The Circular Unitary Ensemble and the Riemann zeta function: the microscopic landscape and a new approach to ratios

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Abstract

We show in this paper that after proper scalings, the characteristic polynomial of a random unitary matrix converges almost surely to a random analytic function whose zeros, which are on the real line, form a determinantal point process with sine kernel. Our scaling is performed at the so-called "microscopic" level, that is we consider the characteristic polynomial at points which are of order $1/n$ distant. We prove this in the framework of virtual isometries to circumvent the fact that the rescaled characteristic polynomial does not even have a moment of order one, hence making the classical techniques of random matrix theory difficult to apply. The strong convergence results in this setup provide us with a new approach to ratios: we are able to solve open problems about the limiting distribution of ratios of characteristic polynomials evaluated at points of the form $\exp(2i\pi\alpha/n)$ and related objects (such as the logarithmic derivative). We also explicitly describe the dependence relation for the logarithm of the characteristic polynomial evaluated at several points on the microscopic scale. On the number theory side, inspired by the Keating-Snaith philosophy, we conjecture some new limit theorems for the value distribution of the Riemann zeta function on the critical line at the stochastic process level.

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1 Introduction

A major breakthrough in the so-called random matrix approach in number theory is the seminal paper of Keating and Snaith [KS00], where they conjecture that the characteristic polynomial of a random unitary matrix, restricted to the unit circle, is a good and accurate model to predict the value distribution of the Riemann zeta function on the critical line. In particular, using this philosophy, they were able to conjecture the exact asymptotics of the moments of the Riemann zeta function, a result which was considered to be out of reach with classical tools from analytic number theory. One simple and naive explanation for the success of the characteristic polynomial as a random model to the Riemann zeta function comes from Montgomery's conjecture that asserts that the zeros of the Riemann zeta function on the critical line (after rescaling) statistically behave like the eigenangles (after rescaling) of large random unitary matrices. Moreover the limiting point process obtained from the eigenvalues is the determinantal sine kernel point process. A natural question which then naturally arose in the community was the existence of a random analytic function with zeros which are this sine kernel process and which would be obtained as a limiting object from characteristic polynomials. As we shall see below, the sequence of characteristic polynomials of random unitary matrices of growing dimensions does not converge. We shall nonetheless prove that after a proper rescaling in "time" (the characteristic polynomial can be viewed as a stochastic process with parameter $z \in \mathbb{C}$, and we shall consider the characteristic polynomial at the scale z/n) and space (that is we normalize with the value of the characteristic polynomial at 1), this sequence converges locally uniformly on compact subsets of the complex plane to a random analytic function with the desired property.

To be more specific, the convergence discussed above will be proved to occur almost surely, thanks to the use of virtual isometries introduced in [BNN12]. The basic idea behind virtual isometries is that of coupling the different dimensions of the unitary groups $U(n)$ together in such a way that marginal distribution on each $U(n)$, for fixed n, is the Haar measure. Along with some of the fine estimates on the eigenvalues from [MNN13] and some new deep estimates related to the logarithmic derivative and ratios of our limiting random analytic function, we establish almost sure convergence. This strong convergence will in turn imply the weak convergence of the same objects. But since our rescaled characteristic polynomials do not even have a moment of order one, proving the weak convergence as stated in Theorem 1.2 with classical methods does not seem to be an easy task. Moreover, this approach based on almost sure convergence will somehow miraculously provide us with straightforward proofs to some known difficult problems on ratios, logarithms and logarithmic derivatives of characteristic polynomials of random unitary matrices. The solutions to these problems always involve the limiting random analytic function discussed above, so that one might think of it as a canonical object in random matrix theory. More generally this approach to ratios of characteristic polynomials brings new insight not only in random matrix theory but also in number theory where we propose a new point of view using random analytic functions to make predictions for the value distribution of the Riemann zeta function. More precisely the contributions of this paper can be summarized as follows:

1. **Ratios and correlations on the microscopic scale:** ratios of characteristic polynomials are relevant objects which have been extensively studied in recent years, for instance in relation with quantum chaotic systems or analytic number theory (see [FS03], [SF03], [BS06], [CS07], [CFZ08], [BG06]), using a wide range of techniques (e.g. classical analysis, representation theory or supersymmetry methods). It has been an open question to characterize the limiting object obtained from ratios of characteristic polynomials evaluated at points of the form $\frac{\alpha}{n}$ for $\alpha \in \mathbb{C}$, when $n \to \infty$.[1] It was more or less observed that the expectation of such ratios converges but the limiting object was not known. In this paper, we shall give an explicit formula for the limit of the ratios and prove that the expectation of the ratios of the characteristic polynomial converges (locally uniformly on compact subsets of $\mathbb{C} - \mathbb{R}$) to the expectation of the corresponding limiting object. We shall also derive the limiting object for the rescaled logarithmic derivative of the characteristic polynomial at the microscopic scale and give two alternative formulas to compute its joint moments evaluated at several points.

   We shall also give a description of the dependence between the log of the characteristic polynomial evaluated at various points distant of $\frac{\alpha}{n}$. To the best of our knowledge this problem was not solved before.

2. **Linear statistics:** we shall see that the logarithmic derivative of our random analytic function is related to linear statistics for the sine kernel point process for test functions in $H^{1/2}$ and we shall prove a convergence result, on the mesoscopic scale, to a holomorphic Gaussian field. It should be noted that such objects, for more general point processes, have been recently studied by Aizenmann and Warzel in [AW13] and our results can be viewed as a complement to the results obtained in there in the special case of the sine kernel determinantal point process.

   Besides we also prove, using a general result from [MNN13], a weak convergence for linear statistics on the microscopic scale for test functions which are only assumed to be integrable and prove that linear statistics on this scale have a natural representation in terms of our limiting random analytic function.

3. **Value distribution of the Riemann zeta function on the critical line:** on the number theory side, we shall state some conjectures relating our limiting random analytic function to the Riemann zeta function. We conjecture that our scaling amounts to eliminating the contribution of prime numbers to keep only those of the Riemann zeros and thus obtain a limiting object whose zeros form a sine kernel determinantal point process, in agreement with the GUE conjecture and the fact that short range statistics conform the GUE predictions (unlike long-range statistics which are better described with primes). We shall also relate the logarithmic derivative of our limiting function to recent conjectures of Goldston, Gonek and Montgomery [GGM01] on the second moment of the logarithmic derivative of the Riemann zeta function. We shall be able to provide a

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[1]This question was asked to A.N. by Alexei Borodin in a private communication.

very general conjecture on the logarithmic derivative of the Riemann zeta function in agreement with the predictions obtained in [GGM01] and in [FGLL13]. Moreover the general formula for the expectation of the ratios provides simpler corresponding conjectures for the Riemann zeta function. The point of view we take is new in the literature on the random matrix approach in number theory: indeed we suggest to model ratios (and not only their expectations) by some natural random analytic functions.

In the sequel, we introduce the main objects and notation and state our main theorem.

1.1 The characteristic polynomial of random unitary matrices and the number theory connections

It is a well known fact in the theory of random unitary matrices that, when properly rescaled, the eigenvalues converge to a determinantal point process with sine kernel:

Proposition 1.1. *Let E_n denote the set of eigenvalues taken in $(-\pi, \pi]$ and multiplied by $n/2\pi$ of a random unitary matrix of size n following the Haar measure. Let us also define, for $y \neq y'$,*

$$K(y, y') = \frac{\sin[\pi(y' - y)]}{\pi(y' - y)}$$

and

$$K(y, y) = 1.$$

Let E_∞ be a determinantal sine-kernel process, i.e. a point process such that for all $r \in \{1, \ldots, n\}$, and for all Borel measurable and bounded functions F with compact support from $\mathbb{R}^r$ to $\mathbb{R}$,

$$\mathbb{E}\left(\sum_{x_1 \neq \cdots \neq x_r \in E_\infty} F(x_1, \ldots, x_r)\right) = \int_{\mathbb{R}^r} F(y_1, \ldots, y_r)\rho_r(y_1, \ldots, y_r)dy_1 \ldots dy_r,$$

where

$$\rho_r(y_1, \ldots, y_r) = \det((K(y_j, y_k))_{1 \leq j,k \leq r}).$$

Then, the point process E_n converges to E_∞ in the following sense: for all Borel measurable bounded functions f with compact support from $\mathbb{R}$ to $\mathbb{R}$,

$$\sum_{x \in E_n} f(x) \underset{n \to \infty}{\longrightarrow} \sum_{x \in E_\infty} f(x),$$

where the convergence above holds in law.

We now recall basic facts about the Riemann zeta function (the reader can find more details in classical textbooks such as [Tit86]). The Riemann zeta function is defined, for $\Re(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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It has a meromorphic continuation to the whole complex plane with a single pole at 1. It also satisfies a functional equation which we can be stated as follows:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma((1-s)/2)\zeta(1-s),$$

and

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(1-s) = \chi(s)^{-1} = 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2).$$

The non-trivial zeros of the zeta function are denoted by $\rho = \sigma + it$, where $0 < \sigma < 1$. The Riemann hypothesis is the assertion that all non trivial zeros satisfy $\sigma = 1/2$ and hence all non trivial zeros are of the form $\rho = 1/2 + it$, with $t \in \mathbb{R}$. If we assume the Riemann hypothesis, then the zeros come in conjugate pairs and we note the zeros in the upper half-plane as $1/2 + i\gamma_j$, where $0 < \gamma_1 \leq \gamma_2 \leq \cdots$. One can count the number of such zeros up to some height T:

$$N(T) := \#\{j;\ 0 \leq \gamma_j \leq T\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

The connection to random matrix theory was conjectured by Montgomery in [Mon73]: it is conjectured that the rescaled zeros of the zeta function $\tilde{\gamma} := \gamma/(2\pi)\log\gamma$ (this rescaling is done in order to obtain an average spacing of order 1) satisfy the same limit theorem as the one given in Proposition 1.1 for the rescaled eigenvalues of random unitary matrices (in fact the conjecture was initially stated for the pair correlation and then extended to all correlations by Rudnik and Sarnak in [RS96]; see the recent paper of Conrey and Snaith [CS14] for a detailed account and new methods).

Another major insight came with the work of Keating and Snaith ([KS00]) where they use the characteristic polynomial of random unitary matrices to model the value distribution of the Riemann zeta function on the critical line (i.e. the family $\{\zeta(1/2 + it),\ t \geq 0\}$) to make spectacular predictions on the moments of the Riemann zeta function. In particular, in [KS00] they computed the moments of the characteristic polynomial of a random unitary matrix following the Haar measure. They deduced that the characteristic polynomial asymptotically behaves like a log-normal distributed random variable when the dimension n goes to infinity: more precisely, its logarithm, divided by $\sqrt{\log n}$, tends to a complex Gaussian random variable Z such that $\mathbb{E}[Z] = \mathbb{E}[Z^2] = 0$ and $\mathbb{E}[|Z|^2] = 1$. This result has been generalized in Hughes, Keating and O'Connell [HKO01], where the authors proved the asymptotic independence of the characteristic polynomial taken at different fixed points. A question which then naturally arises concerns the behavior of the characteristic polynomial at points which vary with the dimension and which are sufficiently close to each other in order to avoid asymptotic independence. The scale we consider in the present paper is the average spacing of the eigenangles of a unitary matrix in dimension n, i.e. $2\pi/n$. More precisely, let $(U_n)_{n\geq 1}$ be a sequence of matrices, U_n being Haar-distributed in $U(n)$, and let Z_n be the characteristic polynomial of U_n:

$$Z_n(X) = \det\left(\mathrm{Id} - U_n^{-1}X\right) = \det\left(\mathrm{Id} - U_n^*X\right). \tag{1.1}$$

For a given $z \in \mathbb{C}$, we consider the value of Z_n at the two points 1 and $e^{2iz\pi/n}$, whose distance is equivalent to $2\pi|z|/n$ when n goes to infinity. We know that the law of $Z_n(1)$ can be approximated by the exponential of a gaussian variable of variance $\log n$, so it does not converge when n goes to infinity. The same is true for $Z_n(e^{2iz\pi/n})$. In order to obtain a convergence in law, it is then natural to consider the ratio $Z_n(e^{2iz\pi/n})/Z_n(1)$, which has order of magnitude 1 and which is well-defined as soon as 1 is not an eigenvalue of U_n, an event occurring almost surely.

If we consider all the values of z together, we obtain a random entire function ξ_n, defined by

$$\xi_n(z) = \frac{Z_n(e^{2iz\pi/n})}{Z_n(1)}. \tag{1.2}$$

Because $(U_n)_{n \geq 1}$ is a sequence of unitary matrices, the following functional equation holds:

$$\overline{\xi_n(z)} = e^{-i2\pi\bar{z}}\xi_n(\bar{z}) \tag{1.3}$$

We will prove that this function has a limiting distribution when n goes to infinity. More precisely, one of the main results of this article is the following:

Theorem 1.2. *In the space of continuous functions from $\mathbb{C}$ to $\mathbb{C}$, endowed with the topology of uniform convergence on compact sets, the random entire function ξ_n converges in law to a limiting entire function ξ_∞. The zeros of ξ_∞ are all real and form a determinantal sine-kernel point process, i.e. for all $r \geq 1$, the r-point correlation function ρ_r corresponding to this point process is given, for all $x_1, \ldots, x_r \in \mathbb{R}$, by*

$$\rho_r(x_1, \ldots, x_r) = \det\left(\frac{\sin[\pi(x_j - x_k)]}{\pi(x_j - x_k)}\right)_{1 \leq j,k \leq r}.$$

Notice that this theorem cannot be straightforwardly deduced from the convergence of the zeroes of ξ_n to a sine-kernel process. Afterall, the convergence of point processes is local in nature while the random analytic function ξ_∞ is certainly not local, being a infinite product over all zeroes.

Taking a finite number of points $z_1, \ldots, z_p \in \mathbb{C}$, we see in particular that the joint law of the mutual ratios of $Z_n(e^{2i\pi z_1/n}), \ldots, Z_n(e^{2i\pi z_p/n})$ converges when n goes to infinity. Now one can hope to gain new insights on the behaviour of ratios of characteristic polynomials on this microscopic scale. More precisely, let us define:

$$R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r) := \frac{Z_n(e^{2i\alpha_1\pi/n}) \cdots Z_n(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\beta_1\pi/n}) \cdots Z_n(e^{2i\beta_r\pi/n})}, \tag{1.4}$$

where $r \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C}$, for all $1 \leq j \leq r$. Ratios such as (1.4), on the macroscopic scale (i.e. without the $1/n$ in the arguments) have been extensively studied in random matrix theory for different random matrix ensembles, e.g. the GUE by Fyodorov and Strahov in [FS03] and [SF03], the COE and the CSE by Borodin and Strahov in [BS06] or in the CUE case by Conrey, Farmer and Zirnbauer ([CFZ08]), by Conrey and Snaith ([CS07]) or Bump and Gamburd ([BG06]). In all cases, one considers the

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expectation of the ratios and the n-limit of this expression. But finding the n-limit of $R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r)$ had remained an open problem. In fact, we shall prove a strong version (i.e. with almost sure convergence) of Theorem 1.2 which will immediately yield the n-limit of $R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r)$ as well as some central limit theorem for the vector $(\log Z_n(e^{2i\pi z/n}), \log Z_n(1))$. The almost sure convergence is established through the machinery of virtual isometries that we recall in the next paragraph.

1.2 Virtual isometries and almost sure convergence

In order to prove Theorem 1.2, we will define the sequence $(U_n)_{n\geq 1}$ of unitary matrices in a common probability space, with a coupling chosen in such a way that an almost sure convergence occurs. An interest of this method is that it is more convenient to deal with pointwise convergence than with convergence in law when we work on a functional space. Moreover, the coupling gives a powerful way to keep track of the sequence $(\xi_n)_{n\geq 1}$ of holomorphic functions, and a deterministic link between this sequence and the limiting function ξ_∞.

Besides it is important to stress that the moments method, which is a classical technique in random matrix theory, seems tedious to implement at best. Indeed the random function at hand ξ_n does not have any integer moment when evaluated on the circle, which makes the use of the formulas on moments of ratios in [BG06] and [CFZ08] difficult to use. For example, in Theorem 3 of the article [BG06], one clearly sees the divergence of moments of ratios, as the evaluation points get close to 1.

The coupling we consider here corresponds to the notion of *virtual isometries*, as defined by Bourgade, Najnudel and Nikeghbali in [BNN12]. The sequence $(U_n)_{n\geq 1}$ can be constructed in the following way:

1. One considers a sequence $(x_n)_{x\geq 1}$ of independent random vectors, x_n being uniform on the unit sphere of $\mathbb{C}^n$.

2. Almost surely, for all $n \geq 1$, x_n is different from the last basis vector e_n of $\mathbb{C}^n$, which implies that there exists a unique $R_n \in U(n)$ such that $R_n(e_n) = x_n$ and $R_n - I_n$ has rank one.

3. We define $(U_n)_{n\geq 1}$ by induction as follows: $U_1 = x_1$ and for all $n \geq 2$,

$$U_n = R_n \begin{pmatrix} U_{n-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

It has already been proven in [BHNY08] that with this construction, U_n follows, for all $n \geq 1$, the Haar measure on $U(n)$. From now on, we always assume that the sequence $(U_n)_{n\geq 1}$ is defined with this coupling.

For each value of n, let $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ be the eigenvalues of U_n, ordered counterclockwise, starting from 1: they are almost surely pairwise distinct and different from 1. If $1 \leq k \leq n$, we denote by $\theta_k^{(n)}$ the argument of $\lambda_k^{(n)}$, taken in the interval $(0, 2\pi)$: $\theta_k^{(n)}$ is the k-th strictly positive eigenangle of U_n. If we consider all the eigenangles of U_n, taken not only in $(0, 2\pi)$ but in the whole real line, we get a (2π)-periodic set with

n points in each period. If the eigenangles are indexed increasingly by $\mathbb{Z}$, we obtain a sequence

$$\cdots < \theta_{-1}^{(n)} < \theta_0^{(n)} < 0 < \theta_1^{(n)} < \theta_2^{(n)} < \cdots,$$

for which $\theta_{k+n}^{(n)} = \theta_k^{(n)} + 2\pi$ for all $k \in \mathbb{Z}$.

It is also convenient to extend the sequence of eigenvalues as a n-periodic sequence indexed by $\mathbb{Z}$, in such a way that for all $k \in \mathbb{Z}$,

$$\lambda_k^{(n)} = \exp\left(i\theta_k^{(n)}\right).$$

With the notation above, the following holds:

Theorem 1.3 (Theorem 7.3 in [MNN13]). *Almost surely, the point process*

$$\left(y_k^{(n)} := \frac{n}{2\pi}\theta_k^{(n)}, \ k \in \mathbb{Z}\right)$$

converges pointwise to a determinantal sine-kernel point process $(y_k, \ k \in \mathbb{Z})$. *And moreover, almost surely, the following estimate holds for all $\varepsilon > 0$:*

$$\forall k \in [-n^{\frac{1}{4}}, n^{\frac{1}{4}}], \ y_k^{(n)} = y_k + O_\varepsilon\left((1 + k^2)n^{-\frac{1}{3}+\varepsilon}\right)$$

Remark 1.4. *The implied constant in O_ε is random: more precisely, it may depend on the sequence $(U_m)_{m \geq 1}$ and on ε. However, it does not depend on k and n.*

We are now able to state the main convergence result of the paper.

