PHASE SPACE FOR GRAVITY WITH BOUNDARIES

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ABSTRACT. This explanatory note, based on the geometrical method by Kijovski and Tulczyjew, describes the construction of the reduced phase space of Lagrangian field theories, i.e., the correct space of initial conditions with its symplectic structure. Several examples and, in particular, the case of four-dimensional gravity in the coframe formalism (Palatini–Cartan theory) are analyzed.

Key points
- Definition of Lagrangian field theories
- Euler–Lagrange equations and Noether 1-form
- Boundary structure
- Evolution relations
- Cauchy data and the reduced phase space
- Some examples: scalar field, electromagnetism, Palatini–Cartan theory.

CONTENTS

1. Introduction

One useful description of a mechanical system is in terms of hamiltonian evolution. This is given by a symplectic manifold and a hamiltonian function—the evolution being given by the flow of its hamiltonian vector field. This description is particulary interesting because it is, in principle, ready-made for quantization, with the symplectic manifold being turned to a Hilbert space and the hamiltonian function to a hamiltonian operator (how this is done, if anyway possible, is another story).

Very often a fundamental system is however introduced via an action functional and its associated Euler–Lagrange (EL) equations. Recasting it as a hamiltonian system—in particular in the case of gravity—is the goal of this note.

The (reduced) phase space of the system refers to the symplectic manifold on which one formulates the hamiltonian evolution of the system, whenever possible, once one chooses a time axis. Roughly speaking, it is the space of Cauchy data for the EL equations. The main issue is that the latter may have also a nonevolutionary content, which then gives rise to constraints. This is typically the case of gauge theories and of gravity.

The traditional way of proceding is due to Dirac [17]. This is an algebraic method with the goal of extending the Legendre transform to the case of degenerate Lagrangians. The method is iterative, with the construction of primary and a sequence of secondary constraints, and can in principle break down at every step.

In this note we will focus on a more geometric approach due to Kijovski and Tulczyjew (KT) [22] (we also rely on the presentation in [14]).[1] The central idea is to reformulate the space of solutions to the EL equations as an isotropic relation between the spaces of initial and final field configurations naturally endowed with a closed 2-form. The flow property is here replaced by the set-theoretic composition of relations, which only in particular cases are graphs of a flow. The initial (or final) space is then naturally endowed with a subset of Cauchy data—the points that can belong to the relation for an arbitrary small interval of time. As the restriction[2] of the 2-form to the space of Cauchy data is usually degenerate, one has to consider an appropriate quotient (by the null directions of the 2-form) to recover the reduced phase space as a symplectic manifold. It may of course happen that the quotient is singular. However, there is a natural way of doing this reduction in stages. In most physical theories, the only problematic quotient turns out to be the last one: the symplectic reduction of a coisotropic (i.e., defined by first-class constraints) submanifold of a symplectic manifold of (appropriately defined) boundary fields.

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[1]A related construction, which we will not discuss here, goes under the name of "covariant phase space" (see [29, 4, 20, 23, 24], and also [25] for an overview). In this approach, the space of solutions, equipped with its symplectic structure, is usually constructed via the bivariational complex (see, e.g., [9] and references therein).

[2]Following the common terminology in symplectic geometry, by "restriction" we mean the "pullback by the inclusion map."

To deal with the last reduction, there are appropriate techniques (e.g., the Batalin–Fradkin–Vilkovisky formalism (BFV) [19, 18], which will not be discussed in this note).

One advantage of the KT method is that it is rather flexible. One does not really have to consider evolution between an initial and a final time, but one can focus on the initial time only. One can even drop the condition that the initial submanifold should be space-like—the formalism working in more general situations—and one does not have to consider cylinders (a product of a manifold by an interval), but one can consider more general manifolds with boundary. One further advantage is that this method can be readily applied to the extension of the BV formalism [6, 7] to manifolds with boundary (and possibly lower-dimensional strata), producing the BV-BFV formalism discussed in [15].

We will start with a warm up dealing with simples cases. Next we will formulate the general theory and will apply it to the already presented examples. Finally, we will discuss gravity in the coframe (a.k.a. Palatini–Cartan) formulation.

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2. Some preliminary examples

We start reviewing some simple examples that will motivate the general theory. We will review in Section 6 how they are described within the general KT method.

2.1. **Mechanics.** Let us start with the familiar example of mechanics in one dimension. The action functional is

$$S[q] = \int_a^b \left(\frac{1}{2} m\dot{q}^2 - V(q) \right) dt,$$

where q (the "field") is a path $[a, b] \to \mathbb{R}$, m is the mass, and V is the potential energy. We want to compute a variation of S isolating in the integral variations of q but not of its derivative(s). We can do this integrating by parts:

$$\delta S = - \int_a^b \left(m\ddot{q} + V'(q) \right) \delta q \, dt + m\dot{q} \, \delta q \Big|_a^b.$$

The bulk term—i.e., the integral over the interval—contains the EL equations as the coefficient of δq:

$$m\ddot{q} + V'(q) = 0.$$

The boundary term can be reinterpreted as the difference, at times b and a, of a 1-form, on the space of positions q and velocities $v = \dot{q}$, that we denote by α and call the Noether 1-form:[3]

$$\alpha = mv \, \delta q.$$

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[3]More precisely, we regard this 1-form as defined on the tangent bundle $T\mathbb{R} = \mathbb{R} \times \mathbb{R} \ni (v, q)$ of the space of fields at a (or b), so δ becomes the usual de Rham differential. The boundary term in the variation of the action is more explicitly written as

$$m\dot{q} \, \delta q \Big|_a^b = \pi_b^* \alpha - \pi_a^* \alpha,$$

where π_a and π_b are the maps from the space of paths to $\mathbb{R} \times \mathbb{R}$ that send the path q to $(\dot{q}(a), q(a))$ and to $(\dot{q}(b), q(b))$, respectively.

We then take its differential

$$\omega := \delta\alpha = m\,\delta v\,\delta q,$$

which is an example of a **symplectic form**. This simply means that ω is closed (i.e., $\delta\omega = 0$) and nondegenerate (i.e., hamiltonian vector fields are uniquely determined). We call the space $T\mathbb{R}$ of positions and velocities together with ω the **phase space** of the theory.[4] (Usually, one calls phase space the space $T^*\mathbb{R}$ of positions q and momenta $p = mv$ with $\omega = \delta p\,\delta q$, but this is just a change of variables.) As the EL equations are of second order, the phase space $T\mathbb{R}$ is the space of possible initial conditions, a.k.a. Cauchy data. Assuming the existence of global solutions, this may also be regarded as the space of solutions to the EL equations.

The dynamics is then given by the Hamiltonian $H = \frac{1}{2}mv^2 + V(q)$ on $T\mathbb{R}$ (or, as usual, $H = \frac{p^2}{2m} + V(q)$ on $T^*\mathbb{R}$) in the sense that the time evolution is governed by the first-order ODE $(\dot{v}, \dot{q}) = X$ where $X = (-\frac{V'}{m}, v)$ is the unique vector field satisfying

$$\iota_X\omega + \mathrm{d}H = 0$$

with ι denoting contraction. The vector field X is called the hamiltonian vector field of H. The graph $L_{[a,b]}$ of the flow of X can be shown to be a lagrangian submanifold of $T\mathbb{R} \times T\mathbb{R}$, upon changing the sign of the symplectic form on one of the two factors. This is an example of what we will in general call the evolution relation of the system.

2.2. **A degenerate example.** Consider now, as action functional, the euclidean length of a curve:

$$S[q] = \int_a^b \|\dot{q}\|\,dt = \int_a^b \sqrt{\dot{q}\cdot\dot{q}}\,dt,$$

where $q\colon [a,b] \to \mathbb{R}^n$ is a path. The first, minor, remark concerning this example is that, to avoid singularities, we have to restrict the space of paths to the regular ones, i.e., those with nonvanishing speed everywhere (this is an open condition, not a constraint). If we compute the variation of the action, we get

$$\delta S = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{q}}{\|\dot{q}\|}\,\delta q\,dt + \frac{\dot{q}\cdot\delta q}{\|\dot{q}\|}\bigg|_a^b.$$

The EL equations are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{q}}{\|\dot{q}\|} = 0,$$

which are solved by straight regular paths, with an unspecified parametrization. The Noether 1-form is

$$\alpha = u \cdot \delta q,$$

where we have introduced the notation u for the normalized velocity $\frac{\dot{q}}{\|\dot{q}\|}$. The corresponding 2-form

$$\omega = \delta u \cdot \delta q$$

is degenerate. To look for its kernel, consider a vector field

$$X = X_q \cdot \frac{\partial}{\partial q} + X_u \cdot \frac{\partial}{\partial u}$$

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[4]Note that the theory produces two copies of the phase space, one for the initial time a and the other for the final time b.

and compute the contraction

$$\iota_X\omega = X_u \cdot \delta q - X_q \cdot \delta u.$$

The first term may vanish only for $X_u = 0$ because the components δq^i of δq are independent. On the other hand, since u is normalized, we have $u \cdot u = 1$ which, differentiated, yields the relation $u \cdot \delta u = 0$. Since δu is orthogonal to u, the last term in $\iota_X\omega$ vanishes exactly when X_q is parallel to u.

