

CONSTRAINED SYSTEMS, GENERALIZED HAMILTON–JACOBI ACTIONS, AND QUANTIZATION

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In memory of Kirill Mackenzie

ABSTRACT. Mechanical systems (i.e., one-dimensional field theories) with constraints are the focus of this paper. In the classical theory, systems with infinite-dimensional targets are considered as well (this then encompasses also higher-dimensional field theories in the hamiltonian formalism). The properties of the Hamilton–Jacobi (HJ) action are described in details and several examples are explicitly computed (including nonabelian Chern–Simons theory, where the HJ action turns out to be the gauged Wess–Zumino–Witten action). Perturbative quantization, limited in this note to finite-dimensional targets, is performed in the framework of the Batalin–Vilkovisky (BV) formalism in the bulk and of the Batalin–Fradkin–Vilkovisky (BFV) formalism at the endpoints. As a sanity check of the method, it is proved that the semiclassical contribution of the physical part of the evolution operator is still given by the HJ action. Several examples are computed explicitly. In particular, it is shown that the toy model for nonabelian Chern–Simons theory and the toy model for 7D Chern–Simons theory with nonlinear Hitchin polarization do not have quantum corrections in the physical part (the extension of these results to the actual cases is discussed in the companion paper [21]). Background material for both the classical part (symplectic geometry, generalized generating functions, HJ actions, and the extension of these concepts to infinite-dimensional manifolds) and the quantum part (BV-BFV formalism) is provided.

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1. INTRODUCTION

The Hamilton–Jacobi (HJ) action for a nondegenerate system is a generating function for the graph of the hamiltonian flow of the system and is obtained by substituting a solution of the Euler–Lagrange (EL) equations into the action functional. We extend this result to degenerate systems (i.e., systems with constraints). More precisely, we consider one-dimensional field theories with constraints in involution (a.k.a. first-class constraints), i.e., theories constrained to live in a coisotropic submanifold of the (possibly infinite-dimensional) target. In the finite-dimensional case, we also study their perturbative quantization.

This paper is mainly based on a discussion of examples. The two main general results, on the role of the HJ action, are presented in Theorem 4.2 on page 21 at the classical level and in Theorem 11.4 on page 65 at the quantum level.

Among the many examples, we discuss in particular Chern–Simons theory and its finite-dimensional toy model versions, get the gauged WZW action as the generalized generating function of the evolution relation via the HJ action, and, in the finite-dimensional version, show that there are no quantum corrections in the physical sector. We will prove the latter result in the infinite-dimensional case in the companion paper [21].

We start with a recollection of the known results for a regular mechanical system (i.e., a one-dimensional field theory with regular lagrangian). In this case, the EL equations define a hamiltonian flow and therefore its graph, which we call the evolution relation, is a lagrangian submanifold of the product of the symplectic manifolds associated to the two endpoints (in the framework of this paper the symplectic manifold associated to an endpoint is the target symplectic manifold, up to the sign of the symplectic form which depends on the orientation of the endpoint). The HJ action, defined as the evaluation of the classical action on solutions of the EL equation for a given choice of endpoint polarizations¹ (boundary conditions), is then a generating function² for the evolution relation. At the quantum level, the HJ action appears as the dominant contribution in the semiclassical expansion of Feynman’s path integral (simply as the evaluation of the action on a background classical solution).

In a constrained system, the following differences occur:

- (1) In the framework of this paper,³ the EL equations split into two classes which we call the evolution equations (those that involve a time derivative)

¹By polarization we mean, in this paper, an explicit realization of a symplectic manifold as a cotangent bundle. Generating functions, wave functions and integral kernels of operators will always be understood as functions on the base of the cotangent bundle.

²A generating function for a lagrangian submanifold L of a cotangent bundle is a function on the base that yields L as the graph of its differential.

³with the exception of the first part of Section 7.4 which portrays a more general situation

and the constraints (those that do not involve time derivatives). The fields actually split into two classes: maps to the target symplectic manifolds and Lagrange multipliers; the evolution equations arise from variations with respect to the former and the constraints from variations with respect to the latter.

- (2) Solutions of the EL equations still define a lagrangian submanifold—which we keep calling the evolution relation—of the product of the symplectic manifolds associated to the two endpoints. Yet, this submanifold is no longer a graph.⁴
- (3) The HJ action, defined as the evaluation of the classical action on solutions of the evolution equation for a given choice of endpoint polarizations, is then a generalized generating function⁵ for the evolution relation. Note that the constraints are not used in the definition of the HJ action but are recovered from it (as they appear in the definition of the evolution relation).
- (4) The HJ action depends parametrically on the Lagrange multipliers but is invariant under a (Lie algebra or, more generally, Lie algebroid) gauge transformation thereof that is trivial at the endpoints.
- (5) At the quantum level, the path integral requires a gauge-fixing procedure one has to keep track of. The formalism we use in this paper is the one due to Batalin and Vilkovisky, in particular in its version coupled to the boundary, referred to as the BV-BFV formalism. The HJ action then appears as the physical (i.e., ghost independent) part of the dominant contribution of the perturbative expansion of the path integral for certain choices of endpoint polarizations and of residual fields (the latter being, in physical parlance, a choice of infrared components, not to be integrated out, of the fields). The HJ action now arises from the resummation of tree diagrams and not from the evaluation of the action on a background classical solution, which is not at hand in the BV-BFV formalism.

Points (3) and (4) are the content of Theorem 4.2 and point (5) of Theorem 11.4—see also equations (101) and (117).

One topic we are interested in is the change of (say, final) endpoint polarization. This is realized by a generating function of the corresponding symplectomorphism. At the classical level, the HJ action for the new polarization is simply the composition of the HJ action for the old polarization with the generating function for the change of polarization. At the quantum level, one has to find an appropriate quantization of the latter generating function. There are two, in general irreconcilable, wishes: unitarity and compatibility with the BV-BFV formalism. In this paper we focus on the latter and show that a BV-BFV quantization is possible if the constraints are linear in the momenta of both endpoint polarizations. This is a serious limitation, yet it applies to some very interesting examples.

1.1. Structure of the paper. This paper consists of three parts.

⁴Geometrically, the constraints define a coisotropic submanifold C , and the evolution equations describe a motion along a leaf of the characteristic distribution. The evolution relation then consists of pairs of points of C that lie on the same leaf.

⁵A generalized generating function is a generating function, see footnote 2, that depends on additional parameters. It defines the lagrangian submanifold of the given cotangent bundle as the intersection of the graph of its differential with respect to the base variables and the vanishing locus of the differential with respect to the additional parameters.

In the first part (Sections 2 to 6), we discuss the HJ action for constrained systems (after briefly reviewing the regular, unconstrained case). We discuss several examples (in particular constraints that are linear or affine in both the positions and the momenta), prove Theorem 4.2, i.e., point (2) of the above list, describe how to incorporate a nontrivial evolution in addition to the constraints, and extend the discussion of generating functions and HJ actions to some bad endpoint conditions.

In the second part (Section 7), we compute the HJ action for several examples with infinite-dimensional target (Chern–Simons theory, BF theory, 2D Yang–Mills theory, abelian Yang–Mills theory in any dimension).

In the third part (Sections 8 to 12), we discuss the BV-BFV quantization. We start with a review of BV and BFV, tailored to the theories we consider in this paper (which are ultimately one-dimensional AKSZ models). Next we discuss the quantization in the case of finite-dimensional targets, both with linear and nonlinear endpoint polarizations, proving in particular Theorem 11.4, i.e., point (5) of the above list, and show that in the linear and affine examples (the toy models for Chern–Simons theory) no quantum corrections arise in the physical sector: see equations (108), (109) and (118).

In Appendix A we review several results on symplectic geometry and on (generalized) generating functions, both in the finite- and in the infinite-dimensional case.

In Appendix B we prove a technical result that ensures that the BV-BFV quantization works in the case at hand.

We defer the BV-BFV quantization with infinite-dimensional target (mainly, Chern–Simons theories) to the companion paper [21].

1.2. How to read this paper. The first part is necessary for the whole paper, with the exception of Section 5 which is relevant only for Sections 7.4, 11.5 and 11.6.

The second and the third part can be read independently from each other, as they focus on different developments of the first part.

Appendix A contains additional material that completes the discussion in the first part and provides a rigorous foundation for the second part. Appendix B contains results for the foundation of the third part.

1.3. Notations. To simplify the reading of the paper, we collect here some of the main notations and some terminology.

Arguments (e.g., of an action, a HJ action, a partition function) are written in square brackets if they are fields (i.e., if they depend on the variable on the source interval) and in round brackets otherwise.

We will use the subscript HJ to denote a HJ action. We will have several versions thereof, with notations like S_{HJ} , S_{HJ}^f or \hat{S}_{HJ}^f referring to different incarnations (change of polarization by f ; dependency on gauge classes of the parameters). This pedantic notation is important to avoid confusion, but the only important point to focus on is that these are all HJ actions in their own way. In the arguments of HJ actions, or more generally of generalized generating functions, we will consistently separate the generalized position variables from the parameters by a semicolon.

In the BFV formalism, the classical master equation (CME) is the equation $\{S, S\} = 0$, where $\{, \}$ is the given even Poisson bracket and S is an odd functional. By Ω we will denote a coboundary operator that quantizes S .

In the BV formalism, the CME is the equation $(S, S) = 0$, where $(,)$ is the given odd Poisson bracket and S is an even functional. By Δ we will denote the BV Laplacian (on functions or on half-densities, depending on the context). The quantum master equation (QME) is the equation $\Delta\psi = 0$, where ψ is an even function or half-density. For ψ of the form $e^{\frac{i}{\hbar}S}$, under the condition $\Delta S = 0$ (which we usually assume), the QME for ψ is equivalent to the CME for S .

In the BV-BFV formalism, where we have both Δ and Ω , the modified quantum master equation (mQME) is the equation $(\Omega + \hbar^2\Delta)\psi = 0$. We often use the abbreviation $\Delta^\Omega := \Delta + \frac{1}{\hbar^2}\Omega$.

For a partition function (or a state), we will use the notation Z , possibly with some label denoting the case we are considering. If we define it as a half-density, we will use the notation \hat{Z} .

1.4. Teaser. We illustrate one of the results of the paper (summarizing the content of Sections 4.2, 11.3, and 12, with simplified, adapted notations for this introduction).

The data of the system are a quadratic Lie algebra \mathfrak{g} , a vector space Z and two derivations: $v: \mathfrak{g} \rightarrow \mathfrak{g} \otimes Z$ and $\bar{v}: \mathfrak{g} \rightarrow \mathfrak{g} \otimes Z^*$. The target symplectic manifold is $T^*(\mathfrak{g} \otimes Z)$ which we identify with $\mathfrak{g} \otimes Z^* \oplus \mathfrak{g} \otimes Z \ni (\bar{q}, q)$ (using the nondegenerate pairing $\langle \cdot, \cdot \rangle$ on \mathfrak{g}).⁶ The fields are a path $(\bar{q}, q): [0, 1] \rightarrow T^*(\mathfrak{g} \otimes Z)$ and a \mathfrak{g} -valued one-form e (i.e., a connection one-form) on $[0, 1]$. The action functional, a toy model for nonabelian Chern–Simons theory (see Section 7.2),⁷ is

$$S[\bar{q}, q, e] = \int_0^1 (\langle \bar{q}, d_e q \rangle - \langle \bar{q}, v(e) \rangle - \langle \bar{v}(e), q \rangle),$$

where d_e denotes the covariant derivative.

Under the assumption that the exponential map to the simply connected Lie group G integrating \mathfrak{g} is surjective, we compute the HJ action—for initial value q_a of q , final value \bar{q}^b of \bar{q} , and a parameter $g = e^\xi \in G$ —as

$$\hat{S}_{\text{HJ}}(q_a, \bar{q}^b; g) = -\langle \bar{q}^b, g^{-1}q_a g \rangle - \langle \bar{q}^b, g^{-1}v(g) \rangle - \langle \bar{v}(g)g^{-1}, q_a \rangle - \text{WZW}(g)$$

with the Wess–Zumino–Witten term

$$\text{WZW}(g) = \frac{1}{2} \langle \bar{v}(g)g^{-1}, v(g)g^{-1} \rangle + \frac{1}{2} \int_0^1 \langle \bar{v}(h)h^{-1}, [v(h)h^{-1}, dh h^{-1}] \rangle,$$

where $h = e^{t\xi}$.

The BV-BFV quantization then provides an integral kernel which, in addition to the classical coordinates (q_a, \bar{q}^b) , also depends on “ghost variables” $c_a, c_b \in \mathfrak{g}[1]$ and is a half-density on $T^*[-1]G \ni (g, g^+)$:

$$\hat{Z}(q_a, c_a, \bar{q}^b, c_b; g, g^+) = e^{\frac{i}{\hbar}\hat{S}_{\text{HJ}}(q_a, \bar{q}^b; g)} e^{-\frac{i}{\hbar}\langle g^+, g c_a - c_b g \rangle} \mu_G,$$

where μ_G is the Haar measure.

This integral kernel satisfies the modified quantum master equation (mQME) $(\Omega + \hbar^2\Delta)\hat{Z} = 0$ —and is defined up to the image of $\Omega + \hbar^2\Delta$ —where Δ is the

⁶We write \bar{q} for the image of the momentum p under the isomorphism $\mathfrak{g}^* \otimes Z^* \rightarrow \mathfrak{g} \otimes Z^*$.

⁷In this case, we have $\mathfrak{g} = \Omega^0(\Sigma, \mathfrak{g}')$ and $Z = \Omega^{0,1}(\Sigma)$, where \mathfrak{g}' is a finite-dimensional quadratic Lie algebra, and Σ is a closed, oriented surface with a choice of complex structure. The derivations v, \bar{v} are now the Dolbeault operators $\partial, \bar{\partial}$.

canonical BV Laplacian on half-densities on $T^*[-1]G$ and

$$\begin{aligned} \Omega = & \left\langle i\hbar \frac{\partial}{\partial q_a}, v(c_a) \right\rangle + \langle \bar{v}(c_a), q_a \rangle - \frac{i\hbar}{2} \left\langle [c_a, c_a], \frac{\partial}{\partial c_a} \right\rangle \\ & - \langle \bar{q}^b, v(c_b) \rangle - \left\langle \bar{v}(c_b), i\hbar \frac{\partial}{\partial \bar{q}^b} \right\rangle - \frac{i\hbar}{2} \left\langle [c_b, c_b], \frac{\partial}{\partial c_b} \right\rangle \end{aligned}$$

is a quantization of the endpoint BFV action describing the coisotropic reduction by the constraints.

In the abelian case, where we use the multiplicative convention for the group G , we write $g = e^T$, and the HJ action becomes

$$\hat{S}_{\text{HJ}}(q_a, \bar{q}^b; T) = -\langle \bar{q}^b, q_a \rangle - \langle \bar{q}^b, v(T) \rangle - \langle \bar{v}(T), q_a \rangle - \frac{1}{2} \langle \bar{v}(T), v(T) \rangle.$$

In this case it is also interesting, as a toy model for 7D Chern–Simons theory with endpoint Hitchin polarization (see Section 7.6), to change the final Darboux coordinates by some generating function $f(q, Q)$ with $\langle \bar{q}, dq \rangle = \langle \bar{Q}, dQ \rangle + df$. The HJ action for initial value q_a of q and final value Q_b of Q is then

$$\hat{S}_{\text{HJ}}^f(q_a, Q_b; T) = -f(q_a + v(T), Q_b) - \langle \bar{v}(T), q_a \rangle - \frac{1}{2} \langle \bar{v}(T), v(T) \rangle.$$

The BV-BFV quantization turns out to be possible when the transformed \mathfrak{g}^* -valued hamiltonian $\tilde{H}(\bar{Q}, Q) = v^*(\bar{q}(\bar{Q}, Q)) + \bar{v}^*(q(\bar{Q}, Q))$, with $*$ denoting transposition, is linear in \bar{Q} . In this case, the integral kernel, with $(T, T^+) \in T^*[-1]\mathfrak{g}$, is

$$\hat{Z}^f(q_a, c_a, Q_b, c_b; T, T^+) = e^{\frac{i}{\hbar} \hat{S}_{\text{HJ}}^f(q_a, Q_b; T)} e^{-\frac{i}{\hbar} T_\alpha^+ (c_a^\alpha - c_b^\alpha)} \sqrt{d^k T d^k T^+},$$

with $k = \dim \mathfrak{g}$. This integral kernel also satisfies the modified quantum master equation (mQME), with Δ the canonical BV Laplacian on half-densities on $T^*[-1]\mathfrak{g}$ and

$$\Omega = \left\langle i\hbar \frac{\partial}{\partial q_a}, v(c_a) \right\rangle + \langle \bar{v}(c_a), q_a \rangle - \left\langle c_b, \tilde{H} \left(-i\hbar \frac{\partial}{\partial Q_b}, Q_b \right) \right\rangle.$$

Remark 1.1 (Parameters and quantum corrections). In the examples just presented, there are no quantum corrections in the physical part of the partition function \hat{Z} . This statement is of course correct only if we compare the classical and quantum theory in the presence of parameters. The point is that usually one can get rid of (some of) the parameters. Let us discuss for simplicity the case when all parameters can be eliminated. In the classical theory, the constraints are recovered by setting to zero the differential of the HJ action as a function of the parameters g or T (parametrized by the generalized position variables). If the Hessian is nondegenerate, we can (locally) solve these equations and write the parameters as functions of the generalized position variables. Inserting this back into the HJ action yields a new, equivalent generating function, now without parameters. At the quantum level, we can instead integrate out the parameters g or T in the partition function, after setting their odd momenta g^+ or T^+ to zero. In the abelian case with linear polarization (generalized position variables q_a and \bar{q}_b), this is equivalent to the classical procedure (apart from an irrelevant constant depending on the determinant of the quadratic form $\langle \bar{v}(T), v(T) \rangle$). In the other cases (abelian case with non-linear polarization and nonabelian case), the quantum procedure yields nontrivial quantum corrections.

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2. HAMILTON–JACOBI FOR NONDEGENERATE ACTIONS

In this section we collect several known facts on the Hamilton–Jacobi action, which will be used in the rest of the paper (for a general reference, see, e.g., [3, Sections 47–48]). We only focus on actions of the form

$$S[p, q] = \int_{t_a}^{t_b} (p_i dq^i - H(p, q) dt),$$

where a sum over repeated indices is understood. Here (p, q) denotes a point (or a path) in the cotangent bundle T^*M of a fixed manifold M , and H is a smooth function on T^*M . By abuse of notation, we denote by dq^i both the coordinate 1-form on M and its pullback by the map $q: [t_a, t_b] \rightarrow M$ (which then actually means $\dot{q}^i dt$). We will later be interested in some generalizations, like adding constraints and working with an infinite-dimensional target.

Recall that the Euler–Lagrange (EL) equations for the above action are Hamilton’s equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Next assume that for every $q_a, q_b \in M$ these equations have a unique solution, which we will denote by $(p^{q_a, q_b}, q_{q_a, q_b})$, satisfying $q_{q_a, q_b}(t_a) = q_a$ and $q_{q_a, q_b}(t_b) = q_b$. Then one defines the Hamilton–Jacobi (HJ) action (a.k.a. Hamilton’s principal function)

$$S_{\text{HJ}}(q_a, q_b) := S[p^{q_a, q_b}, q_{q_a, q_b}].$$

Jacobi’s theorem then, in particular, asserts that

$$(1) \quad p_i^{q_a, q_b}(t_b) = \frac{\partial S_{\text{HJ}}}{\partial q_b^i}, \quad p_i^{q_a, q_b}(t_a) = -\frac{\partial S_{\text{HJ}}}{\partial q_a^i}.$$

In other words, S_{HJ} is a generating function for the graph of the hamiltonian flow of H (from t_a to t_b), see Appendix A (in particular, Remark A.32). Let us briefly review the proof because we will need some generalization thereof in the following. The variation of the action reads

$$\delta S = EL + p_i(t_b) \delta q^i(t_b) - p_i(t_a) \delta q^i(t_a),$$

where EL denotes the bulk term responsible for the EL equations. If we insert a solution of the EL equations, the EL term vanishes. In particular, inserting the solution $(p^{q_a, q_b}, q_{q_a, q_b})$ yields

$$\delta S_{\text{HJ}} = p_i^{q_a, q_b}(t_b) \delta q_b^i - p_i^{q_a, q_b}(t_a) \delta q_a^i,$$

which implies (1).

Remark 2.1 (Time dependency). Above we have considered the time endpoints t_a and t_b as fixed, but one can let them vary as well. We put the explicit dependence in the notation:

$$S[p, q](t_a, t_b) = \int_{t_a}^{t_b} (p_i dq^i - H(p, q) dt).$$

The time-dependent HJ action is then

$$(2) \quad S_{\text{HJ}}(q_a, q_b; t_a, t_b) := S[p^{q_a, q_b}, q_{q_a, q_b}](t_a, t_b),$$

where the solution is still required to satisfy $q_{q_a, q_b}(t_a) = q_a$ and $q_{q_a, q_b}(t_b) = q_b$. As a result, $(p^{q_a, q_b}, q_{q_a, q_b})$ also depends on t_a and t_b (even though we do not include this in the notation). In addition to the formulae in (1), we now also have⁸

$$(3) \quad \frac{\partial S_{\text{HJ}}}{\partial t_b} = -H(p^{q_a, q_b}(t_b), q_b), \quad \frac{\partial S_{\text{HJ}}}{\partial t_a} = H(p^{q_a, q_b}(t_a), q_a).$$

To get these formulae, we have to observe that, e.g., a variation δt_b of t_b produces not only a variation in the final endpoint of the integral, which yields $(p_i^{q_a, q_b}(t_b) \dot{q}_{q_a, q_b}^i(t_b) - H(p^{q_a, q_b}(t_b), q_b)) \delta t_b$, but also an induced variation $-\dot{q}_{q_a, q_b}(t_b) \delta t_b$ of q at the final endpoint.

Remark 2.2 (Uniqueness). For simplicity we have assumed that there is a unique solution $(p^{q_a, q_b}, q_{q_a, q_b})$ satisfying $q_{q_a, q_b}(t_a) = q_a$ and $q_{q_a, q_b}(t_b) = q_b$. This might not be true. Generically one may exclude focal points q_a, q_b connected by a continuous family of solutions,⁹ but typically we may expect a discrete, or finite, family to exist. In this case, the above procedure is actually meant to work in a neighborhood of a given nonfocal pair (q_a, q_b) for a choice of one solution in the discrete set. When one varies q_a or q_b (or the time interval), it is understood that one follows the variation of the chosen solution. The resulting HJ action is then a local generating function. To get the whole graph of the hamiltonian flow of H one needs the various, possibly infinitely many, HJ actions associated to the different solutions.

Example 2.3. A simple example where uniqueness of solutions with given endpoints is not satisfied is the free particle on S^1 , the unit circle.¹⁰ The hamiltonian is simply $H(p, q) = \frac{p^2}{2m}$. The EL equations are $\dot{p} = 0$ and $\dot{q} = \frac{p}{m}$. The graph of the hamiltonian flow of H is then

$$L := \left\{ (p^a, q_a, p^b, q_b) \in T^*S^1 \times T^*S^1 \mid p^b = p^a, q_b = q_a + \frac{p^a}{m}(t_b - t_a) \bmod 2\pi \right\}.$$

On the other hand, if we fix q_a and q_b , we have the solutions

$$p(t) = \frac{m(q_b - q_a + 2k\pi)}{t_b - t_a},$$

$$q(t) = q_a + (q_b - q_a + 2k\pi) \frac{t - t_a}{t_b - t_a} \bmod 2\pi,$$

⁸Equations (1) and (3) are also important because they imply that S_{HJ} is a solution of the Hamilton–Jacobi equation.

⁹For instance, if the target M is a Riemannian manifold with metric g and we take $H(p, q) = \frac{\|p\|_g^2}{2m}$, then the solutions to the EL equations are cotangent lifts of parametrized geodesics. If M is the sphere with round metric and we take antipodal points q_a and q_b , then we have a continuous family of solutions joining them. However, it is enough to perturb one of them, to get a discrete family.

¹⁰The nonuniqueness in this example is due to the fact that S^1 is not simply connected, but it is possible to find examples of nonuniqueness on simply connected configuration spaces as well.

for each $k \in \mathbb{Z}$. It follows that, for a given k , the HJ action reads

$$S_{\text{HJ},k}(q_a, q_b) = \frac{1}{2}m \frac{(q_b - q_a + 2k\pi)^2}{t_b - t_a},$$

which is indeed a generating function for L in a neighborhood of (q_a^0, p_0, q_b^0, p_0) with $p_0 = \frac{m(q_b^0 - q_a^0 + 2k\pi)}{t_b - t_a}$. To generate the whole L , one needs all the $S_{\text{HJ},k}$ s.

Remark 2.4 (Semiclassical approximation). In the path integral approach to quantum mechanics (say, on the configuration space $M = \mathbb{R}^n$), the integral kernel $K(q_a, q_b)$ for the evolution operator is, formally, given by the path integral [26, 28, 29]

$$(4) \quad K(q_a, q_b) = \int_{\substack{q(t_a)=q_a \\ q(t_b)=q_b}} Dp Dq e^{\frac{i}{\hbar} S[p, q]}.$$

Assuming for simplicity that there is a unique solution $(p^{q_a, q_b}, q_{q_a, q_b})$ satisfying $q_{q_a, q_b}(t_a) = q_a$ and $q_{q_a, q_b}(t_b) = q_b$, one usually makes the affine change of variables (translation)

$$(5) \quad p \mapsto p^{q_a, q_b} + \hat{p}, \quad q \mapsto q_{q_a, q_b} + \hat{q},$$

getting

$$K(q_a, q_b) = e^{\frac{i}{\hbar} S_{\text{HJ}}(q_a, q_b)} \int_{\hat{q}(t_a)=\hat{q}(t_b)=0} D\hat{p} D\hat{q} e^{\frac{i}{\hbar} \hat{S}_{(q_a, q_b)}[\hat{p}, \hat{q}]}$$

with

$$\hat{S}_{(q_a, q_b)}[\hat{p}, \hat{q}] = S[p^{q_a, q_b} + \hat{p}, q_{q_a, q_b} + \hat{q}] - S_{\text{HJ}}(q_a, q_b).$$

Since $(p^{q_a, q_b}, q_{q_a, q_b})$ is a solution, $\hat{S}_{(q_a, q_b)}$ starts with a quadratic term in (\hat{p}, \hat{q}) , so one can define the path integral perturbatively. One then sees that the HJ action yields the semiclassical approximation to K . The crucial point here is that the Hessian of S at a solution is nondegenerate (one also says that the action or the lagrangian is regular). This does not happen for degenerate actions (e.g., in gauge theories). In this case one has to use some technique to gauge fix the action, but one also has to take into account that the symplectic space of initial conditions is defined via some symplectic reduction. A cohomological way to resolve these issues is to use the BV formalism [7] in the bulk and the BFV formalism [6, 30] on the boundary. In [19] it was shown how to reconcile the two formalisms, into what was termed the quantum BV-BFV formalism, and it was in particular observed that, in general, extending boundary values to a solution of the EL equations is incompatible with the formalism (we will review this in Sections 9.2.3 and 9.2.6). The above argument to show that the semiclassical contribution to the integral kernel of the evolution operator is the HJ action is therefore no longer valid. We will see in Theorem 11.4 that the result still holds, yet the HJ action is recovered not by evaluating the classical action on a solution but via resumming trees in the Feynman diagrams expansion.

Remark 2.5 (Changing the endpoint conditions). Fixing q at the endpoints yields the EL equations as critical points for S . One may however be interested in choosing different endpoint conditions. In general, they have to correspond to a lagrangian submanifold of T^*M , not necessarily the zero section, but one has to adapt the one-form $p_i dq^i$. For simplicity, we now assume $M = \mathbb{R}^n$. Let (P_i, Q^i) be another set of Darboux coordinates on $T^*\mathbb{R}^n$ and assume (see also Example A.33 for a more

general viewpoint) that we have a generating function $f(q, Q)$ for the transformation: namely,

$$p_i dq^i = P_i dQ^i + df,$$

i.e.,

$$p_i = \frac{\partial f}{\partial q^i}, \quad P_i = -\frac{\partial f}{\partial Q^i}.$$

Assume we want to fix Q at the final endpoint and q at the initial endpoint. We then change the action by a boundary term:

$$S^f[p, q] := -f(q(t_b), Q(p(t_b), q(t_b))) + \int_{t_a}^{t_b} (p_i dq^i - H(p, q) dt).$$

When we take a variation, we now get

$$\delta S^f = EL + P_i(t_b) \delta Q^i(t_b) - p_i(t_a) \delta q^i(t_a),$$

with the same EL term as before. Therefore, the solutions of the EL equations are the critical points of S^f with fixed q at the initial endpoint and Q at the final endpoint. We define the corresponding HJ action as

$$S_{\text{HJ}}^f(q_a, Q_b) := S^f[p^{q_a, Q_b}, q_{q_a, Q_b}],$$

where $(p^{q_a, Q_b}, q_{q_a, Q_b})$ is the (assumedly unique) solution of the EL equations with $q_{q_a, Q_b}(t_a) = q_a$ and $Q(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)) = Q_b$. Exactly as in the case discussed at the very beginning of this section, inserting a solution into the variation of S^f kills the EL term, so we get

$$P_i(p^{q_a, Q_b}, q_{q_a, Q_b})(t_b) = \frac{\partial S_{\text{HJ}}^f}{\partial Q_b^i}, \quad p_i^{q_a, Q_b}(t_a) = -\frac{\partial S_{\text{HJ}}^f}{\partial q_a^i}.$$

In other words, S_{HJ}^f is a generating function for the graph of the hamiltonian flow of H (from t_a to t_b) with respect to the new set of coordinates (see again Remark A.32). The dependency on time can be studied repeating the steps of Remark 2.1 verbatim.

Example 2.6. Consider the case when $H = 0$, i.e., $S[p, q] = \int_{t_a}^{t_b} p_i dq^i$. Since the EL equations have constant solutions, we have $S[p^{q_a, Q_b}, q_{q_a, Q_b}] = 0$ and $q(t_b) = q_a$. Therefore

$$(6) \quad S_{\text{HJ}}^f(q_a, Q_b) = -f(q_a, Q_b).$$

Remark 2.7. For simplicity, we only discuss the change of the final endpoint conditions, but of course the same discussion may be repeated for the initial endpoint and for both endpoints at the same time.

Example 2.8. A very common case is when one wants to fix q at the initial point and p at the final point. This fits in the above construction by choosing $Q = p$ and $P = -q$ and $f(q, Q) = Q_i q^i$. In this case, we simply have

$$S^f[p, q] := -p_i(t_b) q^i(t_b) + \int_{t_a}^{t_b} (p_i dq^i - H(p, q) dt),$$

$$\delta S^f = EL - q^i(t_b) \delta p_i(t_b) - p_i(t_a) \delta q^i(t_a),$$

and

$$(7) \quad q_{q_a, p^b}^i(t_b) = -\frac{\partial S_{\text{HJ}}^f}{\partial p_i^b}, \quad p_i^{q_a, p^b}(t_a) = -\frac{\partial S_{\text{HJ}}^f}{\partial q_a^i}.$$

If we apply this to the Example 2.3 of the free particle on the circle, we get a unique solution for a given (q_a, p^b) and the HJ action

$$S_{\text{HJ}}^f(q_a, p^b) = -p^b q_a - \frac{(p^b)^2}{2m}(t_b - t_a).$$

If we instead consider the case $H = 0$ of Example 2.6, we get, as a particular case of (6),

$$(8) \quad S_{\text{HJ}}^f(q_a, p^b) = -p^b q_a.$$

Remark 2.9 (Legendre transform). One can pass from $S_{\text{HJ}}^f(q_a, p^b)$ to $S_{\text{HJ}}(q_a, q_b)$, and back, by the Legendre transform:

$$S_{\text{HJ}}(q_a, q_b) = \lambda_i^{\text{crit}} q_b^i + S_{\text{HJ}}^f(q_a, \lambda^{\text{crit}}),$$

where λ^{crit} is the critical point (assumed to be unique) of the variable p^b for the function $f(p^b) := p_b^i q_b^i + S_{\text{HJ}}^f(q_a, p^b)$:

$$\frac{\partial f}{\partial p^b} = 0 \text{ at } p^b = \lambda^{\text{crit}}.$$

We will return to this in Section 6.2.

Remark 2.10. In terms of the path-integral computation of the evolution operator, see Remark 2.4, changing the endpoint conditions, as discussed in Remark 2.5, corresponds to assigning different Hilbert space representations, corresponding to different choices of polarization, to the endpoints. For example, fixing p at the final endpoint, as in Example 2.8, corresponds to computing the integral kernel $\hat{K}(q_a, p^b)$ for the position representation at the initial endpoint and the momentum representation at the final one. This is the Fourier transform of $K(q_a, q_b)$ with respect to the second argument. One can see a classical track of this in the $p_i(t_b)q^i(t_b)$ term in S^f (and in the Legendre transform of the previous remark).

Example 2.11. Consider

$$\hat{K}(q_a, p^b) = \int_{\substack{q(t_a)=q_a \\ p(t_b)=p^b}} Dp Dq e^{\frac{i}{\hbar} S^f[p, q]}.$$

with $S^f[p, q] = -p(t_b)q(t_b) + S[p, q]$ for the simple action $S[p, q] = \int_{t_a}^{t_b} p dq$. (Nothing changes if we have many p and q variables: just think of a hidden scalar product $p dq = p_i dq^i$.) As the EL equations imply that the solutions are constant, the change of variables (5) is now

$$p \mapsto p^b + \hat{p}, \quad q \mapsto q_a + \hat{q},$$

with $\hat{p}(t_b) = 0$ and $\hat{q}(t_a) = 0$. Since $S^f[p, q] = -p^b(q_a + \hat{q}(t_b)) + \int_{t_a}^{t_b} (p^b + \hat{p}) d\hat{q}$, we have

$$\hat{K}(q_a, p^b) = e^{-\frac{i}{\hbar} p^b q_a} \int_{\substack{\hat{q}(t_a)=0 \\ \hat{p}(t_b)=0}} D\hat{p} D\hat{q} e^{\frac{i}{\hbar} (-p^b \hat{q}(t_b) + \int_{t_a}^{t_b} p^b d\hat{q} + \int_{t_a}^{t_b} \hat{p} d\hat{q})}.$$

This is a Gaussian integral with quadratic term $\int_{t_a}^{t_b} \hat{p} d\hat{q}$ and no \hat{p} -sources, so it simply yields a constant (which we normalize to one), and we get

$$(9) \quad \hat{K}(q_a, p^b) = e^{-\frac{i}{\hbar} p^b q_a},$$

the integral kernel for the Fourier transform, whose exponent is given by the HJ action (8).

Example 2.12 (Formal quantum change of polarization). The computation of Example 2.11 may be generalized to the general case discussed in Remark 2.5. Namely, we have

$$\hat{K}^f(q_a, Q_b) = \int_{\substack{q(t_a)=q_a \\ Q(p(t_b), q(t_b))=Q_b}} Dp Dq e^{\frac{i}{\hbar} S^f[p, q]},$$

with $S^f[p, q] = -f(q(t_b), Q(p(t_b), q(t_b))) + S[p, q]$. Again we consider $S[p, q] = \int_{t_a}^{t_b} p dq$. The change of variables (5) now becomes

$$p \mapsto p^0 + \hat{p}, \quad q \mapsto q_a + \hat{q},$$

with the constant solution p^0 given by

$$p^0 = \frac{\partial f}{\partial q}(q_a + \hat{q}(t_b), Q_b),$$

and endpoint conditions $\hat{p}(t_b) = 0$ and $\hat{q}(t_a) = 0$.¹¹ In general this change of variables is no longer affine, since p^0 depends on \hat{q} , possibly in a nonlinear way. On the other hand, the Jacobian determinant of the transformation is (formally) 1 (the Jacobian is lower triangular with “1”s on the diagonal). We now have

$$\hat{K}^f(q_a, Q_b) = \int_{\substack{\hat{q}(t_a)=0 \\ \hat{p}(t_b)=0}} D\hat{p} D\hat{q} e^{\frac{i}{\hbar} (-f(q_a + \hat{q}(t_b), Q_b) + \int_{t_a}^{t_b} p^0 d\hat{q} + \int_{t_a}^{t_b} \hat{p} d\hat{q})},$$

which is again a Gaussian integral with quadratic term $\int_{t_a}^{t_b} \hat{p} d\hat{q}$ and no \hat{p} -sources. Therefore,

$$(10) \quad \hat{K}^f(q_a, Q_b) = e^{-\frac{i}{\hbar} f(q_a, Q_b)},$$

the exponential of the HJ action (6).

Digression 2.13 (Unitarity). The result (10) of the previous Example 2.12 is correct insofar as the starting point is the quantization of the action S^f as a partition “function.” This has to be corrected if we understand the integral kernel as a half-density in the boundary variables, which is more appropriate for composition. The idea is that the functional measure is the product over points of the measure $dp dq / (2\pi\hbar)$ (for simplicity we discuss this for the target $T^*\mathbb{R}$). At the end points we then retain only the square root of the fixed boundary variable, as the other corresponding square root will come from another integral kernel we want to compose with. For example, in passing from the q - to the p -representation, the integral kernel half-density will be

$$\mathbb{K}(q_a, p^b) = \sqrt{dq_a} e^{-\frac{i}{\hbar} p^b q_a} \sqrt{\frac{dp^b}{2\pi\hbar}}$$

instead of (9). Note that \mathbb{K} is unitary:

$$\int_{\{p^b\}} \mathbb{K}(q, p^b) \overline{\mathbb{K}}(p^b, q') = \sqrt{dq} \left(\int_{\{p^b\}} e^{\frac{i}{\hbar} p^b (q' - q)} \frac{dp^b}{2\pi\hbar} \right) \sqrt{dq'} = \sqrt{dq} \delta(q' - q) \sqrt{dq'}.$$

Unitarity is somehow expected, since reversing the orientation of the interval on which the theory is defined produces $\overline{\mathbb{K}}(p^b, q')$, and the composition of the quantum

¹¹Note that the endpoint fluctuations of p and q are constrained by the condition $Q(p(t_b), q(t_b)) = Q_b$. We have arranged the change of variables so that, while p^0 may fluctuate at the endpoint, \hat{p} does not.

evolution on an interval with the reversed one should be the identity. In Example 2.12, we could follow the same reasoning and expect unitarity (indeed, at the classical level, composition of the generating functions $f(q', Q_b)$ and $-f(q, Q_b)$, see Remark A.37, yields a generating function for the identity); however, in general, this is not the case at the quantum level as we will see in a moment. In this example as well we split the fields in terms of the initial q and final p variables, but then we expressed the final values in terms of Q_b . The change of variables we performed there entails in particular

$$(11) \quad p(t_b) = \frac{\partial f}{\partial q}(q_a + \hat{q}(t_b), Q_b)$$

and therefore

$$dp(t_b) = \frac{\partial^2 f}{\partial q \partial Q}(q_a + \hat{q}(t_b), Q_b) dQ_b + \frac{\partial^2 f}{\partial q^2}(q_a + \hat{q}(t_b), Q_b) (dq_a + d\hat{q}(t_b)).$$

The terms in dq_a and $d\hat{q}(t_b)$ can be neglected, since they already appear in the functional measure. Therefore, we only retain the first term, of which we eventually have to take the square root, getting the integral kernel half-density

$$\mathbb{K}^f(q_a, Q_b) = \sqrt{dq_a} \left(\int_{\substack{\hat{q}(t_a)=0 \\ \hat{p}(t_b)=0}} D\hat{p} D\hat{q} e^{\frac{i}{\hbar}(-f(q_a + \hat{q}(t_b), Q_b) + \int_{t_a}^{t_b} p^0 d\hat{q} + \int_{t_a}^{t_b} \hat{p} d\hat{q})} \sqrt{\frac{\partial^2 f}{\partial q \partial Q}(q_a + \hat{q}(t_b), Q_b)} \right) \sqrt{\frac{dQ_b}{2\pi\hbar}},$$

where we have assumed the nonzero $\frac{\partial^2 f}{\partial q \partial Q}$ term to be actually positive. Again, since this is a Gaussian integral with quadratic term $\int_{t_a}^{t_b} \hat{p} d\hat{q}$ and no \hat{p} -sources, we get

$$(12) \quad \mathbb{K}^f(q_a, Q_b) = \sqrt{dq_a} e^{-\frac{i}{\hbar}f(q_a, Q_b)} \sqrt{\frac{\partial^2 f}{\partial q \partial Q}(q_a, Q_b)} \sqrt{\frac{dQ_b}{2\pi\hbar}}$$

instead of (10). In a neighborhood of the diagonal $q = q'$, \mathbb{K}^f is almost unitary: i.e.,

$$\int_{\{Q_b\}} \mathbb{K}^f(q, Q_b) \overline{\mathbb{K}^f}(Q_b, q') = \sqrt{dq} (1 + O(\hbar)) \delta(q' - q) \sqrt{dq'},$$

where the $O(\hbar)$ term is a differential operator. This may be seen by the (q and q' dependent) change of variable

$$\lambda(Q_b) = \frac{f(q', Q_b) - f(q, Q_b)}{q' - q}$$

in the dQ_b integral.¹² Note that (12) resembles the starting point of the WKB approximation (see, e.g., [44, Appendix], [25, Section 15], and references therein). Indeed, one possible way to get corrections to it to make it unitary is to regard, if possible, f as the Hamilton–Jacobi action for some hamiltonian system (on the

¹²In some cases, \mathbb{K}^f is exactly unitary. This happens, e.g., when f has product form: $f(q, Q) = \alpha(q)\beta(Q)$. (In particular, this happens when we have a linear symplectomorphism, in which case, moreover, the correction term $\frac{\partial^2 f}{\partial q \partial Q}$ is constant.) In this case, we have

$$\int_{\{Q_b\}} \mathbb{K}^f(q, Q_b) \overline{\mathbb{K}^f}(Q_b, q') = \sqrt{dq} \left(\int_{\{Q_b\}} e^{\frac{i}{\hbar}\beta(Q_b)(\alpha(q') - \alpha(q))} \sqrt{\alpha'(q)\alpha'(q')\beta'(Q)} \frac{dQ_b}{2\pi\hbar} \right) \sqrt{dq'},$$

other hand, the corrections are not uniquely determined, since the choice of this hamiltonian system is not unique).

3. SYSTEMS WITH ONE CONSTRAINT

Our first generalization is an action of the form¹³

$$S[p, q, e] = \int_{t_a}^{t_b} (p_i dq^i - eH(p, q)),$$

where the new field e , the Lagrange multiplier, is a one-form on the interval $[t_a, t_b]$.

We will first consider the case when e is different from zero everywhere. This restriction will allow us to deal with endpoint conditions on the qs . Next we will consider the general case, which, however, will force us to pick different endpoint conditions.

3.1. Nondegenerate Lagrange multiplier. We assume e to be different from zero everywhere. A variation of the action yields

$$\delta S = EL + p_i(t_b)\delta q^i(t_b) - p_i(t_a)\delta q^i(t_a)$$

with the EL equations

$$dq^i = e \frac{\partial H}{\partial p_i}, \quad dp_i = -e \frac{\partial H}{\partial q^i}, \quad H(p, q) = 0.$$

The last equation, the variation with respect to e , is imposed at all times. Note that the endpoint symplectic structures are the same as in the nondegenerate case and depend only on p and q . This in particular means that no endpoint condition on e has to be imposed.