Theorem 1.5. *Almost surely and uniformly on compact subsets of $\mathbb{C}$, we have the convergence:*

$$\xi_n(z) \overset{n\to\infty}{\longrightarrow} \xi_\infty(z) := e^{i\pi z} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{y_k}\right)$$

Here, the infinite product is not absolutely convergent. It has to be understood as the limit of the following product, obtained by regrouping the factors two by two:

$$\left(1 - \frac{z}{y_0}\right) \prod_{k\geq 1} \left[\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right],$$

which is absolutely convergent.

This theorem immediately implies Theorem 1.2, provided that ξ_∞ is entire and that the zeros of ξ_∞ are exactly given by the sequence $(y_k)_{k\in\mathbb{Z}}$. This first point is a direct consequence of the fact that ξ_∞ is the uniform limit on compact sets of the sequence of entire functions $(\xi_n)_{n\geq 1}$, and the second point is a consequence of the fact that the k-th factor of the absolutely convergent product above vanishes at y_k and y_{-k} and only at these points.

Now, thanks to the almost sure convergence, we can state the following corollaries.

Corollary 1.6. *Let $r \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C}$ but $\beta_j \notin (y_k)_{k \in \mathbb{Z}}$, for all $1 \leq j \leq r$. Then the following convergence holds a.s. as $n \to \infty$:*

$$R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r) := \frac{Z_n(e^{2i\alpha_1\pi/n}) \cdots Z_n(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\beta_1\pi/n}) \cdots Z_n(e^{2i\beta_r\pi/n})} \to \frac{\xi_\infty(\alpha_1) \cdots \xi_\infty(\alpha_r)}{\xi_\infty(\beta_1) \cdots \xi_\infty(\beta_r)}$$

In Section 5 we shall establish that the above convergence also holds in expectation locally uniformly. Since the convergence in Theorem 1.5 holds almost surely in the space of holomorphic functions, we immediately obtain:

Corollary 1.7. *We have a.s. uniformly on compact sets, that as $n \to \infty$:*

$$\frac{2i\pi}{n} \frac{Z'_n(e^{2i\pi z/n})}{Z_n(1)} \to \xi'_\infty.$$

The next corollary involves the logarithm of Z_n. We provide a simple proof thanks to our functional convergence result. The determination of this logarithm is the only one such that $\log Z_n$ vanishes at 0 (recall that $Z_n(0) = 1$), and which is continuous on the following maximal simply connected domain

$$\mathcal{D} := \mathbb{C} \setminus \left\{ re^{i\theta_k^{(n)}} \mid k \in \mathbb{Z}, \ r \geq 1 \right\}.$$

Note that for all $z \in \mathcal{D}$, we have:

$$\log Z_n(z) = \sum_{k=1}^{n} \log\left(1 - \frac{z}{\lambda_k^{(n)}}\right),$$

where the principal branch of the logarithm is considered.

Corollary 1.8. *Let $r \in \mathbb{N}$. and fix $(z_1, z_2, \ldots, z_r) \in \mathbb{C}^r$. The following convergence holds in law as $n \to \infty$*

$$\left(\frac{\log Z_n(e^{2i\pi z_1/n})}{\sqrt{\frac{1}{2}\log n}}, \ldots, \frac{\log Z_n(e^{2i\pi z_r/n})}{\sqrt{\frac{1}{2}\log n}} \right) \to \mathcal{N} e$$

where $\mathcal{N}$ stands for a standard complex Gaussian random variable, and e denotes the vector $(1, 1, \ldots, 1) \in \mathbb{C}^r$.

A more general version of the corollary, and a similar result relative to the behavior of the Riemann zeta function near the critical line, have been obtained by Bourgade in [Bou10] (see Theorems 1.1. and 1.4. there).

Proof. Let z be a complex number among $(z_1, z_2, \ldots, z_r)$. One checks that

$$\log Z_n(e^{2i\pi z/n}) - \log Z_n(1) = \log \xi_n(z),$$

where $\log \xi_n$ is the unique determination of the logarithm, vanishing at 0, and continuous in the domain

$$\mathcal{D}'_n := \mathbb{C} \setminus \left\{ y_k^{(n)} - iu \mid k \in \mathbb{Z}, u \geq 0 \right\}.$$

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Let $\log \xi_\infty$ be the similar determination of the logarithm of ξ_∞. Let us fix $z \in \mathbb{C}$, $t > 0$ such that $z + it$ has strictly positive imaginary part, and let L be the line consisting of the two segments from 0 to $z + it$ and from $z + it$ to z. We also recall that the random functions $(\xi_n)_{n \geq 1}$ and ξ_∞ are coupled in such a way that almost surely, ξ_n tends to ξ_∞ uniformly on compact sets of $\mathbb{C}$. Almost surely, for n large enough, 0 and $\Re z$ are not zeros of ξ_n and one deduces that L is included in $\mathcal{D}'_n$. Hence,

$$\log \xi_n(z) = \int_L \frac{\xi'_n(s)}{\xi_n(s)} ds$$

and

$$\log \xi_\infty(z) = \int_L \frac{\xi'_\infty(s)}{\xi_\infty(s)} ds.$$

Now, (ξ_n, ξ'_n) tends to $(\xi_\infty, \xi'_\infty)$ uniformly on L. Moreover, ξ_∞ is continuous and non-vanishing on the compact set L, which implies that $|\xi_\infty|$, and then $|\xi_n|$ for n large enough, are bounded away from zero on L. Hence, ξ'_n/ξ_n tends to ξ'_∞/ξ_∞ uniformly on L, and then $\log \xi_n(z)$ tends to $\log \xi_\infty(z)$. We deduce that

$$\frac{\log Z_n(e^{2i\pi z/n})}{\sqrt{(1/2) \log n}} - \frac{\log Z_n(1)}{\sqrt{(1/2) \log n}} \xrightarrow[n \to \infty]{} 0$$

almost surely with the coupling above, and then in probability. Since we already know that the second term of the difference tends in law to $\mathcal{N}$, we are done. $\qquad\square$

We can use the a.s. convergence of $\log \xi_n$ to $\log \xi_\infty$ that we established above to give a simple proof for the convergence of the number of points in an arc at the microscopic scale:

Corollary 1.9. *For all $z \in \mathbb{R}$, the number of eigenvalues of U_n in the arc between 1 and $e^{2i\pi z/n}$ tends in law to the number of points of a determinantal sine-kernel process in an interval of length $|z|$.*

Proof. For $z \in \mathbb{R}$, the number of eigenvalues of U_n in the arc between 1 and $e^{2i\pi z/n}$, multiplied by the sign of z, is equal to (see Corollary 2.3)

$$N_n(z) := z - \frac{1}{\pi} \Im \log(\xi_n(z)),$$

if 1 and $e^{2i\pi z/n}$ are not eigenvalues of U_n (which holds almost surely). Now we know that $\log(\xi_n)$ tends a.s. to $\log(\xi_\infty)$ when n goes to infinity. The proof of the corollary is completed once one notes that

$$N_\infty(z) := z - \frac{1}{\pi} \Im \log(\xi_\infty(z)),$$

has the same absolute value as the number of zeros of ξ_∞ between 0 and z. $\qquad\square$

Remark 1.10. *We shall prove in Section 3.2 more general results on convergence of linear statistics at the microscopic scale.*

Remark 1.11. *From Corollary 1.7 one can also deduce a joint central limit theorem for the log of the derivative of the characteristic polynomial at $e^{2i\pi z/n}$ and the log of the characteristic polynomial at 1.*

We can eventually easily derive the limiting random analytic function for the logarithmic derivative:

Corollary 1.12. *We have almost surely, for all $z \notin \{y_k, \ k \in \mathbb{Z}\}$:*

$$\frac{2i\pi}{n} \frac{Z_n'(e^{2i\pi z/n})}{Z_n(e^{2i\pi z/n})} \underset{n \to \infty}{\longrightarrow} \frac{\zeta_\infty'(z)}{\zeta_\infty(z)},$$

where

$$\frac{\zeta_\infty'(z)}{\zeta_\infty(z)} = i\pi + \sum_{k \in \mathbb{Z}} \frac{1}{z - y_k} := i\pi + \frac{1}{z - y_0} + \sum_{k=1}^{\infty} \left(\frac{1}{z - y_k} + \frac{1}{z - y_{-k}} \right).$$

Hence, for all $\alpha_1, \ldots, \alpha_r \notin \{y_k, \ k \in \mathbb{Z}\}$,

$$\left(\frac{2i\pi}{n} \right)^r \frac{Z_n'(e^{2i\alpha_1\pi/n})}{Z_n(e^{2i\alpha_1\pi/n})} \frac{Z_n'(e^{2i\alpha_2\pi/n})}{Z_n(e^{2i\alpha_2\pi/n})} \cdots \frac{Z_n'(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\alpha_r\pi/n})} \underset{n \to \infty}{\longrightarrow} \frac{\zeta_\infty'(\alpha_1)}{\zeta_\infty(\alpha_1)} \cdots \frac{\zeta_\infty'(\alpha_r)}{\zeta_\infty(\alpha_r)}.$$

1.3 Outline of the paper

The proof of Theorem 1.5 will be made in several steps in Section 2 , using estimates on the argument of Z_n, stated in Section 2.1, and estimates on the renormalized eigenangles $y_k^{(n)}$, stated in Section 2.2.

In Section 3, we establish some properties of the limiting random function ζ_∞, and prove some general results about convergence of linear statistics at the microscopic scale. Unlike other scales, convergence in law is proved for very general test functions (essentially integrable) and, as expected, no renormalization is needed in the non smooth cases (e.g. indicator functions).

In Section 4, we prove some fine and technical estimates related to the logarithmic derivative that we shall need later for the convergence of moments of ratios and we state some related conjectures on the behavior of the Riemann zeta function in the neighborhood of the critical line.

In Section 5, using estimates from previous sections and the work of Borodin, Olshanski and Strahov ([BOS06]), we prove the convergence of the expectation of ratios of characteristic polynomials to the corresponding expectations of ratios of ζ_∞. This in turn provides simpler formulas for the corresponding conjecture for ratios of the Riemann zeta function. The results in Section 5 complete the convergence obtained in Corollary 1.6 and we can summarize them in the following proposition:

Theorem 1.13. *The following results on ratios hold:*

1. For any $p > 0$ and any compact set $K \subset \mathbb{C} \backslash \mathbb{R}$, we have:

$$\sup_{n \in \mathbb{N} \sqcup \{\infty\}} \mathbb{E} \left(\sup_{(z,z') \in K^2} \left| \frac{\zeta_n(z')}{\zeta_n(z)} \right|^p \right) < \infty.$$

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2. *For $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{C} \backslash \mathbb{R}$, and for all $n \in \mathbb{N} \sqcup \{\infty\}$,*

$$\mathbb{E}\left(\prod_{j=1}^{k}\left|\frac{\xi_n(z'_j)}{\xi_n(z_j)}\right|\right) < \infty$$

Moreover, for every compact set K in $\mathbb{C} \backslash \mathbb{R}$, we have the following convergence, uniformly in $z_1, z_2, \ldots, z_k, z'_1, \ldots, z'_k \in K$:

$$\mathbb{E}\left(\prod_{j=1}^{k}\frac{\xi_n(z'_j)}{\xi_n(z_j)}\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(\prod_{j=1}^{k}\frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right).$$

3. *For all $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{C} \backslash \mathbb{R}$ such that $z_i \neq z'_j$ for $1 \leq i, j \leq n$, we have*

$$\det\left(\frac{1}{z_i - z'_j}\right)_{i,j=1}^{k} \mathbb{E}\left(\prod_{j=1}^{k}\frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right) = \det\left(\frac{1}{z_i - z'_j}\mathbb{E}\left(\frac{\xi_\infty(z'_j)}{\xi_\infty(z_i)}\right)\right)_{i,j=1}^{k}$$

and moreover:

$$\mathbb{E}\left(\frac{\xi_\infty(z')}{\xi_\infty(z)}\right) = \begin{cases} 1 & \text{if } \Im(z) > 0 \\ e^{i2\pi(z'-z)} & \text{if } \Im(z) < 0 \end{cases}$$

And we conjecture that if ω is a uniform random variable on $[0,1]$ and $T > 0$ a real parameter going to infinity, then, for all $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{C} \backslash \mathbb{R}$, such that $z_i \neq z'_j$ for all i, j,

$$\mathbb{E}\left(\prod_{j=1}^{k}\frac{\zeta\left(\frac{1}{2} + iT\omega - \frac{i2\pi z'_j}{\log T}\right)}{\zeta\left(\frac{1}{2} + iT\omega - \frac{i2\pi z_j}{\log T}\right)}\right)$$

$$\xrightarrow{T\to\infty} \det\left(\frac{1}{z_i - z'_j}\right)^{-1}\det\left(\frac{\mathbb{1}_{\Im(z_i)>0} + e^{2i\pi(z'_j - z_i)}\mathbb{1}_{\Im(z_i)<0}}{z_i - z'_j}\right)_{i,j=1}^{k},$$

where the last expression is well-defined where the z_i and the z'_j are all distinct, and is extended by continuity to the case where some of the z_i or some of the z'_j are equal.

This last conjecture looks simpler than the usual one (see e.g. [CFZ08]) which involves complicated sums and difficult combinatorics. Note that this simpler form of the conjecture first appeared in Rodgers' work [Rod15] where he also used the Borodin-Olshanski-Strahov formula. It should be added that it was already observed by the authors in [BOS06] that taking the limit in the expectation of ratios of characteristic polynomials made sense. However, the natural question whether the ratios themselves converge remained open, as well as establishing the convergence stated in the proposition above.

The expectation of products of the logarithmic derivative evaluated at distinct points was also computed in [CFZ08]; we also provide an alternative formula using the determinantal form above.

Eventually, in Section 6, we prove that in a sense which can be made precise, the fluctuations of the determinantal sine-kernel process, viewed at a scale tending to infinity, converge in law to a blue noise, i.e. a noise whose spectral density is proportional to the frequency. In relation with this convergence, we show that the fluctuations of $\zeta'_\infty / \zeta_\infty$, viewed at a large scale, tend to a holomorphic Gaussian process on $\mathbb{C} \backslash \mathbb{R}$, whose covariance structure is explicitly computed. This covariance is consistent with the computation of the two first moments of $\zeta'_\infty / \zeta_\infty$.

Acknowledgement

2 Proof of Theorem 1.5

2.1 On the argument of the characteristic polynomial

In this section, we study the argument of Z_n, in order to deduce estimates on the deviation of $y_k^{(n)}$ from k.

Here, we define the argument as the imaginary part of $\log Z_n$, with the determination of the logarithm given in the previous section.

The next proposition gives a link between the number of eigenvalues of U_n in a given arc of circle, and the variation of the argument of Z_n along this arc. The derivation is relatively standard and we shall not reproduce a proof here (see [Hug01], p. 35-36. or [BHNN13], proof of Proposition 2.2).

Proposition 2.1. *Consider A and B two points on the unit circle. Note $\widehat{AB}$ for the arc joining A and B counterclockwise. Denote by $\ell\left(\widehat{AB}\right)$ the length of the arc and $N\left(\widehat{AB}\right)$ the number of zeros of Z_n in the arc. We assume that A and B are not zeros of Z_n. Then:*

$$N\left(\widehat{AB}\right) = \frac{n\ell\left(\widehat{AB}\right)}{2\pi} - \frac{1}{\pi}\left[\Im\log\left(Z_n(B)\right) - \Im\log\left(Z_n(A)\right)\right].$$

Remark 2.2. *This shows that the imaginary part of the determination of the logarithm $\Im\log Z_n(z)$ increases with speed $n/2$ and jumps by $-\pi$ when encountering a zero.*

Corollary 2.3. *Let $k \in \mathbb{Z}$, and let $\varepsilon > 0$ be small enough so that there are no eigenangles of U_n in $[0, \varepsilon]$ and $(\theta_k^{(n)}, \theta_k^{(n)} + \varepsilon]$. Then:*

$$k = y_k^{(n)} - \frac{1}{\pi}\Im\left(\log\left(Z_n(e^{i(\theta_k^{(n)}+\varepsilon)})\right) - \log\left(Z_n(e^{i\varepsilon})\right)\right)$$

Proof. Notice first that if k is increased by n, $\theta_k^{(n)}$ increases by 2π, $y_k^{(n)}$ increases by n, $\lambda_k^{(n)} = e^{i\theta_k^{(n)}}$ does not change, and the assumption made on ε remains the same. Hence, in the equality we want to prove, the right-hand side and the left-hand side both increase by n, which implies that it is sufficient to show the corollary for $1 \leq k \leq n$. If these inequalities are satisfied, let us choose, in the previous proposition, $A = e^{i\varepsilon}$ and $B = e^{i(\theta_k^{(n)}+\varepsilon)}$. Then we note that

$$N\left(\widehat{AB}\right) = k,$$

and

$$\frac{n\ell\left(\widehat{AB}\right)}{2\pi} = \frac{n\theta_k^{(n)}}{2\pi} = y_k^{(n)},$$

which proves the corollary. $\qquad\square$

This corollary shows that it is equivalent to control the argument of Z_n, and the distance between k and $y_k^{(n)}$. In the remaining of this section, we give some explicit bounds on the distribution of $\Im \log(Z_n)$ on the unit circle.

Proposition 2.4. *For all $x > 0$, one has*

$$\mathbb{P}\left(|\Im\left(\log Z_n(1)\right)| \geq x\right) \leq 2\exp\left(-\frac{x^2}{C + \log n}\right),$$

where $C > 0$ is a universal constant.

