat}_{E,K^\perp}), \Gamma_{TM,F}] \subseteq \Gamma_{TM,F}$,
- (R4) $\Gamma^{flat}_{E,K^\perp}$ *is involutive with respect to the Courant bracket.*

Notice that when F is simple, (R3) is saying that $\rho(\Gamma^{flat}_{E,K^\perp})|_N$ consists of vector fields that are projectable with respect to the quotient map $N \to \underline{N}$.

For the proof, we need some additional observations about the vanishing ideal $\mathcal{I}$ of $\mathcal{N}$. First, we have the following alternative description of $\mathcal{I}_2$:

Lemma 6.6. *For $U \subseteq M$ open,*

$$\mathcal{I}_2(U) = \{(X, D) \in \Gamma_{\mathbb{A}_E}(U) \,|\, X \in \Gamma_{TM,F}(U),\ D(\Gamma^{flat}_{E,K^\perp}(U)) \subseteq \Gamma_{E,K}(U)\}. \tag{6.5}$$

To compare with (4.16), note that the condition $D(\Gamma^{flat}_{E,K^\perp}(U)) \subseteq \Gamma_{E,K}(U)$ implies that $D(\Gamma_{E,K}(U)) \subseteq \Gamma_{E,K}(U)$, so (X, D) must lie in $\Gamma^{N,K}_{\mathbb{A}_E}(U)$. Note also that the same condition says that $[D_N](\Gamma^{flat}_{E_{quot}}) = 0$ (where $[D_N]$ is the image under (4.8) of the restriction of D to N). By Remark 4.7, the descriptions of $\mathcal{I}_2$ in (4.16) and (6.5) are equivalent.

Let I_F denote the subsheaf of C^∞_M consisting of functions whose restrictions to N are basic relative to F (i.e., in $(C^\infty_N)_{bas}$). It now follows from (6.5) that $\widetilde{e} = (X, D) \in \mathcal{I}_2(U)$ if and only if

$$\{\widetilde{e}, I_F(U)\} = \pounds_X(I_F(U)) \subseteq I_N(U), \tag{6.6}$$

$$\{\widetilde{e}, \Gamma^{flat}_{E,K^\perp}(U)\} = D(\Gamma^{flat}_{E,K^\perp}) \subseteq \Gamma_{E,K}(U). \tag{6.7}$$

We will also need the following characterization of $\mathcal{I}_3$.

Lemma 6.7. *Given an open subset $U \subseteq M$, $T \in \mathcal{I}_3(U)$ if and only if $\{T, \Gamma^{flat}_{E,K^\perp}(U)\} \subseteq \mathcal{I}_2(U)$.*

Proof. Recall that $\Gamma^{flat}_{E,K^\perp} = (\mathfrak{N}_\mathcal{I})_1$, see Lemma 4.12. So $T \in \mathcal{I}_3(U)$ implies that $\{T, \Gamma^{flat}_{E,K^\perp}(U)\} \subseteq \mathcal{I}_2(U)$. Conversely, we now check that $\{T, \Gamma^{flat}_{E,K^\perp}(U)\} \subseteq \mathcal{I}_2(U)$ implies that $T \in \mathcal{I}_3(U)$. It suffices to assume that we have local coordinates $\{x^i, e^\mu, p^I\}$ over U, so that $\mathcal{I}$ is generated by $x^1, \dots, x^{r_0}$, $e^1, \dots, e^{r_1}$, $p^1, \dots, p^{r_2}$ (where $r_0 = \mathrm{codim}(N)$, $r_1 = \mathrm{rank}(K)$, and $r_2 = \mathrm{rank}(F)$). Any $T \in (C_\mathcal{M}(U))_3$ is written in coordinates as

$$T = s_I p^I + A_{\mu\nu\eta}(x) e^\mu e^\nu e^\eta, \tag{6.8}$$

where $s^I = B^I_\mu(x) e^\mu$. Given $k^\perp \in \Gamma^{flat}_{E,K^\perp}(U)$, we have

$$\{T, k^\perp\} = p^I\{s_I, k^\perp\} - \{p^I, k^\perp\} s_I + \{A_{\mu\nu\eta}(x) e^\mu e^\nu e^\eta, k^\perp\} \in \mathcal{I}_2(U). \tag{6.9}$$

As a consequence, $\{s_I, k^\perp\} \in \mathcal{I}_0(U)$ for $I > r_2$, i.e., $s_I \in \Gamma_{E,K}(U) = \mathcal{I}_1(U)$ for $I > r_2$. Since $p^I \in \mathcal{I}_2(U)$ for $I \leq r_2$, it follows that $\{p^I, k^\perp\} \in \Gamma_{E,K}(U) = \mathcal{I}_1(U)$ if $I \leq r_2$. So the first two terms on the right-hand side of (6.9) lie in $\mathcal{I}_2(U)$, which forces the term $\{A_{\mu\nu\eta}(x) e^\mu e^\nu e^\eta, k^\perp\}$ to lie in $\mathcal{I}_2(U)$. But this happens if and only if $A_{\mu\nu\eta} \in \mathcal{I}_0(U)$ whenever $\mu, \nu, \eta > r_1$. From the expression in (6.8), we conclude that $T \in \mathcal{I}_3(U)$. □

Proof of Thm. 6.5. Recall that $\mathcal{I}_0 = I_N$, $\mathcal{I}_1 = \Gamma_{E,K}$, and $\mathcal{I}_2$ is given as in (4.16). The reducibility of Θ is equivalent to

(*a*) $\{\Theta, \mathcal{I}_0\} \subseteq \mathcal{I}_1$,
(*b*) $\{\Theta, \mathcal{I}_1\} \subseteq \mathcal{I}_2$,
(*c*) $\{\Theta, \mathcal{I}_2\} \subseteq \mathcal{I}_3$.

For each open $U \subseteq M$, note that $e \in \mathcal{I}_1(U) = \Gamma_{E,K}(U)$ if and only if $\langle e, \Gamma_{E,K^\perp}(U)\rangle = \{e, \Gamma_{E,K^\perp}(U)\} = 0$. So (*a*) is equivalent (on U) to

$$\{\{\Theta, \mathcal{I}_0(U)\}, e'\} = \pounds_{\rho(e')} I_N(U) = 0,$$

for all $e' \in \Gamma_{E,K^\perp}(U)$, i.e., $\rho(K^\perp) \subseteq TN$, which is condition (R1) in the statement of the theorem.

By (6.6) and (6.7), we see that (*b*) is equivalent (on U) to the following two conditions:

$$\{\{\Theta, \mathcal{I}_1(U)\}, f\} \subseteq I_N(U) \,\forall\, f \in C^\infty_M(U) \text{ such that } f|_N \in (C^\infty_N)_{bas}(U \cap N),$$
$$\{\{\Theta, \mathcal{I}_1(U)\}, \Gamma^{flat}_{E,K^\perp}(U)\} \subseteq \Gamma_{E,K}(U).$$

By (6.2), the former is equivalent to condition (R2), while the latter is equivalent to

$$[\![\Gamma_{E,K}, \Gamma^{flat}_{E,K^\perp}]\!] \subseteq \Gamma_{E,K}. \tag{6.10}$$

From Lemma 6.7, along with (6.6) and (6.7), we see that (c) is equivalent to

$$\{\{\{\Theta, \mathcal{I}_2(U)\}, \Gamma^{flat}_{E,K^\perp}(U)\}, I_F(U)\} \subseteq I_N(U), \tag{6.11}$$

$$\{\{\{\Theta, \mathcal{I}_2(U)\}, \Gamma^{flat}_{E,K^\perp}(U)\}, \Gamma^{flat}_{E,K^\perp}(U)\} \subseteq \Gamma_{E,K}(U). \tag{6.12}$$

We first consider (6.11). Let $\widetilde{e} = (X, D) \in \mathcal{I}_2(U)$, $e \in \Gamma^{flat}_{E,K^\perp}(U)$ and $f \in I_F(U)$. By the graded skew-symmetry and Jacobi identity for $\{\cdot,\cdot\}$, we have that

$$\{\{\Theta, \widetilde{e}\}, e\} = \{\Theta, \{\widetilde{e}, e\}\} - \{\widetilde{e}, \{\Theta, e\}\} = \{\{\widetilde{e}, e\}, \Theta\} + \{\{\Theta, e\}, \widetilde{e}\},$$

and

$$\begin{aligned}\{\{\{\Theta, \widetilde{e}\}, e\}, f\} &= \pounds_{\rho(\{\widetilde{e},e\})} f + \{\{\Theta, e\}, \{\widetilde{e}, f\}\} - \{\widetilde{e}, \{\{\Theta, e\}, f\}\} \\ &= \pounds_{\rho(\{\widetilde{e},e\})} f + \pounds_{\rho(e)}(\pounds_X f) - \pounds_X(\pounds_{\rho(e)} f).\end{aligned}$$

By (6.7), $\{\widetilde{e}, e\} \in \Gamma_{E,K}(U)$, so it follows from condition (R2) that $\pounds_{\rho(\{\widetilde{e},e\})} f \in I_N(U)$. So $\{\{\{\Theta, \widetilde{e}\}, e\}, f\} \in I_N(U)$ if and only if $\pounds_{[\rho(e),X]} f \in I_N(U)$, i.e., $[\rho(e), X] \in \Gamma_{TM,F}(U)$ for arbitrary $e \in \Gamma^{flat}_{K^\perp}(U)$ and $X \in \Gamma_{TM,F}(U)$. So (6.11) is equivalent to condition (R3).

In view of (6.12), let us now consider $\widetilde{e} = (X, D) \in \mathcal{I}_2(U)$ and $e_1, e_2 \in \Gamma^{flat}_{E,K^\perp}(U)$. Then

$$\begin{aligned}\{\{\Theta, \widetilde{e}\}, e_1\}, e_2\} &= \{\{\Theta, \{\widetilde{e}, e_1\}\}, e_2\} - \{\{\widetilde{e}, \{\Theta, e_1\}\}, e_2\} \\ &= [\![D(e_1), e_2]\!] - (\{\widetilde{e}, \{\{\Theta, e_1\}, e_2\}\} - \{\{\Theta, e_1\}, \{\widetilde{e}, e_2\}\}) \\ &= [\![D(e_1), e_2]\!] - \{\widetilde{e}, \{\{\Theta, e_1\}, e_2\}\} + [\![e_1, D(e_2)]\!].\end{aligned}$$

By (6.7) and (6.10), we see that both $[\![D(e_1), e_2]\!]$ and $[\![e_1, D(e_2)]\!]$ belong to $\Gamma_{E,K}(U)$. So the condition $\{\{\Theta, \widetilde{e}\}, e_1\}, e_2\} \in \Gamma_{E,K}(U)$ is equivalent to

$$\{\widetilde{e}, \{\{\Theta, e_1\}, e_2\}\} = D([\![e_1, e_2]\!]) \in \Gamma_{E,K}(U).$$

This holds for arbitrary $\widetilde{e} \in \mathcal{I}_2(U)$ if and only if $[\![e_1, e_2]\!] \in \Gamma^{flat}_{E,K^\perp}(U)$. Indeed, note first that $[\![e_1, e_2]\!] \in \Gamma_{E,K^\perp}(U)$, since, for any $e_3 \in \Gamma_{E,K}(U)$,

$$\langle [\![e_1, e_2]\!], e_3\rangle = \pounds_{\rho(e_1)}\langle e_2, e_3\rangle - \langle e_1, [\![e_2, e_3]\!]\rangle \in I_N(U),$$

as a consequence of $\langle e_2, e_3\rangle \in I_N(U)$, and conditions (R1) and (6.10). Then $D([\![e_1, e_2]\!]) \in \Gamma_{E,K}(U)$ for all $(X, D) \in \mathcal{I}_2(U)$ if and only if the projection of $[\![e_1, e_2]\!]|_N$ to $\Gamma_{E_{quot}}$ is ∇-flat (see (4.16)), which is equivalent to $[\![e_1, e_2]\!] \in \Gamma^{flat}_{E,K^\perp}(U)$. So (6.12) is equivalent to condition (R4).

Summing up, we have shown that Θ is reducible if and only if conditions (R1)–(R4) (in the statement of the theorem) as well as (6.10) hold. We finally observe that (6.10) is redundant, since it is implied by condition (R4). Indeed, let $e_1 \in \Gamma_{E,K}(U)$ and $e_2 \in \Gamma^{flat}_{E,K^\perp}(U)$, and notice that $[\![e_1, e_2]\!] = -[\![e_2, e_1]\!]$ by the orthogonality of K and $K^\perp$. For any $e_3 \in \Gamma^{flat}_{E,K^\perp}(U)$ we have

$$\langle [\![e_1, e_2]\!], e_3\rangle = -\langle [\![e_2, e_1]\!], e_3\rangle = 0$$

by property (C3) in the definition of Courant algebroid and condition (R4). This implies that

$$\langle [\![\Gamma_{E,K}, \Gamma^{flat}_{E,K^\perp}]\!], \Gamma_{E,K^\perp}\rangle = 0,$$

since around every point of N there exists a frame of $K^\perp$ consisting of sections that project to ∇-flat sections of $K^\perp/K$ (by the flatness of the F-connection ∇). $\square$

Remark 6.8. Regarding the reducibility conditions (R1)–(R4), in case the following stronger version of (R2) holds,

$$\rho(K) = F,$$

then (R3) becomes redundant: it is a consequence of (6.10) and the fact that $\rho\colon E \to TM$ preserves brackets.

Let us now consider a lagrangian submanifold $\mathcal{N}$ of $\mathcal{M}$, defined by $K \to N$, where $K \subset E|_N$ is a lagrangian subbundle, see Cor. 4.15.

The following is a special case of Thm. 6.5.

Corollary 6.9. *Let $\Theta \in C(\mathcal{M})_3$ be a Courant function. The following are equivalent:*

(a) *Θ is reducible*

(b) *$\rho(K) \subseteq TN$ and, for any sections e_1, $e_2 \in \Gamma(E)$ with $e_1|_N, e_2|_N \in \Gamma(K)$, we have $[\![e_1, e_2]\!]|_N \in \Gamma(K)$.*

Proof. Since $F = TN$, conditions (R1) and (R2) of Thm. 6.5 reduce to $\rho(K) \subseteq TN$. Since $K^\perp = K$, we see that $\Gamma^{flat}_{E,K^\perp} = \Gamma_{E,K}$, so condition (R4) in Thm. 6.5 is equivalent to the second condition in (b). Condition (R3) in Thm. 6.5 is automatically satisfied since $F = TN$ and $\rho(K) \subseteq TN$. □

We are led to the following notion, see, e.g., [AX, BIPŠ09, Šev05].

Definition 6.10. A lagrangian subbundle $K \to N$ of a Courant algebroid $E \to M$ satisfying condition (b) of Corollary 6.9 is called a *Dirac structure supported on N*.

When $N = M$, we have that K is a *Dirac structure* in E in the sense of [LWX97], extending the original definition of [Cou90] when $E = \mathbb{T}M$.

According to the previous corollary, Dirac structures with support are equivalent to lagrangian submanifolds of $\mathcal{M}$ tangent to the hamiltonian vector field of the Courant function Θ.

6.3. **Reduction of Courant algebroids.** Using the equivalence between Courant functions and Courant algebroids, we can derive from Thm. 6.5 a reduction procedure for Courant algebroids. We will express this procedure in classical geometrical terms.

Let $E \to M$ be a Courant algebroid, with pseudo-euclidean structure $\langle\cdot,\cdot\rangle$, anchor ρ and bracket $[\![\cdot,\cdot]\!]$. Let (N, K, F, ∇) be geometric coisotropic data, i.e.,

- N is a submanifold of M,
- $K \subseteq E|_N$ is an isotropic subbundle,
- $F \subseteq TN$ is an integrable subbundle, and
- ∇ is a metric, flat partial F-connection on the vector bundle $E_{quot} = K^\perp/K$ over N.

When F is simple and ∇ has trivial holonomy, we have a surjective submersion $p\colon N \to \underline{N}$ onto the leaf space of F and a quotient of E_{quot} with respect to F and ∇ (see Lemma 5.1), which is a pseudo-euclidean vector bundle $E_{red} \to \underline{N}$. Then $E_{quot} = p^* E_{red}$, and we denote by $p^\sharp\colon \Gamma_{E_{red}} \to p_* \Gamma_{E_{quot}}$ the pullback map of sections.

Theorem 6.11 (Coisotropic reduction of Courant algebroids)**.** *Let $E \to M$ be a Courant algebroid and (N, K, F, ∇) as above. Suppose that F is simple and ∇*

has trivial holonomy. If conditions (R1)–(R4) in Thm. 6.5 hold, then the pseudo-euclidean vector bundle E_{red} inherits a Courant algebroid structure, with anchor ρ_{red} and bracket $[\![\ ,\ ]\!]_{red}$, given as follows: for $\underline{e}$, $\underline{e}_1$, $\underline{e}_2$ sections of E_{red} and f of $C^\infty_{\underline{N}}$,

$$\rho_{red}(\underline{e}) = p_*(\rho(e)), \qquad p^\sharp[\![\underline{e}_1, \underline{e}_2,]\!]_{red} = [[\![e_1, e_2]\!]|_N], \tag{6.13}$$

where e is a section of $K^\perp$ such that $[e] = p^\sharp \underline{e}$, and e_i is a section of $\Gamma^{flat}_{E,K^\perp}$ such that $[e_i|_N] = p^\sharp \underline{e}_i$, $i = 1, 2$.

