GRADED POISSON ALGEBRAS

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Abstract. This note is an expanded and updated version of our entry with the same title for the 2006 Encyclopedia of Mathematical Physics. We give a brief overview of graded Poisson algebras, their main properties and their main applications, in the contexts of super differentiable and of derived algebraic geometry.

Contents

Key words and phrases. Poisson algebras, Poisson manifolds, (graded) symplectic manifolds, Lie algebras, Lie algebroids, BV algebras, Lie-Rinehart algebras, graded manifolds, Schouten–Nijenhuis bracket, Hochschild cohomology, BRST quantization, BV quantization, AKSZ method, derived brackets.

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1. Definitions

1.1. **Graded vector spaces.** By a $\mathbb{Z}$-graded vector space (or simply, graded vector space) we mean a direct sum $A = \oplus_{i\in\mathbb{Z}} A_i$ of vector spaces over a field k of characteristic zero. The A_i are called the components of A of degree i and the degree of a homogeneous element $a \in A$ is denoted by $|a|$. We also denote by $A[n]$ the graded vector space with degree shifted by n, namely, $A[n] = \oplus_{i\in\mathbb{Z}}(A[n])_i$ with $(A[n])_i = A_{i+n}$. The tensor product of two graded vector spaces A and B is again a graded vector space whose degree r component is given by $(A \otimes B)_r = \oplus_{p+q=r} A_p \otimes B_q$.

The symmetric and exterior algebra of a graded vector space A are defined respectively as $S(A) = T(A)/I_S$ and $\bigwedge(A) = T(A)/I_\wedge$, where $T(A) = \oplus_{n\geq 0} A^{\otimes n}$ is the tensor algebra of A and I_S (resp. $I_\wedge$) is the two-sided ideal generated by elements of the form $a \otimes b - (-1)^{|a|\,|b|} b \otimes a$ (resp. $a \otimes b + (-1)^{|a|\,|b|} b \otimes a$), with a and b homogeneous elements of A. The images of $A^{\otimes n}$ in $S(A)$ and $\bigwedge(A)$ are denoted by $S^n(A)$ and $\bigwedge^n(A)$ respectively. Notice that there is a canonical decalage isomorphism $S^n(A[1]) \simeq \bigwedge^n(A)[n]$.

1.2. **Graded algebras and graded Lie algebras.** We say that A is a graded algebra (of degree zero) if A is a graded vector space endowed with a degree zero bilinear associative product $\cdot\colon A\otimes A \to A$. A graded algebra is graded commutative if the product satisfies the condition

$$a \cdot b = (-1)^{|a|\,|b|} b \cdot a$$

for any two homogeneous elements $a, b \in A$ of degree $|a|$ and $|b|$ respectively.

A graded Lie algebra of degree n is a graded vector space A endowed with a graded Lie bracket on $A[n]$. Such a bracket can be seen as a degree $-n$ Lie bracket on A, i.e., as bilinear operation $\{\cdot,\cdot\}\colon A \otimes A \to A[-n]$ satisfying graded antisymmetry and graded Jacobi relations:

$$\{a, b\} = -(-1)^{(|a|+n)(|b|+n)}\{b, a\}$$
$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)}\{b, \{a, c\}\}$$

1.3. **Graded Poisson algebra.** We can now define the main object of interest of this note:

Definition 1.1. A graded Poisson algebra of degree n, or n-Poisson algebra, is a triple $(A, \cdot, \{,\})$ consisting of a graded vector space $A = \oplus_{i\in\mathbb{Z}} A_i$ endowed with a degree zero graded commutative product and with a degree $-n$ Lie bracket. The bracket is required to be a biderivation of the product, namely:

$$\{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|b|(|a|+n)} b \cdot \{a, c\}.$$

Notation. Graded Poisson algebras of degree zero are called Poisson algebras, while for $n = 1$ one speaks of Gerstenhaber algebras [7] or of Schouten algebras.

Sometimes a $\mathbb{Z}_2$-grading is used instead of a $\mathbb{Z}$-grading. In this case, one just speaks of even and odd Poisson algebras.

Example 1.1. Any associative algebra can be seen as a Poisson algebra with the trivial Lie structure, and any graded Lie algebra can be seen as a Poisson algebra with the trivial product.

Example 1.2. The most classical example of a Poisson algebra is the algebra of smooth functions on $\mathbb{R}^{2n}$ endowed with usual multiplication and with the Poisson bracket (already considered by Poisson himself) $\{f,g\} = \partial_{q^i} f \partial_{p_i} g - \partial_{q^i} g \partial_{p_i} f$, where the p_i's and the q^i's, for $i = 1, \ldots, n$, are coordinates on $\mathbb{R}^{2n}$. The bivector field $\partial_{q^i} \wedge \partial_{p_i}$ is induced by the symplectic form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$. An immediate generalization of this example is the algebra of smooth functions on a symplectic manifold $(\mathbb{R}^{2n}, \omega)$ with the Poisson bracket $\{f,g\} = \omega^{ij}\partial_i f \partial_j g$, where $\omega^{ij}\partial_i \wedge \partial_j$ is the bivector field defined by the inverse of the symplectic form $\omega = \omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$; viz.: $\omega_{ij}\omega^{jk} = \delta_i^k$.

