DOOB'S MAXIMAL IDENTITY, MULTIPLICATIVE DECOMPOSITIONS AND ENLARGEMENTS OF FILTRATIONS

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In the memory of J.L. Doob

Abstract. In the theory of progressive enlargements of filtrations, the supermartingale $Z_t = \mathbf{P}(g > t \mid \mathcal{F}_t)$ associated with an honest time g, and its additive (Doob-Meyer) decomposition, play an essential role. In this paper, we propose an alternative approach, using a multiplicative representation for the supermartingale Z_t, based on Doob's maximal identity. We thus give new examples of progressive enlargements. Moreover, we give, in our setting, a proof of the decomposition formula for martingales , using initial enlargement techniques, and use it to obtain some path decompositions given the maximum or minimum of some processes.

1. Introduction

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$ be a filtered probability space satisfying the usual hypotheses (right continuous and complete). Given the end L of an $(\mathcal{F}_t)$ predictable set Γ, i.e

$$L = \sup\{t : (t,\omega) \in \Gamma\},$$

(these times are also refered to as honest times), M. Barlow ([4]) and Jeulin and Yor ([10]) have shown that the supermartingale:

$$Z_t^L = \mathbf{P}(L > t \mid \mathcal{F}_t),$$

chosen to be càdlàg, plays an essential role in the enlargement formulae with respect to L, i.e: in expressing a general $(\mathcal{F}_t)$ martingale (M_t) as a semimartingale in $\left(\mathcal{F}_t^L\right)_{t\geq 0}$, the smallest filtration which contains $(\mathcal{F}_t)$, and makes L a stopping time. This enlargement formula is:

$$M_t = \widetilde{M}_t + \int_0^{t\wedge L} \frac{d < M, Z >_s}{Z_{s_-}} + \int_L^t \frac{d < M, 1 - Z >_s}{1 - Z_{s_-}}, \qquad (1.1)$$

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where $\left(\widetilde{M}_t\right)_{t\geq 0}$ denotes an $\left(\left(\mathcal{F}_t^L\right), \mathbf{P}\right)$ local martingale. Hence it is important to dispose of an explicit formula for $\left(Z_t^L\right)_{t\geq 0}$. In the literature about progressive enlargements of filtrations, not so many examples are fully developed (see e.g. for example [27], [9] or [8]); indeed, the computation of $\left(Z_t^L\right)$ is sometimes difficult. Moreover, the examples are developed essentially in the Brownian setting, where as we shall see, $\left(Z_t^L\right)$ is continuous, and no examples of discontinuous $\left(Z_t^L\right)' s$ are given.

In this paper, we first consider a special family of honest times g, and then we later prove that this family is generic in the sense that every honest time is in fact of this form (under some reasonable assumptions).

More precisely, we consider the following class of local martingales.

Definition 1.1. We say that an $(\mathcal{F}_t)$ local martingale (N_t) belongs to the class $(\mathcal{C}_0)$, if it is strictly positive, with no positive jumps, and $\lim_{t\to\infty} N_t = 0$.

Remark 1.2. Let (N_t) be a local martingale of class $(\mathcal{C}_0)$. Then:

$$S_t \equiv \sup_{s\leq t} N_s,$$

its supremum process, is continuous. This property is essential in our paper. Hence, most of the results we shall state remain valid for positive local martingales, which go to zero at infinity, and whose suprema are continuous.

We associate with a local martingale of class $(\mathcal{C}_0)$, the supermartingale $\left(\frac{N_t}{S_t}\right)_{t\geq 0}$, and the random time g defined as:

$$\begin{aligned} g &\equiv \sup\{t\geq 0: \quad N_t = S_\infty\} \\ &= \sup\{t\geq 0: \quad S_t - N_t = 0\}. \end{aligned}$$

In Section 2, we prove that the associated supermartingale Z satisfies:

$$Z_t \equiv \mathbf{P}\left(g > t \mid \mathcal{F}_t\right) = \frac{N_t}{S_t}, \tag{1.2}$$

and then give the decomposition formula (1.1) in terms of the local martingale (N_t). This will provide us with some new, and explicit examples of such supermartingales (Z_t) which are discontinuous. We also establish some relationship between the multiplicative representation (1.2) and the Doob-Meyer (additive) decomposition of (Z_t).

In Section 3, we study the problem of the initial enlargement of $(\mathcal{F}_t)$ with the variable S_∞, and then give a new proof of (1.1).

In Section 4, we show that the formula (1.2) is in fact very general. More precisely, for any end of a predictable set L, under the assumptions **(CA)**:

- all $(\mathcal{F}_t)$-martingales are **c**ontinuous (e.g: the Brownian filtration);
- L **a**voids every $(\mathcal{F}_t)$ -stopping time T, i.e. $P[L = T] = 0$,

the supermartingale $Z_t^L = \mathbf{P}(L > t \mid \mathcal{F}_t)$ may be represented as (1.2).

In Section 5, we give some new examples of enlargements of filtrations. Moreover, as an illustration of our approach and the method of enlargements of filtrations, we recover and complete some known results of D. Williams ([24]) about path decompositions of some diffusion processes, given their minima. We add a new fragment in these path decompositions, by introducing a new family of random times, as defined in [16] and called pseudo-stopping times, which generalize the fundamental notion of stopping times, introduced by J.L. Doob. We take this opportunity to quote two passages, resp. in the appendix of Meyer's book (1966):

> Les temps d'arrêt ont été utilisés, sans définition formelle, depuis le début de la théorie des processus. La notion apparaît tout à fait clairement pour la première fois chez Doob en 1936.

and in Dellacherie-Meyer's book, volume I ([6]), p.184: 0194

> Il a sans doute fallu autant de génie aux créateurs du calcul différentiel pour expliciter la notion si simple de dérivée, qu'à leurs successeurs pour faire tout le reste. L'invention des temps d'arrêt par Doob est tout à fait comparable.

2. A multiplicative representation formula

2.1. **Doob's maximal identity.** Let $(N_t)_{t\geq 0}$ be a local martingale which belongs to the class $(\mathcal{C}_0)$, with $N_0 = x$. Let $S_t = \sup_{s\leq t} N_s$. We consider:

$$\begin{aligned} g &= \sup\{t \geq 0 : \quad N_t = S_\infty\} \\ &= \sup\{t \geq 0 : \quad S_t - N_t = 0\}. \end{aligned} \tag{2.1}$$

To establish our main proposition, we shall need the following variant of Doob's maximal inequality, which we call Doob's maximal identity:

Lemma 2.1 (Doob's maximal identity). *For any $a > 0$, we have:*

(1)
$$\mathbf{P}(S_\infty > a) = \left(\frac{x}{a}\right) \wedge 1. \tag{2.2}$$
Hence, $\frac{x}{S_\infty}$ is a uniform random variable on $(0,1)$.

(2) *For any stopping time T:*
$$\mathbf{P}\left(S^T > a \mid \mathcal{F}_T\right) = \left(\frac{N_T}{a}\right) \wedge 1, \tag{2.3}$$
where
$$S^T = \sup_{u\geq T} N_u.$$
Hence $\frac{N_T}{S^T}$ is also a uniform random variable on $(0,1)$, independent of $\mathcal{F}_T$.