This theorem can be proven by directly showing that conditions (R1)–(R4) in Thm. 6.5 guarantee that the bracket and anchor can be reduced. But using graded geometry, we will see that the proof is immediate.

Proof. We know that the Courant algebroid $E \to M$ is equivalent to a Courant function Θ on a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, and that the quadruple (N, K, F, ∇) defines a coisotropic submanifold. Further Θ is reducible by Thm. 6.5, and Θ_{red} is a Courant function on the reduction $\underline{\mathcal{N}}$, which is in turn equivalent to a Courant algebroid structure on $E_{red} \to \underline{N}$. The expressions for the reduced anchor and bracket in (6.13) follow from the derived-bracket formulas (6.2) and the definition of $\{\cdot, \cdot\}_{bas}$, see (4.22) and (4.30). □

The previous theorem motivates the following notion.

Definition 6.12. We say that a Courant algebroid $(E, \langle\cdot, \cdot\rangle, \rho, [\![\cdot, \cdot]\!])$ is *reducible* with respect to geometric coisotropic data (N, K, F, ∇) (see Definition 4.6) if conditions (R1)–(R4) in Thm. 6.5 hold.

We illustrate Thm. 6.11 with some special cases. We keep the setup and notation of the theorem.

Example 6.13.

(i) Let $N \subseteq M$ be a submanifold, and set K, F and ∇ to be trivial (i.e., the corresponding coisotropic submanifold in Thm. 4.5 has only constraints in degree 0). Then the reducibility conditions (R1)-(R4) reduce to the single condition that $\rho(E) \subseteq TN$. In this case, the theorem boils down to the simple fact that the restriction $E|_N$ inherits a Courant algebroid structure.

(ii) Consider a submanifold $N \subset M$ and an isotropic subbundle $K \subset E|_N$, setting F and ∇ to be trivial. The reducibility conditions in the theorem become $\rho(K^\perp) \subseteq TN$, $\rho(K) = 0$, and that $\Gamma_{E,K^\perp}$ is closed under the Courant bracket. Thm. 6.11 then says that there is an induced Courant algebroid structure on $K^\perp/K \to N$. This recovers the result in [LBM09, Prop. 2.1], which has as a special case the pullback of Courant algebroids to submanifolds transverse to the anchor [LBM09, Prop 2.4].

(iii) If we set K to be zero in Thm. 6.11, we obtain the following statement: let $N \subseteq M$ be a submanifold with a regular involutive distribution F and a flat, metric F-connection on $E|_N$. Assume that $\rho(E) \subseteq TN$, and that the restricted Courant algebroid $E|_N$ is such that $[\rho(\Gamma^{flat}_{E|_N}), \Gamma_F] \subseteq \Gamma_F$ and $\Gamma^{flat}_{E|_N}$ is involutive. Then, whenever F is simple and ∇ has trivial holonomy, there is an induced Courant algebroid structure on $E_{red} \to \underline{N}$, the quotient of $E|_N$ with respect to F and ∇.

(iv) As a simple special case of (iii), suppose that a (connected) Lie group G acts on the Courant algebroid $E \to M$ by automorphisms. Let N be a G-invariant submanifold of M where the action is free and proper, and such that $\rho(E) \subseteq TN$; the G-action gives rise to a flat, metric partial connection on $E|_N$, see Example 4.9, satisfying the conditions in (iii). It follows that $E_{red} = (E|_N)/G \to N/G$ has an induced Courant algebroid structure.

◇

6.3.1. *The exact case.* Given a Courant algebroid $(E, \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!])$ over M, there is an associated chain complex

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0,$$

where we have used the identification $E \cong E^*$ for the map ρ^*. A Courant algebroid is called *exact* if this sequence is exact. A choice of lagrangian splitting of this sequence identifies E with a twisted Courant algebroid structure on $TM \oplus T^*M$, with respect to a suitable closed 3-form [ŠW01]; this leads to a classification of exact Courant algebroids by elements in $H^3(M)$, the so-called *Ševera classes* [Šev17]

One can refine the coisotropic reduction of Courant algebroid to see when the reduction of an exact Courant algebroid is again exact. Consider the same setup and notation as in Thm. 6.11, but assuming that E is exact.

Proposition 6.14. *The reduced Courant algebroid $E_{red} \to \underline{N}$ is exact if and only if:*

(a) $\dim(N) - \operatorname{rank}(\rho(K^\perp)) = \operatorname{rank}(F) - \operatorname{rank}(\rho(K))$,
(b) $\rho(K^\perp) \cap F = \rho(K)$.

In particular, this holds when $\rho(K^\perp) = TN$ and $\rho(K) = F$.

Proof. We will use the following fact: a Courant algebroid is exact if and only if its rank is twice the dimension of its base manifold and the kernel of its anchor is isotropic.

Let $m_0 = \dim(M) = \frac{1}{2}\operatorname{rank}(E)$ and $r_1 = \operatorname{rank}(K)$. The rank of E_{red} agrees with the rank of $K^\perp/K$, which is $2(m_0 - r_1)$. By properties (R1) and (R2) of Thm. 6.5, we know that $\operatorname{rank}(TN) \geq \operatorname{rank}(\rho(K^\perp))$ and $\operatorname{rank}(F) \geq \operatorname{rank}(\rho(K))$. Since E is exact, we have an exact sequence

$$0 \to \frac{T^*M \cap K^\perp + K}{K} \to \frac{K^\perp}{K} \to \frac{\rho(K^\perp)}{\rho(K)} \to 0.$$

Noticing that $(T^*M \cap K^\perp + K)/K$ is lagrangian in $K^\perp/K$, we conclude that $\operatorname{rank}(K^\perp/K) = 2\operatorname{rank}(\rho(K^\perp)/\rho(K)) = 2(\operatorname{rank}(\rho(K^\perp)) - \operatorname{rank}(\rho(K)))$. It follows that $\operatorname{rank}(K^\perp/K)$ equals $2\dim(\underline{N}) = 2(\operatorname{rank}(TN) - \operatorname{rank}(F))$ if and only if

$$\operatorname{rank}(\rho(K^\perp)) - \operatorname{rank}(\rho(K)) = \operatorname{rank}(TN) - \operatorname{rank}(F),$$

which is condition (a).

If (a) holds, then E_{red} is exact if and only if the kernel of the map $K^\perp/K \to TN/F$, given by $C_1 = (K^\perp \cap \rho^{-1}(F))/K$, is isotropic. We will check that this is equivalent to condition (b).

Note that C_1 contains the lagrangian subbundle $C_2 = (T^*M \cap K^\perp)/K$, so C_1 is isotropic if and only if $C_1 = C_2$. Since C_2 is the kernel of the map $K^\perp/K \to \rho(K^\perp)/\rho(K)$, the condition $C_1 = C_2$ is equivalent to the map $\rho(K^\perp)/\rho(K) \to TN/F$

induced by the inclusion being injective (in fact, an isomorphism by (a)), which is the same as condition (b). □

We now see how the reduction of exact Courant algebroids in [Zam08] is a special case of Corollary 6.11.

Example 6.15 (Reduction by isotropic, involutive subbundles). Consider an exact Courant algebroid E over M. Let $K \to N$ be an isotropic subbundle of E, and suppose that

(a) $\rho(K^{\perp}) = TN$
(b) K is involutive, in the sense that

$$[\![\Gamma_{E,K}, \Gamma_{E,K}]\!] \subseteq \Gamma_{E,K}.$$

We can use $K \to N$ to produce geometric coisotropic data (N, K, F, ∇) as follows. By the exactness of E, we have that

$$K \cap T^*M = \mathrm{Ann}(\rho(K^{\perp})) = \mathrm{Ann}(TN), \tag{6.14}$$

which implies that $\rho(K) \subseteq TN$ has constant rank (since $\ker(\rho|_K) = K \cap T^*M$). It follows from the involutivity condition on K and the bracket-preserving property of ρ that $\rho(K)$ is involutive. We set $F := \rho(K)$. One can now verify that the expression

$$\nabla_X[e|_N] = [[\![k, e]\!]]$$

defines a metric, flat F-connection ∇ on $E_{quot} = K^{\perp}/K$, where $X \in \Gamma(F)$, k is any section of $\Gamma_{E,K}$ satisfying $\rho(k)|_N = X$, and e is a section of $\Gamma_{E,K^{\perp}}$. (The fact that ∇ is well defined follows as in [Zam08, Lemma 3.1], flatness is a consequence of the Jacobi identity (C1) of Courant brackets, while the metric condition follows from (C3).)

One can also check that E is automatically reducible with respect to (N, K, F, ∇) (as in Def. 6.12). Indeed, it remains to verify the reducibility condition (R4) (since (R3) holds as observed in Remark 6.8). Noticing that, for each open $U \subseteq M$,

$$\Gamma^{flat}_{E,K^{\perp}}(U) = \{e \in \Gamma_{E,K^{\perp}}(U) \mid [\![k, e]\!] \in \Gamma_{E,K}(U) \text{ for all } k \in \Gamma_{E,K}(U)\},$$

condition (R4) directly follows from properties (C1), (C3) and (C5) of Courant brackets.

So, assuming that F is simple and ∇ has trivial holonomy, we obtain a reduced exact Courant algebroid E_{red} over $\underline{N}$, the leaf space of N by F, as a consequence of Theorem 6.11 and Proposition 6.14. This construction recovers [Zam08, Thm. 3.7] from a slightly different perspective (see [Gua11, Prop. 7.1] for a special case). ⋄

The following is a particular case of the previous example.

Example 6.16 (Reduction by isotropic, involutive subbundles with an action).
Let $E \to M$ be an exact Courant algebroid, and $K \to N$ be an isotropic, involutive subbundle with $\rho(K^{\perp}) = TN$; let (N, K, F, ∇) be the corresponding geometric coisotropic data, as in the previous example. Assume further that

- a (connected) Lie group G acts on E by Courant-algebroid automorphisms,
- the G-action on E is infinitesimally generated by a bracket-preserving map $\psi\colon \mathfrak{g} \to \Gamma(E)$, in the sense that the $\mathfrak{g}$-action on E by infinitesimal Courant automorphisms is given by

$$u \mapsto (\mathrm{ad}_{\psi(u)} = [\![\psi(u), \cdot]\!]\colon \Gamma(E) \to \Gamma(E)),$$

for $u \in \mathfrak{g}$ (see Remark 6.1),
- for each $x \in N$, the map $\psi_x \colon \mathfrak{g} \to E|_x$, $u \mapsto \psi(u)(x)$ takes values in $K|_x$ and $\rho(\psi_x(\mathfrak{g})) = \rho(K|_x) = F|_x$.

This last condition and the involutivity of K imply that $K \to N$ is a G-invariant subbundle of $E \to M$. Assuming that the G-action on N is free and proper, it is a simple verification that the geometric coisotropic data defined by the G-action on K (as in Example 4.9) coincides with (N, K, F, ∇). Therefore, with this setup, E is reducible with respect to the coisotropic data, F is simple, ∇ has trivial holonomy, and $E_{red} = (K^\perp/K)/G$ is an exact Courant algebroid over N/G.

A special case of this construction is described in [BCG08, § 3.1] (based on [BCG07]), when $K \to N$ and the G-action on E are defined by means of an "extended action" (see § 8.5). $\diamond$

7. Coisotropic reduction of GC and Dirac structures

7.1. **Reduction of GC structures.** Let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold and $(E, \langle\cdot,\cdot\rangle)$ be the corresponding pseudo-euclidean vector bundle. By a *quadratic function* on $\mathcal{M}$ we mean a section of $(C_\mathcal{M})_1 \cdot (C_\mathcal{M})_1 \subseteq (C_\mathcal{M})_2$. With the identification $(C_\mathcal{M})_1 = \Gamma_{E^*}$, quadratic functions are seen as sections of $\Gamma_{\wedge^2 E^*}$, which are equivalent to skew-symmetric endomorphisms $E \to E$ (i.e., sections of $\mathbb{A}_E$ with vanishing symbol). For simplicity, we will keep the same notation for the quadratic function and the corresponding skew-symmetric endomorphism.

Let $\mathcal{N}$ be a coisotropic submanifold of $\mathcal{M}$ defined by (N, K, F, ∇).

Lemma 7.1. *A quadratic function $\mathbb{J}$ on $\mathcal{M}$ is reducible (i.e., it is a section of $\mathfrak{N}_\mathcal{I}$) if and only if $\mathbb{J}(\Gamma^{flat}_{E,K^\perp}) \subseteq \Gamma^{flat}_{E,K^\perp}$.*

This result is an immediate consequence of the description of $(\mathfrak{N}_\mathcal{I})_2$ in Lemma 4.12 once one notices that the the condition in the lemma implies that $\mathbb{J}(K^\perp) \subseteq K^\perp$, and hence $\mathbb{J}(K) \subseteq K$ (so that $\mathbb{J}$ is automatically a section of $\Gamma^{N,K}_{\mathbb{A}_E}$).

Let us recall that a *generalized complex (GC) structure* [Gua11] on a Courant algebroid $E \to M$ is an endomorphism $\mathbb{J} \colon E \to E$ preserving the pairing $\langle\cdot,\cdot\rangle$, such that $\mathbb{J}^2 = -\mathrm{Id}$ and whose Nijenhuis torsion vanishes,

$$[\![\mathbb{J}(e_1), \mathbb{J}(e_2)]\!] - \mathbb{J}([\![\mathbb{J}(e_1), e_2]\!]) - \mathbb{J}([\![e_1, \mathbb{J}(e_2)]\!]) + [\![e_1, e_2]\!] = 0.$$

These objects admit a characterization in terms of the corresponding symplectic degree 2 $\mathbb{N}$-manifold $(\mathcal{M}, \{\cdot,\cdot\})$ with Courant function Θ: it is proven in [Gra06] that generalized complex structures $\mathbb{J} \colon E \to E$ are equivalent to quadratic functions $\mathbb{J}$ on $\mathcal{M}$ satisfying

$$\{\{\Theta, \mathbb{J}\}, \mathbb{J}\} = \Theta. \tag{7.1}$$

Theorem 7.2. *Suppose that the Courant algebroid E is reducible with respect to the geometric coisotropic data (N, K, F, ∇), and let E_{red} be the reduced Courant algebroid.*

If a generalized complex structure $\mathbb{J} \colon E \to E$ satisfies $\mathbb{J}(\Gamma^{flat}_{E,K^\perp}) \subseteq \Gamma^{flat}_{E,K^\perp}$, then it gives rise to a reduced generalized complex structure $\mathbb{J}_{red}$ on E_{red} by

$$p^\sharp \mathbb{J}_{red}(\underline{e}) = [\mathbb{J}(e)],$$

where $\underline{e}$ is a section of E_{red} and e is a section of $K^\perp$ such that $[e] = p^\sharp \underline{e}$.

Proof. Consider the coisotropic submanifold $\mathcal{N}$ corresponding to (N, K, F, ∇), and let $\underline{\mathcal{N}}$ be its reduction. By Lemma 7.1, the condition in the statement says that the quadratic function $\mathbb{J}$ is reducible, so it defines a function $\mathbb{J}_{red}$ on $\underline{\mathcal{N}}$, which is easily seen to be quadratic; moreover, the skew-symmetric endomorphisms corresponding to $\mathbb{J}$ and $\mathbb{J}_{red}$ are related as in the statement of this theorem. To see that J_{red} is generalized complex, notice that (7.1) implies that the same condition holds for Θ_{red} and $\mathbb{J}_{red}$ since both the passage from the Lie normalizer $\mathfrak{N}_{\mathcal{I}}$ to basic functions on $\mathcal{N}$ and the identification of the latter with functions on $\underline{\mathcal{N}}$ preserve Poisson brackets. □

Remark 7.3. One can prove versions of the previous theorem under weaker conditions, e.g., assuming that $\mathbb{J}(K) \cap K^{\perp}$ has constant rank and is contained in K and that $\mathbb{J}$ preserves the flat sections of this bundle, see, e.g., [Zam08, Prop. 6.1]. Graded-geometric interpretations of these results require considering graded submanifolds beyond the coisotropic ones.

Example 7.4. Consider the setup of Example 6.16, with reduced Courant algebroid $E_{red} = (K^{\perp}/K)/G$. In this case, any G-invariant GC structure $\mathbb{J}$ satisfying $\mathbb{J}(K^{\perp}) \subseteq K^{\perp}$ (or, equivalently, $\mathbb{J}(K) \subseteq K$) over N satisfies the condition in Theorem 7.2, and hence can be reduced to a GC structure $\mathbb{J}_{red}$ on E_{red}. This extends [BCG08, Thm. 4.1] (based on [BCG07, Thm. 5.2]). ⋄

7.2. **Reduction of lagrangian submanifolds and Dirac structures.** A well-known construction in classical symplectic geometry is the reduction of lagrangian submanifolds through coisotropic reduction, see, e.g., [Wei79, Lect. 3]. Consider a symplectic manifold M along with a coisotropic submanifold N and a lagrangian submanifold L. Assume that N and L intersect cleanly, i.e., $N \cap L$ is a submanifold and $T(N \cap L) = TN \cap TL$. The latter condition can be equivalently expressed in terms of vanishing ideals as $I_{N \cap L} = I_N + I_L$ (see, e.g., [Li09, Lemma 5.1]). Suppose also that the null foliation of N is simple, so that its leaf space $\underline{N}$ is a smooth symplectic manifold. Then the projection of $N \cap L$ to $\underline{N}$ is, at least locally, a lagrangian submanifold in $\underline{N}$ (sometimes called a lagrangian "sub-immersion").