A further generalization is when the bracket on $\mathcal{C}^\infty(\mathbb{R}^m)$ is defined by $\{f,g\} = \alpha^{ij}\partial_i f \partial_j g$, with the matrix function α not necessarily nondegenerate. The bracket is Poisson if and only if α is skewsymmetric and satisfies

$$\alpha^{ij}\partial_i \alpha^{kl} + \alpha^{il}\partial_i \alpha^{jk} + \alpha^{ik}\partial_i \alpha^{lj} = 0.$$

An example of this, already considered by Lie in [20], is $\alpha^{ij}(x) = f_k^{ij} x^k$, where the f_k^{ij}'s are the structure constants of some Lie algebra.

Example 1.3. Example 1.2 can be generalized to any symplectic manifold (M, ω). To every function $h \in \mathcal{C}^\infty(M)$ one associates the Hamiltonian vector field X_h which is the unique vector field satisfying $\iota_{X_h}\omega = \mathrm{d}h$. The Poisson bracket of two functions f and g is then defined by

$$\{f,g\} = \iota_{X_f}\iota_{X_g}\omega.$$

In local coordinates, the corresponding Poisson bivector field is related to the symplectic form as in Example 1.2.

A generalization is the algebra of smooth functions on a manifold M with bracket $\{f,g\} = \langle \alpha, \mathrm{d}f \wedge \mathrm{d}g \rangle$, where α is a bivector field (i.e., a section of $\bigwedge^2 TM$) such that $\{\alpha, \alpha\}_{SN} = 0$, where $\{\cdot,\cdot\}_{SN}$ is the Schouten–Nijenhuis bracket (see subsection 2.1 below for details, and Example 1.2 for the local coordinate expression). Such a bivector field is called a Poisson bivector field and the manifold M is called a Poisson manifold. Observe that a Poisson algebra structure on the algebra of smooth functions on a smooth manifold is necessarily defined this way. In the symplectic case, the bivector field corresponding to the Poisson bracket is the inverse of the symplectic form (regarded as a bundle map $TM \to T^*M$).

The linear case described at the end of example 1.2 corresponds to $M = \mathfrak{g}^*$ where $\mathfrak{g}$ is a (finite dimensional) Lie algebra. The Lie bracket $\bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ is regarded as an element of $\mathfrak{g} \otimes \bigwedge^2 \mathfrak{g}^* \subset \Gamma(\bigwedge^2 T\mathfrak{g}^*)$ and reinterpreted as a Poisson bivector field on $\mathfrak{g}^*$. The Poisson algebra structure restricted to polynomial functions is described at the beginning of subsection 2.1.

1.4. **Batalin–Vilkovisky algebras.** When n is odd, a generator for the bracket of an n-Poisson algebra A is a degree $-n$ linear map from A to itself,

$$\Delta\colon A \to A[-n],$$

such that

$$\Delta(a \cdot b) = \Delta(a) \cdot b + (-1)^{|a|} a \cdot \Delta(b) + (-1)^{|a|}\{a,b\}.$$

A generator Δ is called exact if and only if it satisfies the condition $\Delta^2 = 0$, and in this case Δ becomes a derivation of the bracket:

$$\Delta(\{a,b\}) = \{\Delta(a), b\} + (-1)^{|a|+1}\{a, \Delta(b)\}.$$

Remark 1.1. Notice that not every odd Poisson algebra A admits a generator. For instance, a nontrivial odd Lie algebra seen as an odd Poisson algebra with trivial multiplication admits no generator. Moreover, even if a generator Δ for an odd Poisson algebra exists, it is far from being unique. In fact, all different generators are obtained by adding to Δ a derivation of A of degree $-n$.

Definition 1.2. An n-Poisson algebra A is called an n-Batalin–Vilkovisky algebra, if it is endowed with an exact generator.

Notation. When $n = 1$ it is customary to speak of Batalin–Vilkovisky algebras, or simply BV algebras; see [2, 9, 17].

There exists a characterization of n-Batalin–Vilkovisky algebras in terms of the product and the generator only [9, 17]. Suppose in fact that a graded vector space A is endowed with a degree zero graded commutative product and a linear map $\Delta\colon A \to A[-n]$ such that $\Delta^2 = 0$, satisfying the following "seven-term" relation:

$$\begin{aligned}\Delta(a\cdot b\cdot c) + \Delta(a)\cdot b\cdot c + (-1)^{|a|}a\cdot\Delta(b)\cdot c + (-1)^{|a|+|b|}a\cdot b\cdot\Delta(c) = \\ = \Delta(a\cdot b)\cdot c + (-1)^{|a|}a\cdot\Delta(b\cdot c) + (-1)^{(|a|+1)|b|}b\cdot\Delta(a\cdot c).\end{aligned}$$

In other words, Δ is a derivation of order 2.

Then, if we define the bilinear operation $\{,\}\colon A\otimes A \to A[-n]$ by

$$\{a,b\} = (-1)^{|a|}\left(\Delta(a\cdot b) - \Delta(a)\cdot b - (-1)^{|a|}a\cdot\Delta(b)\right),$$

we have that the quadruple $(A,\cdot,\{,\},\Delta)$ is an n-Batalin–Vilkovisky algebra. Vice versa, one easily checks that the product and the generator of an n-Batalin–Vilkovisky algebra satisfy the above "seven term" relation.