Remark 2.5. *In the proof below, we prove that one can take $C = \frac{\pi^2}{6} + 1$.*

Proof. Let us note

$$X_n = \Im\left(\log Z_n(1)\right)$$

Thanks to the formula (1.1) in [BHNY08]:

$$\forall \lambda \in \mathbb{R}, \mathbb{E}\left(e^{\lambda X_n}\right) = \prod_{k=1}^{n} \frac{\Gamma(k)^2}{\Gamma\left(k + \frac{i\lambda}{2}\right)\Gamma\left(k - \frac{i\lambda}{2}\right)}$$

Let us start with the standard Chernoff bound:

$$\forall \lambda > 0, \mathbb{P}\left(X_n \geq x\right) \leq e^{-\lambda x}\mathbb{E}\left(e^{\lambda X_n}\right).$$

Now, using the infinite product formula for the Gamma function:

$$\forall z \in \mathbb{C}, \frac{1}{\Gamma(z)} = e^{\gamma z}z\prod_{j=1}^{\infty}\left(1 + \frac{z}{j}\right)e^{-z/j},$$

we have:

$$\mathbb{E}\left(e^{\lambda X_n}\right) = \prod_{k=1}^{n} \frac{\Gamma(k)^2}{\Gamma\left(k + \frac{i\lambda}{2}\right)\Gamma\left(k - \frac{i\lambda}{2}\right)}$$

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$$= \prod_{k=1}^{n} \left(\frac{k^2 + \frac{\lambda^2}{4}}{k^2} \prod_{j=1}^{\infty} \frac{\left(1 + \frac{k + \frac{i\lambda}{2}}{j}\right)\left(1 + \frac{k - \frac{i\lambda}{2}}{j}\right)}{\left(1 + \frac{k}{j}\right)^2} \right)$$

$$= \prod_{k=1}^{n} \left(\frac{k^2 + \frac{\lambda^2}{4}}{k^2} \prod_{j=1}^{\infty} \frac{\left(j + k + \frac{i\lambda}{2}\right)\left(j + k - \frac{i\lambda}{2}\right)}{(j + k)^2} \right)$$

$$= \prod_{k=1}^{n} \prod_{j=0}^{\infty} \frac{(j + k)^2 + \frac{\lambda^2}{4}}{(j + k)^2}$$

$$= \prod_{k=1}^{n} \prod_{j=0}^{\infty} \left(1 + \frac{\lambda^2}{4(j + k)^2}\right)$$

$$\leq \exp\left(\sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{\lambda^2}{4(j + k)^2}\right)$$

$$= \exp\left(\frac{\lambda^2}{4} \sum_{k=1}^{n} \sum_{j=k}^{\infty} \frac{1}{j^2}\right)$$

$$\leq \exp\left(\frac{\lambda^2}{4} \sum_{k=1}^{n} \left(\frac{1}{k^2} + \int_k^{\infty} \frac{dt}{t^2}\right)\right)$$

$$= \exp\left(\frac{\lambda^2}{4} \sum_{k=1}^{n} \left(\frac{1}{k^2} + \frac{1}{k}\right)\right)$$

$$\leq \exp\left(\frac{\lambda^2}{4} \left(\frac{\pi^2}{6} + 1 + \log n\right)\right)$$

Eventually for $C = \frac{\pi^2}{6} + 1$, we obtain

$$\mathbb{P}\left(X_n \geq x\right) \leq \min_{\lambda > 0} e^{-\lambda x + \frac{\lambda^2}{4}(C + \log n)}.$$

The minimum is reached for $\lambda = \frac{2x}{C + \log n}$, giving us the bound:

$$\mathbb{P}\left(\Im\left(\log Z_n(1)\right) \geq x\right) \leq \exp\left(-\frac{x^2}{C + \log n}\right).$$

The desired bound is obtained from the symmetry of $\Im\left(\log Z_n(1)\right)$, as eigenvalues are invariant in law under conjugation:

$$\mathbb{P}\left(|\Im\left(\log Z_n(1)\right)| \geq x\right)$$
$$= \mathbb{P}\left(\Im\left(\log Z_n(1)\right) \geq x\right) + \mathbb{P}\left(-\Im\left(\log Z_n(1)\right) \geq x\right)$$
$$= 2\mathbb{P}\left(\Im\left(\log Z_n(1)\right) \geq x\right)$$

$\square$

We deduce the following estimate on the maximum of the argument of Z_n on the unit circle:

Proposition 2.6. *Almost surely:*

$$\sup_{|z|=1, z\in\mathcal{D}} |\Im \log Z_n(z)| = O(\log n)$$

More precisely, for any $D > \sqrt{2}$:

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \quad \sup_{|z|=1, z\in\mathcal{D}} |\Im \log Z_n(z)| \leq D \log n$$

which means that almost surely:

$$\limsup_n \frac{1}{\log n} \sup_{|z|=1, z\in\mathcal{D}} |\Im \log Z_n(z)| \leq \sqrt{2}$$

Proof. Consider n regularly spaced points on the circle, say:

$$x_{k,n} := e^{i\frac{2\pi k}{n}}, \quad k = 0, 1, 2, \ldots, n-1,$$

and the events:

$$A_{k,n} := \{|\Im \log Z_n(x_{k,n})| \geq D \log n\}$$

Because the law of the spectrum of U_n is invariant under rotation, all the events $A_{k,n}$ have the same probability for different k's. Moreover, thanks to the previous Chernoff bound:

$$n\mathbb{P}(A_{0,n}) \leq 2n \exp\left(-\frac{D^2(\log n)^2}{C + \log n}\right)$$
$$\leq 2n \exp\left(-D^2(\log n - C)\right)$$
$$\leq 2e^{D^2 C} n^{1-D^2}$$

Hence:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{P}(A_{k,n}) = \sum_{n=1}^{\infty} n\mathbb{P}(A_{0,n}) < \infty$$

The Borel-Cantelli lemma ensures that, almost surely:

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall k, \quad |\Im \log Z_n(x_{k,n})| \leq D \log n$$

Now consider a point $z = e^{i\theta} \in \mathcal{D}$. For fixed n, it lies on the arc between $x_{k,n}$ and $x_{k+1,n}$ for a certain k. Because

$$\theta \mapsto \Im \log Z_n(e^{i\theta})$$

is piece-wise linear, increasing with speed $n/2$ and only jumping by $-\pi$, we have:

$$\Im \log Z_n(e^{i\theta}) \leq \Im \log Z_n(x_{k,n}) + \frac{n}{2}\left(\theta - \frac{2\pi k}{n}\right) \leq \Im \log Z_n(x_{k,n}) + \pi$$

In the other direction, we have

$$\Im \log Z_n(e^{i\theta}) \geq \Im \log Z_n(x_{k+1,n}) - \frac{n}{2}\left(\frac{2\pi(k+1)}{n} - \theta\right) \geq \Im \log Z_n(x_{k+1,n}) - \pi$$

So that, almost surely:

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall z \in \mathcal{D}, \quad |\Im \log Z_n(z)| \leq \pi + D \log n$$

The more precise estimate $|\Im \log Z_n(z)| \leq D \log n$ follows after replacing D by $D' \in (\sqrt{2}, D)$ in the previous computation and considering n_0 large enough so that $\pi < (D - D') \log n$. $\qquad\square$

2.2 Precise estimates for the eigenvalues of virtual isometries

The following estimate will reveal crucial for the proof of Theorem 1.5.

Proposition 2.7. *Almost surely and uniformly in n and k:*

$$y_k^{(n)} = k + O\left(\log(2 + |k|)\right)$$

In fact, if $y_k^{(n)}$ is replaced by y_k ($n \to \infty$), this estimate is already easily deduced from existing literature (for example [MM13], [Sos02]). The main tool used here is the following lemma:

Lemma 2.8. *Let E be a point process equal to $\{y_k, k \in \mathbb{Z}\}$ or to $\{y_k^{(n)}, k \in \mathbb{Z}\}$ for some $n \geq 1$. Then, for all finite intervals I, we have*

$$\mathbb{E}(X_I) = |I|, \tag{2.1}$$

$$\mathrm{Var}(X_I) \leq 2 + \frac{2}{\pi^2}\log(1 + |I|), \tag{2.2}$$

where $|I|$ denotes the length of I, X_I the number of points of E in I. Moreover the following tail estimates hold for the random variables $\widetilde{X_I} := X_I - |I|$:

$$\forall t \geq 0, \mathbb{P}\left(\left|\widetilde{X_I}\right| \geq t\right) \leq \exp\left(-\min\left(\frac{t^2}{4\,\mathrm{Var}(X_I)}, \frac{t}{2}\right)\right),$$

and, all the exponential moments of X_I are finite, with the following bound for $0 \leq q < \frac{1}{2}$,

$$\mathbb{E}\left(e^{q\left|\widetilde{X_I}\right|}\right) \leq \frac{1}{1 - 2q} + q\sqrt{4\pi\,\mathrm{Var}(X_I)}e^{4q^2\,\mathrm{Var}(X_I)}.$$

Proof. Equation (2.1) is a consequence of the fact that the 1-point correlation function of the point processes E is identically 1.

Let us now prove the bound (2.2). Let $f = \mathbb{1}_I$; we have $|\hat{f}(k)|^2 = \frac{2\sin^2\left(\frac{|I|k}{2}\right)}{\pi k^2}$, where the Fourier transform of f is normalized as follows:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ikx}dx.$$

We also obviously have

$$\sum_{y \in E} f(y) = X_I.$$

Then, using the 2-point correlation of E, we obtain that

$$\text{Var}\left(\sum_{y \in E} f(y)\right) = \sum_{k \in \mathbb{Z}^*} \frac{2\pi}{n}\left(1 \wedge \frac{|k|}{n}\right)\left|\hat{f}(\frac{2\pi k}{n})\right|^2$$

if E is $\{y_k^{(n)}, k \in \mathbb{Z}\}$, and

$$\text{Var}\left(\sum_{y \in E} f(y)\right) = \int_{\mathbb{R}} 2\pi(1 \wedge |k|)\left|\hat{f}(2\pi k)\right|^2 dk,$$

if E is $\{y_k, k \in \mathbb{Z}\}$. If ν denotes the measure $(1/n)\sum_{k \in \mathbb{Z}^*} \delta_{k/n}$ in the first case and the Lebesgue measure in the second case, we get in both cases:

$$\text{Var}\left(\sum_{y \in E} f(y)\right) = \int_{\mathbb{R}} 2\pi(1 \wedge |k|)\left|\hat{f}(2\pi k)\right|^2 d\nu(k).$$

Hence,

$$\text{Var}(X_I) = \int_{\mathbb{R}} 2\pi(1 \wedge |k|)\frac{2\sin^2(\pi|I|k)}{\pi(2\pi k)^2}d\nu(k)$$

$$= \frac{2}{\pi^2}\left(\int_{[0,1]} \frac{\sin^2(\pi|I|k)}{k}d\nu(k) + \int_{(1,\infty)} \frac{\sin^2(\pi|I|k)}{k^2}d\nu(k)\right).$$

Now,

$$\int_{(1,\infty)} \frac{\sin^2(\pi|I|k)}{k^2}d\nu(k) \le \int_{(1,\infty)} \frac{d\nu(k)}{k^2} \le \int_1^\infty \frac{dk}{k^2} = 1,$$

and, using the inequality $|\sin t| \le 1 \wedge |t|$,

$$\int_{[0,1]} \frac{\sin^2(\pi|I|k)}{k}d\nu(k) \le \int_{[0,1 \wedge (1/\pi|I|)]} \pi^2|I|^2 k\, d\nu(k) + \int_{(1 \wedge (1/\pi|I|),1]} \frac{d\nu(k)}{k}.$$

Now, for $0 < a \le 1$, and for $E = \{y_k^{(n)}, k \in \mathbb{Z}\}$,

$$\int_{[0,a]} k\, d\nu(k) = \frac{1}{n}\sum_{1 \le m \le na} \frac{m}{n} \le \frac{na(na+1)}{2n^2}\mathbb{1}_{na \ge 1} \le \frac{na(2na)}{2n^2}\mathbb{1}_{na \ge 1} \le a^2.$$

and

$$\int_{(a,1]} \frac{d\nu(k)}{k} = \frac{1}{n}\sum_{an < m \le n} \frac{1}{(m/n)} = \left(\sum_{1 \le m \le n} \frac{1}{m}\right) - \left(\sum_{1 \le m \le an} \frac{1}{m}\right)$$

$$\le 1 + \left(\sum_{1 \le m \le n} \frac{1}{m}\right) - \left(\sum_{1 \le m \le an+1} \frac{1}{m}\right)$$

$$\le 1 + (1 + \log n) - \log(an) \le 2 + \log(1/a).$$

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These bounds are obvious for $E = \{y_k, k \in \mathbb{Z}\}$ since ν is the Lebesgue measure in this case, so we get

$$\int_{[0,1]} \frac{\sin^2(\pi|I|k)}{k} d\nu(k) \le \pi^2 |I|^2 \left(1 \wedge (1/\pi|I|)^2\right) + \log(\pi|I| \vee 1) + 2$$

$$\le 1 + \log(\pi(1 + |I|)) + 2 \le 5 + \log(1 + |I|),$$

and then

$$\mathrm{Var}(X_I) \le 2 + \frac{2}{\pi^2} \log(1 + |I|).$$

The estimate of the tail of $\widetilde{X_I}$ can be obtained as follows. If $E = \{y_k^{(n)}, k \in \mathbb{Z}\}$, we can assume $|I| < n$, since any interval of size n has a.s. n points in E. In this case, and also for $E = \{y_k, k \in \mathbb{Z}\}$, the restriction of E to I is determinantal, its kernel is self-adjoint, nonnegative, and locally trace-class with eigenvalues in $[0, 1]$. Thanks to Proposition 2 in [MM13] (which is by the way also a standard result in the theory of point processes), X_I is a sum of independent Bernoulli random variables. We deduce that if $(p_j)_{j \ge 1}$ are the parameters of these variables, and if $q \ge 0$,

$$\mathbb{E}[e^{qX_I}] = \prod_{j \ge 1} \left(1 + p_j(e^q - 1)\right) \le e^{(e^q - 1) \sum_{j \ge 1} p_j} = e^{(e^q - 1)\mathbb{E}[X_I]} = e^{|I|(e^q - 1)} < \infty.$$

Moreover, as in Corollary 4 in [MM13], we can deduce, for $q < 1/2$, the claimed estimate of the tail by using the Bernstein inequality.

We get the bound on the exponential moment as follows. One has

$$\mathbb{E}\left(e^{q|\widetilde{X_I}|}\right) = q \int_{-\infty}^{\infty} e^{qt} \mathbb{P}\left(|\widetilde{X_I}| \ge t\right) dt$$

Then, we split the integral as an integral on $\mathbb{R}_-$, which is bounded by 1, and an integral on $\mathbb{R}_+$. For the integral on $\mathbb{R}_+$, we use the following bound on the tails:

$$q \int_{\mathbb{R}_+} e^{qt} \mathbb{P}\left(|\widetilde{X_I}| \ge t\right) dt \le q \int_{\mathbb{R}_+} e^{qt} \exp\left(-\min\left(\frac{t^2}{4\,\mathrm{Var}(X_I)}, \frac{t}{2}\right)\right) dt$$

$$\le q \int_0^\infty e^{qt - \frac{t^2}{4\,\mathrm{Var}(X_I)}} dt + q \int_0^\infty e^{qt - \frac{t}{2}} dt\,dt$$

$$\le \frac{q}{\frac{1}{2} - q} + q \int_{\mathbb{R}} e^{qt - \frac{t^2}{4\,\mathrm{Var}(X_I)}} dt$$

$$\le \frac{q}{\frac{1}{2} - q} + q\sqrt{4\pi\,\mathrm{Var}(X_I)} e^{q^2\,\mathrm{Var}(X_I)}.$$

Adding 1 to this quantity gives the desired bound. □

Remark 2.9. *In the case where $E = \{y_k, k \in \mathbb{Z}\}$, an asymptotic estimate for the variance of X_I is proven by Costin and Lebowitz [CL95] (see also Soshnikov [Sos02]):*

$$\mathrm{Var}(X_I) = \frac{1}{\pi^2} \log(1 + |I|) + O(1).$$

The bound we have proven here is twice this estimate plus $O(1)$.

Lemma 2.10. *Almost surely:*

$$\forall k \in \mathbb{Z}, y_k = k + O\left(\log(2+|k|)\right)$$

Proof. Consider a sine-kernel process y_k. For $A > 1$ and $a < b$, let $X_{[a,b]}$ be the number of particles y_k in $[a,b]$, and let $X_A := X_{[0,A]}$. From the estimate given in Lemma 2.8,

$$\text{Var}(X_A) \leq \frac{2}{\pi^2}\log A + O(1)$$

Therefore, for all $D > 0$,

$$\mathbb{P}\left(|X_A - A| \geq D\log A\right) \leq 2\exp\left(-(\log A)\min\left(\frac{D^2\pi^2}{8 + O(1/\log A)}, \frac{D}{2}\right)\right).$$

For $D > 2$, and A large enough, $D^2\pi^2/[8 + O(1/\log A)] > D/2$, which implies:

$$\mathbb{P}\left(|X_A - A| \geq D\log A\right) \leq 2\exp\left(-(\log A)(D/2)\right) = 2A^{-D/2}.$$

This quantity is summable for positive integer values of A. By Borel-Cantelli's lemma, we deduce that almost surely, for $A \in \mathbb{N}$:

$$X_A = A + O\left(\log(2+|A|)\right).$$

From the inequality

$$X_{[0,\lfloor A \rfloor]} \leq X_{[0,A]} \leq X_{[0,\lceil A \rceil]},$$

we deduce that the estimate remains true for all $A \geq 0$. Taking $A = y_k$ for $k > 0$ proves the proposition for positive indices. With the same argument one handles the negative ones. $\qquad\square$

In order to prove Proposition 2.7, we will also need the following two lemmas:

Lemma 2.11. *Almost surely:*

$$\forall k \in \mathbb{Z}, y_k^{(n)} = k + O\left(\log n\right)$$

Proof. This is an immediate consequence of Corollary 2.3 and Proposition 2.6. $\qquad\square$

Lemma 2.12. *For every $0 < \eta < \frac{1}{6}$, there exists $\varepsilon > 0$ such that, almost surely:*

$$\forall k \in [-n^\eta, n^\eta], y_k^{(n)} = y_k + O\left(n^{-\varepsilon}\right)$$

Proof. Since $k \in [-n^{1/4}, n^{1/4}]$, we can apply Theorem 1.3, which gives, for all $\delta > 0$,

$$y_k^{(n)} = y_k + O_\delta\left((1+k^2)n^{-\frac{1}{3}+\delta}\right).$$

Since $k = O(n^\eta)$,

$$y_k^{(n)} = y_k + O_\delta\left(n^{2\eta-\frac{1}{3}+\delta}\right),$$

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which, by taking

$$\delta = \frac{1}{6} - \eta > 0,$$

gives the desired result, for

$$\varepsilon = -2\eta + \frac{1}{3} - \delta = 2\delta - \delta = \delta > 0.$$

$\square$

Proof of Proposition 2.7. In the range $|k| \geq n^{1/7}$, it is a consequence of Lemma 2.11. In the range $|k| < n^{1/7}$, it is a consequence of Lemmas 2.10 and 2.12 (for $\eta = 1/7$). $\square$

2.3 Infinite product representation of the ratio and its convergence

First, let us express ξ_n in function of the renormalized eigenangles of U_n.

Proposition 2.13. *One has*

$$\xi_n(z) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}} \right),$$

where the infinite product has to be understood as the limit of the product from $k = -A$ to $k = A$ when the integer A goes to infinity.