To formulate evolution in hamiltonian terms, we have to mod out the kernel of the 2-form ω. This reduction then consists in modding q out along the u direction. The reduced space is the union over $u \in S^{n-1}$, the $(n-1)$-dimensional sphere, of the spaces

$$F_u = \{q \in \mathbb{R}^n\}/[q \sim q + \lambda u, \ \lambda \in \mathbb{R}].$$

As a representative for each equivalence class, one can choose a vector orthogonal to u. Therefore, we may identify F_u with

$$T_u := \{q \in \mathbb{R}^n : q \cdot u = 0\}.$$

But this is just the tangent space of the sphere (viewed as submanifold of $\mathbb{R}^n$) at the point u. We then have that the reduced phase space is $\cup_{u \in S^{n-1}} T_u = TS^{n-1}$, the tangent bundle of the sphere. The symplectic structure is $[\delta u] \cdot \delta q$. Using the round metric, one can actually identify TS^{n-1} with T^*S^{n-1}. Under this transformation, the symplectic form is mapped to the canonical one, $\delta p \cdot \delta q$.

Finally note that the symplectic manifold TS^{n-1} constructed above can also be regarded as the space of solutions to the EL equations, i.e., the space of oriented straight lines in $\mathbb{R}^n$: an oriented line is in fact specified by its direction (the unit tangent vector $u \in S^{n-1}$) and by its position in space (a point $q \in \mathbb{R}^n$ modulo translations along u). Also note that these data do not change along the line, so the evolution on the reduced phase space TS^{n-1} is given by the identity map, and the Hamiltonian vanishes.

2.3. **The scalar field.** Let us now consider one example from field theory in d dimensions. To a d-dimensional Lorentzian manifold (M, g) we associate the space of fields $C^\infty(M)$ and the action functional

$$S_M[\phi] = \frac{1}{2} \int_M g^{\mu\nu} \, \partial_\mu\phi \, \partial_\nu\phi \, \sqrt{|\det g|} \, \mathrm{d}^d x,$$

where $(g^{\mu\nu})$ are the components of the inverse of g. The EL equation is the wave equation on (M, g):

$$\partial_\mu(g^{\mu\nu}\partial_\nu\phi\sqrt{|\det g|}) = 0.$$

We consider $M = \Sigma \times I$, where Σ is space-like and $I = [a, b]$ is a time interval. We also assume for simplicity that near the boundary the metric splits as

$$g = -(\mathrm{d}x^0)^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where 0 denotes the time direction and (h_{ij}) is a euclidean metric. When we compute the variation of S_M, we get a boundary term—the Noether 1-form—on each of the two boundary copies of Σ:

$$\alpha_\Sigma^\partial = \int_\Sigma \phi_0 \, \delta\phi \, \sqrt{\det h} \, \mathrm{d}^{d-1}x,$$

where ϕ_0 denotes $\partial_0\phi$.

In the symplectic setting, we consider ϕ and ϕ_0 as two independent functions on Σ: the initial configuration and the initial velocity. The space F_Σ^∂, with coordinates ϕ and ϕ_0, can be identified with the space of solutions to the wave equation on $\Sigma \times [a, b]$ with ϕ and ϕ_0 as initial conditions at time a. The space F_Σ^∂ with symplectic form

$$(1) \qquad \omega_\Sigma^\partial := \delta\alpha_\Sigma^\partial = \int_\Sigma \delta\phi_0\, \delta\phi\, \sqrt{\det h}\; \mathrm{d}^{d-1}x$$

is then the phase space of the theory.

2.4. **Electromagnetism.** Electromagnetism on a d-dimensional lorentzian manifold (M, g) is described by a 1-form[5] $A = A_\mu \mathrm{d}x^\mu$ on M (called the potential) and the action

$$S_M[A] = \frac{1}{4} \int_M g^{\mu\nu} g^{\rho\sigma}\, F_{\mu\rho}\, F_{\nu\sigma}\, \sqrt{|\det g|}\; \mathrm{d}^d x,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. The EL equations are in this case the Maxwell equations

$$\partial_\mu(g^{\mu\nu} g^{\rho\sigma}\, F_{\nu\sigma}\, \sqrt{|\det g|}) = 0 \quad \forall\rho.$$

Again we consider $M = \Sigma \times I$, where Σ is space-like and $I = [a, b]$ is a time interval with split metric near the boundary. In this case, the boundary contribution— the Noether 1-form—on each copy of Σ is

$$\alpha_\Sigma^\partial = \int_\Sigma h^{ij}\, F_{0i}\, \delta A_j\, \sqrt{\det h}\; \mathrm{d}^{d-1}x,$$

where h_{ij} is the Euclidean metric on Σ obtained by restricting g, and 0 denotes the time component. Again, in the symplectic formalism, we consider F_{0i} and A_j as independent fields. We denote by F_Σ^∂ the space with coordinates F_{0i} and A_j and with symplectic structure

$$\omega_\Sigma^\partial = \delta\alpha_\Sigma^\partial = \int_\Sigma h^{ij}\, \delta F_{0i}\, \delta A_j\, \sqrt{\det h}\; \mathrm{d}^{d-1}x.$$

Remark 2.1. If we introduce the electric field $E^j := h^{ij} F_{0i}$, then we get the usual formula

$$\omega_\Sigma^\partial = \int_\Sigma \delta E^j\, \delta A_j \sqrt{\det h}\; \mathrm{d}^{d-1}x,$$

which simply says that the electric field and the vector potential are canonically conjugate variables.

The crucial point now is that, unlike in the case of the scalar field, the space F_Σ^∂ is not the space of solutions. In fact, near the boundary, the EL equations read

$$\partial_0(g^{\rho j} F_{0j} \sqrt{\det h}) + \partial_i(h^{ij} g^{\rho\sigma} F_{j\sigma} \sqrt{\det h}) = 0 \quad \forall\rho.$$

For $\rho = k$ a space index, we get the evolution equation

$$h^{kj} \dot{F}_{0j} \sqrt{\det h} = \partial_i(h^{ij} h^{kl} F_{jl} \sqrt{\det h}),$$

where we denoted the time derivative by a dot (assuming for simplicity that h is time-independent). The equation for $\rho = 0$,

$$\partial_i(h^{ij} F_{0j} \sqrt{\det h}) = 0,$$

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[5]More generally, a $U(1)$ connection.

is however not an evolution equation and instead defines a constraint on the variables on F_Σ^∂. Note that, with the notations in Remark 2.1, this equation corresponds to the Gauss law $\mathrm{div}\, E = 0$. We then define C as the set of solutions to the Gauss law or, equivalently, as the common zero set for all scalars λ of the functions

$$J_\lambda := -\int_\Sigma \lambda\, \partial_i(h^{ij} F_{0j} \sqrt{\det h})\, \mathrm{d}^{d-1}x = \int_\Sigma \partial_i \lambda\, h^{ij} F_{0j} \sqrt{\det h}\, \mathrm{d}^{d-1}x$$

The restriction of ω_Σ^∂ to C is degenerate. Its kernel is generated by the hamiltonian vector fields X_λ for all J_λs. We immediately get

$$X_\lambda(A_i) = \partial_i \lambda, \qquad X_\lambda(F_{0j}) = 0,$$

which can be interpreted as infinitesimal gauge transformations. The reduced phase space $\underline{C}$ is then defined as the quotient of the space C of solutions to the Gauss law by gauge transformations.

2.5. **Conclusions.** All the above examples have something in common: the evolution can eventually be described in hamiltonian terms, and the symplectic form is ultimately derived from the Noether 1-form that arises as boundary term of a variation of the action.

On the other hand, there are also several differences. In the case of mechanics or of the scalar field this is the end of the story. In the degenerate case of the length functional, the 2-form arising from the the differential of the Noether 1-form is degenerate, so one has to reduce by its kernel in order to get a symplectic form. In the case of electromagnetism (and more generally of Yang–Mills theory), the construction we have presented actually yields a symplectic form on an appropriate space of boundary fields. However, some of the EL equations (the Gauss law) put constraints on them. Imposing the constraints leads however to a degeneracy of the restriction of the symplectic form, so one has now to reduce by its kernel.

In order to put more uniformity to these examples, but also to get ready to deal with more intricated examples, like gravity, we need a more conceptual perspective, which will be developed in the next sections.

3. Lagrangian field theory

In this and in the next two sections, we discuss the general method. We encourage the reader to apply each new construction to the examples of Section 2 (the results will be briefly reviewed in Section 6).

A Lagrangian field theory on a manifold M, possibly with boundary, is specified by a space of fields F_M and a local action functional S_M. The fields are local on M (sections of some sheaf); typically:

- maps from M to some fixed target manifold,
- sections of some vector (or, more generally, fiber) bundle over M,
- connections on M for some principal bundle.

The action functional is of the form $S_M = \int_M L$, where the lagrangian density L is a function of the fields and some of their derivatives (jets) at a point. In this note, we assume M to be compact.[6] To start with, we fix no boundary conditions for the fields.

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[6]If M is not compact, the fields are assumed to vanish fast enough to ensure convergence of the integral. Alternatively, one considers the Lagrangian as a distribution to be applied to test functions with compact support.

Remark 3.1. The lagrangian density L may require some additional structure on M. For example, to define a theory for spinors we need M to allow for spin bundles. To define gravity we need M to allow for lorentzian metrics. For a given theory, we call a manifold that admits the required structures a space–time manifold.

The EL equations are encoded in the variation δS_M of the action. We think of δ as the de Rham differential on F_M. For this, we have to assume that F_M has a suitable structure of infinite-dimensional manifold that allows defining differential forms and the de Rham differential.