To solve the EL equations, we first define

$$\tau(t) := \int_{t_a}^t e.$$

The first two sets of EL equations now read

$$\frac{dq^i}{d\tau} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q^i},$$

so they are the usual Hamilton equations for the hamiltonian H in the time variable $\tau \in [0, T]$, with $T := \int_{t_a}^{t_b} e$. The last EL equation, $H = 0$, is automatically satisfied at any point if we impose it at the initial point (or at the final point). We then have that the endpoint values for (p, q) corresponding to solutions form the set

$$L = \{(p^a, q_a, p^b, q_b) \in T^*M \times T^*M \mid H(p^b, q_b) = 0 \\ \text{and } \exists T \in \mathbb{R}_{\neq 0} : (p^b, q_b) = \Phi_T^H(p^a, q_a)\},$$

where we have assumed the nonzero α' and β' terms to be actually positive. By the change of variable $\mu = \beta(Q_b)$, we get

$$\int_{\{Q_b\}} \mathbb{K}^f(q, Q_b) \overline{\mathbb{K}}^f(Q_b, q') = \sqrt{dq} \left(\int_{\{\mu\}} e^{\frac{i}{\hbar} \mu(\alpha(q') - \alpha(q))} \sqrt{\alpha'(q)\alpha'(q')} \frac{d\mu}{2\pi\hbar} \right) \sqrt{dq'} \\ = \sqrt{dq} \delta(\alpha(q') - \alpha(q)) \sqrt{\alpha'(q)\alpha'(q')} \sqrt{dq'} = \sqrt{dq} \delta(q' - q) \sqrt{dq'}.$$

¹³If one requires $e \neq 0$ everywhere, this action may also be regarded as the first-order version of one-dimensional gravity, described by the coframe e , coupled to matter, described by q . Its BV-BFV quantization is studied in [22].

where Φ_τ^H denotes the hamiltonian flow of H at time τ . Note that in this case L is not the graph of a map $T^*M \rightarrow T^*M$, but just a subset of $T^*M \times T^*M$. We will call it the evolution relation of the system. (See Remark A.38 for a more general perspective.)

For a fixed e , let $(p^{q_a, q_b}, q_{q_a, q_b})$ denote the (assumedly unique) solution to the first two sets of EL equations (i.e., we do **not** impose the constraint $H = 0$) satisfying $q_{q_a, q_b}(t_a) = q_a$ and $q_{q_a, q_b}(t_b) = q_b$. Since we may rewrite the action as $\int_0^T (p_i dq^i - H(p, q) d\tau)$, we get that the HJ action is

$$S_{\text{HJ}}(q_a, q_b)[e] := S[p^{q_a, q_b}, q_{q_a, q_b}, e] = S_{\text{HJ}}^H(q_a, q_b; 0, T),$$

where the last term denotes the HJ action for the system with hamiltonian H over the time interval $[0, T]$.

The first remark is that the HJ action depends on e only via its integral T ; we will then simply write

$$(13) \quad \hat{S}_{\text{HJ}}(q_a, q_b; T) := S_{\text{HJ}}(q_a, q_b)[e], \quad \text{with } T = \int_{t_a}^{t_b} e,$$

so

$$\hat{S}_{\text{HJ}}(q_a, q_b; T) = S_{\text{HJ}}^H(q_a, q_b; 0, T).$$

The second remark, which follows immediately from (1) and (3), is that the evolution relation L is determined by the equations

$$p_i^b = \frac{\partial \hat{S}_{\text{HJ}}}{\partial q_b^i}, \quad p_i^a = -\frac{\partial \hat{S}_{\text{HJ}}}{\partial q_a^i}, \quad \frac{\partial \hat{S}_{\text{HJ}}}{\partial T} = 0.$$

In other words, \hat{S}_{HJ} is a generalized generating function¹⁴ for the lagrangian submanifold L , see Appendix A (in particular, Remark A.11 for the linear case and Remark A.27 for the general case).

3.2. The general case. The assumption $e \neq 0$ everywhere is only needed to ensure that the map $t \mapsto \tau$ is a diffeomorphism and hence to relate \hat{S}_{HJ} to the HJ function for the system with hamiltonian H . The existence of a HJ action, however, holds also in the general case where we make no assumptions on e , provided we choose the endpoint conditions more carefully.

First observe that the first two sets of EL equations, which we will call the evolution equations,

$$dq^i = e \frac{\partial H}{\partial p_i}, \quad dp_i = -e \frac{\partial H}{\partial q^i},$$

can be uniquely solved, for fixed e , in terms of initial conditions p^a, q_a . In fact, in the regions (closed intervals) where $e = 0$, the solution is constant and on the other regions we can make a change of variables $t \mapsto \tau$ as above. Moreover, the function H is constant when evaluated on a solution to the evolution equations, which means that it is enough to impose this constraint at the initial (or final) endpoint to have it satisfied at every time.

Let L still denote the evolution relation of the system, i.e., the set of endpoint values for solutions to the EL equations. Note that, for a given $e = \underline{e} dt$, we have

¹⁴In the terminology of [9, Section 4.3], the triple $(M \times \mathbb{R}_{\neq 0}, \pi_1, \hat{S}_{\text{HJ}})$, where π_1 is the projection to the first factor, is a Morse family generating L .

to compute the “flow” $\Phi_T^{\tilde{H}}$ of the time-dependent hamiltonian $\tilde{H} = \underline{e}H$ up to time $T = \int_{t_a}^{t_b} e$. We then have

$$L = \{(p^a, q_a, p^b, q_b) \in T^*M \times T^*M \mid H(p^b, q_b) = 0 \\ \text{and } \exists e : (p^b, q_b) = \Phi_T^{\tilde{H}}(p^a, q_a)\}.$$

Remark 3.1. We can have a better geometric description if we assume that the differential of H is different from zero at every point of the zero locus C of H . This ensures that C is a submanifold of T^*M and that the hamiltonian vector field X^H of H restricted to C has no zeros, so it defines a distribution of lines. Note that C is an example of what is called a coisotropic submanifold and $\text{span } X^H|_C$ is called its characteristic distribution (see Definition A.19 and Remark A.25 for more on this). In this geometric framework, the evolution relation $L \subset T^*M \times T^*M$ consists of pairs of points on C that can be connected by a path along the characteristic distribution. (It is therefore lagrangian by Lemma A.29.)

Now observe that fixing q_a and q_b is problematic, as for $e = 0$ we have no solution unless $q_a = q_b$. For this reason we have to choose different endpoint conditions, as in Remark 2.5; for instance, we can fix p at one endpoint as in Example 2.8.

We denote by $(p^{q_a, Q_b}, q_{q_a, Q_b}, e)$ a solution $(p^{q_a, Q_b}, q_{q_a, Q_b})$ to the evolution equations for the given e (but we do **not** impose the constraint $H = 0$) satisfying $q_{q_a, Q_b}(t_a) = q_a$ and $Q(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)) = Q_b$. We then set

$$S_{\text{HJ}}^f(q_a, Q_b)[e] := S^f[p^{q_a, Q_b}, q_{q_a, Q_b}, e]$$

with

$$S^f[p, q, e] = -f(q(t_b), Q(p(t_b), q(t_b))) + \int_{t_a}^{t_b} (p_i dq^i - eH(p, q)).$$

We then have

$$(14) \quad \begin{aligned} \delta S_{\text{HJ}}^f &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \int_{t_a}^{t_b} \delta e H(p^{q_a, Q_b}, q_{q_a, Q_b}) \\ &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \left(\int_{t_a}^{t_b} \delta e \right) H(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)) \\ &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \delta T H(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)), \end{aligned}$$

where, as above, we set $T = \int_{t_a}^{t_b} e$. Note that S_{HJ}^f is invariant under variations of e with vanishing integral, so we may define \hat{S}_{HJ}^f like in (13):

$$\hat{S}_{\text{HJ}}^f(q_a, Q_b; T) := S_{\text{HJ}}^f(q_a, Q_b)[e], \quad \text{with } T = \int_{t_a}^{t_b} e.$$

Then the above equation reads

$$\delta \hat{S}_{\text{HJ}}^f = P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \delta T H(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)),$$

and we see that the equations

$$P_i(p^{q_a, Q_b}, q_{q_a, Q_b})(t_b) = \frac{\partial \hat{S}_{\text{HJ}}^f}{\partial Q_b^i}, \quad p_i^a = -\frac{\partial \hat{S}_{\text{HJ}}^f}{\partial q_a^i}, \quad \frac{\partial \hat{S}_{\text{HJ}}^f}{\partial T} = 0$$

define L .

The HJ action, obtained inserting a solution of the evolution equations but ignoring the constraint, is a generalized generating function for the evolution relation determined by all EL equations.

Example 3.2 (Linear case). Consider $M = \mathbb{R}^n$ and suppose H is linear on $T^*\mathbb{R}^n$:

$$H(p, q) = p_i v^i + w_i q^i,$$

where the (v^i, w_i) s are given constants. In this case the evolution equations simply read

$$dq^i = ev^i, \quad dp_i = -ew_i.$$

They can be easily solved in terms of the initial conditions, so we get the evolution relation

$$L = \{(p^a, q_a, p^b, q_b) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n \mid \\ \mid \exists T \in \mathbb{R} : p_i^b v^i + w_i(q_a^i + T v^i) = 0, p^a = p^b + T w, q_b = q_a + T v\},$$

where T corresponds to $\int_{t_a}^{t_b} e$, and we have evaluated the constraint $H = 0$ at the final endpoint. As in Example 2.8 we now fix q_a and p^b . The evolution equations can then be easily integrated for the given endpoint conditions as

$$q_{q_a, p^b}^i(t) = q_a^i + \int_{t_a}^t ev^i = q_a^i + \tau(t)v^i, \\ p_i^{q_a, p^b}(t) = p_i^b + \int_t^{t_b} ew_i = p_i^b + (T - \tau(t))w_i,$$

with $\tau(t) = \int_{t_a}^t e$. For this choice of endpoint conditions we consider

$$S^f[p, q, e] := -p_i(t_b)q^i(t_b) + \int_{t_a}^{t_b} (p_i dq^i - eH(p, q)).$$

Now observe that

$$p_i^{q_a, p^b} dq_{q_a, p^b}^i(t) - eH(p^{q_a, p^b}, q_{q_a, p^b}) = p_i^{q_a, p^b} ev^i - ep_i^{q_a, p^b} v^i - ew_i q_{q_a, p^b}^i \\ = -d\tau w_i (q_a^i + \tau v^i),$$

which implies

$$\int_{t_a}^{t_b} (p_i^{q_a, p^b} dq_{q_a, p^b}^i(t) - eH(p^{q_a, p^b}, q_{q_a, p^b})) = -T w_i q_a^i - \frac{T^2}{2} w_i v^i.$$

On the other hand,

$$p_i^{q_a, p^b}(t_b) q_{q_a, p^b}^i(t_b) = p_i^b (q_a^i + T v^i).$$

In conclusion, we get

$$(15) \quad \hat{S}_{\text{HJ}}^f(q_a, p^b; T) = -p_i^b q_a^i - T(p_i^b v^i + w_i q_a^i) - \frac{T^2}{2} w_i v^i,$$

which is readily seen to be a generalized generating function for L .

Remark 3.3. Sometimes it is possible to solve for T the constraints and then obtain a standard (i.e., without parameters) generating function for the evolution relation L . For instance, in the linear case of Example 3.2, the constraint is

$$p_i^b v^i + w_i q_a^i + T w_i v^i = 0,$$

which can be solved if $w_i v^i \neq 0$. In this case we get

$$T = -\frac{p_i^b v^i + w_i q_a^i}{w_i v^i},$$

which can be inserted back into \hat{S}^f to get the standard generating function

$$\tilde{S}^f(q_a, p^b) = -p_i^b q_a^i + \frac{(p_i^b v^i + w_i q_a^i)^2}{2w_i v^i}.$$

In the one-dimensional case, $n = 1$, we get

$$\tilde{S}^f(q_a, p^b) = -p^b q_a + \frac{(p^b v + w q_a)^2}{2wv} = \frac{1}{2} \frac{v}{w} (p^b)^2 + \frac{1}{2} \frac{w}{v} (q_a)^2,$$

which correctly generates the evolution relation

$$L = \{(p^a, q_a, p^b, q_b) \in T^*\mathbb{R} \times T^*\mathbb{R} \mid p^a v + w q_a = 0 \text{ and } p^b v + w q_b = 0\}.$$

Note that in this very particular example the evolution plays no role and L has the product form $L = L' \times L'$, with $L' = \{(p, q) \in T^*\mathbb{R} \mid pv + wq = 0\}$.

Remark 3.4 (Integrable systems I). The case $n = 1$ can also be regarded as a one-dimensional integrable system with vanishing hamiltonian. In this case, another expression for the HJ action can be derived as follows. Around a point (p_0, q_0) where $dH \neq 0$, by the Carathéodory–Lie–Jacobi theorem¹⁵ we may extend H to a Darboux chart (P, Q) with $P = H$ (P is what is called the action variable and Q the corresponding angle variable). Let $f(q, Q)$ be the generating function satisfying $p dq = P dQ + df$. We may then write

$$S^f[p, q, e] := -f(q(t_b), Q(t_b)) + \int_{t_a}^{t_b} (p dq - eH) = -f(q(t_a), Q(t_a)) + \int_{t_a}^{t_b} (P dQ - eP),$$

where $P = P(p, q)$, $Q = Q(p, q)$. The evolution equations now simply read $dP = 0$, $dQ = e$. Evaluating on a solution of the evolution equations, the action vanishes and we get

$$(16) \quad S_{\text{HJ}}^f(q_a, Q_b; T) = -f(q_a, Q_b - T)$$

for (q_a, Q_b) in a neighborhood of $(q_0, Q(p_0, q_0))$. Alternatively, if $g(q, P)$ is the generating function satisfying $p dq = -Q dP + dg$, we can express the HJ action with final endpoint condition on P by

$$(17) \quad S_{\text{HJ}}^g(q_a, P^b; T) = -TP^b - g(q_a, P^b).$$

Notice also that (17) is the Legendre transform of (16). We will revisit the case of integrable systems in Remarks 4.5 and 5.1.

¹⁵The Carathéodory–Lie–Jacobi theorem says that if H_1, \dots, H_k are functions on a symplectic manifold (M, ω) such that $\{H_i, H_j\} = 0$ and dH_1, \dots, dH_k are linearly independent at some $p \in M$, then there exists a Darboux chart (q_i, p^i) containing p with $q_1 = H_1, \dots, q_k = H_k$. See, for instance, [40, Theorem 13.4.1].

4. SYSTEMS WITH SEVERAL CONSTRAINTS

From now on we consider an action of the form¹⁶

$$(18) \quad S[p, q, e] = \int_{t_a}^{t_b} (p_i dq^i - e^\alpha H_\alpha(p, q)),$$

where the e^α s (the Lagrange multipliers) are one-forms on the interval $[t_a, t_b]$ and the H_α s are given functions (the constraints). (The sum over α , like that over i , is now understood.) The EL equations are

$$dq^i = e^\alpha \frac{\partial H_\alpha}{\partial p_i}, \quad dp_i = -e^\alpha \frac{\partial H_\alpha}{\partial q^i}, \quad H_\alpha(p, q) = 0.$$

Again we will call the first two sets of equations the evolution equations and the last set the constraints. Evaluating the constraints on a solution of the evolution equations yields

$$dH_\alpha = \frac{\partial H_\alpha}{\partial p_i} dp_i + \frac{\partial H_\alpha}{\partial q^i} dq^i = -\{H_\alpha, H_\beta\} e^\beta,$$

where $\{ , \}$ denotes the Poisson bracket.¹⁷ If we assume, as we will do from now on, that the constraints are in involution,¹⁸ i.e., that there are structure functions $f_{\alpha\beta}^\gamma$ such that

$$(19) \quad \{H_\alpha, H_\beta\} = f_{\alpha\beta}^\gamma H_\gamma,$$

then the previous equations become

$$(20) \quad dH_\alpha = -f_{\alpha\beta}^\gamma e^\beta H_\gamma.$$

This implies that, if the constraints are imposed at the initial (or final) endpoint, they will be satisfied at every time.

Remark 4.1. As in Remark 3.1, it may be convenient to assume that the differentials of the H_α s are linearly independent at every point of the common zero locus C of the H_α s. This ensures that C is a submanifold of T^*M and that the hamiltonian vector fields X^{H_α} restricted to C are linearly independent, so they define a regular distribution. It follows that C is also an example of what is called a coisotropic submanifold and $\text{span}\{X_\alpha^H|_C\}$ is called its characteristic distribution (see Definition A.19 and Remark A.25 for more on this).¹⁹ As a consequence of (19), the characteristic distribution is involutive (and hence, by Frobenius' theorem, integrable). In this geometric framework, the evolution relation $L \subset T^*M \times T^*M$ consists of pairs of points on C that can be connected by a path along the characteristic distribution and is therefore lagrangian in $\overline{T^*M} \times T^*M$, where the bar denotes that one takes the opposite symplectic form.

¹⁶Such an action functional is the classical part of an AKSZ theory [2] on the interval $[t_a, t_b]$; see Section 8 for a review. See also [5] for the study of such a theory in the Dirac formalism.

¹⁷We define the Poisson bracket as $\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}$.

¹⁸Using Dirac's terminology, one also says that they are first-class constraints.

¹⁹The characteristic distribution is also the same as the kernel of the restriction of the symplectic form to C .

Like in the case with a single constraint, fixing q_a and q_b is problematic, as for $e^\alpha = 0$ for all α we have no solution unless $q_a = q_b$. Therefore, we choose different endpoint conditions, as in Remark 2.5; for instance, we can fix p at one endpoint as in Example 2.8.

We will denote by $(p^{q_a, Q_b}, q_{q_a, Q_b}, e)$ a solution $(p^{q_a, Q_b}, q_{q_a, Q_b})$ to the evolution equations with the given e (but we do **not** impose the constraints $H_\alpha = 0$) satisfying $q_{q_a, Q_b}(t_a) = q_a$ and $Q(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)) = Q_b$. We then set

$$S_{\text{HJ}}^f(q_a, Q_b)[e] := S^f[p^{q_a, Q_b}, q_{q_a, Q_b}, e]$$

with

$$S^f[p, q, e] = -f(q(t_b), Q(p(t_b), q(t_b))) + \int_{t_a}^{t_b} (p_i dq^i - e^\alpha H_\alpha(p, q)).$$

By taking a variation, we get

$$\delta S_{\text{HJ}}^f = P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \int_{t_a}^{t_b} \delta e^\alpha H_\alpha(p^{q_a, Q_b}, q_{q_a, Q_b}).$$

This shows that S_{HJ}^f is indeed a generalized generating function for the evolution relation L . However, there is a lot of redundancy because the variations with respect to the $e^\alpha(t)$ s yield the constraints for every $t \in [t_a, t_b]$, whereas we have observed that it is enough to impose them at the initial (or final) endpoint. This is related to the fact that, thank to (20) and upon integration by parts, S_{HJ}^f is invariant under a variation of e of the form

$$(21) \quad \delta e^\alpha = d\gamma^\alpha + f_{\alpha\beta}^\gamma e^\alpha \gamma^\beta,$$

with the γ s arbitrary functions on $[t_a, t_b]$ that vanish at both endpoints. We call a transformation as in (21) a gauge transformation. We therefore see that S_{HJ}^f only depends on the $e^\alpha(t)$ s via their gauge classes. Analogously to the case of equation (13), we define

$$(22) \quad \hat{S}_{\text{HJ}}^f(q_a, Q_b; [e]) := S_{\text{HJ}}^f(q_a, Q_b, e), \quad \text{with } e \in [e].$$

In summary, we have the following

Theorem 4.2. *The generalized HJ action \hat{S}_{HJ}^f , obtained by inserting into the classical action S^f the solution of the evolution equations, is a generalized generating functions for the evolution relation, which is defined in terms of all the EL equations (evolution and constrains).*

In the next subsections we will discuss more explicitly the gauge classes and how the HJ action depends on them. We will proceed in increasing order of difficulty. In Section 4.1, we will consider the strictly involutive case where all the structure functions $f_{\alpha\beta}^\gamma$ vanish. This case, which is a straightforward generalization of the single-constraint case, comprises the case where the constraints H_α are linear, see Example 4.3, and will be important for abelian Chern–Simons theory. Next, in Section 4.2, we will consider the case when the structure functions are actually constant, so that there is an underlying Lie algebra structure (the H_α s are in this case the components of an equivariant momentum map). This case will be important for nonabelian Chern–Simons theory. Finally, for completeness, we will treat the general case in Section 4.3: this case requires knowledge of Lie algebroids

and Lie groupoids, see [42] for a general introduction, but will not be used in the rest of the paper, so it may be safely skipped.

4.1. The strictly involutive case. Suppose that all the structure functions $f_{\alpha\beta}^\gamma$ vanish; i.e., $\{H_\alpha, H_\beta\} = 0$ for all α and β . This implies that, evaluating on a solution to the evolution equations, we have $dH_\alpha = 0$ for all α . If we define

$$\tau^\alpha(t) := \int_{t_a}^t e^\alpha,$$

we then have $e^\alpha = d\tau^\alpha$ and $\tau^\alpha(t_a) = 0$. Observe that the computation in (14) can be repeated verbatim yielding

$$\delta S_{\text{HJ}}^f = P_i(t_b)\delta Q_b^i - p_i(t_a)\delta q_a^i - \delta T^\alpha H_\alpha(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)),$$

with $T^\alpha = \tau^\alpha(t_b) = \int_{t_a}^{t_b} e^\alpha$. Again, this first implies that S_{HJ}^f is invariant under variations of the e^α s with vanishing integral, so we may define \hat{S}_{HJ}^f as in (22) as

$$\hat{S}_{\text{HJ}}^f(q_a, Q_b; T) := S_{\text{HJ}}^f(q_a, Q_b)[e], \quad \text{with } T^\alpha = \int_{t_a}^{t_b} e^\alpha.$$

Second, the above computation shows that \hat{S}_{HJ}^f is a generalized generating function for L .

Example 4.3 (Linear case). Assume

$$H_\alpha = p_i v_\alpha^i + w_{i\alpha} q^i = p v_\alpha + w_\alpha q,$$

where the $(v_\alpha^i, w_{i\alpha})$ s are given constants and we use the product of row and column vectors to make the notation a bit lighter. The Poisson bracket of any two constraints is a constant function. The involutivity conditions (19) then imply that all these constant functions vanish, so we actually have

$$\{H_\alpha, H_\beta\} = 0.$$

This is equivalent to the conditions $w_\alpha v_\beta = w_\beta v_\alpha$ for all α and β . For future convenience we define the (symmetric!) matrix

$$(23) \quad A_{\alpha\beta} := w_\alpha v_\beta = w_\beta v_\alpha.$$

We now want to compute \hat{S}_{HJ}^f explicitly as a function of q_a and p^b . First observe that the evolution equations read

$$dq = e^\alpha v_\alpha, \quad dp = -e^\alpha w_\alpha,$$

and can be easily integrated for the given endpoint conditions:

$$\begin{aligned} q_{q_a, p^b}(t) &= q_a + \tau^\alpha(t) v_\alpha, \\ p^{q_a, p^b}(t) &= p^b + (T^\alpha - \tau^\alpha(t)) w_\alpha. \end{aligned}$$

Moreover, we have

$$\begin{aligned} p^{q_a, p^b} dq_{q_a, p^b}(t) - e^\alpha H_\alpha(p^{q_a, p^b}, q_{q_a, p^b}) &= e^\alpha p^{q_a, p^b} v_\alpha - e^\alpha p^{q_a, p^b} w_\alpha - e^\alpha w_\alpha q_{q_a, p^b} \\ &= -d\tau^\alpha w_\alpha (q_a + \tau^\beta v_\beta) = -d\tau^\alpha w_\alpha q_a - d\tau^\alpha \tau^\beta A_{\alpha\beta} \\ &= -d\tau^\alpha w_\alpha q_a - \frac{1}{2} d(\tau^\alpha \tau^\beta) A_{\alpha\beta} \end{aligned}$$

with $A_{\alpha\beta}$ the symmetric matrix defined in (23). This implies

$$\int_{t_a}^{t_b} (p^{q_a, p^b} dq_{q_a, p^b}(t) - e^\alpha H_\alpha(p^{q_a, p^b}, q_{q_a, p^b})) = -T^\alpha w_\alpha q_a - \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta}.$$

On the other hand,

$$p^{q_a, p^b}(t_b) q_{q_a, p^b}(t_b) = p^b(q_a + T^\alpha v_\alpha).$$

In conclusion, we get

$$(24) \quad \hat{S}_{\text{HJ}}^f(q_a, p^b, T) = -p^b q_a - T^\alpha (p^b v_\alpha + w_\alpha q_a) - \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta},$$

which generalizes (15) to the the case of several linear constraints.

Remark 4.4 (General change of endpoint conditions). The above example is simple enough for the answer to be computable for a general change of endpoint conditions as in Remark 2.5. Let $f(q, Q)$ be a generating function for the change and fix q_a and Q_b . By inspection in the previous computation, we see that to compute \hat{S}_{HJ}^f we actually only have to solve $q(t)$ for the given initial condition q_a (this is independent of Q_b), which has already been done above. We then get in general

$$(25) \quad \hat{S}_{\text{HJ}}^f(q_a, Q_b; T) = -f(q_a + T^\alpha v_\alpha, Q_b) - T^\alpha w_\alpha q_a - \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta}.$$

Note that this may also be obtained as the composition of the generating functions (24) and (6), following Remark A.36. Namely, let us write

$$S_1(q_a, p^b; T) = -p^b q_a - T^\alpha (p^b v_\alpha + w_\alpha q_a) - \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta},$$

and

$$S_2(q_a, Q_b) = -f(q_a, Q_b).$$

Let

$$S_3(q_a, p^b, \bar{q}_a, Q_b; T) := S_1(q_a, p^b, T) + S_2(\bar{q}_a, Q_b) + p^b \bar{q}_a.$$

The composition of S_1 and S_2 is the evaluation of S_3 at its critical point in (p^b, \bar{q}_a) :

$$\bar{q}_a + \frac{\partial S_1}{\partial p^b} = 0, \quad p^b + \frac{\partial S_2}{\partial \bar{q}_a} = 0.$$

One easily sees that the result is (25).

Remark 4.5 (Integrable systems II). As a generalization of Remark 3.4, we may consider the case where we have exactly n constraints H_1, \dots, H_n in strict involution - an n -dimensional integrable system with vanishing hamiltonians. Around a point (p_0, q_0) where dH_1, \dots, dH_n are linearly independent, again by the Carathéodory–Jacobi–Lie theorem, see footnote 15, we may find a Darboux chart (P, Q) with $P_i = H_i$ (the action-angle coordinates). By the same computation as in Remark 3.4, in this chart the HJ action is expressed as

$$(26) \quad S_{\text{HJ}}^f(q_a, Q_b; T) = -f(q_a, Q_b - T)$$

and

$$(27) \quad S_{\text{HJ}}^g(q_a, P^b; T) = -T^i P_i^b - g(q_a, P^b).$$

4.2. The Lie algebra case. We now suppose that all the $f_{\alpha\beta}^\gamma$ s are constant (but not necessarily zero). They can be viewed as structure constants of some Lie algebra \mathfrak{g} . In turn the e^α s may be viewed as the components of a \mathfrak{g} valued one-form e on $[t_a, t_b]$ or, equivalently, as a connection one-form. We may also view the H_α s as the components of a map $H: T^*M \rightarrow \mathfrak{g}^*$, where T^*M is the target symplectic manifold. The involutivity conditions (19) say that H is an equivariant momentum map, whereas (20) may be read as the condition $d_e H = 0$ that the composition of H with a solution to the evolution equations is covariantly constant. We rewrite the action as

$$S[p, q, e] = \int_{t_a}^{t_b} (p_i dq^i - \langle H(p, q), e \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of \mathfrak{g}^* with \mathfrak{g} . Next we define

$$(28) \quad h(t) = P e^{\int_{t_a}^t e} \in G,$$

where P denotes the path ordered product and G is the simply connected Lie group with Lie algebra \mathfrak{g} . Note that $h(t_a) = 1$ and $e = h^{-1} dh$.²⁰ We write

$$g := h(t_b) = P e^{\int_{t_a}^{t_b} e}.$$

Finally, we define $\tilde{H} := \text{Ad}_h^* H$. Observe that, when evaluated on a solution to the evolution equations, \tilde{H} is constant. We now repeat the computation as in (14):

$$\begin{aligned} \delta S_{\text{HJ}}^f &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \int_{t_a}^{t_b} \langle H(p^{q_a, Q_b}, q_{q_a, Q_b}), \delta e \rangle \\ &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \int_{t_a}^{t_b} \langle \tilde{H}(p^{q_a, Q_b}, q_{q_a, Q_b}), \text{Ad}_h \delta e \rangle \\ &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \left\langle \tilde{H}(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)), \int_{t_a}^{t_b} d(\delta h h^{-1}) \right\rangle \\ &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \left\langle \tilde{H}(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)), \delta g g^{-1} \right\rangle \\ &= P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \langle H(p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b)), g^{-1} \delta g \rangle. \end{aligned}$$

This explicitly shows that S_{HJ}^f is invariant under variations of e that do not change g . As in (22), we define

$$(29) \quad \hat{S}_{\text{HJ}}^f(q_a, Q_b; g) := S_{\text{HJ}}^f(q_a, Q_b)[e], \quad \text{with } g = P e^{\int_{t_a}^{t_b} e},$$

and it immediately follows that \hat{S}_{HJ}^f is a generalized generating function for the evolution relation L .

Example 4.6 (Biaffine case). Consider a target $T^*V = V^* \oplus V \ni (p, q)$ and constraints that are affine both on V and on V^* ; namely,

$$H_\alpha(p, q) = -p_i (\rho_\alpha)^i_j q^j + p_i v_\alpha^i + w_{i\alpha} q^i = -p \rho_\alpha q + p v_\alpha + w_\alpha q,$$

²⁰We use for simplicity the matrix notation. If G is not a matrix Lie group, the correct formula is

$$e(t) = (d_1 l_h)^{-1} d_t h: T_t I \rightarrow \mathfrak{g},$$

where 1 denotes the unit of G and $l_h: G \rightarrow G$ denotes the left translation $g \mapsto hg$.

where, for each α , ρ_α is a given endomorphism of V , v_α is a given vector in V and w_α is a given vector in V^* (we denote the pairing between V^* and V simply by juxtaposition). The involutivity conditions (19) are equivalent to the relations

$$(30) \quad \begin{aligned} [\rho_\alpha, \rho_\beta] &= f_{\alpha\beta}^\gamma \rho_\gamma, \\ \rho_\alpha v_\beta - \rho_\beta v_\alpha &= f_{\alpha\beta}^\gamma v_\gamma, \\ w_\alpha \rho_\beta - w_\beta \rho_\alpha &= f_{\alpha\beta}^\gamma w_\gamma, \\ w_\beta v_\alpha - w_\alpha v_\beta &= 0. \end{aligned}$$

The first relation says that ρ is a representation of the Lie algebra \mathfrak{g} on V , the second and the third that $v: \mathfrak{g} \rightarrow V$ and $w: \mathfrak{g} \rightarrow V^*$ are Lie algebra 1-cocycles, and the last is what we already saw in Example 4.3. The evolution equations read

$$d_e q = e^\alpha v_\alpha = v(e), \quad d_e p = -e^\alpha w_\alpha = -w(e),$$

where we introduced the covariant derivatives

$$d_e q := dq + e^\alpha \rho_\alpha q, \quad d_e p := dp - e^\alpha p \rho_\alpha.$$

Let $R: G \rightarrow \text{Aut}(V)$ denote the representation that integrates ρ to the simply connected Lie group G . If we introduce $\tilde{q}(t) := R_{h(t)} q(t)$ with h defined in (28), then the first evolution relation becomes

$$d\tilde{q} = R_h v(e),$$

which has solution, for the the initial condition $\tilde{q}(t_a) = q(t_a) = q_a$, given by $\tilde{q}(t) = q_a + \int_{t_a}^t R_h v(e)$. Therefore,

$$q_{q_a, p^b}(t) = R_{h(t)}^{-1} \left(q_a + \int_{t_a}^t R_h v(e) \right).$$

Similarly, we get

$$p^{q_a, p^b}(t) = \bar{R}_{h(t)}^{-1} \left(p^b + \int_t^{t_b} \bar{R}_h w(e) \right),$$

where $\bar{R}_h := (R_h^{-1})^*$ is the dual representation on V^* . For later convenience we introduce the 1-forms on G (valued in V and V^* , respectively)

$$\alpha(h) := R_h v(e), \quad \beta(h) := \bar{R}_h w(e).$$

As a consequence of the second and third relations of (30) (and of the fact that the Maurer–Cartan 1-form e is flat), we get that α and β are closed. This implies that $\int_\gamma \alpha$ and $\int_\gamma \beta$ are invariant under homotopies of the path $\gamma: [t_a, t_b] \rightarrow G$ with fixed endpoints. In particular, if γ is a path joining the identity in G to some element g , then we define

$$\Phi(g) := \int_\gamma \alpha, \quad \Psi(g) := \int_\gamma \beta.$$

Note that, since G is simply connected, we get well-defined maps $\Phi: G \rightarrow V$ and $\Psi: G \rightarrow V^*$.²¹ We now want to compute the HJ action. First observe that

$$\begin{aligned} p^{q_a, p^b} dq_{q_a, p^b} - e^\alpha H_\alpha(p^{q_a, p^b}, q_{q_a, p^b}) &= p^{q_a, p^b} d_e q_{q_a, p^b} - p^{q_a, p^b} v(e) - w(e) q_{q_a, p^b} \\ &= -w(e) q_{q_a, p^b} = -w(e) R_h^{-1} \left(q_a + \int_{t_a}^{\bullet} R_h v(e) \right), \end{aligned}$$

so

$$\int_{t_a}^{t_b} (p^{q_a, p^b} dq_{q_a, p^b} - e^\alpha H_\alpha(p^{q_a, p^b}, q_{q_a, p^b})) = -\Psi(g) q_a - \text{WZW}(g)$$

with $g = h(t_b)$. We introduced the following notation (which will become the WZW term in the case of Chern–Simons theory):

$$(31) \quad \text{WZW}(g) := \int_{t_1 < t_2} \beta_2 \alpha_1$$

with $\alpha_i := \pi_i^* \alpha$, $\beta_i := \pi_i^* \beta$ and $\pi_i: [t_a, t_b]^2 \rightarrow G$, $\pi_i(t_1, t_2) = \gamma(t_i)$, where γ is a(ny) path joining the identity to g (e.g., the path $h(t)$). The fact that $\text{WZW}(g)$ is invariant under homotopies of γ with fixed endpoints follows from the fact that α and β are closed and from the fourth relation in (30). Finally, we have

$$p^{q_a, p^b}(t_b) q_{q_a, p^b}(t_b) = p^b R_g^{-1} (q_a + \Phi(g)),$$

so

$$(32) \quad \hat{S}_{\text{HJ}}^f(q_a, p^b; g) = -p^b R_g^{-1} q_a - p^b R_g^{-1} \Phi(g) - \Psi(g) q_a - \text{WZW}(g),$$

which generalizes (24) to the case of biaffine constraints.

Remark 4.7 (Exponential map). We now want to rewrite the WZW term (31) of the last example to facilitate the comparison with the case of Chern–Simons theory under the assumption that the exponential map $\mathfrak{g} \rightarrow G$ is surjective. To simplify the notation, we also assume $t_a = 0$ and $t_b = 1$. In this case, given $g = e^\xi$, we may choose $h(t) = e^{t\xi}$. It follows that $e = \xi dt$. We introduce

$$(33) \quad \begin{aligned} W_2(g) &:= \Psi(g) \Phi(g) = \int_{t_1 < t_2} \beta_2 \alpha_1 + \int_{t_2 < t_1} \beta_2 \alpha_1, \\ W_3(g) &:= \int_0^1 \Psi(h) \rho_e \Phi(h) = \int_{t_1 < t_2, t_3 < t_2} \beta_3 \rho_{e_2} \alpha_1. \end{aligned}$$

We have

$$\begin{aligned} W_3(g) &= \int_{t_1 < t_2, t_3 < t_2} w(e_3) R_{h_3}^{-1} \rho_{e_2} \alpha_1 = \int_{t_1 < t_2, t_3 < t_2} w(\xi) R_{e^{t_3 \xi}}^{-1} \rho(\xi) dt_3 dt_2 \alpha_1 \\ &= \int_{t_1 < t_2, t_3 < t_2} w(\xi) \rho(\xi) R_{e^{t_3 \xi}}^{-1} dt_3 dt_2 \alpha_1 = - \int_{t_1 < t_2, t_3 < t_2} w(\xi) \frac{d}{dt_3} R_{e^{t_3 \xi}}^{-1} dt_3 dt_2 \alpha_1 \\ &= - \int_{t_1 < t_2} w(\xi) R_{e^{t_2 \xi}}^{-1} dt_2 \alpha_1 + \int_{t_1 < t_2} w(\xi) dt_2 \alpha_1 \\ &= - \int_{t_1 < t_2} \beta_2 \alpha_1 + \int_{t_1 < t_2} w(\xi) dt_2 \alpha_1. \end{aligned}$$

²¹One can easily show that $\Phi(g_1 g_2) = \Phi(g_1) + R_{g_1} \Phi(g_2)$ and $\Psi(g_1 g_2) = \Psi(g_1) + \bar{R}_{g_1} \Psi(g_2)$; i.e., Φ and Ψ are 1-cocycles on G with values in V and V^* , respectively. Our formulae give an explicit integration of the Lie algebra 1-cocycles to the corresponding Lie group 1-cocycles, which is guaranteed by general arguments, see [41, Lemma 2.13].

Therefore,

$$W_2(g) - W_3(g) = 2WZW(g) + \left(\int_{t_2 < t_1} \beta_2 \alpha_1 - \int_{t_1 < t_2} w(\xi) dt_2 \alpha_1 \right).$$

We now want to show that the term in brackets vanishes. In fact,

$$\int_{t_2 < t_1} \beta_2 \alpha_1 = \int_{t_2 < t_1} w(\xi) R_{e^{(t_1 - t_2)\xi}} v(\xi) dt_2 dt_1 \stackrel{s=t_1-t_2}{r=t_1} \int_{s < r} w(\xi) R_{e^{s\xi}} v(\xi) dr ds.$$

On the other hand,

$$\int_{t_1 < t_2} w(\xi) dt_2 \alpha_1 = \int_{t_1 < t_2} w(\xi) R_{e^{t_1\xi}} v(\xi) dt_2 dt_1 \stackrel{s=t_1}{r=t_2} \int_{s < r} w(\xi) R_{e^{s\xi}} v(\xi) dr ds.$$

In conclusion,

$$(34) \quad WZW(g) = \frac{1}{2}W_2(g) - \frac{1}{2}W_3(g).$$

Example 4.8 (Adjoint representation). We now specialize the above example and remark to the case when $(V, \rho) = (\mathfrak{g}, \text{ad})$ is the adjoint representation. Note that the second relation of (30) now says that v is a derivation of \mathfrak{g} . As such it may be extended to the universal enveloping algebra as a derivation²² and then to (the image of the exponential map in) the group. We have

$$v(e^\xi) = \int_0^1 e^{\xi t} v(\xi) e^{\xi(1-t)} dt.$$

In this case, for $g = e^\xi$, we get

$$\Phi(g) = v(g)g^{-1},$$

where again for notational simplicity we have assumed to have a matrix Lie group. We further assume that \mathfrak{g} is a quadratic Lie algebra, i.e., it is equipped with an invariant nondegenerate pairing $\langle \cdot, \cdot \rangle$, which we can use to identify \mathfrak{g}^* with \mathfrak{g} . We will then denote by \bar{q} the image of p under this isomorphism (and we will then take the target to be $\mathfrak{g} \oplus \mathfrak{g}$ with symplectic structure induced from the pairing) and by \bar{v} the composition of w with the isomorphism. We can then rewrite (32) as

$$(35) \quad \hat{S}_{\text{HJ}}^f(q_a, p^b; g) = -\langle \bar{q}^b, g^{-1} q_a g \rangle - \langle \bar{q}^b, g^{-1} v(g) \rangle - \langle \bar{v}(g) g^{-1}, q_a \rangle - WZW(g)$$

and, using (33) and (34), the WZW term as

$$(36) \quad WZW(g) = \frac{1}{2} \langle \bar{v}(g) g^{-1}, v(g) g^{-1} \rangle + \frac{1}{2} \int_0^1 \langle \bar{v}(h) h^{-1}, [v(h) h^{-1}, dh h^{-1}] \rangle.$$

Remark 4.9 (A trivial extension). The last example may be generalized to the case when $V = \mathfrak{g} \otimes Z$, with Z a trivial representation of \mathfrak{g} . We keep denoting the representation by ad : $\text{ad}_x(y \otimes z) = [x, y] \otimes z$. Now $v: \mathfrak{g} \rightarrow \mathfrak{g} \otimes Z$ is a derivation in the sense that $v([x, y]) = \text{ad}_x v(y) - \text{ad}_y v(x)$ for all $x, y \in \mathfrak{g}$. We identify V^* with $\mathfrak{g} \otimes Z^*$ using the nondegenerate pairing on \mathfrak{g} and denote by \bar{q} and \bar{v} the composition of p and w with this isomorphism. Note that \bar{v} is also a derivation in the above sense. The HJ action is again given by (35) and (36) with the obvious understanding of notations.

²²This is well-defined because, if v is a Lie algebra derivation and is extended to the tensor algebra as a derivation, then it preserves the ideal generated by elements of the form $\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]$.

Remark 4.10 (General solution via the exponential map). In the general case, but under the assumption that the exponential map $\mathfrak{g} \rightarrow G$ is surjective, the HJ action can also be characterized as follows. As in (29), we set $g = Pe^{\int_{t_a}^{t_b} e}$. We then take $T \in \mathfrak{g}$ with $e^T = g$ and define

$$h(t) = e^{-\frac{t-t_a}{t_b-t_a}T} Pe^{\int_{t_a}^t e}.$$

We have $e = h^{-1}dh + h^{-1}\frac{Tdt}{t_b-t_a}h$, which shows that we can reduce e to $e_0 := \frac{Tdt}{t_b-t_a}$ by a gauge transformation. Since (29) does not depend on which representative of $[e]$ we take, we may pick e_0 . With the change of variable $s = \frac{t-t_a}{t_b-t_a}$, the action then becomes

$$S[p, q, e_0] = \int_0^1 (p_i dq^i - \tilde{H}(p, q) ds), \quad \tilde{H} := \langle H, T \rangle,$$

As a result,

$$(37) \quad \hat{S}_{\text{HJ}}^f(q_a, Q_b; g) = S_{\text{HJ}}^{f, \tilde{H}}(q_a, Q_b; 0, 1),$$

where $S_{\text{HJ}}^{f, \tilde{H}}$ denotes the HJ action for a system with hamiltonian \tilde{H} . This formula is very general, but, unlike the previous ones for the biaffine case, not explicit.

4.3. The general case. We briefly discuss the general case, where we make no simplifying assumptions on the structure functions $f_{\alpha\beta}^\gamma$, only for completeness, as this will not be needed for the rest of the paper. As a consequence, this section, which requires knowledge of Lie algebroids and Lie groupoids, may be safely skipped by the not interested reader.

We will assume that the differentials of the H_α s on their common zero locus C are linearly independent. This in particular implies that C is a submanifold of $N := T^*M$ and, as observed in Remark 4.1, that the hamiltonian vector fields $X_\alpha := X^{H_\alpha}$ restricted to C are linearly independent, so they define a regular distribution: the characteristic distribution of the coisotropic submanifold C . (See Definition A.19 and Remark A.25.)

The first remark is that the structure functions $f_{\alpha\beta}^\gamma$ and the hamiltonian vector fields X_α of the constraints H_α define a Lie algebroid A over C . As a vector bundle, A is the trivial product $C \times \mathbb{R}^k$, where k is the number of constraints. The anchor map $\rho: A \rightarrow TC$ is given by the vector fields X_α :

$$\rho(x, u_\alpha) = X_\alpha(x),$$

where (u_1, \dots, u_k) is the standard basis of \mathbb{R}^k . We can expand a section $\sigma \in \Gamma(A)$ as $\sigma^\alpha u_\alpha$, where now the u_α s are regarded as constant sections of A . We then have $\rho(\sigma) = \sigma^\alpha X_\alpha$. The Lie bracket is defined as

$$[\sigma, \tau]^\gamma = f_{\alpha\beta}^\gamma \sigma^\alpha \tau^\beta + \sigma^\alpha X_\alpha(\tau^\gamma) - \tau^\alpha X_\alpha(\sigma^\gamma).$$

Remark 4.11. The Lie algebroid A is isomorphic to the characteristic distribution of C viewed as a Lie subalgebroid of TC . Note that in general the conormal bundle N^*C of a coisotropic submanifold of a symplectic manifold N has a canonical Lie algebroid structure, isomorphic to the characteristic distribution, which makes it into a Lie subalgebroid of T^*N . In our particular case, since C is defined by constraints, its normal bundle, and therefore its conormal bundle, can be trivialized, and this is what have done above.

