HOW BADLY ARE THE BURHOLDER-DAVIS-GUNDY INEQUALITIES AFFECTED BY ARBITRARY RANDOM TIMES?

ASHKAN NIKEGHBALI

ABSTRACT. This note deals with the question: what remains of the Burkholder-Davis-Gundy inequalities when stopping times T are replaced by arbitrary random times ρ? We prove that these inequalities still hold when T is a pseudo-stopping time and never holds for ends of predictable sets.

1. INTRODUCTION

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space. We recall the Burkholder-Davis-Gundy inequalities (see [12]):

Proposition 1. *Let $p > 0$. There exist two universal constants c_p and C_p depending only on p, such that for any $(\mathcal{F}_t)$ continuous local martingale (M_t), with $M_0 = 0$, and any $(\mathcal{F}_t)$ stopping time ρ, we have*

$$(1.1) \qquad c_p \mathbb{E}\left[(<M>_\rho)^{\frac{p}{2}}\right] \leq \mathbb{E}\left[(M_\rho^*)^p\right] \leq C_p \mathbb{E}\left[(<M>_\rho)^{\frac{p}{2}}\right]$$

where

$$M_\rho^* = \sup_{t\leq\rho} |M_t|$$

One natural question is: does (1.1) still hold if ρ is replaced by an arbitrary random time? In the sequel, given an arbitrary random time ρ, we shall say that the BDG inequalities hold with ρ if (1.1) holds for every $(\mathcal{F}_t)$ continuous local martingale (M_t), with $M_0 = 0$.

This question has already been studied ([14], [6], [15]) and some partial answers have been given (the title of this note is inspired by a Section of M. Yor's lecture notes ([15]), and more generally, the sequel owes a lot to Marc Yor who introduced the techniques of enlargements of filtrations to me). For example, taking the special case of Brownian motion, it can easily be shown that there cannot exist a constant C such that:

$$\mathbb{E}\left[|B_\rho|\right] \leq C\mathbb{E}\left[\sqrt{\rho}\right]$$

for any random time ρ. For if it were the case, we could take $\rho = \mathbf{1}_A$, for $A \in \mathcal{F}_\infty$, and we would obtain:

$$\mathbb{E}\left[|B_1|\, \mathbf{1}_A\right] \leq C \mathbb{E}\left[\mathbf{1}_A\right]$$

which is equivalent to: $|B_1| \leq C$, a.s., which is absurd.

In the following, we shall consider two families of random times:

(1) pseudo-stopping times;
(2) honest times (or ends of optional sets).

Before proving our theorems, we need to recall a few basic facts and definitions. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space, satisfying the usual hypothese, and $\rho : (\Omega, \mathcal{F}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a random time.

We assume further that the following conditions, which we call conditions (**CA**), are satisfied:

- all $(\mathcal{F}_t)$-martingales are <u>c</u>ontinuous (e.g: in the Brownian filtration).
- the random time ρ <u>a</u>voids every $(\mathcal{F}_t)$-stopping time T, i.e. $\mathbb{P}\left[\rho = T\right] = 0$.

We enlarge the initial filtration $(\mathcal{F}_t)$ with the process $(\rho \wedge t)_{t\geq 0}$, so that the new enlarged filtration $(\mathcal{F}_t^\rho)_{t\geq 0}$ is the smallest filtration which contains $(\mathcal{F}_t)$ and makes ρ a stopping time. A few related processes will play a crucial role in our discussion:

- the $(\mathcal{F}_t)$-supermartingale

(1.2) $$Z_t^\rho = \mathbb{P}\left[\rho > t \mid \mathcal{F}_t\right]$$

  associated to ρ by Azéma (see [7] for detailed references); under (**CA**), (Z_t^ρ) is continuous;
- the $(\mathcal{F}_t)$-dual predictable projection of the process $1_{\{\rho \leq t\}}$, denoted by A_t^ρ (which is also continuous under (**CA**));

The Doob-Meyer decomposition of (1.2) writes:

$$Z_t^\rho = \mu_t^\rho - A_t^\rho,$$

with (μ_t^ρ) a continuous $(\mathcal{F}_t)$ martingale, which is in BMO.

Finally, we recall that every $(\mathcal{F}_t)$-local martingale (M_t), stopped at ρ, is a $(\mathcal{F}_t^\rho)$ semimartingale, with canonical decomposition:

(1.3) $$M_{t\wedge\rho} = \widetilde{M}_t + \int_0^{t\wedge\rho} \frac{d < M, \mu^\rho >_s}{Z_{s-}^\rho}$$

where $\left(\widetilde{M}_t\right)$ is an $(\mathcal{F}_t^\rho)$-local martingale.