2. Examples

2.1. **Schouten–Nijenhuis bracket.** Suppose $\mathfrak{g}$ is a graded Lie algebra of degree zero. Then $A = S(\mathfrak{g}[n])$ is an n-Poisson algebra with its natural multiplication (the one induced from the tensor algebra $T(A)$) and a degree $-n$ bracket, often called the Schouten–Nijenhuis bracket, defined as follows [17, 18]: the bracket on $S^1(\mathfrak{g}[n]) = \mathfrak{g}[n]$ is defined as the suspension of the bracket on $\mathfrak{g}$, while on $S^k(\mathfrak{g}[n])$, for $k > 1$, the bracket is defined inductively by forcing the Leibniz rule

$$\{a, b\cdot c\} = \{a,b\}\cdot c + (-1)^{|b|(|a|+n)}b\cdot\{a,c\}.$$

Moreover, when n is odd, there exists a generator defined as

$$\Delta(a_1\cdot a_2\cdots a_k) = \sum_{i<j}(-1)^{\epsilon}\{a_i,a_j\}\cdot a_1\cdots\widehat{a_i}\cdots\widehat{a_j}\cdots a_k$$

where $a_1,\dots,a_k\in\mathfrak{g}$ and $\epsilon = |a_i| + (|a_i|+1)(|a_1|+\cdots|a_{i-1}|+i-1) + (|a_j|+1)(|a_1|+\cdots|\widehat{a_i}|+\cdots|a_{j-1}|+j-2)$. An easy check shows that $\Delta^2 = 0$, thus $S(\mathfrak{g}[n])$ is an n-Batalin–Vilkovisky algebra for every odd $n\in\mathbb{N}$. For $n = 1$ the Δ-cohomology on $\bigwedge\mathfrak{g}$ is the usual Cartan–Chevalley–Eilenberg cohomology.

In particular, one can consider the Lie algebra $\mathfrak{g} = \mathrm{Der}(B) = \oplus_{j\in\mathbb{Z}}\mathrm{Der}^j(B)$ of derivations of a graded commutative algebra B. More explicitly, $\mathrm{Der}^j(B)$ consists of linear maps $\phi\colon B\to B$ of degree j such that $\phi(ab) = \phi(a)b + (-1)^{j\,|a|}a\phi(b)$ and the bracket is $\{\phi,\psi\} = \phi\circ\psi - (-1)^{|\phi||\psi|}\psi\circ\phi$. The space of multiderivations

$S(\mathrm{Der}(B)[1])$, endowed with the Schouten–Nijenhuis bracket, is a Gerstenhaber algebra.

We can further specialize to the case when B is the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a smooth manifold M; then $\mathfrak{X}(M) = \mathrm{Der}(\mathcal{C}^\infty(M))$ is the space of vector fields on M and $\mathcal{V}(M) = S(\mathfrak{X}(M))[1])$ is the space of multivector fields on M. It is a classical result by Koszul [17] that there is a bijective correspondence between generators for $\mathcal{V}(M)$ and connections on the highest exterior power $\bigwedge^{\dim M} TM$ of the tangent bundle of M. Moreover, flat connections correspond to generators which square to zero.

2.2. **Lie algebroids.** A Lie algebroid E over a smooth manifold M is a vector bundle E over M together with a Lie algebra structure (over $\mathbb{R}$) on the space $\Gamma(E)$ of smooth sections of E, and a bundle map $\rho\colon E \to TM$, called the anchor, extended to a map ρ_* between sections of these bundles, such that

$$\{X, fY\} = f\{X, Y\} + (\rho_*(X)f)Y$$

for any smooth sections X and Y of E and any smooth function f on M. In particular, the anchor map on sections $\rho_*\colon \Gamma(E) \to \mathfrak{X}(M)$ is a morphism of Lie algebras , namely $\rho_*(\{X, Y\}) = \{\rho_*(X), \rho_*(Y)\}$.

The link between Lie algebroids and Gerstenhaber algebras is given by the following Proposition [13, 31]:

Proposition 2.1. *Given a vector bundle E over M, there exists a one-to-one correspondence between Gerstenhaber algebra structures on $A = \Gamma(\bigwedge(E))$ and Lie algebroid structures on E.*

The key of the Proposition is that one can extend the Lie algebroid bracket to a unique graded antisymmetric bracket on $\Gamma(\bigwedge(E))$ such that $\{X, f\} = \rho(X)f$ for $X \in \Gamma(\bigwedge^1(E))$ and $f \in \Gamma(\bigwedge^0(E))$, and that for $Q \in \Gamma(\bigwedge^{q+1}(E))$, $\{Q, \cdot\}$ is a derivation of $\Gamma(\bigwedge(E))$ of degree q.

Example 2.1. A finite dimensional Lie algebra $\mathfrak{g}$ can be seen as a Lie algebroid over a trivial base manifold. The corresponding Gerstenhaber algebra is the one of subsection 2.1.

Example 2.2. The tangent bundle TM of a smooth manifold M is a Lie algebroid with anchor map given by the identity and algebroid Lie bracket given by the usual Lie bracket on vector fields. In this case we recover the Gerstenhaber algebra of multivector fields on M described in subsection 2.1.