The Lie algebroid $A \rightarrow C$ can be extended to the vector bundle $\hat{A} \rightarrow N = T^*M$ with $\hat{A} = N \times \mathbb{R}^k$. We can then view the H_α s as a section H of \hat{A}^* . It is also possible to extend the anchor map and the bracket as defined above, but in general \hat{A} will not be a Lie algebroid (it is, though, when the structure functions are constant).

Next we denote the map $(p, q): I \rightarrow T^*M$, with I the interval $[t_a, t_b]$, as $x: I \rightarrow N$. The field e may be regarded as a section of $T^*I \otimes x^*\hat{A}$. The constraints H_α s composed with x define a section of $x^*\hat{A}^*$, which we keep denoting as H . As a consequence, we may rewrite the action as

$$S[p, q, e] = \int_{t_a}^{t_b} (p_i dq^i - \langle H(p, q), e \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of \hat{A}^* with \hat{A} . A solution (x, e) of the evolution equations is the same as an anchor-preserving morphism $TI \rightarrow \hat{A}$.

The generalized HJ action \hat{S}_{HJ}^f is defined as in (13) and, by Theorem 4.2, is a generalized generating function for the evolution relation.

It is possible to give a more transparent expression for $\hat{S}_{\text{HJ}}^f(q_a, Q_b; [e])$, analogous to the one we have in the Lie algebra case, when the x component of the assumedly unique solution of the evolution equations specified by q_a, Q_b , and e has image in C (note that, if $x(t)$ is in C for some t , then $x(t)$ is in C for every t). In fact, in this case we may view (x, e) as an anchor-preserving morphism $x: TI \rightarrow A$, which is the same as a Lie algebroid morphism $TI \rightarrow A$. Denote by \mathcal{G} the source simply connected Lie groupoid of A (i.e., up to isomorphism, the monodromy groupoid of the characteristic distribution). The Lie algebroid morphism (x, e) , with initial condition $x(t_a) = x_a \in C$, may be then uniquely integrated to a Lie groupoid morphism $E: I \times I \rightarrow \mathcal{G}$. If we denote by α and β the source and target maps of \mathcal{G} , then we have, in particular,

$$x(s) = \alpha(E(s, t)) \quad \forall t \in I, \quad x(t) = \beta(E(s, t)) \quad \forall s \in I.$$

Define $h(t) := E(t_a, t)$. We then have

$$h(t_a) = 1_{x_a}, \quad \alpha(h(t)) = x_a \quad \forall t \in I,$$

where 1_x denotes the unit of \mathcal{G} at $x \in N$. In particular, h is a map $I \rightarrow \alpha^{-1}(x_a)$.

We may now recover e from h , generalizing footnote 20, as follows. First consider the linear map $d_t h: T_t I \rightarrow T_{h(t)} \alpha^{-1}(x_a)$. Next consider the left translation $l_h: \alpha^{-1}(\beta(h)) \rightarrow \alpha^{-1}(\alpha(h))$, $l_h g = hg$. Since $h1_{\beta(h)} = h$, we get the linear map

$$d_{1_{\beta(h)}} l_h: A_{\beta(h)} \rightarrow T_h \alpha^{-1}(\alpha(h)).$$

Using $\beta(h(t)) = x(t)$ and $\alpha(h(t)) = x_a$, we get the linear map

$$e(t) = (d_{1_{x(t)}} l_h)^{-1} d_t h: T_t I \rightarrow A_{x(t)},$$

which may be shown to be the value at t of our section e of $T^*I \otimes x^*\hat{A}$.

The rest of the computation is now like in the Lie algebra case, and we have

$$\hat{S}_{\text{HJ}}^f(q_a, Q_b; g) = S_{\text{HJ}}^f(q_a, Q_b)[e] \quad \text{with } g = h(t_b) \in \alpha^{-1}(x(t_a)) \cap \beta^{-1}(x(t_b)).$$

Finally, we can take variations with respect to q_a, Q_b , and g , getting

$$\delta \hat{S}_{\text{HJ}}^f(q_a, Q_b; g) = P_i(t_b) \delta Q_b^i - p_i(t_a) \delta q_a^i - \left\langle H(x(t_b)), (d_{1_{x(t_b)}} l_g)^{-1} \delta g \right\rangle,$$

with $x(t_b) = (p^{q_a, Q_b}(t_b), q_{q_a, Q_b}(t_b))$. This is consistent with \hat{S}_{HJ}^f being a generalized generating function for the evolution relation L (it is in general a weaker statement,

as we are now only allowed to take variations in the class of variables for which the x component of a solution lies in C).

Remark 4.12. The case of constant structure functions, hence corresponding to a Lie algebra \mathfrak{g} , fits into the general case as the action Lie algebroid $T^*M \times \mathfrak{g}$ with the action Lie groupoid $T^*M \times G$ as its integration. In this case, the Lie algebroid and the Lie groupoid are actually defined over the whole of T^*M and not only over C , so the formula for $\hat{S}_{\text{HJ}}^f(q_a, Q_b; g)$ has no restrictions.

5. SYSTEMS WITH NONTRIVIAL EVOLUTION AND CONSTRAINTS

For completeness, we briefly discuss here the case of systems which, in addition to constraints H_α , $\alpha = 1 \dots k$, also have a nontrivial evolution with hamiltonian H . Namely, we consider an action of the form

$$S[p, q, e] = \int_{t_a}^{t_b} (p_i dq^i - dt H - e^\alpha H_\alpha(p, q)).$$

In addition to assuming involutivity of the constraints, as in (19), we also assume them to be constants of motion for H : namely, $\{H, H_\alpha\} = 0$ for all α .

This system can actually be treated as above simply adding H to the set of the, now $k + 1$, constraints, say, as H_0 . The only difference is that the one-form e^0 instead of being free will be set to be dt . This is a possible choice, as the Lie group \tilde{G} (in the case of structure constants $f_{\alpha\beta}^\gamma$; otherwise, more generally, the Lie groupoid $\tilde{\mathcal{G}}$) corresponding to the rank- $(k + 1)$ Lie algebra (Lie algebroid) factorizes as $\mathbb{R} \times G$ ($\mathbb{R} \times \mathcal{G}$), so we can fix $t_b - t_a$ in the first factor.

The HJ action, which serves as a generalized generating function, is then, in the Lie algebra case,

$$\hat{S}_{\text{HJ}}^f(q_a, Q_b; g) := S_{\text{HJ}}^f(q_a, Q_b)[dt, e], \quad \text{with } g = Pe^{\int_{t_a}^{t_b} e}.$$

In the Lie algebroid case, we get, more generally,

$$\hat{S}_{\text{HJ}}^f(q_a, Q_b; g) := S_{\text{HJ}}^f(q_a, Q_b)[dt, e],$$

with $g := \pi_{\tilde{\mathcal{G}}}(\tilde{h}(t_b)) \in \alpha^{-1}(x(t_a)) \cap \beta^{-1}(x(t_b))$ and $\pi_{\tilde{\mathcal{G}}}$ the projection $\tilde{\mathcal{G}} \rightarrow \mathcal{G}$.

5.1. Classical mechanics. A more conceptual way is to add two new canonically conjugated variables (E, \mathbb{t}) and consider instead the action

$$S[p, q, E, \mathbb{t}, e^0] = \int_{t_a}^{t_b} (p_i dq^i - E d\mathbb{t} - e^0(H - E) - e^\alpha H_\alpha(p, q)).$$

Note that the constraints H_α , $\alpha = 0 \dots k$, with $H_0 = H - E$, are still first class. This is then a system as the ones we have studied in the previous sections. One of the evolution equations is now $dE = 0$, which says that eventually E will be a constant (the energy). Another important evolution equation is $d\mathbb{t} = e^0$, which can be used to substitute $d\mathbb{t}$ to e^0 , so that we may regard \mathbb{t} as time. Note that, if we set endpoint conditions $\mathbb{t}(t_a) = t_a$ and $\mathbb{t}(t_b) = t_b$, we actually get the time evolution from t_a to t_b .

In the following, we focus on the case $H_\alpha = 0$ for all $\alpha > 0$, i.e.,

$$S[p, q, E, \mathbb{t}, e^0] = \int_{t_a}^{t_b} (p_i dq^i - E d\mathbb{t} - e^0(H - E)).$$

The evolution equations are

$$dq^i = e^0 \frac{\partial H}{\partial p_i}, \quad dp_i = -e^0 \frac{\partial H}{\partial q^i}, \quad dE = 0, \quad dt = e^0,$$

and the constraint is $H(p, q) = E$. The evolution relation is then

(38)

$$L = \{(p^a, E^a, q_a, \mathbb{t}_a, p^b, E^b, q_b, \mathbb{t}_b) \mid (p, q)_b = \phi_{\mathbb{t}_b - \mathbb{t}_a}^H((p, q)_a), E^b = E^a = H(p^a, q_a)\}$$

where ϕ_T^H denotes the hamiltonian flow of H for time T .

We now consider endpoint conditions $(q_a, \mathbb{t}_a, q_b, E^b)$. We thus evaluate the modified action $S^f[p, q, E, \mathbb{t}, e^0] := E(t_b)\mathbb{t}(t_b) + S[p, q, E, \mathbb{t}, e^0]$ on the solution, with components $(\tilde{p}, \tilde{q}, \tilde{\mathbb{t}})$, of the evolution equations for fixed $(q_a, \mathbb{t}_a, q_b, E^b)$ and fixed e^0 . The first remark is that $-E d\mathbb{t} + e^0 E$ vanishes on a solution. Therefore,

$$S_{\text{HJ}}^f(q_a, \mathbb{t}_a, q_b, E^b, e^0) = E^b \tilde{\mathbb{t}}(t_b) + \int_{t_a}^{t_b} (\tilde{p}_i d\tilde{q}^i - d\tilde{\mathbb{t}} H(\tilde{p}, \tilde{q})).$$

We can actually pick e^0 constant or at least of constant sign (by a gauge transformation). With this choice we are sure that the map $t \mapsto \tilde{\mathbb{t}}(t) = \mathbb{t}_a + \int_{t_a}^t e^0$ is a diffeomorphism. We can then make this change of variable in the integral, getting

$$\hat{S}_{\text{HJ}}^f(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b; T) = E^b (\mathbb{t}_a + T) + \int_{\mathbb{t}_a}^{\mathbb{t}_a + T} (\check{p}_i d\check{q}^i - d\check{\mathbb{t}} H(\check{p}, \check{q})),$$

with $\check{p}(\check{\mathbb{t}}) := \tilde{p}(t(\mathbb{t}))$, $\check{q}(\check{\mathbb{t}}) := \tilde{q}(t(\mathbb{t}))$ and $T := \int_{t_a}^{t_b} e^0$. But now we recognize in the second term on the right hand side the time-dependent HJ action (2) for the hamiltonian H from time \mathbb{t}_a to time $\mathbb{t}_a + T$. Therefore, we have

$$(39) \quad \begin{aligned} \hat{S}_{\text{HJ}}^f(q_a, \mathbb{t}_a, q_b, E^b; T) &= S_{\text{HJ}}^H(q_a, q_b; \mathbb{t}_a, \mathbb{t}_a + T) + E^b(\mathbb{t}_a + T) \\ &= S_{\text{HJ}}^H(q_a, q_b; 0, T) + E^b(\mathbb{t}_a + T). \end{aligned}$$

One can easily verify that \hat{S}_{HJ}^f is a generating function for the evolution relation (38).

The HJ action \hat{S}_{HJ} for endpoint conditions $(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b)$ can be obtained from (39) by composing, following Remark A.36, \hat{S}_{HJ}^f with the generating function $-E^b \mathbb{t}_b$ of the identity map, and reducing with respect to the intermediate variable E^b ; namely, \hat{S}_{HJ} is the evaluation of $\hat{S}_{\text{HJ}}^f - E^b \mathbb{t}_b$ at its critical point in E^b . We get

$$(40) \quad \hat{S}_{\text{HJ}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b) = S_{\text{HJ}}^H(q_a, q_b; \mathbb{t}_a, \mathbb{t}_b).$$

One can easily verify that \hat{S}_{HJ} is also a generating function for the evolution relation (38). Observe that the dependency on T has disappeared.

Note that we could not have computed \hat{S}_{HJ} directly because fixing the endpoint conditions $(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b)$ yields no solution in general. Namely, choosing e^0 arbitrarily would lead to no solutions if $\mathbb{t}_b - \mathbb{t}_a \neq T := \int_{t_a}^{t_b} e^0$. A way to deal with such a situation will be discussed in Section 6, see Remark 6.2. As a result, T is fixed by the endpoint conditions and therefore is no longer an independent variable for \hat{S}_{HJ} .

The HJ action for the time-independent, constrained system (a parametrization invariant theory) we considered in this section turns out to be the same as the usual HJ action for a time dependent system describing a hamiltonian evolution. In this case, time is restored via the endpoint conditions.

For simplicity we have obtained this result choosing e^0 constant, so that we could make the change of variable globally. We might have also worked with a generic e^0 . In this case we would have made changes of variables in all regions where it is different from zero and ignored the other regions. The “time” variable \mathbb{k} would have suffered slowing down, speeding up, freezing and even rewinding, but with no change in the final result described in (40) by \hat{S}_{HJ} .²³

Remark 5.1 (Integrable systems III). As a generalization of Remarks 3.4 and 4.5, let us consider the case of an integrable system where we have exactly n strictly involutive constraints H_1, \dots, H_n that are integrals of motion for a hamiltonian H , i.e., we have $\{H, H_i\} = \{H_i, H_j\} = 0$ for all i, j . Around a point (p_0, q_0) where the differentials dH_1, \dots, dH_n are linearly independent, again by the Carathéodory–Jacobi–Lie theorem, see fotonote 15, we may find a Darboux chart (P_i, Q^i) with $P_i = H_i$. Changing coordinates from (p_i, q^i) to (P_i, Q^i) in the integral we get

$$S^g[p, q, E, \mathbb{k}, e^0, e] = -g(q(t_a), P(t_a)) + \int_{t_a}^{t_b} (Q^i dP_i - E d\mathbb{k} - e^0(H - E) - e^i P_i).$$

Notice that, since the constraints are integrals of motion, in (P_i, Q^i) coordinates the hamiltonian H depends only on P_i : $0 = \{H, H_i\} = \{H, P_i\} = \partial H / \partial Q_i$. On solutions, we have $dP_i = 0$, $dQ^i = e^i + e^0 \frac{\partial H}{\partial P_i}$ and $dH = 0$. The HJ action then reads

$$(41) \quad \hat{S}_{\text{HJ}}^g(q_a, P^b, \mathbb{k}_a, \mathbb{k}_b; T) = -g(q_a, P^b) - (\mathbb{k}_b - \mathbb{k}_a)H(P^b) - T^i P_i$$

in a neighborhood of (p_0, q_0) .²⁴ We may also compute the HJ action with Q endpoint conditions. We have

$$S^f[p, q, E, \mathbb{k}, e^0, e] = -f(q(t_a), Q(t_a)) + \int_{t_a}^{t_b} (P_i dQ^i - E d\mathbb{k} - e^0(H - E) - e^i P_i).$$

Solving the evolution equation for Q^i we obtain the HJ action²⁵

$$(42) \quad \hat{S}_{\text{HJ}}^f(q_a, Q_b, \mathbb{k}_a, \mathbb{k}_b; T) = -f\left(q_a, Q_b - T - (\mathbb{k}_b - \mathbb{k}_a) \frac{\partial H}{\partial P}\right) + (\mathbb{k}_b - \mathbb{k}_a)H(P(q_a, Q_b)).$$

5.2. The free relativistic particle. The above discussion generalizes easily to the case of a free, relativistic particle. In this case, there is no dynamics but only the constraint $E^2 = m^2 + p^2$. (For simplicity, we omit indices. If we are not in one dimension, p^2 means $\sum_i (p_i)^2$. Similarly, in the action we will simply write pdq instead of $p_i dq^i$). Therefore, we consider the action

$$S[p, q, E, \mathbb{k}, e^0] = \int_{t_a}^{t_b} \left(pdq - E d\mathbb{k} - \frac{e^0}{2}(p^2 + m^2 - E^2) \right),$$

where we have introduced the factor $\frac{1}{2}$ just for convenience. The evolution equations are simply

$$dq = e^0 p, \quad dp = 0, \quad dE = 0, \quad d\mathbb{k} = e^0 E,$$

²³As the prince observed, “Most people think time is like a river that flows swift and sure in one direction, but I have seen the face of time and I can tell you they are wrong. Time is an ocean in a storm,” yet “what is written in the timeline cannot be changed.”

²⁴Here T stands for the collection $T^i := \int_{t_a}^{t_b} e^i$, $i > 0$. The parameter $T^0 := \int_{t_a}^{t_b} e^0$ does not appear, as in (40), because we fix \mathbb{k}_a and \mathbb{k}_b .

²⁵As in footnote 24, T stands for the collection T^i , $i > 0$.

whereas the constraint is

$$p^2 + m^2 - E^2 = 0.$$

We first compute the HJ action, without fixing e^0 , for endpoint conditions given by $(q_a, \mathbb{t}_a, p^b, E^b)$. This means that we have to insert a solution of the evolution equations into

$$S^f[p, q, E, \mathbb{t}, e^0] = E(t_b)\mathbb{t}(t_b) - p(t_b)q(t_b) + S[p, q, E, \mathbb{t}, e^0].$$

The first evolution equation implies $pdq = e^0 p^2$. The second implies that $p(t) = p^b$ for all t , so we actually have $pdq = e^0 (p^b)^2$. The third equation implies $E(t) = E^b$ for all t , which together with the last equation implies $E d\mathbb{t} = e^0 (E^b)^2$. Therefore,

$$S|_{\text{solution}} = \int_{t_a}^{t_b} \frac{e^0}{2} ((p^b)^2 - m^2 - (E^b)^2) = \frac{T}{2} ((p^b)^2 - m^2 - (E^b)^2),$$

where we have set $T := \int_{t_a}^{t_b} e^0$. Solving the evolution equations explicitly yields

$$q(t_b) = q_a + p^b T \quad \text{and} \quad \mathbb{t}(t_b) = \mathbb{t}_a + E^b T,$$

which implies

$$p(t_b)q(t_b) = p^b q_a + (p^b)^2 T \quad \text{and} \quad E(t_b)\mathbb{t}(t_b) = E^b \mathbb{t}_a + (E^b)^2 T$$

on solutions. We then have

$$(43) \quad \hat{S}_{\text{HJ}}^f(q_a, \mathbb{t}_a, p^b, E^b; T) = E^b \mathbb{t}_a - p^b q_a + \frac{(E^b)^2 - (p^b)^2 - m^2}{2} T,$$

and one can easily verify that this is a generating function for the evolution relation.

We may now easily pass to the HJ action for endpoint conditions $(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b)$ simply composing the generating function \hat{S}_{HJ}^f with the generating function $p^b q_b - E^b \mathbb{t}_b$ of the identity map, and reducing with respect to the intermediate variables (p^b, E^b) ; namely, we evaluate $\hat{S}_{\text{HJ}}^f + p^b q_b - E^b \mathbb{t}_b$ at its critical point in (p^b, E^b) . Setting the derivatives with respect to p^b and E^b to zero yields

$$p^b = \frac{\Delta q}{T} \quad \text{and} \quad E^b = \frac{\Delta \mathbb{t}}{T},$$

where we have set

$$\Delta q := q_b - q_a \quad \text{and} \quad \Delta \mathbb{t} := t_b - t_a.$$

Inserting, we get

$$(44) \quad \hat{S}_{\text{HJ}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b; T) = \frac{(\Delta q)^2 - (\Delta \mathbb{t})^2}{2T} - \frac{1}{2} m^2 T.$$

We can finally try to reduce with respect to the variable T , i.e., to evaluate \hat{S}_{HJ} at its critical point in T . Setting the derivative with respect to T to zero yields

$$T^2 = \frac{(\Delta \mathbb{t})^2 - (\Delta q)^2}{m^2}.$$

This shows that we can perform this reduction if and only if the endpoint conditions select a timelike trajectory: $(\Delta \mathbb{t})^2 > (\Delta q)^2$.²⁶ There are two roots in T which yield,

²⁶This is actually a necessary condition for a solution to exist. In fact, solving the evolution equations for \mathbb{t} and q yields $\Delta \mathbb{t} = ET$ and $\Delta q = pT$, so $(\Delta \mathbb{t})^2 - (\Delta q)^2 = (E^2 - p^2)T^2 = m^2 T^2 \geq 0$. The case $(\Delta \mathbb{t})^2 = (\Delta q)^2$ would however imply $T = 0$, so $\Delta \mathbb{t} = 0$ and $\Delta q = 0$, which is excluded because the initial and final conditions must be distinct in the HJ setting.

not unexpectedly,

$$(45) \quad S_{\text{HJ}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b) = \mp m \sqrt{(\Delta \mathbb{t})^2 - (\Delta q)^2},$$

i.e., the Minkowskian length up to the factor $\mp m$. Both choices of sign give a generating function for the evolution relation.²⁷

Remark 5.2. In the massless case $m = 0$, equation (44) becomes

$$\hat{S}_{\text{HJ}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b, T) = \frac{(\Delta q)^2 - (\Delta \mathbb{t})^2}{2T},$$

so the last step—reduction in the variable T —is not possible (the critical point in T does not exist for general values of $q_a, \mathbb{t}_a, q_b, \mathbb{t}_b$). Still, the vanishing of the derivative in T correctly yields the “lightlike displacement” constraint: $(\Delta q)^2 - (\Delta \mathbb{t})^2 = 0$.

6. GENERALIZED GENERATING FUNCTIONS FOR “BAD” ENDPOINT CONDITIONS

The presence of constraints is not the only source for the appearance of extra parameters in the generating function for the evolution relation. Another instance occurs when we choose “bad” endpoint conditions that do not ensure existence of a solution.

6.1. No evolution and no constraints. We discuss a simple example here (more examples, also coupled to the presence of constraints, will appear in the rest of the paper). Consider the action

$$S[p, q] = \int_{t_a}^{t_b} p \dot{q} dt.$$

The EL equations are simply $\dot{p} = \dot{q} = 0$, so the evolution relation is just the diagonal in $T^*\mathbb{R} \times T^*\mathbb{R}$:

$$L = \{(p, q, p, q) \in T^*\mathbb{R} \times T^*\mathbb{R}, (p, q) \in T^*\mathbb{R}\}.$$

If we now choose endpoint conditions q_a and q_b , we will get no solutions unless $q_a = q_b$, in which case we get a whole family of solutions, parametrized by p solving $\dot{p} = 0$. Nevertheless, there is a generalized generating function for L :²⁸

$$S_{\text{gen}}(q_a, q_b; \lambda) = \lambda (q_b - q_a).$$

In fact, $\frac{\partial S_{\text{gen}}}{\partial \lambda} = 0$ yields the condition $q_a = q_b$, whereas $p(t_a) = -\frac{\partial S_{\text{gen}}}{\partial q_a} = \lambda$ and $p(t_b) = \frac{\partial S_{\text{gen}}}{\partial q_b} = \lambda$ imply $p(t_a) = p(t_b)$.

This generating function may also be regarded as a sort of Hamilton–Jacobi action, where we impose only one EL equation: namely, $\dot{p} = 0$. (The other equation, $\dot{q} = 0$, cannot be imposed anyway because generically it has no solutions.) The parameter λ arises here as the constant value of p , so it parametrizes the family of solutions.

We may also obtain the generating function $S_{\text{gen}}(q_a, q_b, \lambda)$ by a partial Legendre transform, i.e., as

$$S_{\text{gen}}(q_a, q_b; \lambda) = \lambda q_b + S_{\text{HJ}}^f(q_a, \lambda),$$

²⁷Note that taking derivatives with respect to \mathbb{t}_a and \mathbb{t}_b yields $E^a = E^b = \pm m \frac{\Delta \mathbb{t}}{\sqrt{(\Delta \mathbb{t})^2 - (\Delta q)^2}}$.

In particular, for $\Delta q = 0$, we get $E^a = E^b = \pm m \operatorname{sgn} \Delta \mathbb{t}$. For $\Delta \mathbb{t} > 0$, it is the plus sign that yields the relation $E = m$ for the relativistic particle, the minus sign corresponding to its antiparticle.

²⁸This generalized generating function can also be obtained as the composition, in the sense of Remark A.37, of the generating functions $\psi_1(q_a, \lambda) = -\lambda q_a$ and $\psi_2(\lambda, q_b) = \lambda q_b$ that do and undo a $-\pi$ -rotation in phase space.

where

$$S_{\text{HJ}}^f(q_a, p^b) = -p^b q_a$$

is the HJ action for the good choice (q_a, p^b) of endpoint conditions. Unlike the true Legendre transform (see Remark 2.9), we do not evaluate at λ^{crit} , which in this case does not exist. We will put this observation more in context in Section 6.2.

The generating function S_{gen} may appear even more naturally from the viewpoint of path integral quantization. In fact, since there is no evolution at all, the evolution operator is the identity operator, so, for the chosen polarizations, its integral kernel is a delta function:

$$K(q_a, q_b) = \delta(q_a - q_b).$$

By Fourier transform, we we can also write

$$K(q_a, q_b) = \int \frac{d\lambda}{2\pi\hbar} e^{\frac{i}{\hbar} S_{\text{gen}}(q_a, q_b, \lambda)}.$$

One formal way to get this result directly from the path integral is as follows. We start with (4) and write $p = \lambda + \hat{p}$, where λ is a constant (a solution to $\dot{p} = 0$). Note that λ here is still a variable to be integrated out, so we have to make sure that $\hat{p}: [t_a, t_b] \rightarrow \mathbb{R}$ is in a complement to the space of constant maps. For example, we may impose $\int_{t_a}^{t_b} \hat{p} dt = 0$.²⁹ Then we have

$$S[\lambda + \hat{p}, q] = \lambda(q_b - q_a) + \int_{t_a}^{t_b} \hat{p} \dot{q} dt.$$

Inserting into (4), we get

$$K(q_a, q_b) = \int d\lambda e^{\frac{i}{\hbar} \lambda (q_b - q_a)} \int_{\substack{q(t_a)=q_a \\ q(t_b)=q_b \\ \int_{t_a}^{t_b} \hat{p} dt=0}} D\hat{p} Dq e^{\frac{i}{\hbar} \int_{t_a}^{t_b} \hat{p} \dot{q} dt}.$$

The second integral formally does not depend on q_a and q_b : in fact, we can make the affine change of variables $q \mapsto \hat{q}$ with

$$\hat{q}(t) = q(t) - \frac{t - t_b}{t_a - t_b} q_a - \frac{t - t_a}{t_b - t_a} q_b.$$

Observe that $\int_{t_a}^{t_b} \hat{p} \dot{q} dt = \int_{t_a}^{t_b} \hat{p} \dot{\hat{q}} dt$, since $\dot{q} - \dot{\hat{q}}$ is constant, and that $\hat{q}(t_a) = \hat{q}(t_b) = 0$. Therefore,

$$\begin{aligned} K(q_a, q_b) &= \int d\lambda e^{\frac{i}{\hbar} \lambda (q_b - q_a)} \int_{\substack{\hat{q}(t_a)=\hat{q}(t_b)=0 \\ \int_{t_a}^{t_b} \hat{p} dt=0}} D\hat{p} D\hat{q} e^{\frac{i}{\hbar} \int_{t_a}^{t_b} \hat{p} \dot{\hat{q}} dt} \\ &\propto \int \frac{d\lambda}{2\pi\hbar} e^{\frac{i}{\hbar} S_{\text{gen}}(q_a, q_b, \lambda)}. \end{aligned}$$

In this example, one might also think of λ as parametrizing the vacua of the theory. Instead of integrating over vacua, one may decide to select just one, labeled by λ , and in this case interpret $S_{\text{gen}}(q_a, q_b, \lambda)$ as the corresponding semiclassical contribution, more in line with Hamilton–Jacobi.

²⁹Let \mathfrak{c} be the space of constant maps and \mathfrak{d} the space of maps $\hat{p}: [t_a, t_b] \rightarrow \mathbb{R}$ with vanishing integral. We have $C^\infty([t_a, t_b]) = \mathfrak{c} \oplus \mathfrak{d}$. In fact, if λ is constant and $(t_a - t_b)\lambda = \int_{t_a}^{t_b} \lambda dt = 0$, then λ vanishes, and we have shown that $\mathfrak{c} \cap \mathfrak{d} = 0$. On the other hand, every map p can be written as $\lambda + \hat{p}$ with $\lambda = \frac{1}{t_b - t_a} \int_{t_a}^{t_b} p(t) dt$. It follows immediately that $\int_{t_a}^{t_b} \hat{p} dt = 0$.

Remark 6.1. For an analysis of this example in the BV-BFV setting, see [19, Sect. 4.4]

Remark 6.2 (Classical mechanics). We may apply the above considerations to the system with one constraint described in Section 5.1. We fix $(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b)$ and let $T = \int_{\mathbb{t}_a}^{\mathbb{t}_b} e^0$. From the evolution equation $d\mathbb{t} = e^0$, we get that there is no solution unless $T = \mathbb{t}_b - \mathbb{t}_a$. Therefore, we have to introduce a new parameter λ to fix this condition, and we get

$$\check{S}_{\text{HJ}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b; T, \lambda) = S_{\text{HJ}}^H(q_a, q_b; \mathbb{t}_a, \mathbb{t}_a + T) + \lambda(T - \mathbb{t}_b + \mathbb{t}_a).$$

Of course this generating function may be reduced. Setting the derivative with respect to λ to zero, we get back $T = \mathbb{t}_b - \mathbb{t}_a$, which may be inserted into \check{S}_{HJ} yielding \hat{S}_{HJ} as in (40).

6.2. The partial Legendre transform. Suppose our system has a unique solution for endpoint conditions (q_a, p^b) , so that we have the HJ action $S_{\text{HJ}}^f(q_a, p^b)$ as in Example 2.8. We define the partial Legendre transform of S_{HJ}^f as

$$S_{\text{gen}}(q_a, q_b, \lambda) := \lambda_i q_b^i + S_{\text{HJ}}^f(q_a, \lambda).$$

It follows from (7) that S_{gen} is a generalized generating function for the same evolution relation as S_{HJ}^f is. In fact, $L_{S_{\text{gen}}}$ is determined by S_{gen} via the equations

$$p_i^b = \frac{\partial S_{\text{gen}}}{\partial q_b^i}, \quad p_i^a = -\frac{\partial S_{\text{gen}}}{\partial q_a^i}, \quad \frac{\partial S_{\text{gen}}}{\partial \lambda} = 0.$$

Since $\frac{\partial S_{\text{gen}}}{\partial q_b^i} = \lambda_i$, we get $p^b = \lambda$. Moreover, by using (7), the last equation reads $q_b - q_{q_a, \lambda}(t_b) = 0$ and the middle equation yields $p^a = p^{q_a, \lambda}(t_a)$. Therefore, $L_{S_{\text{gen}}}$ consists of endpoint values (p^a, q_a, p^b, q_b) of a solution and is thus the evolution relation.³⁰

The above argument extends immediately to the case with constraints. Namely, suppose we have the HJ action $S_{\text{HJ}}^f(q_a, p^b)[e]$. Then we define its partial Legendre transform as

$$S_{\text{gen}}(q_a, q_b, \lambda)[e] := \lambda_i q_b^i + S_{\text{HJ}}^f(q_a, \lambda)[e],$$

and one immediately verifies, as above, that this is also a generalized generating function for the evolution relation.

Example 6.3 (Partial Legendre transform for linear constraints). Consider the case of a system with linear constraints in strict involution as described in Example 4.3. We will suppress the target-space index i . If we apply the partial Legendre transform to (24), we get

$$\hat{S}_{\text{gen}}(q_a, q_b; \lambda, T) := \lambda q_b + \hat{S}_{\text{HJ}}^f(q_a, \lambda; T) = \lambda(q_b - q_a) - T^\alpha(\lambda v_\alpha + w_\alpha q_a) - \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta},$$

which is also a generating function for the evolution relation. We may change this expression a bit by observing that the derivative with respect to λ (i.e., $q_b - q_a - T^\alpha v_\alpha$) will have to be set to zero to define the evolution relation. We can insert this

³⁰It may of course happen that the last equation, $q_b - q_{q_a, \lambda}(t_b) = 0$, can be solved for a unique λ , which we then denote as λ^{crit} . We may then insert this value into $S_{\text{gen}}(q_a, q_b, \lambda)$ and get a generating function without extra parameters, the true Legendre transform $S_{\text{HJ}}(q_a, q_b)$ as in Remark 2.9.

relation into \hat{S}_{gen} (without having to solve with respect to λ). Actually, multiplying the relation with $T^\beta w_\beta$ and using (23), we get

$$T^\alpha T^\beta A_{\alpha\beta} = T^\beta w_\beta q_b - T^\beta w_\beta q_a.$$

We can use this to get rid of the quadratic term in the T s getting the generalized generating function

$$(46) \quad \begin{aligned} \tilde{S}_{\text{gen}}(q_a, q_b, \lambda, T) &= \lambda(q_b - q_a) - T^\alpha \lambda v_\alpha - \frac{1}{2} T^\alpha w_\alpha q_a - \frac{1}{2} T^\alpha w_\alpha q_b \\ &= \left(\lambda - \frac{1}{2} T^\alpha w_\alpha \right) q_b - \left(\lambda + \frac{1}{2} T^\alpha w_\alpha \right) q_a - T^\alpha \lambda v_\alpha. \end{aligned}$$

7. INFINITE-DIMENSIONAL TARGETS

For simplicity, up to now we have only discussed examples where the target symplectic manifold is finite-dimensional. The whole discussion can be generalized, almost verbatim, to the case when the target is infinite-dimensional and there are possibly infinitely many constraints. We will touch upon more analytical issues in Appendix A, and in particular in A.3. Note that the main problem is that, in principle, the evolution relation might fail to be lagrangian.³¹ In this section we will only be considering infinite-dimensional examples where this problem does not actually arise (mainly thanks to the Hodge decomposition of differential forms).

7.1. Three-dimensional abelian Chern–Simons theory. As a warm up, we discuss here the case of abelian Chern–Simons theory in three dimensions. The action for a cylinder $I \times \Sigma$, where $I = [t_a, t_b]$ is an interval and Σ is a closed, oriented surface, is

$$S[A] = \frac{1}{2} \int_{I \times \Sigma} \text{Ad}A$$

with $A \in \Omega^1(I \times \Sigma)$. A variation of the action yields

$$\delta S = \int_{I \times \Sigma} \delta \text{Ad}A - \frac{1}{2} \int_{\{t_b\} \times \Sigma} A \delta A + \frac{1}{2} \int_{\{t_a\} \times \Sigma} A \delta A,$$

from which we read off the EL equations $dA = 0$ and the boundary symplectic structure:

$$\mathcal{F}^\partial = \Omega^1(\Sigma), \quad \omega = -\frac{1}{2} \int_{\Sigma} \delta A \delta A.$$

We may split A into its “horizontal” and “vertical” part

$$A = A_\Sigma + A_I$$

and regard A_Σ as a map $I \rightarrow \Omega^1(\Sigma) = \mathcal{F}^\partial$, with target symplectic form $-\frac{1}{2} \int_{\Sigma} \delta A_\Sigma \delta A_\Sigma$, and A_I as a one-form on I with values in $\Omega^0(\Sigma)$. Splitting also d as $d_\Sigma + d_I$, we then have

$$S'[A_\Sigma, A_I] := S[A_\Sigma + A_I] = \int_{I \times \Sigma} \left(\frac{1}{2} A_\Sigma d_I A_\Sigma + A_I d_\Sigma A_\Sigma \right),$$

from which we see that A_Σ represents the dynamical variables, whereas A_I represents the Lagrange multipliers. Moreover, we now split the EL equations into the evolution equations

$$d_I A_\Sigma = -d_\Sigma A_I$$

³¹This is due to intrinsically analytical problems. See [16, Sect. 5.9] for an example.

and the constraints

$$d_\Sigma A_\Sigma = 0.$$

This is a system with linear constraints (in strict involution, since $d^2 = 0$) that, almost, fits into Example 4.3. What is missing is a splitting of the symplectic space of fields into p and q variables (or, more precisely, a splitting in the sense of Remark A.6). We do it in the complexified version $\mathcal{F}_\mathbb{C}^\partial = \Omega^1(\Sigma) \otimes \mathbb{C}$, which splits as $\Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma)$ in terms of a complex structure on Σ .³² We write accordingly $A_\Sigma = A^{1,0} + A^{0,1}$ and $d_\Sigma = \partial + \bar{\partial}$, so

$$S'[A_\Sigma, A_I] = \int_{I \times \Sigma} \left(\frac{1}{2} A^{1,0} d_I A^{0,1} + \frac{1}{2} A^{0,1} d_I A^{1,0} + A_I (\bar{\partial} A^{1,0} + \partial A^{0,1}) \right).$$

We can integrate by parts the second term, so that the action reduces, up to a boundary term, to

$$S''(A^{1,0}, A^{0,1}, A_I) = \int_{I \times \Sigma} (A^{1,0} d_I A^{0,1} + A_I (\bar{\partial} A^{1,0} + \partial A^{0,1})),$$

where we recognize $A^{1,0}$ as the p variables and $A^{0,1}$ as the q variables. The variation now, correctly, produces the boundary one-form $\int_\Sigma A^{1,0} \delta A^{0,1}$.

We can finally apply the results of Example 4.3 to this case where, up to signs, the matrix v_α^i is now replaced by the operator $\bar{\partial}$ and the matrix $w_{i\beta}$ is replaced by the operator ∂ . We write $\sigma := \int_I A_I \in \Omega^0(\Sigma)$ instead of T , $A_a^{0,1}$ instead of q_a and $A_b^{1,0}$ instead of p^b , so equation (24) becomes

$$(47) \quad \hat{S}_{\text{HJ}}^f = \int_\Sigma \left(A_b^{1,0} A_a^{0,1} + \sigma (\bar{\partial} A_b^{1,0} + \partial A_a^{0,1}) + \frac{1}{2} \sigma \partial \bar{\partial} \sigma \right).$$

Following Example 6.3 we may also get the generating function for the same choice of both endpoint conditions. Namely, extending (46) to this case, we get

$$(48) \quad \tilde{S}_{\text{gen}} = \int_\Sigma \left(A_b^{1,0} \left(\lambda + \frac{1}{2} \bar{\partial} \sigma \right) - A_a^{1,0} \left(\lambda - \frac{1}{2} \bar{\partial} \sigma \right) + \lambda \partial \sigma \right)$$

with $\lambda \in \Omega^{0,1}(\Sigma)$.

7.2. Nonabelian Chern–Simons theory. In this case the action is

$$S[A] = \int_{I \times \Sigma} \left(\frac{1}{2} \langle A, dA \rangle + \frac{1}{6} \langle A, [A, A] \rangle \right),$$

where now A takes values in a quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.

Proceeding as in the abelian case, we get

$$S''(A^{1,0}, A^{0,1}, A_I) = \int_{I \times \Sigma} (\langle A^{1,0}, d_I A^{0,1} \rangle + \langle A_I, \bar{\partial} A^{1,0} + \partial A^{0,1} + [A^{0,1}, A^{1,0}] \rangle),$$

where we recognize the data of Example 4.8 and Remark 4.9 with $-A^{1,0}$ as the \bar{q} , $A^{0,1}$ as the q , A_I as the e variables, ∂ as the derivation \bar{v} , and $\bar{\partial}$ as the derivation

³²This actually corresponds to choosing a complex polarization for the real Chern–Simons theory.

v .³³ We write $A_a^{0,1}$ instead of q_a and $-A_b^{1,0}$ instead of \bar{q}^b , so (35) becomes

$$\hat{S}_{\text{HJ}}^f = \int_{\Sigma} \left(\langle A_b^{1,0}, g^{-1} A_a^{0,1} g \rangle + \langle A_b^{1,0}, g^{-1} \bar{\partial} g \rangle + \langle A_a^{0,1}, \partial g \cdot g^{-1} \rangle \right) + \text{WZW}(g)$$

with $g = \text{Pe}^{\int_I A_I} \in \Omega^0(\Sigma, G)$. For $g = e^{\xi}$, $\xi \in \Omega^0(\Sigma, \mathfrak{g})$, from (36) we get

$$\text{WZW}(g) = -\frac{1}{2} \int_{\Sigma} \langle \partial g \cdot g^{-1}, \bar{\partial} g \cdot g^{-1} \rangle - \frac{1}{12} \int_{\Sigma \times I} \langle dh \cdot h^{-1}, [dh \cdot h^{-1}, dh \cdot h^{-1}] \rangle,$$

with $h = e^{t\xi}$. Thus, the HJ action of Chern–Simons theory can be identified with a “gauged WZW action” (see for instance [33]). This points at a deep relationship between these two theories. We will revisit this correspondence in more detail in [21].

7.3. Nonabelian BF theory. Consider BF theory, defined by the action

$$(49) \quad S_{BF} = \int_M \langle B, F_A \rangle.$$

Here M is an n -manifold and the fields are a \mathfrak{g} -valued 1-form A (viewed as a connection with curvature 2-form $F_A = dA + \frac{1}{2}[A, A]$) and a \mathfrak{g}^* -valued $(n-2)$ -form B ;³⁴ \mathfrak{g} is the Lie algebra of some fixed Lie group G .

On an n -dimensional cylinder $M = I \times \Sigma$, the action becomes

$$(50) \quad S_{BF} = \int_{I \times \Sigma} \langle B_{\Sigma}, d_I A_{\Sigma} \rangle + \langle A_I, d_{A_{\Sigma}} B_{\Sigma} \rangle + \langle B_I, F_{A_{\Sigma}} \rangle,$$

where $d_{A_{\Sigma}} B_{\Sigma} = dB_{\Sigma} + [A_{\Sigma}, B_{\Sigma}]$ is the covariant exterior derivative. Here we expanded the fields according to form degree on I : $A = A_{\Sigma} + A_I$, $B = B_{\Sigma} + B_I$. Note that A_I and B_I in (50) are Lagrange multipliers corresponding to the constraints $d_{A_{\Sigma}} B_{\Sigma} = 0$ and $F_{A_{\Sigma}} = 0$.

Using A_a, B_b boundary conditions, the generalized HJ action reads

$$\hat{S}_{\text{HJ}}(A_a, B_b; g, \beta) = \int_{\Sigma} \langle \beta, F_{A_a} \rangle - (-1)^n \langle B_b, g^{-1} A_a g + g^{-1} d_{\Sigma} g \rangle.$$

Here the auxiliary fields are $g = \text{Pe}^{\int_a^{t_b} A_I} \in \text{Map}(\Sigma, G)$ and $\beta = \int_a^{t_b} \text{Ad}_{\text{Pe}^{\int_a^{t_a} A_I}}^*(B_I) \in \Omega^{n-3}(\Sigma, \mathfrak{g}^*)$. The case of (A_a, A_b) boundary conditions is treated by the partial Legendre transform:

$$S_{\text{gen}}(A_a, A_b; g, \beta, \lambda) = \int_{\Sigma} \langle \beta, F_{A_a} \rangle - (-1)^n \langle \lambda, g^{-1} A_a g + g^{-1} d_{\Sigma} g \rangle + (-1)^n \langle \lambda, A_b \rangle.$$

A similar expression appeared in [43] as a holographic dual of BF theory (in the three-dimensional case and with a different—holomorphic—polarization on the boundary).