Proof. Formula (2.3) is a consequence of (2.2) when applied to the martingale $(N_{T+u})_{u\geq 0}$ and the filtration $(\mathcal{F}_{T+u})_{u\geq 0}$. Formula (2.2) itself is obvious when $a \leq x$, and for $a > x$, it is obtained by applying Doob's optional stopping theorem to the local martingale $(N_{t\wedge T_a})$, where $T_a = \inf\{u \geq 0 : \; N_u > a\}$. □

The next proposition gives an explicit formula for $Z_t \equiv \mathbf{P}(g > t \mid \mathcal{F}_t)$, in terms of the local martingale (N_t). Without loss of generality, **we assume from now on that $\mathbf{x} = \mathbf{1}$**. Indeed, if $N_0 = x$, we consider the local martingale $\left(\frac{N_t}{x}\right)$ which starts at 1.

Proposition 2.2. (1) *In our setting, the formula:*

$$Z_t = \frac{N_t}{S_t}, \; t \geq 0$$

holds.

(2) *The Doob-Meyer additive decomposition of (Z_t) is:*

$$Z_t = \mathbf{E}\left[\log S_\infty \mid \mathcal{F}_t\right] - \log\left(S_t\right). \tag{2.4}$$

Proof. We first note that:

$$\begin{aligned} \{g > t\} &= \{\exists \; u > t : \; S_u = N_u\} \\ &= \{\exists \; u > t : \; S_t \leq N_u\} \\ &= \left\{\sup_{u\geq t} N_u \geq S_t\right\}. \end{aligned}$$

Hence, from (2.3), we get: $\mathbf{P}(g > t \mid \mathcal{F}_t) = \frac{N_t}{S_t}$.

To establish (2.4), we develop $\left(\frac{N_t}{S_t}\right)$ thanks to Ito's formula, to obtain:

$$Z_t = 1 + \int_0^t \frac{1}{S_s} dN_s - \int_0^t \frac{N_s}{(S_s)^2} dS_s.$$

Now, we remark that the measure dS_s is carried by the set $\{s : \; Z_s = 1\}$; hence:

$$\begin{aligned} Z_t &= 1 + \int_0^t \frac{1}{S_s} dN_s - \int_0^t \frac{1}{S_s} dS_s \\ \frac{N_t}{S_t} &= 1 + \int_0^t \frac{1}{S_s} dN_s - \log\left(S_t\right). \end{aligned}$$

From the unicity of the Doob-Meyer decomposition, $\log(S_t)$ is the predictable increasing part of (Z_t) whilst $\left(\int_0^t \frac{1}{S_s} dN_s\right)$ is its martingale part. As (Z_t) is of class (D), $\left(\int_0^t \frac{1}{S_s} dN_s\right)$ is a uniformly integrable martingale. Now, let $t \to \infty$: as $Z_\infty = 0$, $\log S_\infty = 1 + \int_0^\infty \frac{1}{S_s} dN_s$ and thus:

$$1 + \int_0^t \frac{1}{S_s} dN_s = \mathbf{E}\left[\log S_\infty \mid \mathcal{F}_t\right], \tag{2.5}$$

which proves (2). □

Remark 2.3. It is well known, and it follows from (2.4), that the martingale in (2.5) is in fact in BMO.

Corollary 2.4. *Assuming that all $(\mathcal{F}_t)$ martingales are continuous, the following hold:*

(1) *$\log(S_t)$ is the dual predictable projection of $\mathbf{1}_{\{g \leq t\}}$: for any positive predictable process (k_s),*

$$\mathbf{E}(k_g) = \mathbf{E}\left(\int_0^\infty k_s \frac{dS_s}{S_s}\right);$$

(2) *The random time g is honest and avoids any $(\mathcal{F}_t)$stopping time T, i.e. $P[g = T] = 0$.*

Proof. Under our assumptions, the predictable and optional sigma algebras are equal. Thus, it suffices to prove that g avoids stopping times, the other assertions being obvious. Since $\log(S_t)$ is the dual predictable projection of $\mathbf{1}_{\{g \leq t\}}$ and is continuous, then for any $(\mathcal{F}_t)$ stopping time T,

$$\mathbf{E}\left[\mathbf{1}_{\{g=T\}}\right] = \mathbf{E}\left[(\Delta \log(S_\bullet))_T\right] = 0.$$

Thus we get $P(g = T) = 0$. □

We can now write the formula (1.1) in terms of the martingale (N_t).

Proposition 2.5. *Let $(X_t)_{t \geq 0}$ be a local $(\mathcal{F}_t)$ martingale. Then, X has the following decomposition as a semimartingale in $(\mathcal{F}_t^g)$:*

$$X_t = \widetilde{X}_t + \int_0^{t \wedge g} \frac{d < X, N >_s}{N_{s-}} - \int_g^t \frac{d < X, N >_s}{S_\infty - N_{s-}}$$

where $\left(\widetilde{X}_t\right)$ is an $(\mathcal{F}_t^g)$ local martingale.

Proof. This is a consequence of formula (1.1) and Proposition 2.2. □

We shall now give a relationship between (S_t) and $\mathbf{E}[\log S_\infty \mid \mathcal{F}_t]$. For this, we shall need the following easy extension of Skorokhod's reflection lemma (see [12], p.72):

Lemma 2.6. *Let y be a real-valued càdlàg function on $[0, \infty)$, such that y has no negative jumps, and $y(0) = 0$. Then, there exists a unique pair (z, a) of functions on $[0, \infty)$ such that:*

(1) *z=y+a*
(2) *z is positive, càdlàg and has no negative jumps,*
(3) *a is increasing, continuous, vanishing at zero and the corresponding measure da_s is carried by $\{s : z(s) = 0\}$.*

The function a is moreover given by

$$a(t) = \sup_{s \leq t}(-y(s)).$$

Proposition 2.7. *With*

$$\mu_t = \mathbf{E}\left[\log S_\infty \mid \mathcal{F}_t\right],$$

we have:

$$\log\left(S_t\right) = \sup_{s\leq t}\mu_s - 1 \equiv \overline{\mu}_t - 1,$$

or equivalently:

$$S_t = \exp\left(\overline{\mu}_t - 1\right)$$

Proof. From (2.4), we can write:

$$1 - Z_t = \left(1 - \mu_t\right) + \log\left(S_t\right).$$

From Lemma 2.6, we deduce that

$$\log\left(S_t\right) = \sup_{s\leq t}\mu_s - 1.$$

□

2.2. **Some hidden Azéma-Yor martingales.** We shall now associate with the two dimensional process

$$\left(\log\left(S_t\right),\ Z_t\right)_{t\geq 0}$$

a family of martingales reminiscent of Azéma-Yor martingales (see, e.g., [3]) which we shall now discuss. In fact, once again, we have to introduce a slightly generalized version of what are usually called Azéma-Yor martingales. Indeed, these martingales were originally defined for continuous local martingales (see [21], Chapter VI), while we would like to define them for local martingales without positive jumps. This extension can be dealt with the following balayage argument:

Lemma 2.8. *Let $Y = M + A$ be a special semimartingale, where M is a càdlàg local martingale, and A a continuous increasing process. Set $H = \{t :\ Y_t = 0\}$, and define $g_t \equiv \sup\{s < t :\ Y_s = 0\}$. Then, for any locally bounded predictable process (k_t), (k_{g_t}) is predictable and*

$$k_{g_t}Y_t = k_0 Y_0 + \int_0^t k_{g_s}\,dY_s. \tag{2.6}$$

Proof. The proof is the same as the proof for continuous semimartingales. The reader can refer to [5], p.144, for even more general versions of the balayage formula. □

Now, we can state the following generalization of the classical Azéma-Yor martingales:

Proposition 2.9. *Let $(N_t)_{t\geq 0}$ be a local martingale such that its supremum process (S_t) is continuous (this is the case if N_t is in the class $\mathcal{C}_0$). Let f*

be a locally bounded Borel function and define $F(x) = \int_0^x dy f(y)$. *Then,* $X_t \equiv F(S_t) - f(S_t)(S_t - N_t)$ *is a local martingale and:*

$$F(S_t) - f(S_t)(S_t - N_t) = \int_0^t f(S_s)\, dN_s + F(S_0), \tag{2.7}$$

Proof. In (2.6), take $k_t \equiv f(S_t)$, and $Y_t \equiv S_t - N_t$. Then, we have:

$$f(S_{g_t})(S_t - N_t) = \int_0^t f(S_{g_s})\, d(S_s - N_s).$$

But $S_{g_t} = S_t$, hence:

$$F(S_t) - f(S_t)(S_t - N_t) = \int_0^t f(S_s)\, dN_s + F(S_0).$$

In conclusion, for any locally bounded function f,

$$F(S_t) - f(S_t)(S_t - N_t) = \int_0^t f(S_s)\, dN_s + F(S_0),$$

is a local martingale. □

Remark 2.10. Although very simple, these martingales played an essential role in the resolution by Azéma and Yor of Skorokhod's embedding problem (see [21], chapter VI for more details and references).

Remark 2.11. In [15], a special case of Proposition 2.9, for spectrally negative Lévy martingales is obtained by different means.

Now, we associate with the two dimensional process $(\log(S_t), Z_t)_{t\geq 0}$, a canonical family of local martingales which are in fact of the form (2.7).

Proposition 2.12. *Let* f *be a locally bounded and Borel function, and let* $F(x) = \int_0^x dy f(y)$.

(1) *The following processes are local martingales:*

$$F(\log(S_t)) - f(\log(S_t))(1 - Z_t), \; t \geq 0. \tag{2.8}$$

(2) *Denoting* $K(x) = F(x-1)$ *and* $k(x) = f(x-1)$, *then the local martingales in (2.8) are seen to be equal to:*

$$K(\overline{\mu}_t) - k(\overline{\mu}_t)(\overline{\mu}_t - \mu_t), \; t \geq 0. \tag{2.9}$$

Proof. (1). The fact that (2.8) defines a local martingale may be seen as an application of Ito's lemma (when f is regular), followed by a monotone class argument.

(2). Formula (2.9) is obtained by a trivial change of variables, and the fact that: $1 - Z_t = \overline{\mu}_t - \mu_t$, which was derived in Proposition 2.7. □

Remark 2.13. Similar formulas are derived in [18] from different considerations.

3. Initial expansion with S_∞ and enlargement formulae

In this Section, we shall deal with the question of initial enlargement of the filtration $(\mathcal{F}_t)$ with the variable S_∞. This problem cannot be dealt with the powerful enlargement theorem of Jacod (see [9]), but can be treated by a careful combination of different propositions in [8]. However, we shall give a simple proof which can also be adapted to deal with some other situations. Eventually, we will use our result about the initial expansion of $(\mathcal{F}_t)$ with the variable S_∞ to recover formula (1.1).

Let us define the new filtration

$$\mathcal{F}_t^{\sigma(S_\infty)} \equiv \bigcap_{\varepsilon>0} \left(\mathcal{F}_{t+\varepsilon} \vee \sigma\left(S_\infty\right)\right),$$

which satisfies the usual assumptions. The new information $\sigma\left(S_\infty\right)$ is brought in at the origin of time and g is a stopping time for this larger filtration. More precisely:

Lemma 3.1. *The following hold:*

(1)
$$g = \inf\left\{t : \ N_t = S_\infty\right\};$$
and hence g is an $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ stopping time.

(2) *Consequently:*
$$\mathcal{F}_t^{g} \subset \mathcal{F}_t^{\sigma(S_\infty)}.$$

Proof. (1) The measure dS_t is carried by the set $\{t : \ N_t = S_t\}$. As $g = \sup\{t : \ N_t = S_t\}$, the process (S_t) does not grow after g, which also satisfies:

$$g = \inf\left\{t : \ S_t = S_\infty\right\};$$

hence g is an $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ stopping time.

(2) It is obvious. □

Now we introduce some standard terminology.

Definition 3.2. We shall say that the pair of filtrations $\left(\mathcal{F}_t, \mathcal{F}_t^{\sigma(S_\infty)}\right)$ satisfies the (H') hypothesis if every $(\mathcal{F}_t)$ (semi)martingale is a $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ semimartingale.

We shall now show that the pair of filtrations $\left(\mathcal{F}_t, \mathcal{F}_t^{\sigma(S_\infty)}\right)$ satisfies the (H') hypothesis and give the decomposition of a $(\mathcal{F}_t)$ local martingale in $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$. For this, we need to know the conditional law of S_∞ given $\mathcal{F}_t$.

Proposition 3.3. *For any Borel bounded or positive function f, we have:*

$$\begin{aligned}\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right) &= f\left(S_t\right)\left(1-\frac{N_t}{S_t}\right) + \int_0^{N_t/S_t} dx f\left(\frac{N_t}{x}\right) \\ &= f\left(S_t\right)\left(1-\frac{N_t}{S_t}\right) + N_t \int_{S_t}^{\infty} dx \frac{f\left(x\right)}{x^2}.\end{aligned} \tag{3.1}$$

Proof. The proof is based on Lemma 2.1; in the following, U is a random variable, which follows the standard uniform law and which is independent of $\mathcal{F}_t$.

$$\begin{aligned}\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right) &= \mathbf{E}\left(f\left(S_t \vee S^t\right)|\mathcal{F}_t\right)\\ &= \mathbf{E}\left(f\left(S_t\right)\mathbf{1}_{\{S_t\geq S^t\}}|\mathcal{F}_t\right)+\mathbf{E}\left(f\left(S^t\right)\mathbf{1}_{\{S_t<S^t\}}|\mathcal{F}_t\right)\\ &= f\left(S_t\right)\mathbf{P}\left(S_t\geq S^t|\mathcal{F}_t\right)+\mathbf{E}\left(f\left(S^t\right)\mathbf{1}_{\{S_t<S^t\}}|\mathcal{F}_t\right)\\ &= f\left(S_t\right)\mathbf{P}\left(U\leq\frac{N_t}{S_t}|\mathcal{F}_t\right)+\mathbf{E}\left(f\left(\frac{N_t}{U}\right)\mathbf{1}_{\left\{U<\frac{N_t}{S_t}\right\}}|\mathcal{F}_t\right)\\ &= f\left(S_t\right)\left(1-\frac{N_t}{S_t}\right)+\int_0^{N_t/S_t}dx f\left(\frac{N_t}{x}\right).\end{aligned}$$