Example 2.3. If M is a Poisson manifold with Poisson bivector field α, then the cotangent bundle T^*M inherits a natural Lie algebroid structure where the anchor map $\alpha^\#\colon T_p^*M \to T_pM$ at the point $p \in M$ is given by $\alpha^\#(\xi)(\eta) = \alpha(\xi, \eta)$, with $\xi, \eta \in T_p^*M$, and the Lie bracket of the 1-forms ω_1 and ω_2 is given by

$$\{\omega_1, \omega_2\} = \mathrm{L}_{\alpha^\#(\omega_1)}\, \omega_2 - \mathrm{L}_{\alpha^\#(\omega_2)}\, \omega_1 - \mathrm{d}\alpha(\omega_1, \omega_2)$$

The associated Gerstenhaber algebra is the de Rham algebra of differential forms endowed with the bracket defined by Koszul in [17]. As shown in [13], $\Gamma(\bigwedge(T^*M))$ is indeed a BV algebra with an exact generator $\Delta = [\mathrm{d}, \iota_\alpha]$ given by the commutator of the contraction ι_α with the Poisson bivector α and the de Rham differential d. Similar results hold if M is a Jacobi manifold.

It is natural to ask what additional structure on a Lie algebroid E makes the Gerstenhaber algebra $\Gamma(\bigwedge(E))$ into a BV algebra. The answer is given by following result, which is proved in [31]

Proposition 2.2. *Given a Lie algebroid E, there is a one-to-one correspondence between generators for the Gerstenhaber algebra $\Gamma(\bigwedge(E))$ and E-connections on $\bigwedge^{\mathrm{rk}E} E$ (where $\mathrm{rk}E$ denotes the rank of the vector bundle E). Exact generators correspond to flat E-connections, and in particular, since flat E-connections always exist, $\Gamma(\bigwedge(E))$ is always a BV algebra.*

For the appearance of Gerstenhaber algebras in the theory of Lie bialgebras and Lie bialgebroids, see [19].

2.3. **Lie algebroid cohomology.** A Lie algebroid structure on $E \to M$ defines a differential δ on $\Gamma(\bigwedge E^*)$ by

$$\delta f := \rho^* \mathrm{d} f, \quad f \in C^\infty(M) = \Gamma(\Lambda^0 E^*)$$

and

$$\langle \delta\sigma \,,\, X \wedge Y \rangle := \langle \delta \langle \sigma \,,\, X \rangle \,,\, Y \rangle - \langle \delta \langle \sigma \,,\, Y \rangle \,,\, X \rangle - \langle \sigma \,,\, \{X, Y\} \rangle \,, \qquad X,\, Y \in \Gamma(E),\ \sigma \in \Gamma(E^*),$$

where $\rho^* \colon \Omega^1(M) \to \Gamma(E^*)$ is the transpose of $\rho_* \colon \Gamma(E) \to \mathfrak{X}(M)$ and $\langle\ ,\ \rangle$ is the canonical pairing of sections of E^* and E. On $\Gamma(\bigwedge^n E^*)$, with $n \geq 2$, the differential δ is defined by forcing the Leibniz rule.

In example 2.1 we get the Cartan–Chevalley–Eilenberg differential on $\bigwedge \mathfrak{g}^*$; in example 2.2 we recover the de Rham differential on $\Omega^*(M) = \Gamma(\bigwedge T^*M)$, while in example 2.3 the differential on $\mathcal{V}(M) = \Gamma(\bigwedge TM)$ is $\{\alpha,\ \}_{SN}$.

2.4. **Lie–Rinehart algebras.** The algebraic generalization of a Lie algebroid is a Lie–Rinehart algebra. Recall that given a commutative associative algebra B (over some ring R) and a B-module $\mathfrak{g}$, then a Lie–Rinehart algebra structure on $(B, \mathfrak{g})$ is a Lie algebra structure (over R) on $\mathfrak{g}$ and an action of $\mathfrak{g}$ on the left on B by derivations, satisfying the following compatibility conditions:

$$\{\gamma, a\sigma\} = \gamma(a)\sigma + a\{\gamma, \sigma\}$$
$$(a\gamma)(b) = a(\gamma(b))$$

for every $a, b \in B$ and $\gamma, \sigma \in \mathfrak{g}$.

The Lie–Rinehart structures on the pair $(B, \mathfrak{g})$ bijectively correspond to the Gerstenhaber algebra structures on the exterior algebra $\bigwedge_B(\mathfrak{g})$ of $\mathfrak{g}$ in the category of B-modules. When $\mathfrak{g}$ is of finite rank over B, generators for these structures are in turn in bijective correspondence with $(B, \mathfrak{g})$-connections on $\bigwedge_B^{\mathrm{rk}_B \mathfrak{g}} \mathfrak{g}$, and flat connections correspond to exact generators. For additional discussions, see [8, 12].

Lie algebroids are Lie–Rinehart algebras in the smooth setting. Namely, if $E \to M$ is a Lie algebroid, then the pair $(\mathcal{C}^\infty(M), \Gamma(E))$ is a Lie–Rinehart algebra (with action induced by the anchor and the given Lie bracket).

2.5. **Lie–Rinehart cohomology.** Lie algebroid cohomology may be generalized to every Lie–Rinehart algebra $(B, \mathfrak{g})$. Namely, on the complex is $\mathrm{Alt}_B(\mathfrak{g}, B)$ of alternating multilinear functions on $\mathfrak{g}$ with values in B, one can define a differential δ by the rules

$$\langle \delta a, \gamma\rangle = \gamma(a), \qquad a \in B = \mathrm{Alt}^0_B(\mathfrak{g}, B),\ \gamma \in \mathfrak{g},$$

$$\langle \delta a, \gamma \wedge \sigma\rangle = \langle \delta\langle a, \gamma\rangle, \sigma\rangle - \langle \delta\langle a, \sigma\rangle, \gamma\rangle - \langle a, \{\gamma, \sigma\}\rangle, \quad \gamma,\, \sigma \in \mathfrak{g},\ a \in \mathrm{Alt}^1_B(\mathfrak{g}, B),$$

and forcing the Leibniz rule on elements of $\mathrm{Alt}^n_B(\mathfrak{g}, B)$, $n \geq 2$.