We will now discuss an analog of this construction for degree 2 $\mathbb{N}$-manifolds. As a consequence, using the description of Dirac structures in terms of lagrangians submanifolds (explained in the end of § 6.2), we will obtain a reduction procedure for Dirac structures.

Let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold, $\mathcal{N}$ a coisotropic submanifold, and $\mathcal{L}$ a lagrangian submanifold, with bodies N and S, respectively. The corresponding vanishing ideals are denoted by $\mathcal{I}_{\mathcal{N}}$ and $\mathcal{I}_{\mathcal{L}}$. We assume, for simplicity, that

$$N \subseteq S$$

(in particular, one could take $S = M$, which would be enough to treat reduction of ordinary Dirac structures). Let E be the pseudo-euclidean vector bundle corresponding to $\mathcal{M}$, so that $\mathcal{L}$ is described by a lagrangian subbundle $L \to S$ (as in Corollary 4.15), and $\mathcal{N}$ corresponds to geometric coisotropic data (N, K, F, ∇) (as in Theorem 4.5).

7.2.1. Clean intersection. We start by considering the intersection of $\mathcal{N}$ and $\mathcal{L}$ in $\mathcal{M}$. Following the classical case, we say that $\mathcal{N}$ and $\mathcal{L}$ *intersect cleanly* if the sheaf of ideals $\mathcal{I}_{\mathcal{N}} + \mathcal{I}_{\mathcal{L}}$ is regular (in the sense of §2.3).

We have the following geometric characterization of the clean-intersection condition.

Proposition 7.5. *The sheaf of ideals $\mathcal{I}_\mathcal{N}+\mathcal{I}_\mathcal{L}$ is regular if and only if*

i) *$K\cap L|_N$ has constant rank, and*

ii) *$\nabla_Y(\Gamma_{L_{quot}})\subseteq\Gamma_{L_{quot}}$, for any section Y of Γ_F,*

where $L_{quot}=\frac{(K^\perp\cap L|_N)+K}{K}$.

Note that condition i) ensures that

$$L_{quot}\subseteq E_{quot}=K^\perp/K$$

is a smooth lagrangian subbundle.

Proof. The sheaf of ideals $\mathcal{I}_\mathcal{N}+\mathcal{I}_\mathcal{L}$ is locally generated in degrees 0, 1 and 2, and it satisfies

$$(\mathcal{I}_\mathcal{N}+\mathcal{I}_\mathcal{L})_0=I_N,\quad (\mathcal{I}_\mathcal{N}+\mathcal{I}_\mathcal{L})_1=\Gamma_{E,K+L|_N},\quad (\mathcal{I}_\mathcal{N}+\mathcal{I}_\mathcal{L})_2=\Gamma_{\mathbb{A}_E,\widetilde{K}+\widetilde{L}|_N},$$

recalling that $(\mathcal{I}_\mathcal{L})_1=\Gamma_{E,L}$, $(\mathcal{I}_\mathcal{N})_1=\Gamma_{E,K}$, and the vector bundles $\widetilde{K}\to N$ and $\widetilde{L}\to S$ are such that $(\mathcal{I}_\mathcal{N})_2=\Gamma_{\mathbb{A}_E,\widetilde{K}}$ and $(\mathcal{I}_\mathcal{L})_2=\Gamma_{\mathbb{A}_E,\widetilde{L}}$. It follows that $\mathcal{I}_\mathcal{N}+\mathcal{I}_\mathcal{L}$ is regular if and only if the following conditions are satisfied: $K+L|_N$ and $\widetilde{K}+\widetilde{L}|_N$ have constant rank, and

$$(\widetilde{K}+\widetilde{L}|_N)\cap\wedge^2E|_N=(K+L|_N)\wedge E|_N, \tag{7.2}$$

see Lemma 2.11 (upon using the isomorphism $E\cong E^*$). Notice that the condition that $\widetilde{K}+\widetilde{L}|_N$ has constant rank is implied by the other two. Indeed the sequence

$$0\to(\widetilde{K}+\widetilde{L}|_N)\cap\wedge^2E|_N\to\widetilde{K}+\widetilde{L}|_N\to TS|_N\to 0,$$

obtained by restricting (3.7), is exact, because $\widetilde{L}$ maps surjectively onto TS under the symbol map (to verify this last claim, recall that $\Gamma_{\widetilde{L}}=\Gamma^L_{\mathbb{A}_{E|_N}}$ and the surjectivity of (4.8), shown in Prop. A.2). If $K+L|_N$ has constant rank and (7.2) holds, then the first (nontrivial) term in the previous sequence has constant rank, and the exactness implies that the middle term has constant rank as well. Hence, to prove the proposition, we may assume that item i) holds and show that, in this case, item ii) is equivalent to (7.2). This will be verified in the following two claims.

Claim: *Condition* (7.2) *holds if and only if the symbol map $\sigma\colon\widetilde{K}\cap\widetilde{L}|_N\to F$ is onto.*

To prove the claim, recall (from Prop. 2.10 and Lemma 2.11) that

$$\widetilde{K}\cap\wedge^2E|_N=K\wedge E|_N\quad\text{and}\quad\widetilde{L}|_N\cap\wedge^2E|_N=L|_N\wedge E|_N, \tag{7.3}$$

so we have that

$$(K+L|_N)\wedge E|_N\subseteq\widetilde{K}\cap\wedge^2E+\widetilde{L}|_N\cap\wedge^2E\subseteq(\widetilde{K}+\widetilde{L}|_N)\cap\wedge^2E.$$

It follows that (7.2) is equivalent to the opposite inclusion, namely

$$(\widetilde{K}+\widetilde{L}|_N)\cap\wedge^2E\subseteq(K+L|_N)\wedge E|_N.$$

Suppose that this last condition holds, and let $X\in F$. One can take $\widetilde{k}\in\widetilde{K}$ and $\widetilde{l}\in\widetilde{L}|_N$ with $\sigma(\widetilde{k})=X$ and $\sigma(\widetilde{l})=-X$, so that $\widetilde{k}+\widetilde{l}\in\wedge^2E$. Then $\widetilde{k}+\widetilde{l}=$

$k_i \wedge e^i + l_i \wedge e^i$, for $k_i \in K$, $l_i \in L$, and $e^i \in E$. Hence $\widetilde{k} - k_i \wedge e^i = -\widetilde{l} + l_i \wedge e^i$ lies in $\widetilde{K} \cap \widetilde{L}|_N$ and has symbol X, showing that the symbol map $\widetilde{K} \cap \widetilde{L}|_N \to F$ is onto.

Conversely, suppose now that the symbol map $\widetilde{K} \cap \widetilde{L}|_N \to F$ is onto, and let $\widetilde{k} + \widetilde{l} \in \wedge^2 E|_N$, with $\widetilde{k} \in \widetilde{K}$ and $\widetilde{l} \in \widetilde{L}|_N$. Then $\sigma(\widetilde{k}) = -\sigma(\widetilde{l}) =: X$. Let $u \in \widetilde{K} \cap \widetilde{L}|_N$ be such that $\sigma(u) = X$. Then $\widetilde{k} + \widetilde{l}$ is the sum of $\widetilde{k} - u \in \widetilde{K} \cap \wedge^2 E|_N$ and $\widetilde{l} + u \in \widetilde{L}|_N \cap \wedge^2 E|_N$, and (7.3) shows implies that $\widetilde{k} + \widetilde{l} \in (K + L|_N) \wedge E|_N$, proving the claim.

Claim: *The symbol map $\widetilde{K} \cap \widetilde{L}|_N \to F$ is onto if and only if*

$$\nabla_Y(\Gamma_{L_{quot}}) \subseteq \Gamma_{L_{quot}}, \;\; \textit{for any section } Y \textit{ of } \Gamma_F. \tag{7.4}$$

From (4.16) it follows that $\Gamma_{\widetilde{L}} = \Gamma^L_{\mathbb{A}_{E|_S}}$ and, for each open subset $V \subseteq N$,

$$\Gamma_{\widetilde{K}}(V) = \{(Y, D) \in \Gamma^K_{\mathbb{A}_{E|_N}}(V) \,|\, Y \in \Gamma_F(V),\, [D] = \nabla_Y\}.$$

Therefore

$$\Gamma_{\widetilde{K} \cap \widetilde{L}|_N}(V) = \{(Y, D) \in \Gamma^{K, L|_N}_{\mathbb{A}_{E|_N}}(V) \,|\, Y \in \Gamma_F(V),\, [D] = \nabla_Y\}.$$

By Prop. A.3, the map

$$\Gamma^{K, L|_N}_{\mathbb{A}_{E|_N}} \to \Gamma^{L_{quot}}_{\mathbb{A}_{E_{quot}}}, \;\; (Y, D) \mapsto (Y, [D]),$$

is onto, and it is clear that it restricts to a surjective map

$$\Gamma_{\widetilde{K} \cap \widetilde{L}|_N} \twoheadrightarrow \Gamma^{L_{quot}}_{\mathbb{A}_{E_{quot}}} \cap \nabla(\Gamma_F),$$

where the sheaf on the right-hand side has sections (over an open $V \subseteq N$) of the form $(Y, \nabla_Y) \in \Gamma_{\mathbb{A}_{E_{quot}}}(V)$, with $Y \in \Gamma_F(V)$ and $\nabla_Y(\Gamma_{L_{quot}}) \subseteq \Gamma_{L_{quot}}$. Considering symbol maps we obtain the commutative diagram

$$\begin{array}{ccc} \Gamma_{\widetilde{K} \cap \widetilde{L}|_N} & \twoheadrightarrow & \Gamma^{L_{quot}}_{\mathbb{A}_{E_{quot}}} \cap \nabla(\Gamma_F) \\ & \searrow \quad \swarrow & \\ & \Gamma_F & \end{array}$$

Now the surjectivity of the horizontal map implies that each symbol map is onto if and only if the other one is, which is the statement in the claim. □

7.2.2. *Reduction of lagrangian submanifolds.* Assuming that the coisotropic submanifold $\mathcal{N}$ and the lagrangian submanifold $\mathcal{L}$ (whose body contains the body of $\mathcal{N}$) intersect cleanly, and that $\mathcal{N}$ is reduced to $\underline{\mathcal{N}}$, we now describe the reduction of $\mathcal{L}$ to $\underline{\mathcal{N}}$.

Denoting by $\iota \colon N \hookrightarrow M$ the inclusion of bodies, we have that $C_{\mathcal{N}} = \iota^{-1}(C_{\mathcal{M}}/\mathcal{I}_N)$. Hence $\mathcal{I} := \iota^{-1}((\mathcal{I}_{\mathcal{N}} + \mathcal{I}_{\mathcal{L}})/\mathcal{I}_{\mathcal{N}})$ is a sheaf of ideals in $C_{\mathcal{N}}$, to be understood as the vanishing ideal of the submanifold $\mathcal{N} \cap \mathcal{L}$ in $\mathcal{N}$. This sheaf is locally generated in degrees 0, 1 and 2, and we have

$$\mathcal{I}_0 = 0, \quad \mathcal{I}_1 = \Gamma_{(K + L|_N)/K}, \quad \mathcal{I}_2 = \Gamma_{(\widetilde{K} + \widetilde{L}|_N)/\widetilde{K}}.$$

Recall from § 7.2.1 that, by the clean-intersection condition, the quotients

$$\frac{(K+L|_N)}{K}=\frac{L|_N}{K\cap L|_N}\quad\text{and}\quad\frac{(\widetilde{K}+\widetilde{L}|_N)}{\widetilde{K}}=\frac{\widetilde{L}|_N}{\widetilde{K}\cap\widetilde{L}|_N}$$

are vector bundles over N.

Now consider the subsheaf (of algebras) of the sheaf of basic functions on $\mathcal{N}$ (see § 4.3) given by

$$\mathcal{I}_{quot}:=\mathcal{I}\cap(C_{\mathcal{N}})_{bas}\subseteq(C_{\mathcal{N}})_{bas},$$

which is locally generated in degrees 0, 1 and 2, and satisfies

$$(\mathcal{I}_{quot})_0=0,\qquad(\mathcal{I}_{quot})_1=\Gamma_{(K+L|_N)/K}\cap\Gamma^{flat}_{E_{quot}}=\Gamma^{flat}_{L_{quot}},\tag{7.5}$$

and, for each open $V\subseteq N$, $(\mathcal{I}_{quot})_2(V)$ is given by operators $(Y,D)\in\Gamma^{K,L|_N}_{\mathbb{A}_{E|_N}}(V)$ satisfying

$$[Y,\Gamma_F(V)]\subseteq\Gamma_F(V),\quad[D](\Gamma^{flat}_{E_{quot}}(V))\subseteq\Gamma^{flat}_{E_{quot}}(V)$$

modulo those such that $Y\in\Gamma_F(V)$, and $[D]=\nabla_Y$. We will give another characterization of $(\mathcal{I}_{quot})_2$.

Consider the vector bundle $\mathbb{A}^{\nabla}_{E_{quot}}$ (see (4.28)) with its natural flat partial connection. Let $\Gamma^{L_{quot},flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$ be the subsheaf of $\Gamma^{flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$ whose sections $\overline{(Y,\Delta)}$ satisfy the additional property that

$$\Delta(\Gamma_{L_{quot}})\subseteq\Gamma_{L_{quot}}.$$

(Note that this last condition is well defined for the class $\overline{(Y,\Delta)}$ by the clean-intersection assumption and Prop. 7.5 part (ii).) Then the map (4.29) giving the identification $((C_{\mathcal{N}})_{bas})_2\cong\Gamma^{flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$ restricts to an injective map

$$(\mathcal{I}_{quot})_2\to\Gamma^{L_{quot},flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$$

which is also surjective by (A.3), so

$$(\mathcal{I}_{quot})_2\cong\Gamma^{L_{quot},flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}.\tag{7.6}$$

Suppose that $\mathcal{N}$ is reducible to $\underline{\mathcal{N}}$, with projection $(p,p^\sharp)\colon\mathcal{N}\to\underline{\mathcal{N}}$, as in § 5.2. Recall that we have an isomorphism $p^\sharp\colon C_{\underline{\mathcal{N}}}\to p_*(C_{\mathcal{N}})_{bas}$, and let $\mathcal{I}_{red}\subseteq C_{\underline{\mathcal{N}}}$ be defined by

$$p^\sharp(\mathcal{I}_{red})=p_*(\mathcal{I}_{quot}).$$

Consider the pseudo-euclidean vector bundle $E_{red}\to\underline{N}$ corresponding to $\underline{\mathcal{N}}$, given by the quotient of $E_{quot}=K^\perp/K$ with respect to F and ∇ (see Theorem 5.4). Due to the geometric description of the clean-intersection condition in Prop. 7.5, one may also consider the quotient of the lagrangian vector subbundle $L_{quot}=\frac{(K^\perp\cap L|_N)+K}{K}$ of E_{quot} with respect F and ∇, which is a lagrangian subbundle

$$L_{red}\subseteq E_{red}.$$

Theorem 7.6. *Suppose that a coisotropic submanifold $\mathcal{N}$ and a lagrangian submanifold $\mathcal{L}$ (with body containing the one of $\mathcal{N}$) intersect cleanly, and that $\mathcal{N}$ reduces to $\underline{\mathcal{N}}$. Then:*

(a) *$\mathcal{I}_{red}$ is the vanishing ideal of a lagrangian submanifold $\underline{\mathcal{L}}$ of $\underline{\mathcal{N}}$, corresponding to the lagrangian subbundle L_{red} (in the sense of Cor. 4.15).*

(b) *If Θ is a Courant function on $\mathcal{M}$ that is reducible for $\mathcal{N}$ and $\mathcal{L}$, then Θ_{red} satisfies $\{\Theta_{red}, \mathcal{I}_{red}\} \subseteq \mathcal{I}_{red}$ (i.e., it is reducible for $\underline{\mathcal{L}}$).*

Proof. We know that $L_{red} \to \underline{N}$ is a lagrangian subbundle of E_{red}, so it gives rise to a lagrangian submanifold $\underline{\mathcal{L}}$ of $\underline{\mathcal{N}}$ by Cor. 4.15. To prove (a), we must check that its vanishing ideal equals $\mathcal{I}_{red}$, and it is enough to verify this fact in degrees 0, 1 and 2. From the expressions for $(\mathcal{I}_{quot})_0$ and $(\mathcal{I}_{quot})_1$ in (7.5), it is clear that

$$(\mathcal{I}_{red})_0 = 0 = (\mathcal{I}_{\underline{\mathcal{L}}})_0, \qquad (\mathcal{I}_{red})_1 = \Gamma_{L_{red}} = (\mathcal{I}_{\underline{\mathcal{L}}})_1.$$

Verifying the remaining case in degree 2 amounts to checking that $(\mathcal{I}_{red})_2$ coincides with $\Gamma^{L_{red}}_{\mathbb{A}_{E_{red}}}$, i.e., the sheaf of sections of $\mathbb{A}_{E_{red}}$ that preserve $\Gamma_{L_{red}}$.