A straightforward change of variable in the last integral also gives:

$$\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right)=f\left(S_t\right)\left(1-\frac{N_t}{S_t}\right)+N_t\int_{S_t}^\infty dy\frac{f\left(y\right)}{y^2}.$$

□

One may now ask if $\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right)$ is of the form (2.7). The answer to this question is positive. Indeed:

$$\begin{aligned}\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right) &= f\left(S_t\right)\left(1-\frac{N_t}{S_t}\right)+N_t\int_{S_t}^\infty dy\frac{f\left(y\right)}{y^2}\\ &= S_t\int_{S_t}^\infty dy\frac{f\left(y\right)}{y^2}-\left(S_t-N_t\right)\left(\int_{S_t}^\infty dy\frac{f\left(y\right)}{y^2}-\frac{f\left(S_t\right)}{S_t}\right).\end{aligned}$$

Hence,

$$\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right)=H\left(1\right)+H\left(S_t\right)-h\left(S_t\right)\left(S_t-N_t\right),$$

with

$$H\left(x\right)=x\int_x^\infty dy\frac{f\left(y\right)}{y^2},$$

and

$$h\left(x\right)=h_f\left(x\right)\equiv\int_x^\infty dy\frac{f\left(y\right)}{y^2}-\frac{f\left(x\right)}{x}=\int_x^\infty\frac{dy}{y^2}\left(f\left(y\right)-f\left(x\right)\right).$$

Moreover, again from formula (2.7), we have the following representation of $\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right)$ as a stochastic integral:

$$\mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right)=\mathbf{E}\left(f\left(S_\infty\right)\right)+\int_0^t h\left(S_s\right)dN_s. \tag{3.2}$$

Let us sum up these results, introducing some notations:

$$\lambda_t\left(f\right) \equiv \mathbf{E}\left(f\left(S_\infty\right)|\mathcal{F}_t\right) \tag{3.3}$$

$$\qquad = f\left(S_t\right)\left(1-\frac{N_t}{S_t}\right)+N_t\int_{S_t}^\infty dx\frac{f\left(x\right)}{x^2}; \tag{3.4}$$

and

$$\lambda_t(f) = \mathbf{E}(f(S_\infty)) + \int_0^t \dot{\lambda}_s(f)\, dN_s, \tag{3.5}$$

where:

$$\dot{\lambda}_s(f) = h_f(S_s). \tag{3.6}$$

Moreover, there exist two families of random measures $(\lambda_t(dx))_{t\geq 0}$ and $\left(\dot{\lambda}_t(dx)\right)_{t\geq 0}$, with

$$\lambda_t(dx) = \left(1 - \frac{N_t}{S_t}\right)\delta_{S_t}(dx) + N_t \mathbf{1}_{\{x>S_t\}}\frac{dx}{x^2} \tag{3.7}$$

$$\dot{\lambda}_t(dx) = -\frac{1}{S_t}\delta_{S_t}(dx) + \mathbf{1}_{\{x>S_t\}}\frac{dx}{x^2}, \tag{3.8}$$

such that

$$\lambda_t(f) = \int \lambda_t(dx) f(x) \tag{3.9}$$

$$\dot{\lambda}_t(f) = \int \dot{\lambda}_t(dx) f(x). \tag{3.10}$$

Eventually, we notice that there is an absolute continuity relationship between $\lambda_t(dx)$ and $\dot{\lambda}_t(dx)$; more precisely,

$$\dot{\lambda}_t(dx) = \lambda_t(dx)\, \rho(x,t), \tag{3.11}$$

with

$$\rho(x,t) = \frac{-1}{S_t - N_t}\mathbf{1}_{\{S_t = x\}} + \frac{1}{N_t}\mathbf{1}_{\{S_t < x\}}. \tag{3.12}$$

Now, we can state the main theorem of this section.

Theorem 3.4. *Let $(N_t)_{t\geq 0}$ be a local martingale in the class $\mathcal{C}_0$ (recall $N_0 = 1$). Then, the pair of filtrations $\left(\mathcal{F}_t, \mathcal{F}_t^{\sigma(S_\infty)}\right)$ satisfies the (H') hypothesis and every $(\mathcal{F}_t)$ local martingale (X_t) is an $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ semimartingale with canonical decomposition:*

$$X_t = \widetilde{X}_t + \int_0^t \mathbf{1}_{\{g>s\}} \frac{d < X, N >_s}{N_{s-}} - \int_0^t \mathbf{1}_{\{g\leq s\}} \frac{d < X, N >_s}{S_\infty - N_{s-}},$$

where $\left(\widetilde{X}_t\right)$ is a $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ local martingale.

Remark 3.5. The following proof is tailored on the arguments found in [27], although our framework is more general: we do not assume that our filtration has the predictable representation property with respect to some martingale nor that all martingales are continuous.

Proof. We can first assume that X is in $\mathcal{H}^1$; the general case follows by localization. Let Λ_s be an $\mathcal{F}_s$ measurable set, and take $t > s$. Then, for any bounded test function f, $\lambda_t(f)$ is a bounded martingale, hence in BMO, and we have:

$$\begin{aligned}\mathbf{E}\left(\mathbf{1}_{\Lambda_s} f\left(A_\infty\right)\left(X_t - X_s\right)\right) &= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\lambda_t(f) X_t - \lambda_s(f) X_s\right)\right)\\ &= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(<\lambda(f), X>_t - <\lambda(f), X>_s\right)\right)\\ &= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\int_s^t \dot{\lambda}_u(f)\, d<X,N>_u\right)\right)\\ &= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\int_s^t \int \lambda_u(dx)\,\rho(x,u)\, f(x)\, d<X,N>_u\right)\right)\\ &= \mathbf{E}\left(\mathbf{1}_{\Lambda_s}\left(\int_s^t d<X,N>_u\, \rho\left(S_\infty, u\right)\right)\right).\end{aligned}$$

But from (3.12), we have:

$$\rho\left(S_\infty, t\right) = \frac{-1}{S_t - N_t}\mathbf{1}_{\{S_t = S_\infty\}} + \frac{1}{N_t}\mathbf{1}_{\{S_t < S_\infty\}}.$$

It now suffices to note (from Lemma 3.1) that (S_t) is constant after g and g is the first time when $S_\infty = S_t$, or in other words:

$$\mathbf{1}_{\{S_\infty > S_t\}} = \mathbf{1}_{\{g > t\}}, \text{ and } \mathbf{1}_{\{S_\infty = S_t\}} = \mathbf{1}_{\{g \leq t\}}.$$

This completes the proof. □

Theorem 3.4 yields a new proof of the decomposition formula in the progressive enlargement case. More precisely, we have:

Corollary 3.6. *The pair of filtrations $(\mathcal{F}_t, \mathcal{F}_t^g)$ satisfies the (H') hypothesis. Moreover, every $(\mathcal{F}_t)$ local martingale X decomposes as:*

$$X_t = \widetilde{X}_t + \int_0^t \mathbf{1}_{\{g>s\}} \frac{d<X,N>_s}{N_s} - \int_0^t \mathbf{1}_{\{g \leq s\}} \frac{d<X,N>_s}{S_\infty - N_s},$$

where $\left(\widetilde{X}_t\right)$ is a $(\mathcal{F}_t^g)$ local martingale.