2.6. **Hochschild cohomology.** Let A be an associative algebra with product μ, and consider the Hochschild cochain complex $\mathsf{Hoch}(A) = \prod_{n\geq 0} \mathrm{Hom}(A^{\otimes n}, A)[-n+1]$. There are two basic operations between two elements $f \in \mathrm{Hom}(A^{\otimes k}, A)[-k+1]$ and $g \in \mathrm{Hom}(A^{\otimes l}, A)[-l+1]$, namely a degree zero product

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{k+l}) = (-1)^{kl} f(a_1 \otimes \cdots \otimes a_l)\, g(a_{l+1} \otimes \cdots \otimes a_{k+l})$$

and a degree -1 bracket $\{f, g\} = f \circ g - (-1)^{(k-1)(l-1)} g \circ f$, where

$$\begin{aligned}(f \circ g)(a_1 \otimes \cdots \otimes a_{k+l-1}) &= \\ &= \sum_{i=1}^{k-1} (-1)^{i(l-1)} f(a_1 \otimes \cdots \otimes a_i \otimes g(a_{i+1} \otimes \cdots \otimes a_{i_l}) \otimes \cdots \otimes a_{k+l-1}).\end{aligned}$$

It is well known from [7] that the cohomology $\mathsf{HHoch}(A)$ of the Hochschild complex with respect to the differential $\mathrm{d}_{\mathsf{Hoch}} = \{\mu, \cdot\}$ has the structure of a Gerstenhaber algebra. More generally, there is a Gerstenhaber algebra structure on Hochschild cohomology of differential graded associative algebras [21] and of A_∞ algebras. Moreover, if A is endowed with a symmetric, invariant and non-degenerate inner product, then $\mathsf{HHoch}(A)$ is also a BV algebra [28].

2.7. **Graded symplectic manifolds.** The construction of Example 1.3 can be extended to graded symplectic manifolds; see [1, 9, 26]. Recall that a symplectic structure of degree n on a graded manifold N is a closed nondegenerate two-form ω such that $L_E\omega = n\omega$ where L_E is the Lie derivative with respect to the Euler field of N (see [24] for details). Let us denote by X_h the vector field associated to the function $h \in \mathcal{C}^\infty(N)$ by the formula $\iota_{X_h}\omega = \mathrm{d}h$. Then the bracket

$$\{f, g\} = \iota_{X_f}\iota_{X_g}\omega$$

gives $\mathcal{C}^\infty(N)$ the structure of a graded Poisson algebra of a degree n.

If the symplectic form has odd degree and the graded manifold has a volume form, then it is possible to construct an exact generator defined by

$$\Delta(f) = \frac{1}{2}\mathrm{div}(X_f)$$

where div is the divergence operator associated to the given volume form [9, 15].

An explicit characterization of graded symplectic manifolds has been given in [24]. In particular it is proved there that every symplectic form of degree n with $n \geq 1$ is necessarily exact. More precisely, one has $\omega = \mathrm{d}(\iota_E\omega/n)$.

2.8. **Shifted cotangent bundle.** The main examples of graded symplectic manifolds are given by shifted cotangent bundles. If N is a graded manifold then the shifted cotangent bundle $T^*[n]N$ is the graded manifold obtained by shifting by n the degrees of the fibers of the cotangent bundle of N. This graded manifold possesses a non degenerate closed two-form of degree n, which can be expressed in local coordinates as

$$\omega = \sum_i \mathrm{d}x^i \wedge \mathrm{d}x_i^\dagger$$

where $\{x^i\}$ are local coordinates on N and $\{x_i^\dagger\}$ are coordinate functions on the fibers of $T^*[n]N$. In local coordinates, the bracket between two homogeneous functions f and g is given by

$$\{f, g\} = -(-1)^{|x_i^\dagger|\,|f|} \frac{\partial f}{\partial x_i^\dagger} \frac{\partial g}{\partial x^i} - (-1)^{(|f|+n)(|g|+n)+|x_i^\dagger|\,|g|} \frac{\partial g}{\partial x_i^\dagger} \frac{\partial f}{\partial x^i}$$

If in addition the graded manifold N is orientable, then $T^*[n]N$ has a volume form too; when n is odd, the exact generator $\Delta(f) = \frac{1}{2}\mathrm{div}X_f$, already considered in [17], is written in local coordinates as

$$\Delta = \frac{\partial}{\partial x_i^\dagger} \frac{\partial}{\partial x^i}.$$

In the case $n = -1$, we have a natural identification between functions on $T^*[-1]N$ and multivector fields $\mathcal{V}(N)$ on N and we recover again the Gerstenhaber algebra of subsection 2.1. Moreover it is easy to see that, under the above identification, Δ applied to a vector field of N is the usual divergence operator.