Recall the map (5.7) that identifies $\Gamma_{\mathbb{A}_{E_{red}}}$ with

$$p_*((C_{\mathcal{N}})_{bas})_2 = p_*\Gamma^{flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}.$$

A section $(\underline{Y}, \underline{\Delta})$ of $\Gamma_{\mathbb{A}_{E_{red}}}$ corresponds to $\overline{(Y, \Delta)}$ in $p_*\Gamma^{flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$ if and only if

$$\overline{Y} = p_*(Y), \qquad p_1^\sharp \circ \underline{\Delta} \circ (p_1^\sharp)^{-1} = \Delta|_{\Gamma^{flat}_{E_{quot}}},$$

with $p_1^\sharp$ as in (5.6) (see Prop. A.7). Recall that $(\mathcal{I}_{red})_2 \subseteq \Gamma_{\mathbb{A}_{E_{red}}}$ is defined by the condition that it agrees with $p_*(\mathcal{I}_{quot})_2 = p_*\Gamma^{L_{quot},flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$ under this identification. We will check that $\Gamma^{L_{red}}_{\mathbb{A}_{E_{red}}}$ has this property and hence coincides with $(\mathcal{I}_{red})_2$.

Note that a section $(\underline{Y}, \underline{\Delta})$ of $\Gamma_{\mathbb{A}_{E_{red}}}$ satisfying $\underline{\Delta}(\Gamma_{L_{red}}) \subseteq \Gamma_{L_{red}}$ corresponds to a section $\overline{(Y, \Delta)}$ such that $\Delta(\Gamma^{flat}_{L_{quot}}) \subseteq \Gamma^{flat}_{L_{quot}}$. The fact that flat sections locally generate $\Gamma_{L_{quot}}$ implies that

$$\Delta(\Gamma^{flat}_{L_{quot}}) \subseteq \Gamma^{flat}_{L_{quot}} \iff \Delta(\Gamma_{L_{quot}}) \subseteq \Gamma_{L_{quot}}.$$

Hence $\overline{(Y, \Delta)}$ is a section of $\Gamma^{L_{quot},flat}_{\mathbb{A}^{\nabla}_{E_{quot}}}$, proving (a).

To prove (b), recall that Θ being reducible for $\mathcal{N}$ says that it is a section of $\mathfrak{N}_{\mathcal{I}_{\mathcal{N}}}$, and hence defines a section $\Theta_{\mathcal{N}}$ of $(C_{\mathcal{N}})_{bas} = \iota^{-1}(\mathfrak{N}_{\mathcal{I}_{\mathcal{N}}}/\mathcal{I}_{\mathcal{N}})$. Assuming that Θ is also reducible for $\mathcal{L}$, i.e., $\{\Theta, \mathcal{I}_{\mathcal{L}}\} \subseteq \mathcal{I}_{\mathcal{L}}$, directly implies that

$$\{\Theta_{\mathcal{N}}, \mathcal{I}_{quot}\} \subseteq \mathcal{I}_{quot},$$

which proves (b) upon the identification $p^\sharp : C_{\underline{\mathcal{N}}} \xrightarrow{\sim} p_*(C_{\mathcal{N}})_{bas}$. □

7.2.3. Reduction of Dirac structures. We now use Theorem 7.6 to obtain a reduction procedure for Dirac structures, phrased in classical geometric terms. Let E be a Courant algebroid over M, with anchor $\rho\colon E \to TM$, pseudo-euclidean structure $\langle \cdot, \cdot \rangle$ and bracket $[\![\cdot, \cdot]\!]$.

Let us consider the setup for Courant algebroid reduction of § 6.3: geometric coisotropic data (N, K, F, ∇) with respect to which E is reducible, as in Definition 6.12. We also assume that F is simple and ∇ has trivial holonomy, so that we have a reduced Courant algebroid $E_{red} \to \underline{N}$, as in Thm. 6.11. Let L be a Dirac structure in E with support on a submanifold S containing N (see Def. 6.10).

Theorem 7.7. *In the setup above, suppose that*

(a) *$L|_N \cap K$ has constant rank and*

(b) *the lagrangian subbundle* $L_{quot} = \frac{(K^\perp \cap L|_N)+K}{K}$ *in* $E_{quot} = K^\perp/K$ *is* ∇-*invariant, i.e.,* $\nabla_Y(\Gamma_{L_{quot}}) \subseteq \Gamma_{L_{quot}}$ *for any section* Y *of* F.

Then the quotient of L_{quot} *with respect to* F *and* ∇ *is a Dirac structure* $L_{red} \subset E_{red}$.

The proof is just a translation of Thm. 7.6. The Courant algebroid E corresponds to a symplectic degree 2 $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a Courant function Θ, the coisotropic data (N, K, F, ∇) gives a coisotropic submanifold $\mathcal{N}$ with respect to which Θ is reducible, and the Dirac structure L corresponds to a lagrangian submanifold $\mathcal{L}$ (with body $S \supseteq N$) with respect to which Θ is reducible. The assumptions in the theorem amount to the clean intersection condition, and the fact that L_{red} is a Dirac structure is equivalent to the property in part (b) of Thm. 7.6 (by Cor. 6.9).

Building on Example 6.15, we see that Thm. 7.7 recovers the following reduction construction from [Zam08].

Example 7.8 (Reduction of Dirac structures with support). Let E be an exact Courant algebroid over M, and let L be a Dirac structure supported on a submanifold $N \subseteq M$ such that $\rho(L) = TN$. Suppose that N is equipped with an involutive distribution F that is simple, with leaf space $\underline{N}$. With this setup, we now recall how to canonically obtain an exact Courant algebroid E_{red} over $\underline{N}$ together with a Dirac structure L_{red} therein.

We set $K := L \cap \rho^{-1}(F) \to N$, which is an isotropic subbundle of E with $\rho(K) = F$, and moreover satisfies the condition

$$\rho(K^\perp) = TN,$$

since $K^\perp = L + \rho^*(\mathrm{Ann}(F))$. Then, as explained in Example 6.15, one canonically obtains coisotropic data (N, K, F, ∇) with respect to which E is reducible. Moreover, it is proven in [Zam08, Lemma 5.4, Prop. 5.5] that the F-connection ∇ obtained in this case automatically has trivial holonomy. (This relies on the fact that any splitting of $\rho|_L\colon L \to TN$, applied to projectable vector fields on N, yields flat sections of E_{quot}.) Hence by Thm. 6.11 we obtain a reduced Courant algebroid $E_{red} \to \underline{N}$, which is exact by Prop. 6.14.

Using that $K \subseteq L \subseteq K^\perp$ and the involutivity of K and L, one can directly verify that the two conditions in Thm. 7.7 are satisfied. Therefore L reduces to a Dirac structure L_{red} in $E_{red} \to \underline{N}$. ◇

By combining the above example with Thm. 7.2 one recovers the "reduction of branes" in generalized complex geometry described in [Zam08, Thm. 7.4] (it is shown in part b) of the proof of [Zam08, Thm. 7.4] that the hypothesis of Thm. 7.2 is satisfied).

8. Momentum maps and hamiltonian reduction

In this section we discuss symplectic reduction of hamiltonian actions in the context of symplectic degree 2 $\mathbb{N}$-manifolds. We recall the main ingredients of the classical procedure [MW74].

Let M be a symplectic manifold, with Poisson bracket $\{\cdot,\cdot\}$, and let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-action on M,

$$\mathfrak{g} \to \mathfrak{X}(M), \quad u \mapsto u_M,$$

is called *hamiltonian* if there is a smooth map $\mu\colon M \to \mathfrak{g}^*$ satisfying the following properties: its dual map

$$\mu^*\colon \mathfrak{g} \to C^\infty(M), \quad (\mu^* u)(x) = \langle \mu(x), u\rangle,$$

is a Lie algebra homomorphism such that

$$u_M = X_{\mu^* u} = \{\mu^* u, \cdot\}.$$

The map $\mu\colon M \to \mathfrak{g}^*$ is called a *momentum map*. The bracket-preserving property of μ^* is equivalent to the $\mathfrak{g}$-equivariance of μ with respect to the co-adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

We will restrict ourselves to the simplest formulation of symplectic reduction, by assuming that 0 is a regular value of μ, so that $N := \mu^{-1}(0)$ is a $\mathfrak{g}$-invariant submanifold of M; letting G be any connected Lie group with Lie algebra $\mathfrak{g}$, we also assume that the $\mathfrak{g}$-action on N integrates to a G-action that is free and proper. Then $M_{red} := N/G$ carries a natural symplectic structure. In this setting, N is a coisotropic submanifold of M, and M_{red} agrees with its coisotropic reduction. In particular, $C^\infty(N)^G = C^\infty(N)_{bas}$ is a Poisson algebra, and the reduced symplectic form on M_{red} is characterized by the fact that the identification $q^*\colon C^\infty(M_{red}) \xrightarrow{\sim} C^\infty(N)^G$ induced by the quotient map $q\colon N \to M_{red}$ is an isomorphism of Poisson algebras.

We will present an analogue of this construction for degree 2 $\mathbb{N}$-manifolds.

8.1. **Differential graded Lie algebras of degree 2.** A *graded Lie algebra* is a (real) graded vector space $\widetilde{\mathfrak{g}} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_k$, equipped with a degree-preserving graded Lie bracket $[\cdot,\cdot]\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$. This means that, for homogeneous elements $x \in \mathfrak{g}_k$ and $y \in \mathfrak{g}_l$,

$$[x, y] = -(-1)^{kl}[y, x]$$

and $[x,\cdot]$ is a degree k derivation of the bracket (graded Jacobi identity).

For a degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, we consider the space of global sections of the sheaf of vector fields $\mathcal{TM}$, denoted by $\mathfrak{X}(\mathcal{M})$, equipped with its graded Lie bracket. An *action* of a graded Lie algebra $\widetilde{\mathfrak{g}}$ on $\mathcal{M}$ is a graded Lie algebra morphism $\widetilde{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M})$.

A *differential graded Lie algebra (DGLA)* is a graded Lie algebra equipped with a coboundary operator $\delta\colon \mathfrak{g} \to \mathfrak{g}$ that is a degree 1 derivation of the Lie bracket,

$$\delta[x, y] = [\delta x, y] + (-1)^k[x, \delta y],$$

for x of degree k. Such a δ is referred to as a *differential.*

For the purpose of studying hamiltonian actions on symplectic degree 2 $\mathbb{N}$-manifolds (see § 8.2), we will focus on (differential) graded Lie algebras that are concentrated in degrees 0, -1, and -2; we will say that such a (differential) graded Lie algebra is *of degree* 2.

For ordinary vector spaces $\mathfrak{h}$, $\mathfrak{a}$, and $\mathfrak{g}$, consider the graded vector space

$$\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}. \tag{8.1}$$

Recall that the notation means that elements in $\mathfrak{h}$ have degree -2, and elements in $\mathfrak{a}$ have degree -1.

Proposition 8.1. *A graded Lie algebra structure on* $\widetilde{\mathfrak{g}}$ *is equivalent to the following data:*

- *a Lie algebra structure on* $\mathfrak{g}$*,*

- *a representation* $\tau\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{a})$,
- *a representation* $\lambda\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$, *and*
- *a symmetric bilinear map* $\varpi\colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{h}$ *that is* $\mathfrak{g}$*-equivariant, i.e.,* $u \cdot \varpi(a_1, a_2) = \varpi(u \cdot a_1, a_2) + \varpi(a_1, u \cdot a_2)$ *for all* $v \in \mathfrak{g}$, $a_1, a_2 \in \mathfrak{a}$.

Proof. The correspondence is given by

$$\tau(u)(a) = [u, a], \quad \lambda(u)(h) = [u, h], \quad \varpi(a_1, a_2) = [a_1, a_2],$$

for $u \in \mathfrak{g}$, $a, a_1, a_2 \in \mathfrak{a}$, and $h \in \mathfrak{h}$. The fact that τ and λ are representations, as well as the $\mathfrak{g}$-equivariance property of ϖ, follow from the graded Jacobi identity for the bracket on $\widetilde{\mathfrak{g}}$. □

Now suppose that $\widetilde{\mathfrak{g}}$ as in (8.1) is equipped with a graded Lie algebra structure as well as a coboundary operator δ, so that we have a 3-term chain complex

$$\mathfrak{h} \xrightarrow{\delta} \mathfrak{a} \xrightarrow{\delta} \mathfrak{g}.$$

Proposition 8.2. $(\widetilde{\mathfrak{g}}, [\cdot,\cdot], \delta)$ *is a DGLA if and only if the following conditions hold:*

(a) δ *is* $\mathfrak{g}$*-equivariant with respect to* λ, τ, *and the adjoint action of* $\mathfrak{g}$ *on itself.*
(b) $\delta\varpi(a_1, a_2) = \tau(\delta a_1)(a_2) + \tau(\delta a_2)(a_1)$ *for all* $a_1, a_2 \in \mathfrak{a}$.
(c) $\lambda(\delta a)(h) = \varpi(a, \delta h)$ *for all* $a \in \mathfrak{a}, h \in \mathfrak{h}$.

The proof is a direct verification.

8.2. **Hamiltonian actions and reduction in degree 2.** We start with some general considerations. Given a graded vector space

$$\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g},$$

we will regard the graded vector space $\widetilde{\mathfrak{g}}^*[2] = \mathfrak{g}^*[2] \oplus \mathfrak{a}^*[1] \oplus \mathfrak{h}^*$ as a split degree 2 $\mathbb{N}$-manifold determined by the vector bundles $E_1 = \mathfrak{a}^* \times \mathfrak{h}^* \to \mathfrak{h}^*$ and $E_2 = \mathfrak{g}^* \times \mathfrak{h}^* \to \mathfrak{h}^*$, as in Example 2.8. In particular, given a degree 2 $\mathbb{N}$-manifold $\mathcal{M}$, any morphism $\widetilde{\mu} = (\mu, \mu^\sharp)\colon \mathcal{M} \to \widetilde{\mathfrak{g}}^*[2]$ is determined by three maps,

$$\mu\colon M \to \mathfrak{h}^*, \qquad \varrho\colon \mathfrak{a} \to C(\mathcal{M})_1, \qquad \varphi\colon \mathfrak{g} \to C(\mathcal{M})_2, \tag{8.2}$$

where μ is the map between bodies, ϱ and φ are defined by the components of $\mu^\sharp$ in degrees 1 and 2, and we denote by $C(\mathcal{M})$ the graded algebra of global sections of the sheaf $C_{\mathcal{M}}$. Let $\mu^*\colon \mathfrak{h} \to C^\infty(M)$ be given by

$$\mu^*(h)(x) = \langle \mu(x), h \rangle.$$

With a slight abuse of notation, we will denote by

$$\widetilde{\mu}^\sharp\colon \widetilde{\mathfrak{g}} \to C(\mathcal{M})[2] \tag{8.3}$$

the morphism of graded vector spaces defined by μ^*, ϱ and φ in (8.2).

Suppose now that $\widetilde{\mathfrak{g}}$ is a graded Lie algebra of degree 2 and $\mathcal{M}$ is a symplectic degree 2 $\mathbb{N}$-manifold, so that $C(\mathcal{M})[2]$ is a graded Lie algebra with respect to the Poisson bracket. A $\widetilde{\mathfrak{g}}$-action on $\mathcal{M}$,

$$\widetilde{\mathfrak{g}} \to \mathfrak{X}(\mathcal{M}), \quad \xi \mapsto \xi_{\mathcal{M}},$$

is called *hamiltonian* if there is a morphism of degree 2 $\mathbb{N}$-manifold,

$$\widetilde{\mu}\colon \mathcal{M} \to \widetilde{\mathfrak{g}}^*[2],$$

so that the induced map $\widetilde{\mu}^\sharp \colon \widetilde{\mathfrak{g}} \to C(\mathcal{M})[2]$ is a morphism of graded Lie algebras and determines the action via

$$\xi_{\mathcal{M}} = X_{\widetilde{\mu}^\sharp \xi} = \{\widetilde{\mu}^\sharp \xi, \cdot\}, \tag{8.4}$$

for all $\xi \in \widetilde{\mathfrak{g}}$. As in the classical case we refer to $\widetilde{\mu}$ as a *momentum map*.

Similarly to the classical case, one can perform reduction with respect to a hamiltonian action with suitable regularity assumptions. Suppose that 0 is a regular value for the momentum map (in the sense of § 2.4), so that

$$\mathcal{N} = \widetilde{\mu}^{-1}(0)$$

is a submanifold of $\mathcal{M}$ with body $N = \mu^{-1}(0)$ and sheaf of vanishing ideals $\mathcal{I}$ generated by the image of the maps μ^*, ϱ and φ in (8.2).

Remark 8.3. Since the sheaf of vanishing ideals of a submanifold is locally generated in degrees 0, 1 and 2, our assumption that 0 is a regular value for $\widetilde{\mu}$ justifies why $\widetilde{\mathfrak{g}}$ was taken to be of degree 2.

A direct consequence of $\widetilde{\mu}^\sharp$ being bracket preserving is that $\mathcal{N}$ is coisotropic (see (4.2)), and condition (8.4) implies that the vector fields $\xi_{\mathcal{M}}$, for $\xi \in \widetilde{\mathfrak{g}}$, preserve $\mathcal{I}$. Hence each $\xi_{\mathcal{M}}$ induces a vector field $\xi_{\mathcal{N}}$ on $\mathcal{N}$ (see Prop. 2.20), in such a way that the map $\xi \mapsto \xi_{\mathcal{N}}$ defines a $\widetilde{\mathfrak{g}}$-action on $\mathcal{N}$. Moreover, the momentum map condition (8.4) implies that the vector fields $\xi_{\mathcal{N}}$, for $\xi \in \widetilde{\mathfrak{g}}$, span the null distribution of $\mathcal{N}$ (see Lemma 4.3). Hence the sheaf of invariant functions on $\mathcal{N}$, defined on each open subset $V \subseteq N$ by

$$C_{\mathcal{N}}^{\widetilde{\mathfrak{g}}}(V) = \{f \in C_{\mathcal{N}}(V) \mid \xi_{\mathcal{N}}(f) = 0 \ \ \forall \xi \in \widetilde{\mathfrak{g}}\},$$

agrees with the sheaf of basic functions, $C_{\mathcal{N}}^{\widetilde{\mathfrak{g}}} = (C_{\mathcal{N}})_{bas}$, and therefore is a sheaf of Poisson algebras.