Proof. Let X be an $(\mathcal{F}_t)$ martingale which is in $\mathcal{H}^1$; the general case follows by localization. From Theorem 3.4

$$X_t = \widetilde{X}_t + \int_0^t \mathbf{1}_{\{g>s\}} \frac{d<X,N>_s}{N_s} - \int_0^t \mathbf{1}_{\{g \leq s\}} \frac{d<X,N>_s}{S_\infty - N_s},$$

where $\left(\widetilde{X}_t\right)_{t\geq 0}$ denotes an $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ martingale. Thus, $\left(\widetilde{X}_t\right)$, which is equal to:

$$X_t - \left(\int_0^t \mathbf{1}_{\{g>s\}} \frac{d<X,N>_s}{N_s} - \int_0^t \mathbf{1}_{\{g \leq s\}} \frac{d<X,N>_s}{S_\infty - N_s},\right),$$

is $(\mathcal{F}_t^g)$ adapted (recall that $\mathcal{F}_t^g \subset \mathcal{F}_t^{\sigma(S_\infty)}$), and hence it is an $(\mathcal{F}_t^g)$ martingale. □

4. A multiplicative characterization of Z_t

Usually, in the literature about progressive enlargements of filtrations, it is assumed that the conditions **(CA)** are satisfied. Now, we shall prove that under this assumption the supermartingale $Z_t^L = \mathbf{P}(L > t \mid \mathcal{F}_t)$, associated with an honest time, can be represented as $\left(\frac{N_t}{S_t}\right)_{t\geq 0}$, where N_t is a positive local martingale. More precisely, we have the following:

Theorem 4.1. *Let L be an honest time. Then, under the conditions* ***(CA)****, there exists a continuous and nonnegative local martingale $(N_t)_{t\geq 0}$, with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$, such that:*

$$Z_t = \mathbf{P}(L > t \mid \mathcal{F}_t) = \frac{N_t}{S_t}$$

Proof. Under the conditions **(CA)**, $(Z_t)_{t\geq 0}$ is continuous and can be written as (see [1] or [5] for details):

$$Z_t = M_t - A_t,$$

where (M_t) and (A_t) are continuous, $Z_0 = 1$ and dA_t is carried by $\{t : \ Z_t = 1\}$. Then, for $t < T_0 \equiv \inf\{t : \ Z_t = 0\}$, we have:

$$\log(Z_t) = \int_0^t \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d < M >_s}{Z_s^2} - A_t,$$

hence:

$$-\log(Z_t) = -\left(\int_0^t \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d < M >_s}{Z_s^2}\right) + A_t; \tag{4.1}$$

and, from Skorokhod's reflection lemma, we have:

$$A_t = \sup_{u\leq t}\left(\int_0^u \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d < M >_s}{Z_s^2}\right). \tag{4.2}$$

Now, combining (4.1) and (4.2), we obtain:

$$Z_t = \frac{N_t}{S_t},$$

where

$$N_t = \exp\left(\int_0^t \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d < M >_s}{Z_s^2}\right)$$

is a local martingale starting from 1, and

$$\begin{aligned} S_t &= \sup_{u\leq t}\left(\exp\left(\int_0^u \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d < M >_s}{Z_s^2}\right)\right) \\ &= \exp\left(\sup_{u\leq t}\left(\int_0^u \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d < M >_s}{Z_s^2}\right)\right) \\ &= \exp(A_t). \end{aligned}$$

We finally note that, since $Z_{T_0} = 0$, $\lim_{t\uparrow T_0} N_t = 0$, which allows to define N_t for all $t \geq 0$. □

Corollary 4.2. *The supermartingale* $Z_t = \mathbf{P}(L > t \mid \mathcal{F}_t)$ *admits the following additive and multiplicative representations:*

$$\begin{aligned} Z_t &= \frac{N_t}{S_t} \\ Z_t &= M_t - A_t. \end{aligned}$$

Moreover, these two representations are related as follows:

$$\begin{aligned} N_t &= \exp\left(\int_0^t \frac{dM_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d<M>_s}{Z_s^2}\right) \\ S_t &= \exp(A_t); \end{aligned}$$

and

$$\begin{aligned} M_t &= 1 + \int_0^t \frac{dN_s}{S_s} = \mathbf{E}(\log S_\infty \mid \mathcal{F}_t), \\ A_t &= \log S_t. \end{aligned}$$

Proof. It is a consequence of Proposition 2.2 and Theorem 4.1. □

Now, as a consequence of Theorem 4.1, we can recover the enlargement formulae and the fact that the pair of filtrations $(\mathcal{F}_t, \mathcal{F}_t^L)$ satisfies the (H') hypothesis:

Corollary 4.3. *Let* L *be an honest time. Then under the conditions* ***(CA)****, the pair of filtrations* $(\mathcal{F}_t, \mathcal{F}_t^L)$ *satisfies the* (H') *hypothesis and every* $(\mathcal{F}_t)$ *local martingale* X *is an* $(\mathcal{F}_t^L)$ *semimartingale with canonical decomposition:*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge L} \frac{d<X,Z>_s}{Z_s} + \int_L^t \frac{d<X,1-Z>_s}{1-Z_s},$$

where $\left(\widetilde{X}_t\right)_{t\geq 0}$ *denotes an* $\left((\mathcal{F}_t^L)\right)$ *local martingale.*

Proof. It is a combination of Theorem 4.1 and Corollary 3.6. □

Remark 4.4. We then see that under the assumptions **(CA)**, the initial enlargement of filtrations with A_∞ amounts to enlarging initially the filtration with S_∞, the terminal value of the supremum process of a continuous local martingale in $\mathcal{C}_0$.

We shall now outline another nontrivial consequence of Theorem 4.1 here. In [2], the authors are interested in giving explicit examples of dual predictable projections of processes of the form $\mathbf{1}_{g\leq t}$, where g is an honest time. Indeed, these dual projections are natural examples of increasing injective processes (see [2] for more details and references). With Theorem 4.1, we have a complete characterization of such projections:

Corollary 4.5. *Assume the assumption* ***(C)*** *holds, and let* (C_t) *be an increasing process. Then* C *is the dual predictable projection of* $\mathbf{1}_{g\leq t}$*, for some* *honest time g that avoids stopping times, if and only if there exists a continuous local martingale N_t in the class $\mathcal{C}_0$ such that*

$$C_t = \log S_t.$$

The previous results can be naturally extended to the case where the supermartingale Z_t has only negative jumps; we gave a special treatment under the hypothesis **(CA)** because of its practical importance. We just give here the extension of Theorem 4.1; the corollaries are easily deduced.