2.9. **Examples from algebraic topology.** For any $n > 1$, the homology of the n-fold loop space $\Omega^n(M)$ of a topological space M has the structure of an $(n-1)$-Poisson algebra [22]. In particular the homology of the double loop space $\Omega^2(M)$ is a Gerstenhaber algebra, and has an exact generator defined using the natural circle action on this space [9]. The homology of the free loop space $\mathcal{L}(M)$ of a closed oriented manifold M is also a BV algebra when endowed with the "Chas–Sullivan intersection product" and with a generator defined again using the natural circle action on the free loop space. Recently it has been shown that the homology of the space of framed embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{n+k}$ is an n-Poisson algebra.

2.10. **Shifted Poisson structures on derived stacks.** In recent years, a vast generalization of the notion of Poisson structure on a smooth manifold has been introduced in the context of derived algebraic geometry by Calaque, Pantev, Töen, Vaquié and Vezzosi who considered shifted Poisson structures on derived Artin stacks [6]. Basic examples are the 1-shifted and 2-shifted Poisson structures on BG given by elements in $(\wedge^3\mathfrak{g})^{\mathfrak{g}}$ and in $(S^2\mathfrak{g})^{\mathfrak{g}}$, respectively, where G is a reductive algebraic group and $\mathfrak{g}$ is its Lie algebra. The corresponding notion for smooth stacks presented by Lie groupoids has then been investigated by Bonechi, Ciccoli, Laurent-Gengoux and Xu in [5], and by Ginot, Ortiz and Stefani in [10].

One starts by considering, for any $m \in \mathbb{Z}$ the differential graded commutative algebra

$$\mathsf{MultiVect}(X, m) = \bigoplus_{p \geq 0} \Gamma(X, S^p(\mathbb{T}_X[-m])$$

of m-shifted multivector fields on X. Here $\mathbb{T}_X$ denotes the tangent complex of the stack X and Γ is the derived global sections (i.e., hypercohomology) functor. The algebra $\mathsf{MultiVect}(X, m)$ is bigraded, with one grading given by the cohomological degree and the other, called the weight given by the exponent p. The main result is then that $\mathsf{MultiVect}(X, m)$ carries a natural $\mathbb{P}_{m+1}$-commutative differential graded algebra structure, where $\mathbb{P}_{m+1}$ is the dg-operad whose algebras are Poisson cdga's with a bracket of degree $-m$. Equivalently, (commutative) $\mathbb{P}_{m+1}$-algebras can be inductively defined as (commutative) associative algebras in $\mathbb{P}_m$-algebras [25]. Moreover, the Poisson bracket on $\mathbb{P}_{m+1}$ has weight -1. Therefore, $\mathsf{MultiVect}(X, m)[m]$ is a dg-Lie algebra with a Lie bracket of weight -1, and one can define the space $\mathsf{Poiss}(X, n)$ of n-shifted Poisson structures on X as

$$\mathsf{Poiss}(X, n) = \mathsf{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(\mathbb{K}(2)[-1], \mathsf{MultiVect}(X, n+1)[n+1]),$$

where $\mathsf{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}$ is the Hom-space in the $(\infty, 1)$-category of graded dg-Lie algebras over the (characteristic zero) field of definition $\mathbb{K}$ of the stack X, and $\mathbb{K}(2)[-1]$ is the graded dg-Lie algebra consisting of $\mathbb{K}$ in pure cohomological degree 1, pure weight 2, and trivial bracket. When X is a smooth underived scheme, $\mathsf{Poiss}(X, 0)$ is the usual set of Poisson bivectors on X. Moreover, one has a natural notion of nondegenerate shifted Poisson strutures and an equivalence $\mathsf{Poiss}(X, n)^{\mathsf{nd}} \simeq \mathsf{Symp}(X, n)$ between the space of nondegenerate n-shifted Poisson structures on X and the space of n-shifted symplectic structures on X as defined in [23]

3. Applications

3.1. **BRST quantization in the Hamiltonian formalism.** The BRST procedure, after Becchi–Rouet–Stora and Tyutin [3, 4, 29], is a method for quantizing classical mechanical systems or classical field theories in the presence of symmetries. We describe here its classical, Hamiltonian counterpart following [27]. The starting point is a symplectic manifold M (the "phase space"), a function H (the "Hamiltonian" of the system) governing the evolution of the system, and the "constraints" given by several functions g_i which Poisson commute with H and among each other up to a $\mathcal{C}^\infty(M)$-linear combination of the g_i's.

Then the dynamics is constrained on the locus V of common zeroes of the g_i's. When V is a submanifold, the g_i's are a set of generators for the ideal I of functions vanishing on V. Observe that I is closed under the Poisson bracket. Functions in I are called "first class constraints." The Hamiltonian vector fields of first class constraints, which are by construction tangential to V, are the "symmetries" of the system.

When V is smooth, then it is a coisotropic submanifold of M and the Hamiltonian vector fields determined by the constraints give a foliation $\mathcal{F}$ of V. In the nicest case V is a principal bundle with $\mathcal{F}$ its vertical foliation and the algebra of functions $\mathcal{C}^\infty(V/\mathcal{F})$ on the "reduced phase space" $V/\mathcal{F}$ is identified with the I-invariant subalgebra of $\mathcal{C}^\infty(M)/I$.