Remark 3.2. In this note, we prefer to consider smooth fields and give F_M the structure of a Fréchet manifold. One may also prescribe a different regularity on the fields and give F_M the structure of a Banach manifold.

Digression 3.3 (A categorical approach)**.** A different option, which is actually closer to the physical viewpoint but which we will not explore in this note, consists in regarding F_M as an internal hom space in the category of smooth manifolds. Let us explain this in the case of a sigma model where F_M is the space of maps from M to a fixed target manifold N (possibly $\mathbb{R}$, as in scalar field theory). We then regard F_M as the functor that to a finite-dimensional manifold Z associates the set of smooth maps from $Z \times M$ to N. We think of this as a way of exploring F_M by manifolds Z (it is the notion of Grothendiek's functor of points, but it is also the usual textbook way of parametrizing the fields when computing functional derivatives). The natural context is that of "presheaves" over the category **Mfld** of finite-dimensional smooth manifolds, namely, of functors from **Mfld**$^{\mathrm{op}}$ to the category **Set** of sets. A finite-dimensional manifold X can also be regarded as a presheaf $\underline{X}$, namely, the functor that to a finite-dimensional manifold Z associates the set of smooth maps from Z to X. We regard presheaves, e.g., F_M or $\underline{X}$, as generalized manifolds. A natural transformation among the corresponding functors is regarded as a map between the generalized manifolds. Particularly useful is the generalized manifold Ω^k, namely, the functor that to a finite-dimensional manifold Z associates the set $\Omega^k(Z)$ of smooth k-forms on Z. A k-form on X is now equivalently described as a map from $\underline{X}$ to Ω^k. One then defines k-forms on a generalized manifold, e.g., F_M, as maps to Ω^k. For every k, we have the universal de Rham differential δ, the natural transformation from Ω^k to Ω^{k+1}, which, when instantiated on a given Z, is the usual de Rham differential $\mathrm{d} \colon \Omega^k(Z) \to \Omega^{k+1}(Z)$. The action functional S_M is now a map from F_M to Ω^0, namely, the natural tranformation, which, when instantiated on a given Z, is the smooth function on Z given by $\int_M L$ (recall that the fields are now parametrized by Z). The variational derivative δS_M is now just the composition of the universal de Rham differential δ with the map S_M, a generalized map from F_M to Ω^1, so a 1-form on F_M.

3.1. Euler–Lagrange spaces. If M has no boundary, solutions to the EL equations are, by definition, the critical points of S_M, viz., the fields at which $\delta S_M = 0$. If we introduce the Euler–Lagrange 1-form $el_M := \delta S_M$, then the solutions to the EL equations are the zeros of el_M. We denote by EL_M the space of solutions.

Note that there is an ambiguity in writing el_M, as there was already in writing S_M, because of total derivatives, whose integral is zero. If M has nonempty boundary, the different ways are no longer equivalent but differ by boundary terms.

We take the definition of S_M as fixed (we will return to this later on), but in el_M we only want variations of the fields, and not of their derivatives, to appear. This

requires integrating by parts (several times if necessary). Finally, we may write (in a unique way)

$$\delta S_M = el_M + \alpha_M, \tag{2}$$

where α_M, also a 1-form on F_M, collects the boundary terms. It follows that

$$\alpha_M = \int_{\partial M} a_M, \tag{3}$$

where a_M is a density on the boundary ∂M depending on the fields and their jets (including transversal ones) at a point on ∂M.

We keep denoting by EL_M the zero locus of el_M. In this note, for simplicity, we will assume that EL_M is a submanifold of F_M. Note that (2) has the following consequences:

(i) The restriction of α_M to EL_M is an exact 1-form.
(ii) If we change the definition of S_M to $\widetilde{S}_M$ via the addition of total derivatives to the Lagrangian L_M, we have $\widetilde{S}_M = S_M + \phi_M$, where $\phi_M = \int_{\partial M} f_M$ collects the boundary terms. Since our convention for el_M is fixed (namely, we write $\delta\widetilde{S}_M = el_M + \widetilde{\alpha}_M$), we get

$$\widetilde{\alpha}_M = \alpha_M + \delta\phi_M.$$

Changing the action as in (ii) should be regarded as an equivalence (anyway, the space EL_M is does not change).[7] We then define the 2-form

$$\omega_M \coloneqq \delta\alpha_M$$

which has invariant meaning ($\delta\alpha_M = \delta\widetilde{\alpha}_M$). Point (i) above then becomes:[8]

The restriction of ω_M to EL_M is zero.

Borrowing the terminology of symplectic geometry, we say that EL_M is an isotropic submanifold of (F_M, ω_M).

3.2. **Reduction.** The 2-form ω_M is closed by construction but in general is degenerate, so it is not a symplectic form. We can remedy for this quotienting F_M by the kernel of ω_M, i.e., by the distribution of vector fields X on F_M satisfying $\iota_X\omega = 0$.[9]

We will assume that the quotient $F_{\partial M}^{\partial}$ has a smooth manifold structure such that the canonical projection $\pi\colon F_M \to F_{\partial M}^{\partial}$ is a surjective submersion.[10]

The reason for the notation using the boundary symbol ∂ is that, because of (3), a vector field X that changes the bulk fields preserving their boundary values (including their transversal jets) is necessarily in the kernel of ω_M. As a result,

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[7]In the quantum theory, what matters is the Gibbs weight $e^{\frac{i}{\hbar}S_M}$, which, in view of this remark, should be considered not as a function but as a section of a line bundle over F_M: the change in the definition of S_M is the action of the gauge transformation $e^{\frac{i}{\hbar}\phi_M}$. From this point of view, α_M should be considered as a 1-form connection.

[8]Explicitly, this means that ω_M evaluated on a solution and contracted with a solution of the linearized equations vanishes. See footnote 2.

[9]Here ι denotes contraction. Namely, $\iota_X\omega$ is the 1-form that satisfies $\iota_X\omega(Y) = \omega(X,Y)$ for every vector field Y.

[10]We first have to assume that this distribution is regular, i.e., that the kernel distribution consists of sections of a vector bundle. Since ω is closed, this distribution is automatically involutive (viz., $\iota_X\omega = 0$ and $\iota_Y\omega = 0$ imply $\iota_{[X,Y]}\omega = 0$), but, in the Fréchet context, the Frobenius theorem does not hold, so we have to assume explicitly that the distribution is integrable. Finally, we have to assume that the leaf space $F_{\partial M}^{\partial}$ has smooth structure as in the text. All this turns out to work in the case of usual field theories.

$F_{\partial M}^{\partial}$ only depends on "boundary data." We call an element of $F_{\partial M}^{\partial}$ a **boundary field**.

By construction, there is a unique symplectic form $\omega_{\partial M}^{\partial}$ on $F_{\partial M}^{\partial}$ such that $\pi^* \omega_{\partial M}^{\partial} = \omega_M$.

A central concept is the space of boundary fields that arise from solutions to EL equations:

$$(4) \qquad\qquad L_M := \pi(EL_M).$$

Note that L_M is automatically isotropic with respect to $\omega_{\partial M}^{\partial}$. We will later require further structure on L_M (viz., that it is a split lagrangian submanifold; see Definition 5.2).

To make further structures clearer, but also to present a version that still works when $F_{\partial M}^{\partial}$ is singular, it is convenient to perform the reduction in stages, with the first reduction, which we will denote by $\widetilde{F}_{\partial M}^{\partial}$, always possible. If also $F_{\partial M}^{\partial}$ is smooth, we will then have the following commutative diagram of surjective submersions:

$$(5) \qquad\qquad \begin{array}{ccc} F_M & \xrightarrow{\ \widetilde{\pi}\ } & \widetilde{F}_{\partial M}^{\partial} \\ & {}^{\pi}\searrow & \downarrow{\scriptstyle p} \\ & & F_{\partial M}^{\partial} \end{array}$$

We will call points of $\widetilde{F}_{\partial M}^{\partial}$ **preboundary fields**. We study them in the next section.

4. Preboundary fields

We already observed that a vector field X on F_M that changes the bulk fields preserving their boundary values (including their transversal jets, which encode the inward-pointing derivative data of the fields at the boundary) is necessarily in the kernel of ω_M (actually, also in that of α_M).[11] We now consider the distribution consisting of only such vector fields and denote by $\widetilde{F}_{\partial M}^{\partial}$ its leaf space and by

$$\widetilde{\pi} \colon F_M \to \widetilde{F}_{\partial M}^{\partial}$$

the canonical projection. This is a space of fields on ∂M (with the transversal jets now being new fields). As already mentioned, we call an element of $\widetilde{F}_{\partial M}^{\partial}$ a **preboundary field**.

Note that $\widetilde{\pi}$ is a surjective submersion (corresponding to restricting fields and their transversal jets to the boundary) and that there is a unique closed local 2-form $\widetilde{\omega}_{\partial M}^{\partial}$ on $\widetilde{F}_{\partial M}^{\partial}$ such that $\widetilde{\pi}^* \widetilde{\omega}_{\partial M}^{\partial} = \omega_M$.

The 2-form $\widetilde{\omega}_{\partial M}^{\partial}$ is in general still degenerate, so a further reduction will be needed, leading again to $F_{\partial M}^{\partial}$ with canonical projection $p \colon F_M \to F_{\partial M}^{\partial}$.