Proposition 4.6. *Let L be an honest time that avoids stopping times. Assume that Z_t^L has no positive jumps. Then, there exists a local martingale $(N_t)_{t\geq 0}$, in the class $\mathcal{C}_0$, with $N_0 = 1$, such that:*

$$\left(Z_t^L =\right) Z_t = \mathbf{P}\left(L > t \mid \mathcal{F}_t\right) = \frac{N_t}{S_t}$$

Proof. We use the same notations as in the proof of Theorem 4.1. For $t < T_0 \equiv \inf\{t : \; Z_t = 0\}$, we have:

$$-\log\left(Z_t\right) = -\left(\int_0^t \left(\frac{dM_s}{Z_{s-}} - \frac{1}{2}\frac{d < M^c >_s}{Z_{s-}^2}\right) + \sum_{0<s\leq t}\left(\log\left(1 + \frac{\Delta Z_s}{Z_{s-}}\right) - \frac{\Delta Z_s}{Z_{s-}}\right)\right) + A_t.$$

Now, from Lemma 2.6,

$$A_t = \sup_{s\leq t}\left(\int_0^t \left(\frac{dM_s}{Z_{s-}} - \frac{1}{2}\frac{d < M^c >_s}{Z_{s-}^2}\right) + \sum_{0<s\leq t}\left(\log\left(1 + \frac{\Delta Z_s}{Z_{s-}}\right) - \frac{\Delta Z_s}{Z_{s-}}\right)\right).$$

Now, combining the last two equalities, we obtain:

$$Z_t = \frac{N_t}{S_t},$$

where

$$N_t = \exp\left(\int_0^t \left(\frac{dM_s}{Z_{s-}} - \frac{1}{2}\frac{d < M^c >_s}{Z_{s-}^2}\right)\right) \prod_{0<s\leq t}\left(1 + \frac{\Delta Z_s}{Z_{s-}}\right)\exp\left(-\frac{\Delta Z_s}{Z_{s-}}\right).$$

□

5. Examples and applications

In this section, we look at some specific local martingales N_t, and use the initial enlargement formula with S_∞, to get some path decompositions, given the maximum or the minimum of some stochastic processes. Our aim here is to illustrate how techniques from enlargement of filtrations can be applied. To have a complete description for the path decompositions, we associate with g a random time, called pseudo-stopping time, which occurs before g. Eventually, we give some explicit examples of supermartingales Z_t with jumps.

5.1. **Pseudo-stopping times.** In [16], we have proposed the following generalization of stopping times:

Definition 5.1. Let $\rho : (\Omega, \mathcal{F}) \to \mathbf{R}_+$ be a random time; ρ is called a pseudo-stopping time if for every bounded $(\mathcal{F}_t)$ martingale we have:

$$\mathbf{E}(M_\rho) = \mathbf{E}(M_0).$$

David Williams ([25]) gave the first example of such a random time and the following systematic construction is established in [16]:

Proposition 5.2. *Let L be an honest time. Then, under the conditions **(CA)**,*

$$\rho \equiv \sup\left\{t < L : \; Z_t^L = \inf_{u \le L} Z_u^L\right\},$$

is a pseudo-stopping time, with

$$Z_t^\rho \equiv \mathbf{P}(\rho > t \mid \mathcal{F}_t) = \inf_{u \le t} Z_u^L,$$

and Z_ρ^ρ follows the uniform distribution on $(0,1)$.

The following property, also proved in [16], is essential in studying path decompositions:

Proposition 5.3. *Let ρ be a pseudo-stopping time and let M_t be an $(\mathcal{F}_t)$ local martingale. Then $(M_{t\wedge\rho})$ is an $(\mathcal{F}_t^\rho)$ local martingale.*

In our setting, Proposition 5.2 gives:

Proposition 5.4. *Define the nonincreasing process (r_t) by:*

$$r_t \equiv \inf_{u \le t} \frac{N_u}{S_u}.$$

Then,

$$\rho \equiv \sup\left\{t < g : \; \frac{N_t}{S_t} = \inf_{u \le g} \frac{N_u}{S_u}\right\},$$

is a pseudo-stopping time and r_ρ follows the uniform distribution on $(0,1)$.

5.2. **Path decompositions given the maxima or the minima of a diffusion.** Now, we shall apply the techniques of enlargements of filtrations to establish some path decompositions results. Some of the following results have been proved by David Williams in [24], using different methods. Jeulin has also given a proof based on enlargements techniques in the case of transient diffusions (see [8]). Here, we complete the results of David Williams by introducing the pseudo-stopping times ρ defined in Proposition 5.4, and we detail some interesting examples.

5.2.1. *The killed Brownian Motion.* Let

$$N_t \equiv B_t,$$

where $(B_t)_{t\geq 0}$ is a Brownian Motion starting at 1, and stopped at $T_0 = \inf\{t: \ B_t = 0\}$. Let

$$S_t \equiv \sup_{s\leq t} B_s.$$

Let

$$g = \sup\{t : B_t = S_t\}$$

and

$$\rho = \sup\left\{t < g : \ \frac{B_t}{S_t} = \inf_{u\leq g} \frac{B_u}{S_u}\right\}.$$

From Doob's maximal identity, $S_{T_0} = S_g$ is distributed as the reciprocal of a uniform distribution $(0,1)$, i.e. it has the density: $\mathbf{1}_{[1,\infty)}(x)\,\dfrac{1}{x^2}$.

Proposition 5.5. *Let $(B_t)_{t\geq 0}$ be a Brownian Motion starting at 1 and stopped when it first hits 0. Then:*

- $\dfrac{B_\rho}{S_\rho}$ *follows the uniform law on $(0,1)$, and conditionally on $\dfrac{B_\rho}{S_\rho} = r$, (B_t) is a Brownian Motion up to the first time when $B_t = rS_t$.*
- *(B_t) is an $(\mathcal{F}^g_t)$ and $\left(\mathcal{F}_t^{\sigma(S_{T_0})}\right)$ semimartingale with canonical decomposition:*

$$B_t = \widetilde{B}_t + \int_0^{t\wedge g} \frac{ds}{B_s} - \int_g^{t\wedge T_0} \frac{ds}{S_{T_0} - B_s}, \tag{5.1}$$

where $\left(\widetilde{B}_t\right)$ is an $\mathcal{F}_t^{\sigma(S_{T_0})}$ Brownian Motion, stopped at T_0 and independent of S_{T_0}. Consequently, we have the following path decomposition: conditionally on $S_{T_0} = m$:

  (1) *the process $(B_t;\ t\leq g)$ is a Bessel process of dimension 3, started from 1, considered up to T_m, the first time when it hits m;*

  (2) *the process $(S_g - B_{g+t};\ t \leq T_0 - g)$ is a $(\mathcal{F}_{g+t})$ three dimensional Bessel process, started from 0, considered up to T_m, the first time when it hits m, and is independent of $(B_t;\ t\leq g)$.*

Proof. The results concerning the decomposition until ρ are consequences of the results of Subsection 5.1. The decomposition formula is a consequence of Theorem 3.4. Since $\left(\widetilde{B}_t\right)$ is an $\mathcal{F}_t^{\sigma(S_{T_0})}$ local martingale, with $t\wedge T_0$ as its bracket, it follows from Lévy's theorem that it is an $\mathcal{F}_t^{\sigma(S_{T_0})}$ Brownian Motion. Moreover, it is independent of $\mathcal{F}_0^{\sigma(S_{T_0})} = \sigma(S_{T_0})$. Now, conditionally on $S_{T_0} = m$, with $T_m = \inf\{t: \ B_t = m\}$, (B_t) satisfies the following stochastic differential equation:

$$B_t = \widetilde{B}_t + \int_0^{t\wedge T_m} \frac{ds}{B_s}.$$

Hence it is a three dimensional Bessel process up to T_m.