Assuming that the regularity conditions for coisotropic reduction are satisfied (as in Theorem 5.4), one obtains a degree 2 symplectic $\mathbb{N}$-manifold $\mathcal{M}_{red}$, along with a surjective submersion $\mathcal{N} \to \mathcal{M}_{red}$ that identifies $C_{\mathcal{M}_{red}}$ with $C_{\mathcal{N}}^{\widetilde{\mathfrak{g}}}$ as sheaves of Poisson algebras (and this identification uniquely characterizes the symplectic structure on $\mathcal{M}_{red}$). As in the classical case, one refers to $\mathcal{M}_{red}$ as the *symplectic reduction* of $\mathcal{M}$ with respect to the momentum map $\widetilde{\mu}$.

Suppose now, additionally, that $\mathcal{M}$ is equipped with a Courant function Θ. Then $C(\mathcal{M})[2]$ is not only a graded Lie algebra but a DGLA, with differential given by $\{\Theta, \cdot\}$. A natural way to ensure that Θ is reducible with respect to $\mathcal{N} = \widetilde{\mu}^{-1}(0)$ is assuming that $\widetilde{\mathfrak{g}}$ is also a DGLA, with differential δ, and that $\widetilde{\mu}^\sharp \colon \widetilde{\mathfrak{g}} \to C(\mathcal{M})[2]$ is a morphism of DGLAs, since in that case the momentum map satisfies

$$\{\Theta, \widetilde{\mu}^\sharp(\xi)\} = \widetilde{\mu}^\sharp(\delta\xi), \quad \forall \xi \in \widetilde{\mathfrak{g}}. \tag{8.5}$$

Since this condition implies that Θ is reducible, it follows, as explained in § 6.2, that Θ gives rise to a Courant function Θ_{red} on $\mathcal{M}_{red}$.

We now give the classical geometric descriptions of the momentum map $\widetilde{\mu}$, of the coisotropic submanifold $\mathcal{N} = \widetilde{\mu}^{-1}(0)$, and of the symplectic reduced manifold $\mathcal{M}_{red}$ and Courant function Θ_{red}.

8.3. **Degree 2 hamiltonian actions in classical terms.** Let $\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}$ be a graded Lie algebra of degree 2, as in (8.1), and let $\mathcal{M}$ be a symplectic degree 2 $\mathbb{N}$-manifold equipped with a hamiltonian $\widetilde{\mathfrak{g}}$-action with momentum map $\widetilde{\mu}\colon \mathcal{M} \to \widetilde{\mathfrak{g}}^*[2]$.

Suppose that $\mathcal{M}$ corresponds to the pseudo-euclidean vector bundle $(E, \langle\cdot,\cdot\rangle)$. Then by (8.2) (and (3.12)) the momentum map $\widetilde{\mu}$ is described by maps

$$\mu\colon M \to \mathfrak{h}^*, \qquad \varrho\colon \mathfrak{a} \to \Gamma(E), \qquad \varphi\colon \mathfrak{g} \to \Gamma(\mathbb{A}_E). \tag{8.6}$$

Recall the characterization of the graded Lie algebra structure on $\widetilde{\mathfrak{g}}$ in Prop. 8.1 as a Lie algebra $\mathfrak{g}$, together with representations τ and λ on $\mathfrak{a}$ and $\mathfrak{h}$, respectively, and an equivariant symmetric map $\varpi\colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{h}$.

One can directly verify the next result.

Lemma 8.4. *The property that $\widetilde{\mu}^\sharp$ in* (8.3) *preserves graded Lie brackets is equivalent to*

(a) $[\varphi(u), \varphi(v)] = \varphi([u, v])$,
(b) $\varrho(\tau(u)a) = \varphi(u)(\varrho(a))$,
(c) $\mu^*(\lambda(u)h) = \pounds_{u_M}(\mu^* h)$,
(d) $\mu^*(\varpi(a_1, a_2)) = \langle \varrho(a_1), \varrho(a_2)\rangle$,

for all $u, v \in \mathfrak{g}$, $a, a_1, a_2 \in \mathfrak{a}$, $h \in \mathfrak{h}$, and where $u_M \in \mathfrak{X}(M)$ is the symbol of the differential operator $\varphi(u)$.

By (a) in the previous lemma, φ defines an action of $\mathfrak{g}$ on $E \to M$ by derivations preserving the pseudo-euclidean metric, and the composition of φ with the symbol map,

$$\mathfrak{g} \xrightarrow{\varphi} \Gamma(\mathbb{A}_E) \xrightarrow{\sigma} \mathfrak{X}(M), \quad u \mapsto u_M,$$

defines a $\mathfrak{g}$-action on M. Conditions (b) and (c) say that the maps $\mu\colon M \to \mathfrak{h}^*$ and $\varrho\colon \mathfrak{a} \to \Gamma(E)$ in (8.6) are $\mathfrak{g}$-equivariant, where $\mathfrak{h}^*$ is equipped with the linear $\mathfrak{g}$-action given by the dual of the representation of $\mathfrak{g}$ on $\mathfrak{h}$. Condition (d), which can be written as

$$\langle \varrho(a_1), \varrho(a_2)\rangle(x) = \langle \mu(x), \varpi(a_1, a_2)\rangle, \qquad x \in M, \tag{8.7}$$

shows how ϱ is related to the pseudo-euclidean metric on E.

The first assumption for reduction is that 0 is a regular value for the momentum map $\widetilde{\mu}$. Denote by $\mathfrak{a}_M$ and $\mathfrak{g}_M$ the trivial bundles $\mathfrak{a} \times M \to M$ and $\mathfrak{g} \times M \to M$, respectively, and keep the same notation ϱ and φ for the maps

$$\mathfrak{a}_M \to E,\ (a, x) \mapsto \varrho(a)|_x, \qquad \mathfrak{g}_M \to \mathbb{A}_E,\ (u, x) \mapsto \varphi(u)|_x.$$

We have the following direct consequence of Cor. 2.24.

Lemma 8.5. *The origin $0 \in \mathfrak{h}^*$ is a regular value of the momentum map $\widetilde{\mu}\colon \mathcal{M} \to \widetilde{\mathfrak{g}}^*[2]$ if and only if*

(a) 0 *is a regular value of $\mu\colon M \to \mathfrak{h}^*$,*
(b) *$\varrho\colon \mathfrak{a}_M \to E$ is fiber-wise injective at every point of $\mu^{-1}(0)$,*
(c) *$\sigma \circ \varphi\colon \mathfrak{g}_M \to TM$ is fiber-wise injective at every point of $\mu^{-1}(0)$.*

Notice that condition (c) states that the action of $\mathfrak{g}$ on $\mu^{-1}(0)$ is locally free.

Assuming that $0 \in \mathfrak{h}^*$ is a regular value of $\widetilde{\mu}$, we saw in §8.2 that $\mathcal{N} = \widetilde{\mu}^{-1}(0)$ is a coisotropic submanifold with body $N = \mu^{-1}(0)$. We will now describe the corresponding geometric coisotropic data (N, K, F, ∇) (as in Thm. 4.5).

By the equivariance of $\mu\colon M \to \mathfrak{h}^*$, the submanifold N is $\mathfrak{g}$-invariant, and we denote the restricted action of $\mathfrak{g}$ on N by $u \mapsto u_N := u_M|_N$. As a consequence, for each $u \in \mathfrak{g}$, the derivation $\varphi(u) \in \Gamma(\mathbb{A}_E)$ restricts to a derivation

$$\varphi(u)|_N \in \Gamma(\mathbb{A}_{E|_N}).$$

Note also that the vector subbundle

$$K := \varrho(\mathfrak{a}_M)|_N \subseteq E|_N$$

is isotropic (by (8.7)) and invariant by the derivations $\varphi(u)|_N$ (by part (b) of Lemma 8.4), i.e., $\varphi(u)|_N(K) \subseteq K$. Recall that the induced derivation of $E_{quot} = K^\perp/K$ (see (4.8)) is denoted by $[\varphi(u)|_N]$.

Lemma 8.6. *The geometric coisotropic data (N, K, F, ∇) corresponding to the coisotropic submanifold $\mathcal{N} = \widetilde{\mu}^{-1}(0)$ are given as follows:*

- $N = \mu^{-1}(0)$,
- $K = \varrho(\mathfrak{a}_M)|_N$,
- $F = \sigma(\varphi(\mathfrak{g}_M))|_N = \{u_N, : u \in \mathfrak{g}\}$,
- $\nabla_{u_N} = [\varphi(u)|_N]$, *for* $u \in \mathfrak{g}$.

Proof. The geometric characterization of submanifolds defined by regular values of maps is given in Prop. 2.26, from where the descriptions of N and K immediately follow. By (2.39), the vector bundle $\widetilde{K}$ is given by $\varphi(\mathfrak{g}_M)|_N + K \wedge E|_N$, which can be checked to correspond to F and ∇ as in the statement using (4.17). □

Let us now assume that $\widetilde{\mathfrak{g}}$ is a DGLA, with differential δ, see Prop. 8.2, and that $\mathcal{M}$ is equipped with a Courant function Θ, so that E acquires a Courant algebroid structure with anchor ρ and bracket $[\![\cdot,\cdot]\!]$ (see § 6.1).

Lemma 8.7. *The map $\widetilde{\mu}^\sharp\colon \widetilde{\mathfrak{g}} \to C(\mathcal{M})[2]$ preserves differentials (as in (8.5), implying that Θ is reducible) if and only if*

(a) $\varrho(\delta h) = \rho^*(d(\mu^* h))$,
(b) $\varphi(\delta a) = [\![\varrho(a), \cdot]\!]$,
(c) $\varphi(u)([\![e_1, e_2]\!]) = [\![\varphi(u)(e_1), e_2]\!] + [\![e_1, \varphi(u)(e_2)]\!]$,

for $h \in \mathfrak{h}$, $a \in \mathfrak{a}$, $u \in \mathfrak{g}$ and $e_1, e_2 \in \Gamma(E)$.

Proof. The conditions for $\widetilde{\mu}^\sharp$ to preserve differentials are

$$\{\Theta, \mu^* h\} = \varrho(\delta h), \qquad \{\Theta, \varrho(a)\} = \varphi(\delta a), \qquad \{\Theta, \varphi(u)\} = 0,$$

for all $h \in \mathfrak{h}$, $a \in \mathfrak{a}$, $u \in \mathfrak{g}$. The first two conditions coincide with (a) and (b) since

$$\{\Theta, f\} = \rho^* df, \qquad \{\Theta, e\} = [\![e, \cdot]\!],$$

for $f \in C^\infty(M)$ and $e \in \Gamma(E)$. The last condition is the vanishing of the degree 3 function $\{\Theta, \varphi(u)\}$, for any $u \in \mathfrak{g}$. This condition is equivalent to

$$\{\{\{\Theta, \varphi(u)\}, e_1\}, e_2\} = 0$$

for all $e_1, e_2 \in \Gamma(E)$. Indeed, for any degree 3 function S, the coordinate expression (6.8) shows that the derived brackets $\{\{S, e_1\}, e_2\}$ and $\{\{S, e\}, f\}$ determine S, where e_1, e_2 range over $\Gamma(E)$ and f over $C^\infty(M)$; further, the vanishing of the first expression for all e_1, e_2 implies the vanishing of the second, by the Leibniz rule. Now, by the Jacobi identity, note that

$$\begin{aligned}\{\{\{\Theta,\varphi(u)\},e_1\},e_2\} &= \{\{\Theta,\{\varphi(u),e_1\}\},e_2\} - \{\{\varphi(u),\{\Theta,e_1\}\},e_2\}\\ &= [\![\varphi(u)(e_1),e_2]\!] - (\{\varphi(u),\{\{\Theta,e_1\},e_2\}\} - \{\{\Theta,e_1\},\{\varphi(u),e_2\}\})\\ &= [\![\varphi(u)(e_1),e_2]\!] - \varphi(u)([\![e_1,e_2]\!]) + [\![e_1,\varphi(u)(e_2)]\!].\end{aligned}$$

□

8.4. **Hamiltonian reduction of Courant, Dirac and GC structures.** Using hamiltonian symplectic reduction in degree 2 and the results in §8.3, we now formulate *in classical terms* reduction procedures for Courant, Dirac and GC structures.

We start by recalling the general setup.

(A) *Objects to be reduced*: We consider a Courant algebroid $E \to M$ with pairing $\langle\cdot,\cdot\rangle$, anchor map $\rho\colon E \to TM$ and bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$. We may equip E with a Dirac structure $L \subset E$, or with a generalized complex structure $\mathbb{J}\colon E \to E$.

(B) *Objects that act (DGLAs of degree 2)*: Following Propositions 8.1 and 8.2, we consider a Lie algebra $\mathfrak{g}$, along with $\mathfrak{g}$-modules $\mathfrak{a}$ and $\mathfrak{h}$, and a $\mathfrak{g}$-equivariant symmetric bilinear map $\varpi\colon \mathfrak{a}\otimes\mathfrak{a} \to \mathfrak{h}$. Suppose that there are operators

$$\mathfrak{h} \xrightarrow{\delta} \mathfrak{a} \xrightarrow{\delta} \mathfrak{g}, \tag{8.8}$$

with $\delta^2 = 0$, and that preserve the $\mathfrak{g}$-module structures (where $\mathfrak{g}$ is viewed as a $\mathfrak{g}$-module with respect to the adjoint representation). Assume moreover that

$$\delta\varpi(a_1,a_2) = (\delta a_1)\cdot a_2 + (\delta a_2)\cdot a_1, \quad (\delta a)\cdot h = \varpi(a,\delta h), \tag{8.9}$$

for a, a_1, $a_2 \in \mathfrak{a}$ and $h \in \mathfrak{h}$.

(C) *Infinitesimal hamiltonian actions*: Following Lemmas 8.4 and 8.7, we suppose that the vector bundle E is equipped with a Lie algebra morphism $\varphi\colon \mathfrak{g} \to \Gamma(\mathbb{A}_E)$, that makes E into a $\mathfrak{g}$-equivariant vector bundle and induces a $\mathfrak{g}$-action on M. We further assume that there are $\mathfrak{g}$-equivariant maps $\mu\colon M \to \mathfrak{h}^*$ and $\varrho\colon \mathfrak{a} \to \Gamma(E)$ such that

$$\mu^*(\varpi(a_1,a_2)) = \langle\varrho(a_1),\varrho(a_2)\rangle, \tag{8.10}$$

for $a_1, a_1 \in \mathfrak{a}$, and that

$$\begin{aligned}&\varrho(\delta h) = \rho^*(d(\mu^* h)),\\ &(\delta a)\cdot e = [\![\varrho(a),e]\!],\\ &u\cdot[\![e_1,e_2]\!] = [\![u\cdot e_1,e_2]\!] + [\![e_1,u\cdot e_2]\!],\end{aligned} \tag{8.11}$$

for $u \in \mathfrak{g}$, $a \in \mathfrak{a}$, $h \in \mathfrak{h}$ and $e, e_1, e_2 \in \Gamma(E)$. (Here we used the notation $\varphi(u)(e) = u\cdot e$.)

Definition 8.8. We say that the maps φ, ϱ and μ define a *hamiltonian action* of a DGLA of degree 2 (as in **(B)**) on the Courant algebroid $E \to M$.

Remark 8.9. (*i*) The first and second identities in (8.11) amount to the commutativity of the following diagram:

$$\begin{array}{ccccc} \mathfrak{h} & \xrightarrow{\delta} & \mathfrak{a} & \xrightarrow{\delta} & \mathfrak{g} \\ {\scriptstyle \mu^*}\downarrow & & \downarrow{\scriptstyle \varrho} & & {\scriptstyle \varphi}\downarrow \\ C^\infty(M) & \xrightarrow{\rho^* d} & \Gamma(E) & \xrightarrow{\mathrm{ad}} & \Gamma(\mathbb{A}_E). \end{array}$$

The third identity states that φ takes values in infinitesimal Courant algebroid automorphisms.

(*ii*) It follows from the first two conditions in (8.11), the first identity in (8.9), the equivariance of ϱ and axiom (C5) of Courant algebroids that

$$\rho^* d(\mu^*(\varpi(a_1, a_2))) = \rho^* d(\langle \varrho(a_1), \varrho(a_2) \rangle),$$

for $a_1, a_1 \in \mathfrak{a}$, which is a weaker version of (8.10).

(D) *Regularity assumptions*: Following Lemma 8.5, we assume that $0 \in \mathfrak{h}^*$ is a regular value for $\mu \colon M \to \mathfrak{h}^*$ and that, for each $x \in \mu^{-1}(0)$, the map

$$\mathfrak{a} \to E|_x, \qquad a \mapsto \varrho(a)|_x, \tag{8.12}$$

is injective. Note that, by the $\mathfrak{g}$-equivariance of μ, the $\mathfrak{g}$-action on E restricts to a $\mathfrak{g}$-action on $E|_{\mu^{-1}(0)}$. For a connected Lie group G integrating $\mathfrak{g}$, we will assume that this infinitesimal action integrates to a G-action on $E|_{\mu^{-1}(0)}$ for which the restricted G-action on $\mu^{-1}(0)$ is free and proper.