³³To be more precise, the Lie algebra of Remark 4.9 is now $\Omega^0(\Sigma, \mathfrak{g})$ with pointwise Lie bracket induced from that of \mathfrak{g} , the trivial representation space Z is $\Omega^{0,1}(\Sigma)$, and their tensor product, over $\Omega^0(\Sigma)$, is $V = \Omega^{0,1}(\Sigma, \mathfrak{g})$. The dual space V^* is now replaced by $\Omega^{1,0}(\Sigma, \mathfrak{g})$ with pairing to V induced by the pairing $\langle \cdot, \cdot \rangle$ on \mathfrak{g} and integration on Σ .

³⁴More generally, A is a connection in a principal G -bundle \mathcal{P} over M and B is an $(n-2)$ -form valued in the coadjoint bundle of \mathcal{P} . In the context of a cylinder $M = I \times \Sigma$, we would take $\mathcal{P} = \pi^* \mathcal{P}_{\Sigma}$ where \mathcal{P}_{Σ} is a G -bundle over Σ and $\pi: I \times \Sigma \rightarrow \Sigma$ is the projection. The results of this subsection and of subsection 7.4 generalize immediately to this setup.

Remark 7.1 (Relation to Group Quantum Mechanics). Specializing to $n = 2$, and assuming Σ to be connected, we have $\Sigma = S^1$. On $\Omega^\bullet(S^1, \mathfrak{g})$ we can consider a set of canonical coordinates given by $Q = B$, $P = A - *_{S^1}B$, where $*_{S^1}$ denotes the Hodge star for some choice of Riemannian metric on S^1 . We choose the coordinates $q = A, p = B$ at t_a and Q, P at t_b . The corresponding generating function is $f(A, B) = \int_{S^1} \langle B, A \rangle - \frac{1}{2} \langle B, *_{S^1}B \rangle$ and the HJ action is given by

$$(51) \quad \hat{S}_{\text{HJ}}^f(A_a, B_b, g) = - \int_{S^1} \langle B_b, g^{-1}A_a g + g^{-1}d_{S^1}g \rangle - \frac{1}{2} \langle B_b, *_{S^1}B_b \rangle.$$

One may further specialize by choosing the boundary condition $A_a = 0$ and imposing the equation of motion $B_b = - *_{S^1} g^{-1}dg$ for B_b . The action then reduces to

$$(52) \quad S_{GQM}(g) = \frac{1}{2} \int_{S^1} \langle g^{-1}dg, *_{S^1}g^{-1}dg \rangle,$$

the action for a free particle moving in the group G , studied for instance in [32]. We thus recover a relation between a 2D theory and group quantum mechanics which could be seen as an instance of holographic correspondence. In the case $G = PSL(2, \mathbb{R})$ (corresponding to a model of 2D gravity whose classical solutions are constant negative curvature geometries) it was discussed in the literature on AdS₂/CFT₁ correspondence. In [48] a very interesting subsequent reduction from S_{GQM} was considered, in the case $G = PSL(2, \mathbb{R})$, leading to the Schwarzian 1D theory.

7.4. More examples: 2D Yang–Mills theory and electrodynamics in general dimension. In this section we discuss two examples of systems with nontrivial evolution and constraints: two special cases of Yang–Mills theory on a cylinder. They provide (generalized) examples of the setup of Section 5.

Consider the first-order formulation of Yang–Mills theory,

$$(53) \quad S_{\text{YM}}[A, B] = \int_M \langle B, F_A \rangle - \frac{1}{2} \langle B, *B \rangle,$$

where M is a pseudo-Riemannian n -manifold which we will take to be a cylinder $M = I \times \Sigma$ with product metric $g = -(dt)^2 + g_\Sigma$ and g_Σ a Riemannian metric on Σ ; here $*$ is the associated Hodge star operator. Note that the action (53) is an extension of the BF action (49) by a metric-dependent term.

Expanding the fields on the cylinder according to form degree along I as $A = A_\Sigma + A_I$, $B = B_\Sigma + B_I$, we can write the action (53) as

$$(54) \quad S_{\text{YM}} = \int_{I \times \Sigma} \langle B_\Sigma, d_I A_\Sigma - \frac{1}{2} * B_\Sigma \rangle + \langle B_I, F_{A_\Sigma} - \frac{1}{2} * B_I \rangle + \langle A_I, d_{A_\Sigma} B_\Sigma \rangle.$$

The EL equations read

$$(55a) \quad d_I A_\Sigma + d_{A_\Sigma} A_I - *B_\Sigma = 0,$$

$$(55b) \quad d_I B_\Sigma + [A_I, B_\Sigma] + d_{A_\Sigma} B_I = 0,$$

$$(55c) \quad F_{A_\Sigma} - *B_I = 0,$$

$$(55d) \quad d_{A_\Sigma} B_\Sigma = 0.$$

Here A_I is a Lagrange multiplier and (55d) is the corresponding constraint; B_I is not a Lagrange multiplier (enters the action quadratically) and although the corresponding equation (55c) does not contain time derivatives, it does not impose

constraints on the fields in the phase space (which B_I is not a part of). Therefore, (55c) is treated as part of the evolution relations, alongside (55a) and (55b), for the purpose of computing the HJ action.

Introducing a group-valued variable $h(t) = Pe^{\int_{t_a}^t A_I} \in \text{Map}(I \times \Sigma, G)$ and performing a change of variables

$$A_\Sigma = h^{-1} \tilde{A}_\Sigma h + h^{-1} d_\Sigma h, \quad B_\Sigma = h^{-1} \tilde{B}_\Sigma h, \quad B_I = h^{-1} \tilde{B}_I h$$

we can rewrite the evolution equations as

$$(56) \quad \begin{aligned} \partial_t^2 \tilde{A}_\Sigma + (-1)^n *_\Sigma d_{\tilde{A}_\Sigma} *_\Sigma F_{\tilde{A}_\Sigma} &= 0, \\ \tilde{B}_I &= *F_{\tilde{A}_\Sigma}, \quad \tilde{B}_\Sigma = *_\Sigma \partial_t \tilde{A}_\Sigma. \end{aligned}$$

Thus, essentially, one has to solve the (generally very complicated) nonlinear equation (56) and recover the remaining fields using the other two equations. Equation (56) becomes linear in two cases: the case $n = 2$ (as the second term in (56) vanishes by a degree reason) and the abelian case.

7.4.1. 2D Yang–Mills theory. In the two-dimensional case (with $\Sigma = S^1$), we have $B_I = 0$ for a degree reason (and thus $B = B_\Sigma$), so we are exactly in the setting of Section 5. The action (54) on $I \times S^1$ becomes

$$S_{\text{YM}} = \int_{I \times S^1} \langle B, d_I A_\Sigma - \frac{1}{2} * B \rangle + \langle A_I, d_{A_\Sigma} B \rangle.$$

In terms of the boundary conditions (A_a, B_b) , we obtain the following generalized HJ action:

$$\hat{S}_{\text{HJ}}^f(A_a, B_b; g) = \int_\Sigma -\langle B_b, g^{-1} A_a g + g^{-1} d_\Sigma g \rangle - \frac{\tau}{2} \langle B_b, *_\Sigma B_b \rangle,$$

where $\tau = t_b - t_a$. Here $g = h(t_b) = Pe^{\int_{t_a}^{t_b} A_I} \in \text{Map}(\Sigma, G)$, the group-valued auxiliary variable for the generalized HJ action. We can also consider (A_a, A_b) boundary conditions, and then the answer is

$$S_{\text{gen}}(A_a, A_b; g, \lambda) = \int_\Sigma -\langle \lambda, g^{-1} A_a g + g^{-1} d_\Sigma g \rangle - \frac{\tau}{2} \langle \lambda, *_\Sigma \lambda \rangle + \langle \lambda, A_b \rangle.$$

Here $\lambda \in \Omega^1(\Sigma, \mathfrak{g}^*)$ is the new auxiliary field.

7.4.2. Electrodynamics (general dimension). Next consider Yang–Mills theory (53) on a cylinder of general dimension in the abelian case, $G = \mathbb{R}$. Solving the evolution equations and plugging the result into the action, we find, for boundary conditions (A_a, B_b) , the following generalized HJ action:³⁵

$$(57) \quad \begin{aligned} \hat{S}_{\text{HJ}}(A_a, B_b; \gamma) &= \int_\Sigma \left(-\frac{\tau}{2} B_b^{\text{cocl}} \wedge *_\Sigma B_b^{\text{cocl}} - A_a^{\text{cl}} \wedge B_b^{\text{cocl}} - d_\Sigma \gamma \wedge B_b \right) + \\ &+ \sum_\sigma \left(-\frac{1}{2} \frac{\tan \omega_\sigma \tau}{\omega_\sigma} (B_b^\sigma)^2 - \frac{1}{2} \omega_\sigma \tan \omega_\sigma \tau (A_a^\sigma)^2 - \frac{1}{\cos \omega_\sigma \tau} A_a^\sigma B_b^\sigma \right). \end{aligned}$$

Here:

³⁵In particular, equation (56) reads $\partial_t^2 \tilde{A}_\Sigma + (-1)^n *_\Sigma d_\Sigma *_\Sigma d_\Sigma \tilde{A}_\Sigma = 0$ (where the change of variables is $A_\Sigma = \tilde{A}_\Sigma + d_\Sigma \int_{t_a}^t A_I$). It breaks into two equations for the closed and coexact parts of \tilde{A}_Σ : $\partial_t^2 \tilde{A}_\Sigma^{\text{cl}} = 0$ and $\partial_t^2 \tilde{A}_\Sigma^{\text{coex}} = -\Delta_\Sigma A_\Sigma^{\text{coex}}$. The solution of the boundary problem is then $\tilde{A}_\Sigma^{\text{cl}} = A_a^{\text{cl}} + (-1)^n *_\Sigma B_b^{\text{cocl}} t$, $\tilde{A}_\Sigma^{\text{coex}} = \sum_\sigma \chi_\sigma \left(\frac{\sin \omega_\sigma t}{\omega_\sigma \cos \omega_\sigma (t_b - t_a)} B_b^\sigma + \frac{\cos \omega_\sigma (t_b - t)}{\cos \omega_\sigma (t_b - t_a)} A_a^\sigma \right)$.

- $\tau = t_b - t_a$.
- $\gamma = \int_{t_a}^{t_b} A_I \in C^\infty(\Sigma)$ is an auxiliary field.
- We introduce $\{\chi_\sigma\}$, an orthonormal basis of coexact 1-forms on Σ which diagonalize the Laplacian: $\Delta_\Sigma \chi_\sigma = \omega_\sigma^2 \chi_\sigma$. We split the field A_a , according to the Hodge decomposition, into its closed and coexact parts: $A_a = A_a^{\text{cl}} + \sum_\sigma \chi_\sigma A_a^\sigma$.
- Similarly, $\{*\Sigma\chi_\sigma\}$ is an orthonormal basis of exact $(n-2)$ -forms on Σ and we have a splitting $B_b = B_b^{\text{cocl}} + \sum_\Sigma * \chi_\sigma B_b^\sigma$ into coclosed and exact parts.

Note that the term $\int_\Sigma d_\Sigma \gamma \wedge B_b$ in \hat{S}_{HJ} depends in fact only on the coexact part of B_b .

We also remark that the summand in the sum over σ in (57) is the HJ action of the harmonic oscillator with frequency ω_σ . Here the oscillator corresponds to the field $A^\sigma(t)$ (the coefficient of the expansion of A_Σ^{coex} in the basis $\{\chi_\sigma\}$) with corresponding momentum $B^\sigma(t)$ (the coefficient of the expansion of B_Σ^{ex} in the basis $\{*\Sigma\chi_\sigma\}$).

Remark 7.2 (p -form electrodynamics). The case of abelian Yang–Mills theory generalizes straightforwardly to “ p -form electrodynamics”—the theory defined by the action

$$S_{p\text{-form}} = \int_M B \wedge dA - \frac{1}{2} B \wedge *B,$$

where the fields are $A \in \Omega^p(M)$, $B \in \Omega^{n-p-1}(M)$. The generalized HJ action on a cylinder is given by an expression identical to (57) up to signs:

$$\begin{aligned} \hat{S}_{\text{HJ}}(A_a, B_b; \gamma) &= \int_\Sigma \left(-\frac{\tau}{2} B_b^{\text{cocl}} \wedge * \Sigma B_b^{\text{cocl}} + \epsilon A_a^{\text{cl}} \wedge B_b^{\text{cocl}} + \epsilon d_\Sigma \gamma \wedge B_b \right) + \\ &+ \sum_\sigma \left(-\frac{1}{2} \frac{\tan \omega_\sigma \tau}{\omega_\sigma} (B_b^\sigma)^2 - \frac{1}{2} \omega_\sigma \tan \omega_\sigma \tau (A_a^\sigma)^2 + \frac{\epsilon}{\cos \omega_\sigma \tau} A_a^\sigma B_b^\sigma \right). \end{aligned}$$

Here $\epsilon = (-1)^{np+n+p}$; A_a^σ are the coefficients of the expansion on A_a^{coex} in the orthonormal basis χ_σ of coexact p -forms on Σ ; B_b^σ are the coefficients of the expansion of B_b^{ex} in the orthonormal basis $*\Sigma\chi_\sigma$ of exact $(n-p-1)$ -forms on Σ ; the auxiliary field γ is a $(p-1)$ -form on Σ . Note that the case $p=0$ corresponds to the massless scalar theory which has no constraints (and the field γ disappears for a degree reason).

7.5. Higher-dimensional Chern–Simons theory. The results of Section 7.1 are easily generalized to higher-dimensional abelian Chern–Simons theories. Consider again a cylinder $I \times M$, where M is now a $4k+2$ -dimensional manifold ($k=0, 1, 2, \dots$) with a complex structure and

$$(58) \quad S[A] = \frac{1}{2} \int_{I \times M} \text{Ad}A,$$

for $A \in \Omega^{2k+1}(M)$. Proceeding as above, we obtain $A = A_M + A_I$ with A_M a map $I \rightarrow \Omega^{2k+1}(M)$ and A_I a one-form on I with values in $\Omega^{2k}(M)$, and we have

$$(59) \quad S'[A_M, A_I] = \int_{I \times M} \frac{1}{2} A_M d_I A_M + A_I d_M A_M.$$

To obtain a splitting of the symplectic space of fields $\Omega^{2k+1}(M)$, we pass to the complexification and employ a decomposition by complex bidegree, namely

$$(60) \quad \mathcal{F}_{\mathbb{C}}^{\partial} = \Omega^{2k+1}(M, \mathbb{C}) = \underbrace{\Omega^{2k+1,0}(M) \oplus \dots \oplus \Omega^{k+1,k}(M)}_{\Omega_{+}^{2k+1}(M)} \oplus \underbrace{\Omega^{k,k+1}(M) \oplus \dots \oplus \Omega^{0,2k+1}(M)}_{\Omega_{-}^{2k+1}(M)}$$

and accordingly we write $A_M = A_M^{+} + A_M^{-}$. Plugging this decomposition into (59), we obtain, up to a boundary term,

$$(61) \quad S''(A_M^{+}, A_M^{-}, A_I) = \int_{I \times M} A_M^{-} d_I A_M^{+} + A_I (d_M A_M^{+} + d_M A_M^{-})$$

with corresponding boundary 1-form $A_M^{-} \delta A_M^{+}$. We will choose the endpoint conditions A_M^{+} on $M \times \{0\}$ and A_M^{-} on $M \times \{1\}$.³⁶ Notice that for the component of A_I of Hodge type (k, k) , the last term reads $A_I^{k,k} (\bar{\partial} A_M^{k+1,k} + \partial A_M^{k,k+1})$. This gives rise to the only nonvanishing “component” of the matrix $A_{\alpha\beta}$ of Example 4.3, the other components vanishing because of $d_M^2 = 0$. Relabeling $\int A_I =: \sigma \in \Omega^{2k}(M)$ we obtain the HJ action

$$(62) \quad \hat{S}_{\text{HJ}}^f = \int_M \mathbf{A}_b^{-} \mathbf{A}_a^{+} + \sigma (d_M \mathbf{A}_a^{+} + d_M \mathbf{A}_b^{-}) - \frac{1}{2} \sigma^{k,k} \partial \bar{\partial} \sigma^{k,k}.$$

In applications it is interesting to consider more general coordinates on the space $\Omega^{2k+1}(M)$. Assume that (A^Q, A^P) is such a set of coordinates and that $f(A_M^{+}, A^Q)$ is the corresponding generating function. Then, according to Remark 4.4, the HJ action is

$$(63) \quad \hat{S}_{\text{HJ}}^f = f(\mathbf{A}_a^{+} + d_M^{+} \sigma, \mathbf{A}_b^Q) + \int_M \sigma d_M \mathbf{A}_a^{+} - \int_M \frac{1}{2} \sigma^{k,k} \partial \bar{\partial} \sigma^{k,k},$$

where d_M^{+} is the composition of d_M with the projection to $\Omega_{+}^{2k+1}(M)$, and the sign in front of the last term is due to using the $+$ -representation on the a -boundary.

7.6. A nonlinear polarization in 7D Chern–Simons theory and the Kodaira–Spencer action. Let us now focus on the 7D case ($k = 1$ in the section above): i.e., M is now a 6-dimensional manifold with a complex structure. We will now actually assume that it is Kähler, and fix a reference nonvanishing holomorphic 3-form ω_0 . Let us fix some notation. We denote by $\Omega^{-p,q}(M)$ sections of the bundle $\wedge^p(T_{\mathbb{C}}M)^{1,0} \wedge \wedge^q(T_{\mathbb{C}}M)^{0,1}$, i.e., $(0, q)$ -forms with values in $(p, 0)$ -vector fields. Contraction with the reference holomorphic 3-form provides a map

$$\begin{aligned} \Omega^{-p,q}(M) &\rightarrow \Omega^{3-p,q}(M) \\ A &\mapsto A^{\vee} =: A\omega_0 \end{aligned}$$

(we omit symbols for wedge products and contractions). For $A, B, C \in \Omega^{-1,1}(M)$, we further define the operations

$$(64a) \quad A^{\vee} \circ B^{\vee} = (AB)\omega_0 \in \Omega^{1,2}(M),$$

$$(64b) \quad \langle A^{\vee}, B^{\vee}, C^{\vee} \rangle = A^{\vee} (B^{\vee} \circ C^{\vee}) \in \Omega^{3,3}(M),$$

$$(64c) \quad \langle A^3 \rangle = \frac{1}{6} \frac{\langle A^{\vee}, A^{\vee}, A^{\vee} \rangle}{\omega_0 \bar{\omega}_0} \in \Omega^0(M),$$

³⁶The attentive reader will have noticed that for the 3D case $k = 0$ this gives the opposite boundary conditions than in Section 7.1. This flips the sign of the last term in (62) below.

and similarly for $A, B, C \in \Omega^{1,-1}(M)$ by replacing ω_0 with its conjugate $\bar{\omega}_0$.

The symplectic space of boundary fields is $\mathcal{F}^\partial = \Omega^3(M)_\mathbb{C}$. The geometry of this space was studied by Hitchin in [35]. The main point of interest for us is that any complex 3-form admits a splitting³⁷ $A_M = A_M^{+,nl} + A_M^{-,nl}$ where $A_M^{\pm,nl}$ are decomposable three-forms. Here, a 3-form A is called decomposable if around every point there exists a local coframe θ_i in which $A = \theta_1 \wedge \theta_2 \wedge \theta_3$. We will call A nondegenerate if $A_M^{+,nl} \wedge A_M^{-,nl}$ is everywhere nonvanishing. All three-forms which are not themselves decomposable are nondegenerate and for nondegenerate 3-forms $A^{\pm,nl}$ are uniquely determined by A [35]. This splitting defines a polarization $\mathcal{P}^{nl,-}$ on³⁸ $\Omega^3(M)_\mathbb{C}$ by letting $\mathcal{P}_{A_M} \subset T_{A_M} \Omega^3(M)_\mathbb{C} \cong \Omega^3(M)_\mathbb{C}$ be the subspace spanned by the $A_M^{+,nl}$, whose leaf space is parametrized by $A_M^{-,nl}$. To write down a generating function $f(A_M^+, A_M^{-,nl})$, we parametrize $A_M^{-,nl}$ by

$$(65) \quad A^{-,nl} = \bar{\rho} e^{\bar{\mu}} \bar{\omega}_0 = \bar{\rho} \left(\bar{\omega}_0 + \bar{\mu} \bar{\omega}_0 + \frac{\bar{\mu}^2}{2} \bar{\omega}_0 + \frac{\bar{\mu}^3}{6} \bar{\omega}_0 \right),$$

where $\bar{\rho} \in \Omega_\mathbb{C}^0(M)$ and $\bar{\mu} \in \Omega^{1,-1}(M)$, and $\mu \omega_0$ should be interpreted as extension of contraction to forms with values in vector fields. An expression for the generating function is then [34]

$$(66) \quad f(A^{3,0}, A^{2,1}, \bar{\rho}, \bar{\mu}) = \int_M \bar{\rho} (A^{3,0} \bar{\omega}_0 + A^{2,1} \bar{\mu} \bar{\omega}_0) + \bar{\rho}^2 \langle \bar{\mu}^3 \rangle \omega_0 \bar{\omega}_0 - \frac{\left\langle \left((A^{2,1} - \frac{1}{2} \bar{\rho} \bar{\mu}^2 \bar{\omega}_0)^\vee \right)^3 \right\rangle}{(A^{3,0})^\vee - \bar{\rho} \langle \bar{\mu}^3 \rangle} \omega_0 \bar{\omega}_0.$$

The HJ action is then

$$(67) \quad \hat{S}_{\text{HJ}}^f = -\frac{1}{2} \int_M \sigma^{1,1} \partial \bar{\partial} \sigma^{1,1} + \int_M \sigma^{1,1} \bar{\partial} A_a^{2,1} + \sigma^{0,2} (\partial A_a^{2,1} + \bar{\partial} A_a^{3,0}) + f(A_a^{3,0} + \partial \sigma^{2,0}, A_a^{2,1} + \partial \sigma^{1,1} + \bar{\partial} \sigma^{2,0}, \bar{\rho}, \bar{\mu}).$$

To see the connection to the Kodaira–Spencer action functional introduced in [8], take the following quantum state considered in [34]:

$$(68) \quad \psi_{\text{GS}}(\bar{\rho}, \bar{\mu}) = \delta(\bar{\mu}) \exp \frac{i}{\hbar} \int_M \bar{\rho} \omega_0 \bar{\omega}_0.$$

ψ_{GS} is the quantization of the lagrangian $L_{\text{GS}} \subset \Omega_\mathbb{C}^3(M)$ given by $\bar{\mu} = 0, p_{\bar{\rho}} = \omega_0 \bar{\omega}_0$ (here $p_{\bar{\rho}}$ is the canonical momentum coordinate associated to $\bar{\rho}$). L_{GS} has the following generalized generating function, with $\lambda \in \Omega^{-1,1}(M)$ an auxiliary parameter:

$$(69) \quad S_{\text{GS}}(\bar{\mu}, \bar{\rho}; \lambda) = \int_M (\lambda \bar{\mu} + \bar{\rho}) \omega_0 \bar{\omega}_0.$$

Let $\text{ev}: \Omega_\mathbb{C}^3(M) \rightarrow \Omega_\mathbb{C}^3(M)$ denote³⁹ the Chern–Simons evolution relation and interpret L_{GS} as a canonical relation $\{pt\} \rightarrow \Omega_\mathbb{C}^3(M)$. We want to compute the

³⁷The superscript nl stands for “nonlinear” and indicates that the map $A_M \mapsto (A_M^{+,nl}, A_M^{-,nl})$ is nonlinear, as opposed to $A_M \mapsto (A_M^+, A_M^-)$, which is linear.

³⁸Strictly speaking, the polarization is defined only on the open subset of nondegenerate forms.

³⁹We use the notations of equation (124) of Appendix A.1.4 for canonical relations.

composition⁴⁰ $L := \text{ev}^T \circ L_{\text{GS}}: \{pt\} \dashrightarrow \Omega_{\mathbb{C}}^3(M)$. This, following Remark A.37, has the generalized generating function $S = -\hat{S}_{\text{HJ}}^f + S_{\text{GS}}$, where $\bar{\mu}, \bar{\rho}$ now become auxiliary fields. We can immediately solve the constraint $\bar{\mu} = 0$ to obtain a new generalized generating function for L ,:

$$(70) \quad S' = \left[-\hat{S}_{\text{HJ}}^f + S_{\text{GS}} \right]_{\bar{\mu}=0} = \frac{1}{2} \int_M \sigma^{1,1} \partial \bar{\partial} \sigma^{1,1} - \int_M \sigma^{1,1} \bar{\partial} A_a^{2,1} - \int_M \sigma^{0,2} (\partial A_a^{2,1} + \bar{\partial} A_a^{3,0}) - \int_M \bar{\rho}_b (A_a^{3,0} + \partial \sigma^{2,0} - \omega_0) \bar{\omega}_0 + \int_M \frac{\langle ((A_a^{2,1} + \partial \sigma^{1,1} + \bar{\partial} \sigma^{2,0})^\vee)^3 \rangle}{(A_a^{3,0} + \partial \sigma^{2,0})^\vee} \omega_0 \bar{\omega}_0.$$

We now proceed with solving the constraint equations by setting derivatives with respect to auxiliary fields to zero. The equation $\delta S' / \delta \bar{\rho} = 0$ is

$$(71) \quad -\frac{\delta S'}{\delta \bar{\rho}} = A_a^{3,0} + \partial \sigma^{2,0} - \omega_0 = 0$$

which implies $\bar{\partial} A_a^{3,0} = -\bar{\partial} \partial \sigma^{2,0}$. The equation $\delta S' / \delta \sigma^{0,2} = 0$ then gives

$$(72) \quad -\frac{\delta S'}{\delta \sigma^{0,2}} = \partial A_a^{2,1} + \bar{\partial} A_a^{3,0} = \partial (A_a^{2,1} + \bar{\partial} \sigma^{2,0}) = 0.$$

Evaluating S' on the set given by those constraints we obtain

$$S'' = \frac{1}{2} \int_M \sigma^{1,1} \partial \bar{\partial} \sigma^{1,1} - \int_M \sigma^{1,1} \bar{\partial} A_a^{2,1} + \int_M \langle ((A_a^{2,1} + \partial \sigma^{1,1} + \bar{\partial} \sigma^{2,0})^\vee)^3 \rangle \omega_0 \bar{\omega}_0.$$

By (72) and Hodge decomposition for ∂ , there are a unique harmonic $(2, 1)$ -form x and a $(1, 1)$ -form β such that $x + \partial \beta = A_a^{2,1} + \bar{\partial} \sigma^{2,0}$. Letting $b = \sigma^{1,1} + \beta$, we have $\bar{\partial} A_a^{2,1} = \bar{\partial} \partial \beta = \partial \bar{\partial} (\sigma^{1,1} - b)$. Plugging this into S'' , we obtain

$$(73) \quad S''' = \frac{1}{2} \int_M \sigma^{1,1} \partial \bar{\partial} \sigma^{1,1} - \int_M \sigma^{1,1} \partial \bar{\partial} \sigma^{1,1} + \int_M \sigma^{1,1} \partial \bar{\partial} b + \frac{1}{6} \langle x + \partial b, x + \partial b, x + \partial b \rangle.$$

After imposing the constraint

$$\frac{\delta S'''}{\delta \sigma^{1,1}} = 0 = -\partial \bar{\partial} \sigma^{1,1} + \partial \bar{\partial} b,$$

the action becomes

$$(74) \quad S_{\text{KS}}(x, b) = \frac{1}{2} \int_M b \partial \bar{\partial} b + \frac{1}{6} \langle x + \partial b, x + \partial b, x + \partial b \rangle,$$

the action functional for Kodaira–Spencer gravity defined in [8]. For any $A \in \Omega^{2,1}(M)$ we have

$$\frac{\delta}{\delta A} \langle (A^\vee)^3 \rangle \omega_0 \bar{\omega}_0 = \frac{1}{2} A \circ A \in \Omega^{1,2}.$$

Thus, the equation for b finally becomes

$$(75) \quad \partial \bar{\partial} b + \frac{1}{2} \partial (x + \partial b) \circ (x + \partial b) = 0,$$

the Kodaira–Spencer equation for $A = x + \partial b$.

Finally, we can characterize the lagrangian L as follows:

⁴⁰The relation ev goes from $A^{+,1}$ to $A^{-,n1}$, in the composition we require the transpose relation ev^T . This changes the sign of the generating function: the generalized generating function of ev^T is $-\hat{S}_{\text{HJ}}^f$.

Proposition 7.3. *Let ω_0 be a reference holomorphic 3-form on a compact Calabi–Yau manifold M . Then the composition $L = \text{ev}^T \circ L_{GS} \subset \Omega_{\mathbb{C}}^3(M)$ of the Chern–Simons evolution relation ev on $M \times I$ and the lagrangian L_{GS} defined by (69) is given by complex 3-forms with decomposition $A = A^{3,0} + A^{2,1} + A^{1,2} + A^{0,3}$ satisfying*

$$(76a) \quad A^{3,0} = \omega_0 - \partial\sigma^{2,0},$$

$$(76b) \quad A^{2,1} = x + \partial b - \bar{\partial}\sigma^{2,0} - \partial\sigma^{1,1},$$

$$(76c) \quad A^{1,2} = \frac{1}{2}(x + \partial b) \circ (x + \partial b) - \bar{\partial}\sigma^{1,1} - \partial\sigma^{0,2},$$

$$(76d) \quad A^{0,3} = c\bar{\omega}_0 - \bar{\partial}\sigma^{0,2},$$

where x is a harmonic $(2,1)$ form and $b \in \Omega^{1,1}$ such that $x + \partial b$ satisfies the Kodaira–Spencer equation in the form (75), $c \in \mathbb{C}$ is a constant and $\sigma = \sigma^{2,0} + \sigma^{1,1} + \sigma^{0,2} \in \Omega_{\mathbb{C}}^2(M)$ is any 2-form.

Proof. Equation (76a) is (71). Equation (76b) follows from (72) and the definition of b . Equation (76c) follows from $A^{1,2} = \delta S' / \delta A^{2,1}$ together with the constraint equation (71). Finally, we have

$$A^{0,3} = \frac{\delta S'}{\delta A^{3,0}} = -\bar{\partial}\sigma^{0,2} - \bar{\rho}\bar{\omega}_0 - \frac{\langle ((A_a^{2,1} + \partial\sigma^{1,1} - \bar{\partial}\sigma^{2,0})^\vee)^3 \rangle}{((A_a^{3,0} + \partial\sigma^{2,0})^\vee)^2} \bar{\omega}_0,$$

and again the denominator is 1 by (71). Now, the equation

$$0 = \frac{\delta S'}{\delta \sigma^{2,0}} = \partial \bar{\rho} \bar{\omega}_0 + \partial \frac{\langle ((A_a^{2,1} + \partial\sigma^{1,1} - \bar{\partial}\sigma^{2,0})^\vee)^3 \rangle}{((A_a^{3,0} + \partial\sigma^{2,0})^\vee)^2} \bar{\omega}_0 + \bar{\partial} \frac{(A_a^{2,1} + \partial\sigma^{1,1} - \bar{\partial}\sigma^{2,0}) \circ (A_a^{2,1} + \partial\sigma^{1,1} - \bar{\partial}\sigma^{2,0})}{((A_a^{3,0} + \partial\sigma^{2,0})^\vee)^2} \bar{\omega}_0$$

implies, again using (71),

$$\bar{\partial} \partial (\bar{\rho} + \langle ((A_a^{2,1} + \partial\sigma^{1,1} - \bar{\partial}\sigma^{2,0})^\vee)^3 \rangle) \bar{\omega}_0 = 0,$$

which means that the expression in brackets is constant. \square

Remark 7.4. It follows from the general discussion of Appendix A that L is isotropic, but one can also check it directly. To this end, compute the restriction of the symplectic form $\int_M \delta A^{3,0} \wedge \delta A^{0,3} + \delta A^{2,1} \wedge \delta A^{1,2}$ to L . Denoting $B = x + \partial b$, we get

$$\begin{aligned} \delta A^{3,0} \wedge \delta A^{0,3} &= \int_M \partial \bar{\partial} \delta \sigma^{2,0} \delta \sigma^{0,2} \\ \delta A^{2,1} \wedge \delta A^{1,2} &= \int_M \delta \sigma^{1,1} \wedge \delta (\bar{\partial} B + \frac{1}{2} B \circ B) + \int_M \bar{\partial} \delta \sigma^{2,0} \delta \sigma^{0,2} \end{aligned}$$

(other terms cancel due to $\bar{\partial}^2 = \partial^2 = 0$ or integration by parts). The terms containing $\sigma^{0,2} \sigma^{2,0}$ cancel, while the remaining term is zero by (75).

Remark 7.5. Given a harmonic $(2,1)$ -form x , solutions $b \in \Omega^{1,1}(M)$ of the Kodaira–Spencer equation are determined only up to a ∂ -exact term. However, for fixed x one can find a unique solution b_x by demanding that $\partial^* b_x = 0$, see [47] or [8], where ∂^* denotes the formal adjoint of ∂ with respect to the Kähler metric. Let

$\underline{L} = \{(\omega_0, x + \partial b_x, \frac{1}{2}(x + \partial b_x) \circ (x + \partial b_x), c\bar{\omega}_0)\}$ with notation as in Propostion 7.3. We can then rewrite Proposition 7.3 as saying that

$$(77) \quad L = \underline{L} + d\Omega_{\mathbb{C}}^2(M)$$

(\underline{L} is a submanifold of $\Omega_{\mathbb{C}}^3(M)$ but not a subspace). At any point $(y, d\sigma) \in L$, the tangent space is $T_y \underline{L} \oplus d\Omega_{\mathbb{C}}^2(M)$. For $y = (\omega_0, x + \partial b_x + \frac{1}{2}(x + \partial b_x) \circ (x + \partial b_x), c\bar{\omega}_0)$, the tangent space is

$$T_y \underline{L} = \{(0, \dot{x}, x \circ \dot{x}, \dot{c}), \dot{x} \in H^{2,1}(M), \dot{c} \in \mathbb{C}\}.$$

Thus, L is actually split lagrangian, with isotropic complement given by $L' = H^{3,0}(M) \oplus H^{1,2}(M) \oplus d^*\Omega_{\mathbb{C}}^4(M)$.

8. BFV, AKSZ AND BV

The degenerate action (18) we have been focusing on so far can be put into BV form, which is suitable for the perturbative quantization (which we will consider in the next sections).

The first step consists in resolving the constraint manifold $C = \{(p, q) \mid H_{\alpha}(p, q) = 0 \forall \alpha\}$ in terms of the BFV formalism [6, 30]. For this we introduce new odd variables c^{α} , one for each constraint, of ghost number 1 and their momenta b_{α} , odd and of ghost number -1 ;⁴¹ i.e., we extend the symplectic structure to the BFV symplectic structure

$$\omega_{\text{BFV}} = dp_i dq^i + db_{\alpha} dc^{\alpha}.$$

The BFV action is then defined as the odd function of ghost number 1

$$S_{\text{BFV}}(p, q, b, c) = c^{\alpha} H_{\alpha}(p, q) + \dots,$$

where the dots denote terms in the ghost momenta b_{α} such that the Poisson bracket of S_{BFV} with itself vanishes (this is called the classical master equation). The BFV action exists and is uniquely defined up to symplectomorphisms [6, 30, 46]. Note that the linear term in the b s depends on the structure functions,

$$S_{\text{BFV}}(p, q, b, c) = c^{\alpha} H_{\alpha}(p, q) - \frac{1}{2} f_{\alpha\beta}^{\gamma}(p, q) b_{\gamma} c^{\alpha} c^{\beta} + \dots,$$

and that the higher-order terms are not needed in the Lie algebra case (in which the $f_{\alpha\beta}^{\gamma}$ s are constant).

The next step is the AKSZ construction [2], which in this special case takes the following very simple form. We consider the replacements⁴²

$$p_i \rightarrow p_i + q_i^+, \quad q^i \rightarrow q^i - p_+^i, \quad c^{\alpha} \rightarrow c^{\alpha} - e^{\alpha}, \quad b_{\alpha} \rightarrow -e_{\alpha}^+ + c_{\alpha}^+,$$

where, in each assignment, the first term on the right hand side is a function on $[t_a, t_b]$ with the same Grassmann parity and ghost number as the variable on the left hand side, whereas the second term is a 1-form on $[t_a, t_b]$ with opposite Grassmann parity and with ghost number reduced by 1. Note, in particular, that we recover the even 1-forms e^{α} of ghost number zero. With this notation we define the BV action as

⁴¹Here odd means a Grassmann variable, whereas even refers to an ordinary variable; ghost number is an additional degree which is helpful for bookkeeping. The original variables p and q are even and are assigned ghost number zero.

⁴²The “weird” choice of signs is purely conventional. It is done in such a way that *i*) the classical part of the BV action S_{BV} introduced below agrees with (18), and *ii*) the BV form ω_{BV} introduced a bit later has the usual form.

$$\begin{aligned}
S_{\text{BV}}[p, q, e, c, p_+, q^+, e^+, c^+] &:= \\
\int_{t_a}^{t_b} ((p_i + q_i^+) dq^i - p_+^i) + (-e_\alpha^+ + c_\alpha^+) d(c^\alpha - e^\alpha) + S_{\text{BFV}}(p + q^+, q - p^+, -e^+ + c^+, c - e) \\
&= \int_{t_a}^{t_b} (p_i dq^i - e_\alpha^+ dc^\alpha + S_{\text{BFV}}(p + q^+, q - p^+, -e^+ + c^+, c - e)),
\end{aligned}$$

where only the one-form components get integrated.

Example 8.1. In the main cases of interest for this paper, the structure functions $f_{\alpha\beta}^\gamma$ are constant. We then get

$$\begin{aligned}
S_{\text{BFV}}(p, q, b, c) &= c^\alpha H_\alpha(p, q) - \frac{1}{2} f_{\alpha\beta}^\gamma b_\gamma c^\alpha c^\beta, \\
S_{\text{BV}}[p, q, e, c, p_+, q^+, e^+, c^+] &= \int_{t_a}^{t_b} \left(p_i dq^i - e_\alpha^+ dc^\alpha - e^\alpha H_\alpha(p, q) \right. \\
&\quad \left. + c^\alpha \left(q_i^+ \frac{\partial H_\alpha}{\partial p_i}(p, q) - p_+^i \frac{\partial H_\alpha}{\partial q^i}(p, q) \right) \right. \\
&\quad \left. - \frac{1}{2} f_{\alpha\beta}^\gamma c_\gamma^+ c^\alpha c^\beta + f_{\alpha\beta}^\gamma e_\gamma^+ c^\alpha e^\beta \right).
\end{aligned}$$

These are the BFV and BV actions that we are going to consider in the next sections.

Let us return to the general case for the remaining part of this section. The BV action satisfies the following properties:

- (1) It is even and of ghost number zero.
- (2) The terms independent of the antifields (i.e., the fields with a + label) are the action (18).
- (3) If we ignore boundary terms (or impose periodic boundary conditions for all fields), its Poisson bracket with itself, with respect to the odd symplectic⁴³ form of ghost number -1

$$\omega_{\text{BV}} := \int_{t_a}^{t_b} (\delta q_i^+ \delta q^i + \delta p_+^i \delta p_i + \delta e_\alpha^+ \delta e^\alpha + \delta c_\alpha^+ \delta c^\alpha),$$

vanishes.

This is the starting point for the BV quantization.

Remark 8.2 (Chern–Simons). If we apply this construction to the Chern–Simons case described in the previous section, we get the BV formalism for Chern–Simons theory in $[t_a, t_b] \times \Sigma$. We will return to this in the forthcoming paper [21].

Remark 8.3. The case where in addition to constraints there is also a nontrivial hamiltonian H can be treated as at the beginning of Section 5, as originally proposed in [24], or, more conceptually, as in Section 5.1.

⁴³The nondegeneracy of ω_{BV} is actually not needed for classical considerations. At the quantum level only a partial nondegeneracy is actually needed, namely the one along the directions for which one considers integration; see more on this in Section 9.2.3.

9. AN OUTLINE OF ELEMENTS OF BV-BFV QUANTIZATION

The BV and BFV structures discussed in Section 8 can be put together, and this may be used towards quantization [17, 19]. We briefly review this story in the case at hand assuming a finite-dimensional target. The case of infinite-dimensional targets discussed in Section 7 will be addressed in the forthcoming paper [21].

9.1. The classical BV-BFV setting. As described in Section 8 we have a space \mathcal{F} of BV fields $(p, q, e, c, p_+, q^+, e^+, c^+)$ associated to the bulk $[t_a, t_b]$ and a space of BFV variables (p, q, b, c) associated to each boundary point. As the boundary of $[t_a, t_b]$ actually consists of two endpoints (with opposite orientation), we define the space \mathcal{F}^∂ of BFV variables by doubling the (p, q, b, c) s:

$$\mathcal{F}^\partial \ni (p^a, q_a, b^a, c_a, p^b, q_b, b^b, c_b).$$

The BFV symplectic form and the BFV action are now

$$\omega_{\text{BFV}} = dp_i^a dq_a^i + db_a^a dc_a^\alpha - dp_i^b dq_b^i - db_a^b dc_b^\alpha$$

and

$$\begin{aligned} S_{\text{BFV}} &= c_a^\alpha H_\alpha(p^a, q_a) - \frac{1}{2} f_{\alpha\beta}^\gamma(p^a, q_a) b_\gamma^\alpha c_a^\beta + \cdots \\ &\quad - c_b^\alpha H_\alpha(p^b, q_b) + \frac{1}{2} f_{\alpha\beta}^\gamma(p^b, q_b) b_\gamma^\alpha c_b^\beta - \cdots. \end{aligned}$$

The bulk setting $(\mathcal{F}, S_{\text{BV}}, \omega_{\text{BV}})$ and the boundary BFV theory $(\mathcal{F}^\partial, S_{\text{BFV}}, \omega_{\text{BFV}})$ are related as follows:

- (1) We have a surjective submersion $\pi: \mathcal{F} \rightarrow \mathcal{F}^\partial$ given by

$$\begin{array}{llll} p^b = p(t_b) & q_b = q(t_b) & b^b = -e^+(t_b) & c_b = c(t_b) \\ p^a = p(t_a) & q_a = q(t_a) & b^a = -e^+(t_a) & c_a = c(t_a) \end{array}$$

- (2) If we denote by Q_{BV} the hamiltonian vector field of S_{BV} with respect to ω_{BV} , obtained by ignoring boundary terms (or by imposing periodic boundary conditions for all fields), and by Q_{BFV} the hamiltonian vector field of S_{BFV} with respect to ω_{BFV} we get

$$(78) \quad [Q_{\text{BV}}, Q_{\text{BV}}] = 0, \quad [Q_{\text{BFV}}, Q_{\text{BFV}}] = 0, \quad d\pi(Q_{\text{BV}}) = Q_{\text{BFV}},$$

where $d\pi$ denotes the differential of the map π (the last equation may be equivalently formulated, more algebraically, as the property that $\pi^*(Q_{\text{BFV}}h) = Q_{\text{BV}}\pi^*h$ for every function h on \mathcal{F}^∂).

- (3) If we introduce the potential

$$\alpha_{\text{BFV}} = p_i^a dq_a^i + b_a^\alpha dc_a^\alpha - p_i^b dq_b^i - b_a^b dc_b^\alpha$$

for the BFV form ω_{BFV} , then the modified classical master equation (mCME)

$$(79) \quad \iota_{Q_{\text{BV}}}\omega_{\text{BV}} = \delta S_{\text{BV}} + \pi^*\alpha_{\text{BFV}}$$

is satisfied. Here the variation δ of the BV action is computed by taking the boundary terms into account (they are precisely compensated by $\pi^*\alpha_{\text{BFV}}$) and is regarded as producing a 1-form on \mathcal{F} .