It also follows from the decomposition formula that:

$$B_{g+t} = \widetilde{B}_{g+t} + \int_0^g \frac{ds}{B_s} - \int_0^{t\wedge(T_0-g)} \frac{ds}{S_g - B_{g+s}}.$$

This equation can also be written as:

$$S_g - B_{g+t} = -\left(\widetilde{B}_{g+t} - \widetilde{B}_g\right) + \int_0^{t\wedge(T_0-g)} \frac{ds}{S_g - B_{g+s}}.$$

Now, $\left(\widetilde{B}_{g+t} - \widetilde{B}_g\right)$ is an $(\mathcal{F}_{g+t})$ Brownian Motion, starting from 0, and is independent of $\mathcal{F}_g$. Taking $\widetilde{\beta}_t \equiv -\left(\widetilde{B}_{g+t} - \widetilde{B}_g\right)$, which is also an $(\mathcal{F}_{g+t})$ Brownian Motion, starting from 0, independent of $\mathcal{F}_g$, the process $\xi_t \equiv S_g - B_t$ satisfies the stochastic differential equation:

$$\xi_t = \widetilde{\beta}_t + \int_0^{t\wedge(T_0-g)} \frac{ds}{\xi_s};$$

hence it is a three dimensional Bessel process, started at 0, and considered up to T_m, and conditionally on S_g, is independent of $\mathcal{F}_g$. □

5.2.2. *Some recurrent diffusions.* The previous example can be generalized to a wider class of recurrent diffusions (X_t), satisfying the stochastic differential equation:

$$X_t = x + B_t + \int_0^t b(X_s)\,ds,\ x > 0 \tag{5.2}$$

where (B_t) is the standard Brownian Motion, and b is a Borel integrable function which allows existence and uniqueness for equation (5.2) (for example b bounded or Lipschitz continuous). The infinitesimal generator L of this diffusion is:

$$L = \frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

Let $T_0 \equiv \inf\{t:\ X_t = 0)$, and denote by s the scale function of X, which is strictly increasing and which vanishes at zero, i.e:

$$s(z) = \int_0^z \exp\left(-2\widehat{b}(y)\right) dy,$$

where

$$\widehat{b}(y) = \int_0^y b(u)\,du.$$

Hence,

$$N_t \equiv \frac{s(X_{t\wedge T_0})}{s(x)}$$

is a continuous local martingale belonging to the class $\mathcal{C}_0$. If S_t denotes the supremum process of N_t and $\overline{X}_t$ the supremum process of X_t, we have:

$$S_t = \frac{s\left(\overline{X}_{t\wedge T_0}\right)}{s(x)}.$$

Now, let

$$g = \sup\left\{t < T_0 :\ X_t = \overline{X}_t\right\},$$

and

$$\rho = \sup\left\{t < g :\ \frac{X_t}{\overline{X}_t} = \inf_{u\leq g}\frac{X_u}{\overline{X}_u}\right\}.$$

Proposition 5.6. *Let (X_t) be a diffusion process satisfying equation (5.2). Then:*

- $\dfrac{X_\rho}{\overline{X}_\rho}$ *follows the uniform law on $(0,1)$, and conditionally on* $\dfrac{X_\rho}{\overline{X}_\rho} = r$, $(X_t,\ t \leq \rho)$ *is a diffusion process, up to the first time when $X_t = r\overline{X}_t$, with the same infinitesimal generator as X.*
- (X_t) *is an* $(\mathcal{F}_t^g)$ *and an* $\left(\mathcal{F}_t^{\sigma(\overline{X}_{T_0})}\right)$ *semimartingale with canonical decomposition:*

$$X_t = \widetilde{B}_t + \int_0^t b(X_u)\,du + \int_0^{t\wedge g}\frac{s'(X_u)}{s(X_u)}du - \int_g^{t\wedge T_0}\frac{s'(X_u)}{s\left(\overline{X}_{T_0}\right) - s(X_u)}du, \quad (5.3)$$

where $\left(\widetilde{B}_t\right)$ *is an* $\mathcal{F}_t^{\sigma(\overline{X}_{T_0})}$ *Brownian Motion, stopped at T_0 and independent of $\overline{X}_{T_0}$. Consequently, we have the following path decomposition: conditionally on $\overline{X}_{T_0} = m$:*

(1) *the process $(X_t;\ t \leq g)$ is a diffusion process started from $x > 0$, considered up to T_m, the first time when it hits m, with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(b(x) + \frac{s'(x)}{s(x)}\right)\frac{d}{dx}.$$

(2) *the process $(X_{g+t};\ t \leq T_0 - g)$ is a $(\mathcal{F}_{g+t})$ diffusion process, started from m, considered up to T_0, the first time when it hits 0, and is independent of $(X_t;\ t \leq g)$; its infinitesimal generator is given by:*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(b(x) + \frac{s'(x)}{s(x) - s(m)}\right)\frac{d}{dx}.$$

(3) $\overline{X}_{T_0}$ *follows the same law as* $s^{-1}\left(\dfrac{1}{U}\right)$, *where U follows the uniform law on $(0,1)$.*

Proof. The proof is exactly the same as the proof of Proposition 5.5, so we will not reproduce it here. □

5.2.3. *Geometric Brownian Motion with negative drift.* Let

$$N_t \equiv \exp\left(2\nu B_t - 2\nu^2 t\right),$$

where (B_t) is a standard Brownian Motion, and $\nu > 0$. With the notation of Theorem 3.4, we have:

$$S_t = \exp\left(\sup_{s\leq t} 2\nu\left(B_s - \nu s\right)\right),$$

and

$$g = \sup\left\{t:\ (B_t - \nu t) = \sup_{s\geq 0}\left(B_s - \nu s\right)\right\}.$$

Before stating our proposition, let us mention that we could have worked with more general continuous exponential local martingales, but we preferred to keep the discussion as simple as possible (the proof for more general cases is exactly the same).

Proposition 5.7. *With the assumptions and notations used above, we have:*

(1) *The variable* $\sup_{s\geq 0}\left(B_s - \nu s\right)$ *follows the exponential law of parameter* 2ν.
(2) *Every local martingale* X *is an* $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ *semimartingale and decomposes as:*

$$X_t = \widetilde{X}_t + 2\nu < X, B >_{t\wedge g} - 2\nu \int_g^t \frac{N_s}{S_\infty - N_s} d < X, B >_s,$$

where $\widetilde{X}_t$ *is an* $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ *local martingale.*
(3) *Conditionally on* $S_\infty = m$, *the process* $(B_t - \nu t;\ t \leq g)$ *is a Brownian Motion with drift* $+\nu$ *up to the first hitting time of its maximum* $m/2\nu$.

Proof. From Doob's maximal equality, $\left(\exp\left(\sup_{s\leq g}\left(2\nu B_s - 2\nu^2 s\right)\right)\right)^{-1}$ follows the uniform law and hence $\sup_{s\geq 0}\left(B_s - \nu s\right)$ follows the exponential law of parameter 2ν.