There is also a unique local 1-form $\widetilde{\alpha}_{\partial M}^{\partial}$ on $\widetilde{F}_{\partial M}^{\partial}$ such that $\widetilde{\pi}^* \widetilde{\alpha}_{\partial M}^{\partial} = \alpha_M$. Moreover, we have $\widetilde{\omega}_{\partial M}^{\partial} = \delta \widetilde{\alpha}_{\partial M}^{\partial}$ and

$$(6) \qquad\qquad \delta S_M = el_M + \widetilde{\pi}^* \widetilde{\alpha}_{\partial M}^{\partial}.$$

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[11]Alternatively, we may consider vector fields that preserve germs of boundary values. Namely, we require that, for every field ϕ, X evaluated at ϕ is compactly supported in the interior of M. This construction produces a different version of the intermediate space $\widetilde{F}_{\partial M}^{\partial}$, but of course the final reduction $F_{\partial M}^{\partial}$ is the same.

We now define the analogue of (4), namely, the space of preboundary fields that can be extended to a, not necessarily unique, solution to the EL equations:

$$\widetilde{L}_M := \widetilde{\pi}(EL_M).$$

Note that $\widetilde{L}_M$ is automatically isotropic with respect to $\widetilde{\omega}^\partial_{\partial M}$ and that $L_M = p(\widetilde{L}_M)$. It is convenient to assume that $\widetilde{L}_M$ is a submanifold of $\widetilde{F}^\partial_{\partial M}$.

4.1. **Composition of evolution relations.** Note that $\widetilde{F}^\partial_\Sigma$ is actually defined for every manifold Σ that can arise as a boundary component of a space–time manifold M. Moreover, if we denote by $\{\partial_s M\}_{s=1,\dots,k}$ the connected components of ∂M, we have[12]

$$\widetilde{F}^\partial_{\partial M} = \prod_{s=1}^{k} \widetilde{F}^\partial_{\partial_s M}$$

and

$$\widetilde{\alpha}^\partial_{\partial M} = \sum_{s=1}^{k} \widetilde{\alpha}^\partial_{\partial_s M}.$$

Suppose now we cut the manifold M along a hypersurface Σ into two pieces M_1 and M_2, so we can recover M as the gluing of M_1 and M_2 along their common boundary component Σ (we assume here that Σ does not cut the boundary of M). We have

$$\partial M = \partial_1 M \sqcup \partial_2 M, \quad \partial M_1 = \partial_1 M \sqcup \Sigma, \quad \partial M_2 = \partial_2 M \sqcup \Sigma,$$

where $\partial_i M$ denotes the component of the boundary of M that is also a component of the boundary of M_i.

A solution to the EL equations on M is then uniquely given by a pair of solutions to the EL equations on M_1 and M_2 that match on Σ. More precisely, denote by $\widetilde{\pi}_i \colon F_{M_i} \to \widetilde{F}^\partial_\Sigma$, $i = 1, 2$, the map that corresponds to restricting fields on M_i and their transversal jets to the boundary component Σ. We then have the fiber product formula

(7) $\quad EL_M = EL_{M_1} \times_{\widetilde{F}^\partial_\Sigma} EL_{M_2} = \{(\phi_1, \phi_2) \in EL_{M_1} \times EL_{M_2} : \widetilde{\pi}_1(\phi_1) = \widetilde{\pi}_2(\phi_2)\}.$

Analogously, we have

(8) $\qquad\qquad \widetilde{L}_M = \{(\psi_1, \psi_2) \in \widetilde{L}_{M_1} \times \widetilde{L}_{M_2} : \pi_{\Sigma,1}(\psi_1) = \pi_{\Sigma,2}(\psi_2)\},$

where $\pi_{\Sigma,i}$ is the canonical projection from $\widetilde{F}^\partial_{\partial M_i}$ to $\widetilde{F}^\partial_\Sigma$.

It is more instructive to view the above formulae using the language of relations and correspndences.

Definition 4.1. A relation from the set A to the set B is a subset of $A \times B$. If R_{AB} is a relation from A to B and R_{BC} is a relation from B to C, one defines the composition

$$R_{BC} \circ R_{AB} := \{(a, c) \in A \times C : \exists b \in B \ (a, b) \in R_{AB} \ (b, c) \in R_{BC}\}$$

as a relation from A to C.[13] A correspondence from the set A to the set B is a map $C \to A \times B$ where C is some set. The composition of correspondences is

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[12]Here $\prod$ denotes the cartesian product of the given spaces. In the second formula, the pullback of $\widetilde{\alpha}^\partial_{\partial_s M}$ by the projection to the corresponding factor is understood.

[13]A particular example of a relation is the graph of a map. The composition of graphs, as relations, is the same as the graph of the usual composition of maps.

defined as their fibered product. The image of a correspondence is a relation, and the composition of such images, as relations, is the image of the composition of the correspondences.

In this setting we may view $\widetilde{L}_{M_1}$ (EL_{M_1}) as a relation (correspondence) from $\widetilde{F}^{\partial}_{\partial_1 M}$ to $\widetilde{F}^{\partial}_{\Sigma}$ and $\widetilde{L}_{M_2}$ (EL_{M_2}) as a relation (correspondence) from $\widetilde{F}^{\partial}_{\Sigma}$ to $\widetilde{F}^{\partial}_{\partial_2 M}$. In this language, (8) is the composition of relations, whereas (7) is the composition of correspondences.

For this reason, given a space–time manifold M and a decomposition of its boundary into two components, we call $\widetilde{L}_M$ (EL_M) the evolution relation (evolution correspondence).

Remark 4.2 (Evolutionary flow). A particular case is $M = \Sigma \times [t_1, t_3]$, with Σ having empty boundary and $t_1 < t_3$. We now pick $t_2 \in (t_1, t_3)$ and cut M along $\Sigma \times \{t_2\}$. In this case, $M_i = \Sigma \times [t_i, t_{i+1}]$, and we have

$$\widetilde{L}_{\Sigma \times [t_1, t_3]} = \widetilde{L}_{\Sigma \times [t_2, t_3]} \circ \widetilde{L}_{\Sigma \times [t_1, t_2]}.$$

This law may be thought of as the general version of time evolution (we will see that in many cases there is a further reduction in which the evolution relation actually becomes the graph of a flow).

Remark 4.3 (Relative structures). Suppose again we cut the manifold M along a hypersurface Σ into two pieces M_1 and M_2. Since $S_M = S_{M_1} + S_{M_2}$, equation (6) for M, M_1, and M_2 implies

$$\widetilde{\alpha}^{\partial}_{\Sigma, 1} = -\widetilde{\alpha}^{\partial}_{\Sigma, 2},$$

where $\widetilde{\alpha}^{\partial}_{\Sigma, i}$ denotes the 1-form on $\widetilde{F}^{\partial}_{\Sigma}$ viewed as a boundary component of M_i. Therefore, with analogue notation, we have

$$\widetilde{\omega}^{\partial}_{\Sigma, 1} = -\widetilde{\omega}^{\partial}_{\Sigma, 2}.$$

This shows that choosing a boundary component Σ as the target or the source of the evolution relation yields opposite sign to the associated 2-form.[14]

4.2. **Cauchy data.** Let Σ be a manifold that can appear as a boundary of a space–time manifold. We then consider the theory on $M_\epsilon = \Sigma \times [0, \epsilon]$ with ϵ a positive real number. We view $\widetilde{L}_{M_\epsilon}$ as a relation from $\widetilde{F}^{\partial}_{\Sigma} = \widetilde{F}^{\partial}_{\Sigma \times \{0\}}$ to $\widetilde{F}^{\partial}_{\Sigma \times \{\epsilon\}}$. We define

$$\widetilde{C}_{\Sigma} := \{c \in \widetilde{F}^{\partial}_{\Sigma} : \exists \epsilon > 0 \; \exists u \in F^{\partial}_{\Sigma \times \{\epsilon\}} \; (c, u) \in \widetilde{L}_{M_\epsilon}\}.$$

The space $\widetilde{C}_{\Sigma}$ consists of all the preboundary fields that can be extended to solutions on some cylinder. Note that in general $\widetilde{C}_{\Sigma}$ is not the whole of $\widetilde{F}^{\partial}_{\Sigma}$ (in the examples we will see that $\widetilde{C}_{\Sigma}$ is determined by the constraints of the theory, viz., by those EL equations that do not describe an evolution in the transverse direction). We call $\widetilde{C}_{\Sigma}$ the space of Cauchy data.

More generally, we denote by $\widetilde{C}_{\partial M}$ the product of the spaces of Cauchy data associated to each boundary component of M, viewed as a subset of $\widetilde{F}^{\partial}_{\partial M}$.