Some consequences of the setup: The images of (8.12) for each $x \in \mu^{-1}(0)$ define a vector subbundle

$$K \subseteq E|_{\mu^{-1}(0)} \tag{8.13}$$

that is isotropic (by (8.10)) and G-invariant (as a consequence of the $\mathfrak{g}$-equivariance of ϱ and the fact that G is connected). Note that the geometric coisotropic data (N, K, F, ∇) defined in Lemma 8.6 coincides with the one of Example 4.9. As seen in Example 5.6, in the present setup F is simple and ∇ has trivial holonomy, and the reduction of the pseudo-euclidean vector bundle E is given by $E_{red} = (K^\perp/K)/G$ over $M_{red} = \mu^{-1}(0)/G$,

$$\begin{array}{ccc} K^\perp/K & \longrightarrow & E_{red} \\ \downarrow & & \downarrow \\ \mu^{-1}(0) & \longrightarrow & M_{red}. \end{array}$$

By Thm. 5.4, E_{red} is the pseudo-euclidean vector bundle corresponding to $\mathcal{M}_{red}$, the symplectic degree 2 $\mathbb{N}$-manifold obtained by hamiltonian reduction described in § 8.2.

Theorem 8.10. *Consider a hamiltonian action of a DGLA of degree 2 (as in* **(B)***) on a Courant algebroid $E \to M$ satisfying the regularity conditions in* **(D)***. Then the following conclusions hold:*

(a) *The pseudo-euclidean vector bundle $E_{red} \to \mu^{-1}(0)/G$ inherits a Courant algebroid structure (with anchor and bracket defined as in* (6.13)*);*

(b) *If $L \subseteq E$ is a G-invariant Dirac structure such that $L|_{\mu^{-1}(0)} \cap K$ has constant rank, then it gives rise to a Dirac structure L_{red} in E_{red} (as in Thm. 7.7);*

(c) *If $\mathbb{J}$ is a G-invariant GC structure on E such that $\mathbb{J}(K) \subseteq K$, then E_{red} inherits a GC structure $\mathbb{J}_{red}$ (as in Thm. 7.2).*

In part (a), the reducibility of the Courant algebroid structure is ensured by (8.11) (following Lemma 8.7 and §8.2). Parts (b) and (c) are special cases of Theorems 7.7 and 7.2, respectively.

Remark 8.11. Prop. 6.14 gives conditions for E_{red} in part (a) to be an exact Courant algebroid; e.g., it is enough that $\rho(K^{\perp}) = T(\mu^{-1}(0))$ and that $\rho(K)$ agrees with the distribution tangent to the G-orbits on $\mu^{-1}(0)$.

Remark 8.12. One can slightly weaken the invariance hypotheses in the previous theorem: in (b) it is enough to require the G-invariance of $L_{quot} \subset K^{\perp}/K$, and in (c) the G-invariance of the map $\mathbb{J}_{quot} \colon K^{\perp}/K \to K^{\perp}/K$, induced by $\mathbb{J}$.

We now illustrate the specific roles of $\mathfrak{h}, \mathfrak{a}$ and $\mathfrak{g}$ in (**B**) (i.e., the components of the DGLA) in the reduction procedure of a Courant algebroid E, described in part (a) of the previous theorem. We consider situations where precisely one of $\mathfrak{h}, \mathfrak{a}$ or $\mathfrak{g}$ is non-zero.

Example 8.13.

- When only $\mathfrak{h}$ is non-zero, it is just a vector space, and the previous setup consists of a smooth map $\mu \colon M \to \mathfrak{h}^*$ such that $\rho(E) \subseteq \ker(d\mu)$. If 0 is a regular value of μ, then $\mu^{-1}(0)$ is a submanifold with the property that $\rho(E) \subseteq T(\mu^{-1}(0))$. The reduced Courant algebroid in this case is the restriction $E|_{\mu^{-1}(0)}$ (see Example 6.13 (i)).
- If only $\mathfrak{a}$ is nonzero, then it is a vector space, and the previous setup consists of a linear map $\varrho \colon \mathfrak{a} \to \Gamma(E)$ such that $\langle \varrho(a_1), \varrho(a_2) \rangle = 0$ and $[\![\varrho(a), \cdot]\!] = 0$ for all a_1, a_2 and $a \in \mathfrak{a}$ (by (8.10) and the second equation in (8.11)). Assuming that ϱ is fiberwise injective, its image defines an isotropic subbundle $K \subset E$, and the reduction scheme in this case says that $K^{\perp}/K \to M$ has an induced Courant algebroid structure.
- When only $\mathfrak{g}$ is non-zero, the setup consists of a Lie algebra morphism $\mathfrak{g} \to \Gamma(\mathbb{A}_E)$ acting by infinitesimal Courant algebroid automorphisms (the third equation in (8.11)). Assume that G is a connected Lie group integrating $\mathfrak{g}$ and that the infinitesimal action on M integrates to a G-action that is free and proper. In this case, the reduced Courant algebroid is $E/G \to M/G$, which is simply the quotient of E by an action of G by Courant algebroid automorphisms.

◇

In the full reduction scheme for the Courant algebroid $E \to M$ in Theorem 8.10, we see that $\mathfrak{h}$ is used to cut out the submanifold $\mu^{-1}(0)$, $\mathfrak{a}$ is used to define the vector bundle $K^{\perp}/K \to \mu^{-1}(0)$, and $\mathfrak{g}$ is used to quotient this bundle.

8.5. **The case of exact DGLAs.** We will now consider the reduction setup of the previous subsection in the special case where the DGLA of degree 2 is *exact*, i.e., the complex (8.8) defines a short exact sequence

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\delta} \mathfrak{a} \xrightarrow{\delta} \mathfrak{g} \longrightarrow 0.$$

In the context of exact Courant algebroids, we will see how this special setup recovers the reduction schemes of [BCG07, BCG08].

8.5.1. *Courant algebras.* We now show that exact DGLAs of degree 2 admit an alternative description in terms of the following objects, introduced in [BCG07, Definitions 2.6 and 2.7].

Definition 8.14. A *Courant algebra* over a Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ with a bilinear bracket $[\![\cdot,\cdot]\!]$ and a map $p : \mathfrak{a} \to \mathfrak{g}$ such that, for all $a_1, a_2, a_3 \in \mathfrak{a}$,

(a) $[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!]$, that is, $(\mathfrak{a}, [\![\cdot,\cdot]\!])$ is a Leibniz algebra;
(b) $p([\![a_1, a_2]\!]) = [p(a_1), p(a_2)]$.

We define a Courant algebra to be *exact* if p is surjective and $\mathfrak{h} := \ker(p)$ is left-central, i.e. $[\![h, a]\!] = 0$ for all $h \in \mathfrak{h}$, $a \in \mathfrak{a}$.

Just as in [BCG07, BCG08], we will only be concerned with exact Courant algebras. We note that the previous definition is a mild modification of the original one in [BCG07, Def, 2.7], where $\mathfrak{h}$ was just required to be abelian (i.e., $[\![\mathfrak{h}, \mathfrak{h}]\!] = 0$) rather than left-central. The next result indicates that our slightly stronger condition is actually more natural, and as we will see in § 8.5.2 below, not restrictive.

Proposition 8.15. *There is a canonical one-to-one correspondence between exact DGLAs of degree 2 and exact Courant algebras.*

Proof. Let $\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}$ be a DGLA of degree 2, given in terms of the data in (**B**) of §8.4, that we assume to be exact. We immediately obtain a surjective map $p := \delta\colon \mathfrak{a} \to \mathfrak{g}$. Define a bracket on $\mathfrak{a}$ by the derived bracket formula

$$[\![a_1, a_2]\!] = [\delta a_1, a_2] = \delta a_1 \cdot a_2. \tag{8.14}$$

The fact that $\delta\colon \mathfrak{a} \to \mathfrak{g}$ is a morphism of $\mathfrak{g}$-modules implies that

$$p([\![a_1, a_2]\!]) = \delta(\delta a_1 \cdot a_2) = \delta a_1 \cdot \delta a_2 = [p(a_1), p(a_2)].$$

Using this last condition and that $\mathfrak{a}$ is a $\mathfrak{g}$-module, we see that

$$\begin{aligned} [\![a_1, [\![a_2, a_3]\!]]\!] &= \delta a_1 \cdot (\delta a_2 \cdot a_3) = [\delta a_1, \delta a_2] \cdot a_3 + \delta a_2 \cdot (\delta a_1 \cdot a_3) \\ &= (\delta(\delta a_1 \cdot a_2)) \cdot a_3 + \delta a_2 \cdot (\delta a_1 \cdot a_3) \\ &= [\![[\![a_1, a_3]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!]. \end{aligned}$$

So $[\![\cdot,\cdot]\!]$ satisfies both conditions displayed in Def. 8.14. Note that $[\![a_1, \cdot]\!] = 0$ whenever $\delta a_1 = p(a_1) = 0$, hence $\ker(p) \cong \mathfrak{h}$ is left-central. So $p\colon \mathfrak{a} \to \mathfrak{g}$ and the bracket (8.14) make $\mathfrak{a}$ into an exact Courant algebra over $\mathfrak{g}$.

Conversely, consider an exact Courant algebra $p\colon \mathfrak{a} \to \mathfrak{g}$, with bracket $[\![\cdot,\cdot]\!]$ on $\mathfrak{a}$ and $\mathfrak{h} = \ker(p)$. We can now produce the data described in **(B)** of § 8.4 as follows. Take $\mathfrak{g}$ with the given Lie-algebra structure. Then $\mathfrak{a}$ acquires a $\mathfrak{g}$-module structure via

$$u \cdot a := [\![\widehat{u}, a]\!],$$

where $\widehat{u} \in \mathfrak{a}$ is such that $p(\widehat{u}) = u$. (This is well-defined since $\mathfrak{h}$ is left-central; the two conditions in Def. 8.14 ensure that it is a $\mathfrak{g}$-module.) Note that $\mathfrak{h}$ is a $\mathfrak{g}$-submodule of $\mathfrak{a}$. We finally define the symmetric bilinear map $\varpi\colon \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{h}$ by

$$\varpi(a_1, a_2) := [\![a_1, a_2]\!] + [\![a_2, a_1]\!] = p(a_1) \cdot a_2 + p(a_2) \cdot a_1. \tag{8.15}$$

(The fact that ϖ takes values in $\mathfrak{h}$ follows from $p\colon \mathfrak{a} \to \mathfrak{h}$ being bracket preserving.) Using the first condition in Def. 8.14, one can check that ϖ is $\mathfrak{g}$-equivariant. The operators δ in (8.8) are defined as the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{a}$ and projection $p\colon \mathfrak{a} \to \mathfrak{g}$. The conditions in (8.9) are both satisfied: the first is just by the definition of ϖ above, and the second follows from $\mathfrak{h}$ being left central. By Propositions 8.1 and 8.2, $\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}$ becomes a DGLA of degree 2, with bracket given by

$$[h_1 + a_1 + u_1, h_2 + a_2 + u_2] =$$
$$= ([\![\widehat{u}_1, h_2]\!] + [\![a_1, a_2]\!] + [\![a_2, a_1]\!] - [\![\widehat{u}_2, h_1]\!]) + ([\![\widehat{u}_1, a_2]\!] - [\![\widehat{u}_2, a_1]\!]) + [u_1, u_2],$$

where $\widehat{u}_i \in \mathfrak{a}$ are such that $p(\widehat{u}_i) = u_i$. □

8.5.2. *Infinitesimal hamiltonian actions and reduction data.* We now see how the data for an infinitesimal hamiltonian action on a Courant algebroid, described in (**C**) of § 8.4, can be simplified if we assume the DGLA to be exact.

Proposition 8.16. *A hamiltonian action of an exact DGLA of degree 2, with corresponding Courant algebra $p\colon \mathfrak{a} \to \mathfrak{g}$, on a Courant algebroid $E \to M$ is equivalent to:*

- *a bracket preserving map $\varrho\colon \mathfrak{a} \to \Gamma(E)$, such that $[\![\varrho(\mathfrak{h}), \cdot]\!] = 0$,*
- *a $\mathfrak{g}$-equivariant map $\mu\colon M \to \mathfrak{h}^*$ such that*
  - *(i) $\varrho(h) = \rho^*(d(\mu^* h))$, for all $h \in \mathfrak{h}$,*
  - *(ii) $\mu^*([\![a, a]\!]) = \frac{1}{2}\langle \varrho(a), \varrho(a)\rangle$, for all $a \in \mathfrak{a}$.*

Before proving the proposition, we need to explain the equivariance condition on μ. Note that the condition $[\![\varrho(\mathfrak{h}), \cdot]\!] = 0$ implies that $\rho(\varrho(\mathfrak{h})) = 0$ (by the Leibniz identity (C2) of Courant brackets), so $\rho \circ \varrho$ induces a Lie-algebra map $\mathfrak{g} = \mathfrak{a}/\mathfrak{h} \to \Gamma(TM)$, i.e., a $\mathfrak{g}$-action on M. The equivariance of μ is with respect to this action.

Proof. Assume that (8.8) defines a short exact sequence. We must check that the data in (**C**) of § 8.4, given by

- a Lie algebra morphism $\varphi\colon \mathfrak{g} \to \Gamma(\mathbb{A}_E)$,
- $\mathfrak{g}$-equivariant maps $\varrho\colon \mathfrak{a} \to \Gamma(E)$ and $\mu\colon M \to \mathfrak{h}^*$

and satisfying conditions (8.10) and (8.11), reduce to the data in this proposition.

By exactness, the second condition in (8.11) says that $[\![\varrho(\mathfrak{h}), \cdot]\!] = 0$ and that φ is completely determined by ϱ via

$$\varphi(u) = [\![\varrho(\widehat{u}), \cdot]\!], \tag{8.16}$$

where $\widehat{u} \in \mathfrak{a}$ satisfies $\delta\widehat{u} = p(\widehat{u}) = u$. In particular, the third condition in (8.11) is automatically satisfied (by axiom (C1) of Courant brackets). By (8.14) and (8.16), we see that in terms of the Courant algebra $\mathfrak{a} \to \mathfrak{g}$, the $\mathfrak{g}$-equivariance of $\varrho\colon \mathfrak{a} \to \Gamma(E)$ is equivalent to ϱ being bracket preserving. Further, (8.10) can be writtten as the second item of the proposition, while the first item agrees with the first condition in (8.11). This shows that the maps ϱ and μ in (**C**) satisfy the conditions stated in the proposition. On the other hand, we also see that starting with the data in this proposition, by setting $\varphi\colon \mathfrak{g} \to \Gamma(\mathbb{A}_E)$ as in (8.16) we obtain the data as in (**C**), yielding the claimed equivalence. □

To obtain a further simplification of the setup, we will consider the following concrete Courant algebra, which is the main example in [BCG07, BCG08].

Example 8.17 (Hemisemidirect product). Given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $\mathfrak{h}$, there is a natural exact Courant algebra structure on $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ with $p\colon \mathfrak{a} \to \mathfrak{g}$ the canonical projection and bracket defined by

$$[\![(u_1, h_1), (u_2, h_2)]\!] = ([u_1, u_2], u_1 \cdot h_2).$$

The previous bracket is the hemisemidirect product of $\mathfrak{g}$ and $\mathfrak{h}$, first considered in [KW01, Example 2.2]. The corresponding exact DGLA $\widetilde{\mathfrak{g}} = \mathfrak{h}[2] \oplus \mathfrak{a}[1] \oplus \mathfrak{g}$ is determined by the $\mathfrak{g}$-module structure on $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ given by $u\cdot(u_1, h_1) = ([u, u_1], u\cdot h_1)$ and

$$\varpi((u_1, h_1), (u_2, h_2)) = u_1 \cdot h_2 + u_2 \cdot h_1.$$

$\diamond$

Suppose that the Courant algebroid $E \to M$ has the additional property that $\rho^* d(C^\infty(M)) \subseteq \Gamma(E)$ is left central, i.e.,

$$[\![\rho^* d(C^\infty(M)), \cdot]\!] = 0. \tag{8.17}$$

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a $\mathfrak{g}$-module.

Definition 8.18. A pair $(\widetilde{\psi}, \mu)$ consisting of

- a bracket preserving map $\widetilde{\psi}\colon \mathfrak{g} \to \Gamma(E)$ with isotropic image, i.e., $\left\langle \widetilde{\psi}(u), \widetilde{\psi}(v) \right\rangle = 0$ for all $u, v \in \mathfrak{g}$,
- a $\mathfrak{g}$-equivariant map $\mu\colon M \to \mathfrak{h}^*$, where M is equipped with the action
$$\psi = \rho \circ \widetilde{\psi}\colon \mathfrak{g} \to \Gamma(TM),$$

is called *reduction data* for E.

The previous definition extends the notion of reduction data in [BCG08, § 3.2].

The importance of reduction data is that they give rise to hamiltonian actions on Courant algebroids. Let $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ be the Courant algebra of Example 8.17.

Proposition 8.19. *Let E be a Courant algebroid satisfying* (8.17) *and equipped with reduction data $(\widetilde{\psi}, \mu)$. Then $\varrho\colon \mathfrak{a} \to \Gamma(E)$,*

$$\varrho(u, h) := \widetilde{\psi}(u) + \rho^* d(\mu^* h),$$

and μ define a hamiltonian action of $\mathfrak{a}$ on E.