Pseudo-stopping times, which are natural extensions of stopping times, have been introduced in [9]. Let us recall some results about them:

Definition 1. *We say that ρ is a $(\mathcal{F}_t)$ pseudo-stopping time if for every bounded $(\mathcal{F}_t)$-martingale (M_t), we have*

$$(1.4) \qquad\qquad \mathbb{E}M_\rho = \mathbb{E}M_0$$

Theorem 1 ([9]). *Let ρ be a random time which satisfies $\mathbb{P}\left(\rho = \infty\right) = 0$. Then the following properties are equivalent:*

(1) *ρ is a $(\mathcal{F}_t)$ pseudo-stopping time, i.e (1.4) is satisfied;*
(2) *every $(\mathcal{F}_t)$ local martingale (M_t) satisfies*

$$(M_{t\wedge\rho})_{t\geq 0} \ \ is \ a \ local \ (\mathcal{F}_t^\rho) \ martingale$$

When ρ is the end of a predictable set Γ, i.e

$$(1.5) \qquad\qquad \rho = \sup\left\{t : (t,\omega) \in \Gamma\right\}$$

it has been shown by Yor (see [14]) that there exists a universal constant C, and Φ_ρ an $\mathbb{R}_+$-valued random variable depending on ρ, such that, for every $(\mathcal{F}_t)$ local martingale (M_t) :

$$(1.6) \qquad\qquad \mathbb{E}\left[M_\rho^*\right] \leq C\mathbb{E}\left[\Phi_\rho\sqrt{<M>_\rho}\right],$$

Moreover, for any $q > 1$, $\|\Phi_\rho\|_q \leq C_q$, for a universal constant C_q, so that from (1.6):

$$\mathbb{E}\left[M_\rho^*\right] \leq C_q' \|<M>_\rho\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $C_q' = CC_q$.

However, it was not clear that the "strict" BDG inequalities (1.1) might hold for some larger class of random times than the class of stopping times.

In this note, using techniques of progressive enlargement of filtrations, we prove two theorems for the special, but important, cases when ρ is a pseudo-stopping time ([9]) or the end of a predictable set. We show that the BDG inequalities for continuous local martingales still hold when we stop these processes at a pseudo-stopping time, whereas they never hold for ends of predictable sets (under the conditions (**CA**).

2. The Burkholder-Davis-Gundy inequalities for local martingales stopped at a pseudo-stopping time or at the end of a predictable set

Theorem 2. *Let $p > 0$. There exist two universal constants c_p and C_p depending only on p, such that for any $(\mathcal{F}_t)$ local martingale (M_t), with $M_0 = 0$, and any $(\mathcal{F}_t)$ pseudo-stopping time ρ, such that $\mathbb{P}\left(\rho = \infty\right) = 0$, we have*

$$c_p\mathbb{E}\left[(<M>_\rho)^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\left(M_\rho^*\right)^p\right] \leq C_p\mathbb{E}\left[(<M>_\rho)^{\frac{p}{2}}\right].$$

Proof. It suffices, with the previous Theorem, to notice that in the enlarged filtration $(\mathcal{F}_t^\rho)$, $(M_{t\wedge\rho})$ is a martingale and ρ is a stopping time in this filtration; then, we apply the classical BDG inequalities. $\qquad\square$

Remark 1. *The constants c_p and C_p are the same as those obtained for martingales in the classical framework; in particular the asymptotics are the same (see [3]).*

Remark 2. *It would be possible to show the above Theorem, just using the definition of pseudo-stopping times (as random times for which the optional stopping theorem holds); but the proof is much longer.*

There exists a version of the BDG inequalities for discontinuous martingales, due to Meyer (see [5]), which involves the bracket $[M]_t$ of the local martingale M. The same proof as the proof of Theorem 2 also applies to show that Meyer's extension of the BDG inequalities holds for discontinuous martingales stopped at a pseudo-stopping time. More precisely,

Proposition 2. *Let $p > 0$. There exist two universal constants c_p and C_p depending only on p, such that for any $(\mathcal{F}_t)$ local martingale (M_t), with $M_0 = 0$, and any $(\mathcal{F}_t)$ pseudo-stopping time ρ we have*

$$c_p \mathbb{E}\left[([M]_\rho)^{\frac{p}{2}} \right] \leq \mathbb{E}\left[(M_\rho^*)^p \right] \leq C_p \mathbb{E}\left[([M]_\rho)^{\frac{p}{2}} \right].$$

Next, we give a negative result, which emphasizes the necessity of introducing other type of inequalities (see [14, 15]). Indeed, we show that the BDG inequalities do not hold for ends of predictable sets.

Theorem 3. *The BDG inequalities never hold for ends of predictable sets under the conditions (**CA**).*

Proof. Suppose BDG inequalities hold up to ρ, the end of a predictable set. For any local martingale in the filtration $(\mathcal{F}_t)$, we have the decomposition formula (1.3). We can apply the BDG inequalities (for the special case $p = 1$) to the local martingales $(M_{t\wedge\rho})$ and $\left(\widetilde{M}_t \right)$ in the filtration $(\mathcal{F}_t^\rho)$. This leads to the fact that there exists a constant C such that

$$\mathbb{E}\left[\sup_{t \geq 0} \left| \int_0^{t\wedge\rho} \frac{d < \widetilde{M}, \widetilde{\mu}^\rho >_s}{Z_s^\rho} \right| \right] \leq C\mathbb{E}\left[\sqrt{< \widetilde{M} >_\rho} \right]$$

It is equivalent to prove that

$$(2.1) \qquad \mathbb{E}\left[\int_0^\rho \frac{\left| d < \widetilde{M}, \widetilde{\mu}^\rho >_s \right|}{Z_s^\rho} \right] \leq C\mathbb{E}\left[\sqrt{< \widetilde{M} >_\rho} \right]$$

cannot hold to get a contradiction.