A simple consequence of the above, see [19], is that we also have the equation

$$(80) \quad \frac{1}{2} \iota_{Q_{\text{BV}}}\iota_{Q_{\text{BV}}}\omega_{\text{BV}} = \pi^*S_{\text{BFV}},$$

where the left hand side may be read as the correct replacement for the otherwise ill-defined Poisson bracket, induced by ω_{BV} , of S_{BV} with itself, whereas the right hand side yields the error boundary terms. This equation will play an important role in quantization.

Remark 9.1 (Changing the endpoint conditions). As in the purely classical setting, see Remark 2.5, we might want to use other variables, say P and Q , at the final endpoint via a generating function $f(q, Q)$. We can arrange for this by subtracting f from the BV action,

$$S_{\text{BV}}^f[p, q, e, c, p_+, q^+, e^+, c^+] := S_{\text{BV}}[p, q, e, c, p_+, q^+, e^+, c^+] - f(q(t_b), Q(p(t_b), q(t_b))),$$

and at the same time adding df to the BFV potential,

$$\alpha_{\text{BFV}}^f := \alpha_{\text{BFV}} + df(q_b, Q(p^b, q_b)) = p_i^a dq_a^i + b_\alpha^a dc_\alpha^a - P_i^b dQ_b^i - b_\alpha^b dc_\alpha^b.$$

This way we get the wished-for classical part of the symplectic form at the final endpoint, yet preserving the mCME (79), now in the form

$$\iota_{Q_{\text{BV}}} \omega_{\text{BV}} = \delta S_{\text{BV}}^f + \pi^* \alpha_{\text{BFV}}^f.$$

Remark 9.2 (Changing the endpoint field and ghost conditions). We might also want to exchange the role of b and c at the final endpoint. To do so, we consider the extended generating function

$$\tilde{f}(q, Q, c, C) = f(q, Q) + cC.$$

We then get

$$S_{\text{BV}}^{\tilde{f}}[p, q, e, c, p_+, q^+, e^+, c^+] = S_{\text{BV}}^f[p, q, e, c, p_+, q^+, e^+, c^+] + c_\alpha(t_b) e^+(t_b)$$

and

$$\alpha_{\text{BFV}}^{\tilde{f}} = p_i^a dq_a^i + b_\alpha^a dc_\alpha^a - P_i^b dQ_b^i - c_\alpha^b db_\alpha^b.$$

The mCME (79) now reads

$$(81) \quad \iota_{Q_{\text{BV}}} \omega_{\text{BV}} = \delta S_{\text{BV}}^{\tilde{f}} + \pi^* \alpha_{\text{BFV}}^{\tilde{f}}.$$

Remark 9.3. The induced equation (80) holds as it is, since it is independent of the choice of the 1-form.

9.2. The quantum BV-BFV setting. In this section, we give a very short overview of the BV-BFV quantization, tailored for the examples of this paper. For a more general exposition, see [19].

9.2.1. Boundary polarization. For the quantum setting, we first have to choose a polarization in \mathcal{F}^∂ . Concretely, in the case at hand, we write \mathcal{F}^∂ as $T^*\mathcal{B}$, for a suitable lagrangian submanifold \mathcal{B} , with canonical symplectic form. We require that the BFV 1-form of (79) should be the canonical 1-form of the cotangent bundle. This means that the choice of \mathcal{B} is related to the choice of boundary polarization. (Recall that, on the initial endpoint, we choose q_a and c_a as coordinates on \mathcal{B} ; on the final endpoint, we choose Q_b and c_b or Q_b and b^b —i.e., we work in the setting of Remark 9.1 or 9.2, and we consider the corresponding 1-form α_{BFV}^f or $\alpha_{\text{BFV}}^{\tilde{f}}$.)

9.2.2. *Splitting.* For simplicity, we assume the target (for the (p, q) fields) to be $T^*\mathbb{R}^n$. In this case \mathcal{B} is a (graded) vector space, and $\mathcal{F}^\partial = \mathcal{B}^* \oplus \mathcal{B}$. The crucial point is to write the space of bulk fields as

$$(82) \quad \tilde{\mathcal{F}} = \mathcal{Y} \oplus \mathcal{B},$$

where $\tilde{\mathcal{F}}$ is a suitable replacement (regularization) of the space of bulk fields, compatible with the boundary polarization by this splitting. Note that \mathcal{B}^* is now a subspace of \mathcal{Y} . We assume that equations (78) and (81), and therefore also (80), are satisfied (now working on $\tilde{\mathcal{F}}$ instead of \mathcal{F} and with a suitable replacement $\tilde{S}_{\text{BV}}^{\tilde{f}}$ of $S_{\text{BV}}^{\tilde{f}}$).

Example 9.4. The q -field component of \mathcal{B} is q_a . As q -component of \mathcal{Y} we can then take the space $C_0^\infty([t_a, t_b], \mathbb{R}^n) \ni \hat{q}$ of maps that vanish at t_a . If we define

$$q(t) = q_a + \hat{q}(t),$$

as we did in Remark 2.4, then we see that the q -component of $\tilde{\mathcal{F}}$ is the space of smooth maps $[t_a, t_b] \rightarrow \mathbb{R}^n$, as in \mathcal{F} . If on the other hand we set

$$q(t) = \begin{cases} q_a & t = t_a \\ \hat{q}(t) & t > t_a \end{cases},$$

then we get, in $\tilde{\mathcal{F}}$, maps with a possible discontinuity at t_a , unlike in \mathcal{F} . In this case, the quantum space of bulk fields $\tilde{\mathcal{F}}$ is slightly different from its classical counterpart \mathcal{F} . It turns out that, in general, these discontinuous extensions are more suitable for the BV-BFV formalism, as we explain below (see Section 11 and 12 for examples and also for the correct definition of \tilde{S}_{BV} in these cases).

9.2.3. *Good splittings.* Given a splitting (82), we decompose

$$Q_{\text{BV}} = Q_{\mathcal{Y}} + Q_{\mathcal{B}}$$

and

$$\omega_{\text{BV}} = \omega_{\mathcal{Y}\mathcal{Y}} + \omega_{\mathcal{Y}\mathcal{B}} + \omega_{\mathcal{B}\mathcal{B}}$$

according to which (co)tangent space(s) the vector field/2-form belongs. We assume $\omega_{\mathcal{Y}\mathcal{Y}}$ to be closed and (weakly) nondegenerate as a 2-form on \mathcal{Y} . The mCME (81) now reads, in \mathcal{Y} - and \mathcal{B} -components,⁴⁴

$$\begin{aligned} \iota_{Q_{\mathcal{Y}}}\omega_{\mathcal{Y}\mathcal{Y}} + \iota_{Q_{\mathcal{B}}}\omega_{\mathcal{Y}\mathcal{B}} &= \delta_{\mathcal{Y}}\tilde{S}_{\text{BV}}^{\tilde{f}}, \\ \iota_{Q_{\mathcal{Y}}}\omega_{\mathcal{Y}\mathcal{B}} + \iota_{Q_{\mathcal{B}}}\omega_{\mathcal{B}\mathcal{B}} &= \delta_{\mathcal{B}}\tilde{S}_{\text{BV}}^{\tilde{f}} + \alpha_{\text{BFV}}^{\tilde{f}}, \end{aligned}$$

where we have omitted writing π^* . Note that, by assumption, $\alpha_{\text{BFV}}^{\tilde{f}}$ only has \mathcal{B} -components.

We say that we have a good splitting if $\iota_{Q_{\mathcal{B}}}\omega_{\mathcal{Y}\mathcal{B}} = 0$, for in this case $Q_{\mathcal{Y}}$ is the hamiltonian vector field, with respect to $\omega_{\mathcal{Y}\mathcal{Y}}$, of $\tilde{S}_{\text{BV}}^{\tilde{f}}$ on \mathcal{Y} (with everything parametrized by \mathcal{B}). We assume from now on that we have a good splitting. We denote by $(\ , \)_{\mathcal{Y}}$ the Poisson bracket induced by $\omega_{\mathcal{Y}\mathcal{Y}}$, so we have

$$(\tilde{S}_{\text{BV}}^{\tilde{f}}, \tilde{S}_{\text{BV}}^{\tilde{f}})_{\mathcal{Y}} = \iota_{Q_{\mathcal{Y}}}\iota_{Q_{\mathcal{Y}}}\omega_{\mathcal{Y}\mathcal{Y}} = \iota_{Q_{\text{BV}}}\iota_{Q_{\text{BV}}}\omega_{\text{BV}} - \iota_{Q_{\mathcal{B}}}\iota_{Q_{\mathcal{B}}}\omega_{\mathcal{B}\mathcal{B}}.$$

⁴⁴We write the formulae with generating function \tilde{f} . Of course, they hold also with f instead.

By the induced equation (80), we then get

$$(\tilde{S}_{\text{BV}}^{\tilde{f}}, \tilde{S}_{\text{BV}}^{\tilde{f}})_{\mathcal{Y}} = 2S_{\text{BFV}} - \iota_{Q_{\mathcal{B}}} \iota_{Q_{\mathcal{B}}} \omega_{\mathcal{B}\mathcal{B}}.$$

If the latter term depends only on the $T^*\mathcal{B}$ variables, which we assume from now on, we may define a new function

$$S^{\partial} := S_{\text{BFV}} - \frac{1}{2} \iota_{Q_{\mathcal{B}}} \iota_{Q_{\mathcal{B}}} \omega_{\mathcal{B}\mathcal{B}}$$

on $T^*\mathcal{B}$, and we have

$$(83) \quad (\tilde{S}_{\text{BV}}^{\tilde{f}}, \tilde{S}_{\text{BV}}^{\tilde{f}})_{\mathcal{Y}} = 2S^{\partial}.$$

Remark 9.5. Usually we do not have a good splitting. A way out, as also suggested in Remark 2.33 of [19], is to consider a family of splittings such that there is a limit in which the theory exists and corresponds to a good splitting (but of a different space of fields). For instance, one can define a section s_{ε} of the short exact sequence $0 \rightarrow \mathcal{Y} \xrightarrow{\iota} \mathcal{F} \rightarrow \mathcal{B} \rightarrow 0$ by multiplying fields in \mathcal{B} by a smooth function equal to 1 close to the boundary and with support contained in a collar of length ε . We thus obtain a family of BV-BFV theories

$$\mathcal{F}_{\varepsilon} = \left(\mathcal{Y} \oplus \mathcal{B}, (\iota \oplus s_{\varepsilon})^* \omega_{\text{BV}}, (\iota \oplus s_{\varepsilon})^* S_{\text{BV}}^{\tilde{f}}, (\iota \oplus s_{\varepsilon})^* Q_{\text{BV}}, \pi \circ (\iota \oplus s_{\varepsilon}) \right)$$

over $(\mathcal{F}^{\partial}, S_{\text{BFV}}, \omega_{\text{BFV}})$ isomorphic to $(\mathcal{F}, \omega_{\text{BV}}, S_{\text{BV}}^{\tilde{f}}, Q_{\text{BV}}, \pi)$ for $\varepsilon > 0$. The limit as $\varepsilon \rightarrow 0$ gives a well-defined theory \mathcal{F}_0 on $\mathcal{Y} \oplus \mathcal{B}$ (one just integrates by parts the terms containing ds_{ε}), but it does not arise from a well-defined section s - rather, it corresponds to the discontinuous extension in Example 9.4.⁴⁵ This theory on $\mathcal{Y} \oplus \mathcal{B}$ is then the replacement as in (82) which is suitable for quantization. The space \tilde{F} can be constructed by embedding \mathcal{F} in a larger space $\tilde{\tilde{F}}$ where the limit $s_0: \mathcal{B} \rightarrow \tilde{\tilde{F}}$ exists, and then we let $\tilde{F} = (\iota \oplus s_0)(\mathcal{Y} \oplus \mathcal{B})$. For our purposes one can typically let $\tilde{\tilde{F}}$ be smooth functions on the interval with a discontinuity at the endpoints.

9.2.4. *The modified quantum master equation.* In the BV setting, one has a second-order differential operator $\Delta_{\mathcal{Y}}$, the BV Laplacian, acting on functions on \mathcal{Y} with the properties

$$\Delta_{\mathcal{Y}}^2 = 0, \quad \Delta_{\mathcal{Y}}(fg) = \Delta_{\mathcal{Y}}(f)g \pm f\Delta_{\mathcal{Y}}g \pm (f, g)_{\mathcal{Y}}.$$

In the finite-dimensional case, this operator is actually canonically defined on half-densities [37, 45]. It is then transferred to functions upon the choice of a non-degenerate, $\Delta_{\mathcal{Y}}$ -closed, reference half-density. In the infinite-dimensional setting, the definition of $\Delta_{\mathcal{Y}}$ requires a regularization, which we presently assume to have.

Moreover, for simplicity, we assume $\Delta_{\mathcal{Y}} \tilde{S}_{\text{BV}}^{\tilde{f}} = 0$. Therefore, we have

$$\Delta_{\mathcal{Y}} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}} = -\frac{1}{2\hbar^2} (\tilde{S}_{\text{BV}}^{\tilde{f}}, \tilde{S}_{\text{BV}}^{\tilde{f}})_{\mathcal{Y}} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}} = -\frac{1}{\hbar^2} S^{\partial} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}}.$$

The final step consists in finding some operator Ω on the space of functions on \mathcal{B} satisfying

$$(84) \quad \Omega e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}} = S^{\partial} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}}.$$

⁴⁵In this theory the BV form is nondegenerate along \mathcal{Y} , but degenerate otherwise. This is all what is needed to define BV quantization.

If we find such an operator, then we get the modified Quantum Master Equation (mQME)

$$(85) \quad (\Omega + \hbar^2 \Delta_{\mathcal{Y}}) e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}} = 0,$$

which is a quantization of (81).

Note that Ω commutes with $\Delta_{\mathcal{Y}}$ (the former acts on functions on \mathcal{B} , the latter on functions on \mathcal{Y}), so the operator

$$(86) \quad \Delta_{\mathcal{Y}}^{\Omega} := \Delta_{\mathcal{Y}} + \frac{1}{\hbar^2} \Omega$$

squares to zero if and only if Ω does so, which we are going to assume. The reason for this normalization, in which we prefer to regard $\Delta_{\mathcal{Y}}^{\Omega}$ as a modification of $\Delta_{\mathcal{Y}}$ by a contribution due to the boundary,⁴⁶ will become clear in Section 9.2.7, where we will address gauge fixing and gauge-fixing independence. The mQME can now be rewritten as

$$\Delta_{\mathcal{Y}}^{\Omega} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}} = 0.$$

There are two main settings for the construction of Ω , which we describe in the next two sections. (In principle, there may be more choices available, but we are not aware of them.)

9.2.5. *Case I: No boundary symmetries.* Suppose $Q^{\partial} = 0$. This implies $Q_{\mathcal{B}} = 0$ and $S^{\partial} = S_{\text{BFV}} = 0$. In this case, every splitting is good and, moreover, we have $(\tilde{S}_{\text{BV}}^{\tilde{f}}, \tilde{S}_{\text{BV}}^{\tilde{f}})_{\mathcal{Y}} = 0$. Therefore, we obtain the mQME by simply setting $\Omega = 0$.

Note, in particular, that we might choose a splitting as in the first part of Example 9.4 for all fields, so that $\tilde{\mathcal{F}} = \mathcal{F}$ and $\tilde{S}_{\text{BV}}^{\tilde{f}} = S_{\text{BV}}^{\tilde{f}}$. We may also choose, say for the q part, to split $q(t) = q_0(t) + \hat{q}(t)$, where q_0 is the solution to the EL equations with the given boundary conditions. We then recover the usual procedure for dealing with a path integral without gauge symmetries, as in equation (5) of Remark 2.4.

9.2.6. *Case II: Discontinuous splitting.* A rather general procedure for dealing with the BV-BFV quantization, proposed in [19], is based on a discontinuous splitting of the fields, as in the second part of Example 9.4 and in Remark 9.5. In this case, $\tilde{\mathcal{F}} \neq \mathcal{F}$ and $\tilde{S}_{\text{BV}}^{\tilde{f}} \neq S_{\text{BV}}^{\tilde{f}}$. Moreover, the regularization of $\Delta_{\mathcal{Y}}$ requires some care, so we will be rather sketchy here. The main advantage of such a choice is that $\omega_{\mathcal{Y}\mathcal{B}} = \omega_{\mathcal{B}\mathcal{B}} = 0$. This is then a good splitting in which $S^{\partial} = S_{\text{BFV}}$ and

$$\delta_{\mathcal{B}} \tilde{S}_{\text{BV}}^{\tilde{f}} = -\alpha_{\text{BFV}}^{\tilde{f}}.$$

⁴⁶One might also as well argue that the coboundary operator should be defined as $\Omega_{\mathcal{Y}} := \Omega + \hbar^2 \Delta_{\mathcal{Y}}$ —i.e., as a modification of Ω by a contribution due to the bulk—stressing that what matters more are the boundary fields, as the bulk fields will eventually be integrated out in the functional integration. One might also argue that, since, without boundary, one has $-i\hbar\Delta \left(e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}} \mathcal{O}} \right) = e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}} (Q_{\text{BV}} \mathcal{O} - i\hbar\Delta)$, where on the right hand side we see the quantum modification of the BV operator Q_{BV} , then $-i\hbar\Delta$ is the correct coboundary operator in the bulk, so the correct coboundary operator in the presence of a boundary should be $-i\hbar\Delta_{\mathcal{Y}}^{\Omega} = -i\hbar\Delta_{\mathcal{Y}} - \frac{i}{\hbar}\Omega$.

Since $\alpha_{\text{BFV}}^{\tilde{f}}$ is by assumption the canonical 1-form of the cotangent bundle $T^*\mathcal{B} = \mathcal{B}^* \oplus \mathcal{B} \ni (z^*, z)$, we write $\alpha_{\text{BFV}}^{\tilde{f}} = z_\mu^* dz^\mu$. Therefore, the last equation becomes

$$\frac{\partial \tilde{S}_{\text{BV}}^{\tilde{f}}}{\partial z^\mu} = -z_\mu^*.$$

It then follows that, if we define Ω to be the Schrödinger quantization of S_{BFV} ,

$$(87) \quad \Omega := S_{\text{BFV}} \left(i\hbar \frac{\partial}{\partial z}, z \right)$$

with all the derivatives to the right (standard ordering), then we get

$$\Omega e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}} = S_{\text{BFV}}(z^*, z) e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}},$$

which is (84) (recall that we now have $S^\partial = S_{\text{BFV}}$).

Remark 9.6. A general observation at this point is that Ω is a quantization of S_{BFV} and that the classical master equation $\{S_{\text{BFV}}, S_{\text{BFV}}\} = 0$ is the lowest-order condition for $\Omega^2 = 0$. Note that nothing guarantees that this condition is satisfied at all orders. This actually becomes a consistency condition for the theory. (See more, e.g., in Section 10).

In this presentation, we treated $\Delta_{\mathcal{Y}}$ rather formally. The correct argument requires a regularization and a careful analysis of the Feynman diagrams, which we present in Appendix B, and relies on certain assumptions.

The simplest case when the above argument turns out to be correct is when the initial and final polarizations are related by a linear transformation (i.e., $f(q, Q)$ is linear in the qs and in the Qs). This is the setting of [19], which we will use in Section 10 (in the particular case $f(q, Q) = Q_i q^i$ as, e.g., in Example 2.8).

For a more general f , a compatibility conditions between the constraints and f is required, see Assumption 12.1 or B.1. In particular, the above procedure actually works for a system as in Section 4 with the conditions that *i*) all the hamiltonians H_α are linear in the p variables and *ii*) all the hamiltonians \tilde{H}_α are linear in the P variables, where $\tilde{H}_\alpha(P, Q) := H_\alpha(p(P, Q), q(P, Q))$, and we use the symplectomorphism induced by f (no matter how nonlinear it is; what saves the game here is the linearity of the hamiltonians in the momenta). Note that this is not an artificial, quirky situation, but it actually occurs in important examples, the main one (although with infinite-dimensional target) being described in Section 7.6.

9.2.7. The BV pushforward. One of the properties of the BV formalism is the following. Assume we can write $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ as a sum of symplectic subspaces. If \mathcal{L} is a lagrangian submanifold of \mathcal{Y}_2 and ψ is in the kernel of $\Delta_{\mathcal{Y}}^\Omega$, then the integral ψ_1 of ψ over \mathcal{L} is in the kernel of $\Delta_{\mathcal{Y}_1}^\Omega := \Delta_{\mathcal{Y}_1} + \frac{1}{\hbar^2} \Omega$; moreover, if we deform \mathcal{L} , the result ψ_1 will change by a $\Delta_{\mathcal{Y}_1}^\Omega$ -exact term.

By the mQME (85), $e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}}$ is $\Delta_{\mathcal{Y}}^\Omega$ -closed, so we may apply this procedure, which is known as the BV pushforward. The result of the pushforward can again be recast in the form $e^{\frac{i}{\hbar} S_1}$, where S_1 is a formal power series in \hbar . The choice of \mathcal{L} is called a gauge fixing, and the last property in the previous paragraph is referred to as gauge-fixing independence. With the normalization in (86), the argument of $\Delta_{\mathcal{Y}_1}^\Omega$ under a change of gauge fixing is a formal power series in \hbar times $e^{\frac{i}{\hbar} S_1}$, which is appropriate in perturbation theory.

We may, in particular, choose $\mathcal{Y}_2 = \mathcal{Y}$ (assuming we can find a lagrangian \mathcal{L} on which the integral of ψ converges). In this case, ψ_1 will be a function on \mathcal{B} satisfying $\Omega\psi_1 = 0$. Note that in this case Δ^Ω is just Ω/\hbar^2 .

Two remarks are here in order. The first is that this procedure is more invariantly defined in terms of half-densities instead of functions. The second is that there might be lagrangian submanifolds \mathcal{L} and \mathcal{L}' that are not deformations of one another (at least through a path of lagrangian submanifolds on which the integral of ψ converges); in this case, the two results of integration might not be $\Delta_{\mathcal{Y}_1}^\Omega$ -cohomologous; see, e.g., Example 9.8 below.

Remark 9.7. The BV pushforward may of course be iterated. The setting is then the following. We have a fixed space \mathcal{B} with a fixed coboundary operator Ω acting on its functions. We then have, for some ordered indices k , an odd symplectic space \mathcal{Y}_k and a $\Delta_{\mathcal{Y}_k}^\Omega$ -closed function ψ_k on $\mathcal{Y}_k \times \mathcal{B}$. The $\Delta_{\mathcal{Y}_{k+1}}^\Omega$ -closed function ψ_{k+1} on $\mathcal{Y}_{k+1} \times \mathcal{B}$ is obtained by BV-pushforward. On the new spaces $\mathcal{Y}_k \times \mathcal{B}$ one can recover some derived structures reminiscent of the original one on $\mathcal{Y} \times \mathcal{B}$; see [19, Remark 2.17] and [20, Proposition 9.1].

Example 9.8. As a toy example (actually a BV-BFV quantization of Example 3.2 with $n = 1$, $v = 0$ and $w = 1$ after a first BV pushforward), we consider $\mathcal{B} = \mathbb{R} \oplus \mathbb{R}[1] \oplus \mathbb{R} \oplus \mathbb{R}[1] \ni (q_a, c_a, p^b, c_b)$, $\mathcal{Y}_1 = \mathbb{R} \oplus \mathbb{R}[-1] \ni (T, T^+)$, $\Delta_{\mathcal{Y}_1} = \frac{\partial^2}{\partial T^+ \partial T}$, and

$$\Omega = c_a q_a - i\hbar c_b \frac{\partial}{\partial p^b}.$$

The state

$$\psi_1(q_a, c_a, p^b, c_b, T, T^+) = e^{-\frac{i}{\hbar}(p^b q_a + T q_a + T^+(c_a - c_b))}$$

is $\Delta_{\mathcal{Y}_1}^\Omega$ -closed. If we consider the lagrangian submanifold $\mathcal{L} = \{T^+ = 0\}$ of \mathcal{Y}_1 , we get⁴⁷

$$\psi_2^{\mathcal{L}}(q_a, c_a, p^b, c_b) = \int_{\mathcal{L}} \psi_1(q_a, c_a, p^b, c_b, T, T^+) = \int \frac{dT}{2\pi\hbar} \psi_1(q_a, c_a, p^b, c_b, T, 0) = \delta(q_a),$$

which is clearly Ω -closed. If, on the other hand, we consider the lagrangian submanifold $\tilde{\mathcal{L}} = \{T = 0\}$, which cannot be deformed to \mathcal{L} , we get

$$\begin{aligned} \psi_2^{\tilde{\mathcal{L}}}(q_a, c_a, p^b, c_b) &= \int_{\tilde{\mathcal{L}}} \psi_1(q_a, c_a, p^b, c_b, T, T^+) = \int i\hbar dT^+ \psi_1(q_a, c_a, p^b, c_b, 0, T^+) \\ &= (c_a - c_b) e^{-\frac{i}{\hbar} p^b q_a} = \delta(c_a - c_b) e^{-\frac{i}{\hbar} p^b q_a}, \end{aligned}$$

which is also Ω -closed but certainly not Ω -cohomologous to $\psi_2^{\mathcal{L}}$ (the two have different ghost numbers).

9.2.8. Composition. The space \mathcal{B} we consider is actually a product $\mathcal{B}_a \times \mathcal{B}_b$, with \mathcal{B}_a having coordinates q_a and c_a and with \mathcal{B}_b having coordinates Q_b and c_b or Q_b and b^b . We assume that Ω is local, i.e., $\Omega = \Omega_a - \Omega_b$, where each summand is an operator on the space of functions of the corresponding space. In addition, we assume Ω_a and Ω_b to be self-adjoint operators on functions on \mathcal{B}_a and \mathcal{B}_b , respectively.⁴⁸

For instance, see Example 9.8 where we have $\Omega_a = c_a q_a$ and $\Omega_b = i\hbar c_b \frac{\partial}{\partial p^b}$. We will see more examples in Sections 10 and 12.1.

⁴⁷We conventionally normalize the measures dT and dT^+ in order to get nicer looking formulae.

⁴⁸This is defined in terms of some measure on \mathcal{B}_a . A more invariant formulation regards the states as half-densities, so this choice of measure is not necessary.

Suppose we have a $\Delta_{\mathcal{Y}}^{\Omega_a - \Omega_b}$ -closed state ψ on $\mathcal{B}_a \times \mathcal{B}_b \times \mathcal{Y}$ and a $\Delta_{\mathcal{Y}}^{\Omega_b}$ -closed state ψ_b on $\mathcal{B}_b \times \tilde{\mathcal{Y}}$. We can then pair ψ with ψ_b integrating over \mathcal{B}_b (we assume this integration to converge). Then⁴⁹

$$\psi_a := \int_{\mathcal{B}_b} \psi \psi_b$$

is a $\Delta_{\mathcal{Y} \times \tilde{\mathcal{Y}}}^{\Omega_a}$ -closed state on $\mathcal{B}_a \times \mathcal{Y} \times \tilde{\mathcal{Y}}$. A further reduction by BV pushforward, as in Section 9.2.7, is of course possible if we have $\mathcal{Y} \times \tilde{\mathcal{Y}} = \mathcal{Y}_1 \times \mathcal{Y}_2$.

Example 9.9. Take $\psi = \psi_1$ as in Example 9.8 and $\tilde{\mathcal{Y}}$ to be a point. The ghost-number-one state

$$\psi_b(p^b, c_b) = \phi(p^b) \delta(c_b) = \phi(p^b) c_b,$$

where ϕ is an arbitrary function, is obviously Ω_b -closed. After pairing, it yields, up to a multiplicative constant that depends on the choice of measure,

$$\psi_a(q_a, c_a, T, T^+) = e^{-\frac{1}{\hbar}(Tq_a + T^+c_a)} \hat{\phi}(q_a),$$

where $\hat{\phi}$ is the Fourier transform of ϕ . Note that

$$-\hbar^2 \Delta_{\mathcal{Y}_1} \psi_a = c_a q_a \psi_a = \Omega_a \psi_a,$$

so ψ_a is $\Delta_{\mathcal{Y}_1}^{\Omega_a}$ -closed. We may now integrate over the lagrangian submanifolds $\mathcal{L} = \{T^+ = 0\}$ or $\tilde{\mathcal{L}} = \{T = 0\}$ getting

$$\psi_a^{\mathcal{L}}(q_a, c_a) = \delta(q_a) \hat{\phi}(0) \quad \text{and} \quad \psi_a^{\tilde{\mathcal{L}}}(q_a, c_a) = \delta(c_a) \hat{\phi}(q_a),$$

which are clearly Ω_a -closed. If, on the other hand, we consider the ghost-number-zero state

$$\tilde{\psi}_b(p^b, c_b) = 1,$$

which is obviously Ω_b -closed, then we get, again up to a multiplicative constant,

$$\tilde{\psi}_a(q_a, c_a, T, T^+) = \delta(q_a) \delta(T^+) = \delta(q_a) T^+,$$

which is obviously $\Delta_{\mathcal{Y}_1}$ - and Ω_a -closed, and therefore $\Delta_{\mathcal{Y}_1}^{\Omega_a}$ -closed. By a further BV pushforward as above, we get

$$\tilde{\psi}_a^{\mathcal{L}}(q_a, c_a) = 0 \quad \text{and} \quad \tilde{\psi}_a^{\tilde{\mathcal{L}}}(q_a, c_a) = \delta(q_a),$$

which are clearly Ω_a -closed.

Remark 9.10. A generalization of the above is when we have a $\Delta_{\mathcal{Y}}^{\Omega_a - \Omega_b}$ -closed state ψ on $\mathcal{B}_a \times \mathcal{B}_b \times \mathcal{Y}$ and a $\Delta_{\tilde{\mathcal{Y}}}^{\Omega_b - \Omega_{b'}}$ -closed state $\tilde{\psi}$ on $\mathcal{B}_b \times \mathcal{B}_{b'} \times \tilde{\mathcal{Y}}$. We can then pair them on \mathcal{B}_b , getting a $\Delta_{\mathcal{Y} \times \tilde{\mathcal{Y}}}^{\Omega_a - \Omega_{b'}}$ -closed state

$$\hat{\psi} = \int_{\mathcal{B}_b} \psi \tilde{\psi}$$

on $\mathcal{B}_a \times \mathcal{B}_{b'} \times \mathcal{Y} \times \tilde{\mathcal{Y}}$ (which could possibly be further reduced by BV pushforwards). If we interpret the above states ψ and $\tilde{\psi}$ as evolution operators corresponding to the intervals $[t_a, t_b]$ and $[t_b, t_{b'}]$, respectively, then $\hat{\psi}$ can be interpreted as the corresponding evolution operator on $[t_a, t_{b'}]$.

⁴⁹In the composition integral we assume that there is a measure on \mathcal{B}_b or that the states are half-densities, see footnote 48.

10. BV-BFV BOUNDARY STRUCTURES FOR LINEAR POLARIZATIONS

In this section we prepare for the application of the quantum BV-BFV formalism to the case of one-dimensional systems with constraints, whose classical treatment was the focus of Sections 2 to 6, with linear polarizations. We discuss here only the quantization of the boundary structure, whereas the quantization of the bulk will be discussed in Section 11. We defer the case of nonlinear polarizations to Section 12 and the case of infinite-dimensional targets, discussed at the classical level in Section 7, to [21]. In many places, we also make use of deformation quantization, for which we refer to the valuable review [27] and references therein.

10.1. Three cases. We will work out some representative cases of linear polarizations. Namely, we assume that the coordinates in \mathcal{B}_b are either

Case I: (p^b, c_b) , or

Case II: (p^b, b^b) ,

with (q_a, c_a) as the coordinates in \mathcal{B}_a (see Remark 9.2). For completeness, and for later applications, we will also consider:

Case III: (q_a, b^a) as the coordinates in \mathcal{B}_a and (p^b, c_b) as the coordinates in \mathcal{B}_b .

For the quantization we will consider a discontinuous splitting of the bulk fields (see Example 9.4).

Finally, we confine ourselves to the case when the structure functions are constant (see Example 8.1); we then have

$$\begin{aligned} S_{\text{BFV}} &= c_a^\alpha H_\alpha(p^a, q_a) - \frac{1}{2} f_{\alpha\beta}^\gamma b_\gamma^a c_a^\alpha c_b^\beta - c_b^\alpha H_\alpha(p^b, q_b) + \frac{1}{2} f_{\alpha\beta}^\gamma b_\gamma^b c_b^\alpha c_b^\beta, \\ S_{\text{BV}} &= \int_{t_a}^{t_b} \left(p_i dq^i - e_\alpha^+ dc^\alpha - e^\alpha H_\alpha(p, q) + c^\alpha \left(q_i^+ \frac{\partial H_\alpha}{\partial p_i}(p, q) - p_+^i \frac{\partial H_\alpha}{\partial q^i}(p, q) \right) \right. \\ &\quad \left. - \frac{1}{2} f_{\alpha\beta}^\gamma c_\gamma^+ c^\alpha c^\beta + f_{\alpha\beta}^\gamma e_\gamma^+ c^\alpha e^\beta \right). \end{aligned}$$

Remark 10.1 (Unimodularity). The BV action satisfies the classical master equation. To ensure $\Delta S_{\text{BV}} = 0$, although at a formal level, we assume the Lie algebra \mathfrak{g} , of which the $f_{\alpha\beta}^\gamma$ s are the structure constants, to be unimodular: i.e., $f_{\alpha\beta}^\alpha = 0$.

Remark 10.2 (The coboundary operator). In all the cases we consider, the operator Ω is of the form $\Omega_a - \Omega_b$ with Ω_a and Ω_b acting on different spaces, so $\Omega^2 = \Omega_a^2 + \Omega_b^2$ and we have $\Omega^2 = 0$ if and only if $\Omega_a^2 = 0$ and $\Omega_b^2 = 0$.

10.2. The boundary structure in Case I. We use $\tilde{f} = f = p_i q^i$, so

$$\alpha_{\text{BFV}}^{\tilde{f}} = \alpha_{\text{BFV}} + df(q_b, p^b) = p_i^a dq_a^i + b_\alpha^a dc_a^\alpha + q_b^i dp_i^b - b_\alpha^b dc_b^\alpha.$$

We now proceed as in Section 9.2.6. If we write $\alpha_{\text{BFV}}^{\tilde{f}} = z_\mu^* dz^\mu$, we get

$$p^a = q_a^*, \quad b^a = c_a^*, \quad q_b = (p^b)^*, \quad b^b = -c_b^*.$$

The recipe (87) for constructing Ω is to replace z_μ^* with $i\hbar \frac{\partial}{\partial z^\mu}$ in S_{BFV} . Therefore, we get

$$\Omega_a = c_a^\alpha H_\alpha \left(i\hbar \frac{\partial}{\partial q_a}, q_a \right) - \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_a^\alpha c_a^\beta \frac{\partial}{\partial c_a^\gamma},$$

$$\Omega_b = c_b^\alpha H_\alpha \left(p^b, i\hbar \frac{\partial}{\partial p^b} \right) + \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_b^\alpha c_b^\beta \frac{\partial}{\partial c_b^\gamma}.$$

For example, if the index sets are both $\{1\}$, so $f_{\alpha\beta}^\gamma = 0$, and we take $H_1(p, q) = q$, we get the Ω of Example 9.8.

We can easily see that $\Omega^2 = 0$ if and only if we have, at both endpoints, the following quantization of (19):

$$(88) \quad \left[\widehat{H}_\alpha, \widehat{H}_\beta \right] = i\hbar f_{\alpha\beta}^\gamma \widehat{H}_\gamma.$$

Here \widehat{H}_α denotes the corresponding quantization of H_α at the given endpoint. Equivalently, we may reformulate this as

$$(89) \quad [H_\alpha, \star H_\beta] = i\hbar f_{\alpha\beta}^\gamma H_\gamma,$$

where $[\phi \star, \psi] := \phi \star \psi - \psi \star \phi$ denotes the commutator with respect to either star product induced from the above quantization (if the condition is satisfied for one star product, it is automatically satisfied also for the other). Namely, at the initial endpoint we have

$$(90) \quad \phi \star \psi(p, q) = e^{i\hbar \frac{\partial^2}{\partial \tilde{q}^i \partial \tilde{p}_i}} \phi(\tilde{p}, q) \psi(p, \tilde{q}) \Big|_{\tilde{p}=p, \tilde{q}=q},$$

whereas at the final endpoint we have

$$(91) \quad \phi \star \psi(p, q) = e^{i\hbar \frac{\partial^2}{\partial \tilde{q}^i \partial \tilde{p}_i}} \phi(p, \tilde{q}) \psi(\tilde{p}, q) \Big|_{\tilde{p}=p, \tilde{q}=q}.$$

In principle, if (89) is not satisfied on the nose, one may look for \hbar -deformations of the H_α s that satisfy it. However, there are in general obstructions to do it. For the purposes of the present paper the following remark is important.

Remark 10.3. In both the linear case of Example 4.3 and the biaffine case of Example 4.6, condition (89) is automatically satisfied, so we have $\Omega^2 = 0$.

Digression 10.4. Lemma 4.14 of [19] states that $\Omega^2 = 0$ in case of BF -like theories (like the theories at hand). The regularization was assumed there to be in terms of the Fulton–MacPherson–Axelrod–Singer compactified configurations spaces [31, 4] (which is also the case we will consider in the next sections). What was also implicitly assumed there, although unfortunately not made clear, was that the bulk dimension should be larger than 1.⁵⁰ For this reason that lemma does not apply to the present case. We will briefly discuss in Appendix B the correct version of the lemma in the case at hand. Note that Lemma 4.14 of [19] also does not apply directly to the case of infinite-dimensional targets that will be treated in [21], since the appropriate gauge fixing needed to reduce to the present considerations (the axial gauge fixing) is not compatible with the aforementioned compactification of configuration spaces. Actually, the failure of (89) has not only the consequence that the boundary operator Ω does not square to zero, but also that the classical master equation $(S_{\text{BV}}, S_{\text{BV}}) = 0$ is not enough for the regularized QME to hold in

⁵⁰The proof of that lemma relied in particular on the contribution of the principal faces—i.e., collapses of two points in the bulk—of the boundary of compactified configurations spaces. In d dimensions, such faces have dimension $d-1$, whereas propagators in BF -like theories are $(d-1)$ -forms. For $d > 1$, there must then be exactly one propagator between the two collapsing points; the sum over all these contributions is zero as a consequence of the classical master equation. For $d = 1$, however, there is no bound on the number of propagators between two collapsing points.

the bulk. What will save the game in [21], where we will consider Chern–Simons theories on cylinders with the axial gauge, is that the theories considered there are examples of the linear or of the biaffine case.

10.3. The boundary structure in Case II. We use $\tilde{f} = p_i q^i + c^\alpha b_\alpha$, so

$$\alpha_{\text{BFV}}^{\tilde{f}} = \alpha_{\text{BFV}} + d\tilde{f}(q_b, c_b, p^b, b^b) = p_i^a dq_a^i + b_\alpha^a dc_a^\alpha + q_b^i dp_i^b - c_b^\alpha db_b^\alpha.$$

In this case, the prescriptions of Section 9.2.6 yield

$$p^a = q_a^*, \quad b^a = c_a^*, \quad q_b = (p^b)^*, \quad c_b = -(b^b)^*$$

and

$$\begin{aligned} \Omega_a &= c_a^\alpha H_\alpha \left(i\hbar \frac{\partial}{\partial q_a}, q_a \right) - \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_a^\alpha c_a^\beta \frac{\partial}{\partial c_a^\gamma}, \\ \Omega_b &= -i\hbar H_\alpha \left(p^b, i\hbar \frac{\partial}{\partial p^b} \right) \frac{\partial}{\partial b_\alpha^b} + \frac{\hbar^2}{2} f_{\alpha\beta}^\gamma b_\gamma^b \frac{\partial^2}{\partial b_\alpha^b \partial b_\beta^b}. \end{aligned}$$

Again $\Omega^2 = 0$ if and only if (89) is satisfied at both endpoints.

10.4. The boundary structure in Case III. In this case have to use the one-form

$$\alpha_{\text{BFV}}^{\hat{f}} = p_i^a dq_a^i + c_a^\alpha db_\alpha^a + q_b^i dp_i^b - b_\alpha^b dc_b^\alpha.$$

To achieve this we add $p_i q^i$ at the final endpoint of the BV action, like in Cases I and II, but now we subtract $c^\alpha b_\alpha$ at the initial endpoint. The function \hat{f} is now more precisely written as $p_i^b q_b^i - c_a^\alpha b_\alpha^a$ and we have $\alpha_{\text{BFV}}^{\hat{f}} = \alpha_{\text{BFV}} + d\hat{f}$. In this case, the prescriptions of Section 9.2.6 yield

$$p^a = q_a^*, \quad c_a = (b^a)^*, \quad q_b = (p^b)^*, \quad b^b = -c_b^*,$$

and

$$\begin{aligned} \Omega_a &= i\hbar H_\alpha \left(i\hbar \frac{\partial}{\partial q_a}, q_a \right) \frac{\partial}{\partial b_\alpha^a} + \frac{\hbar^2}{2} f_{\alpha\beta}^\gamma b_\gamma^a \frac{\partial^2}{\partial b_\alpha^a \partial b_\beta^a}, \\ \Omega_b &= c_b^\alpha H_\alpha \left(p^b, i\hbar \frac{\partial}{\partial p^b} \right) + \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_b^\alpha c_b^\beta \frac{\partial}{\partial c_b^\gamma}. \end{aligned}$$

Also in this case $\Omega^2 = 0$ if and only if (89) is satisfied at both endpoints.

11. BV-BFV QUANTIZATION WITH LINEAR POLARIZATIONS

In this section we quantize the constrained one-dimensional systems with linear polarizations on the endpoints. We refer to the three cases whose boundary structure has been discussed in Section 10 (see Section 10.1 for the list of the three case). We postpone the more complicated Case I and start with Cases II and III.

After that, we will discuss the gluing of intervals and the composition of the corresponding states. We will conclude the section with the quantization of the systems with nontrivial evolution discussed in Section 5 at the classical level: in particular, classical mechanics, Section 5.1, and the free relativistic particle, Section 5.2.

11.1. **Quantization in Case II.** First we have to subtract the final-endpoint pull-back of $\tilde{f} = p_i q^i + c^\alpha b_\alpha$ from the BV action getting

$$S_{\text{BV}}^{\tilde{f}} = S_{\text{BV}} - p_i(t_b)q^i(t_b) + c^\alpha(t_b)e_\alpha^+(t_b).$$

Next we pick the discontinuous splitting $\tilde{\mathcal{F}} = \mathcal{Y} \oplus \mathcal{B}$ of Example 9.4 for all bulk fields. Namely, by decorating the elements of \mathcal{Y} with a hat, we set

$$\begin{aligned} q(t) &= \begin{cases} q_a & t = t_a \\ \hat{q}(t) & t > t_a \end{cases}, & c(t) &= \begin{cases} c_a & t = t_a \\ \hat{c}(t) & t > t_a \end{cases}, \\ p(t) &= \begin{cases} \hat{p}(t) & t < t_b \\ p^b & t = t_b \end{cases}, & e^+(t) &= \begin{cases} \hat{e}^+(t) & t < t_b \\ -b^b & t = t_b \end{cases}, \end{aligned}$$

with boundary conditions

$$\hat{q}(t_a) = 0, \quad \hat{c}(t_a) = 0, \quad \hat{p}(t_b) = 0, \quad \hat{e}^+(t_b) = 0.$$

We decorate the remaining fields, all belonging to \mathcal{Y} , also with a hat: $\hat{p}^+, \hat{q}_+, \hat{e}, \hat{c}^+$.