We may restrict the closed 2-form $\widetilde{\omega}^{\partial}_{\Sigma}$ to $\widetilde{C}_{\Sigma}$ and denote this restriction by $\widetilde{\omega}^{C}_{\Sigma}$. In general, it will be degenerate. In some good cases, $\widetilde{C}_{\Sigma}$ is a submanifold and

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[14]If our theory is defined on oriented space–time manifolds and the Lagrangian is viewed as a top form, then we view a_M in (3) also as a top form (now on the boundary). In this setting, α_M depends on a choice of orientation, and we choose the orientation of Σ in opposite ways depending on its being viewed as the source or the target of the evolution relation.

the leaf space $\underline{C_\Sigma}$ of the distribution given by the vector fields in the kernel of $\widetilde{\omega}_\Sigma^C$ has a smooth structure such the the canonical projection $\widetilde{p}_C \colon \widetilde{C}_\Sigma \to \underline{C_\Sigma}$ is a surjective submersion. In this case, there is a unique symplectic structure $\underline{\omega}_\Sigma^C$ on $\underline{C_\Sigma}$ satisfying $\widetilde{\omega}_\Sigma^C = \widetilde{p}_C^* \underline{\omega}_\Sigma^C$. We call the symplectic manifold $(\underline{C_\Sigma}, \underline{\omega}_\Sigma^C)$ the reduced phase space associated to Σ. If this constuction works for every boundary component of M, we write $\underline{C_{\partial M}}$ for the product of the symplectic manifolds associated to each component. The evolution relation $\widetilde{L}_M$ may be intersected with $\widetilde{C}_{\partial M}$ and projected to $\underline{C_{\partial M}}$ giving rise to the reduced evolution relation $\underline{L}_M$. This describes the correct evolution of the system. (In some particularly good cases, $\underline{L}_{\Sigma \times [a,b]}$ will be the graph of a hamiltonian flow.)

5. Boundary fields

We now return to the full reduction $F_M \to F_{\partial M}^\partial$ mentioned in Section 3.2; see the diagram (17) for reference.

We assume here that the quotient $F_{\partial M}^\partial$ has a smooth manifold structure such that the canonical projection $\pi \colon F_M \to F_{\partial M}^\partial$ is a surjective submersion and recall that an element of $F_{\partial M}^\partial$ is called a boundary field.

The excursus through the space of preboundary fields in Section 4 and the fact that we can obtain $F_{\partial M}^\partial$ from $\widetilde{F}_{\partial M}^\partial$ by a further reduction allow us to see more structure.

The first consequence is that, for every manifold Σ that can arise as a boundary component of a space–time manifold M, we may define $F_\Sigma^\partial := p(\widetilde{F}_\Sigma^\partial)$. This implies that, if we denote by $\{\partial_s M\}_{s=1,\dots,k}$ the connected components of ∂M, we have

$$F_{\partial M}^\partial = \prod_{s=1}^k F_{\partial_s M}^\partial$$

and

$$\omega_{\partial M}^\partial = \sum_{s=1}^k \omega_{\partial_s M}^\partial.$$

The second consequence is that $L_M = \pi(EL_M) = p(\widetilde{L}_M)$ is also a relation, which we keep calling the evolution relation, once we choose a decomposition of ∂M into two components. Moreover, the evolution relation of a manifold cut along a hypersurface is the composition of the evolution relations of its parts. Remark 4.3 now implies that

$$\omega_{\Sigma,1}^\partial = -\omega_{\Sigma,2}^\partial,$$

so F_Σ^∂ has opposite symplectic structure if it is regarded as target instead of source space.

Digression 5.1 (Composition of isotropic relations)**.** Following [30, 31], we can give a better description of the composition of evolution relations, which we now assume to be submanifolds, in terms of symplectic geometry. Observe that L_M can be obtained as follows: first, we consider $L_{M_1} \times L_{M_2}$ as a submanifold of $F_{\partial M_1}^\partial \times F_{\partial M_2}^\partial = F_{\partial_1 M}^\partial \times F_\Sigma^\partial \times F_\Sigma^\partial \times F_{\partial_2 M}^\partial$; then we intersect it with $F_{\partial_1 M}^\partial \times \Delta_{F_\Sigma^\partial} \times F_{\partial_2 M}^\partial$, where $\Delta_{F_\Sigma^\partial}$ denotes the diagonal in $F_\Sigma^\partial \times F_\Sigma^\partial$; finally we project to $F_{\partial M}^\partial$ by the map

$\pi_\Delta \colon F^\partial_{\partial_1 M} \times \Delta_{F^\partial_\Sigma} \times F^\partial_{\partial_2 M} \to F^\partial_{\partial M}$ that forgets the middle factor. Therefore, we have

$$L_M = \pi_\Delta \left(L_{M_1} \times L_{M_2} \cap F^\partial_{\partial_1 M} \times \Delta_{F^\partial_\Sigma} \times F^\partial_{\partial_2 M} \right).$$

The point of this construction is that the map π_Δ performs the symplectic reduction of the restriction of $\omega^\partial_{F^\partial_{\partial M_1} \times F^\partial_{\partial M_2}}$ to $F^\partial_{\partial_1 M} \times \Delta_{F^\partial_\Sigma} \times F^\partial_{\partial_2 M}$. The crucial point here is that $\Delta_{F^\partial_\Sigma}$ is a lagrangian submanifold of $F^\partial_\Sigma \times F^\partial_\Sigma$ when the two factors are given, as is the case here, opposite symplectic structures.

5.1. **Boundary conditions.** In this note we assume that there is a local 1-form $\alpha^\partial_{\partial M}$ such that $\pi^* \alpha^\partial_{\partial M} = \alpha_M$.[15] It follows that

  (i) $\alpha^\partial_{\partial M}$ is uniquely determined,
  (ii) $\omega^\partial_{\partial M} = \delta \alpha^\partial_{\partial M}$, and
  (iii) equation (2) yields

(9) $$\delta S_M = el_M + \pi^* \alpha^\partial_{\partial M}.$$

In order to select solutions to the EL equations that could also be regarded as critical points of S_M, one may then choose a submanifold B of $F^\partial_{\partial M}$ with the following two properties:

  (i) The restriction of $\alpha^\partial_{\partial M}$ to B vanishes, so $\delta S_M = el_M$ on $\pi^{-1}(B)$.
  (ii) B intersects L_M transversally, so each solution is isolated.

We will see that such submanifolds B are actually related to possible boundary conditions of the theory.

The first condition implies that B should also be isotropic. The second condition in particular entails that $T_u B$ and $T_u L_M$ are complementary subspaces of $T_u F^\partial_{\partial M}$, for every intersection point u (i.e., the boundary value in B of a solution). Note that $\omega^\partial_{\partial M}$ at u defines a symplectic form on the vector space $T_u F^\partial_{\partial M}$, and that $T_u B$ and $T_u L_M$ are isotropic with respect to it. We say that $T_u B$ and $T_u L_M$ are split lagrangian subspaces, according to the following

Definition 5.2. [13] An isotropic subspace of a symplectic vector space is called split lagrangian if it admits an isotropic complement.

The motivation for this terminology is that an isotropic subspace of a finite-dimensional symplectic space is split lagrangian precisely when it is half-dimensional, so lagrangian according to the usual definition. In the infinite-dimensional symplectic case, there are several definitions, of different strength, of what lagrangian should be, split lagrangian being the strongest.[16]

If we want that every point u of L_M could be obtained by such an intersection, we have to require that $T_u L_M$ should be split lagrangian for every $u \in L_M$: we say, in this case, that L_M is a split lagrangian submanifold.[17]

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[15]This condition actually fails in some interesting examples (like a charged particle in a magnetic field or the WZW model). What still happens in these examples however is that $\alpha^\partial_{\partial M}$ exists as a 1-form connection instead of a global 1-form.

[16]Note that, if L is split lagrangian in a symplectic vector space V, then its symplectic orthogonal $L^\perp$ is equal to L, so the symplectic reduction of L is a point. The condition $L = L^\perp$ is usually taken to define lagrangian subspaces. In particular, split lagrangian implies lagrangian.

[17]The submanifolds B that select isolated solutions have also to be split lagrangian, but in principle it is enough to require this in a neighborhood of the intersection points.

By construction, L_M is always isotropic. On the other hand, the condition that L_M should be split lagrangian is required for the theory to behave well when boundary conditions are imposed and does not necessarily hold in general. Also note that this may depend on the choice of space–time manifold M. For a Lagrangian theory to be good, we have to require that there is a class of manifolds with boundary for which the condition holds (this class should at least contain manifolds of the form $\Sigma \times I$, with I a time interval and Σ belonging to a physically interesting class, e.g., being space-like). The theories encountered in physics (scalar or spinor field theories and Yang–Mills theories, pure or coupled to the former) are good in this sense.

In the case of gravity, we will see that one has to be a bit more careful in the definition of F_M in order for the theory to be well behaved on a large class of space–time manifolds with boundary: namely, one has to impose some open condition for the fields in F_M on the boundary of M.[18] We do not view this as a boundary condition (such as the one imposed by the intersection with an isotropic B) which would instead correspond to a closed condition.

Remark 5.3 (Coisotropic Cauchy spaces). One can show that, if $L_{\Sigma \times [0,\epsilon]}$ is lagrangian for all $\epsilon > 0$, then C_Σ is coisotropic (i.e., the symplectic orthogonal of its tangent bundle is contained in the tangent bundle itself). In this case, C_Σ is locally given by first-class constraints (using Dirac's terminology). Moreover, the reduced phase space $\underline{C}_\Sigma$ is obtained modding out by the hamiltonian vector fields of the constraints (which one can regard as gauge transformations). This is the typical case we will analyze in the rest of this note.

6. Examples

We now briefly return to the examples discussed in Section 2, in light of the general KT method.