Proof. We must verify that the conditions in Prop. 8.16 hold. It is clear that $\varrho(0, h) = \rho^* d(\mu^* h)$ for all $h \in \mathfrak{h}$. By (8.17), it follows that $[\![\varrho(\mathfrak{h}), \cdot]\!] = 0$. To check that ϱ is bracket preserving, notice that

$$\begin{aligned}
[\![\varrho(u_1, h_1), \varrho(u_2, h_2)]\!] &= [\![\widetilde{\psi}(u_1) + \rho^* d(\mu^* h_1), \widetilde{\psi}(u_2) + \rho^* d(\mu^* h_2)]\!] \\
&= \widetilde{\psi}([u_1, u_2]) + \rho^* d(\pounds_{\psi(u_1)} \mu^* h_2) \\
&= \widetilde{\psi}([u_1, u_2]) + \rho^* d(\mu^*(u_1 \cdot h_2)) \\
&= \varrho([\![(u_1, h_1), (u_2, h_2)]\!]),
\end{aligned}$$

where we used that $\widetilde{\psi}$ preserves brackets as well as axiom (C5) of Courant algebroids in the second equality, and the equivariance of μ in the third equality. It remains to verify that

$$\mu^*([\![(u, h), (u, h)]\!]) = \frac{1}{2} \langle \varrho(u, h), \varrho(u, h) \rangle$$

for all $(u,h) \in \mathfrak{g} \oplus \mathfrak{h}$. By equivariance, the left-hand side equals

$$\mu^*(u \cdot h) = \pounds_{\psi(u)}\mu^* h.$$

On the other hand, we have

$$\langle \varrho(u,h), \varrho(u,h)\rangle = 2\Big\langle \widetilde{\psi}(u), \rho^* d(\mu^* h)\Big\rangle = 2\pounds_{\psi(u)}\mu^* h,$$

where we have used that $\widetilde{\psi}$ has isotropic image and that $\rho \circ \rho^* = 0$ on Courant algebroids. This completes the proof. □

A class of Courant algebroids satisfying (8.17) can be obtained from Lie algebroids, as explained in Example 6.2. Recall that for a Lie algebroid $A \to M$, with anchor ρ_A, bracket $[\cdot,\cdot]_A$, and differential d_A, and any d_A-closed element $\chi \in \Gamma(\wedge^3 A^*)$, we have a Courant algebroid structure on the pseudo-euclidean vector bundle $E = A \oplus A^*$ with anchor $\rho(a,\xi) = \rho_A(a)$ and bracket

$$[\![(a_1,\xi_1),(a_2,\xi_2)]\!] = ([a_1,a_1], \pounds_{a_1}\xi_2 - i_{a_2} d_A\xi_1 + i_{a_2} i_{a_1}\chi).$$

It is a direct verification that (8.17) holds for such Courant algebroids. Particular examples are given by doubles of Lie algebroids (when $\chi = 0$) and by exact Courant algebroids (when $A = TM$), see Example 6.3; in this last setting, the previous proposition recovers [BCG08, Prop. 2.3].

Example 8.20. When $E = A \oplus A^*$ is the double Courant algebroid of a Lie algebroid $A \to M$, reduction data for E amount to

(1) a Lie algebra map $\eta\colon \mathfrak{g} \to \Gamma(A)$,
(2) a linear map $\nu\colon \mathfrak{g} \to \Gamma(A^*)$ such that, for $u, v \in \mathfrak{g}$,
  (a) $\nu([u,v]) = \pounds_{\eta(u)}\nu(v) - i_{\eta(v)} d_A \nu(u)$,
  (b) $i_{\eta(u)}\nu(v) = -i_{\eta(v)}\nu(u)$,
(3) a $\mathfrak{g}$-equivariant map $\mu\colon M \to \mathfrak{h}^*$, where M carries the action $\rho_A \circ \eta\colon \mathfrak{g} \to \Gamma(TM)$.

Here η and ν are components of a map $\widetilde{\psi}\colon \mathfrak{g} \to \Gamma(E)$, $\widetilde{\psi}(u) = \eta(u) + \nu(u)$, which is bracket preserving with isotropic image, by (a) and (b) in (2) above. According to Prop. 8.19, η, ν and μ yield a hamiltonian action of the hemisemidirect product Courant algebra $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ on E.

A very special case is when $\nu = 0$ and $\mathfrak{h} = 0$, so that the only non-trivial map is the Lie algebra map $\eta\colon \mathfrak{g} \to \Gamma(A)$. This defines a (inner) $\mathfrak{g}$-action on A by $u \mapsto [u,\cdot]$ lifting the $\mathfrak{g}$-action on M. Suppose that the $\mathfrak{g}$-action on A integrates to a global action of a connected Lie group G, and that the G-action on M is free and proper. The map $\mathfrak{g} \times M \to A$, $(u,x) \mapsto \eta(u)|_x$ is injective (by freeness of the action) and its image, denoted by $\eta(\mathfrak{g}) \subseteq A$, is a G-invariant subbundle. Then the quotient of $A/\eta(\mathfrak{g})$ by the G-action defines a reduced Lie algebroid A_{red} over M/G, as considered, e.g., in [MPRO12, Thm. 3.6] (see also [BIPL, § 2.2] and references therein). In this case, the hamiltonian reduction of the Courant algebroid $E = A \oplus A^*$ (as described in §8.4) is $E_{red} = A_{red} \oplus A^*_{red}$, the double of the reduced Lie algebroid A_{red}. ⋄

The next remark describes the graded hamiltonian action (in the sense of § 8.2) corresponding to the previous example. It is a refinement of a cotangent lifted action on $T^*[2]A[1]$.

Remark 8.21 (Graded cotangent bundle reduction). We will be concerned with the graded counterpart of the following construction in ordinary symplectic geometry. Let N be a manifold and $\kappa\colon T^*N \to N$ its cotangent bundle, endowed with its canonical symplectic structure. We regard $C^\infty(N)$ as an abelian Lie subalgebra of $C^\infty(T^*N)$ via κ^*. By identifying vector fields on N with linear functions on T^*N, we view $\mathfrak{X}(N) \subseteq C^\infty(T^*N)$ as a Lie subalgebra. Hence any Lie algebra map $\mathfrak{g} \to \mathfrak{X}(N)$ may be seen as a Lie algebra map into $C^\infty(T^*N)$, we see that any $\mathfrak{g}$-action on N automatically defines a hamiltonian $\mathfrak{g}$-action on T^*N, known as its *cotangent lift*. A natural way to obtain more general hamiltonian actions on T^*N is by combining cotangent lifts with "fiber translations". Consider

(i) a Lie algebra map $\eta\colon \mathfrak{g} \to \mathfrak{X}(N) \subseteq C^\infty(T^*N)$;
(ii) a linear map $\nu\colon \mathfrak{g} \to C^\infty(N)$ satisfying the cocycle condition

$$\nu([u_1, u_2]) = \pounds_{\eta(u_1)}(\nu(u_2)) - \pounds_{\eta(u_2)}(\nu(u_1)), \quad \forall u_1, u_2 \in \mathfrak{g}$$

(this condition is saying that $\eta + \kappa^*\nu\colon \mathfrak{g} \to C^\infty(T^*N)$ is a Lie algebra map);
(iii) a $\mathfrak{g}$-module $\mathfrak{h}$ and a $\mathfrak{g}$-equivariant map $\mu\colon N \to \mathfrak{h}^*$, with dual map $\mu^* : \mathfrak{h} \to C^\infty(N)$.

Let $\mathfrak{g} \ltimes \mathfrak{h}$ denote the semidirect product Lie algebra. Then

$$\widetilde{\mu}\colon \mathfrak{g} \ltimes \mathfrak{h} \to C^\infty(T^*N), \ \ (u, w) \to \big(\eta(u) + \kappa^*(\nu(u))\big) + \kappa^*(\mu^* w) \tag{8.18}$$

is a Lie algebra map, defining a hamiltonian $\mathfrak{g} \ltimes \mathfrak{h}$-action on T^*N. We now consider the analogous construction in the graded context.

Let $A[1]$ be a degree 1 $\mathbb{N}$-manifold and $\mathcal{M} = T^*[2]A[1]$. Assume that $A[1]$ is equipped with a Q-structure, see Remark 2.16, so that $A \to M$ is a Lie algebroid. As seen in Example 6.2, the Courant function on $\mathcal{M}$ defined by Q via (6.3) corresponds to the Courant algebroid $A \oplus A^*$, the double of A.

Let $\mathfrak{g}$ be a Lie algebra and consider the DGLA $\mathfrak{g}[1] \oplus \mathfrak{g}$ with differential given by the identity, and graded Lie bracket given by the one of $\mathfrak{g}$ and its adjoint representation. For an element $u \in \mathfrak{g}$, we write $u[1]$ for the same element in $\mathfrak{g}[1]$ (with this notation, the differential is given by $u[1] \mapsto u$). Any DGLA morphism $\mathfrak{g}[1] \oplus \mathfrak{g} \to \mathfrak{X}(A[1])$ (where the differential in $\mathfrak{X}(A[1])$ is $[Q, \cdot]$) is determined by its restriction $\eta\colon \mathfrak{g}[1] \to \mathfrak{X}(A[1])_{-1} = \Gamma(A)$ via

$$(w, u) \mapsto \eta(w) + [Q, \eta(u[1])].$$

Since the natural embedding $\mathfrak{X}(A[1]) \hookrightarrow C(\mathcal{M})[2]$ in (6.3) is a DGLA morphism, we conclude that a Lie algebra map $\eta\colon \mathfrak{g} \to \Gamma(A)$ yields a DGLA morphism

$$\overline{\eta}\colon \mathfrak{g}[1] \oplus \mathfrak{g} \to C(\mathcal{M})[2],$$

thereby defining a hamiltonian $\mathfrak{g}[1] \oplus \mathfrak{g}$-action on $\mathcal{M}$ (the "cotangent lift" of the given $\mathfrak{g}[1] \oplus \mathfrak{g}$-action on $A[1]$).

For the analogue of (ii) above, we consider a map of complexes $\mathfrak{g}[1] \oplus \mathfrak{g} \to C(A[1])[2]$ (the latter is endowed with the differential $\pounds_Q$); as before, such a map is determined by its restriction $\nu\colon \mathfrak{g}[1] \to C(A[1])_1 = \Gamma(A^*)$ via

$$(w, u) \mapsto \nu(w) + \pounds_Q \nu(u[1]).$$

Since the embedding $\kappa^\sharp\colon C(A[1]) \to C(\mathcal{M})$ is a chain map (recall that $\kappa^\sharp$ was defined in Example 3.7), we see that any linear map $\nu\colon \mathfrak{g} \to \Gamma(A^*)$ yields a map of complexes

$$\overline{\nu}\colon \mathfrak{g}[1] \oplus \mathfrak{g} \to C(\mathcal{M})[2].$$

The analogue of the cocycle condition in (ii) (making $\overline{\eta}+\overline{\nu}\colon \mathfrak{g}[1]\oplus\mathfrak{g}\to C(\mathcal{M})[2]$ into a DGLA morphism) is

$$\nu([u,w]) = \pounds_{[Q,\eta(u[1])]}\nu(w) - \pounds_{\eta(w)}(\pounds_Q \nu(u[1])), \tag{8.19}$$

$$\pounds_{\eta(w_1)}\nu(w_2) = -\pounds_{\eta(w_2)}\nu(w_1), \tag{8.20}$$

for all $w, w_1, w_2 \in \mathfrak{g}[1]$, and $u \in \mathfrak{g}$.

For the analogue of (iii) above, let $\mathfrak{h}$ be a $\mathfrak{g}$-module. The complex $\mathfrak{h}[2]\oplus\mathfrak{h}[1]$, with the identity map as differential, then carries a natural $\mathfrak{g}[1]\oplus\mathfrak{g}$-module structure. A chain map $\mathfrak{h}[2]\oplus\mathfrak{h}[1]\to C(A[1])[2]$ is determined by its restriction to $\mathfrak{h}[2]$, which is a linear map $\mu^*\colon \mathfrak{h}\to C^\infty(M)$. We assume that the chain map is $\mathfrak{g}[1]\oplus\mathfrak{g}$-equivariant, which amounts to the $\mathfrak{g}$-equivariance of $\mu\colon M\to\mathfrak{h}^*$, the map dual to μ^* (here M carries the action given by $\rho_A\circ\eta$). The semidirect product $\widetilde{\mathfrak{g}} = (\mathfrak{g}[1]\oplus\mathfrak{g})\ltimes(\mathfrak{h}[2]\oplus\mathfrak{h}[1])$ can be checked to agree with the (exact) DGLA of Example 8.17, corresponding to the hemisemidirect product of $\mathfrak{g}$ and $\mathfrak{h}$. In analogy with (8.18), the following map can be shown to be a morphism of DGLAs:

$$\begin{aligned}\widetilde{\mu}\colon \mathfrak{h}[2]\oplus(\mathfrak{h}[1]\oplus\mathfrak{g}[1])\oplus\mathfrak{g} &\to C(\mathcal{M})[2]\\ (h_a, h_b, w, u) &\mapsto (\overline{\eta}+\overline{\nu})(w,u) + \kappa^\sharp(\mu^* h_a) + \kappa^\sharp(\pounds_Q(\mu^*(h_b[1]))).\end{aligned}$$

Hence the map $\widetilde{\mu}$ satisfies (8.5) and yields a hamiltonian $\widetilde{\mathfrak{g}}$-action on $\mathcal{M} = T^*[2]A[1]$ for which the Courant function Q is reducible with respect to $\widetilde{\mu}^{-1}(0)$ (see § 8.2). It is a direct verification that (8.19) and (8.20) translate into equations (a) and (b) in (2) of Example 8.20, and this hamiltonian $\widetilde{\mathfrak{g}}$-action on $T^*[2]A[1]$ is precisely the one corresponding to the hamiltonian action on the Courant algebroid $A\oplus A^*$ described in that example. ⋄

8.5.3. *Extended actions on exact Courant algebroids.* We now explain how our general reduction setup relates to the framework of Courant reduction in [BCG07], which is formulated in terms of a suitable notion of action of an exact Courant algebra on an *exact* Courant algebroid.

Let E be an exact Courant algebroid. Recall that, in this case, the anchor $\rho\colon E\to TM$ defines an inclusion

$$\rho^*\colon \Omega^1(M)\hookrightarrow\Gamma(E).$$

The next definition is found in [BCG07, § 2.2 and 2.3] and [BCG08, § 2.3 and 2.4].

Definition 8.22. An *extended action* of an exact Courant algebra $\mathfrak{a}\to\mathfrak{g}$ on an exact Courant algebroid $E\to M$ is given by bracket-preserving maps Ψ and ψ making the following diagram commute,

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & \mathfrak{g} \\ \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\psi} \\ \Gamma(E) & \longrightarrow & \Gamma(TM), \end{array}$$

and such that $\Psi(\mathfrak{h})\subseteq\Omega^1_{cl}(M)$, where $\mathfrak{h} = \ker(\mathfrak{a}\to\mathfrak{g})$. A *momentum map* for the extended action is a $\mathfrak{g}$-equivariant map $\mu\colon M\to\mathfrak{h}^*$ such that $\Psi(h) = \rho^*(d(\mu^* h))$, for all $h\in\mathfrak{h}$.

Note that, in the previous definition, ψ is completely determined by Ψ via $\psi(u) = \rho(\Psi(\widehat{u}))$, where $\widehat{u}\in\mathfrak{a}$ is any lift of $u\in\mathfrak{g}$ (this is well defined since $\rho(\Psi(\mathfrak{h})) = 0$). So an extended action with momentum map is simply given by

- a bracket-preserving map $\Psi\colon \mathfrak{a} \to \Gamma(E)$, and
- a $\mathfrak{g}$-equivariant map $\mu\colon M \to \mathfrak{h}^*$ such that $\Psi(h) = \rho^*(d(\mu^* h))$, for all $h \in \mathfrak{h}$.

By comparison with Prop. 8.16, we immediately obtain

Proposition 8.23. *Hamiltonian actions of an exact DGLA of degree 2, with corresponding Courant algebra $\mathfrak{a} \to \mathfrak{g}$, on an exact Courant algebroid E are equivalent to extended actions $\Psi\colon \mathfrak{a} \to \Gamma(E)$ with momentum map $\mu\colon M \to \mathfrak{h}^*$ satisfying the additional property that*

$$\mu^*([\![a,a]\!]) = \frac{1}{2}\langle \Psi(a), \Psi(a)\rangle, \qquad \forall a \in \mathfrak{a}. \tag{8.21}$$

With the notation of Prop. 8.16, we just set $\Psi = \varrho$, noticing that the condition $[\![\Psi(\mathfrak{h}), \cdot]\!] = 0$ is automatic since $\Psi(\mathfrak{h}) \subseteq \rho^* d(C^\infty(M))$ and the Courant algebroid E is assumed to be exact.