The decomposition formula is a consequence of Theorem 3.4 and the fact that: $dN_t = 2\nu N_t dB_t$.

To show (3), it suffices to notice that $B_t - \nu t$ is equal to $\widetilde{B}_t + \nu t$ in the filtration $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$, with $\left(\widetilde{B}_t\right)$ an $\left(\mathcal{F}_t^{\sigma(S_\infty)}\right)$ Brownian Motion which is independent of S_∞. □

5.2.4. *General transient diffusions.* Now, we consider (R_t), a transient diffusion with values in $[0, \infty)$, which has $\{0\}$ as entrance boundary. Let s be a scale function for R, which we can choose such that:

$$s(0) = -\infty, \text{ and } s(\infty) = 0.$$

Then, under the law $\mathbf{P}_x$, for any $x > 0$, the local martingale $\left(N_t = \dfrac{s(R_t)}{s(x)},\ t \geq 0 \right)$ satisfies the conditions of Theorem 3.4, and we have:

$$\mathbf{P}_x(g > t | \mathcal{F}_t) = \frac{s(R_t)}{s(I_t)}$$

where

$$g = \sup\{t : \ R_t = I_t\},$$

and

$$I_t = \inf_{s \leq t} R_s.$$

We thus recover results of Jeulin ([8], Proposition 6.29, p.112) by other means. Jeulin used this formula and gave a quick proof of a theorem of David Williams ([24]), using initial enlargement of filtrations arguments. Our proof would follow the same lines and so we refer to the book of Jeulin. We would rather detail an interesting example: the three dimensional Bessel process.

Proposition 5.8. *Let (R_t) be a three dimensional Bessel process starting from 1, and set, as above, $I_t = \inf_{s \leq t} R_s$, and $g = \sup\{t : \ R_t = I_t\}$. Define ρ by: $\rho = \sup\left\{ t < g : \ \dfrac{I_t}{R_t} = \inf_{u \leq g} \dfrac{I_u}{R_u} \right\}$. Then:*

1. *The variable $\dfrac{I_\rho}{R_\rho}$ follows the uniform law on $(0,1)$ and, conditionally on $I_\rho = rR_\rho$, $(R_t, t \leq T_r)$ is a three dimensional Bessel process starting from 1, up to the first time T_r when $I_t = rR_t$.*
2. *$I_\infty \equiv I_g$ follows the uniform law on $(0,1)$;*
3. *Conditionally on $I_\infty = r$, the process $(R_t,\ t \leq g)$ is a Brownian Motion starting from 1 and stopped when it first hits r.*

Proof. There exists $(\beta)_{t \geq 0}$, a Brownian Motion, such that

$$R_t = 1 + \beta_t + \int_0^t \frac{ds}{R_s}.$$

(1) follows easily from the results of Subsection 5.1. Now, from Ito's formula, it follows that

$$\frac{1}{R_t} = 1 - \int_0^t \frac{d\beta_s}{R_s^2};$$

hance, it is a local martingale. In $\left(\mathcal{F}_t^{\sigma(I_\infty)}\right)$,

$$\beta_{t \wedge g} = \widetilde{\beta}_t - \int_0^{t \wedge g} \frac{ds}{R_s},$$

where $\left(\widetilde{\beta}_t\right)$ is an $\left(\mathcal{F}_t^{\sigma(I_\infty)}\right)$ Brownian Motion independent of I_∞. Hence, $R_{t \wedge g}$ decomposes as

$$R_{t \wedge g} = \widetilde{\beta}_t$$

in $\left(\mathcal{F}_t^{\sigma(I_\infty)}\right)$, and this completes the proof for (3), and (2) is an immediate consequence of Doob's maximal identity. $\square$

Remark 5.9. The previous method applies to any transient diffusion $(R_t)_{t\geq 0}$, with values in $(0,\infty)$, and which satisfies:

$$R_t = x + B_t + \int_0^t du\, c\,(R_u)\,,$$

where $c : \mathbb{R}_+ \to \mathbb{R}$ allows uniqueness in law for this equation. These diffusions were studied in [23] to obtain some extension of Pitman's theorem (see also [27]).

5.3. **Some examples of Z_t with jumps.** We shall conclude this paper by giving some explicit examples of discontinuous $Z's$. Let X be a Poisson process with parameter c and let $N_t = X_t - ct$. N is a martingale in the natural filtration $(\mathcal{F}_t)$ of X. Every local martingale Y in this filtration may be written as:

$$Y_t = Y_0 + \int_0^t k_s dN_s,$$

where k is an $(\mathcal{F}_t)$ predictable process. Now, for $f : \mathbb{R}_+ \to \mathbb{R}_+$ a locally bounded and Borel function, let

$$\mathcal{E}_t^f = \exp\left(-\int_0^t f(s)\, dX_s + c\int_0^t (1-\exp(-f(s)))\, ds\right)$$

$\mathcal{E}_t^f$ is an $\mathcal{F}_t$ local martingale which can be represented as:

$$\mathcal{E}_t^f = 1 + \int_0^t \mathcal{E}_{s-}^f \left(\exp(-f(s)) - 1\right) dN_s.$$

If $\int_0^\infty f(s)\, ds = \infty$, then $\lim_{t\to\infty} \mathcal{E}_t^f = 0$.

Proposition 5.10. *Let f be a nonnegative locally bounded and Borel function on $\mathbf{R}_+$, such that $\lim_{t\to\infty} \mathcal{E}_t^f = 0$. Define:*

$$g = \sup\left\{t :\ \mathcal{E}_t^f = \overline{\mathcal{E}}_t^f\right\},$$

where

$$\overline{\mathcal{E}}_t^f = \sup_{s\leq t} \mathcal{E}_s^f.$$

Then:

1. $\sup_{s\geq 0}\left(-\int_0^t f(s)\, dX_s + c\int_0^t (1-\exp(-f(s)))\, ds\right)$ *is distributed as a random variable with the exponential law with parameter* 1*;*
2. *The supermartingale Z_t^g associated with g is given by:*

$$\mathbf{P}\left(g > t \mid \mathcal{F}_t\right) = \frac{\mathcal{E}_t^f}{\overline{\mathcal{E}}_t^f};$$

(3) *Every* $\mathcal{F}_t$ *local martingale* $Y_t \left(= \int_0^t k_s dN_s\right)$ *is a semimartingale in the filtration* $\mathcal{F}_t^{\sigma\left(\overline{\mathcal{E}}_\infty^f\right)}$, *with canonical decomposition:*

$$Y_t = \widetilde{Y}_t + c \int_0^{t \wedge g} k_s \left(\exp\left(-f\left(s\right)\right) - 1\right) ds - c \int_g^t k_s \left(\exp\left(-f\left(s\right)\right) - 1\right) \frac{\mathcal{E}_s^f}{\overline{\mathcal{E}}_\infty^f - \mathcal{E}_s^f} ds,$$

where $\widetilde{Y}_t$ *is an* $\mathcal{F}_t^{\sigma\left(\overline{\mathcal{E}}_\infty^f\right)}$ *local martingale.*

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