In the case of mechanics, with target $\mathbb{R}$ as in Section 2.1, we have $\underline{C} = C = F^\partial = T\mathbb{R}$, with coordinates interpreted as initial position q and initial velocity v. The symplectic form is $m\,\delta v\,\delta q$. If we set $p := mv$, we identify these spaces with $T^*\mathbb{R}$ with canonical symplectic form $\delta p\,\delta q$. The evolution relation $\underline{L}_{[a,b]} \subset \underline{C} \times \underline{C} = T^*\mathbb{R} \times T^*\mathbb{R}$ is the graph of the hamiltonian flow from time a to time b.

In the degenerate example of Section 2.2, we have $\underline{C} = C = F^\partial = T^*S^{n-1}$ with canonical symplectic structure. The evolution relation $\underline{L}_{[a,b]}$ is just the graph of the identity.

In the case of the scalar field of Section 2.3, we have $\underline{C}_\Sigma = C_\Sigma = F^\partial_\Sigma = C^\infty(\Sigma) \oplus C^\infty(\Sigma) \ni (\phi, \phi_0)$ and symplectic form as in (1). The evolution relation $\underline{L}_{\Sigma \times [a,b]}$ consists of pairs $((\phi^a, \phi_0^a), (\phi^b, \phi_0^b))$ with (ϕ^b, ϕ_0^b) the evaluation at time b of the solution ϕ and its time derivative $\partial_0 \phi$ of the Cauchy problem with initial condition (ϕ^a, ϕ_0^a) at time a.

In the case of electromagnetism of Section 2.4, we have $F^\partial_\Sigma = \mathfrak{X}(\Sigma) \oplus \mathfrak{X}(\Sigma) \ni (A, E)$ and symplectic form as in Remark 2.1.[19] The Cauchy space C_Σ consists of the pairs (A, E) with E satisfying the Gauss law $\mathrm{div}\, E = 0$. The reduced phase space $\underline{C}_\Sigma$ is the quotient of C_Σ by gauge transformations $A_i \mapsto A_i + \partial_i \lambda$. The evolution relation $\underline{L}_{\Sigma \times [a,b]}$ consists of pairs $(([A]^a, E^a), ([A]^b, E^b))$ with $([A]^b, E^b)$ the

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[18]We call this the boundary metric nondegenerate PC theory in Section 7.

[19]$\mathfrak{X}(\Sigma)$ denotes the vector fields on Σ.

evaluation at time b of the solution of the Cauchy problem for the time-dependent Maxwell equations with initial condition $([A]^a, E^a)$ at time a.[20]

7. Gravity in the coframe formalism

We now come to the example of the reduced phase space we want to discuss in this note: gravity. In its usual formulation, gravity is described by a Lorentzian metric g and by the Einstein–Hilbert action. The construction of the reduced phase space turns out to be more convenient and elegant in its formulation via (co)frames (using ideas mainly stemming from Cartan and Weyl), usually called the Palatini–Cartan (PC) theory or the first-order formulation of gravity (see, e.g., the reviews [21] and [28] and references therein). We will mainly follow [11].[21]

The first dynamical field in this formulation is a smooth field of frames on our space–time manifold M. The dynamical metric g is recovered using the frame starting from some reference metric (whose choice is irrelevant). A frame field may also be viewed as an isomorphism between a reference vector bundle $\mathcal{V}$ on M and the tangent bundle TM. It turns out that the computations are handier if one uses the inverse of a frame field, which we call a coframe (field). It turns out that to construct the action one also needs a second dynamical field, a connection for the orthogonal frame bundle. This construction renders gravity closer in spirit to gauge theories. More precisely, given a d-dimensional manifold M, the data are

- a vector bundle $\mathcal{V}$ over M isomorphic to TM (over the identity map on M), and
- a smooth family η of inner products with signature $(1, d-1)$ on the fibers of $\mathcal{V}$.[22]

The first dynamical field is an isomorphism $e\colon TM \to \mathcal{V}$ (over the identity map on M), called the coframe. The dynamical gravity field g is recovered pulling η back by e. Namely, if we pick coordinates on M (we use Greek indices for them) and a local basis for the fiber of $\mathcal{V}$ (we use the first lower case letters of the Latin alphabet for its components), then the coframe e is locally specified by its components e_μ^a (an invertible $d \times d$ matrix at each point of M).[23] If we denote the components of η by η_{ab}, we get the components of the space–time metric g as

$$(10) \qquad\qquad g_{\mu\nu} = e_\mu^a \, e_\nu^b \, \eta_{ab}.$$

Note that g has only $d(d+1)/2$ independent components, so the d^2 components of e are $d(d-1)/2$ too many. Essentially, we have to consider the internal orthogonal group to mod out the extra components.

In particular, we have to introduce a connection A to compute the covariant derivative $\boldsymbol{D}$ of sections of $\mathcal{V}$. If we denote by ϕ^a the components of a section ϕ of $\mathcal{V}$, we have

$$(\boldsymbol{D}_\mu \phi)^a = \partial_\mu \phi^a + A_{\mu b}^a \phi^b.$$

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[20][A] denotes the gauge equivalence class of A.

[21]For the treatment with Dirac's method, see, e.g., [2, 5]. For the covariant phase space formalism, see [26, 3].

[22]Note that this is possible if and only if M admits a Lorentzian structure.

[23]We denote by $\bar{e}\colon \mathcal{V} \to TM$, with components $\bar{e}_a^\mu$, the inverse of e. At each point $x \in M$, for each a we have a tangent vector $\bar{e}_a(x)$ with components $(\bar{e}_a^1(x), \ldots, \bar{e}_a^d(x))$. The vectors $\bar{e}_1(x), \ldots, \bar{e}_d(x)$ are linearly independent, so they yield a frame for the tangent space $T_x M$. For this reason, $\bar{e}$ is called a frame and e a coframe.

In particular we want this connection to be compatible with the metric η or, equivalently, its inverse, whose components we denote by η^{ab}. Namely,

$$0 = (\boldsymbol{D}_\mu \eta)^{ab} = \partial_\mu \eta^{ab} + A^a_{\mu c} \eta^{cb} + A^b_{\mu c} \eta^{ac}.$$

It is convenient to decompose

$$A^a_{\mu b} = \eta_{bc}(\alpha^{ac}_\mu + \omega^{ac}_\mu)$$

with α symmetric and ω antisymmetric in the upper indices. The above equations then read

$$0 = \partial_\mu \eta^{ab} + \alpha^{ab}_\mu + \alpha^{ba}_\mu,$$

which can be solved as

$$\alpha^{ab}_\mu = -\frac{1}{2} \partial_\mu \eta^{ab}.$$

Note that there are no further conditions on ω. The covariant derivative can then be written as

$$(\boldsymbol{D}_\mu \phi)^a = (D_\mu \phi)^a + \omega^{ab}_\mu \eta_{bc} \phi^c$$

with

$$(D_\mu \phi)^a = \partial_\mu \phi^a - \frac{1}{2} \partial_\mu \eta^{ab} \, \eta_{bc} \phi^c.$$

The second field of the theory is ω. If we pick a reference ω_0, then $\omega - \omega_0$ is a 1-form taking values in $\Lambda^2 \mathcal{V}$.

It is convenient to introduce the following piece of terminology. A k-form on M taking values in $\Lambda^l \mathcal{V}$ will be called a (k, l)-form or a form of type (k, l). In particular, e is a $(1, 1)$-form and $\omega - \omega_0$ is a $(1, 2)$-form. The operator $\boldsymbol{D}$ acts on (k, l)-forms producing $(k + 1, l)$-forms by skew-symmetrizing the lower space–time indices. We call this operation the covariant differential and denote it d_ω.

The curvature 2-form F_ω is the $(2, 2)$-form defined via $\mathrm{d}^2_\omega \phi = F_\omega \cdot \phi$, for any section ϕ of $\mathcal{V}$, with $(F_\omega \cdot \phi)^a = \eta_{bc} F^{ac}_\omega \phi^b$ in local coordinates.

From now on, we focus on the four-dimensional case, $d = 4$, with cosmological constant Λ. For a space–time manifold M, admitting $\mathcal{V}$ as above, we define the action functional (the Palatini–Cartan action)

$$S_M[e, \omega] := \int_M \left(\frac{1}{2} e^2 F_\omega + \frac{\Lambda}{24} e^4 \right),$$

where e^k denotes the (k, k)-form obtained by taking the wedge product of k factors e. Here the wedge product is done both with respect to the form and to the $\mathcal{V}$-components.[24] It can be checked [11] that $(4, 4)$-forms, in particular the integrand in the action, are canonically identified with densities. By taking a variation of S_M, as in (2), we get

$$\alpha_M = \int_{\partial M} \frac{1}{2} e^2 \, \delta \omega$$

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[24] In local coordinates, the integrand is then given, up to a normalization factor, by

$$\epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} \left(\frac{1}{2} e^a_\mu e^b_\nu (F_\omega)^{cd}_{\rho\sigma} + \frac{\Lambda}{24} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma \right).$$

and the EL equations

(11a) $$eF_\omega + \frac{\Lambda}{6}e^3 = 0,$$

(11b) $$ed_\omega e = 0.$$

One can check that the second equation is actually equivalent to

(11c) $$d_\omega e = 0,$$

as a consequence of the fact that e is nondegenerate. Moreover, for a given e, there is a unique connection $\omega(e)$ satisfying this equation. The connection $\omega(e)$ corresponds, via e, to the Levi-Civita connection for g as in (10). Finally, the equation $eF_{\omega(e)} + \frac{\Lambda}{6}e^3 = 0$ corresponds to the Einstein equation, with cosmological constant Λ, for g.