Proof.

$$\xi_n(z) = \frac{Z_n\left(\exp\left(\frac{i2\pi z}{n}\right)\right)}{Z_n(1)}$$

$$= \prod_{k=1}^{n} \frac{1 - \frac{\exp\left(\frac{i2\pi z}{n}\right)}{\lambda_k^{(n)}}}{1 - \frac{1}{\lambda_k^{(n)}}}$$

$$= \prod_{k=1}^{n} \frac{1 - \exp\left(\frac{i2\pi z}{n} - i\theta_k^{(n)}\right)}{1 - \exp\left(-i\theta_k^{(n)}\right)}$$

$$= \prod_{k=1}^{n} \frac{\exp\left(\frac{i2\pi z}{2n} - \frac{1}{2}i\theta_k^{(n)}\right)}{\exp\left(-\frac{1}{2}i\theta_k^{(n)}\right)} \frac{\exp\left(-\frac{i2\pi z}{2n} + \frac{1}{2}i\theta_k^{(n)}\right) - \exp\left(-\frac{1}{2}i\theta_k^{(n)} + \frac{i2\pi z}{2n}\right)}{\exp\left(\frac{1}{2}i\theta_k^{(n)}\right) - \exp\left(-\frac{1}{2}i\theta_k^{(n)}\right)}$$

$$= \prod_{k=1}^{n} \exp\left(\frac{i\pi z}{n}\right) \frac{\sin\left(\frac{\pi z}{n} - \frac{1}{2}\theta_k^{(n)}\right)}{\sin\left(-\frac{1}{2}\theta_k^{(n)}\right)}$$

$$= \exp(i\pi z) \prod_{k=1}^{n} \frac{\sin\left(\frac{1}{2}\theta_k^{(n)} - \frac{\pi z}{n}\right)}{\sin\left(\frac{1}{2}\theta_k^{(n)}\right)}$$

Now, the standard product formula for the sine function can be written as follows:

$$\forall \alpha \in \mathbb{C}, \sin(\alpha) = \alpha \lim_{A \to \infty} \prod_{0 < |j| \le A} \left(1 - \frac{\alpha}{\pi j}\right).$$

We then have:

$$\xi_n(z) = \exp(i\pi z) \prod_{k=1}^{n} \left(\frac{\frac{1}{2}\theta_k^{(n)} - \frac{\pi z}{n}}{\frac{1}{2}\theta_k^{(n)}} \lim_{A \to \infty} \prod_{0 < |j| \le A} \frac{1 - \frac{\frac{1}{2}\theta_k^{(n)} - \frac{\pi z}{n}}{\pi j}}{1 - \frac{\frac{1}{2}\theta_k^{(n)}}{\pi j}} \right)$$

$$= \exp(i\pi z) \prod_{k=1}^{n} \left(\left(1 - \frac{z}{y_k^{(n)}}\right) \lim_{A \to \infty} \prod_{0 < |j| \le A} \left(1 - \frac{z}{nj + y_k^{(n)}}\right) \right)$$

$$= \exp(i\pi z) \prod_{k=1}^{n} \lim_{A \to \infty} \prod_{0 \le |j| \le A} \left(1 - \frac{z}{nj + y_k^{(n)}}\right)$$

Using the periodicity of the eigenangles, we have:

$$y_{k+jn}^{(n)} = jn + y_k^{(n)},$$

and then

$$\xi_n(z) = \exp(i\pi z) \lim_{A \to \infty} \prod_{1 - nA \le k \le n + nA} \left(1 - \frac{z}{y_k^{(n)}}\right).$$

Now, for $B \ge 2n$, $A \ge 2$ integers such that $An \le B \le An + n - 1$, the product of $1 - \frac{z}{y_k^{(n)}}$ from $1 - nA$ to $n + nA$ and the product from $-B$ to B differ by at most $2n$ factors, which are all $1 + O(|z|/y_{nA}^{(n)}) + O(|z|/|y_{1-nA}^{(n)}|) = 1 + O(|z|/nA)$. The quotient between these two products is then well-defined and $\exp[O(|z|/A)] = \exp[O(n|z|/B)]$ for B large enough, which implies that it tends to one when B goes to infinity. Hence,

$$\xi_n(z) = \exp(i\pi z) \lim_{B \to \infty} \prod_{-B \le k \le B} \left(1 - \frac{z}{y_k^{(n)}}\right).$$

$\square$

We are now ready to prove Theorem 1.5.

Proof of theorem 1.5. Thanks to the estimate from Proposition 2.7:

$$y_k^{(n)} = k + O\left(\log(2 + |k|)\right)$$

We have that, for $k \ge 1$ and z in a compact K:

$$\left(1 - \frac{z}{y_k^{(n)}}\right)\left(1 - \frac{z}{y_{-k}^{(n)}}\right) = 1 - z\frac{O(\log(2 + |k|))}{k^2} + O\left(\frac{|z|^2}{k^2}\right)$$

$$= 1 + \frac{O_K \left(\log(2 + |k|) \right)}{k^2}$$

Hence:

$$\xi_n \left(z \right) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}} \right)$$

is a sequence of entire functions uniformly bounded on compact sets. Therefore, by Montel's theorem, uniform convergence on compact sets is implied by pointwise convergence. Let us then focus on proving pointwise convergence.

Fix $A \geq 2$. Let us prove that:

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}} \right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}} \right) = O_K \left(\frac{\log A}{A} \right), \tag{2.3}$$

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k} \right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k} \right) = O_K \left(\frac{\log A}{A} \right). \tag{2.4}$$

Here, the infinite products are, as before, the limits of the products from $-B$ to B for B going to infinity. Note that the existence of the infinite product involving y_k is an immediate consequence of the absolute convergence of the product

$$\left(1 - \frac{z}{y_0} \right) \prod_{k \geq 1} \left[\left(1 - \frac{z}{y_k} \right) \left(1 - \frac{z}{y_{-k}} \right) \right],$$

stated in Theorem 1.5, and following from the estimate:

$$\left(1 - \frac{z}{y_k} \right) \left(1 - \frac{z}{y_{-k}} \right) = 1 - z \frac{O(\log(2 + |k|))}{k^2} + O \left(\frac{|z|^2}{k^2} \right) = 1 + \frac{O_K \left(\log(2 + |k|) \right)}{k^2}.$$

We now prove (2.3): a proof of (2.4) is simply obtained by removing the indices n. We have:

$$\prod_{|k| \geq A} \left(1 - \frac{z}{y_k^{(n)}} \right) = 1 + O_K \left(\sum_{k \geq A} \frac{\log(2 + |k|)}{k^2} \right) = 1 + O_K \left(\frac{\log A}{A} \right)$$

and

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}} \right) = O_K \left(1 \right)$$

Therefore:

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}} \right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}} \right)$$

$$= \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}} \right) \left(1 - \prod_{|k| > A} \left(1 - \frac{z}{y_k^{(n)}} \right) \right)$$

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$$= \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right)\left(1 - \left(1 + O_K\left(\frac{\log A}{A}\right)\right)\right)$$

$$= O_K\left(\frac{\log A}{A}\right)$$

Because errors are uniform in n, this is saying:

$$\sup_n \left| \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right) \right| \xrightarrow[A \to \infty]{} 0$$

Now:

$$\left| \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k}\right) \right|$$

$$\leq \left| \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{|k| \leq A} \left(1 - \frac{z}{y_k}\right) \right|$$

$$+ \left| \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) \right|$$

$$+ \left| \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k}\right) - \prod_{|k| \leq A} \left(1 - \frac{z}{y_k}\right) \right|$$

$$\leq \left| \prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{|k| \leq A} \left(1 - \frac{z}{y_k}\right) \right| + O_K\left(\frac{\log A}{A}\right)$$

Hence, as $y_k^{(n)} \to y_k$ pointwise:

$$\limsup_{n \to \infty} \left| \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k}\right) \right| = O_K\left(\frac{\log A}{A}\right)$$

Taking $A \to \infty$ completes the proof. $\qquad\square$

3   First properties of ξ_∞ and linear statistics

3.1   The order of ξ_∞ as an entire function

We first start with a simple statement on the order of ξ_∞ as an entire function:

Proposition 3.1. *Almost surely, ξ_∞ is of order 1. More precisely, there exists a.s. a random $C > 0$, such that for all $z \in \mathbb{C}$.*

$$|\xi_\infty(z)| \leq e^{C|z| \log(2+|z|)}.$$

On the other hand, there exists a.s. a random $c > 0$ such that for all $x \in \mathbb{R}$,

$$|\xi_\infty(ix)| \geq c e^{c|x|}.$$

Proof. We have:

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 - z\frac{O(\log(2 + |k|))}{k^2} + O\left(\frac{|z|^2}{k^2}\right)$$

with errors being uniform in z and $k \geq 1$. We distinguish between three regimes for $k \in \mathbb{Z}$ different from zero: $|k| \geq e^{|z|}$, $|z| \leq |k| < e^{|z|}$, $1 \leq |k| < |z|$. In the first regime,

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 + O\left(\frac{|z|(\log(2 + |k|))}{k^2}\right),$$

which implies

$$\left|\prod_{k \geq e^{|z|}} \left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right| \leq \exp\left(O\left(|z| \sum_{k \geq e^{|z|}} \frac{\log(2 + k)}{k^2}\right)\right)$$

$$= \exp\left(O\left(|z| \sum_{k \geq e^{|z|}} k^{-3/2}\right)\right)$$

$$= \exp\left(O\left(|z|e^{-|z|/2}\right)\right) = O(1).$$

In the second regime,

$$\log(2 + |k|) \leq \log(e^{|z|} + 2) \leq \log(3e^{|z|}) \leq |z| + 2,$$

and then

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 + O\left(\frac{|z|(|z| + 2)}{k^2}\right),$$

which implies

$$\left|\prod_{|z| \leq k < e^{|z|}} \left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right| \leq \exp\left(O\left(|z|(|z| + 2) \sum_{k \geq |z| \vee 1} \frac{1}{k^2}\right)\right) = \exp O(|z|).$$

Finally, in the third regime, we have, since $|y_k/k|$ is a.s. bounded from below,

$$1 - \frac{z}{y_k} = 1 + O(|z/k|),$$

which in turn implies

$$\left|\prod_{1 \leq k < |z|} \left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right| \leq \exp\left(O\left(|z| \sum_{1 \leq k < |z|} (1/k)\right)\right) = \exp O\left(|z|\log(2 + |z|)\right).$$

Since

$$\left|1 - \frac{z}{y_0}\right| \leq \exp(|z|/y_0) = \exp O(|z|),$$

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we deduce by combining the three regimes, the following upper bound:

$$|\xi_\infty(z)| \leq \exp O\left(|z|\log(2+|z|)\right).$$

In order to prove the lower bound, we first use the equality:

$$|\xi_\infty(ix)|^2 = \prod_{k\in\mathbb{Z}}\left(1+\frac{x^2}{y_k^2}\right).$$

Since $|y_k| = O(|k|)$ for $k \neq 0$, we deduce that there exists a random $c > 0$ such that

$$|\xi_\infty(ix)|^2 \geq \prod_{k\neq 0}\left(1+\frac{x^2}{ck^2}\right),$$

and then

$$|\xi_\infty(ix)| \geq \prod_{k\geq 1}\left(1+\frac{x^2}{ck^2}\right) = \frac{\sinh(\pi x/\sqrt{c})}{\pi x/\sqrt{c}},$$

which shows the lower bound given in the proposition.

$\square$

3.2 Convergence of linear statistics

We proved in [MNN13] that if E_n is the set of zeros of ξ_n (i.e. the set of eigenvalues of U_n, multiplied by $n/2\pi$), and if E_∞ is the set of zeros of ξ_∞, then, for all measurable and bounded functions f from $\mathbb{R}$ to $\mathbb{C}$, with compact support, the following convergence in law holds:

$$\sum_{x\in E_n} f(x) \xrightarrow[n\to\infty]{} \sum_{x\in E_\infty} f(x).$$

We now improve this result by showing that it holds for more general test functions and show how linear statistics can be expressed in terms of ξ_∞.

Proposition 3.2. *Let E_n be the set of zeros of ξ_n (i.e. the set of eigenvalues of U_n, multiplied by $n/2\pi$), and E_∞ the set of zeros of ξ_∞. Then, for all integrable functions f from $\mathbb{R}$ to $\mathbb{C}$,*

$$\sum_{x\in E_n} f(x) \xrightarrow[n\to\infty]{} \sum_{x\in E_\infty} f(x)$$

in distribution.

Proof. For $A > 0$, let f_A be the function given by $f_A(x) := f(x)\mathbb{1}_{|x|\leq A, |f(x)|\leq A}$ and let $g_A := f - f_A$. Proposition 4.1 of [MNN13] implies that the proposition is true when f is replaced by f_A, i.e. for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_n} f_A(x)\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_\infty} f_A(x)\right)\right]. \tag{3.1}$$

On the other hand, since the one-point correlation function of E_n and E_∞ is equal to 1, we have

$$\mathbb{E}\left[\sum_{x\in E_n}|g_A(x)|\right] = \mathbb{E}\left[\sum_{x\in E_n}|g_A(x)|\right] = \int_{\mathbb{R}}|g_A|.$$

Hence,

$$\mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_n}f(x)\right)\right] = \mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_n}f_A(x)\right)\right] + O\left(|\lambda|\mathbb{E}\left[\sum_{x\in E_n}|g_A(x)|\right]\right)$$

$$= \mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_n}f_A(x)\right)\right] + O\left(|\lambda|\int_{\mathbb{R}}|g_A|\right)$$

and the similar estimate with E_n replaced by E_∞. Taking the limsup of the difference when n goes to infinity gives, using (3.1):

$$\limsup_{n\to\infty}\left|\mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_n}f(x)\right)\right] - \mathbb{E}\left[\exp\left(i\lambda\sum_{x\in E_\infty}f(x)\right)\right]\right| = O\left(|\lambda|\int_{\mathbb{R}}|g_A|\right)$$

for all $A > 0$. Now, by dominated convergence, the integral of $|g_A|$ goes to zero when A goes to infinity, which gives the desired result. $\qquad\square$

It is natural to conjecture that something similar happens for the zeros of the Riemann zeta function:

Conjecture 3.3. *Assume the Riemann hypothesis. For all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$\int_{\mathbb{R}}|f(x)|\log(2+|x|)dx < \infty,$$

$$\sum_{x\in E_T^\zeta}f(x) \xrightarrow[T\to\infty]{} \sum_{x\in E_\infty}f(x)$$

in distribution, where E_T^ζ denotes the non-trivial zeros of $z\mapsto\zeta\left(\frac{1}{2}+iT\omega-\frac{2i\pi z}{\log T}\right)$, and ω is a uniform variable in $[0,1]$.

The extra factor $\log(2+|x|)$ in the integrability condition is due to the fact that we sum f over all zeros of ζ, who have a logarithmically increasing average density.

One can also express linear statistics of E_n in terms of the logarithm of ξ_∞. We have the following:

Proposition 3.4. *Let f be a $\mathcal{C}^1$ function from $\mathbb{R}$ to $\mathbb{R}$, integrable, such that*

$$|xf(x)| \xrightarrow[|x|\to\infty]{} 0, \quad \int_{\mathbb{R}}|xf'(x)| < \infty.$$

Then we have a.s.

$$\sum_{x\in E_n}f(x) - \int_{\mathbb{R}}f(x)dx = \frac{1}{\pi}\int_\infty^\infty \Im\log(\xi_n(y))f'(y)dy.$$

Of course, a similar result holds with E_∞ instead of E_n.

Proof. For all $A > 0$,

$$\sum_{x \in E_n \cap [-A,A]} f(x) = f(-A) \operatorname{Card}(E_n \cap [-A, A]) + \sum_{x \in E_n \cap [-A,A]} \int_{-A}^{x} f'(y) dy$$

$$= f(-A) \operatorname{Card}(E_n \cap [-A, A]) + \int_{-A}^{A} f'(y) \operatorname{Card}(E_n \cap [y, A]) dy$$

$$= f(-A)(N_n(A) - N_n(-A)) + \int_{-A}^{A} (N_n(A) - N_n(y)) f'(y) dy$$

$$= f(A) N_n(A) - f(-A) N_n(-A) - \int_{-A}^{A} N_n(y) f'(y) dy.$$

Since $N_n(y) = O(1 + |y|)$ a.s., we deduce, from the assumptions made on f, that almost surely:

$$\sum_{x \in E_n} f(x) = -\int_{-\infty}^{\infty} N_n(y) f'(y) dy = \frac{1}{\pi} \int_{\infty}^{\infty} \Im \log(\xi_n(y)) f'(y) dy - \int_{-\infty}^{\infty} y f'(y) dy.$$

Doing an integration by parts gives the desired result. $\qquad\square$

4 Fine estimates for the logarithimic derivative and related conjectures for the Riemann zeta function

We now state a conjecture which relates the random function ξ_∞ to the behavior of the Riemann zeta function ζ close to the critical line:

Conjecture 4.1. *Let ω be a uniform random variable on $[0, 1]$ and $T > 0$ a real parameter going to infinity. Our random limiting function should be related to the renormalized zeta function with randomized argument. We conjecture the following convergence in law, uniformly in the parameter z on every compact set:*

$$\left(\frac{\zeta \left(\frac{1}{2} + iT\omega - \frac{i2\pi z}{\log T} \right)}{\zeta \left(\frac{1}{2} + iT\omega \right)}; z \in \mathbb{C} \right) \overset{T \to \infty}{\longrightarrow} (\xi_\infty(z); z \in \mathbb{C})$$

By taking logarithmic derivatives, it is natural also to conjecture the following convergence

$$\left(\frac{-i2\pi}{\log T} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + iT\omega - \frac{i2\pi z}{\log T} \right); z \in \mathbb{C} \right) \overset{T \to \infty}{\longrightarrow} \left(\frac{\xi_\infty'}{\xi_\infty}(z); z \in \mathbb{C} \right)$$

on compact sets bounded away from the real line.

This conjecture is supported by the following lemma:

Lemma 4.2. *We have, for $z \notin \mathbb{R}$,*

$$\frac{\zeta_\infty'}{\zeta_\infty}(z) = i\pi + \sum_{k \in \mathbb{Z}} \frac{1}{z - y_k} =: i\pi + \frac{1}{z - y_0} + \sum_{k=1}^\infty \left(\frac{1}{z - y_k} + \frac{1}{z - y_{-k}} \right),$$

and when the random variable ω is fixed:

$$\frac{-i2\pi}{\log T} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + iT\omega - \frac{i2\pi z}{\log T} \right) = i\pi + \sum_{\tilde{\gamma} \in E_T^\zeta} \frac{1}{z - \tilde{\gamma}} + o(1)$$

where E_T^ζ are the non-trivial zeros of the Riemann zeta function centered around $\frac{1}{2} + i\omega T$ and renormalized so that their average spacing around the origin is $\mathcal{O}(1)$. More precisely, if $\tilde{\gamma} \in E_T^\zeta$, then:

$$\tilde{\gamma} := \frac{-\log T}{2\pi i} \left(\rho - \frac{1}{2} - i\omega T \right)$$

with ρ a zero of ζ. The infinite sum on $\tilde{\gamma}$ has to be understood as follows:

$$\sum_{\tilde{\gamma}} \frac{1}{z - \tilde{\gamma}} = \frac{1}{z - \tilde{\gamma}_0} + \sum_{k=1}^\infty \left(\frac{1}{z - \tilde{\gamma}_k} + \frac{1}{z - \tilde{\gamma}_{-k}} \right),$$

where $(\tilde{\gamma}_k)_{k \in \mathbb{Z}}$ are ordered by increasing real part, increasing imaginary part if they have the same real part, and counted with multiplicity.