From a physical point of view, the points of $V/\mathcal{F}$ are the interesting states at a classical level, and a quantization of this system means a quantization of $\mathcal{C}^\infty(V/\mathcal{F})$. The BRST procedure gives a method of quantizing $\mathcal{C}^\infty(V/\mathcal{F})$ starting from the (known) quantization of $\mathcal{C}^\infty(M)$. Notice that these notions immediately generalize to graded symplectic manifolds.

From an algebraic point of view, one starts with a graded Poisson algebra P and a multiplicative ideal I which is closed under the Poisson bracket. The algebra of functions on the "reduced phase space" is replaced by $(P/I)^I$, the I-invariant subalgebra of P/I. This subalgebra inherits a Poisson bracket even if P/I does not. Moreover the pair $(B, \mathfrak{g}) = (P/I, I/I^2)$ inherits a graded Lie–Rinehart structure. The "Rinehart complex" $\mathrm{Alt}_{P/I}(I/I^2, P/I)$ of alternating multilinear functions on I/I^2 with values in P/I, endowed with the differential described in 2.5, plays the role of the de Rham complex of vertical forms on V with respect to the foliation $\mathcal{F}$ determined by the constraints.

In case V is a smooth submanifold, we also have the following geometric interpretation: Let N^*V denote the conormal bundle of V (i.e., the annihilator of TV in T_VP). This is a Lie subalgebroid of T^*P if and only if V is coisotropic. Since we may identify I/I^2 with sections of N^*C (by the de Rham differential), $(P/I, I/I^2)$ is the corresponding Lie–Rinehart pair. The Rinehart complex is then the corresponding Lie algebroid complex $\Gamma(\bigwedge(N^*V)^*)$ with differential described in 2.3. The image of the anchor map $N^*V \to TV$ is the distribution determining $\mathcal{F}$, so by duality we get an injective chain map form the vertical de Rham complex to the Rinehart complex.

The main point of the BRST procedure is to define a chain complex $C^\bullet = \bigwedge(\Psi^* \oplus \Psi) \otimes P$, where Ψ is a graded vector space, with a coboundary operator D (the "BRST operator"), and a quasi-isomorphism (i.e., a chain map that induces an isomorphism in cohomology)

$$\pi \colon (C^\bullet, D) \to \left(\mathrm{Alt}_{P/I}(I^2/I, P/I), \mathrm{d}\right).$$

This means in particular that the zeroth cohomology $H^0_D(C)$ gives the algebra $(P/I)^I$ of functions on the "reduced phase space". Observe that there is a natural symmetric inner product on $\Psi^* \oplus \Psi$ given by the evaluation of Ψ^* on Ψ. This inner product, as an element of $S^2(\Psi \oplus \Psi^*) \simeq S^2(\Psi) \oplus (\Psi \otimes \Psi^*) \oplus S^2(\Psi^*)$, is concentrated in the component $\Psi \otimes \Psi^*$, and so it defines an element in $\bigwedge^2(\Psi[1] \oplus \Psi^*[-1]) \simeq S^2(\Psi)[2] \oplus (\Psi \otimes \Psi^*) \oplus S^2(\Psi^*)[-2]$, i.e., a degree zero bivector field on $\Psi[1] \oplus \Psi^*[-1]$. It is easy to see that this bivector field induces a degree zero Poisson structure on $S(\Psi^*[-1] \oplus \Psi[1])$. From another viewpoint this is the Poisson structure corresponding to the canonical symplectic structure on $T^*\Psi[1]$. Finally, we have that $S(\Psi^*[-1] \oplus \Psi[1]) \otimes P$ is a degree zero Poisson algebra. Note that the superalgebra underlying the graded algebra $S(\Psi^*[-1] \oplus \Psi[1]) \otimes P$ is canonically isomorphic to the complex $C^\bullet = \bigwedge(\Psi^* \oplus \Psi) \otimes P$. When $P = \mathcal{C}^\infty(M)$, we can think of $S(\Psi^*[-1] \oplus \Psi[1]) \otimes \mathcal{C}^\infty(M)$ as the algebra of functions on the graded symplectic manifold $N = (\Psi[1] \oplus \Psi^*[-1]) \times M$ (the "extended phase space"). In physical language, coordinate functions on $\Psi[1]$ are called "ghost fields" while coordinate functions on $\Psi^*[-1]$ are called "ghost momenta" or, by some authors, "antighost fields" (not to be confused with the antighosts of the Lagrangian functional-integral approach to quantization).

Suppose now that there exists an element $\Theta \in S(\Psi^*[-1] \oplus \Psi[1]) \otimes P$ such that $\{\Theta, \cdot\} = D$, that one can extend the "known" quantization of P to a quantization of $S(\Psi^*[-1] \oplus \Psi[1]) \otimes P$ as operators on some (graded) Hilbert space $\mathcal{T}$ and that the operator Q which quantizes Θ has square zero. Then one can consider the "true space of physical states" $H^0_Q(\mathcal{T})$ on which the ad_Q-cohomology of operators will act. This provides one with a quantization of $(P/I)^I$.

For further details on this procedure, and in particular for the construction of D, we refer to [11, 16, 27] and references therein. Observe that some authors refer to this method as BVF [Batalin–Vilkovisky–Fradkin] and reserve the name BRST to the case when the g_i's are the components of an equivariant moment map.

For a generalization to graded manifolds different from $(\Psi[1] \oplus \Psi^*[-1]) \times M$ we refer to [24]. There it is proved that the element Θ exists if the graded symplectic form has degree different from -1.