Notation 7.1. From now on we will put a tilde on the name of the bulk fields, so we will write $\widetilde{e} = \widetilde{e}_\mu dx^\mu$ and $\widetilde{\omega} = \widetilde{\omega}_\mu dx^\mu$ (with μ running from 1 to 4). We will reserve the notation without the tilde for the fields on the boundary: $e = e_i dx^i$ and $\omega = \omega_i dx^i$ (with i running from 1 to 3 and $x \in \partial M$). Note that the fields on the boundary take values in the restriction $\mathcal{V}|_{\partial M}$ of $\mathcal{V}$ to the boundary. The nondegeneracy condition now says that the three components $e_1(x)$, $e_2(x)$, and $e_3(x)$ are linearly independent in $\mathcal{V}_x$ for every $x \in \partial M$.

From the form of α_M, it might seem that the space of boundary fields consists of e and ω as 1-forms on the boundary (taking values in $\mathcal{V}|_{\partial M}$ and $\Lambda^2 \mathcal{V}|_{\partial M}$, respectively). Let us denote this space by $\widehat{F}^\partial_{\partial M}$. It turns out however that the 2-form

(12) $$\omega_M = \delta\alpha_M = \int_{\partial M} e\, \delta e\, \delta\omega$$

is degenerate, so some reduction is still needed in order to get the space $F^\partial_{\partial M}$ of boundary fields. More precisely, if X is a vector field on $\widehat{F}^\partial_{\partial M}$ with component u along e and v along ω, we get that $\iota_X \omega_M = 0$ iff $eu = 0$ and $ev = 0$. One can check that nondegeneracy of e implies $u = 0$. However, v may be nonzero.[25] In conclusion, the space $F^\partial_{\partial M}$ of boundary fields consists of coframes e on the boundary and equivalence classes $[\omega]$ of connections under the equivalence $\omega \sim \omega + v$ with $ev = 0$. We denote by $\widehat{p}$ the canonical projection $\widehat{F}^\partial_{\partial M} \to F^\partial_{\partial M}$.

We now want to discuss the Cauchy data. We then assume $M = \Sigma \times [0, \epsilon]$ and consider the first boundary component $\Sigma = \Sigma \times \{0\}$. We have to split the EL equations (11a) and (11c) into constraints and evolution equations. The constraints are just their restrictions, as differential forms, to the boundary; this way, using Notation 7.1, we can define the space of Cauchy data $\widehat{C}_\Sigma$ inside $\widehat{F}^\partial_\Sigma$ as

$$\widehat{C}_\Sigma = \left\{ (e, \omega) \in \widehat{F}^\partial_\Sigma : eF_\omega + \frac{\Lambda}{6}e^3 = 0,\ d_\omega e = 0 \right\}.$$

The space of Cauchy data C_Σ is then obtained as $\widehat{p}(\widehat{C}_\Sigma)$.

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[25]One can check that, for each $x \in \partial M$, the space of $v(x)$s satisfying $e(x)\, v(x) = 0$ is six-dimensional.

Proceeding this way is however a bit difficult, since the constraint function $\mathrm{d}_\omega e$ is not invariant under the equivalence relation.[26] It would indeed be simpler to consider instead the space

$$(13) \qquad \widehat{C}'_\Sigma = \left\{ (e,\omega) \in \widehat{F}^\partial_\Sigma : eF_\omega + \frac{\Lambda}{6}e^3 = 0, \ e\mathrm{d}_\omega e = 0 \right\} \supsetneq \widehat{C}_\Sigma.$$

Note that we have replaced the condition $\mathrm{d}_\omega e = 0$ with the strictly weaker condition

$$(14) \qquad\qquad\qquad\qquad e\mathrm{d}_\omega e = 0,$$

and it is not difficult to show that this is invariant under the equivalence relation. It turns out that replacing $\widehat{C}_\Sigma$ with $\widehat{C}'_\Sigma$ is indeed possible under a simple assumption.

Definition 7.2. We say that e is metric nondegenerate if $g^\partial_{ij} := e^a_i\, e^b_i\, \eta_{ab}$ is nondegenerate.

In order to get to this situation, we have to define the space of fields F_M accordingly. Namely, we have to put the condition that the coframe $\widetilde{e}$ becomes metric nondegenerate when restricted to the boundary. A stricter condition would be that the coframe $\widetilde{e}$ produces a riemannian metric on ∂M (i.e., ∂M is space-like). These are open conditions on the space of fields, so they are part of the definition of the theory.[27] We call the PC theory with coframe field required to be metric nondegenerate on the boundary the boundary metric nondegenerate PC theory. We then have [16, 11] the

Theorem 7.3. *In the boundary metric nondegenerate PC theory, $C_\Sigma = \widehat{p}(\widehat{C}'_\Sigma)$.*

This means that in studying C_Σ we can instead use $\widehat{C}'_\Sigma$ and work up to the equivalence relation. This turns out to be very convenient, e.g., in proving that C_Σ is coisotropic (see Theorem 7.4 below).

Proof. Instead of restricting (11c) to the boundary, it is more convenient to go back to (11b) and separate the constraints and the evolution equations there. It is clear that (14) is a constraint. On the other hand, there are more constraints hidden in the transversal part of the equation. We get, using Notation 7.1 and denoting the transversal index by n,

$$(15) \qquad\qquad\qquad \widetilde{e}_n(\mathrm{d}_{\widetilde{\omega}}\widetilde{e}) + \widetilde{e}(\mathrm{d}_{\widetilde{\omega}}\widetilde{e})_n = 0.$$

The transversal derivative of $\widetilde{e}$ in the last term shows that this equation also describes evolution. The problem is that it is not in normal form. In particular, it might happen that the first term is not of the form $\widetilde{e}$ times something, which would prevent the evolution equation to be solved. The condition for this not to happen is also a constraint.

From now on we focus for simplicity on the case when the space of fields F_M is defined by the condition that the coframe $\widetilde{e}$ produces a riemannian metric g^∂ on ∂M (i.e., ∂M is space-like).[28] This condition in particular implies that the restriction $\mathcal{V}|_{\partial M}$ of $\mathcal{V}$ admits a global time-like section ε. We now fix such an ε.

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[26]The other constraint function, $eF_\omega + \frac{\Lambda}{6}e^3$, is instead invariant upon using $\mathrm{d}_\omega e = 0$. In fact, upon sending ω to $\omega + v$, it becomes $e\mathrm{d}_\omega v$ which is equal to $\mathrm{d}_\omega(ev) - \mathrm{d}_\omega e\,v$. The first term vanishes by the condition on v and the second by the other constraint.

[27]Reference [12] studies the case when ∂M is light-like. For the treatment of this case with Dirac's method, see [1].

[28]The general case is treated in [11].

By nondegeneracy, the three components of e together with ε form a basis of $\mathcal{V}_x$ at every point x of the boundary. This means that on the boundary we can expand $\widetilde{e}_n = \rho\epsilon + \xi^i e_i = \rho\varepsilon + \iota_\xi e$, where ρ is a (nowhere vanishing) function and ξ is interpreted as a vector field on Σ (ρ and ξ are uniquely determined). Inserting this into (15) evaluated on points on the boundary yields, after some algebra,

$$\iota_\xi(e\mathrm{d}_\omega e) + \rho\varepsilon\mathrm{d}_\omega e + e((\mathrm{d}_{\widetilde{\omega}}\widetilde{e})_n - \iota_\xi\mathrm{d}_\omega e) = 0.$$

The first term vanishes upon using the constraint (14), so we can ignore it. We then see that the additional constraint, callled the structural constraint in [11], is

(16)
$$\varepsilon\mathrm{d}_\omega e = e\sigma$$

for some (1,1)-form σ.[29] We then have, for whatever ε we chose,

$$\widehat{C}_\Sigma = \left\{ (e,\omega) \in \widehat{F}_\Sigma^\partial : eF_\omega + \frac{\Lambda}{6}e^3 = 0, \ e\mathrm{d}_\omega e = 0, \ \exists\sigma \ \varepsilon\mathrm{d}_\omega e = e\sigma \right\}.$$

The crucial observation now is Theorem 17 in [11] which asserts that, for every metric nondegenerate e, there is a unique ω in each equivalence class satisfying the structural constraint (16).[30] This means that we can identify the quotient space F_Σ^∂ with the subspace

$$F_\Sigma^\varepsilon := \left\{ (e,\omega) \in \widehat{F}_\Sigma^\partial : \exists\sigma \ \varepsilon\mathrm{d}_\omega e = e\sigma \right\}$$

of $\widehat{F}_\Sigma^\partial$. We therefore get the diagram

(17)
$$\begin{array}{ccc} F_\Sigma^\varepsilon & \overset{i}{\lhook\joinrel\longrightarrow} & \widehat{F}_\Sigma^\partial \\ & {\scriptstyle \tau^\varepsilon}\searrow & \downarrow{\scriptstyle \widehat{p}} \\ & & F_\Sigma^\partial \end{array}$$

where i is the inclusion map and τ^ε is a diffeomorphism. We then obtain the space of Cauchy data $C_\Sigma = \widehat{p}(\widehat{C}_\Sigma)$ as $\tau^\varepsilon(C_\Sigma^\varepsilon)$ with

$$C_\Sigma^\varepsilon := i(\widehat{C}_\Sigma) = \left\{ (e,\omega) \in F_\Sigma^\varepsilon : eF_\omega + \frac{\Lambda}{6}e^3 = 0, \ e\mathrm{d}_\omega e = 0 \right\}.$$

However, it is clear that $C_\Sigma^\varepsilon = i(\widehat{C}_\Sigma')$, with $\widehat{C}_\Sigma'$ defined in (13), so $C_\Sigma = \tau^\varepsilon(C_\Sigma^\varepsilon) = \tau^\varepsilon(i(\widehat{C}_\Sigma')) = \widehat{p}(\widehat{C}_\Sigma')$. $\qquad\square$

We conclude this section with the following important

Theorem 7.4. *In the boundary metric nondegenerate PC theory, C_Σ is coisotropic.*

Proof. Thanks to Theorem 7.3, we can work with $\widehat{C}_\Sigma'$ which we can rewrite as the common zero locus of the functions

$$P_c = \int_\Sigma c\,e\mathrm{d}_\omega e,$$

$$T_\mu = \int_\Sigma \mu\left(eF_\omega + \frac{\Lambda}{6}e^3\right),$$

---

[29]The evolution equation at boundary points can now be written in normal form as $(\mathrm{d}_{\widetilde{\omega}}\widetilde{e})_n = \iota_\xi\mathrm{d}_\omega e - \rho\sigma$.