Remark 4.3. *The absolute convergence of the last sum can be easily deduced from the classical estimate, for $A > 2$, on the number of nontrivial zeros $N(A)$ with imaginary part in $[0, A]$, or in $[-A, 0]$:*

$$N(A) = \varphi(A) + O(\log A),$$

for

$$\varphi(A) = \frac{A}{2\pi} \log \left(\frac{A}{2\pi e} \right).$$

Indeed, all the ways to number the renormalized zeros $\tilde{\gamma}$ consistently with the statement of the lemma are deduced from each other by translation of the indices, and for any such numbering one checks that

$$\tilde{\gamma}_k = \mathrm{sgn}(k) \frac{\log T}{2\pi} \varphi^{(-1)}(|k|) + O(\log(2 + |k|)),$$

where $\varphi^{(-1)}$ is the inverse of the bijection from $[2\pi e, \infty)$ to $\mathbb{R}_+$, induced by φ. The implicit constant depends on T, ω and the precise numbering of the zeros, but not on k. This estimate is sufficient to ensure the convergence of the last series in the lemma, when one takes into account that $\varphi^{(-1)}(k) \geq k / \log k$ for all $k \geq 2$. The sum of the series does not depend on the numbering of the $\tilde{\gamma}$'s, since any translation of the indices change the partial sums by a bounded number of terms, which tend to zero. Note that the $\tilde{\gamma}$'s are all real if and only if the Riemann hypothesis is satisfied.

Proof. The convergence of the first series in the lemma is easily deduced from the estimate in Proposition 2.7. The partial sums are the logarithmic derivatives of the corresponding partial products associated to ξ_∞. Since uniform convergence on compact sets of non-vanishing holomorphic functions implies the corresponding convergence of the logarithmic derivative, we get the part of the lemma related to ξ'_∞/ξ_∞. For the formula involving ζ, we start by the Hadamard product formula for the zeta function:

$$\forall s \in \mathbb{C}\backslash\{1\}, \zeta(s) = \pi^{s/2} \frac{\prod_\rho \left(1 - \frac{s}{\rho}\right)}{2(s-1)\Gamma\left(1 + \frac{s}{2}\right)}.$$

The product has to be computed by grouping pairs of conjugate non-trivial zeros of zeta. Hence, for s not a zero nor a pole:

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{2}\log\pi + \sum_\rho \frac{1}{s-\rho} - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1 + \frac{s}{2}\right)$$

Take $s = \frac{1}{2} + iT\omega - \frac{i2\pi z}{\log T}$ with $T \to \infty$ and use the asymptotics $\frac{\Gamma'}{\Gamma}\left(1 + \frac{s}{2}\right) = \log T + O(1)$. The error is uniform in z on compact sets away from the real line. Then:

$$\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iT\omega - \frac{i2\pi z}{\log T}\right) = \frac{-i2\pi}{\log T}\sum_\rho \frac{1}{s-\rho} + \frac{i2\pi}{\log T}\frac{1}{2}(\log T + O(1)) + o(1)$$

$$= i\pi + \frac{-i2\pi}{\log T}\sum_\rho \frac{1}{-\frac{i2\pi z}{\log T} - \left(\rho - \frac{1}{2} - i\omega T\right)} + o(1)$$

Here, all the sums on ρ are obtained by grouping pairs of conjugate values of ρ. Writing the last sum as a function of the sequence $(\tilde{\gamma}_k)_{k\in\mathbb{Z}}$ gives

$$\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iT\omega - \frac{i2\pi z}{\log T}\right) = i\pi + \sum_{k=1}^\infty \left(\frac{1}{z - \tilde{\gamma}_{a+k}} + \frac{1}{z - \tilde{\gamma}_{a+1-k}}\right) + o(1),$$

where a depends only on the way to number the $\tilde{\gamma}_k$'s. Changing the partial sums by at most $2|a| + 1$ terms, all tending to zero, gives the partial sums of the series in the lemma. □

Our formulation can be easily related to the GUE conjectures [RS96], which is the natural extension of Montgomery's conjecture [Mon73] on pair correlations. Indeed, the previous lemma gives a good heuristic of Conjecture 4.1: since the randomized and renormalized zeros $\tilde{\gamma}$ are expected to behave like a sine kernel point process, the two expressions should match in law when $T \to \infty$. It is interesting to notice that the term $i\pi$ in the expression of ζ'/ζ is due to the Archimedian gamma factor in the Hadamard product of ζ. With the same renormalization corresponding to the average spacing of the zeros, we get the same term for the logarithmic derivative of the characteristic polynomial of the CUE.

We will now compute the first two moments of $\frac{\xi'_\infty}{\xi_\infty}$, which will naturally give a conjecture on the corresponding moments of $\frac{\zeta'}{\zeta}$. A particular case of our conjecture is in fact equivalent to the pair correlation conjecture under Riemann hypothesis, thanks

to results by Goldston, Gonek and Montgomery [GGM01]. One should also note that recently Farmer, Gonek, Lee and Lester obtain in [FGLL13] an equivalent formulation, with different methods, for the moments of the logarithmic derivative of the Riemann zeta function in terms of the correlation functions of the sine kernel: the objects that are introduced there are different but our formulation is essentially the same as theirs. The main difference is that we propose to consider directly a random meromorphic function which follows from a conjecture for the ratios of the zeta function itself (in particular there is no more n-limit to consider on the random matrix side) and that the logarithmic derivative ξ'_∞ / ξ_∞ seems to carry some spectral interpretation (see the last section and the reference there to the recent work by Aizenman and Warzel [AW13]).

As shown in Lemma 4.2, ξ'_∞ / ξ_∞ can be written as an infinite sum indexed by $\mathbb{Z}$ which is not absolutely convergent, but which converges if we cut the sum at $-k$ and k for $k \in \mathbb{N}$, and then let $k \to \infty$. Instead of considering the terms indexed by $m \in \{-k, -k+1, \dots, k\}$, it can be more convenient to take all the terms of index m such that $|y_m| \leq A$, and then to let $A \to \infty$. The following result says that the two ways to consider the infinite sum give the same result.

Proposition 4.4. *Almost surely, for all $z \notin \{y_k, k \in \mathbb{Z}\}$,*

$$\frac{\xi'_\infty(z)}{\xi_\infty(z)} = i\pi + \lim_{A \to \infty} \sum_{|y_k| < A} \frac{1}{z - y_k}.$$

Proof. By Lemma 2.10, there exists almost surely $C > 0$ such that

$$|y_k - k| \leq C \log(2 + |k|)$$

for all $k \in \mathbb{Z}$. It is sufficient to show that almost surely, for all $z \notin \{y_k, k \in \mathbb{Z}\}$,

$$\left(i\pi + \sum_{|y_k| < A} \frac{1}{z - y_k} \right) - \left(\sum_{|k| < A - C \log(2+A)} \frac{1}{z - y_k} + i\pi \right) \xrightarrow[A \to \infty]{} 0.$$

Indeed, the second term of the difference is already known to converge to $\xi'_\infty(z)/\xi_\infty(z)$. Now, $|k| < A - C \log(2 + A)$ implies that

$$|y_k| \leq |k| + C \log(2 + |k|) \leq |k| + C \log(2 + A) < A,$$

and then we have to show that

$$\sum_{|k| \geq A - C \log(2+A), |y_k| < A} \frac{1}{z - y_k} \xrightarrow[A \to \infty]{} 0.$$

Since $|y_k| \geq |k| - C \log(2 + |k|)$, it is sufficient to prove

$$\sum_{|k| \geq A - C \log(2+A), |k| - C \log(2+|k|) < A} \frac{1}{|z - y_k|} \xrightarrow[A \to \infty]{} 0.$$

Now, this convergence holds since for C, z and $(y_k)_{k \in \mathbb{Z}}$ fixed, the number of terms of the sum is $O(\log A)$ when A goes to infinity, and all the terms are $O(1/A)$.

$\square$

We will now bound some exponential moments related to ξ'_∞/ξ_∞. In order to apply this bound later to convergence results, it will also be useful to consider ξ'_n/ξ_n for finite n. The infinite product given in Proposition 2.13 is clearly uniformly convergent in compact sets if we regroup the terms of indices k and $-k$, and ξ_n does not vanish outside the real axis. Hence, we can take the logarithmic derivative:

$$\frac{\xi'_n(z)}{\xi_n(z)} = i\pi + \frac{1}{z - y_0^{(n)}} + \sum_{k \geq 1} \left(\frac{1}{z - y_k^{(n)}} + \frac{1}{z - y_{-k}^{(n)}} \right).$$

Since $y_k^{(n)} - k$ is n-periodic and then bounded, one deduces that we also have

$$\frac{\xi'_n(z)}{\xi_n(z)} = i\pi + \lim_{A \to \infty} \sum_{|y_k^{(n)}| < A} \frac{1}{z - y_k^{(n)}}.$$

From now, we will allow n to be either ∞ or a strictly positive integer, and we will write by convention $y_k^{(\infty)} := y_k$. Moreover, we define:

$$\sum_{|y_k^{(n)}| > A} \frac{1}{z - y_k^{(n)}} := \frac{\xi'_n(z)}{\xi_n(z)} - i\pi - \sum_{|y_k^{(n)}| \leq A} \frac{1}{z - y_k^{(n)}}.$$

Then, we have the following estimate:

Proposition 4.5. *Let $K \subset \mathbb{C} \backslash \mathbb{R}$ be a compact set. Then, there exists $C_K > 0$, depending only on K, such that for all $p \geq 0$ and for all $A \geq C_K(1 + p^2 \log(2 + p))$,*

$$\sup_{n \in \mathbb{N} \sqcup \{\infty\}} \mathbb{E} \left[\sup_{z \in K} e^{p \left| \sum_{|y_k^{(n)}| > A} \frac{1}{z - y_k^{(n)}} \right|} \right] \leq 1 + \frac{C_K p \log A}{\sqrt{A}}$$

In particular, for all fixed $p > 0$, we have:

$$\limsup_{A \to \infty} \sup_{n \in \mathbb{N} \sqcup \{\infty\}} \mathbb{E} \left[\sup_{z \in K} e^{p \left| \sum_{|y_k^{(n)}| > A} \frac{1}{z - y_k^{(n)}} \right|} \right] = 1$$

Proof. Let $\alpha > 1$ be an exponent to be decided later and denote for every $\ell \in \mathbb{Z}$ the intervals:

$$I_\ell := \left(|\ell|^\alpha, (|\ell| + 1)^\alpha \right]$$

$$I_\ell^A := \text{sgn}(\ell) \left(I_\ell \cap [A, \infty) \right)$$

First there is a deterministic constant $C_{K,\alpha} > 0$ such that for $\ell \geq 0$:

$$|y_k^{(n)}| \in I_\ell \Rightarrow \left| \frac{1}{z - y_k^{(n)}} - \frac{\text{sgn} \, y_k^{(n)}}{(1 + \ell)^\alpha} \right| \leq \frac{C_{K,\alpha}}{(1 + \ell)^{\alpha + 1}}$$

Then, thanks to the triangular inequality and Proposition 4.4, and using the notation X and $\widetilde{X}$ given in Lemma 2.8,

$$
\left| \sum_{|y_k^{(n)}| > A} \frac{1}{z - y_k^{(n)}} \right| \leq \sum_{\ell \geq 0} \left| \sum_{|y_k^{(n)}| \in I_\ell^A} \frac{1}{z - y_k^{(n)}} \right|
$$

$$
\leq \sum_{\ell \geq 0} \left(\left| \sum_{|y_k^{(n)}| \in I_\ell^A} \frac{\operatorname{sgn} y_k^{(n)}}{(1 + \ell)^\alpha} \right| + \sum_{|y_k^{(n)}| \in I_\ell^A} \frac{C_{K,\alpha}}{(1 + \ell)^{\alpha+1}} \right)
$$

$$
\leq \sum_{\ell \geq 0} \left(\frac{\left| X_{I_\ell^A} - X_{-I_\ell^A} \right|}{(1 + \ell)^\alpha} + C_{K,\alpha} \frac{X_{I_\ell^A} + X_{-I_\ell^A}}{(1 + \ell)^{\alpha+1}} \right)
$$

$$
\leq \sum_{\ell \geq 0} \left(\frac{\left| \widetilde{X_{I_\ell^A}} \right| + \left| \widetilde{X_{-I_\ell^A}} \right|}{(1 + \ell)^\alpha} + C_{K,\alpha} \frac{X_{I_\ell^A} + X_{-I_\ell^A}}{(1 + \ell)^{\alpha+1}} \right)
$$

$$
\leq 2 C_{K,\alpha} \sum_{\ell \geq 0} \frac{\left| I_\ell^A \right|}{(1 + \ell)^{\alpha+1}} + \sum_{\ell \geq 0} \frac{1}{(1 + \ell)^\alpha} \left(\left| \widetilde{X_{I_\ell^A}} \right| + \left| \widetilde{X_{-I_\ell^A}} \right| \right) \left(1 + \frac{C_{K,\alpha}}{1 + \ell} \right)
$$

Notice that I_ℓ^A is empty when $|\ell| < A^{\frac{1}{\alpha}} - 1$. Thanks to that, we will now prove that:

$$
\left| \sum_{|y_k^{(n)}| > A} \frac{1}{z - y_k^{(n)}} \right| = \mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right) + \left(1 + \mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right) \right) \sum_{\ell \geq 0} \frac{\left(\left| \widetilde{X_{I_\ell^A}} \right| + \left| \widetilde{X_{-I_\ell^A}} \right| \right)}{(1 + \ell)^\alpha} \qquad (4.1)
$$

Indeed, from the previous equation, the first sum can be written as:

$$
2 C_{K,\alpha} \sum_{\ell \geq A^{\frac{1}{\alpha}} - 1} \frac{\left| I_\ell^A \right|}{(1 + \ell)^{\alpha+1}} \ll_{K,\alpha} \sum_{\ell \geq A^{\frac{1}{\alpha}} - 1} \frac{(1 + \ell)^\alpha - \ell^\alpha}{(1 + \ell)^{\alpha+1}}
$$

$$
\ll_\alpha \sum_{\ell \geq A^{\frac{1}{\alpha}} - 1} \frac{(1 + \ell)^{\alpha-1}}{(1 + \ell)^{\alpha+1}}
$$

$$
= \sum_{\ell \geq A^{\frac{1}{\alpha}} - 1} \frac{1}{(1 + \ell)^2}
$$

$$
\ll_\alpha \frac{1}{A^{\frac{1}{\alpha}}}
$$

And, in the second sum, write $\left(1 + \frac{C_{K,\alpha}}{1 + \ell} \right) = 1 + \mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right)$ to deduce inequality (4.1).

Now, we are ready to exponentiate the inequality (4.1) after multiplication by $p \geq 0$. Let $(\beta_\ell)_{\ell \in \mathbb{Z}}$ be the probability weights given by:

$$
\beta_\ell = \frac{1}{Z_{A,\alpha}} \mathbb{1}_{\left\{ 1 + |\ell| \geq A^{\frac{1}{\alpha}} \right\}} \frac{1}{(1 + |\ell|)^\alpha}
$$

where $Z_{A,\alpha}$ is the normalisation constant, chosen in such a way that the sum of β_ℓ for $\ell \in \mathbb{Z}$ is equal to 1. One easily checks that

$$Z_{A,\alpha} = \mathcal{O}_\alpha \left(\frac{1}{A^{1-(1/\alpha)}} \right).$$

We have:

$$\exp \left(p \left| \sum_{|y_k^{(n)}|>A} \frac{1}{z-y_k^{(n)}} \right| \right)$$

$$\leq \quad \exp \left(p\mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right) \right) \exp \left(p \left(1 + \mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right) \right) Z_{A,\alpha} \sum_{\ell \in \mathbb{Z}} \beta_\ell \left| \widetilde{X_{I_\ell^A}} \right| \right)$$

$$\overset{\text{(Jensen)}}{\leq} \quad e^{p\mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right)} \sum_{\ell \in \mathbb{Z}} \beta_\ell \exp \left(p \left(1 + \mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right) \right) \mathcal{O}_\alpha \left(\frac{1}{A^{1-(1/\alpha)}} \right) \left| \widetilde{X_{I_\ell^A}} \right| \right)$$

$$= \quad e^{p\mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right)} \sum_{\ell \in \mathbb{Z}} \beta_\ell \exp \left(q \left| \widetilde{X_{I_\ell^A}} \right| \right)$$

$$= \quad e^{p\mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right)} \left(\sum_{\ell \in \mathbb{Z}} \beta_\ell \left[\exp \left(q \left| \widetilde{X_{I_\ell^A}} \right| \right) - 1 \right] + \sum_{\ell \in \mathbb{Z}} \beta_\ell \right)$$

$$= \quad e^{p\mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right)} + e^{p\mathcal{O}_{K,\alpha} \left(\frac{1}{A^{\frac{1}{\alpha}}} \right)} \sum_{\ell \in \mathbb{Z}} \beta_\ell \left[\exp \left(q \left| \widetilde{X_{I_\ell^A}} \right| \right) - 1 \right].$$

where

$$q = \mathcal{O}_{K,\alpha} \left(\frac{p}{A^{1-(1/\alpha)}} \right).$$

If $0 < q \leq 1/4$, we get $1/(1-2q) \leq 1+4q$ and then by Lemma 2.8:

$$\mathbb{E} \left[e^{q|\widetilde{X_{I_\ell^A}}|} - 1 \right] \leq \frac{1}{1-2q} - 1 + q\sqrt{4\pi \operatorname{Var}(X_{I_\ell^A})} e^{4q^2 \operatorname{Var}(X_{I_\ell^A})}$$

$$\leq q \left(4 + \sqrt{4\pi \operatorname{Var}(X_{I_\ell^A})} \right) e^{4q^2 \operatorname{Var}(X_{I_\ell^A})}$$

$$\leq q \left(4 + \frac{1 + 4\pi \operatorname{Var}(X_{I_\ell^A})}{2} \right) e^{4q^2 \operatorname{Var}(X_{I_\ell^A})}$$

$$\leq 7q \left(1 + \operatorname{Var}(X_{I_\ell^A}) \right) e^{4q^2 \operatorname{Var}(X_{I_\ell^A})}.$$

Using the estimate of the variance given in Lemma 2.8, we deduce

$$\mathbb{E} \left[e^{q|\widetilde{X_{I_\ell^A}}|} - 1 \right] \ll q \log(2 + |I_\ell^A|) e^{\mathcal{O}(q^2 \log(2+|I_\ell^A|))}$$

$$= q \log \left(2 + \mathcal{O}_\alpha \left((1+|\ell|)^{\alpha-1} \right) \right) e^{\mathcal{O}\left(q^2 \log \left(2 + \mathcal{O}_\alpha \left((1+|\ell|)^{\alpha-1} \right) \right) \right)}$$

$$\ll_\alpha q \log(2+|\ell|)e^{\mathcal{O}_\alpha(q^2 \log(2+|\ell|))} = q(2+|\ell|)^{\mathcal{O}_\alpha(q^2)}\log(2+|\ell|)$$

Hence, in the region where $q \leq 1/4$, we get

$$\mathbb{E}\left[\sup_{z\in K} e^{p\left|\sum_{|y_k^{(n)}|>A} \frac{1}{z-y_k^{(n)}}\right|}\right] \leq e^{\mathcal{O}_{K,\alpha}\left(\frac{p}{A^{\frac{1}{\alpha}}}\right)} + \mathcal{O}_\alpha\left(qe^{\mathcal{O}_{K,\alpha}\left(\frac{p}{A^{\frac{1}{\alpha}}}\right)}\sum_{\ell\in\mathbb{Z}}\beta_\ell(2+|\ell|)^{\mathcal{O}_\alpha(q^2)}\log(2+|\ell|)\right)$$