Remark 8.24. For an extended action $\Psi\colon \mathfrak{a} \to \Gamma(E)$ with momentum map $\mu\colon M \to \mathfrak{h}^*$, let $K \subseteq E|_{\mu^{-1}(0)}$ be as in (8.13): for each $x \in \mu^{-1}(0)$, $K|_x$ is the image of the map $a \mapsto \Psi(a)|_x$. Then condition (8.21) admits an alternative formulation in terms of K being isotropic. Indeed, it is immediate that (8.21) implies that K is isotropic. On the other hand, assuming that K is isotropic, (8.21) holds at all $x \in \mu^{-1}(0)$. But, by Remark 8.9, part (*ii*), and the fact that the anchor ρ is surjective (since E is exact), we have that

$$d(\mu^*([\![a,a]\!])) = \frac{1}{2} d\langle \Psi(a), \Psi(a)\rangle, \qquad \forall a \in \mathfrak{a},$$

which implies that condition (8.21) holds at every connected component of M that contains a point in $\mu^{-1}(0)$. ◇

Although (8.21) is not part of the original definition of momentum maps for extended actions in [BCG08, § 2.4], the associated reduction procedure (see [BCG08, § 3.2]) includes the extra assumption that $K \subseteq E|_{\mu^{-1}(0)}$ be isotropic. As explained in the previous remark, when considering Courant reduction at the value $0 \in \mathfrak{h}^*$, it makes no difference which of the two assumptions is made.

In conclusion, the Courant reductions for extended actions with momentum maps of [BCG07, BCG08] are special cases of the setup in § 8.4, namely the cases in which both the DGLA and the Courant algebroid are exact. When restricted to this context, the reduction of Courant algebroids, Dirac and GC structures in Theorem 8.10 coincide with the reduction schemes described in [BCG08, § 3 and § 4], following [BCG07]. (In this special setup, one can directly check that $\rho(K)$ agrees with the $\mathfrak{g}$-orbits on $\mu^{-1}(0)$, and $\rho(K^\perp) = T\mu^{-1}(0)$, so reduced Courant algebroids are again exact, see Remark 8.11.)

Back to the graded-geometric perspective, the reduction of exact Courant algebroids in [BCG08] is nothing but usual symplectic reduction for cotangent bundles $T^*[2]T[1]M$, equipped with a suitable Courant function, with respect to a hamiltonian action of an exact DGLA of degree 2 (cf. Remark 8.21).

Remark 8.25. More general reduction setups, not requiring that K is isotropic, can be found in [BCG07]. For their graded geometric descriptions, one must consider symplectic reduction by submanifolds which are not necessarily coisotropic, such as arbitrary momentum levels in the case of hamiltonian actions.

Appendix A. Atiyah algebroids and quotients

We will consider two types of quotient constructions for (pseudo-euclidean) vector bundles and discuss their effects on the corresponding Atiyah algebroids.

A.1. **Pseudo-euclidean reduction.** Let $E \to N$ be a pseudo-euclidean vector bundle, and $K \to N$ be an isotropic subbundle. Then $E_{quot} := K^\perp/K$ is also a pseudo-euclidean vector bundle over N. We will collect in this appendix some results relating the Atiyah algebroids $\mathbb{A}_E$ and $\mathbb{A}_{E_{quot}}$. (Comparing to the setup and notation in the paper, the pseudo-euclidean vector bundle E in this appendix is to be thought of as the restriction of a pseudo-euclidean vector bundle on a manifold M to a submanifold $N \subseteq M$.)

We will also consider a lagrangian subbundle $L \subset E$. Assuming that $L \cap K$ has constant rank (which is equivalent to $L \cap K^\perp$ having constant rank), then $L_{quot} = (L \cap K^\perp + K)/K$ is a lagrangian subbundle of E_{quot}.

We will make use of the following facts from linear algebra.

Lemma A.1. *Let $E \to N$ be a pseudo-euclidean vector bundle, and $K \to N$ be an isotropic subbundle.*

(i) *There is an identification*

$$E \cong R \oplus (K \oplus K^*)$$

that preserves K, where the right-hand side is the direct sum of the pseudo-euclidean vector bundles R and $(K \oplus K^*)$.

(ii) *Given a lagrangian subbundle $L \subseteq E$ such that $L \cap K$ has constant rank, one can choose the identification in (i) such that $L = (L \cap R) \oplus (L \cap K) \oplus (L \cap K^*)$.*

Proof. One can always find a subbundle T that is an isotropic complement for $K^\perp$ in E. Then $R := (T \oplus K)^\perp \subset K^\perp$ is a complement for K in $K^\perp$, $K^\perp = K \oplus R$, and the pairing $\langle \cdot, \cdot \rangle$ on E is non-degenerate on R. Hence the pairing on $R^\perp = K \oplus T$ is also non-degenerate, and K and T are lagrangian subbundles therein, so there is a natural identification $R^\perp = K \oplus K^*$. It follows that $E = R \oplus R^\perp = R \oplus (K \oplus K^*)$, proving (i).

Note that an isotropic complement T can be canonically constructed from any choice of complement $\widetilde{T}$ of $K^\perp$ in E following a procedure analogous to the one described in [CdS01, Proposition 8.2] in symplectic linear algebra. Fix now a lagrangian subbundle L in E, and take a subbundle I such that $(L \cap K^\perp) \oplus I = L$. Then starting from a complement $\widetilde{T}$ of $K^\perp$ so that $I \subset \widetilde{T} \subset I^\perp$, we obtain from the aforementioned procedure an isotropic complement T of $K^\perp$ in E such that $L = (L \cap K^\perp) \oplus (L \cap T)$. Moreover, for $R = (T \oplus K)^\perp$ (which, as we saw, satisfies $K^\perp = R \oplus K$), we have that $L \cap K^\perp = (L \cap R) \oplus (L \cap K)$, proving that (ii) holds. □

In particular, in the decomposition of E in (i) of the previous lemma, the factor R is isomorphic to the quotient $E_{quot} = K^\perp/K$ as pseudo-euclidean bundles. Moreover, if (ii) holds, then $L \cap R$ is isomorphic to the lagrangian subbundle L_{quot} in E_{quot}.

Consider the natural projection map $\Gamma_{K^\perp} \to \Gamma_{E_{quot}}$, denoted by $e \mapsto [e]$, and let $\Gamma^K_{\mathbb{A}_E}$ be the subsheaf of $\Gamma_{\mathbb{A}_E}$ given by metric-preserving derivations (Y, D) of E that satisfy $D(\Gamma_K) \subseteq \Gamma_K$ (and hence $D(\Gamma_{K^\perp}) \subseteq \Gamma_{K^\perp}$ since D preserves the metric).

Proposition A.2. *The map*

$$\Gamma^K_{\mathbb{A}_E} \to \Gamma_{\mathbb{A}_{E_{quot}}}, \quad (Y, D) \mapsto (Y, [D]), \tag{A.1}$$

where $[D]([e]) = [De]$, *for* e *a local section of* $\Gamma_{K^\perp}$, *is onto.*

Proof. We use the decomposition of E in Lemma A.1, part (i), so that $R \cong E_{quot}$ and

$$\Gamma_{\mathbb{A}_{E_{quot}}} \cong \Gamma_{\mathbb{A}_R}.$$

Fix $(Y, \Delta) \in \Gamma_{\mathbb{A}_R}(V)$, where $V \subseteq N$ is open. Let ∇ be any connection on the vector bundle $K \to N$, with dual connection ∇^* on $K^* \to N$. Then $\nabla_Y \colon \Gamma_K(V) \to \Gamma_K(V)$ and $\nabla^*_Y : \Gamma_{K^*}(V) \to \Gamma_{K^*}(V)$ are derivations of K and K^*, respectively, and

$$(Y, \nabla_Y \oplus \nabla^*_Y) \in \Gamma_{\mathbb{A}_{K \oplus K^*}}(V).$$

Letting $D := \Delta \oplus \nabla_Y \oplus \nabla^*_Y$, it follows that $(Y, D) \in \Gamma^K_{\mathbb{A}_E}(V)$ is such that $[D] = \Delta$. □

For an open subset $V \subseteq N$, note that $(Y, D) \in \Gamma^K_{\mathbb{A}_E}(V)$ is in the kernel of (A.1) if and only if $Y = 0$ and $D(\Gamma_{K^\perp}(V)) \subseteq \Gamma_K(V)$. In other words, the kernel of (A.1) is

$$\{T \in \Gamma_{\wedge^2 E}(V) \mid T(\Gamma_{K^\perp}(V)) \subseteq \Gamma_K(V)\} = \Gamma_{K \wedge E}(V).$$

So we have an exact sequence

$$0 \to \Gamma_{K \wedge E} \to \Gamma^K_{\mathbb{A}_E} \to \Gamma_{\mathbb{A}_{E_{quot}}} \to 0. \tag{A.2}$$

Given a lagrangian subbundle $L \subset E$ such that $L \cap K$ has constant rank, let $\Gamma^{K,L}_{\mathbb{A}_E}$ be the subsheaf of $\Gamma_{\mathbb{A}_E}$ defined by metric-preserving derivations (Y, D) of E satisfying $D(\Gamma_K) \subseteq \Gamma_K$ and $D(\Gamma_L) \subseteq \Gamma_L$. Let $\Gamma^{L_{quot}}_{\mathbb{A}_{E_{quot}}}$ be the subsheaf of $\Gamma_{\mathbb{A}_{E_{quot}}}$ given by metric-preserving derivations (Y, Δ) of E_{quot} such that $\Delta(\Gamma_{L_{quot}}) \subseteq \Gamma_{L_{quot}}$.

Proposition A.3. *The map* (A.1) *restricts to a map*

$$\Gamma^{K,L}_{\mathbb{A}_E} \to \Gamma^{L_{quot}}_{\mathbb{A}_{E_{quot}}} \tag{A.3}$$

that is onto.

Proof. We write E as in Lemma A.1, part (i), in such a way that L can be written as in part (ii). The identification $R \cong E_{quot}$ is such that $L \cap R \cong L_{quot}$. Given an open subset $V \subseteq N$, take $(Y, \Delta) \in \Gamma^{L_{quot}}_{\mathbb{A}_{E_{quot}}}(V)$, which can be seen as an element in $\Gamma_{\mathbb{A}_R}(V)$ preserving $\Gamma_{L \cap R}(V)$. Let ∇ be a connection on K preserving the subbundle $L \cap K$. Then the dual connection ∇^* on K^* automatically preserves $L \cap K^*$ (since $L \cap K^* \subseteq K^*$ coincides with the annihilator of $L \cap K \subseteq K$). Letting $D = (\Delta \oplus \nabla_Y \oplus \nabla^*_Y)$, we have that $(Y, D) \in \Gamma^{K,L}_{\mathbb{A}_E}$ and $[D] = \Delta$. □

A.2. **Quotients of vector bundles.** We will consider the setup of § 5. Let $A \to N$ be a vector bundle, let F be a simple distribution on N with leaf space $\underline{N}$ and projection $p \colon N \to \underline{N}$. Suppose that A is equipped with a flat F-connection ∇ with trivial holonomy. As recalled in Lemma 5.1, we have a quotient vector bundle $\underline{A} \to \underline{N}$, whose sheaf of sections $\Gamma_{\underline{A}}$ is naturally identified with $p_* \Gamma^{flat}_A$.

We denote by $\mathbb{A}_A$ the Atiyah algebroid of A, i.e., the vector bundle whose sections are the derivations of A. In this subsection we explain how the Atiyah algebroids of A and $\underline{A}$ are related, see Proposition A.7.

Remark A.4. The whole discussion that follows carries over to the case where A has an additional pseudo-euclidean structure, ∇ is metric preserving (so that $\underline{A}$ is also pseudo-euclidean), and $\mathbb{A}_A$ and $\mathbb{A}_{\underline{A}}$ are defined in terms of metric-preserving derivations.

Recall that the F-connection ∇ can be thought of as a vector-bundle map $\nabla\colon F \to \mathbb{A}_A$ such that $\sigma \circ \nabla = \mathrm{Id}_F$, where $\sigma\colon \mathbb{A}_A \to TN$ is the symbol map. Consider the vector bundle

$$\mathbb{A}_A^{\nabla} := \mathbb{A}_A/\nabla(F).$$

Then ∇ induces a natural flat F-connection $\overline{\nabla}$ on $\mathbb{A}_A^{\nabla} \to N$ via

$$\overline{\nabla}_Z(\overline{(Y,\Delta)}) = \overline{([Z,Y],[\nabla_Z,\Delta])}, \tag{A.4}$$

where Z is a section of F, and we denote the class of a section (Y,Δ) of $\mathbb{A}_A$ in $\mathbb{A}_A^{\nabla}$ by $\overline{(Y,\Delta)}$. The connection $\overline{\nabla}$ is well defined due to the flatness of ∇. Note also that the symbol map induces a map $\overline{\sigma}\colon \mathbb{A}_A^{\nabla} \to TN/F$ which relates the connection $\overline{\nabla}$ with the usual Bott connection ∇^{Bott} on TN/F via $\overline{\sigma} \circ \overline{\nabla}_Z = \nabla_Z^{Bott} \circ \overline{\sigma}$.

Lemma A.5. *A section $\overline{(Y,\Delta)}$ of $\Gamma_{\mathbb{A}_A^{\nabla}}$ is flat if and only if*

(a) *Y satisfies $[Y,\Gamma_F] \subseteq \Gamma_F$, and*
(b) *Δ satisfies $\Delta(\Gamma_A^{flat}) \subseteq \Gamma_A^{flat}$.*

Note that the first condition in the lemma just says that Y defines a ∇^{Bott}-flat section of TN/F.

We will give an alternative description of the sheaf of flat sections of $\mathbb{A}_A^{\nabla}$. Note that $\Gamma_{TN/F}^{flat}$ is a sheaf of $(C_N^\infty)_{bas}$-modules and that there is a natural action of $\Gamma_{TN/F}^{flat}$ on $(C_N^\infty)_{bas}$ by Lie derivatives. We will consider another sheaf of $(C_N^\infty)_{bas}$-modules on N, denoted by $\mathcal{A}$. Its sections are given, on each open subset $V \subseteq N$, by pairs (ξ,δ), where $\xi \in \Gamma_{TN/F}^{flat}(V)$ and $\delta\colon \Gamma_A^{flat}(V) \to \Gamma_A^{flat}(V)$ satisfy

$$\delta(fe) = \pounds_\xi(f)e + f\delta(e),$$

for $f \in (C_N^\infty)_{bas}(V)$ and $e \in \Gamma_A^{flat}(V)$.

Lemma A.6. *For each open $V \subseteq N$, the map*

$$\Gamma_{\mathbb{A}_A^{\nabla}}^{flat}(V) \to \mathcal{A}(V), \quad \overline{(Y,\Delta)} \mapsto (\overline{Y},\overline{\Delta}), \tag{A.5}$$

where $\overline{Y}$ is the projection of Y to $\Gamma_{TM/F}(V)$ and $\overline{\Delta}$ is the restriction of Δ to $\Gamma_A^{flat}(V)$, is an isomorphism.

Proof. The map (A.5) is well defined by Lemma A.5, and clearly injective. It remains to show that it is surjective. For a given section $(\xi,\delta) \in \mathcal{A}(V)$, one can pick $Y \in \Gamma_{TN}(V)$ such that $[Y,\Gamma_F(V)] \subseteq \Gamma_F(V)$ and $\xi = \overline{Y}$. Using the fact that A admits local frames of ∇-flat sections, one can directly check that the choice of Y uniquely determines an extension of δ to an element (Y,Δ) in $\Gamma_{\mathbb{A}_A}(V)$. (This extension is constructed using local flat frames applying the Leibniz rule, and is well defined since the transition matrix of two flat frames consists of basic functions; uniquess is immediate.) □

Since $\underline{A}$ is the quotient of A with respect to F and ∇, the pullback of sections gives us an isomorphism

$$p_1^\sharp\colon \Gamma_{\underline{A}} \xrightarrow{\sim} p_*\Gamma_A^{flat}. \tag{A.6}$$

Proposition A.7. *There is a natural identification*

$$\Gamma_{\mathbb{A}_{\underline{A}}} \cong p_* \Gamma^{flat}_{\mathbb{A}^{\nabla}_A}$$

defined as follows: a section $(\underline{Y}, \underline{\Delta})$ *of* $\Gamma_{\mathbb{A}_{\underline{A}}}$ *corresponds to a section* $\overline{(Y, \Delta)}$ *of* $p_* \Gamma^{flat}_{\mathbb{A}^{\nabla}_A}$ *if and only if* $\underline{Y} = p_* Y$ *and* $p_1^\sharp \circ \underline{\Delta} = \Delta \circ p_1^\sharp$.

Proof. We have an identification

$$p_* \Gamma^{flat}_{TN/F} \xrightarrow{\sim} \Gamma_{T\underline{N}} \tag{A.7}$$

through the natural projection map, taking a section $\xi = \overline{Y}$ to $\underline{Y} = p_*(Y)$, that satisfies the relation

$$p^*(\pounds_{\underline{Y}} f) = \pounds_Y (p^* f),$$

for any section f of $C^\infty_{\underline{N}}$. From the isomorphisms (A.6) and (A.7) we obtain an isomorphism

$$p_* \mathcal{A} \xrightarrow{\sim} \Gamma_{\mathbb{A}_{\underline{A}}},$$

given on sections by $(\overline{Y}, \delta) \mapsto (p_*(Y), \underline{\Delta})$, where $\underline{\Delta}$ and δ are related by $p_1^\sharp \circ \underline{\Delta} = \delta \circ p_1^\sharp$. Composing this last isomorphism with the one in Lemma A.6, we obtain the identification in the statement. □

With the previous proposition, one can in fact verify that the quotient of $\mathbb{A}^{\nabla}_A$ with respect to F and $\overline{\nabla}$ is $\mathbb{A}_{\underline{A}}$.

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