[30]In [12] it is shown how to modify the result in the light-like case.

with c a $(0,2)$-form and μ a $(0,1)$-form. We now compute their hamiltonian vector fields (X_c and Y_μ) with respect to the degenerate 2-form ω_M defined in (12): $\iota_{X_c}\omega_M = \delta P_c$, $\iota_{Y_\mu}\omega_M = \delta T_\mu$. The nontrivial fact—a consequence of Theorem 7.3—is that they exist. They are not unique, but their ambiguity lies in the direction of the equivalence relation, along which the functions are invariant. As a consequence, their Poisson brackets are well-defined. We will only compute their restrictions to C_Σ^ε and show that they vanish. This is enough to prove the theorem.

We present the hamiltonian vector fields and leave the rest of the computation to the reader:

$$X_c(e) = c \cdot e, \qquad\qquad X_c(\omega) = \mathrm{d}_\omega c + v,$$

$$Y_\mu(e) \approx \mathrm{d}_\omega \mu \qquad\qquad e Y_\mu(\omega) = \mu \left(F_\omega + \frac{\Lambda}{2} e^2 \right),$$

where $(c \cdot e)_i^a = \eta_{rs} c^{ar} e_i^s$, v satisfies $ev = 0$, and $\approx$ means upon restriction to C_Σ^ε. $\qquad\square$

Remark 7.5. Looking into the proof, we see that X_c generates the internal gauge transformations. On the other hand, one can see that Y_μ generates diffeomorphisms, both tangential and transversal to Σ. To see this, it is better to expand $\mu = \lambda\varepsilon + \iota_\xi e$, with ε as in the proof to Theorem 7.3, where λ is a function and ξ a vector field on Σ. We then have $T_\mu = H_\lambda + P_\xi$ with

$$H_\lambda = \int_\Sigma \lambda\varepsilon \left(eF_\omega + \frac{\Lambda}{6} e^3 \right),$$

$$P_\xi = \int_\Sigma \iota_\xi e\, e F_\omega.$$

Computing the hamiltonian vector fields separately, one can see that that of P_ξ generates the tangential vector fields and that of H_λ the transversal ones. One can also recognize H_λ and P_ξ as the hamiltonian and momentum constraints. The full structure of the Poisson brackets among P_C, H_λ, and P_ξ is described in [11].

Remark 7.6. In the computations in the proof of Theorem 7.4, several integration by parts are involved. If one wants to extend this construction to the case when Σ has a boundary, the boundary terms generate an interesting "corner" structure. This is better studied using the BFV formalism [19, 18]; see [10]. An alternative approach would consist in extending to the context of gravity the results of [27].

8. Conclusion

In this note, we have described the construction of the reduced phase space for Lagrangian fields theories following the geometrical method by Kijowski and W. M. Tulczyjew [22] and, in particular, we have applied it to the case of coframe gravity in four dimensions.

References

[1] S. Alexandrov and S. Speziale, "First order gravity on the light front," Phys. Rev. **D91**, 064043 (2015).

[2] A. Ashtekar, "New hamiltonian formulation of general relativity," Phys. Rev. **D36**, 1587–1602 (1987).

[3] J. F. Barbero, J. Margalef-Bentabol, and E. J. S. Villaseñor, "Palatini gravity with non-metricity, torsion, and boundaries in metric and connection variables," Phys. Rev. **D104**, 044046 (2021).

[4] G. Barnich and F. Brandt, "Covariant theory of asymptotic symmetries, conservation laws and central charges," Nucl. Phys. **B633**, 3–82 (2002).

[5] N. Barros e Sa, "Hamiltonian analysis of general relativity with the Immirzi parameter," Int. J. Mod. Phys. **D10**, 261–272 (2001).

[6] I. Batalin and G. Vilkovisky, "Relativistic S-matrix of dynamical systems with boson and fermion constraints," Phys. Lett. **B69**, 309–312 (1977).

[7] I. Batalin and G. Vilkovisky, "Gauge algebra and quantization," Phys. Lett. **B102**, 27–31 (1981).

[8] M. Bojowald and R. J. Szabo (eds.), *Encyclopedia of Mathematical Physics.*

[9] T. J. Bridges, P.E. Hydon, and J. K. Lawson, "Multisymplectic structures and the variational bicomplex," Mathematical Proceedings of the Cambridge Philosophical Society, Volume 148, 159–178 (2010).

[10] G. Canepa and A. S. Cattaneo, "Corner structure of four-dimensional general relativity in the coframe formalism," arXiv:2202.08684

[11] G. Canepa, A. S. Cattaneo, and M. Schiavina, "Boundary structure of general relativity in tetrad variables," Adv. Theor. Math. Phys. **25**, 327–377 (2021).

[12] G. Canepa, A. S. Cattaneo, and M. Tecchiolli, "Gravitational constraints on a lightlike boundary," Ann. Henri Poincaré **22**, 3149–3198 (2021).

[13] A. S. Cattaneo and I. Contreras Palacios, "Split canonical relations," Ann. Henri Poincaré **4**, 155–185 (2021)

[14] A. S. Cattaneo, P. Mnev, and N. Reshetikhin "Classical and quantum Lagrangian field theories with boundary," PoS(CORFU2011)044

[15] A. S. Cattaneo, P. Mnev, and N. Reshetikhin, "Classical BV theories on manifolds with boundaries," Commun. Math. Phys. **332**, 535–603 (2014).

[16] A. S. Cattaneo and M. Schiavina, "The reduced phase space of Palatini–Cartan–Holst theory," Ann. Henri Poincaré **20**, 445–480 (2019).

[17] P. A. M. Dirac, "Generalized Hamiltonian dynamics," Canad. J. Math. **2**, 129–148 (1950).

[18] E. S. Fradkin and G. A. Vilkovisky, "Quantization of relativistic systems with constraints: Equivalence of canonical and covariant formalisms in quantum theory of gravitational field," CERN Preprint CERN-TH-2332 (1977).

[19] E. Fradkin and G. Vilkovisky, "Quantization of relativistic systems with constraints," Phys. Lett. **B55**, 224–226 (1975).

[20] L. Freidel, R. Oliveri, D. Pranzetti, and S. Speziale, "Extended corner symmetry, charge bracket and Einstein's equations," J. High Energ. Phys. **83**, 083 (2021).

[21] F. W. Hehl, P. Von Der Heyde, GD. Kerlick, and J. M. Nester, "General relativity with spin and torsion: Foundations and prospects," Rev.Mod.Phys. **48**, 393–416 (1976).

[22] J. Kijowski and W. M. Tulczyjew, *A Symplectic Framework for Field Theories*, Lecture notes in Physics **107**, Springer Verlag (Berlin, Heidelberg, 1979).

[23] J. Margalef-Bentabol and E. J. S. Villaseñor, "Geometric formulation of the covariant phase space methods with boundaries," Phys. Rev. **D103**, 025011 (2021).

[24] J. Margalef-Bentabol and E. J. S. Villaseñor, "Proof of the equivalence of the symplectic forms derived from the canonical and the covariant phase space formalisms," Phys. Rev. **D105**, L101701 (2022).

[25] nLab authors, "Phase space," nLab

[26] R. Oliveri and S. Speziale, "Boundary effects in general relativity with tetrad variables," Gen. Rel. Grav. **52**, 83 (2020).

[27] A. Riello and M. Schiavina, "Hamiltonian gauge theory with corners: Constraint reduction and flux superselection," arXiv:2207.00568

[28] M. Tecchiolli, "On the mathematics of coframe formalism and Einstein–Cartan theory—A brief review," Universe **2019**, 5, 206. https://doi.org/10.3390/universe5100206

[29] R. M. Wald and A. Zoupas, "A general definition of 'conserved quantities' in general relativity and other theories of gravity," Phys. Rev. **D61**, 084027 (2000).

[30] A. Weinstein, "Symplectic manifolds and their lagrangian submanifolds," Adv. Math. **6** 329–346 (1971).

[31] A. Weinstein, "Symplectic categories," Proceedings of the Summer School, IST, Lisbon, July 13–17, 2009, Port. Math. **67** 119 (2010), arXiv:0911.4133

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