The sum in ℓ is smaller than or equal to

$$\frac{1}{Z_{A,\alpha}}\sum_{|\ell|\geq A^{1/\alpha}-1}(1+|\ell|)^{-\alpha}(2+|\ell|)^{\mathcal{O}_{K,\alpha}(q^2)}\log(2+|\ell|).$$

If the exponent $\mathcal{O}_{K,\alpha}(q^2)$ is strictly smaller than $(\alpha-1)/2$, then the terms of the last series are bouded by $2^\alpha(2+|\ell|)^{-\beta}\log(2+|\ell|)$, where $\beta > 1+(\alpha-1)/2$ and $\beta = \alpha - \mathcal{O}_{K,\alpha}(q^2)$. Hence, in this case,

$$\sum_{|\ell|\geq A^{1/\alpha}-1}(1+|\ell|)^{-\alpha}(2+|\ell|)^{\mathcal{O}_{K,\alpha}(q^2)}\log(2+|\ell|) \ll_\alpha \sum_{\ell\geq A^{1/\alpha}-1}(2+|\ell|)^{-\beta}\log(2+|\ell|)$$

Now, since $\beta > 1$, the function $x \mapsto x^{-\beta}\log x$ is nonincreasing on $[e,\infty)$. Hence, for $A \geq e^\alpha$,

$$\sum_{|\ell|\geq A^{1/\alpha}-1}(1+|\ell|)^{-\alpha}(2+|\ell|)^{\mathcal{O}_{K,\alpha}(q^2)}\log(2+|\ell|) \ll_\alpha \int_{A^{1/\alpha}}^\infty x^{-\beta}\log x\,dx$$

$$\ll_\alpha \left[\frac{x^{1-\beta}\log x}{1-\beta}\right]_{A^{1/\alpha}}^\infty - \frac{1}{1-\beta}\int_{A^{1/\alpha}}^\infty x^{-\beta}dx$$

$$\leq \frac{1}{\beta-1}A^{(1-\beta)/\alpha}\log(A^{1/\alpha}) + \frac{1}{(\beta-1)^2}A^{(1-\beta)/\alpha}$$

$$\leq \frac{2}{\alpha-1}A^{(1-\beta)/\alpha}\left(\frac{2}{\alpha-1}+\log(A^{1/\alpha})\right) \ll_\alpha A^{(1-\beta)/\alpha}\log A$$

$$= \frac{\log A}{A^{1-(1/\alpha)+\mathcal{O}_{K,\alpha}(q^2)}}.$$

Moreover,

$$Z_{A,\alpha} = \sum_{|\ell|\geq A^{1/\alpha}-1}\frac{1}{(1+|\ell|)^\alpha} \geq \int_{A^{1/\alpha}+1}^\infty u^{-\alpha} \gg_\alpha \frac{1}{(1+A^{1/\alpha})^{\alpha-1}} \gg_\alpha \frac{1}{A^{1-(1/\alpha)}},$$

if $A \geq 1$. The condition $\mathcal{O}_{K,\alpha}(q^2) < (\alpha-1)/2$ is satisfied as soon as $q \ll_{K,\alpha} 1$, and since q is dominated by $p/A^{1-(1/\alpha)}$, as soon as $A \gg_{K,\alpha} p^{\alpha/(\alpha-1)}$. Hence, if $A \gg_{K,\alpha} 1+p^{\alpha/(\alpha-1)}$,

$$\mathbb{E}\left[\sup_{z\in K} e^{p\left|\sum_{|y_k^{(n)}|>A} \frac{1}{z-y_k^{(n)}}\right|}\right] = e^{\mathcal{O}_{K,\alpha}\left(\frac{p}{A^{\frac{1}{\alpha}}}\right)} + \mathcal{O}_\alpha\left(qe^{\mathcal{O}_{K,\alpha}\left(\frac{p}{A^{\frac{1}{\alpha}}}\right)}A^{\mathcal{O}_{K,\alpha}(q^2)}\log A\right)$$

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$$= e^{\mathcal{O}_{K,\alpha}\left(\frac{p}{A^{\frac{1}{\alpha}}}\right)} + \mathcal{O}_{K,\alpha}\left(\frac{p\log A}{A^{1-(1/\alpha)}}e^{\mathcal{O}_{K,\alpha}\left(\frac{p}{A^{\frac{1}{\alpha}}}+\frac{p^2\log A}{A^{2-(2/\alpha)}}\right)}\right).$$

Let us now choose $\alpha = 2$. For $A \gg_K 1 + p^2$, we get

$$\mathbb{E}\left[\sup_{z\in K} e^{p\left|\sum_{|y_k^{(n)}|>A}\frac{1}{z-y_k^{(n)}}\right|}\right] = e^{\mathcal{O}_K\left(\frac{p}{\sqrt{A}}\right)} + \mathcal{O}_K\left(\frac{p\log A}{\sqrt{A}}e^{\mathcal{O}_K\left(\frac{p}{\sqrt{A}}+\frac{p^2\log A}{A}\right)}\right).$$

Now, by assumption, $p/\sqrt{A} \ll_K 1$, which implies that

$$e^{\mathcal{O}_K\left(\frac{p}{\sqrt{A}}\right)} = 1 + \mathcal{O}_K\left(\frac{p}{\sqrt{A}}\right) = \mathcal{O}_K(1)$$

and

$$\mathbb{E}\left[\sup_{z\in K} e^{p\left|\sum_{|y_k^{(n)}|>A}\frac{1}{z-y_k^{(n)}}\right|}\right] = 1 + \mathcal{O}_K\left(\frac{p\log A}{\sqrt{A}}e^{\mathcal{O}_K\left(\frac{p^2\log A}{A}\right)}\right).$$

Now, $\log A / A$ is nonincreasing in $A \geq e$, so if $A \geq e + p^2 \log(2 + p)$, we get

$$\mathcal{O}_K\left(\frac{p^2\log A}{A}\right) \leq \mathcal{O}_K\left(\frac{p^2\log(e + p^2\log(2+p))}{e + p^2\log(2+p)}\right) = \mathcal{O}_K(1),$$

which gives

$$\mathbb{E}\left[\sup_{z\in K} e^{p\left|\sum_{|y_k^{(n)}|>A}\frac{1}{z-y_k^{(n)}}\right|}\right] = 1 + \mathcal{O}_K\left(\frac{p\log A}{\sqrt{A}}\right).$$

$\square$

As a consequence of the above bound we have the following estimates on the L^p norms of ξ'_∞/ξ_∞.

Proposition 4.6. *For any compact set K of $\mathbb{C}\backslash\mathbb{R}$, and for all $p \geq 1$, there exists an absolute constant $C_{p,K}$ such that:*

$$\forall A \geq 0, \sup_{z\in K}\mathbb{E}\left(\left|\sum_{|y_k|>A}\frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} \leq C_{p,K}\frac{\log(2+A)}{\sqrt{1+A}}$$

and in particular,

$$\sup_{z\in K}\mathbb{E}\left(\left|\sum_{|y_k|>A}\frac{1}{z-y_k}\right|^p\right) \xrightarrow[A\to\infty]{} 0,$$

Moreover, $\xi'_\infty(z)/\xi_\infty(z)$ is in L^p for all $z \notin \mathbb{R}$ and $p \geq 1$.

Proof. For $A \geq 2$, let us define $q = \sqrt{A}/\log A$. For A large enough, $2 + q \leq A$ and then

$$C_K(1 + q^2 \log(2 + q)) \leq C_K\left(1 + \frac{A}{\log A}\right),$$

which is smaller than A if A is large enough depending on K. By, Proposition 4.5, we deduce that there exists $D_K \geq 2$ such that for $A \geq D_K$,

$$\mathbb{E}\left[e^{q\left|\sum_{|y_k|>A} \frac{1}{z-y_k}\right|}\right] \leq 1 + \frac{C_K q \log A}{\sqrt{A}} = 1 + C_K.$$

Now, we have $x^p \ll_p e^x$, and then for $A \geq D_K \geq 2$,

$$\mathbb{E}\left[\left|q \sum_{|y_k|>A} \frac{1}{z-y_k}\right|^p\right] \ll_{p,K} 1,$$

i.e.

$$\left(\mathbb{E}\left[\left|\sum_{|y_k|>A} \frac{1}{z-y_k}\right|^p\right]\right)^{1/p} \ll_{p,K} 1/q = \frac{\log A}{\sqrt{A}} \ll \frac{\log(2+A)}{\sqrt{1+A}}$$

In order to remove the condition $A \geq D_K$, it suffices, by using the Minkowski inequality, to check that for $A < D_K$,

$$\mathbb{E}\left(\left|\sum_{A<|y_k|\leq D_K} \frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} \ll_{p,K} 1.$$

Now, each term $1/(z - y_k)$ is bounded by $1/(\inf_{z\in K}|\Im z|)$, and then

$$\mathbb{E}\left(\left|\sum_{A<|y_k|\leq D_K} \frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} \ll_K ||X_{[-D_K,D_K]}||_{L_p}.$$

Now, this last bound is finite since $X_{[-D_K,D_K]}$ admits exponential moments by Lemma 2.8, and since it depends only on K and p, we get the desired bound.

The fact that ξ'_∞/ξ_∞ is in L^p is immediately obtained by taking $A = 0$ and by observing that the restriction $|y_k| > 0$ in the sum is irrelevant, since 0 is a.s. not a point in $\{y_k, k \in \mathbb{Z}\}$. $\qquad\square$

The preceding result allows to compute the moments of ξ'_∞/ξ_∞ by first restricting the infinite sums to the y_k's between $-A$ and A, and then by letting $A \to \infty$. More precisely, for all fixed $z_1, z_2, \ldots, z_p \notin \mathbb{R}$,

$$\forall p \geq 1, \frac{\xi'_\infty}{\xi_\infty}(z_1) \ldots \frac{\xi'_\infty}{\xi_\infty}(z_p) \in L^p$$

and

$$\mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z_1) \ldots \frac{\xi'_\infty}{\xi_\infty}(z_p)\right) = \lim_{A\to\infty} \mathbb{E}\left(\prod_{j=1}^{p}\left(i\pi + \sum_{|y_k|<A} \frac{1}{z_j-y_k}\right)\right).$$

The last quantity can be computed thanks to the sine kernel correlation functions of order less or equal than p, on the segment $[-A, A]$. We will now perform the computation of the two first moments.

Remark 4.7. *Before proceeding we should mention that since we have been able to prove the convergence of the rescaled logarithmic derivative of the characteristic polynomial to $\frac{\zeta'_\infty}{\zeta_\infty}$, we should also be able to obtain an alternative expression for the moments using the formulas in [CS08] for the moments of ratios of the logarithmic derivative of the characteristic polynomial. Although the combinatorial expressions there provide closed formulas, we do not find them easier to handle than the method we have described above. As we shall see it below, the formulas for the second moments are already very involved.*

First moment $M_1(z), z \notin \mathbb{R}$:

$$M_1(z) := \mathbb{E}\left(\frac{\zeta'_\infty}{\zeta_\infty}(z)\right)$$

$$= i\pi + \lim_{A \to \infty} \mathbb{E}\left(\sum_{|y_k| \leq A} \frac{1}{z - y_k}\right)$$

$$= i\pi + \lim_{A \to \infty} \int_{[-A,A]} dy \frac{\rho_1(y)}{z - y}$$

$$= i\pi \left(1 - \mathrm{sgn}\left(\Im(z)\right)\right)$$

$$= i2\pi \mathbb{1}_{\{\Im(z) < 0\}}$$

Second moment $M_2(z, z'); z, z' \notin \mathbb{R}$: Let us first assume that z and z' have not the same real part, in particular $z_1 \neq z_2$. One has:

$$M_2(z, z') := \mathbb{E}\left(\frac{\zeta'_\infty}{\zeta_\infty}(z)\frac{\zeta'_\infty}{\zeta_\infty}(z')\right)$$

$$= -\pi^2 + \pi^2 \left(\mathrm{sgn}\left(\Im(z)\right) + \mathrm{sgn}\left(\Im(z')\right)\right) + \mathbb{E}\left(\sum_{k,l} \frac{1}{z - y_k}\frac{1}{z' - y_l}\right)$$

$$= -\pi^2 + \pi^2 \left(\mathrm{sgn}\left(\Im(z)\right) + \mathrm{sgn}\left(\Im(z')\right)\right) + \lim_{A \to \infty} \mathbb{E}\left(\sum_{|y_k|,|y_l| \leq A} \frac{1}{z - y_k}\frac{1}{z' - y_l}\right)$$

Moreover:

$$\mathbb{E}\left(\sum_{|y_k|,|y_l| \leq A} \frac{1}{z - y_k}\frac{1}{z' - y_l}\right) = \int_{[-A,A]} \frac{dy}{(z - y)(z' - y)} + \int_{[-A,A]^2} \frac{dy_1 dy_2 \left(1 - S(y_1 - y_2)^2\right)}{(z - y_1)(z' - y_2)},$$

where

$$S(x) = \frac{\sin(\pi x)}{\pi x}$$

The first integral corresponds to the indices $k = l$ while the second integral corresponds to $k \neq l$. The former is handled by a partial fraction decomposition (recall that $z \neq z'$):

$$\lim_{A \to \infty} \int_{[-A,A]} \frac{dy}{(z-y)(z'-y)} = i\pi \frac{\mathrm{sgn}\left(\Im(z)\right) - \mathrm{sgn}\left(\Im(z')\right)}{z - z'}$$

The second integral can be written as $I_1 - I_2$, where

$$I_1 = \int_{[-A,A]^2} \frac{dy_1 dy_2}{(z-y_1)(z'-y_2)},$$

and

$$I_2 = \int_{[-A,A]^2} \frac{S(y_1 - y_2)^2}{(z-y_1)(z'-y_2)} dy_1 dy_2.$$

One has immediately

$$\lim_{A \to \infty} I_1 = \lim_{A \to \infty} \left(\int_{[-A,A]} \frac{dy}{z-y} \right) \left(\int_{[-A,A]} \frac{dy}{z'-y} \right) = -\pi^2 \, \mathrm{sgn}\left(\Im(z)\right) \mathrm{sgn}\left(\Im(z')\right).$$

For fixed z and z', the integral I_2 is dominated by

$$\int_{\mathbb{R}^2} \frac{1}{(1+|y_1|)(1+|y_2|)[1+(y_1-y_2)^2]} dy_1 dy_2$$
$$\leq \frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{1+(y_1-y_2)^2} \left(\frac{1}{(1+|y_1|)^2} + \frac{1}{(1+|y_2|)^2} \right) dy_1 dy_2$$
$$= \int_{\mathbb{R}} \frac{dy}{1+y^2} \int_{\mathbb{R}} \frac{du}{(1+|u|)^2} < \infty.$$

Hence,

$$\lim_{A \to \infty} I_2 = \int_{\mathbb{R}^2} \frac{S(y_1 - y_2)^2}{(z-y_1)(z'-y_2)} dy_1 dy_2,$$

where the last integral is absolutely convergent. The change of variable $u = y_2$, $v = y_1 - y_2$ gives

$$\lim_{A \to \infty} I_2 = \int_{\mathbb{R}} dv S(v)^2 \int_{\mathbb{R}} \frac{du}{(z - u - v)(z' - u)}.$$

The integral in u can again be computed by a partial fraction decomposition, and one gets

$$\int_{\mathbb{R}} \frac{du}{(z - u - v)(z' - u)} = i\pi \frac{\mathrm{sgn}\left(\Im(z)\right) - \mathrm{sgn}\left(\Im(z')\right)}{z - z' - v}.$$

Note that since z and z' are assumed to have different imaginary parts, the denominator does not vanish. One then has

$$\lim_{A \to \infty} I_2 = i\pi \left[\mathrm{sgn}\left(\Im(z)\right) - \mathrm{sgn}\left(\Im(z')\right) \right] \int_{\mathbb{R}} \frac{S(v)^2}{z - z' - v} dv,$$

where

$$\int_{\mathbb{R}} \frac{S(v)^2}{z - z' - v} dv = \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{2 - e^{2i\pi v} - e^{-2i\pi v}}{v^2(z - z' - v)} dv.$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{2i\pi v} + 2i\pi v}{v^2(z - z' - v)} dv + \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{-2i\pi v} - 2i\pi v}{v^2(z - z' - v)} dv,$$

In the two last integrals, the integrands are bounded near zero and dominated by $1/v^2$ at infinity, and then the integrals are absolutely convergent. Moreover, the integrands can be extended to meromorphic functions of v, with the unique pole $v = z - z'$. Note that because of the addition of the terms $\pm 2i\pi v$, there is no pole at $v = 0$. In the first integral, if we replace $\mathbb{R}$ by the contour given by the union of $(-\infty, -R]$, $[-R, -R + iR]$, $[-R + iR, R + iR]$, $[R + iR, R]$ and (R, ∞), the modified integral tends to zero when R goes to infinity. One deduces that the initial integral is equal to $2i\pi$ times the sum of the residues of the integrand at the poles in the upper half plane:

$$\frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{2i\pi v} + 2i\pi v}{v^2(z - z' - v)} dv = \frac{1 - e^{2i\pi(z - z')} + 2i\pi(z - z')}{2i\pi(z - z')^2} \mathbb{1}_{\Im(z - z') > 0}$$

Changing v in $-v$ and exchanging z and z', we deduce

$$\frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{-2i\pi v} - 2i\pi v}{v^2(z - z' - v)} dv = -\frac{1 - e^{-2i\pi(z - z')} - 2i\pi(z - z')}{2i\pi(z - z')^2} \mathbb{1}_{\Im(z - z') < 0},$$

and by adding the equalities:

$$\int_{\mathbb{R}} \frac{S(v)^2}{z - z' - v} dv = \frac{\text{sgn}\left(\Im(z - z')\right)\left(1 - e^{2i\pi(z - z')\,\text{sgn}(\Im(z - z'))}\right)}{2i\pi(z - z')^2} + \frac{1}{z - z'}.$$

By noting that

$$i\pi \left[\text{sgn}\left(\Im(z)\right) - \text{sgn}\left(\Im(z')\right)\right] \text{sgn}\left(\Im(z - z')\right) = 2i\pi \mathbb{1}_{\Im(z)\Im(z') < 0},$$

we deduce

$$\lim_{A \to \infty} I_2 = \frac{1 - e^{2i\pi(z - z')\,\text{sgn}(\Im(z - z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z') < 0} + i\pi \frac{\text{sgn}\left(\Im(z)\right) - \text{sgn}\left(\Im(z')\right)}{z - z'}.$$

Hence,

$$\lim_{A \to \infty} (I_1 - I_2) = -\pi^2 \text{sgn}\left(\Im(z)\right) \text{sgn}\left(\Im(z')\right) - \frac{1 - e^{2i\pi(z - z')\,\text{sgn}(\Im(z - z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z') < 0}$$
$$- i\pi \frac{\text{sgn}\left(\Im(z)\right) - \text{sgn}\left(\Im(z')\right)}{z - z'},$$

and

$$\lim_{A \to \infty} \mathbb{E}\left(\sum_{|y_k|, |y_l| \leq A} \frac{1}{z - y_k} \frac{1}{z' - y_l} \right) = -\pi^2 \text{sgn}\left(\Im(z)\right) \text{sgn}\left(\Im(z')\right)$$
$$- \frac{1 - e^{2i\pi(z - z')\,\text{sgn}(\Im(z - z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z') < 0}.$$

Hence

$$M_2(z,z') = -4\pi^2 \mathbb{1}_{\Im(z)<0,\Im(z')<0} - \frac{1 - e^{2i\pi(z-z')\operatorname{sgn}(\Im(z-z'))}}{(z-z')^2}\mathbb{1}_{\Im(z)\Im(z')<0}.$$

This formula has been proven for $\Im(z) \neq \Im(z')$. It remains true without this assumption. Indeed, the L^2 convergence of $i\pi + \sum_{|y_k|\leq A}\frac{1}{z-y_k}$ towards $\xi'(z)/\xi(z)$ for $A \to \infty$ has been proven uniformly in compact sets away from the real line. Since the joint moments of the former quantity are easily proven to be continuous, one deduces that M_2 is continuous with respect to $z, z' \notin \mathbb{R}$.