3.2. **BV quantization in the Lagrangian formalism.** The BV formalism is a procedure for the quantization of physical systems with symmetries in the Lagrangian formalism, see [2, 11]. As a first step, the "configuration space" M of the system is augmented by the introduction of "ghosts". If G is the group of symmetries, this means that one has to consider the graded manifold $W = \mathfrak{g}[1] \times M$. The second step is to double this space by introducing "antifields for fields and ghosts", namely one has to consider the "extended configuration space" $T^*[-1]W$, whose space of functions is a BV algebra by subsection 2.8. The algebra of "observables" of this physical system is by definition the cohomology $H^*_\Delta(\mathcal{C}^\infty(T^*[-1]W))$ with respect to the exact generator Δ.

4. Related topics

4.1. **AKSZ.** The graded manifold $T^*[-1]W$ considered in 3.2 is a particular example of a QP-manifold, i.e., of a graded manifold M endowed with an integrable (i.e., selfcommuting) vector field Q of degree 1 and a graded Q-invariant symplectic structure P. In quantization of classical mechanical theories, the graded symplectic manifold of interest is the space of fields and antifields with symplectic form of degree 1, while Q is the Hamiltonian vector field defined by the action functional S; the integrability of Q is equivalent to the classical master equation $\{S, S\} = 0$ for the action functional. Quantization of the theory is then reduced to the computation of the functional integral $\int_{\mathcal{L}} \exp(iS/\hbar)$, where $\mathcal{L}$ is a Lagrangian submanifold of M. This functional integral actually depends only on the homology class of the Lagrangian. Locally, a QP-manifold is a shifted cotangent bundle $T^*[-1]N$ and a Lagrangian submanifold is the graph of an exact 1-form. In the notations of 2.8, a Lagrangian submanifold $\mathcal{L}$ is therefore locally defined by equations $x_i^\dagger = \partial\Phi/\partial x^i$, and the function Φ is called a fixing fermion. The action functional of interest is then the gauge-fixed action $S\big|_{\mathcal{L}} = S(x^i, \partial\Phi/\partial x^i)$.

The language of QP-manifolds has powerful applications to sigma-models: if Σ is a finite-dimensional graded manifold equipped with a volume element, and M is a QP manifold, then the graded manifold $C^\infty(\Sigma, M)$ of smooth maps from Σ to M has a natural structure of QP-manifold which describes some field theory if one arranges for the symplectic structure to be of degree 1. As an illustrative example, if $\Sigma = T[1]X$, for a compact oriented 3-dimensional smooth manifold X, and $M = \mathfrak{g}[1]$, where $\mathfrak{g}$ is the Lie algebra of a compact Lie group, the QP-manifold $C^\infty(\Sigma, M)$ is relevant to Chern-Simons theory on X. Similarly, if $\Sigma = T[1]X$, for a compact oriented 2-dimensional smooth manifold X and $M = T[1]N$ is the shifted tangent bundle of a symplectic manifold, then the QP-structure on $C^\infty(\Sigma, M)$ is related to the A-model with target N; if the symplectic manifold N is of the form $N = T^*K$ for a complex manifold K, then one can endow $C^\infty(\Sigma, M)$ with a complex QP-manifold structure, which is related to the B-model with target K; this shows that, in some sense, the B-model can be obtained from the A-model by "analytic continuation" [1]. If $\Sigma = T[1]X$, for a compact oriented 2-dimensional smooth manifold X and $M = T^*[-1]N$ with canonical symplectic structure, then the QP-structure on $C^\infty(\Sigma, M)$ is related to the Poisson sigma model (QP-structures on $T^*[-1]N$ with canonical symplectic structure are in one-to-one correspondence with Poisson structures on N). The study of QP manifolds is sometimes referred to as "the AKSZ formalism". In [24] QP-manifolds with symplectic structure of degree 2 are studied and shown to be in one-to-one correspondence with Courant algebroids.

4.2. **Graded Poisson algebras from cohomology of P_∞.** The Poisson bracket on a Poisson manifold can be derived from the Poisson bivector field α using the Schouten–Nijenhuis bracket as follows [14]:

$$\{f, g\} = \{\{\alpha, f\}_{SN}, g\}_{SN}.$$

This may be generalized [30] to the case of a graded manifold M endowed with a multivector field α of total degree 2 (i.e., $\alpha = \sum_{i=0}^{\infty} \alpha_i$, where α_i is an i-vector field of degree $2-i$) satisfying the equation $\{\alpha, \alpha\}_{SN} = 0$. One then has the derived multibrackets

$$\lambda_i \colon A^{\otimes i} \to A,$$
$$\lambda_i(a_1, \ldots, a_i) := \{\{\ldots\{\{\alpha_i, a_1\}_{SN}, a_2\}_{SN} \ldots\}_{SN}, a_i\}_{SN},$$

with $A = \mathcal{C}^\infty(M)$. Observe that λ_i is a multiderivation of degree $2-i$. The operations λ_i define the structure of an L_∞-algebra on A. Such a structure is called a P_∞-algebra (P for Poisson) since the λ_i's are multiderivations. If $\lambda_0 = \alpha_0$ vanishes, then λ_1 is a differential, and the λ_1-cohomology inherits a graded Poisson algebra structure. This structure can be used to describe the deformation quantization of coisotropic submanifolds and to describe their deformation theory.

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