We now formally integrate by parts⁵¹ the terms $p_i dq^i$ and $e_\alpha^+ dc^\alpha$ in $S_{\text{BV}}^{\tilde{f}}$,

$$\int_{t_a}^{t_b} p_i dq^i \rightarrow -\hat{p}_i(t_a)q_a^i + \int_{t_a}^{t_b} \hat{p}_i d\hat{q}^i, \quad \int_{t_a}^{t_b} e_\alpha^+ dc^\alpha \rightarrow \hat{e}_\alpha^+(t_a)c_a^\alpha + \int_{t_a}^{t_b} \hat{e}_\alpha^+ d\hat{c}^\alpha,$$

in order to get the BV action adapted to this splitting:

$$\begin{aligned} \tilde{S}_{\text{BV}}^{\tilde{f}} &:= -p_i^b \hat{q}^i(t_b) - \hat{c}^\alpha(t_b)b_\alpha^b - \hat{p}_i(t_a)q_a^i - \hat{e}_\alpha^+(t_a)c_a^\alpha \\ &\quad + \int_{t_a}^{t_b} \left(\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha - \hat{e}^\alpha H_\alpha(\hat{p}, \hat{q}) \right. \\ &\quad \left. + \hat{c}^\alpha \left(\hat{q}_i^+ \frac{\partial H_\alpha}{\partial p_i}(\hat{p}, \hat{q}) - \hat{p}_i^+ \frac{\partial H_\alpha}{\partial q^i}(\hat{p}, \hat{q}) \right) \right. \\ &\quad \left. - \frac{1}{2} f_{\alpha\beta}^\gamma \hat{c}_\gamma^+ \hat{c}^\alpha \hat{c}^\beta + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha \hat{e}^\beta \right). \end{aligned}$$

We finally pick the gauge-fixing lagrangian

$$(92) \quad \mathcal{L} = \{\hat{p}^+ = \hat{q}^+ = \hat{e} = \hat{c}^+ = 0\},$$

getting

$$\tilde{S}_{\text{BV}}^{\tilde{f}} \Big|_{\mathcal{L}} = -p_i^b \hat{q}^i(t_b) - \hat{c}^\alpha(t_b)b_\alpha^b - \hat{p}_i(t_a)q_a^i - \hat{e}_\alpha^+(t_a)c_a^\alpha + \int_{t_a}^{t_b} (\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha).$$

The propagators defined by the kinetic terms and compatible with the boundary conditions are

$$(93) \quad \langle \hat{q}^i(s) \hat{p}_j(t) \rangle = i\hbar \delta_j^i \theta(s-t) \quad \text{and} \quad \langle \hat{c}^\alpha(s) \hat{e}_\beta^+(t) \rangle = i\hbar \delta_\beta^\alpha \theta(s-t),$$

where

$$\theta(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

is the Heaviside step function. We can then compute

$$Z_{\text{II}} := \int_{\mathcal{L}} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^{\tilde{f}}}$$

⁵¹See Remark 9.5.

as

$$(94) \quad Z_{\text{II}}(q_a, c_a, p^b, b^b) = e^{-\frac{i}{\hbar}(p_i^b q_a^i - b_\alpha^b c_a^\alpha)}.$$

One can immediately check that $\Omega Z_{\text{II}} = 0$.

The striking result is that Z_{II} is extremely simple and does not depend on the constraints H_α . Still its definition requires the whole BV-BFV machinery and, in particular, condition (89). The reason is that we have chosen a clever gauge fixing that simplifies the computation, but we want to be sure that the result is invariant (up to Ω -coboundary terms) under deformations of the gauge-fixing lagrangian (see, in case, the following Digression).

As we discuss in the following Digression, just deforming the gauge fixing, picking e different from zero, does not change the result (in Δ^Ω -cohomology). Still one can get a less trivial answer for Case I, as we will discuss in Section 11.4. The present “trivial” answer is anyway useful for gluing solutions on different intervals along the lines of Section 9.2.8. We will see in Section 11.4 how to get a nontrivial answer.

Digression 11.1 (Change of gauge fixing). By the general BV-BFV philosophy—proved in the case at hand in Appendix B—we know that deformations of \mathcal{L} produce Ω -cohomologous results. Instead of using this general result, as a matter of example we explicitly consider a particular case. Let

$$\mathcal{L}_u := \left\{ \hat{p}^+ = \hat{q}^+ = \hat{c}^+ = 0, \hat{e} = e_0 := \frac{u dt}{t_b - t_a} \right\},$$

where, for simplicity, we assume u to be a constant function on $[t_a, t_b]$. We now have

$$\begin{aligned} \tilde{S}_{\text{BV}}^f \Big|_{\mathcal{L}_u} &= -p_i^b \hat{q}^i(t_b) - \hat{c}^\alpha(t_b) b_\alpha^b - \hat{p}_i(t_a) q_a^i - \hat{e}_\alpha^+(t_a) c_a^\alpha \\ &\quad + \int_{t_a}^{t_b} \left(\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha - e_0^\alpha H_\alpha(\hat{p}, \hat{q}) + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha e_0^\beta \right). \end{aligned}$$

Let us first consider the case when the structure constants vanish. In this case, we compute, following [19, Section 4.4],

$$Z_{\text{II}}^u := \int_{\mathcal{L}_u} e^{\frac{i}{\hbar} \tilde{S}_{\text{BV}}^f} = e^{-\frac{i}{\hbar}(p_i^b q_a^i - b_\alpha^b c_a^\alpha)} e_{\star}^{-\frac{i}{\hbar} u^\alpha H_\alpha}(p^b, q_a),$$

where e_\star is the star exponential with respect to the initial star product (90). We have

$$\Omega_a Z_{\text{II}}^u = c_a^\alpha Z_{\text{II}}^u \star H_\alpha, \quad \Omega_b Z_{\text{II}}^u = c_\alpha^b H_\alpha \star Z_{\text{II}}^u,$$

so the mQME $\Omega Z_{\text{II}}^u = 0$ is satisfied, provided that (89), which now simply reads $[H_\alpha, \star H_\beta] = 0$, holds. Let us now consider a family of gauge fixings as above, parametrized by a path $u(x)$, x is some interval, and check gauge-fixing independence under the assumption (89). We have

$$\frac{\partial Z_{\text{II}}^u}{\partial x} = -\frac{i}{\hbar} \frac{du^\alpha}{dx} H_\alpha \star Z_{\text{II}}^u;$$

that is,

$$(95) \quad \frac{\partial Z_{\text{II}}^u}{\partial x} = \Delta^\Omega \psi$$

with $\psi = -b_\alpha^b \frac{du^\alpha}{dx} Z_{\text{II}}^u$ and $\Delta^\Omega = \frac{1}{\hbar^2} \Omega$ in accordance with (86). The general case with nonvanishing structure constants is a bit more complicated. We only report

the final computations. As in Section 4.2, we view the $f_{\alpha\beta}^\gamma$ s as structure constants of a Lie algebra \mathfrak{g} for a certain basis; we then view objects with an upper Greek index as taking value in \mathfrak{g} and objects with a lower Greek index as taking values in \mathfrak{g}^* ; finally, we denote by $\langle \cdot, \cdot \rangle$ the pairing of \mathfrak{g}^* with \mathfrak{g} . We then get

$$Z_{\text{II}}^u = e^{-\frac{i}{\hbar}(p_i^b q_a^i - \langle b^b, \text{Ad}_{e^u} c_a \rangle)} e_{\star}^{-\frac{i}{\hbar}\langle H, u \rangle}(p^b, q_a).$$

Therefore, we have

$$\begin{aligned} \Omega_a Z_{\text{II}}^u &= Z_{\text{II}}^u \star \langle H, c_a \rangle - \frac{1}{2} \langle b^b, \text{Ad}_{e^u} [c_a, c_a] \rangle Z_{\text{II}}^u, \\ \Omega_b Z_{\text{II}}^u &= \langle H, \text{Ad}_{e^u} c_a \rangle \star Z_{\text{II}}^u - \frac{1}{2} \langle b^b, \text{Ad}_{e^u} [c_a, c_a] \rangle Z_{\text{II}}^u, \end{aligned}$$

and hence

$$\Omega Z_{\text{II}}^u = Z_{\text{II}}^u \star \left(\langle H, c_a \rangle - \left\langle e_{\star}^{\frac{i}{\hbar}\langle H, u \rangle} \star H \star e_{\star}^{-\frac{i}{\hbar}\langle H, u \rangle}, \text{Ad}_{e^u} c_a \right\rangle \right).$$

Using (89), one can then easily prove that

$$\chi(s) := \left\langle e_{\star}^{\frac{i}{\hbar}s\langle H, u \rangle} \star H \star e_{\star}^{-\frac{i}{\hbar}s\langle H, u \rangle}, \text{Ad}_{e^{su}} c_a \right\rangle$$

is constant in s , and therefore $\Omega Z_{\text{II}}^u = 0$. If we consider again a family of gauge fixings parametrized by a path $u(x)$ and assume (89), we get again (95), now with $\psi = -\langle b^b, \text{Ad}_{e^u} \frac{du}{dx} \rangle Z_{\text{II}}^u$.

11.2. Quantization in Case III. Case III is very similar to Case II. We proceed very quickly, just outlining the differences. First, the BV action for the given boundary polarizations is

$$S_{\text{BV}}^{\hat{f}} = S_{\text{BV}} - p_i(t_b)q^i(t_b) - c^\alpha(t_a)e_\alpha^+(t_a).$$

In picking the discontinuous extension, p and q are realized as in Case II, with the same boundary conditions, whereas

$$(96) \quad e^+(t) = \begin{cases} -b^a & t = t_a \\ \hat{e}^+(t) & t > t_a \end{cases}, \quad c(t) = \begin{cases} \hat{c}(t) & t < t_b \\ c_b & t = t_b \end{cases},$$

with boundary conditions

$$(97) \quad \hat{e}^+(t_a) = 0, \quad \hat{c}(t_b) = 0.$$

The BV action adapted to this splitting is then

$$\begin{aligned} \tilde{S}_{\text{BV}}^{\hat{f}} &= -p_i^b \hat{q}^i(t_b) - \hat{e}_\alpha^+(t_b)c_b^\alpha - \hat{p}_i(t_a)q_a^i + \hat{c}^\alpha(t_a)b_\alpha^a \\ &+ \int_{t_a}^{t_b} \left(\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha - \hat{e}^\alpha H_\alpha(\hat{p}, \hat{q}) \right. \\ (98) \quad &+ \hat{c}^\alpha \left(\hat{q}_i^+ \frac{\partial H_\alpha}{\partial p_i}(\hat{p}, \hat{q}) - \hat{p}_i^+ \frac{\partial H_\alpha}{\partial q^i}(\hat{p}, \hat{q}) \right) \\ &\left. - \frac{1}{2} f_{\alpha\beta}^\gamma \hat{c}_\gamma^+ \hat{c}^\alpha \hat{c}^\beta + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha \hat{e}^\beta \right). \end{aligned}$$

We pick again the gauge-fixing lagrangian (92). This produces the same $\hat{p}\hat{q}$ propagator as in Case II, see (93), but now we have

$$\langle \hat{e}_\beta^+(s) \hat{c}^\alpha(t) \rangle = i\hbar \delta_\beta^\alpha \theta(s-t).$$

We then get

$$Z_{\text{III}}(q_a, b^a, p^b, c_b) = e^{-\frac{i}{\hbar}(p_i^b q_a^i - c_b^a b_a^a)}.$$

One can immediately check that $\Omega Z_{\text{III}} = 0$.

The considerations of Digression 11.1 of course apply also to this case.

11.3. Quantization in Case I. Finally, we consider Case I. Now we have to subtract the final-endpoint pullback of $f = p_i q^i$ from the BV action getting

$$S_{\text{BV}}^f = S_{\text{BV}} - p_i(t_b)q^i(t_b).$$

Since the boundary values c_a and c_b are given (as coordinates of \mathcal{B}), the fluctuations \hat{c} of the ghost fields \hat{c} , in the discontinuous splitting, will have vanishing boundary conditions at both endpoints. On the other hand, there are no endpoint conditions on the fluctuations \hat{e}^+ . Proceeding this way, we would get, as in the previous cases, a term $\int_{t_a}^{t_b} \hat{e}_\alpha^+ d\hat{c}^\alpha$ in the action. This term is not well suited for the perturbative expansion as d now has a kernel on the space of the \hat{e}^+ fields (namely, the constant ones). Since we compute the functional integral perturbatively, we have to remove this constant term, call it T^+ . More precisely, we are going to proceed as in Section 9.2.7; namely, in addition to a splitting $\tilde{F} = \mathcal{Y} \oplus \mathcal{B}$, we also split $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, where \mathcal{Y}_2 contains the fluctuations (which we will denote with a hat) over which we integrate by perturbation theory, and \mathcal{Y}_1 is a finite-dimensional, odd symplectic space of which the T^+ s are half of the coordinates. The other half will have to be variables T which are part of the fields e . In general, we will not integrate over \mathcal{Y}_1 , but in some cases this is possible (yet one has to be careful that there are in general nonequivalent choices of gauge-fixing lagrangians: see Example 9.8 and Remark 11.5). In accordance with [19], we call \mathcal{Y}_1 the space of residual fields.

Explicitly, the spaces $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{B}$ are defined as follows. In \mathcal{Y}_1 we have coordinates (T^α, T_α^+) with canonical odd symplectic structure $\delta T_\alpha^+ \delta T^\alpha$. In \mathcal{Y}_2 we have the hat fields $\hat{p}, \hat{q}, \hat{e}, \hat{c}, \hat{p}_+, \hat{q}^+, \hat{e}^+, \hat{c}^+$ that must obey the following conditions (99)

$$\hat{q}(t_a) = 0, \quad \hat{c}(t_a) = 0, \quad \hat{p}(t_b) = 0, \quad \hat{c}(t_b) = 0, \quad \int_{t_a}^{t_b} \hat{e}^+ dt = 0, \quad \int_{t_a}^{t_b} \hat{e} = 0.$$

Finally, \mathcal{B} has coordinates q_a, c_a, p^b, c_b . We define $\tilde{F} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{B}$ via

$$q(t) = \begin{cases} q_a & t = t_a \\ \hat{q}(t) & t > t_a \end{cases}, \quad p(t) = \begin{cases} \hat{p}(t) & t < t_b \\ p^b & t = t_b \end{cases}, \quad c(t) = \begin{cases} c_a & t = t_a \\ \hat{c}(t) & t_a < t < t_b \\ c_b & t = t_b \end{cases},$$

$$e(t) = \frac{T dt}{t_b - t_a} + \hat{e}(t), \quad e^+(t) = T^+ + \hat{e}^+(t),$$

and

$$p_+(t) = \hat{p}_+(t), \quad q^+(t) = \hat{q}^+(t), \quad c^+(t) = \hat{c}^+(t).$$

To get the adapted BV action, we use the replacements

$$\int_{t_a}^{t_b} p_i dq^i \rightarrow -\hat{p}_i(t_a)q_a^i + \int_{t_a}^{t_b} \hat{p}_i d\hat{q}^i, \\ \int_{t_a}^{t_b} e_\alpha^+ dc^\alpha \rightarrow -T_\alpha^+ (c_b^\alpha - c_a^\alpha) - \hat{e}_\alpha^+(t_b)c_b^\alpha + \hat{e}_\alpha^+(t_a)c_a^\alpha + \int_{t_a}^{t_b} \hat{e}_\alpha^+ d\hat{c}^\alpha,$$

which yield

$$\begin{aligned} \tilde{S}_{\text{BV}}^f &= T_\alpha^+ (c_b^\alpha - c_a^\alpha) - p_i^b \hat{q}^i(t_b) + \hat{e}_\alpha^+(t_b) c_b^\alpha - \hat{p}_i(t_a) q_a^i - \hat{e}_\alpha^+(t_a) c_a^\alpha \\ &+ \int_{t_a}^{t_b} \left(\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha - \frac{T^\alpha dt}{t_b - t_a} H_\alpha(\hat{p}, \hat{q}) - \hat{e}^\alpha H_\alpha(\hat{p}, \hat{q}) \right. \\ &+ \hat{c}^\alpha \left(\hat{q}_i^+ \frac{\partial H_\alpha}{\partial p_i}(\hat{p}, \hat{q}) - \hat{p}_+^i \frac{\partial H_\alpha}{\partial q^i}(\hat{p}, \hat{q}) \right) - \frac{1}{2} f_{\alpha\beta}^\gamma \hat{c}_\gamma^+ \hat{c}^\alpha \hat{c}^\beta \\ &\left. - f_{\alpha\beta}^\gamma T_\gamma^+ \hat{c}^\alpha \frac{T^\beta dt}{t_b - t_a} + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha \frac{T^\beta dt}{t_b - t_a} - f_{\alpha\beta}^\gamma T_\gamma^+ \hat{c}^\alpha \hat{e}^\beta + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha \hat{e}^\beta \right). \end{aligned}$$

On the gauge-fixing lagrangian (92) we have

$$\begin{aligned} \tilde{S}_{\text{BV}}^f \Big|_{\mathcal{L}} &= T_\alpha^+ (c_b^\alpha - c_a^\alpha) - p_i^b \hat{q}^i(t_b) + \hat{e}_\alpha^+(t_b) c_b^\alpha - \hat{p}_i(t_a) q_a^i - \hat{e}_\alpha^+(t_a) c_a^\alpha \\ &+ \int_{t_a}^{t_b} \left(\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha - \frac{T^\alpha dt}{t_b - t_a} H_\alpha(\hat{p}, \hat{q}) \right. \\ &\left. - f_{\alpha\beta}^\gamma T_\gamma^+ \hat{c}^\alpha \frac{T^\beta dt}{t_b - t_a} + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha \frac{T^\beta dt}{t_b - t_a} \right). \end{aligned}$$

The $\hat{p}\hat{q}$ propagator is the same as in Case II, see (93), but the $\hat{c}\hat{e}^+$ propagator now has to take into account the conditions (99) on \hat{c} and \hat{e}^+ . We get

$$(100) \quad \langle \hat{c}^\alpha(s) \hat{e}_\beta^+(t) \rangle = i\hbar \delta_\beta^\alpha (\theta(s-t) - \phi(s))$$

with

$$\phi(s) = \frac{s - t_a}{t_b - t_a}.$$

Note that there are no mixed terms involving the physical fields $\hat{p}\hat{q}$ and the ghost fields $\hat{c}\hat{e}^+$ in $\tilde{S}_{\text{BV}}^f \Big|_{\mathcal{L}}$, so the partition function $Z_I := \int_{\mathcal{L}} e^{\frac{1}{\hbar} \tilde{S}_{\text{BV}}^f}$ is the product of a “physical part” Z_I^{phys} and a “ghost part” Z_I^{gh} . The physical part is computed following [19, Section 4.4]:

$$Z_I^{\text{phys}}(q_a, p^b; T) = e^{-\frac{1}{\hbar} p_i^b q_a^i} e_\star^{-\frac{1}{\hbar} T^\alpha H_\alpha(p^b, q_a)}.$$

Remark 11.2. The \star -exponential arises from Feynman diagrams consisting of multi-edge linear graphs of arbitrary length. The lowest-order terms are the linear trees. Note that each vertex (either in the bulk or at one endpoint) contributes with a factor $\frac{1}{\hbar}$, whereas each edge contributes with a factor \hbar , so a tree of any length yields a global factor $\frac{1}{\hbar}$. Moreover, such trees simply define the (semi)classical evolution operator, i.e., the Hamilton–Jacobi action. Therefore,

$$(101) \quad Z_I^{\text{phys}}(q_a, p^b; T) = e^{\frac{1}{\hbar} (\tilde{S}_{\text{HJ}}^f(q_a, p^b, e^T) + O(\hbar))}$$

with the HJ action as in (37).

Remark 11.3 (Semiclassical expansion). The last formula can also be obtained as follows. As we have seen, in the Lie algebra case and in the chosen gauge fixing, the physical part completely decouples from the ghost part. Moreover, as in Remark 4.10, we may view the gauge-fixed action for the physical part as the action for a mechanical system with hamiltonian $\tilde{H} = \langle H, T \rangle$ (and time variable $s = \frac{t}{t_b - t_a}$). For such a system we may go back, via a translation in the functional integral, to the usual noncontinuous choice of fields, $p(t) = p_{q_a, p^b}(t) + \check{p}(t)$ and $q(t) = q_{q_a, p^b}(t) + \check{q}(t)$, and integrate over the fluctuations \check{p} and \check{q} , getting the

semiclassical expansion (101). Note that, on the other hand, the discontinuous extension, besides being the correct one in general for BV-BFV, is also interesting because it produces the \star -product representation of the partition function.⁵²

We may summarize the content of equation (101) and of the two above remarks in the following

Theorem 11.4. *The physical part of the dominant contribution of the perturbative expansion of the BV-BFV partition function, in the polarization of Case I, is the HJ action.*

The ghost part is very easy to compute in the abelian case, $f_{\alpha\beta}^\gamma = 0$, since the interaction terms vanish and the boundary source terms cannot be coupled to each other:

$$(102) \quad Z_I^{\text{gh,abelian}}(c_a, c_b; T^+) = e^{-\frac{i}{\hbar} T_\alpha^+ (c_a^\alpha - c_b^\alpha)}.$$

Therefore,

$$(103) \quad Z_I^{\text{abelian}}(q_a, c_a, p^b, c_b; T, T^+) = e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} T^\alpha H_\alpha} (p^b, q_a) e^{-\frac{i}{\hbar} T_\alpha^+ (c_a^\alpha - c_b^\alpha)}.$$

It is very easy in this case to check that the mQME holds. In fact, we have

$$\Omega_a Z_I^{\text{abelian}} = c_a^\alpha Z_I^{\text{abelian}} \star H_\alpha, \quad \Omega_b Z_I^{\text{abelian}} = c_b^\alpha H_\alpha \star Z_I^{\text{abelian}}.$$

Therefore, since (89) implies in this case that the H s \star -commute,

$$\Omega Z_I^{\text{abelian}} = (c_a^\alpha - c_b^\alpha) H_\alpha \star Z_I^{\text{abelian}}.$$

Since the right hand side is also clearly equal to $-\hbar^2 \frac{\partial^2}{\partial T^\alpha \partial T_\alpha^+} Z_I^{\text{abelian}}$, the mQME is satisfied.

In the nonabelian case, the ghost part Z_I^{gh} can be computed in terms of Feynman diagrams. In Figure 1 we introduce a notation for the ghost propagator. In $\tilde{S}_{\text{BV}}^f|_{\mathcal{L}}$,

$$\begin{array}{c} \leftarrow \text{-----} \\ \hat{e}^+ \qquad \qquad \hat{c} \end{array}$$

FIGURE 1. Ghost propagator

we have three univalent vertices (one at each endpoint, each containing \hat{e}^+ , and one in the bulk, containing \hat{c}) and one bulk $\hat{e}^+ \hat{c}$ -bivalent vertex; see Figure 2. If a univalent \hat{c} -vertex is present, we can only build a linear tree, of any length,

$$c_a \begin{array}{c} | \\ \times \text{---} \\ | \end{array} \quad , \quad \begin{array}{c} | \\ \text{---} \times \\ | \end{array} c_b \quad , \quad \begin{array}{c} \leftarrow \text{---} \circ \\ T^+ T \\ | \end{array} \quad , \quad \begin{array}{c} \leftarrow \text{---} \circ \text{---} \\ T \\ | \end{array}$$

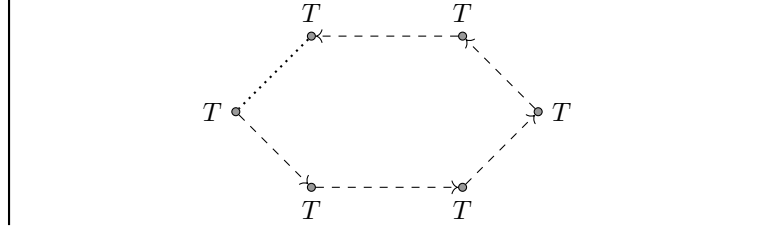
FIGURE 2. Ghost vertices

connecting it to one of the endpoint \hat{e}^+ -vertices via the bivalent vertices: these trees will give rise to the F -terms below (in which we include also the “abelian” term $T_\alpha^+ (c_a^\alpha - c_b^\alpha)$); see Figure 3 (where the grey vertices are integrated along the interval; the dots denote other possible vertices). In addition, we have wheels of any length

⁵²Cf. the Clifford-exponential presentation of the nonabelian 1D Chern–Simons partition function in [1].

FIGURE 3. F -diagrams

using only bivalent vertices which produce the W -terms below (note that, thanks to the unimodularity assumption of Remark 10.1, there are no tadpoles, which would otherwise have to be regularized); see Figure 4 (where we use a two-dimensional representation just for pictorial reasons: the grey vertices are still integrated along the interval). To get a more readable expression, we use again the notations of

FIGURE 4. W -diagrams

Section 4.2, and of the second part of Digression 11.1, in terms of the Lie algebra \mathfrak{g} , its dual \mathfrak{g}^* and their pairing $\langle \cdot, \cdot \rangle$. We get

$$(104) \quad Z_I(q_a, c_a, p^b, c_b; T, T^+) = e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} \langle H, T \rangle} e^{-\frac{i}{\hbar} \langle T^+, F_-(\text{ad}_T) c_b + F_+(\text{ad}_T) c_a \rangle} e^{W(T)}.$$

The series F_+ , F_- , W can be explicitly computed, although we do not present the details here:

$$F_+(x) = \frac{x}{1 - e^{-x}} = \sum_{n \geq 0} (-1)^n \frac{B_n}{n!} x^n,$$

$$F_-(x) = -\frac{x}{e^x - 1} = -\sum_{n \geq 0} \frac{B_n}{n!} x^n,$$

$$W(T) = \sum_{n \geq 1} \frac{B_n}{n \cdot n!} \text{tr}(\text{ad}_T)^n = \text{tr} \log \frac{\sinh \frac{\text{ad}_T}{2}}{\frac{\text{ad}_T}{2}},$$

where the B_n s are the Bernoulli numbers: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}, \dots$. The mQME can also be directly verified.

Remark 11.5 (Reduction of the residual fields). We may sometimes perform a further BV pushforward on \mathcal{Y}_1 . One possibility, which leads to a less interesting result as in Case II, consists in choosing the lagrangian $\mathcal{L}^{\text{triv}} = \{T = 0\}$. We get in this case

$$(105) \quad Z_I^{\mathcal{L}^{\text{triv}}}(q_a, c_a, p^b, c_b) = e^{-\frac{i}{\hbar} p_i^b q_a^i} \delta(c_b - c_a).$$

More interesting choices consist in taking \mathcal{L} to be the conormal bundle, shifted by -1 , of a submanifold of the Lie algebra \mathfrak{g} . For example, we can take the whole \mathfrak{g}

as such a submanifold, and therefore $\mathcal{L} = \{T^+ = 0\}$, which yields

$$(106) \quad Z_{\Gamma}^{\mathcal{L}}(q_a, p^b) = \int_{\mathfrak{g}} e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} \langle H, T \rangle} d^k T,$$

where $k = \dim \mathfrak{g}$ and $d^k T$ is the Lebesgue measure (of course under the assumption that the integral converges). Note that $Z_{\Gamma}^{\mathcal{L}}$ does no longer depend on the ghost variables.

Remark 11.6 (Passing to the group). In the nonabelian case, assuming the exponential map to be surjective, it is convenient to make a change of variables, setting $g = \psi(T) := e^T$. A first advantage of this transformation is that the semiclassical term of the physical part of Z_{Γ} is brought to the form of Section 4.2. Since we are in the BV setting, we have to complete this transformation to a symplectomorphism. We do it by taking the cotangent lift:

$$\begin{aligned} \Psi: T^*[-1]\mathfrak{g} &\rightarrow T^*[-1]G \\ (T, T^+) &\mapsto \left(e^T, (d_e l_{e^T})^{*, -1} \frac{\text{ad}_T^*}{1 - e^{-\text{ad}_T^*}} T^+ \right), \end{aligned}$$

where e denotes the identity element of G and l_g the left multiplication by g : $l_g(h) = gh$. The first term of the ghost part now simplifies as follows:

$$\langle T^+, F_-(\text{ad}_T)c_b + F_+(\text{ad}_T)c_a \rangle = \langle (d_e l_g)^* g^+, c_a - \text{Ad}_g^{-1} c_b \rangle.$$

Finally, we have to observe that, more appropriately for the BV formalism [37, 45], the partition function should be regarded as a half-density:

$$\hat{Z}_{\Gamma} = Z_{\Gamma} \sqrt{d^k T d^k T^+},$$

with $k = \dim \mathfrak{g}$, $d^k T$ the Lebesgue measure, and $d^k T^+$ the standard Berezinian measure. We have, again under the assumption of unimodularity (see Remark 10.1),⁵³ that

$$\psi^* \mu_G = \det \frac{\sinh \frac{\text{ad}_T}{2}}{\frac{\text{ad}_T}{2}} d^k T = e^{\mathbb{W}(T)} d^k T,$$

where μ_G is the Haar measure. Therefore,

$$\hat{Z}_{\Gamma}(q_a, c_a, p^b, c_b; g, g^+) = e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} \langle H, \log g \rangle} e^{-\frac{i}{\hbar} \langle (d_e l_g)^* g^+, c_a - \text{Ad}_g^{-1} c_b \rangle} \mu_G,$$

where we reinterpret the Haar measure μ_G on G as a half-density on $T^*[-1]G$. We can now “improve” (106) by taking $\mathcal{L} := \{g^+ = 0\} = G$ —the zero section of $T^*[-1]G$ —getting

$$(107) \quad Z_{\Gamma}^{\mathcal{L}}(q_a, p^b) = \int_G e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} \langle H, \log g \rangle} d\mu_G,$$

which converges for G compact. The integral in (106) usually does not converge, as it corresponds to summing infinitely many copies of the integral over G (this is an instance of what is known as Gribov’s ambiguity in the physics literature). The

⁵³In the case of a nonunimodular Lie algebra, one may extend the $\hat{c}\hat{e}^+$ propagator of (100) to the diagonal as $\langle \hat{c}^\alpha(s) \hat{e}_\beta^+(s) \rangle = -i\hbar \delta_\beta^\alpha \phi(s)$. With this regularization, the tadpoles produce the additional factor $e^{-\frac{1}{2} \text{tr ad}_T}$ in Z_{Γ} . On the other hand, without the assumption of unimodularity, one has $\psi^* \mu_G = e^{\mathbb{W}(T)} e^{-\frac{1}{2} \text{tr ad}_T} d^k T$, so the results discussed in this remark in principle also hold without the assumption of unimodularity.

“improved” formula (107) corresponds to choosing one single domain in \mathfrak{g} diffeomorphic to G (up to a measure zero subset). A similar procedure was developed in [36].

We conclude by discussing two important examples.

Example 11.7 (Linear case). Consider linear constraints as in Example 4.3. There are then two simplifications. The first is that in the computation of Z_I^{phys} we only have linear vertices in \hat{q} and \hat{p} , so the only connected Feynman graphs have one single edge: connecting two boundary points, or one boundary and a bulk point, or two bulk points. See Figure 5 for our notation for the physical propagator and Figure 6 for the connected Feynman graphs. These graphs correspond exactly to



FIGURE 5. Physical propagator

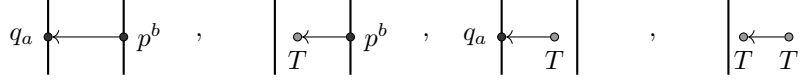


FIGURE 6. Linear case

the terms in (24) (this result is of course consistent with Remarks 11.2 and 11.3.) The second simplification is that the structure functions vanish, so the ghost part is simply the one of equation (102). In conclusion, we get

$$(108) \quad \begin{aligned} Z_I(q_a, c_a, p^b, c_b; T, T^+) &= e^{-\frac{i}{\hbar}(p_i^b q_a^i + T^\alpha (p_i^b v_\alpha^i + w_{\alpha,i} q_a^i) + \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta} + T_\alpha^+ (c_a^\alpha - c_b^\alpha))} \\ &= e^{\frac{i}{\hbar} \hat{S}_{\text{HJ}}^f(q_a, p^b, T)} e^{-\frac{i}{\hbar} T_\alpha^+ (c_a^\alpha - c_b^\alpha)}, \end{aligned}$$

with \hat{S}_{HJ}^f as in (24). This result is the prototype for abelian Chern–Simons theory, see [21, Sections 5.3.1 and 6]. We may also integrate over the residual fields with the gauge fixing $T^+ = 0$ as in (106). In general, there will be some linear terms in T , corresponding to the kernel of A , which produce delta functions. The remaining terms yield a Gaussian integration which produces quadratic terms in the p, q variables. This is the quantum analogue of solving some of the constraints. For example, in the case of Remark 3.3, we get

$$Z_I^{\mathcal{L}}(q_a, p^b) = e^{\frac{i}{\hbar} \hat{S}^f(q_a, p^b)} = e^{\frac{i}{\hbar} (\frac{1}{2} \frac{w}{v} (p^b)^2 + \frac{1}{2} \frac{w}{v} (q_a)^2)}.$$

Example 11.8 (Biaffine case). In the case of constraints that are affine both in the p and in the q variables, see Example 4.6, the physical part of the partition function can still be computed explicitly, as no multiple edges appear. We then end up having only linear trees in the computation of the \star -exponential; see Figure 7. Therefore, we get (37) with HJ action (32), putting $g = e^T$, and with no $O(\hbar)$ corrections. The partition half-density is then more naturally expressed in group variables as in Remark 11.6. We get

$$(109) \quad \hat{Z}_I(q_a, c_a, p^b, c_b; g, g^+) = e^{\frac{i}{\hbar} \hat{S}_{\text{HJ}}^f(q_a, p^b, g)} e^{-\frac{i}{\hbar} \langle (d_e l_g)^* g^+, c_a - \text{Ad}_g^{-1} c_b \rangle} \mu_G$$

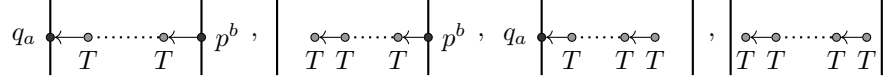


FIGURE 7. Biaffine case

$$= e^{-\frac{i}{\hbar}(p^b R_g^{-1} q_a + p^b R_g^{-1} \Phi(g) + \Psi(g) q_a + \text{WZW}(g) + \langle (de l_g)^* g^+, c_a - \text{Ad}_g^{-1} c_b \rangle)} \mu_G.$$

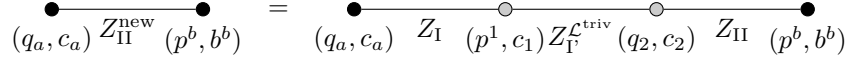
In the particular case of the adjoint representation, using the results of Example 4.8, we may also write

$$\begin{aligned} \hat{Z}_1(q_a, c_a, \bar{q}^b, c_b; g, g^+) &= \\ &= e^{-\frac{i}{\hbar}(\langle \bar{q}^b, g^{-1} q_a g \rangle + \langle \bar{q}^b, g^{-1} v(g) \rangle + \langle \bar{v}(g) g^{-1}, q_a \rangle + \text{WZW}(g) + \langle (de l_g)^* g^+, c_a - \text{Ad}_g^{-1} c_b \rangle)} \mu_G \end{aligned}$$

with the WZW term as in (36). This result is the prototype for nonabelian Chern–Simons theory, see [21, Sections 5.3 and 5.4]. Also in this case we may integrate over the residual fields, now with the gauge fixing $g^+ = 0$, as in (107). We get

$$Z_1^{\mathcal{L}}(q_a, \bar{q}^b) = \int_G e^{-\frac{i}{\hbar}(\langle \bar{q}^b, g^{-1} q_a g \rangle + \langle \bar{q}^b, g^{-1} v(g) \rangle + \langle \bar{v}(g) g^{-1}, q_a \rangle + \text{WZW}(g))} d\mu_G.$$

11.4. Gluing. Partition functions may be composed as described in Section 9.2.8. Here we are interested in composing a partition function from (q_a, c_a) to (p^1, c_1) with a partition function from (p^1, c_1) to (q_2, c_2) with a partition function from (q_2, c_2) to (p^b, b^b) , where (p^1, c_1) and (q_2, c_2) are some intermediate boundary fields. The first partition function belongs to Case I and the last to Case II. The intermediate one will be discussed below. The result will be a partition function from (q_a, c_a) to (p^b, b^b) , so of type II. See Figure 8.


 FIGURE 8. Composition of the partition function $Z_{\text{II}}^{\text{new}}$

The intermediate partition function can be computed along the lines of Case I, with the only difference that we now fix p at the initial endpoint and q at the final one (instead of the other way around). We are actually interested in the version where we integrate out the residual fields, as we did in Remark 11.5 for Case I. The analogue of (105) is now

$$(110) \quad Z_{\text{I}}^{\mathcal{L}^{\text{triv}}}(p^a, c_a, q_b, c_b) = e^{\frac{i}{\hbar} p_i^a q_b^i} \delta(c_b - c_a),$$

and we refer to this as Case I', in the trivial gauge fixing.

We now proceed to the computation of the “new” version of Case II as

$$\begin{aligned} Z_{\text{II}}^{\text{new}}(q_a, c_a, p^b, b^b; T, T^+) &:= \int Z_{\text{I}}(q_a, c_a, p^1, c_1; T, T^+) \\ &Z_{\text{I}}^{\mathcal{L}^{\text{triv}}}(p^1, c_1, q_2, c_2) Z_{\text{II}}(q_2, c_2, p^b, b^b) \frac{d^n p^1 d^n q_2}{(2\pi\hbar)^n} d^k c_1 d^k c_2 (-i\hbar)^k, \end{aligned}$$

where n is the dimension of the target configuration space, k is the dimension of the Lie algebra, and we have conveniently normalized the measure. A simple computation yields

$$Z_{\text{II}}^{\text{new}}(q_a, c_a, p^b, b^b; T, T^+) = e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} \langle H, T \rangle} e^{\frac{i}{\hbar} \langle T^+, F_+(\text{ad}_T) c_a \rangle} e^{\mathbb{W}(T)} \delta(b^b + F_-(\text{ad}_T)^* T^+).$$

If we now reduce it using the “trivial” gauge-fixing lagrangian $\{T = 0\}$, we get exactly Z_{II} as in (94).

If we instead choose the gauge-fixing lagrangian $\{T^+ = 0\}$, possibly passing to group variables, typically we get a less trivial, nonequivalent result.

This shows that the results of Section 11.1 actually correspond to choosing a somewhat trivial gauge fixing, but that more interesting results can be obtained also in this case. We saw in Digression 11.1 that just deforming the gauge fixing, picking e different from zero, does not change the result (in Δ^Ω -cohomology). To get interesting results, we should actually split the space \mathcal{Y} as $\mathcal{Y}_1 \oplus \mathcal{Y}_2$, make (the analogue of) gauge fixing (92) and then choose a nontrivial gauge fixing on \mathcal{Y}_1 . To get a glimpse of (the outcome of) this procedure, take the formula that defined $Z_{\text{II}}^{\text{new}}$ but now without integrating over c_1 . We get

$$\tilde{Z}_{\text{II}}^{\text{new}}(q_a, c_a, p^b, b^b; T, T^+, c_1) = e^{-\frac{i}{\hbar} p_i^b q_a^i} e_{\star}^{-\frac{i}{\hbar} \langle H, T \rangle} e^{\frac{i}{\hbar} \langle T^+, F_-(\text{ad}_T) c_1 + F_+(\text{ad}_T) c_a \rangle} e^{\mathbb{W}(T)} e^{\frac{i}{\hbar} \langle b^b, c_1 \rangle}.$$

We should view this as a BV pushforward, where we have set to zero the momenta c_1^\pm of c_1 . The residual fields T, T^+, c_1, c_1^\pm arise as part of the fields e, e^+, c, c^+ .

Remark 11.9. The above discussion shows that leaving some residual fields is important not to end up with a trivial answer. It moreover teaches us that, in gluing several intervals together, it is enough to retain residual fields in the partition function associated to one interval, whereas on each of the others we can safely consider a fully reduced, “trivial” partition function. Similar considerations were presented in [36, Remark 3.13].

11.5. Quantum mechanics. The system described in Section 5.1 is a particular case of what we have considered above. The only, notational, difference is that we have an additional position variable \mathbb{k} and its momentum E ; moreover, they appear in the action with an extra minus sign. Since there is a single hamiltonian, $H - E$, we are in the abelian case of (103). This taken into account, we get

$$Z_1^{\text{QM}}(q_a, \mathbb{k}_a, c_a, p^b, E^b, c_b; T, T^+) = e^{\frac{i}{\hbar} (E^b \mathbb{k}_a - p_i^b q_a^i)} e_{\star}^{-\frac{i}{\hbar} T(H-E)}(p^b, E^b, q_a, \mathbb{k}_a) e^{\frac{i}{\hbar} T^+ (c_b - c_a)}.$$

Since E commutes with the (p, q) variables and H does not depend on \mathbb{k} , we actually have

$$e_{\star}^{-\frac{i}{\hbar} T(H-E)}(p^b, E^b, q_a, \mathbb{k}_a) = e^{\frac{i}{\hbar} T E^b} e_{\star}^{-\frac{i}{\hbar} T H}(p^b, q_a),$$

so

$$Z_1^{\text{QM}}(q_a, \mathbb{k}_a, c_a, p^b, E^b, c_b; T, T^+) = e^{-\frac{i}{\hbar} p_i^b q_a^i} e^{\frac{i}{\hbar} E^b (\mathbb{k}_a + T)} e_{\star}^{-\frac{i}{\hbar} T H}(p^b, q_a) e^{\frac{i}{\hbar} T^+ (c_b - c_a)},$$

which is the quantum version of (39).

We are also interested in the partition function from (q_a, \mathbb{k}_a, c_a) to (q_b, \mathbb{k}_b, c_b) . We can obtain it, following the strategy of Section 11.4, by composing a partition function from (q_a, \mathbb{k}_a, c_a) to (p^1, E^1, c_1) with a partition function from (p^1, E^1, c_1) to (q_b, \mathbb{k}_b, c_b) . The first is Case I, Z_1^{QM} , with a relabeling of the endpoint variables.

The second is of type Γ' , which again we compute with trivial gauge fixing as in (110):

$$(111) \quad Z_{\Gamma'}^{\text{QM}, \mathcal{L}^{\text{triv}}} (p^a, E^a, c_a, q_b, \mathbb{t}_b, c_b) = e^{\frac{i}{\hbar} p_i^a q_b^i} e^{-\frac{i}{\hbar} E^a \mathbb{t}_b} \delta(c_b - c_a).$$

We then get

$$\begin{aligned} \tilde{Z}^{\text{QM}}(q_a, \mathbb{t}_a, c_a, q_b, \mathbb{t}_b, c_b; T, T^+) &:= \int Z_{\Gamma'}^{\text{QM}}(q_a, \mathbb{t}_a, c_a, p^1, E^1, c_1; T, T^+) \\ &Z_{\Gamma'}^{\text{QM}, \mathcal{L}^{\text{triv}}}(p^1, E^1, c_1, q_b, \mathbb{t}_b, c_b) \frac{d^n p^1}{(2\pi\hbar)^n} \frac{dE^1}{2\pi\hbar} dc_1, \end{aligned}$$

that is,

$$\begin{aligned} \tilde{Z}^{\text{QM}}(q_a, \mathbb{t}_a, c_a, q_b, \mathbb{t}_b, c_b; T, T^+) &= \int e^{\frac{i}{\hbar} p_i^1 (q_b^i - q_a^i)} e_{\star}^{-\frac{i}{\hbar} T H}(p^1, q_a) \\ &e^{\frac{i}{\hbar} T^+ (c_b - c_a)} \delta(T + \mathbb{t}_a - \mathbb{t}_b) \frac{d^n p^1}{(2\pi\hbar)^n}. \end{aligned}$$

In this case, using the gauge-fixing lagrangian $\mathcal{L} = \{T^+ = 0\}$,

$$Z^{\text{QM}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b) := \int \tilde{Z}^{\text{QM}}(q_a, \mathbb{t}_a, c_a, q_b, \mathbb{t}_b, c_b; T, 0) dT,$$

yields the interesting result

$$Z^{\text{QM}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b) = \int e^{\frac{i}{\hbar} p_i^1 (q_b^i - q_a^i)} e_{\star}^{-\frac{i}{\hbar} (\mathbb{t}_b - \mathbb{t}_a) H}(p^1, q_a) \frac{d^n p^1}{(2\pi\hbar)^n},$$

which is the quantum version of (40). To see this more clearly, observe that we have the relation

$$\langle p | e^{-\frac{i}{\hbar} T \hat{H}} | q \rangle = e^{-\frac{i}{\hbar} p_i q^i} e_{\star}^{-\frac{i}{\hbar} T H}(p, q)$$

between the operator formalism and the \star -product formalism. Therefore,

$$Z^{\text{QM}}(q_a, \mathbb{t}_a, q_b, \mathbb{t}_b) = \int e^{\frac{i}{\hbar} p_i^1 q_b^i} \langle p^1 | e^{-\frac{i}{\hbar} (\mathbb{t}_b - \mathbb{t}_a) \hat{H}} | q_a \rangle \frac{d^n p^1}{(2\pi\hbar)^n} = \langle q_b | e^{-\frac{i}{\hbar} (\mathbb{t}_b - \mathbb{t}_a) \hat{H}} | q_a \rangle.$$

Therefore, we eventually get the standard quantum mechanics evolution with hamiltonian \hat{H} and time lapse $\mathbb{t}_b - \mathbb{t}_a$. In our picture, however, the original theory was parametrization invariant and $(\mathbb{t}_b, \mathbb{t}_a)$ are endpoint variables (and not fixed time endpoints).