Second moment with a conjugate $\tilde{M}_2(z,z'); z,z' \notin \mathbb{R}$**:** Let us now define

$$\tilde{M}_2(z,z') := \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\overline{\frac{\xi'_\infty}{\xi_\infty}(z')}\right)$$

Since

$$\overline{\frac{\xi'_\infty}{\xi_\infty}(z')} = -2i\pi + \frac{\xi'_\infty}{\xi_\infty}(\overline{z'}),$$

one gets

$$\tilde{M}_2(z,z') = M_2(z,\overline{z'}) - 2i\pi M_1(z),$$

and then

$$\tilde{M}_2(z,z') = 4\pi^2 \mathbb{1}_{\Im(z)<0,\Im(z')<0} - \frac{1 - e^{2i\pi(z-\overline{z'})\operatorname{sgn}(\Im(z-\overline{z'}))}}{(z-\overline{z'})^2}\mathbb{1}_{\Im(z)\Im(z')>0}.$$

In particular, we get the L^2 norm:

$$\mathbb{E}\left(\left|\frac{\xi'_\infty}{\xi_\infty}(z)\right|^2\right) = 4\pi^2 \mathbb{1}_{\Im(z)<0} + \frac{1 - e^{-4\pi|\Im(z)|}}{4\Im^2(z)}.$$

As a consequence of the previous computation, if our conjecture is true and moments are also controlled then:

Conjecture 4.8.

$$\lim_{T\to\infty}\frac{1}{\log^2 T}\mathbb{E}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a}{\log T}\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a'}{\log T}\right)\right)$$
$$=\mathbb{1}_{\Re(a)<0,\Re(a')<0} - \frac{1 - e^{-(a'-a)\operatorname{sgn}\Re(a'-a)}}{(a-a')^2}\mathbb{1}_{\Re(a)\Re(a')<0}$$

$$\lim_{T\to\infty}\frac{1}{\log^2 T}\mathbb{E}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a}{\log T}\right)\overline{\frac{\zeta'}{\zeta}\left(\frac{1}{2}+i\omega T+\frac{a'}{\log T}\right)}\right)$$
$$=\mathbb{1}_{\Re(a)<0,\Re(a')<0} + \frac{1 - e^{-(a+\overline{a'})\operatorname{sgn}\Re(a+\overline{a'})}}{(a+\overline{a'})^2}\mathbb{1}_{\Re(a)\Re(a')>0}$$

Remark 4.9. *In Lemma 4.2, we see that there is a correspondance between a and $-2i\pi z$ in this conjecture and the computations just above. This explains the signs of the terms involved in the conjecture, and the fact the imaginary parts of z and z' are replaced by the real parts of a and a'.*

For $a = a'$, one recovers the first statement of theorem 3 in [GGM01], which is equivalent to the pair correlation conjecture under Riemann hypothesis. Higher moments formulas are also expected to be equivalent to the convergence of higher correlation functions of ζ zeros towards the corresponding correlations for the sine-kernel process.

5 The moments of ratios related to $\tilde{\zeta}_\infty$

5.1 Expectation of ratios

For $z \in \mathbb{C}$, the random variable $\tilde{\zeta}_\infty(z)$ has no moment of order 1. However, if we consider the ratio of products of values of $\tilde{\zeta}_\infty$ at points outside the real axis, and if there are the same number of factors in the numerator and in the denominator, then the ratio is integrable. This result is a consequence of the following theorem:

Theorem 5.1. *For any $p > 0$ and any compact set $K \subset \mathbb{C}\backslash\mathbb{R}$, we have:*

$$\sup_{n\in\mathbb{N}\sqcup\{\infty\}} \mathbb{E}\left(\sup_{(z,z')\in K^2} \left|\frac{\tilde{\zeta}_n(z')}{\tilde{\zeta}_n(z)}\right|^p\right) < \infty$$

Proof. Let $(z, z') \in K^2$. Without loss of generality, one can enlarge the compact set K to a compact that is symmetric with respect to the real line, and whose part above the real line is convex. Using the functional equation (1.3) if necessary, we can then assume that z and z' are both in the upper half-plane.

Since this part of K is supposed to be convex, $[z, z'] \subset K$. Therefore, the segment $[z, z']$ does not cut the real line, where zeros lie. Hence for $n \in \mathbb{N} \sqcup \{\infty\}$:

$$\left|\frac{\tilde{\zeta}_n(z')}{\tilde{\zeta}_n(z)}\right|^p \leq \left|\exp\left(p\Re\left(\int_{[z,z']} \frac{\tilde{\zeta}'_n}{\tilde{\zeta}_n}\right)\right)\right| \leq e^{p|z-z'|\sup_{u\in K}\left|\frac{\tilde{\zeta}'_n}{\tilde{\zeta}_n}(u)\right|}$$

By absorbing the quantity $|z - z'| = \mathcal{O}_K(1)$ in the exponent p, we only have to prove that for all $p > 0$:

$$\sup_{n\in\mathbb{N}\sqcup\{\infty\}} \mathbb{E}\left(\sup_{z\in K} e^{p\left|\sum_{k\in\mathbb{Z}}\frac{1}{z-y_k^{(n)}}\right|}\right) = \sup_{n\in\mathbb{N}\sqcup\{\infty\}} \mathbb{E}\left(\sup_{z\in K} e^{p\left|\frac{\tilde{\zeta}'_n}{\tilde{\zeta}_n}(z)\right|}\right) < \infty.$$

By Proposition 4.5, we know that for $A := C_K(1 + 4p^2\log(2 + 2p))$,

$$\sup_{n\in\mathbb{N}\sqcup\{\infty\}} \mathbb{E}\left(\sup_{z\in K} e^{2p\left|\sum_{|y_k^{(n)}|>A}\frac{1}{z-y_k^{(n)}}\right|}\right) < \infty.$$

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By the Cauchy-Schwarz inequality, it is then sufficient to check that

$$\sup_{n \in \mathbb{N} \sqcup \{\infty\}} \mathbb{E} \left(\sup_{z \in K} e^{2p \left| \sum_{|y_k^{(n)}| \leq A} \frac{1}{z - y_k^{(n)}} \right|} \right) < \infty.$$

Now,

$$\left| \sum_{|y_k^{(n)}| \leq A} \frac{1}{z - y_k^{(n)}} \right| \leq X_{[-A,A]} \sup_{z \in K, t \in \mathbb{R}} \frac{1}{|z - t|} \leq \frac{1}{\inf_{z \in K} |\Im z|} X_{[-A,A]}$$

and all the exponential moments of this last variable are finite, thanks to Lemma 2.8.

$\square$

From this bound, we are able to deduce the following convergence result:

Proposition 5.2. *For $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{C} \backslash \mathbb{R}$, and for all $n \in \mathbb{N} \sqcup \{\infty\}$,*

$$\mathbb{E} \left(\prod_{j=1}^{k} \left| \frac{\xi_n(z'_j)}{\xi_n(z_j)} \right| \right) < \infty$$

Moreover, for every compact set K in $\mathbb{C} \backslash \mathbb{R}$, we have the following convergence, uniformly in $z_1, z_2, \ldots, z_k, z'_1, \ldots, z'_k \in K$:

$$\mathbb{E} \left(\prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)} \right) \xrightarrow[n \to \infty]{} \mathbb{E} \left(\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)} \right).$$

Proof. The finiteness of the expectation is a direct consequence of Theorem 5.1 and the Hölder inequality. The convergence we want to prove can be written as follows:

$$\sup_{z_1, z_2, \ldots, z_k, z'_1, \ldots z'_k \in K} \left| \mathbb{E} \left[\prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)} - \prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)} \right] \right| \xrightarrow[n \to \infty]{} 0,$$

which is implied by

$$\mathbb{E} \left[\sup_{z_1, z_2, \ldots, z_k, z'_1, \ldots z'_k \in K} \left| \prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)} - \prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)} \right| \right] \xrightarrow[n \to \infty]{} 0.$$

Now, we have

$$\left| \prod_{j=1}^{k} \frac{\xi_n(z'_j)}{\xi_n(z_j)} - \prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)} \right| \leq \sum_{m=1}^{k} \prod_{1 \leq j < m} \left| \frac{\xi_n(z'_j)}{\xi_n(z_j)} \right| \prod_{m < j \leq k} \left| \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)} \right| \left| \frac{\xi_n(z'_m)}{\xi_n(z_m)} - \frac{\xi_\infty(z'_m)}{\xi_\infty(z_m)} \right|$$

It is then sufficient to show, for $1 \leq m \leq k$,

$$\mathbb{E} \left[\left(\sup_{z, z' \in K} \left| \frac{\xi_n(z')}{\xi_n(z)} \right| \right)^{m-1} \left(\sup_{z, z' \in K} \left| \frac{\xi_\infty(z')}{\xi_\infty(z)} \right| \right)^{k-m} \sup_{z, z' \in K} \left| \frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)} \right| \right] \xrightarrow[n \to \infty]{} 0,$$

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which is implied (after two applications of the Cauchy-Schwarz inequality) by

$$\sup_{r \in \mathbb{N} \sqcup \infty} \mathbb{E}\left[\left(\sup_{z,z' \in K}\left|\frac{\xi_r(z')}{\xi_r(z)}\right|\right)^{4m-4}\right]^{1/4} \sup_{r \in \mathbb{N} \sqcup \infty} \mathbb{E}\left[\left(\sup_{z,z' \in K}\left|\frac{\xi_r(z')}{\xi_r(z)}\right|\right)^{4k-4m}\right]^{1/4}$$

$$\times \mathbb{E}\left[\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right]^{1/2} \xrightarrow[n \to \infty]{} 0.$$

From Theorem 5.1, it is then sufficient to show that

$$\mathbb{E}\left[\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right] \xrightarrow[n \to \infty]{} 0.$$

Now,

$$\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right| = \sup_{z,z' \in K}\left|\frac{\xi_n(z')[\xi_\infty(z) - \xi_n(z)] + \xi_n(z)[\xi_n(z') - \xi_\infty(z')]}{\xi_n(z)\xi_\infty(z)}\right|$$

$$\leq \frac{2\left(\sup_{r \in \mathbb{N} \sqcup \{\infty\}} \sup_{z \in K}|\xi_r(z)|\right)\left(\sup_{z \in K}|\xi_n(z) - \xi_\infty(z)|\right)}{\inf_{r \in \mathbb{N} \sqcup \{\infty\}} \inf_{z \in K}|\xi_r(z)|^2}.$$

Almost surely, ξ_n converges uniformly to ξ_∞ on K. Hence, the numerator of the last fraction converges to zero when n goes to infinity. On the other hand, since ξ_∞ does not vanish on K (all its zeros as real), its infimum a on K is strictly positive. By the uniform convergence of $(\xi_r)_{r \geq 1}$ towards ξ_∞, there exists $r_0 \geq 1$ such that $r \geq r_0$ implies $\inf_K |\xi_r| \geq a/2$. Moreover, since ξ_r also has only real zeros, $\inf_K |\xi_r| > 0$ for all $r < r_0$. We deduce that the denominator of the fraction above is strictly positive. Since it does not depend on n, whereas the numerator goes to zero, we get almost surely:

$$\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2 \xrightarrow[n \to \infty]{} 0.$$

By dominated convergence, for all $B > 0$,

$$\mathbb{E}\left[B \wedge \sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right] \xrightarrow[n \to \infty]{} 0.$$

and then

$$\limsup_{n \to \infty} \mathbb{E}\left[\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right]$$

$$\leq \limsup_{n \to \infty} \mathbb{E}\left[\mathbb{1}_{\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2 \geq B} \sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right]$$

$$\leq \frac{1}{B}\limsup_{n \to \infty} \mathbb{E}\left[\sup_{z,z' \in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^4\right]$$

$$\leq \frac{1}{B} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\sup_{z,z'\in K}\left|\frac{\xi_n(z')}{\xi_n(z)}\right| + \sup_{z,z'\in K}\left|\frac{\xi_\infty(z')}{\xi_\infty(z)}\right|\right)^4\right]$$

$$\leq \frac{16}{B} \sup_{n\in\mathbb{N}\sqcup\{\infty\}} \mathbb{E}\left[\sup_{z,z'\in K}\left|\frac{\xi_n(z')}{\xi_n(z)}\right|^4\right]$$

By Theorem 5.1, the last quantity is $\mathcal{O}_K(1/B)$. Since B can be chosen arbitrarily large, we get

$$\mathbb{E}\left[\sup_{z,z'\in K}\left|\frac{\xi_n(z')}{\xi_n(z)} - \frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right] \xrightarrow[n\to\infty]{} 0.$$

$\square$

Now, the joint moments of ratios of ξ_n can be explicitly computed, by using tools given by Borodin, Olshanski and Strahov. In [BS06] and [BOS06], they established that certain determinantal formulas for ratios of characteristic polynomials are equivalent to a certain property regarding the underlying point process of zeros[2]. This property was named *Giambelli compatibility* (equation 0.2 in [BOS06]). We are now concerned with a particular case of that general framework. Consider a point process $\Lambda = \Lambda_n = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n-point configurations in $\mathbb{C}$. We assume that the underlying probability distribution is of the form:

$$\mathbb{P}\left(\Lambda \in dx\right) = \frac{1}{C_n} |\Delta(x)|^2 \prod_{i=1}^n \alpha(dx_i) \tag{5.1}$$

where $\Delta(x) = \prod_{1\leq i<j\leq n}(x_i - x_j)$ is the Vandermonde determinant, α is a reference measure on $\mathbb{C}$ whose moments are all finite, and C_n is a normalisation constant. We then have the following result:

Theorem 5.3. *If for $u \in \mathbb{C}$ we note*

$$D(u) = \prod_{i=1}^n (u - \lambda_i),$$

then the following formal identity holds for all $k \geq 1$,

$$\det\left(\frac{1}{u_i - v_j}\right) \mathbb{E}\left(\prod_{j=1}^k \frac{D(v_j)}{D(u_j)}\right) = \det\left(\frac{1}{u_i - v_j}\mathbb{E}\left(\frac{D(v_j)}{D(u_i)}\right)\right)_{i,j=1}^k. \tag{5.2}$$

This identity has to be understood as follows. Writing

$$\frac{1}{u_j - \lambda_i} = \sum_{m=1}^\infty \frac{\lambda_i^m}{u_j^{m+1}},$$

we deduce an expression of $\prod_{j=1}^k \frac{D(v_j)}{D(u_j)}$ and $\frac{D(v_j)}{D(u_i)}$ as multivariate power series in the variables $u_1,\ldots,u_k,v_1,\ldots,v_k$ for which all the nonnegative exponents are bounded by n, and whose

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[2]The authors are grateful to Brad Rodgers for many insightful discussions on the subject

coefficients are polynomial functions of $\lambda_1, \ldots, \lambda_n$. The fact that the moments of α are all finite implies that one can take, term by term, the expectation of these power series. The two sides of (5.2) can then both be written as power series in $u_1, \ldots, u_k, v_1, \ldots, v_k$ with exponents bounded from above, divided by $\prod_{1 \le i,j \le k}(u_i - v_j)$. The formula (5.2) says that these two power series coincide.

Pointers to the proof. This result is proven in [BOS06], up to small changes. Comparing our notation with [BOS06], we have

$$D(v_j) = v_j^n E(-v_j), \ [D(u_j)]^{-1} = u_j^{-n} H(u_j),$$

and then we deduce immediately our formula from Proposition 2.2. of [BOS06], by changing the sign of the v_j's and by multiplying both sides by $\prod_{j=1}^k (v_j/u_j)^n$. Note that this multiplication is the reason why we allow here nonnegative exponents up to n in the formal power series. In [BOS06], the results are proved for α carried by $\mathbb{R}$, however, they can immediately be extended to $\mathbb{C}$: the only change occurs in the proof of Theorem 3.1 in [BOS06]. We have to replace $A_{\lambda_i+N-i+N-j}$ by $A_{\lambda_i+N-i,N-j}$, where

$$A_{p,q} := \int_{\mathbb{C}} x^p \bar{x}^q \alpha(dx),$$

the conjugate coming from the fact that the joint density of Λ involves $|\Delta(x)|^2$ instead of $(\Delta(x))^2$. $\qquad\square$

We specialise α to be the Lebesgue measure on the circle $S^1 = \{|z| = 1\}$ and, thanks to the Weyl integration formula, equation (5.1) becomes the density of eigenvalues for the CUE. The random vector $\Lambda^{(n)}$ can therefore be seen as the zeros of the characteristic polynomial Z_n (equation (1.1)). The following corollary is intuitive, although the proof requires some care in passing from a statement on formal power series to a statement on actual analytic functions:

Theorem 5.4. *For $(z_1, \ldots, z_k) \in (\mathbb{C}\backslash\mathbb{R})^k$ and $(z_1', \ldots, z_k') \in \mathbb{C}^k$, such that for $1 \le i,j \le k$, $z_i - z_j'$ is not an integer multiple of n,*

$$\det\left(\frac{1}{e^{\frac{i2\pi z_i}{n}} - e^{\frac{i2\pi z_j'}{n}}}\right)_{i,j=1}^k \mathbb{E}\left(\prod_{j=1}^k \frac{\xi_n(z_j')}{\xi_n(z_j)}\right) = \det\left(\frac{1}{e^{\frac{i2\pi z_i}{n}} - e^{\frac{i2\pi z_j'}{n}}}\mathbb{E}\left(\frac{\xi_n(z_j')}{\xi_n(z_i)}\right)\right)_{i,j=1}^k \quad (5.3)$$

and moreover:

$$\mathbb{E}\left(\frac{\xi_n(z')}{\xi_n(z)}\right) = \begin{cases} 1 & \text{if } \Im(z) > 0 \\ e^{i2\pi(z'-z)} & \text{if } \Im(z) < 0 \end{cases} \quad (5.4)$$

Proof. Recall that

$$\xi_n(z) = \frac{Z_n(e^{2i\pi z/n})}{Z_n(1)} = \frac{D(e^{2i\pi z/n})}{D(1)}.$$

When forming a ratio, simplifications occur and give:

$$\forall 1 \le j \le k, \ \frac{\xi_n(z_j')}{\xi_n(z_j)} = \frac{D(e^{\frac{i2\pi z_j'}{n}})}{D(e^{\frac{i2\pi z_j}{n}})},$$

Now we set $(u,v) \in \mathbb{C}^k \times \mathbb{C}^k$ such that $v_j = e^{\frac{i2\pi z_j'}{n}}$ and $u_j = e^{\frac{i2\pi z_j}{n}}$. The result we have to prove is equivalent to the following: the equation (5.2) holds as an equality of complex numbers for all $(u_1, \ldots, u_k, v_1, \ldots v_k) \in (\mathbb{C}\backslash S^1)^k \times \mathbb{C}^k$ such that $u_i \neq v_j$ for all $i, j \in \{1, \ldots, k\}$.