The inequality (2.1) means that

$$\int_0^{\rho\wedge t} \frac{d\widetilde{\mu}_s^\rho}{Z_s^\rho} \in BMO$$

for the filtration $\left(\mathcal{F}_{t\wedge\rho}^\rho \right)$. Indeed, from the representation theorem for $(\mathcal{F}_t^\rho)$ martingales proved by Barlow in [2], we know that every continuous local martingale in the filtration $\left(\mathcal{F}_{t\wedge\rho}^\rho \right)$ is of the form $\widetilde{M}_t$.

Now, from the following extension of Fefferman's inequality: there exists a universal constant K such that, if (X_t) and (Y_t) are two continuous semimartingales (with respect to a certain filtration, which, here, we take to be $(\mathcal{F}_t^\rho)$), then:

$$\mathbb{E}\left[\int_0^\infty |d < X, Y >_s|\right] \leq K\mathbb{E}\left[\sqrt{< X >_\infty \, \rho_2(Y)}\right]$$

where

$$\rho_2(Y) = \operatorname{ess\,sup}_t \mathbb{E}\left[< Y >_\infty - < Y >_t | \mathcal{F}_t^\rho\right]$$

we deduce that

$$(2.2) \qquad \rho_2(Y) = \operatorname{ess\,sup}_t \mathbb{E}\left[\int_t^\rho \frac{d < \mu^\rho >_s}{(Z_s^\rho)^2} \mid \mathcal{F}_t^\rho\right] \leq C$$

for $t < \rho$ and some constant C.

Moreover, it was proved by Yor in [8] that:

$$\rho_2(Y) = 2\left(1 + \log\frac{1}{I_\rho}\right)$$

where

$$I_\rho = \inf_{u \leq \rho} Z_u^\rho.$$

Hence, from (2.2):

$$\log\frac{1}{I_\rho} \leq C$$

must hold. But it is known (see [8] or [9]) that I_ρ is uniformly distributed on $(0, 1)$, so this inequality cannot hold. Hence the BDG inequalities up to ends of predictable sets do not hold in their original form. $\square$

The theorems emphasizes once again the difference between stopping times, or more generally pseudo-stopping times and ends of predictable sets (see [10] for more discussions).

We have not been able to find a random time ρ which is not a pseudo-stopping time or the end of a predictable set, and for which the BDG inequalities hold.

References

[1] J. Azéma: *Quelques applications de la théorie générale des processus I*, Invent. Math. **18** (1972) 293-336.

[2] M.T. Barlow, *Study of a filtration expanded to include an honest time*, ZW, **44**, 1978, 307-324.

[3] M.T. Barlow, M. Yor: *(Semi)-martingale inequalities and local times*, Z.W. **55** (1981), 237-254.

[4] C. Dellacherie, B. Maisonneuve, P.A. Meyer: *Probabilités et potentiel*, Chapitres XVII-XXIV: Processus de Markov (fin), Compléments de calcul stochastique, Hermann (1992).

[5] C. Dellacherie, P.A. Meyer: *Probabilités et potentiel*, Hermann, Paris, vol.II 1980.

[6] T. Jeulin, M. Yor: *Grossissement d'une filtration et semimartingales: formules explicites*, Sém.Proba. XII, Lecture Notes in Mathematics **649**, (1978), 78-97.

[7] T. Jeulin: *Semi-martingales et grossissements d'une filtration*, Lecture Notes in Mathematics **833**, Springer (1980).

[8] T. Jeulin, M. Yor (eds): *Grossissements de filtrations: exemples et applications*, Lecture Notes in Mathematics **1118**, Springer (1985).

[9] A. Nikeghbali, M. Yor: *A definition and some characteristic properties of pseudo-stopping times*, to appear in Annals of Prob.

[10] A. Nikeghbali, M. Yor: *Non stopping times and stopping theorems*, submitted, available on ArXiv.

[11] P.E. Protter: *Stochastic integration and differential equations*, Springer. Second edition (2005).

[12] D. Revuz, M. Yor: *Continuous martingales and Brownian motion*, Springer. Third edition (1999).

[13] D. Williams: *A non stopping time with the optional-stopping property*, Bull. London Math. Soc. **34** (2002), 610-612.

[14] M. Yor: *Some aspects of Brownian motion*, Part II. Some recent martingale problems. Birkhauser, Basel (1997).

[15] M. Yor: *Random times and (enlargement of filtrations) in a Brownian setting*, to be published in Lecture Notes in Mathematics, Springer (2005).

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, et CNRS UMR 7599, 175, rue du Chevaleret F-75013 Paris, France, and C.R.E.S.T, (Centre de Recherche en Economie et Statistique Théorique)

E-mail address: nikeghba@ccr.jussieu.fr