11.6. The quantum relativistic particle. We can adapt the discussion of Section 11.5 to the case of the relativistic particle introduced in Section 5.2. Again we have an additional position variable \mathbb{t} and its momentum E , and we are in the abelian case, now with hamiltonian $\frac{1}{2}(p^2 + m^2 - E^2)$. Therefore, we have, see (103),

$$\begin{aligned} Z_{\Gamma}^{\text{QRP}}(q_a, \mathbb{t}_a, c_a, p^b, E^b, c_b; T, T^+) &= \\ &= e^{\frac{i}{\hbar} (E^b \mathbb{t}_a - p_i^b q_a^i)} e_{\star}^{-\frac{i}{\hbar} \frac{T}{2} (p^2 + m^2 - E^2)}(p^b, E^b, q_a, \mathbb{t}_a) e^{\frac{i}{\hbar} T^+ (c_b - c_a)}. \end{aligned}$$

The simplification now is that the hamiltonian only depends on momentum variables, so its \star -exponential is just the usual exponential, and we get

$$Z_{\Gamma}^{\text{QRP}}(q_a, \mathbb{t}_a, c_a, p^b, E^b, c_b; T, T^+) = e^{\frac{i}{\hbar} (E^b \mathbb{t}_a - p_i^b q_a^i)} e^{-\frac{i}{\hbar} \frac{T}{2} ((p^b)^2 + m^2 - (E^b)^2)} e^{\frac{i}{\hbar} T^+ (c_b - c_a)}.$$

This shows that the physical part of the partition function, in this polarization, is exactly the exponential of i/\hbar times the HJ action (43) with no quantum corrections.

We can obtain the partition function from (q_a, \mathbb{k}_a, c_a) to (q_b, \mathbb{k}_b, c_b) exactly as in Section 11.5:

$$\begin{aligned} \tilde{Z}^{\text{QRP}}(q_a, \mathbb{k}_a, c_a, q_b, \mathbb{k}_b, c_b; T, T^+) &:= \int Z_{\Gamma}^{\text{QRP}}(q_a, \mathbb{k}_a, c_a, p^1, E^1, c_1; T, T^+) \\ &Z_{\Gamma}^{\text{QRP}, \mathcal{L}^{\text{triv}}}(p^1, E^1, c_1, q_b, \mathbb{k}_b, c_b) \frac{d^n p^1}{(2\pi\hbar)^n} \frac{dE^1}{2\pi\hbar} dc_1, \end{aligned}$$

with $Z_{\Gamma}^{\text{QRP}, \mathcal{L}^{\text{triv}}} = Z_{\Gamma}^{\text{QM}, \mathcal{L}^{\text{triv}}}$, see (111). We get

$$\tilde{Z}^{\text{QRP}}(q_a, \mathbb{k}_a, c_a, q_b, \mathbb{k}_b, c_b; T, T^+) = \frac{e^{i\frac{\pi}{4}(1-n)}}{(2\pi\hbar T)^{\frac{n+1}{2}}} e^{\frac{i}{\hbar} \left(\frac{(\Delta q)^2 - (\Delta \mathbb{k})^2}{2T} - \frac{1}{2} m^2 T \right)} e^{\frac{i}{\hbar} T^+ (c_b - c_a)},$$

with $\Delta q = q_b - q_a$ and $\Delta \mathbb{k} = t_b - t_a$. Again we have that, apart from the prefactor $T^{-\frac{n+1}{2}}$, the physical part of the partition function, in this polarization, is the exponential of i/\hbar times the HJ action (44).

Finally, we may integrate over the residual fields using the gauge-fixing Lagrangian $\mathcal{L} = \{T^+ = 0\}$. This yields

$$Z^{\text{QRP}}(q_a, \mathbb{k}_a, q_b, \mathbb{k}_b) = \int_{-\infty}^{\infty} \frac{e^{i\frac{\pi}{4}(1-n)}}{(2\pi\hbar T)^{\frac{n+1}{2}}} e^{\frac{i}{\hbar} \left(\frac{(\Delta q)^2 - (\Delta \mathbb{k})^2}{2T} - \frac{1}{2} m^2 T \right)} dT.$$

To make the integral well-defined, we actually have to deform the integration contour, the real line, to avoid the singularity at $T = 0$: namely, we replace the interval $(-\epsilon, \epsilon)$ with a half circle of radius ϵ centered at zero in the upper or in the lower half plane, and take the limit for $\epsilon \rightarrow 0$ after integrating. We distinguish two cases:

- (1) In the timelike case $(\Delta \mathbb{k})^2 > (\Delta q)^2$, we have to pick the half circle in the upper half plane to tame the singularity of $\frac{1}{T}$ in the exponent. The semiclassical asymptotics of Z^{QRP} can be computed by the saddle-point approximation around the two critical points of the exponent. Each expansion gives semiclassically the exponential of i/\hbar times one of the two HJ actions in (45), but now there are also quantum corrections.
- (2) In the spacelike case $(\Delta q)^2 > (\Delta \mathbb{k})^2$, we have on the other hand to pick the half circle in the lower half plane to tame the singularity of $\frac{1}{T}$ in the exponent. In this case, we get $Z^{\text{QRP}} = 0$, since we can close the contour with a half circle at infinity in the lower half plane, on which the integrand is holomorphic. (Note that we cannot use this argument in the timelike case, for such a contour would bound $T = 0$. On the other hand, we cannot pick a half circle at infinity in the upper half plane because of the $m^2 T$ term in the exponent.)

Also note that Z^{QRP} satisfies the mQME, which, in this case, is equivalent to saying that it satisfies the Klein–Gordon equation with mass square m^2 both with respect to (q_a, \mathbb{k}_a) and with respect to (q_b, \mathbb{k}_b) .

12. BV-BFV QUANTIZATION WITH NONLINEAR POLARIZATIONS

In this section we discuss quantization with a nonlinear change of polarization at the final endpoint, still with finite-dimensional target. As in Remark 2.5, we assume we have a generating function $f(q, Q)$ such that

$$p_i dq^i = P_i dQ^i + df.$$

We then assume that the coordinates in \mathcal{B}_a are (q_a, c_a) and the coordinates in \mathcal{B}_b are (Q^b, c_b) . This is then a generalization of Case I, according to the terminology at the beginning of Section 10. We will refer to it as INL (NL for nonlinear). Our goal is to compute the corresponding partition function Z_{INL} . Even though this can be computed directly generalizing what we did in Section 11.3, we will present a simpler computation that follows the ideas of Section 11.4.

We start introducing some notation. If $H_\alpha(p, q)$ is one of the constraints (or, more generally, an arbitrary function of the p, q variables), we write

$$(112) \quad \tilde{H}_\alpha(P, Q) := H_\alpha(p(P, Q), q(P, Q)),$$

where on the right hand side we used the symplectomorphism induced by the generating function f . It turns out, as briefly announced in Section 9.2.6 (and proved in Appendix B) that the construction of a boundary operator Ω so that the partition function satisfies the mQME is possible if we make the following assumption.

Assumption 12.1. We assume

$$(113) \quad H_\alpha \left(i\hbar \frac{\partial}{\partial q}, q \right) e^{-\frac{i}{\hbar}f} = \tilde{H}_\alpha \left(-i\hbar \frac{\partial}{\partial Q}, Q \right) e^{-\frac{i}{\hbar}f}, \quad \forall \alpha.$$

Remark 12.2. The assumption is automatically satisfied if all the hamiltonians H_α are linear in the p variables and all the hamiltonians \tilde{H}_α are linear in the P variables. Note that these are precisely the quantizability conditions in geometric quantization. This case is relevant for 7D Chern–Simons theory as in [34].

That this is a sufficient condition follows from a delicate analysis of the Feynman diagrams near the boundary, generalizing [19, Section 4.2]. We will perform this analysis in Appendix B. As for this section we will simply check that, under this assumption, the partition functions we construct indeed satisfy the mQME with Ω constructed as in Section 9.2.6.

A minimalistic reading of this section, skipping the intricacies of Appendix B, can of course just be that we produce a solution of the mQME, which is enough for the applications. We stress here that Assumption 12.1 is satisfied by the important example of Section 7.6. Therefore, what we discuss here is a toy model for the quantization of [34], to which we will return in [21].

12.1. Boundary structure. In the present Case INL, we use f to change the potential α_{BFV} to

$$\alpha_{\text{BFV}}^f = \alpha_{\text{BFV}} + df(q_b, Q_b) = p_a^i dq_a^i + b_a^\alpha dc_a^\alpha - P_b^i dQ_b^i - b_b^\alpha dc_b^\alpha.$$

Following the recipe (87), we get

$$\begin{aligned} \Omega_a &= c_a^\alpha H_\alpha \left(i\hbar \frac{\partial}{\partial q_a}, q_a \right) - \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_a^\alpha c_a^\beta \frac{\partial}{\partial c_a^\gamma}, \\ \Omega_b &= c_b^\alpha \tilde{H}_\alpha \left(-i\hbar \frac{\partial}{\partial Q_b}, Q_b \right) + \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_b^\alpha c_b^\beta \frac{\partial}{\partial c_b^\gamma}. \end{aligned}$$

The first consequence of Assumption 12.1 is that Ω_a and Ω_b square to zero, so $\Omega^2 = 0$. In fact, since the hamiltonians are linear in the momentum variables, their \star -commutators, in the induced \star -products, are the same as $i\hbar$ times their Poisson brackets, so (89) is automatically satisfied (for the H s and for the \tilde{H} s).

12.2. Gluing. As announced, we will compute Z_{INL} via a gluing procedure, similarly to what we did in Section 11.4. Namely, we will compose a partition function from (q_a, c_a) to (p^1, c_1) with a partition function from (p^1, c_1) to (q_2, b^2) with a partition function from (q_2, b^2) to (Q_b, c_b) , where (p^1, c_1) and (q_2, b^2) are some intermediate boundary fields. See Figure 9. The first partition function belongs to Case I. The second, which we will denote as Case II', can be computed along the lines of Case II with the only difference that we now fix p at the initial endpoint and q at the final one (instead of the other way around). The analogue of (94) is

$$Z_{\text{II}'}(p^a, c_a, q_b, b^b) = e^{\frac{i}{\hbar}(p_i^a q_b^i + b_a^b c_a^a)}.$$

The third partition function belongs to the analogue of Case III with Q_b instead of p^b . We will call it Case IIIINL and compute it in Section 12.3. Equipped with all this, we have

$$(114) \quad Z_{\text{INL}}(q_a, c_a, Q_b, c_b; T, T^+) = \int Z_{\text{I}}(q_a, c_a, p^1, c_1; T, T^+) \\ Z_{\text{II}'}(p^1, c_1, q_2, b^2) Z_{\text{IIIINL}}(q_2, b^2, Q_b, c_b) \frac{d^n p^1 d^n q_2}{(2\pi\hbar)^n} d^k c_1 d^k b^2 (-i\hbar)^k,$$

where again n is the dimension of the target configuration space, k is the dimension of the Lie algebra, and we have conveniently normalized the measure.

FIGURE 9. Composition of the partition function Z_{INL}

As a composition of partition functions that satisfy the mQME, Z_{INL} satisfies it as well.

12.3. The Case IIIINL. In this case, we have to use the function $\check{f}(b^a, c_a, p^b, q_b) = f(q_b, Q(p^b, q_b)) - c_a^\alpha b_\alpha^a$ to modify the potential

$$\alpha_{\text{BFV}}^{\check{f}} = \alpha_{\text{BFV}} + d\check{f}(q_b, Q_b) = p_i^a dq_a^i + c_a^\alpha db_\alpha^a - P_i^b dQ_b^i - b_a^b dc_b^a.$$

Following again the recipe (87), we get

$$(115a) \quad \Omega_a = i\hbar H_\alpha \left(i\hbar \frac{\partial}{\partial q_a}, q_a \right) \frac{\partial}{\partial b_\alpha^a} + \frac{\hbar^2}{2} f_{\alpha\beta}^\gamma b_\gamma^a \frac{\partial^2}{\partial b_\alpha^a \partial b_\beta^a},$$

$$(115b) \quad \Omega_b = c_b^\alpha \tilde{H}_\alpha \left(-i\hbar \frac{\partial}{\partial Q_b}, Q_b \right) + \frac{i\hbar}{2} f_{\alpha\beta}^\gamma c_b^\alpha c_b^\beta \frac{\partial}{\partial c_b^\gamma},$$

and we clearly have $\Omega^2 = 0$ (again by Assumption 12.1). The BV action for the given boundary polarizations is then

$$S_{\text{BV}}^{\check{f}} = S_{\text{BV}} - f(q(t_b), Q(p(t_b), q(t_b))) - c^\alpha(t_a) e_\alpha^+(t_a).$$

Next we choose the discontinuous splitting of the fields e^+ , c as in (96) with boundary conditions (97). For the p , q fields we choose instead

$$q(t) = \begin{cases} q_a & t = t_a \\ \hat{q}(t) & t > t_a \end{cases}, \quad p(t) = \begin{cases} \hat{p}(t) & t < t_b \\ p^0 & t = t_b \end{cases},$$

with

$$(116) \quad p^0 := \frac{\partial f}{\partial q}(\hat{q}(t_b), Q_b)$$

and boundary conditions

$$\hat{q}(t_a) = 0, \quad \hat{p}(t_b) = 0.$$

The BV action adapted to this splitting is then the same as (98) with only the first term changed:

$$\begin{aligned} \tilde{S}_{\text{BV}}^f &= -f(\hat{q}(t_b), Q_b) - \hat{e}_\alpha^+(t_b)c_b^\alpha - \hat{p}_i(t_a)q_a^i + \hat{c}^\alpha(t_a)b_\alpha^a \\ &+ \int_{t_a}^{t_b} \left(\hat{p}_i d\hat{q}^i - \hat{e}_\alpha^+ d\hat{c}^\alpha - \hat{e}^\alpha H_\alpha(\hat{p}, \hat{q}) \right. \\ &+ \hat{c}^\alpha \left(\hat{q}_i^+ \frac{\partial H_\alpha}{\partial p_i}(\hat{p}, \hat{q}) - \hat{p}_+^i \frac{\partial H_\alpha}{\partial q^i}(\hat{p}, \hat{q}) \right) \\ &\left. - \frac{1}{2} f_{\alpha\beta}^\gamma \hat{c}_\gamma^+ \hat{c}^\alpha \hat{c}^\beta + f_{\alpha\beta}^\gamma \hat{e}_\gamma^+ \hat{c}^\alpha \hat{e}^\beta \right). \end{aligned}$$

We pick again the gauge-fixing lagrangian (92). This produces the same $\hat{p}\hat{q}$ and $\hat{e}^+\hat{c}$ propagators as in Case III (since the hat fields here and there have exactly the same boundary conditions), and we can then easily compute

$$Z_{\text{IIINL}}(q_a, b^a, Q_b, c_b) = e^{-\frac{i}{\hbar}(f(q_a, Q_b) - c_b^\alpha b_\alpha^a)}.$$

One can immediately check that $\Omega Z_{\text{IIINL}} = 0$. Assumption 12.1 plays here a fundamental role.

Remark 12.3 (mQME vs unitarity). The physical part of the partition function, $e^{-\frac{i}{\hbar}f(q_a, Q_b)}$, is exactly the exponential of the HJ action (6) as in (10). This is not unexpected because, since the physical part decouples from the ghost part, in the physical part we can go back to the continuous splitting, as in Example 2.12. Note that the partition function we get is not unitary, as it misses the corrections discussed in Digression 2.13. In principle, we might add such corrections—e.g., as in (12)—if possible, but this would spoil the mQME unless we were able to find corrections to the H_α s and \tilde{H}_α s that satisfy (113) with the corrected f . There is no guarantee that such corrections exist. This is the price we have to pay in the quantum BV-BFV formalism in order to get a solution to the mQME. The mQME is fundamental to ensure that the result of a composition, as in the procedure of Section 12.2, also solves the mQME. As a consequence, if there is a conflict, we prefer to have Z_{INL} solve the mQME at the expense of not being unitary. We will return to this in Remark 12.7.

12.4. The computation of Z_{INL} . We can now use (114) to compute

$$\begin{aligned} Z_{\text{INL}}(q_a, c_a, Q_b, c_b; T, T^+) &= \int e^{\frac{i}{\hbar}(p_1^i(q_2^i - q_a^i) - f(q_2, Q_b))} e_\star^{-\frac{i}{\hbar}\langle H, T \rangle}(p^1, q_a) \frac{d^n p^1 d^n q_2}{(2\pi\hbar)^n} \\ &e^{\frac{i}{\hbar}\langle T^+, F_-(\text{ad}_T)c_b + F_+(\text{ad}_T)c_a \rangle} e^{\mathcal{W}(T)}. \end{aligned}$$

Remark 12.4. If $f(q, Q) = Q_i q^i$, then one easily sees that $Z_{\text{INL}} = Z_{\text{I}}$ with $p^b = Q_b$.

Remark 12.5. At the leading order in \hbar , the physical part of Z_{I} is the exponential of i/\hbar times the HJ action, see (101). The integral in the p^1, q_2 variables then

produces, also at the leading order, the composition of the generating functions f and \hat{S}_{HJ} . Therefore, we have

$$(117) \quad Z_{\text{INL}}^{\text{phys}}(q_a, Q_b; T) = e^{\frac{i}{\hbar}(\hat{S}_{\text{HJ}}^f(q_a, Q_b, e^T) + O(\hbar))}.$$

Example 12.6 (Linear case). In the case of linear constraints of Example 4.3, there are no corrections in the physical part to the HJ action, see Example 11.7. Moreover, since the HJ action (24) is at most linear in the momenta, the integral in the p^1, q_2 variables produces exactly the composition of the generating functions f and \hat{S}_{HJ} . Finally, the theory is abelian in this case. Therefore, we get

$$(118) \quad \begin{aligned} Z_{\text{INL}}(q_a, c_a, Q_b, c_b; T, T^+) &= e^{-\frac{i}{\hbar}(f(q_a + T^\alpha v_\alpha, Q_b) + T^\alpha w_{\alpha, i} q_a^i + \frac{1}{2} T^\alpha T^\beta A_{\alpha\beta} + T_\alpha^+ (c_a^\alpha - c_b^\alpha))} \\ &= e^{\frac{i}{\hbar} \hat{S}_{\text{HJ}}^f(q_a, Q_b; T)} e^{-\frac{i}{\hbar} T_\alpha^+ (c_a^\alpha - c_b^\alpha)}, \end{aligned}$$

with \hat{S}_{HJ}^f as in (25). This result is the prototype for abelian 7D Chern–Simons theory with nonlinear polarization, see [21, Sections 6], i.e., for the quantization of [34].

Remark 12.7. As a consequence of Remark 12.3, the partition function Z_{INL} is in general nonunitary if we consider a nonlinear change of polarization. On the other hand, it satisfies the mQME. An application of this, following Section 9.2.8, consists in composing Z_{INL} with an Ω -closed state ψ in the final variables (Q_b, c_b) . The result is a state ψ' in the initial variables (q_a, c_a) and in the residual fields (T, T^+) which satisfies the mQME. The state ψ' will not be unitarily equivalent to the state ψ in general, but the idea is that we introduce the state ψ just to produce the solution ψ' of the mQME. In the abelian case, for ψ we may take an arbitrary function of Q_b times a delta function of all the ghost variables. This is automatically Ω closed. As an application, in [21] we will use the state (68) (times the delta function of the ghosts) as the final state ψ . This will show that the result of [34], which connects the Kodaira–Spencer [39] action [8] to 7D Chern–Simons theory (with the nonlinear Hitchin polarization [35] at the final endmanifold), survives at the quantum level with no corrections.

APPENDIX A. SYMPLECTIC GEOMETRY AND GENERATING FUNCTIONS

In this appendix we recall some basic facts on symplectic geometry (also in the infinite-dimensional case) and on (generalized) generating functions for lagrangian submanifolds (and canonical relations). Our main reference will be [9]. For the infinite-dimensional case, we will also follow [13].

We will discuss first the linear case, where actually most of the subtleties are already present, and then briefly review the extension to manifolds.

A.1. Symplectic spaces. A symplectic form on a vector space V is a skew-symmetric bilinear form ω with the property of being nondegenerate:

$$(119) \quad \omega(v, w) = 0 \quad \forall w \in V \Rightarrow v = 0.$$

The pair (V, ω) is called a symplectic space.

A symplectomorphism between symplectic spaces (V, ω_V) and (W, ω_W) is an isomorphism $\phi: V \rightarrow W$ such that

$$(120) \quad \omega_W(\phi(v_1), \phi(v_2)) = \omega_V(v_1, v_2), \quad \forall v_1, v_2 \in V.$$

Remark A.1. Note that to a bilinear form ω on V one can associate the linear map $\omega^\sharp: V \rightarrow V^*$,

$$\omega^\sharp(v)(w) := \omega(v, w), \quad v, w \in V.$$

The nondegeneracy of ω is equivalent to the injectivity of ω^\sharp . If V is finite-dimensional, then this implies that ω^\sharp is actually a bijection, but this is not guaranteed in the infinite-dimensional case.

Remark A.2. In the infinite-dimensional case, the above condition is often referred to as weak nondegeneracy and ω is called a weak symplectic form, reserving the term symplectic form to the case when ω^\sharp is a bijection. In this paper we will only consider weak symplectic forms, and we will just call them symplectic forms.

Example A.3. One of the infinite-dimensional examples for this paper is the symplectic space associated to abelian Chern–Simons theory. Let Σ be a closed, oriented $2(2k+1)$ -manifold. Let $V = \Omega^{2k+1}(\Sigma)$. Then the bilinear form

$$\omega(\alpha, \beta) = \int_{\Sigma} \alpha \wedge \beta, \quad \alpha, \beta \in V,$$

is skew-symmetric and nondegenerate, hence symplectic.

A.1.1. *Special subspaces.* Given a subspace W of V , one defines its orthogonal space as

$$W^\perp := \{v \in V : \omega(v, w) = 0, \forall w \in W\}.$$

Note that $V^\perp = \{0\}$.

Lemma A.4. *Let W and Z be two subspaces. Then*

- (1) $W \subseteq Z \Rightarrow Z^\perp \subseteq W^\perp$;
- (2) $(W + Z)^\perp = W^\perp \cap Z^\perp$.
- (3) $W \subseteq W^{\perp\perp}$ and $W^\perp = W^{\perp\perp\perp}$.
- (4) *If V is finite-dimensional, then $\dim V = \dim W + \dim W^\perp$ and $W = W^{\perp\perp}$.*

Proof. For (1), let $v \in Z^\perp$. By definition $\omega(v, z) = 0 \forall z \in Z$. A fortiori $\omega(v, w) = 0 \forall w \in W$. So $v \in W^\perp$.

As for (2), observe that $v \in (W + Z)^\perp$ if and only if $\omega(v, w + z) = 0 \forall w \in W, \forall z \in Z$. In particular, taking either w or z equal to zero, we see that v belongs to both W^\perp and Z^\perp . On the other hand, if $v \in W^\perp \cap Z^\perp$, by linearity we get $\omega(v, w + z) = \omega(v, w) + \omega(v, z) = 0 \forall w \in W, \forall z \in Z$; so $v \in (W + Z)^\perp$.

As for (3), let $w \in W$. Then $\omega(w, v) = -\omega(v, w) = 0 \forall v \in W^\perp$. Hence, $w \in W^{\perp\perp}$. For the second equation, taking orthogonals in the first yields $W^{\perp\perp\perp} \subseteq W^\perp$. On the other hand, applying the first equation to the subspace W^\perp yields $W^\perp \subseteq W^{\perp\perp\perp}$.

In finite dimensions, ω yields an isomorphism $V/W^\perp \rightarrow W^*$, so $\dim V - \dim W^\perp = \dim W^* = \dim W$. Moreover,

$$\dim W^{\perp\perp} = \dim V - \dim W^\perp = \dim V - (\dim V - \dim W) = \dim W.$$

□

A subspace W is called isotropic if $W \subseteq W^\perp$, coisotropic if $W^\perp \subseteq W$ and lagrangian if $W = W^\perp$.

Note that a subspace W is isotropic if and only if $\omega(w_1, w_2) = 0$ for all $w_1, w_2 \in W$ and that a subspace is lagrangian if and only if it is at the same time isotropic and coisotropic.

In the finite-dimensional case, L is lagrangian if and only if L is isotropic and $2 \dim L = \dim V$. Finally, an isotropic subspace L is called split lagrangian if it admits an isotropic complement. (For more on lagrangian and split lagrangian subspaces, we refer to [13]).

Lemma A.5.

$$\textit{split lagrangian} \implies \textit{lagrangian}$$

In the finite-dimensional case, the two concepts are equivalent.

Proof. Suppose that L is split lagrangian. Then we can find an isotropic subspace W such that $V = L \oplus W$. In particular, $V = L + W$. By (2) in Lemma A.4, we then have $L^\perp \cap W^\perp = \{0\}$. Since $W \subseteq W^\perp$, we also have $L^\perp \cap W = \{0\}$, which implies $L^\perp \subseteq L$. Therefore, L is lagrangian.

Suppose on the other hand that L is lagrangian and V is finite-dimensional. Suppose W is isotropic but not lagrangian and $L \cap W = \{0\}$. Observe that by dimensional reasons, $L + W$ cannot be the whole V . Therefore $(L + W)^\perp$ cannot be $\{0\}$. Using (2) in Lemma A.4, we then have $W^\perp \cap L \neq \{0\}$.

We now want to show that there is a nonzero vector $v \in W^\perp$ that does not belong to L nor to W . In fact, if this were not the case, we would have $W^\perp \subseteq W + L$. By (1) and (2) in Lemma A.4, we would then have $W \supseteq W^\perp \cap L^\perp = W^\perp \cap L$. Intersecting with L , we would conclude $\{0\} = W^\perp \cap L$, which is a contradiction.

Let finally \mathcal{I} be the set of isotropic subspaces of V whose intersection with L is $\{0\}$. Note that $\{0\} \in \mathcal{I}$. If $W \in \mathcal{I}$ is not lagrangian, by the above argument we can find a nonzero vector $v \in W^\perp$ that does not belong to L nor to W . But then $W \oplus \text{span } v$ is also in \mathcal{I} and of dimension strictly higher than that of W . After finitely many steps, we end up with a $W \in \mathcal{I}$ which is lagrangian. But since $\dim L = \dim W = \dim V/2$, we see that W is a complement to L . \square

Remark A.6 (Split symplectic spaces). For applications in field theory, we will only consider symplectic spaces that admit split lagrangian subspaces. In particular, we fix a lagrangian subspace W and a lagrangian complement W' and write

$$(121) \quad V = W \oplus W'.$$

We will call this a split symplectic space.

Note that in the finite-dimensional case, ω^\sharp establishes an isomorphism between W' and W^* , which in turns establishes a symplectomorphism between V and $W \oplus W^*$ with the canonical symplectic form

$$(122) \quad \omega(v \oplus a, w \oplus b) = b(v) - a(w), \quad v, w \in W, a, b \in W^*.$$

In the infinite-dimensional case, $\omega^\sharp|_{W'}: W' \rightarrow W^*$ is only injective. Still, we may regard $W \oplus W'$ as a symplectic subspace of $W \oplus W^*$, which is what we will do in the following.

Example A.7. Consider Example A.3 with $k = 0$. If we work with complex valued forms, then we have the splitting of $V = \Omega^1(\Sigma) \otimes \mathbb{C}$ into $W = \Omega^{1,0}(\Sigma)$ and $W' = \Omega^{0,1}(\Sigma)$, with respect to some complex structure on Σ .

A.1.2. *Generating functions.* Assume we have a split symplectic space as in (121). Consider the graph of a linear map $\phi: W \rightarrow W'$:

$$L := \{w \oplus \phi(w), w \in W\} \subset V.$$

Since

$$\omega(w_1 \oplus \phi(w_1), w_2 \oplus \phi(w_2)) = \omega(w_1, \phi(w_2)) + \omega(\phi(w_1), w_2),$$

we see that L is isotropic, hence split lagrangian because W' is an isotropic complement to it, if and only if

$$\omega(w_1, \phi(w_2)) = -\omega(\phi(w_1), w_2), \quad \forall w_1, w_2 \in W.$$

In this case, we can use ϕ and ω to define a quadratic form on W :

$$\psi(w) := \frac{1}{2}\omega(w, \phi(w)).$$

Note then that

$$\delta\psi(w) = \omega(\phi(w), \delta w) = \omega^\sharp(\phi(w))(\delta w),$$

so

$$\delta\psi = \omega^\sharp \circ \phi: W \rightarrow W^*$$

as a one-form on W . Since ω^\sharp is injective, it is possible to recover ϕ out of $\delta\psi$.

Vice versa, suppose we have a splitting $V = W \oplus W'$, already realized as a subspace of $W \oplus W^*$ with canonical symplectic structure, and a quadratic form ψ on W . If $\delta\psi(W) \subseteq W'$, then the graph L_ψ of $\delta\psi$ is a split lagrangian subspace of V and ψ is called a generating function for it.

Example A.8. Let I be the closed interval $[-1/2, 1/2]$ and $V := C^0(I) \oplus C^0(I)$ as a real vector space. Define

$$\omega(f \oplus g, h \oplus k) = \int_I (fk - gh) dx.$$

Then (V, ω) is split symplectic with $W = W' = C^0(I)$ (more properly $W = C^0(I) \oplus \{0\}$ and $W' = \{0\} \oplus C^0(I)$). The quadratic form $\psi(f) = \frac{1}{2} \int_I f^2 dx$ has differential $\delta_f \psi = f$, so it is a generating function for the diagonal $L_\psi = \{f \oplus f, f \in C^0(I)\}$. On the other hand, the quadratic form $\tilde{\psi}(f) = \frac{1}{2} f(0)^2$, $f \in W$, has differential $\delta_f \tilde{\psi} = f(0)\delta$, with δ the delta function at 0, which is an element of W^* but not of W' , so it is not an allowed generating function.

A.1.3. Symplectic reduction. We will only consider the particular case of coisotropic reduction, which is of interest for this paper.

Let C be a coisotropic subspace of (V, ω) . Then the kernel of the restriction of ω to C is precisely the orthogonal subspace C^\perp . It follows that $\underline{C} := C/C^\perp$, the symplectic reduction of C , is endowed with the symplectic form

$$(123) \quad \underline{\omega}([v], [w]) = \omega(v, w), \quad v \in [v], w \in [w].$$

Denote by π the canonical projection $C \rightarrow \underline{C}$. If L is a subspace of V , we set $\underline{L} := \pi(L \cap C) \subseteq \underline{C}$.

Lemma A.9 (Relative linear reduction). *If L is isotropic in V , then \underline{L} is isotropic in \underline{C} . In the finite-dimensional case, if L is lagrangian in V , then \underline{L} is lagrangian in \underline{C} .*

Proof. The first statement may be easily proved observing that, if $[v]$ and $[w]$ are in \underline{L} , then every $v \in [v]$ and every $w \in [w]$ lie in $L \cap C$ and hence in L . Therefore, $\omega(v, w) = 0$ and, by (123), we get $\underline{\omega}([v], [w]) = 0$, so \underline{L} is isotropic.

A different proof, which also leads to the second statement, is based on the following:

$$\begin{aligned}\pi^{-1}(\underline{L}^\perp) &= (\pi^{-1}(\underline{L}))^\perp \cap C = (L \cap C + C^\perp)^\perp \cap C = \\ &= (L \cap C)^\perp \cap C \supset (L^\perp + C^\perp) \cap C = L^\perp \cap C + C^\perp.\end{aligned}$$

In the last equality we have used the obvious fact that $(A+B) \cap C = A \cap C + B$ for any three subspaces A , B , and C with $B \subset C$. Thus, $\underline{L}^\perp \supset \pi(L^\perp \cap C) \supset \pi(L \cap C) = \underline{L}$. All inclusions are equalities under the assumptions that V is finite-dimensional and L is lagrangian. \square

In infinite dimensions, it may happen that L is lagrangian but \underline{L} is not:

Example A.10. Consider the symplectic space of Example A.8 and

$$C := \{f \oplus g \in V : g(0) = 0\}.$$

One has that $C^\perp = \{0\}$, so C is coisotropic and $\underline{C} = C$. Let

$$L := \left\{ f \oplus g \in V : \int_I f \, dx = 0, \, g \text{ constant} \right\}.$$

One can easily show that L is lagrangian, actually split lagrangian, in V , but \underline{L} is not lagrangian in \underline{C} .

Remark A.11 (Generalized generating functions in the linear case). Suppose we have a sum $V \oplus Z$ of symplectic spaces with $Z = B \oplus B'$ split. First observe that $C := V \oplus B \oplus \{0\}$ is coisotropic in $V \oplus Z$ with $C^\perp = \{0\} \oplus B \oplus \{0\}$, so $\underline{C} = V$. Now assume that also $V = W \oplus W'$ is split and rearrange $V \oplus Z$ as the split symplectic space $(W \oplus B) \oplus (W' \oplus B')$. Let L_ψ be the split lagrangian subspace of $V \oplus Z$ with generating function a quadratic function ψ on $W \oplus B$. It then follows that \underline{L}_ψ is isotropic in V . If it is split lagrangian (as it automatically happens in the finite-dimensional case), then one says that ψ is a generalized generating function for \underline{L}_ψ , depending on extra parameters in B .

Example A.12. Consider $V = W \oplus W^*$ and $Z = B \oplus B^*$ with W and B finite-dimensional. Let $F: W \rightarrow B^*$ be a linear map and let $F^*: B \rightarrow W^*$ denote its dual. Then the quadratic form $\psi(w \oplus b) = (Fw, b)$, with (\cdot, \cdot) denoting the pairing of a space with its dual, is a generalized generating function for $\underline{L}_\psi = \ker F \oplus \text{im } F^* = \ker F \oplus (\ker F)^0$, with 0 denoting the annihilator space. Unless F is the zero map, \underline{L}_ψ cannot be generated by a function on W only, so extra parameters are indeed necessary.

Example A.13. The above example may be extended to the infinite-dimensional case as follows. Let $F: W \rightarrow B' \subseteq B^*$ and $F': B \rightarrow W' \subseteq W^*$ be linear maps such that $\iota_{W'} F' = (\iota_{B'} F)^* \iota$, where $\iota_{B'}: B' \rightarrow B^*$ and $\iota_{W'}: W' \rightarrow W^*$ are the given inclusions and $\iota: B \rightarrow B^{**}$ is the canonical inclusion. Then the quadratic form $\psi(w \oplus b) = (\iota_{B'} F w, b) = (\iota_{W'} F' b, w)$ is a generating function for a lagrangian subspace L_ψ of $V \oplus Z$. The corresponding isotropic subspace \underline{L}_ψ of V is $\ker F \oplus \text{im } F'$.

Example A.14. A simple illustration of the last example is the following. Let Σ be a closed, oriented two-manifold and let $W = W' = \Omega^1(\Sigma)$, $B = \Omega^0(\Sigma)$, $B' = \Omega^2(\Sigma)$, with symplectic forms induced by integration over Σ :

$$\omega(\alpha \oplus \alpha', \beta \oplus \beta') = \int_\Sigma (\alpha \wedge \beta' - \beta \wedge \alpha').$$

The linear maps $F = d: \Omega^1(\Sigma) \rightarrow \Omega^2(\Sigma)$ and $F' = d: \Omega^0(\Sigma) \rightarrow \Omega^1(\Sigma)$ satisfy the assumptions, so we have the generalized generating function $\psi(w \oplus b) = \int_{\Sigma} dw \wedge b$, with $w \in \Omega^1(\Sigma)$ and $b \in \Omega^0(\Sigma)$. We get $\underline{L}_{\psi} = \Omega_{\text{cl}}^1(\Sigma) \oplus \Omega_{\text{ex}}^1(\Sigma)$, where the subscripts denote closed and exact forms. In this example \underline{L}_{ψ} is actually a split lagrangian. This follows from the Hodge decomposition theorem. In fact, putting a metric on Σ one can define the Hodge $*$ operator and the codifferential d^* . As an isotropic complement of \underline{L}_{ψ} one can then take $\Omega_{\text{coex}}^1(\Sigma) \oplus \Omega_{\text{cocl}}^1(\Sigma)$.

A.1.4. Canonical relations. Canonical relations are a generalization of symplectomorphisms.

Recall that a linear map ϕ between (V, ω_V) and (W, ω_W) is called a symplectomorphism if it is an isomorphism and respects the symplectic forms as in (120). It follows that the graph of ϕ ,

$$L_{\phi} = \{v \oplus \phi(v), v \in V\},$$

is a split lagrangian subspace of $\bar{V} \oplus W$, where \bar{V} denotes V with symplectic form $-\omega_V$. In fact, we have

$$\omega_{\bar{V} \oplus W}(v_1 \oplus \phi(v_1), v_2 \oplus \phi(v_2)) = -\omega_V(v_1, v_2) + \omega_W(\phi(v_1), \phi(v_2)) = 0,$$

so L_{ϕ} is isotropic. Note that $L_{-\phi}$ is also isotropic and, since ϕ is an isomorphism, it is a complement to L_{ϕ} .

More generally, one calls any (split) lagrangian subspace L of $\bar{V} \oplus W$ a (split) canonical relation from V to W . One also writes

$$(124) \quad L: V \dashrightarrow W.$$

If $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$ are split symplectic spaces, then a lagrangian subspace L of $\bar{V} \oplus W$ might have a (generalized) generating function ψ , a quadratic form on $V_1 \oplus W_1(\oplus B)$, which is then called a (generalized) generating function for the canonical relation L .

Here is an example of a canonical relation that is important for our applications:

Lemma A.15. *Let C be a coisotropic subspace of V . Then the subspace $L_C := \{(v, v') \in C \oplus C \mid v' - v \in C^{\perp}\}$ is isotropic in $\bar{V} \oplus V$. If V is finite-dimensional, then L is lagrangian, hence a canonical relation from V to itself.*

Proof. For $v_1, v_2 \in C$ and $w_1, w_2 \in C^{\perp}$ we have $\omega(v_1 + w_1, v_2 + w_2) - \omega(v_1, v_2) = 0$, so L_C is isotropic. If $\dim V = 2n$ and $\dim C = k$, then $\dim L_C = \dim C + \dim C^{\perp} = k + (2n - k) = 2n$, so $2 \dim L_C = \dim(\bar{V} \oplus V)$. \square

For an example where L_C is not lagrangian, consider C as in Example A.10. Since $C^{\perp} = \{0\}$, we have that $L_C = \Delta_C$, the diagonal in $C \oplus C$. On the other hand, $L_C^{\perp} = \Delta_V$, the diagonal in $V \oplus V$, which properly contains Δ_C .

Another relation between canonical relations and coisotropic submanifolds is the following result (observed in [18] in the context of field theory).

Lemma A.16. *Let L be a lagrangian subspace of $\bar{V} \oplus V$. Define $C := \pi_1(L) \subseteq V$, with π_1 the projection to the first summand. Then C is coisotropic.*

Note that this result holds also in the infinite-dimensional case.

Proof. By definition we have that v belongs to C iff there is a $w \in V$ such that (v, w) belongs to L . Now let \tilde{v} be in C^{\perp} , i.e., $\omega(\tilde{v}, v) = 0 \forall v \in C$. This implies

$$\omega(\tilde{v}, v) - \omega(0, w) = 0 \forall (v, w) \in L.$$

Therefore, $(\tilde{v}, 0) \in L^\perp = L$, which implies $\tilde{v} \in C$. \square

A.2. Symplectic manifolds. We now extend the above results to manifolds.

Definition A.17 (Symplectic manifolds). A symplectic form ω on a smooth manifold M is a nondegenerate closed two-form on it. By nondegenerate we mean that for each $x \in M$ the bilinear form ω_x on $T_x M$ satisfies (119). Note that, for each $x \in M$, the pair $(T_x M, \omega_x)$ is then a symplectic space. The pair (M, ω) is called a symplectic manifold.

We allow M to be an infinite-dimensional manifold (for the examples in this paper a Fréchet manifold). In almost all the examples of this paper the symplectic manifold M is actually a vector space, but we also consider nonlinear maps and nonlinear lagrangian submanifolds.

Definition A.18 (Symplectomorphisms). A symplectomorphism between symplectic manifolds (M, ω_M) and (N, ω_N) is a diffeomorphism $\phi: M \rightarrow N$ such that $\phi^* \omega_N = \omega_M$.

Definition A.19 (Special subspaces). A submanifold S of a symplectic manifold M is called isotropic / coisotropic / lagrangian / split lagrangian if $T_x S$ is a subspace of this class in $T_x M$ for every $x \in S$. Note that, as these concepts are defined in terms of tangent spaces, they make sense both for embedded and for immersed submanifolds. (Unless we explicitly say otherwise, by submanifold we then mean an immersed submanifold.)

Definition A.20 (The characteristic distribution). The characteristic distribution of a coisotropic submanifold C of M is the subbundle $C^\perp := \cup_{x \in C} (T_x C)^\perp$ of TC . It is a subbundle, hence a regular distribution, because it is the kernel of $\omega_C^\sharp: TC \rightarrow T^*C$, where ω_C is the restriction of the symplectic form to C .

Lemma A.21. *Let C be a coisotropic submanifold of M . Then*

- (1) *For every $X \in \Gamma(C^\perp)$, we have $L_X \omega_C = 0$, where L denotes the Lie derivative;*
- (2) *The characteristic distribution C^\perp is involutive.*

Proof. By definition $\iota_X \omega_C = 0$. Since $d\omega_C = 0$, we get $L_X \omega_C = 0$ by Cartan's magic formula.

Next suppose that Y is also a section of C^\perp . By Cartan's calculus we have

$$\iota_{[Y, X]} \omega_C = [\iota_Y, L_X] \omega_C,$$

where we have also used the above statement. Therefore, $[Y, X] \in \Gamma(C^\perp)$. \square

Remark A.22 (Split symplectic manifolds). We are now interested in the generalization of the concept of split symplectic space of Remark A.6: a split symplectic manifold. In the finite-dimensional case, this is the notion of a cotangent bundle T^*M with its canonical symplectic form (this is the two-form which in local coordinates q^i for the base and p_i for the fiber reads $dp_i dq^i$).⁵⁴ In the infinite-dimensional case, more generally, we consider subbundles \tilde{T}^*M of T^*M with symplectic form given by the restriction of the canonical symplectic form.

⁵⁴More intrinsically, the canonical symplectic form is the differential of the canonical one-form θ with $\theta_{(p, q)}(v) := (p, d_{(p, q)}\pi(v))$, with $q \in M$ and $p \in T_q^*M$, where π is the projection $T^*M \rightarrow M$.

Remark A.23 (Polarizations). A more flexible notion of splitting than that of Remark A.22, which is used in the classical setting, is that of polarization, which is used for geometric quantization. Namely, a (real) polarization on a symplectic manifold M is an integrable smooth distribution D (i.e., a subbundle of TM) such that D_x is split lagrangian in T_xM for every $x \in M$. One also uses the notion of complex polarization, which is an integrable subbundle D of the complexification $TM \otimes \mathbb{C}$ of the tangent bundle such that D_x is split lagrangian in $T_xM \otimes \mathbb{C}$ for every $x \in M$.

Remark A.24 (Generating functions). Let \tilde{T}^*M be a split symplectic manifold and ϕ a section (in particular ϕ is a one-form on M). Its graph L is then lagrangian, actually embedded split lagrangian, if and only if ϕ is closed. If $\phi = d\psi$, then ψ is called a generating function for L , which we also denote as L_ψ . Note that a function ψ on M defines an embedded split lagrangian submanifold L_ψ of \tilde{T}^*M under the condition that $d\psi$ is a section of it.

Remark A.25 (Symplectic reduction). Next we consider symplectic reduction of coisotropic submanifolds. If C is coisotropic in M , then its characteristic distribution $C^\perp := \cup_{x \in C} (T_x C)^\perp$ is a regular involutive distribution on C by Lemma A.21. If C is defined by constraints, one can verify that the characteristic distribution is spanned by the hamiltonian vector fields of the constraints (which therefore must be in involution under the Poisson bracket).⁵⁵ If M is a Banach manifold (e.g., it is finite-dimensional), then the characteristic distribution is also integrable. In the Fréchet case, integrability is not guaranteed. Assuming that the characteristic distribution is integrable, we denote by \underline{C} its leaf space and by $\pi: C \rightarrow \underline{C}$ the canonical projection. If \underline{C} is smooth, then it has a unique symplectic structure $\underline{\omega}$ such that $\pi^*\underline{\omega} = \iota_C^*\omega$, with $\iota_C: C \rightarrow M$ the inclusion map. In particular, for each $\underline{x} \in \underline{C}$, the symplectic form $\underline{\omega}_{\underline{x}}$ is defined as in Section A.1.3.

Remark A.26 (Relative reduction). Assume \underline{C} to be smooth. If L is an isotropic submanifold of M and $\underline{L} := \pi(L \cap C)$ is a submanifold of \underline{C} , then, by Lemma A.9, \underline{L} is an isotropic submanifold of \underline{C} . If in addition M is finite-dimensional and L is lagrangian, then \underline{L} is also lagrangian.

Remark A.27 (Generalized generating functions). Our main example, which extends Remark A.11 to the nonlinear case, will be the product of two split symplectic manifolds, $N = \tilde{T}^*M \times \tilde{T}^*B = \tilde{T}^*(M \times B)$, with $C = \tilde{T}^*M \times B$ and hence $\underline{C} = \tilde{T}^*M$. Let ψ be a function on $M \times B$ such that the one-form $d\psi$ is a section of $\tilde{T}^*(M \times B)$ and let L_ψ the split lagrangian submanifold in N it generates. If \underline{L}_ψ is a split lagrangian submanifold of $\underline{C} = \tilde{T}^*M$, then we say that ψ is a generalized generating function for \underline{L}_ψ . In the finite-dimensional case, one only has to check that that \underline{L}_ψ is a submanifold because it is then automatically split lagrangian. Some conditions for this to happen are presented in [9, Section 4.3]. There, more generally, B is just assumed to be the typical fiber of a submersion $\pi: E \rightarrow M$ and the triple (E, π, ψ) , such that \underline{L}_ψ is a submanifold, is called a Morse family for \underline{L}_ψ .

⁵⁵The hamiltonian vector field of a function H is the unique vector field X_H such that $\iota_{X_H}\omega + dH = 0$. (Note that in the infinite-dimensional case the existence of a hamiltonian vector field is an assumption, but its uniqueness is guaranteed.) The Poisson bracket of two functions H and G , with hamiltonian vector fields X_H and X_G , is the function $\{H, G\} := X_G(H) = \iota_{X_H}\iota_{X_G}\omega$. Its hamiltonian vector field is $[X_H, X_G]$.

Explicitly, if we denote by q^i and T^α the coordinates on M and B , respectively, and by p_i and E_α their momenta, the lagrangian submanifold L_ψ is given by the equations $p_i = \frac{\partial \psi}{\partial q^i}$ and $E_\alpha = \frac{\partial \psi}{\partial T^\alpha}$. Intersecting with $C = \tilde{T}^*M \times B$ corresponds to imposing $\frac{\partial \psi}{\partial T^\alpha} = 0$. Therefore, \underline{L}_ψ is given by the equations

$$p_i = \frac{\partial \psi}{\partial q^i}, \quad \frac{\partial \psi}{\partial T^\alpha} = 0.$$

Definition A.28 (Canonical relations). A (split) canonical relation L between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a (split) lagrangian submanifold of $\bar{M}_1 \times M_2$ (i.e., $M_1 \times M_2$ with symplectic form $\pi_2^* \omega_2 - \pi_1^* \omega_1$ with $\pi_i: M_1 \times M_2 \rightarrow M_i$ the canonical projections). In particular, the graph of a symplectomorphism from M_1 to M_2 is an embedded split canonical relation.

Here is an example (generalizing Lemma A.15):

Lemma A.29. *Let C be a coisotropic submanifold of M and assume that its involutive characteristic distribution is actually integrable. We say that two points $(x, x') \in C \times C$ lie on the same leaf of the characteristic distribution if there is a path in C with endpoints x and x' and tangent vector at each point lying in the characteristic distribution. Then, if*

$$L_C := \{(x, x') \in C \times C \mid x, x' \text{ lie on the same leaf of the characteristic distribution}\}$$

is a submanifold, it is isotropic in $\bar{M} \times M$. If M is finite-dimensional, then L is lagrangian, hence a canonical relation from M to itself.

Proof. Let (x, x') be a point in L_C . By integrability, there is a neighborhood U of x that is mapped to a convex open subspace of a vector space in which the distribution corresponds to a family of linear subspaces. We first consider the case where $x' \in U$. The images y and y' of x and x' can then be actually connected by a straight line. We consider the constant vector field corresponding to this direction and its flow Φ at time 1, a translation mapping a neighborhood of y to a neighborhood of y' satisfying $\Phi(y) = y'$. Since Φ is a flow along a vector field in the kernel of the image of the restriction of the symplectic form, it leaves the latter invariant by Lemma A.21. Let A be the differential of Φ at y pulled back to x . Therefore, A is a linear isomorphism $T_x C \rightarrow T_{x'} C$ satisfying

$$\omega_x(v, w) = \omega_{x'}(Av, Aw) \quad \forall v, w \in T_x C.$$

By construction $A(T_x C^\perp) = T_{x'} C^\perp$. Therefore, A induces a symplectomorphism \underline{A} between $\underline{T_x C} := T_x C / T_x C^\perp$ and $\underline{T_{x'} C} := T_{x'} C / T_{x'} C^\perp$. The first consequence of this is that

$$(v, v') \in T_{(x, x')} L_C \iff v' - Av \in T_{x'} C^\perp \iff [v'] = \underline{A}[v].$$

The second consequence is that $T_{(x, x')} L_C$ is isotropic. In fact, for $(v, v'), (w, w') \in T_{(x, x')} L_C$ we get

$$\omega_{x'}(v', w') = \omega_{x'}(Av, Aw) = \omega_x(v, w).$$

For a generic point (x, x') in L_C , we have by definition a path $\gamma: [0, 1] \rightarrow C$ such that $\gamma(0) = x$, $\gamma(1) = x'$ and $d_t \gamma \in T_{\gamma(t)} C^\perp \forall t \in [0, 1]$. By compactness of $[0, 1]$, we may find a finite partition $0 = t_0 < t_1 < t_2 < \dots < t_n = 1$ such that, for each $i = 0, \dots, n-1$, the points $\gamma(t_i)$ and $\gamma(t_{i+1})$ lie in a neighborhood U_i as in

the previous paragraph. Set $x_i := \gamma(t_i)$ and proceed as above. We then have that $(v_i, v_{i+1}) \in T_{(x_i, x_{i+1})}L_C$ iff $[v_{i+1}] = \underline{A}_i[v_i]$. Therefore,

$$(v, v') \in T_{(x, x')}L_C \iff [v'] = \underline{A}_{n-1} \dots \underline{A}_1 \underline{A}_0[v] \iff v' - \underline{A}_{n-1} \dots \underline{A}_1 \underline{A}_0 v \in T_{x'}C^\perp.$$

Given $v \in T_x C$, we set $[v_{i+1}] := \underline{A}_i \dots \underline{A}_1 \underline{A}_0[v]$ and choose representatives $v_{i+1} \in [v_{i+1}]$. For $(v, v'), (w, w') \in T_{(x, x')}L$, we then have

$$\omega_{x_{i+1}}(v_{i+1}, w_{i+1}) - \omega_{x_i}(v_i, w_i) = 0 \quad \forall i = 0, \dots, n-1.$$

Summing over i , we get

$$\omega_{x'}(v', w') - \omega_x(v, w) = 0,$$

which shows that $T_{(x, x')}L_C$ is isotropic.

In the finite-dimensional case, we easily check, as in the proof of Lemma A.15, that $\dim L = \frac{1}{2} \dim(\bar{M} \times M)$. \square

We also have the following straightforward generalization of Lemma A.16.

Lemma A.30. *Let L be a lagrangian submanifold of $\bar{M} \times M$. If $C := \pi_1(L)$, where π_1 is the projection to the first factor, is a submanifold of M , then C is coisotropic.*

Proof. By definition we have that x belongs to C iff there is a $y \in M$ such that (x, y) belongs to L . Moreover, v belongs to $T_x C$ iff there is a $w \in T_y M$ such that (v, w) belongs to $T_{(x, y)}L$. Now let \tilde{v} be in $T_x C^\perp$, i.e., $\omega_x(\tilde{v}, v) = 0 \quad \forall v \in T_x C$. This implies

$$\omega_x(\tilde{v}, v) - \omega_y(0, w) = 0 \quad \forall (v, w) \in T_{(x, y)}L.$$

Therefore, $(\tilde{v}, 0) \in T_{(x, y)}L^\perp = T_{(x, y)}L$, which implies $\tilde{v} \in T_x C$. \square

Remark A.31 (Applications in field theory). The above two lemmata, A.29 and A.30, are important for the applications in field theory. In the finite-dimensional case, we have assumed in Section 4, see Remark 4.1, that the constraints define a coisotropic submanifold C . Then the evolution relation L coincides with L_C and is therefore lagrangian by Lemma A.29. In the infinite-dimensional case, it is instead better to start from the evolution relation L , which is automatically isotropic. If we can prove that it is lagrangian (e.g., by the Hodge decomposition theorem), as we can do in all the examples in Section 7, we then get by Lemma A.30 that C is coisotropic. Finally, we just have to check that $L = L_C$.

If $M_i = \tilde{T}^*N_i$, $i = 1, 2$, then we may identify $\bar{M}_1 \times M_2$ with $\tilde{T}^*(N_1 \times N_2)$ (changing the sign of the \tilde{T}^*N_1 components of the canonical symplectic form). A (split) lagrangian submanifold L of $\bar{M}_1 \times M_2$ might have a (generalized) generating function ψ on $N_1 \times N_2$, which is then called a (generalized) generating function for the (split) canonical relation L from M_1 to M_2 .

Remark A.32 (Generating function for a canonical relation). Explicitly, if we denote by q^i and Q^i the coordinates on N_1 and N_2 , respectively, and by p_i and P_i their momenta, the canonical relation defined by a generating function ψ is given by the equations

$$P_i = \frac{\partial \psi}{\partial Q^i}, \quad p_i = -\frac{\partial \psi}{\partial q^i}.$$

Example A.33 (Generating function for a change of polarization). A particular case is when $M_1 = M_2$ and L is the graph of the identity map, which is obviously a symplectomorphism, but we choose N_1 and N_2 to be different. If there is a generating function ψ for L , with respect to these splittings, then the coordinates on \tilde{T}^*N_1 and \tilde{T}^*N_2 are related by the last equation.

Remark A.34 (Generalized generating function for a canonical relation). If we have instead a generalized generating function ψ on $N_1 \times N_2 \times B$, with coordinates q^i, Q^i, T^α , then the corresponding canonical relation is defined by the equations

$$P_i = \frac{\partial \psi}{\partial Q^i}, \quad p_i = -\frac{\partial \psi}{\partial q^i}, \quad \frac{\partial \psi}{\partial T^\alpha} = 0.$$

Remark A.35 (Composition of canonical relations). If we have a relation L_1 from M_1 to M_2 and a relation L_2 from M_2 to M_3 (i.e., $L_1 \subseteq M_1 \times M_2$ and $L_2 \subseteq M_2 \times M_3$), we may set-theoretically compose them to a relation $L_2 \circ L_1$ from M_1 to M_3 :

$$(125) \quad L_2 \circ L_1 := \{(x_1, x_3) \in M_1 \times M_3 \mid \exists x_2 \in M_2 : (x_1, x_2) \in L_1 \text{ and } (x_2, x_3) \in L_2\}.$$

If M_1, M_2 and M_3 are manifolds and L_1 and L_2 are submanifolds, there is no guarantee that $L_2 \circ L_1$ is also a submanifold. Let us assume this to be the case and also that L_1 and L_2 are (split) canonical relations—i.e., M_1, M_2 and M_3 are symplectic manifolds and L_1 and L_2 are (split) lagrangian submanifolds of $\bar{M}_1 \times M_2$ and $\bar{M}_2 \times M_3$, respectively. Then the composition (125) may be realized as a particular case of Remark A.26. Namely, we consider the (split) lagrangian submanifold $L := L_1 \times L_2$ and the coisotropic submanifold $C := M_1 \times \Delta_{M_2} \times M_3$ in $\bar{M}_1 \times M_2 \times \bar{M}_2 \times M_3$, where $\Delta_{M_2} := \{(x_2, x_2) \in M_2 \times M_2\}$ denotes the diagonal submanifold. We clearly have $\underline{C} = \bar{M}_1 \times M_3$ and $\underline{L} = L_2 \circ L_1$. Therefore, in the finite-dimensional case L is lagrangian, whereas in the infinite-dimensional case, at least it is isotropic and one has to check separately if it is also (split) lagrangian.

Remark A.36 (Composition of generating functions I: mixed polarizations). Let us assume that the composition of the (split) canonical relations L_1 and L_2 of Remark A.35 is a (split) canonical relation. Let us also assume that L_1 and L_2 have (generalized) generating functions ψ_1 and ψ_2 in the following way. We assume $M_i = \tilde{T}^*N_i$, $i = 1, 3$, $M_2 = U \times U'$ with U and U' open subsets of V and $V' \subseteq V^*$, respectively, and that $\psi_1 \in C^\infty(N_1 \times U' \times B)$ and $\psi_2 \in C^\infty(U \times N_3 \times \tilde{B})$, where B and \tilde{B} are parameter spaces. For notational simplicity, we work as in the finite-dimensional case and avoid writing component indices. We denote the coordinates as follows: (p, q) for \tilde{T}^*N_1 , (\tilde{p}, \tilde{q}) for \tilde{T}^*N_3 , T for B , and \tilde{T} for \tilde{B} . For the first copy of M_2 we denote the coordinates in $U \times U'$ by (Q, P) and for the second copy by (\hat{Q}, \hat{P}) . Then ψ_1 is a function of q, P, T and L_1 is given by the equations

$$Q = -\frac{\partial \psi_1}{\partial P}, \quad p = -\frac{\partial \psi_1}{\partial q}, \quad \frac{\partial \psi_1}{\partial T} = 0,$$

whereas ψ_2 is a function of $\hat{Q}, \tilde{q}, \tilde{T}$ and L_2 is given by the equations

$$\tilde{p} = \frac{\partial \psi_2}{\partial \tilde{q}}, \quad \hat{P} = -\frac{\partial \psi_2}{\partial \hat{Q}}, \quad \frac{\partial \psi_2}{\partial \tilde{T}} = 0.$$

Since C is defined by the equations $\hat{P} = P$ and $\hat{Q} = Q$, we see that L has generalized generating function, on $N_1 \times N_3 \times B \times \tilde{B} \times U' \times U$,

$$\psi_3(q, \tilde{q}; T, \tilde{T}, P, Q) = \psi_1(q, P; T) + \psi_2(Q, \tilde{q}; \tilde{T}) + PQ,$$

where the last term is defined via the pairing. Note that this is a generalized generating function with parameter space $B \times \tilde{B} \times M_2$. The conditions with respect to M_2 , given by

$$\frac{\partial \psi_1}{\partial P} + Q = 0 \quad \text{and} \quad \frac{\partial \psi_2}{\partial Q} + P = 0,$$

are precisely the first condition for L_1 and the second condition for L_2 . If we can solve these equations with respect to (P, Q) , we also get a (generalized) generating function $\underline{\psi}_3 \in C^\infty(N_1 \times N_3 \times B \times \tilde{B})$, which is the critical value of ψ_3 at this point. Note that if ψ_1 and ψ_2 are not generalized (i.e., B and \tilde{B} are one-point sets), then $\underline{\psi}_3$ is also not generalized.

Remark A.37 (Composition of generating functions II: same polarization). In the setting of the preceding Remark A.36, we now assume that $\psi_1 \in C^\infty(N_1 \times U \times B)$ and $\psi_2 \in C^\infty(U \times N_3 \times \tilde{B})$. Now ψ_1 is a function of q, Q, T and L_1 is given by the equations

$$P = \frac{\partial \psi_1}{\partial Q}, \quad p = -\frac{\partial \psi_1}{\partial q}, \quad \frac{\partial \psi_1}{\partial T} = 0,$$

whereas, as before, ψ_2 is a function of $\hat{Q}, \tilde{q}, \tilde{T}$ and L_2 is given by the equations

$$\tilde{p} = \frac{\partial \psi_2}{\partial \tilde{q}}, \quad \hat{P} = -\frac{\partial \psi_2}{\partial \hat{Q}}, \quad \frac{\partial \psi_2}{\partial \tilde{T}} = 0.$$

Setting $\hat{P} = P$ and $\hat{Q} = Q$, yields

$$\frac{\partial \psi_1}{\partial Q} + \frac{\partial \psi_2}{\partial Q} = 0.$$

Therefore, the composition L has generalized generating function, on $N_1 \times N_3 \times B \times \tilde{B} \times U$,

$$\psi_3(q, \tilde{q}; T, \tilde{T}, Q) = \psi_1(q, Q; T) + \psi_2(Q, \tilde{q}; \tilde{T}).$$

Note that now the parameter space is $B \times \tilde{B} \times U$. Also in this case, it might be possible to solve the critical equation in Q and get a (generalized) generating function $\underline{\psi}_3 \in C^\infty(N_1 \times N_3 \times B \times \tilde{B})$, which is not generalized if ψ_1 and ψ_2 are not.

Remark A.38 (Evolution relations and generalized generating functions in field theory). Let us conclude with the application to field theory [38, 18]. Under some regularity assumptions, a lagrangian field theory in $d + 1$ dimensions produces a (usually infinite-dimensional) symplectic manifold $\mathcal{F}_\Sigma^\partial$ of boundary fields on every closed d -manifold Σ (roughly speaking, this is the space of fields and some of their normal jets on which the boundary one-form obtained after integration by parts from the variation of the action functional is the potential for a symplectic form). The field theory on $I \times \Sigma$, where I is an interval, produces a subset L , which we call the evolution relation, of $\mathcal{F}_\Sigma^\partial \times \mathcal{F}_\Sigma^\partial$ given by the boundary data corresponding to possible solutions to the EL equations. If L is a submanifold, it is then automatically isotropic in $\overline{\mathcal{F}_\Sigma^\partial} \times \mathcal{F}_\Sigma^\partial$. In a good field theory, we require L to be split lagrangian, so a canonical relation from $\mathcal{F}_\Sigma^\partial$ to itself. (In the case of a field theory with nondegenerate lagrangian, L is the graph of the associated hamiltonian flow, so it is an embedded split canonical relation.) In this case, there might be a (generalized) generating function for L with respect to a split symplectic structure on $\mathcal{F}_\Sigma^\partial$. Throughout this paper we have constructed (generalized) generating functions for several examples from field theory. The discussion of the first part of this paper shows how to

construct a generalized generating function for the evolution relation in terms of the HJ action. Note that in the infinite-dimensional case one has to show separately (e.g., by Hodge decomposition theorem) that it is actually split lagrangian.

Remark A.39 (Gauge-fixing fermion). Another application of the formalism of (generalized) generating functions occurs in the gauge-fixing procedure in the BRST or BV formalism, where the symplectic manifold is actually a supermanifold and the symplectic form is odd with respect to the internal grading. In this case a generating function ψ for a gauge-fixing lagrangian L_ψ must be odd, and for this reason it is usually called a gauge-fixing fermion. Usually, there is no generating function on the base \mathcal{F} of the BV space of fields $\Pi\tilde{T}^*\mathcal{F}$, so one really has to resort to a generalized generating function on $\mathcal{F} \times B$, where B is a space of parameters. For example, in abelian gauge theories (on a trivial principal bundle) on a manifold M , one has $\mathcal{F} = \Omega^1(M) \times \Pi\Omega^0(M) \ni (A, c)$, where A is the connection and c is the ghost. We denote the corresponding momenta, with opposite parity, by A^+ and c^+ . To impose the Lorenz gauge fixing $d^*A = 0$ (working in the complement of cohomology) actually means defining L as the conormal bundle of the submanifold of \mathcal{F} defined by this equation, which clearly does not have a generating function. One therefore introduces extra parameters $\bar{c} \in B = \Pi\Omega^{\text{top}}(M)$ and the generalized generating function (the gauge-fixing fermion) $\psi = \int_M \bar{c} d^*A$. The corresponding gauge-fixing lagrangian $L = \underline{L}_\psi$ is then correctly given by the equations

$$A^+ = d^*\bar{c}, \quad c^+ = 0, \quad d^*A = 0.$$

In field theory, one prefers to introduce the last condition (namely, $\bar{c}^+ = \frac{\partial\psi}{\partial\bar{c}} = 0$) in terms of a Lagrange multiplier $\lambda \in B' = \Omega^0(M)$,

$$\delta(\bar{c}^+) = \int D\lambda e^{\frac{i}{\hbar} \int_M \lambda \bar{c}^+},$$

and then to regard $\int_M \lambda \bar{c}^+$ as a new term to be added to the action. The extended BV space of fields eventually is $\Pi\tilde{T}^*(\mathcal{F} \times B \times B')$. (For a slightly more general presentation of this, see [12, Sect. 3.2.1].)

A.3. Generalized Hamilton–Jacobi actions with infinite-dimensional targets. We now give more details on the constructions of Section 7, in the case of an infinite-dimensional target. This is based on results of [17] and [23].

The first thing we have to generalize is the target $T^*\mathbb{R}^n$ and the term $p_i dq^i$ in the action. We focus on the case when the target is linear (as this is the case for all the examples in this paper) and we take it to be a split symplectic space $V = W \oplus W'$ as in Remark A.6. More concretely, we assume we have it realized with $W' \subseteq W^*$ and canonical symplectic form ω induced from the pairing as in (122). We may also write ω as the differential of the 1-form α on V defined by

$$(\alpha(v \oplus a), w \oplus b) := a(w).$$

The map $(p, q): [t_a, t_b] \rightarrow T^*\mathbb{R}^n$ is now generalized to a map $\phi: [t_a, t_b] \rightarrow V$ and the term in the action $\int_{t_a}^{t_b} p_i dq^i$ is generalized to $\int_{t_a}^{t_b} \phi^* \alpha$.

Next we have to generalize the Lagrange multipliers e^α , the constraints H_α and the second term in the action: $\int_{t_a}^{t_b} e^\alpha H_\alpha$. For this we take a (possibly infinite-dimensional) vector space U and a subspace U' of its dual U^* such that the canonical pairing $\langle \cdot, \cdot \rangle$ of U' with U is still weakly nondegenerate. We assume we are given a

(possibly nonlinear) map $H: V \rightarrow U'$ and introduce e as a 1-form on $[t_a, t_b]$ taking values in U . We can then write the action as

$$S[\phi, e] = \int_{t_a}^{t_b} (\phi^* \alpha - \langle e, H \circ \phi \rangle).$$

We denote by \mathcal{F} the space of fields (ϕ, e) as an infinite-dimensional Fréchet manifold and assume S to be a smooth function on \mathcal{F} . Integrating by parts, we may express its differential as

$$(126) \quad \delta S = EL + \pi_b^* \alpha - \pi_a^* \alpha,$$

where EL is again expressed as an integral over $[t_a, t_b]$ and $\pi_b, \pi_a: \mathcal{F} \rightarrow V$ are the surjective submersions $\pi_b: (\phi, e) \mapsto \phi(t_b)$ and $\pi_a: (\phi, e) \mapsto \phi(t_a)$.

To define the EL equations we have to assume that, for every $x \in V$, $d_x H \in V^* \otimes U'$ lies in the image of $\omega^\sharp \otimes \text{Id}: V \otimes U' \rightarrow V^* \otimes U'$. We then have a unique $X_x \in V \otimes U'$ such that $\omega^\sharp(X_x) = d_x H$. The EL equations then read

$$d\phi = \langle e, X \rangle, \quad H \circ \phi = 0.$$

Again we call the first equation the evolution equation and the second the constraint.

Remark A.40. In the case of field theory, the existence of the family X of hamiltonian vector fields is ensured by locality. Namely, V and U are (modeled on) spaces of sections of fiber bundles over a closed manifold Σ . The fields (ϕ, e) are then sections of fiber bundles over $[t_a, t_b] \times \Sigma$. Finally, H and α are assumed to be local (over Σ).

We can now define the evolution relation L as the set of pairs of points in V that can be connected by solutions of the EL equations. We will assume L to be a (possibly immersed) lagrangian submanifold.⁵⁶ Note that L does not depend on the choice of splitting of V , since the EL equations are defined only in terms of H and the symplectic form.

Remark A.41. That L is isotropic is guaranteed, under some regularity assumptions. Indeed, we have $L = \pi_a \times \pi_b(\mathcal{EL})$, where $\mathcal{EL} \subset \mathcal{F}$ is the set of solutions to the EL equations. If we assume that \mathcal{EL} is a (possibly immersed) submanifold, then (126) implies that L is isotropic. In fact, with $\pi := \pi_a \times \pi_b$ and $\alpha_{V \oplus V} := (-\alpha, \alpha)$, we have

$$\iota_{\mathcal{EL}}^* \delta S = \pi^* \iota_L^* (\alpha_{V \oplus V}),$$

with ι_S denoting the inclusion maps. This implies $\pi^* \iota_L^* (\delta \alpha_{V \oplus V}) = 0$ and, therefore, $\iota_L^* (\delta \alpha_{V \oplus V}) = 0$ because π is a submersion.

Let $\pi_i: V \oplus V \rightarrow V$, $i = 1, 2$, denote the two projections and set $C_i = \pi_i(L)$. It then follows that $C_1 = C_2 = C := H^{-1}(0)$. In fact, since the constraints have to be satisfied everywhere, they are satisfied in particular at the endpoints, so $C_i \subseteq C$. On the other hand, for $e = 0$, we have the constant solutions (c, c) with $c \in C$, so $C \subseteq C_i$. Since, by definition, $L \subset C_1 \times C_2$, we then have $L \subset C \times C$.⁵⁷ Next we assume that C is a (possibly immersed) submanifold of V .

⁵⁶For the boundary value problem to be well-defined, one should actually ask for L to be split lagrangian, but this is not needed in the present section.

⁵⁷Note that, as a relation on V , L is symmetric and transitive, but in general not reflexive, and that C is the largest subset of V on which L restricts as an equivalence relation.

Lemma A.42. *C is coisotropic in V .*

Proof. Fix $u \in C$ and $\hat{u} \in \pi_1^{-1}(u) \subset L$. We have assumed that $T_{\hat{u}}L$ is lagrangian, and we want to prove that $(T_u C)^\perp \subseteq T_u C$. We have

$$T_u C = \{v \in V \mid \exists w \in V : (v, w) \in T_{\hat{u}}L\}.$$

By definition, if $\tilde{v} \in V$ belongs to $(T_u C)^\perp$, then $\omega(\tilde{v}, v) = 0$ for every $v \in T_u C$, i.e., $\omega(\tilde{v}, v) = 0$ for every $(v, w) \in T_{\hat{u}}L$. This implies $(\tilde{v}, 0) \in (T_{\hat{u}}L)^\perp$. Since L is lagrangian, we get $(\tilde{v}, 0) \in T_{\hat{u}}L$, so $\tilde{v} \in T_u C$. \square

Observe that, for every $t \in [t_a, t_b]$, the right hand side of the evolution equation lies in $(T_{\phi(t)}C)^\perp$. Since C is coisotropic, this is contained in $T_{\phi(t)}C$. As a consequence, a solution of the evolution equation starting at some point of C will never leave C , i.e., if the constraints are imposed at the initial (or final) endpoint, they will be satisfied at every time. (In the finite-dimensional case, this statement followed from (20).)

Remark A.43. The above assumptions that V is split and L is lagrangian are satisfied in the case of Chern–Simons theory thanks to the Hodge decomposition of differential forms (actually one even proves that L is split lagrangian). One can also easily check that L and C are submanifolds.

We now move to the HJ action. For this we might want to choose a different splitting $V = Z \oplus Z'$ at the final endpoint. The HJ action as a generalized generating function for L will depend on the choices of splitting, even though L does not. We denote by β the 1-form corresponding to this splitting. We assume that we have a generating function f on $W \oplus Z$ for the change of polarization, as in Example A.33. In particular, $\alpha = \beta + \delta f$. If we define

$$S^f := S - \pi_b^* f,$$

we get

$$\delta S^f = EL + \pi_b^* \beta - \pi_a^* \alpha.$$

Finally we define S_{HJ}^f , as a function on $W \times Z \times \Omega^1([t_a, t_b], U)$, as S^f evaluated on the assumedly unique solution $\bar{\phi}$ of the evolution equation with given endpoint conditions for ϕ in W and Z . We then get

$$\delta S_{\text{HJ}}^f = - \int_{t_a}^{t_b} \langle \delta e, H \circ \bar{\phi} \rangle + \pi_b^* \beta - \pi_a^* \alpha,$$

which shows that S_{HJ}^f is a generalized generating function for L . In general, it may be difficult to characterize the variations of e that leave S_{HJ}^f invariant, since we cannot use (20). This may be however done explicitly, e.g., when U is a Lie algebra and $H: V \rightarrow U'$ is an equivariant momentum map. In this case, which is enough to treat Chern–Simons theory, one easily sees that $\delta e = d\gamma + [e, \gamma]$, with γ a map $[t_a, t_b] \rightarrow U$ that vanishes at the endpoints, leaves S_{HJ}^f invariant.

APPENDIX B. THE MODIFIED DIFFERENTIAL QUANTUM MASTER EQUATION

Following [15] and [19, Section 4.2], we explain how the mQME and the (appropriate notion of) invariance under deformations of gauge fixing arise from the perturbative treatment of the theory. We will be rather sketchy and rely on some knowledge of such references.

B.1. Assumptions. We focus on the case where the constraints are in involution with structure constants; i.e., the constraints H_α are the components of an equivariant momentum map as in (19):

$$(127) \quad \{H_\alpha, H_\beta\} = f_{\alpha\beta}^\gamma H_\gamma, \quad \forall \alpha, \beta.$$

Moreover, if we change the polarization via a generating function $f(q, Q)$, we define the transformed constraints \tilde{H}_α as in (112), so we have the relations

$$(128) \quad H_\alpha \left(\frac{\partial f}{\partial q}, q \right) = \tilde{H}_\alpha \left(-\frac{\partial f}{\partial Q}, Q \right), \quad \forall \alpha.$$

For example, if we just interchange the p and q variables— $Q = p, P = -q, f = qQ$ —then we have the definition $\tilde{H}_\alpha(P, Q) := H_\alpha(Q, -P)$ and the relation $H_\alpha(Q, q) = \tilde{H}_\alpha(-q, Q)$.

As already mentioned several times in the paper, our result is based on the following

Assumption B.1. We assume that the constraints H_α are quantized to the components of a quantum equivariant momentum map as in (88), i.e.,

$$(129) \quad [\hat{H}_\alpha, \hat{H}_\beta] = i\hbar f_{\alpha\beta}^\gamma \hat{H}_\gamma,$$

with

$$\hat{H}_\alpha = H_\alpha \left(i\hbar \frac{\partial}{\partial q}, q \right),$$

and that they are compatible, at the quantum level, with the generating function $f(q, Q)$ of the change of polarization as in (113), i.e.,

$$(130) \quad H_\alpha \left(i\hbar \frac{\partial}{\partial q}, q \right) e^{-\frac{i}{\hbar}f} = \tilde{H}_\alpha \left(-i\hbar \frac{\partial}{\partial Q}, Q \right) e^{-\frac{i}{\hbar}f}, \quad \forall \alpha.$$

Remark B.2. Observe the following:

- (1) In the above formulae, the standard ordering is assumed, i.e., all the derivatives are placed to the right.
- (2) The assumptions (129) and (130) can equivalently be written using star products.
- (3) The classical relations (127) automatically imply the quantum relations (129) if the constraints H_α are linear or biaffine.
- (4) The classical relations (128) automatically imply the quantum relations (130) if
 - (a) the endpoint polarizations are linear (in particular, f is bilinear in (q, Q)), or
 - (b) the constraints $H_\alpha(p, q)$ are linear in p and the transformed constraints $\tilde{H}_\alpha(P, Q)$ are linear in P .
- (5) If the assumptions (129) and (130) are not satisfied on the nose, one may try to deform the H_α s and the \tilde{H}_α s with \hbar -corrections in order to impose the relations, but there is no guarantee that this is possible.
- (6) If the change of polarization is not linear, $e^{-\frac{i}{\hbar}f}$ might not be the integral kernel of a unitary operator. One might try to deform f , if possible, with \hbar -corrections in order to make the operator unitary. One should then try to find deformations of the H_α s and the \tilde{H}_α s so as to achieve the relations (129) and (130). Note that the classical condition of point (4b) that the

nondeformed H_α s and \tilde{H}_α s are linear in the respective momenta is no longer enough for the relation (130) to be satisfied on the nose, since the \tilde{H}_α are defined using the nondeformed f .

B.2. The propagator and the mdQME. The starting point for our discussion is the propagator η , which is an inverse of (more precisely, a parametrix for) the de Rham differential d , compatible with the endpoint conditions.

For example, in the quantization in Case II, the propagator is $\eta(s, t) = \theta(s - t)$ —the Heaviside step function—for all fields, see (93). In Case I, because of the different endpoint conditions (and the choice of e), the propagator in the ghost sector is $\eta(s, t) = \theta(s - t) - \phi(s)$, where ϕ is a chosen function with the property $\phi(t_b) - \phi(t_a) = 1$.

In general, the propagator η is a smooth function on the compactified configuration space, see [31, 4], of two points on the interval $I = [t_a, t_b]$, i.e.,

$$C_2(I) = \{(s, t) \in I \times I \mid s \leq t\} \sqcup \{(s, t) \in I \times I \mid s \geq t\},$$

that satisfies the endpoint conditions and the limit $\eta|_{s \rightarrow t^+} - \eta|_{s \rightarrow t^-} = 1$ and such that $d\eta = \chi$, where χ is a smooth 1-form on $I \times I$ containing information on the cohomology and the endpoint conditions.

The endpoint conditions are fixed throughout. However, the choices of χ and η are not unique and correspond to different gauge fixings and choices of residual fields. We can actually change η by adding to it a smooth function g on $I \times I$ that satisfies the endpoint conditions and simultaneously change χ by adding dg to it. It is convenient to interpolate between the two propagators using a parameter $u \in J := [0, 1]$ defining $\eta_u := \eta + ug$ and $\chi_u = \chi + udg$. It is even better to define $\check{\eta} \in C^\infty(C_2(I) \times J)$ as $\check{\eta}(s, t, u) = \eta_u(s, t)$ and $\check{\chi} \in \Omega^1(C_2(I) \times J)$ via⁵⁸

$$(131) \quad d\check{\eta} = \check{\chi},$$

where now d denotes the total differential on $\Omega^\bullet(C_2(I) \times J)$.

As the choice of χ_u is related to the choice of the residual fields, we have to deform them accordingly. We next compute the partition “form” for these choices and we denote it as $\check{Z} \in \Omega^\bullet(J)$. Note that the zero-form component is the usual partition function, depending on the parameter u .

We claim that (131) and Assumption B.1 imply that the partition form satisfies the differential mQME (mdQME)

$$(\Omega + i\hbar d_J + \hbar^2 \Delta)\check{Z} = 0,$$

where $d_J = du \frac{\partial}{\partial u}$ is the de Rham differential on J . The zero-form part of the mdQME is the mQME for the partition function (for every value of u), whereas the one-form part implies that the partition function changes by an $(\Omega + \hbar^2 \Delta)$ -exact term under a deformation of the data. These are the two statements we wanted to prove. Therefore, we are just left with showing that the mdQME holds.

⁵⁸In our theory we use two propagators—the propagator in the physical sector (p, q, p^+, q^+) and the propagator in the ghost sector (e^+, c, c^+, e) —and we may want to take different choices for them. This means that we will have a pair $(\check{\eta}, \check{\chi})$ for the physical sector and a possibly different one for the ghost sector.

B.3. Proof of the mdQME. To prove this we simply apply $i\hbar d_J$ to \check{Z} and observe that, for every integral in the Feynman diagram expansion, we can use the generalized Stokes’s formula

$$d_J \int = \pm \int d \pm \int_{\partial},$$

where the signs depend on dimensions (number of vertices) and on the choices of orientation.

The first contribution on the right hand side replaces the propagators, one by one, by $\hat{\chi}$. The sum of all these diagrams corresponds to the application of $i\hbar\Delta$.

The second contribution is related to Ω , as we are going to see. First observe that we are considering integrals over boundaries of compactified configuration spaces $C_n(I)$ of n points on the interval $I = [t_a, t_b]$. Such boundaries are of two types.

Internal boundaries: In this case $k > 1$ points collapse together in the interior of I . Such a boundary component fibers over $C_{n-k+1}(I)$ with a $(k-2)$ -dimensional fiber.

Endpoint boundaries: In this case $k \geq 1$ points collapse together at one endpoint of I . Such a boundary component fibers over $C_{n-k}(I)$ with a $(k-1)$ -dimensional fiber.

The crucial point is that the residual fields and the forms $\hat{\chi}$, which have no discontinuities, are basic in the above fibrations. Along the fibers we only have to integrate the product of the propagators $\hat{\eta}$ whose both vertices are in the fiber. As the propagators are zero-forms, we may get a nonzero integral only if

- (1) we are at an internal boundary component with $k = 2$, or
- (2) we are at an endpoint boundary component with $k = 1$.

In the first case, we have three kinds of collapsing diagrams (we call physical vertices those containing the constraint hamiltonians—and their derivatives—and ghost vertices those containing the structure constants):

- i. Two physical vertices collapse together: these diagrams—which may have any positive number of physical propagators—produce a contribution of the form $[H_\alpha, \star H_\beta]$.
- ii. One physical vertex collapses with a ghost vertex: these diagrams produce a contribution of the form $f_{\alpha\beta}^\gamma H_\gamma$.
- iii. Two ghost vertices collapse together: these diagrams sum up to zero thanks to the Jacobi identity.

The contributions in i. and ii. cancel each other thanks to (129) in Assumption B.1.

Therefore, we are only left with the case of a single bulk vertex collapsing at one endpoint. We consider the following cases:

- (a) Consider first the case of a ghost vertex collapsing, say, at the initial endpoint. It is a very simple situation, as the fiber contains one or two ghost propagators, connecting the vertex to initial endpoint insertions. This corresponds to the application of $f_{\alpha\beta}^\gamma c_a^\alpha c_a^\beta \frac{\partial}{\partial c_a^\gamma}$ or $f_{\alpha\beta}^\gamma b_\gamma^a \frac{\partial^2}{\partial b_\alpha^a \partial b_\beta^a}$, depending on the choice of endpoint condition. The case of the final endpoint is similar.
- (b) Next consider the case of a physical vertex collapsing at the initial endpoint, where, by convention, we always choose a linear polarization. Depending on the ghost initial endpoint conditions, the ghost propagator stemming from the vertex is either connected to an initial endpoint insertion—corresponding to the

insertion of a c_a —or leaves the collapsing diagram—corresponding to the application of a derivative with respect to a b^a . Besides, there is an arbitrary number of physical propagators stemming from the collapsing vertex, some connecting it to an initial endpoint insertion and some leaving the collapsing diagram. Depending on the physical initial endpoint conditions, summing over all such diagrams corresponds to the application of $H_\alpha\left(i\hbar\frac{\partial}{\partial q_a}, q_a\right)$ or $H_\alpha\left(p^a, -i\hbar\frac{\partial}{\partial p^a}\right)$. This together with the result of (a) for the initial endpoint corresponds to the application of Ω_a .

- (c) Finally consider the case of a physical vertex collapsing at the final endpoint. If the polarization is linear, this is very similar to case (b) and, together with the result of (a) for the final endpoint, corresponds to the application of Ω_b . In general, we have instead several propagators connecting the collapsing vertex to several f insertions. This sums up to a final endpoint insertion of $H_\alpha\left(i\hbar\frac{\partial}{\partial q}, q\right) e^{-\frac{i}{\hbar}f}|_{q=\hat{q}(t_b), Q=Q_b}$.⁵⁹ By the assumed relation (130), we have⁶⁰

$$\tilde{H}_\alpha\left(-i\hbar\frac{\partial}{\partial Q_b}, Q_b\right) e^{-\frac{i}{\hbar}f(\hat{q}(t_b), Q_b)} = H_\alpha\left(i\hbar\frac{\partial}{\partial q}, q\right) e^{-\frac{i}{\hbar}f}|_{q=\hat{q}(t_b), Q=Q_b},$$

so this contribution, together with the result of (a) for the final endpoint, corresponds to the application of Ω_b as in (115b).

B.4. Historical remarks. The first instance of the method described in this appendix for tracking the changes under deformation of the gauge fixing goes back to [4] in the case of Chern–Simons theory without boundary, without residual fields and with gauge fixings parametrized by a Riemannian metric. The method was extended in [11] to more general propagators. In [15] the case with residual fields was first considered, and in [19, Section 4.2] the general method delineated here—for general AKSZ theories, in presence of boundary and with residual fields—was introduced. By this method the boundary-related operator Ω is constructed in terms of boundary terms as explained above. The main difference with the present study is that there the case of dimension larger than one was considered, where some vanishing theorems simplify the computations. A differential, compatible with the BV structure, was explicitly introduced in [10] to keep track of changes in the choice of background, leading to the differential QME (dQME). The incorporation of the boundary, along the lines of [19], leading to the mdQME was finally presented in [14].

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⁵⁹If the constraint hamiltonians $H_\alpha(p, q)$ are linear in p , we have at most one physical propagator connecting the collapsing vertex to a final endpoint insertion of f . This then simply sums up to a final endpoint insertion of $H_\alpha(p^0, \hat{q})$, with $p^0 = \frac{\partial f}{\partial q}(\hat{q}(t_b), Q_b)$ as in (116), times the insertion of $e^{-\frac{i}{\hbar}f(\hat{q}(t_b), Q_b)}$.

⁶⁰If the constraint hamiltonians are linear in the respective momenta, this is simply

$$\begin{aligned} \tilde{H}_\alpha\left(-i\hbar\frac{\partial}{\partial Q_b}, Q_b\right) e^{-\frac{i}{\hbar}f(\hat{q}(t_b), Q_b)} &= \tilde{H}_\alpha(P(p^0, \hat{q}(t_b)), Q_b) e^{-\frac{i}{\hbar}f(\hat{q}(t_b), Q_b)} \\ &= H_\alpha(p^0, \hat{q}(t_b)) e^{-\frac{i}{\hbar}f(\hat{q}(t_b), Q_b)}, \end{aligned}$$

where we have used the classical relation (128) in the second equality.

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