Now, all the computations in Theorem 5.3, implicitly needed in order to write an equality of formal series divided by $\prod_{1 \leq i, j \leq n}(u_i - v_j)$, can be translated to get an equality of complex numbers, provided that the formal series converge absolutely and that the denominator does not vanish. This last condition is satisfied since we assume $u_i \neq v_j$ for all $i, j \in \{1, \ldots, k\}$. Now, if $|u_i| > 1$ for all i, each term of the power series corresponding to $D(v_j)/D(u_i)$ is dominated by the corresponding term of the power series

$$\prod_{m=1}^{n} \left(|v_j| + |\lambda_m|\right) \prod_{m=1}^{n} \left(\sum_{p=0}^{\infty} \frac{|\lambda_m|^p}{|u_i|^{p+1}} \right) = \left(\frac{|v_j| + 1}{|u_i| - 1} \right)^n.$$

We deduce that the power series involved in the left-hand side of (5.2) after removing the denominator $\prod_{1 \leq i, j \leq n}(u_i - v_j)$ is term by term majorized by the series corresponding to

$$\prod_{j=1}^{k} \left(\frac{|v_j| + 1}{|u_j| - 1} \right)^n \sum_{\sigma \in \mathfrak{S}_k} \prod_{1 \leq i, j \leq k, j \neq \sigma(i)} (|u_i| + |v_j|),$$

which is convergent since this quantity is finite. Similarly, the series in the right-hand side is bounded by

$$\sum_{\sigma \in \mathfrak{S}_k} \prod_{1 \leq i, j \leq k, j \neq \sigma(i)} (|u_i| + |v_j|) \prod_{i=1}^{k} \left(\frac{|v_{\sigma(i)}| + 1}{|u_i| - 1} \right)^n.$$

Hence, we have proven that (5.2) holds under the assumption that u_i is outside the unit disc for all i. Now to extend the result to inside the circle, we shall remove an arc from the circle so that our working domain becomes connected.[3] Going back to the definition of the distribution of $\Lambda^{(n)}$, assume that α is not the uniform measure on the circle, but the uniform mesure on $\mathcal{D}_\varepsilon := \{z \in \mathbb{C} \mid |z| = 1, |z - 1| \geq \varepsilon\}$ for a certain $\varepsilon \in (0,1)$. In that setting, equation (5.2) still occurs for $|u_i| > 1$, since our proof is available as soon as the measure α is supported by the unit circle. Now, the expectations involved in (5.2) are integrals, with respect to the distribution of $\Lambda^{(n)}$, of rational functions of $(u_1, \ldots, u_k, v_1, \ldots, v_k, \lambda_1, \ldots, \lambda_n)$. If $u_1, \ldots, u_k$ are in a compact set K_1 of $\mathbb{C}\backslash\mathcal{D}_\varepsilon$, and $v_1, \ldots, v_k$ are in a compact set K_2 of $\mathbb{C}$, then these rational functions are bounded by a quantity depending only on K_1 and K_2, since almost surely on the law of $\Lambda^{(n)}$,

$$\left| \frac{D(v_j)}{D(u_i)} \right| = \prod_{m=1}^{n} \left| \frac{v_j - \lambda_m}{u_i - \lambda_m} \right| \leq \left(\frac{1 + \sup_{v \in K_2} |v|}{\operatorname{dist}(K_1, \mathcal{D}_\varepsilon)} \right)^n < \infty.$$

Hence, using dominated convergence, one deduces that the expectations in (5.2) are holomorphic functions of $(u_1, \ldots, u_k, v_1, \ldots, v_k)$ on $(\mathbb{C}\backslash\mathcal{D}_\varepsilon)^k \times \mathbb{C}^k$. Hence, the two sides of (5.2) can be written as quotients by $\prod_{1 \leq i, j \leq k}(u_i - v_j)$ of holomorphic functions. Since

---

[3]This idea was suggested to us by Brad Rodgers

these holomorphic functions coincide on $(\{z \in \mathbb{C}, |z| > 1\})^k \times \mathbb{C}^k$, and $(\mathbb{C}\backslash \mathcal{D}_\varepsilon)^k \times \mathbb{C}^k$ is connected, they coincide on $(\mathbb{C}\backslash \mathcal{D}_\varepsilon)^k \times \mathbb{C}^k$, and in particular, (5.2) holds for all $u_1, \ldots, u_k \in \mathbb{C}\backslash S^1$ and $v_1, \ldots, v_k \in \mathbb{C}$. Now, if $u_1, \ldots, u_k \in \mathbb{C}\backslash S^1$, $v_1, \ldots, v_k \in \mathbb{C}$ are fixed, the left-hand side of (5.2) is the integral, with respect to the law of $\Lambda^{(n)}$, of a continuous, bounded function of $(\lambda_1, \ldots, \lambda_n) \in (S^1)^k$, and the right-hand side is a linear combination of products of such integrals. Hence, the two sides of (5.2) are continuous with respect to the law of $\Lambda^{(n)}$. Now, it is easy to check that the law of $\Lambda^{(n)}$ for α uniform on $\mathcal{D}_\varepsilon$ tends to the law for α uniform on S^1 when ϵ goes to zero. Hence, since (5.2) holds for α uniform on $\mathcal{D}_\varepsilon$, it also occurs for α uniform on S^1.

It remains to prove (5.4). Using the change of variables $u = e^{2i\pi z/n}$ and $v = e^{2i\pi z'/n}$, we have to check

$$\mathbb{E}\left(\frac{D(v)}{D(u)}\right) = \left\{ \begin{array}{ll} 1 & \text{if } |u| < 1 \\ (v/u)^n & \text{if } |u| > 1 \end{array}\right. \tag{5.5}$$

If $|u| < 1$, we can write

$$\frac{D(v)}{D(u)} = \prod_{m=1}^n \frac{1 - v\lambda_m^{-1}}{1 - u\lambda_m^{-1}} = \prod_{m=1}^n (1 - v\lambda_m^{-1}) \left(\sum_{p=0}^\infty u^p \lambda_m^{-p}\right).$$

If we expand this expression as a power series in u and v with polynomial coefficients in $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$, this series is term by term dominated by

$$\left[(1 + |v|)\left(\sum_{p=0}^\infty |u|^p\right)\right]^n < \infty.$$

Hence, the expectation of $D(v)/D(u)$ can be obtained by adding the expectations of each term of the corresponding power series. For all nonnegative integers $p, q \geq 0$, the term in $u^p v^q$ is a polynomial in $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ with total degree $p + q$. Now, the law of $\Lambda^{(n)}$ remains invariant if we multiply $(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$ by any $z \in S^1$, and then the expectation of the term in $u^p v^q$ is invariant by multiplication by z^{-p-q}, which implies that it is zero for all $(p, q) \neq (0, 0)$. Hence the expectation of $D(v)/D(u)$ is equal to the constant term of the corresponding series, which is equal to 1, and then we get (5.5) for $|u| < 1$. The case $|u| > 1$, $v \neq 0$ is the deduced as follows: we have

$$D(u) = \prod_{m=1}^n (u - \lambda_m) = (-u)^n \left(\prod_{m=1}^n \lambda_m\right) \prod_{m=1}^n (u^{-1} - \overline{\lambda_m}) = (-u)^n \left(\prod_{m=1}^n \lambda_m\right) \overline{D(\bar{u}^{-1})},$$

$$\frac{D(v)}{D(u)} = (v/u)^n \frac{\overline{D(\bar{v}^{-1})}}{\overline{D(\bar{u}^{-1})}},$$

and then, since $|\bar{u}^{-1}| < 1$,

$$\mathbb{E}\left(\frac{D(v)}{D(u)}\right) = (v/u)^n.$$

Using dominated convergence, it is easy to check that $\mathbb{E}[D(v)/D(u)]$ is continuous with respect to $(u, v) \in (\mathbb{C}\backslash S^1) \times \mathbb{C}$, which allows to extend (5.5) to the case $|u| > 1$, $v = 0$.

$\square$

As Borodin, Olshanski and Strahov have in fact noticed, taking the limit for n going to infinity is meaningful. Here, we can go further since we have now constructed the limiting object ξ_∞. More precisely, using the convergence proven in Proposition 5.2, we easily get the following:

Theorem 5.5 (Ratio formula). *For all $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{C} \backslash \mathbb{R}$ such that $z_i \neq z'_j$ for $1 \leq i, j \leq n$, we have*

$$\det\left(\frac{1}{z_i - z'_j}\right)^k_{i,j=1} \mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right) = \det\left(\frac{1}{z_i - z'_j} \mathbb{E}\left(\frac{\xi_\infty(z'_j)}{\xi_\infty(z_i)}\right)\right)^k_{i,j=1}$$

and moreover:

$$\mathbb{E}\left(\frac{\xi_\infty(z')}{\xi_\infty(z)}\right) = \begin{cases} 1 & \text{if } \Im(z) > 0 \\ e^{i2\pi(z'-z)} & \text{if } \Im(z) < 0 \end{cases}$$

The condition $z_i \neq z'_j$ is not really restrictive, since for $z_i = z'_j$, the ratio inside the expectation can be immediatly simplified by removing the factor $\xi_\infty(z_i) = \xi_\infty(z'_j)$ in the numerator and the denominator. If the z_i and the z'_j are all pairwise distinct, we can divide by the Cauchy determinant in the left-hand side, in order to get the joint moment of ratios $\xi_\infty(z'_j)/\xi_\infty(z_j)$. If some of the z_i or some of the z'_j are equal, the Cauchy determinant is zero, so the ratio formula does not give the moment directly: however, the moment can be recovered from the fact that it is continuous with respect to $z_1, \ldots, z_k, z'_1, \ldots, z'_k \notin \mathbb{R}$, this property of continuity coming from the uniformity of the convergence in Proposition 5.2. The joint moments of ratios of the form $\frac{\xi_\infty(z')}{\xi_\infty(z)}$ and conjugates of such ratios can then be easily deduced from the following:

$$\overline{\xi_\infty(z)} = e^{-i\pi\bar{z}} \prod_{k\in\mathbb{Z}} \left(1 - \frac{\bar{z}}{y_k}\right) = e^{-2i\pi\bar{z}}\left[e^{i\pi\bar{z}} \prod_{k\in\mathbb{Z}} \left(1 - \frac{\bar{z}}{y_k}\right)\right] = e^{-2i\pi\bar{z}}\xi_\infty(\bar{z}).$$

In this way, we get for all $z, z' \notin \mathbb{R}$,

$$\mathbb{E}\left[\left|\frac{\xi_\infty(z')}{\xi_\infty(z)}\right|^2\right] = e^{-4\pi\Im(z'-z)\mathbb{1}_{\Im(z)<0}}\left(1 + (1 - e^{-4\pi\Im(z')\,\text{sgn}(\Im(z))})\frac{|z-z'|^2}{4\Im(z)\Im(z')}\right).$$

Given Conjecture 4.1, it is natural to expect the following:

Conjecture 5.6. *Let ω be a uniform random variable on $[0,1]$ and $T > 0$ a real parameter going to infinity. Then, for all $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{C} \backslash \mathbb{R}$, such that $z_i \neq z'_j$ for all i, j,*

$$\mathbb{E}\left(\prod_{j=1}^{k} \frac{\zeta\left(\frac{1}{2} + iT\omega - \frac{i2\pi z'_j}{\log T}\right)}{\zeta\left(\frac{1}{2} + iT\omega - \frac{i2\pi z_j}{\log T}\right)}\right)$$

$$\xrightarrow{T\to\infty} \det\left(\frac{1}{z_i - z'_j}\right)^{-1} \det\left(\frac{\mathbb{1}_{\Im(z_i)>0} + e^{2i\pi(z'_j - z_i)}\mathbb{1}_{\Im(z_i)<0}}{z_i - z'_j}\right)^k_{i,j=1},$$

where the last expression is well-defined where the z_i and the z'_j are all distinct, and is extended by continuity to the case where some of the z_i or some of the z'_j are equal.

Remark 5.7. *In a recent work [Rod15], Rodgers has shown that the GUE conjectures and the Riemann hypothesis imply the above conjecture.*

5.2   Moments of the logarithmic derivative

We have seen in Section 4 how to compute the expectation of products of the logarithmic derivative of the characteristic polynomial evaluated at different points at the microscopic scale. In particular, it appeared that this method is hard to exploit when one considers a product with three or more factors. On the other hand, one might try to compute such expectations using the ratios formula (5.5). Indeed, it follows from Theorem 5.1 that we can differentiate $\mathbb{E}\left(\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right)$ with respect to z'_j and then take $z'_j = z_j$, with the z''_j's all distinct. As an application one can see that the formulas given Section 4 can be obtained with this method in a much quicker way. We shall use this approach to establish a general formula for the moments of the logarithmic derivative.

Before proceeding, let us mention again that the moments of the logarithmic derivative of the characteristic polynomial as well as their asymptotic behavior have already been studied in the random matrix literature in relation with the Riemann zeta function (see e.g. [CFZ08], [CS07] or [CS08]). Since the formula for the ratios that is usually used in this literature is different from our formula (5.5), the formula we shall establish will look different as well.

We will state our main formula at the end of the section after discussing several computational steps.

We assume for the moment that the z_j's and the z'_j's are all distinct and not on the real line. If A denote the set of indexes j such that z_j has negative real part, we get

$$
\mathbb{E}\left[\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right] = \frac{\det\left(\frac{e^{2i\pi(z'_j - z_i)\mathbb{1}_{i\in A}}}{z_i - z'_j}\right)_{i,j=1}^{k}}{\det\left(\frac{1}{z_i - z'_j}\right)_{i,j=1}^{k}}.
$$

The denominator is the Cauchy determinant:

$$
\det\left(\frac{1}{z_i - z'_j}\right)_{i,j=1}^{k} = \frac{\prod_{i<j}(z_j - z_i)\prod_{i<j}(z'_i - z'_j)}{\prod_{i,j}(z_i - z'_j)}.
$$

Expanding the numerator then gives, after dividing by the Cauchy determinant:

$$
\mathbb{E}\left[\prod_{j=1}^{k} \frac{\xi_\infty(z'_j)}{\xi_\infty(z_j)}\right] = \sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)\prod_{i\in A} e^{2i\pi(z'_{\sigma(i)} - z_i)} \prod_{i,j\neq\sigma(i)}(z_i - z'_j)\prod_{i<j}(z_j - z_i)^{-1}\prod_{i<j}(z'_i - z'_j)^{-1}.
$$

This expression is proven for z_j, z'_j all distinct: by continuity, it also holds for the z_j's distinct on one hand, and the z''_j's distinct on the other hand. Now, it is possible, in the last expression, to differentiate inside the expectation, with respect to any set of variables. Indeed, since the product of ratios of ξ_∞ is holomorphic with respect to all

the variables on $\mathbb{C}\backslash\mathbb{R}$, differentiating is equivalent to taking suitable integrals on small circles, by using the formula

$$f'(z) = \frac{1}{2\pi\epsilon} \int_0^{2\pi} e^{-i\theta} f(z + \epsilon e^{i\theta}) d\theta,$$

and the integrals can be exchanged with the expectation, because all the moments of ratios of ξ_∞ are uniformly bounded on compact sets of $\mathbb{C}\backslash\mathbb{R}$, by Theorem 3.11. We deduce that for $z_1, \dots, z_k$ pairwise distinct and not real:

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\xi_\infty'(z_j)}{\xi_\infty(z_j)}\right] = \sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma) \frac{\partial^k}{\partial z_1' \dots \partial z_k'} \left(\prod_{i\in A} e^{2i\pi(z_{\sigma(i)}' - z_i)}\right.$$

$$\left.\prod_{i,j\neq\sigma(i)} (z_i - z_j') \prod_{i<j}(z_j - z_i)^{-1} \prod_{i<j}(z_i' - z_j')^{-1}\right)_{z_1'=z_1,\dots,z_k'=z_k}.$$

For each permutation σ, we have a multiple derivative of a product, so we have to add all the possible terms obtained by distributing the derivations on the different factors. If j is not in the set $F(\sigma)$ of fixed points of σ, then the product contains the factor $z_j - z_j'$. If this factor is not differentiated with respect to z_j', the corresponding term vanishes by taking $z_j' = z_j$. If this factor is differentiated with respect to z_j', it becomes equal to -1. Hence, we get

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\xi_\infty'(z_j)}{\xi_\infty(z_j)}\right] = \prod_{i<j}(z_j - z_i)^{-1} \sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)(-1)^{k-|F(\sigma)|}\dots$$

$$\dots \times \frac{\partial^{|F(\sigma)|}}{\prod_{j\in F(\sigma)} \partial z_j'} \left(\prod_{i\in A} e^{2i\pi(z_{\sigma(i)}' - z_i)} \prod_{i,j\neq i,\sigma(i)} (z_i - z_j') \prod_{i<j}(z_i' - z_j')^{-1}\right)_{z_1'=z_1,\dots,z_k'=z_k}.$$

We have to differentiate a product of three factors (which are products themselves). Hence, we can write the result as a sum of terms indexed by partitions of $F(\sigma)$ into three subsets U, V, W. For a given partition, the corresponding term can be nonzero only if the first product depends on all the variables indexed by U, which means that for all $j \in U$, there exists $i \in A$ such that $\sigma(i) = j$, i.e. $i = \sigma^{-1}(j)$. Since U is included in $F(\sigma)$, we have $j \in F(\sigma)$ and then $\sigma^{-1}(j) = j$, i.e. $i = j$ and $j \in A$. Hence, we only need to consider partitions for which $U \subset A$. Moreover, if this condition is satisfied, each derivation of the first term simply multiplies it by $2i\pi$, and then we get:

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\xi_\infty'(z_j)}{\xi_\infty(z_j)}\right] = \prod_{i<j}(z_j - z_i)^{-1} \sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)(-1)^{k-|F(\sigma)|} \prod_{i\in A} e^{2i\pi(z_{\sigma(i)} - z_i)}\dots$$

$$\dots \times \sum_{U\sqcup V\sqcup W=F(\sigma), U\subset A} (2i\pi)^{|U|} \frac{\partial^{|V|}}{\prod_{j\in V} \partial z_j'} \left(\prod_{i,j\neq i,\sigma(i)} (z_i - z_j')\right)_{z_1'=z_1,\dots,z_k'=z_k} \dots$$

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$$\dots \times \frac{\partial^{|W|}}{\prod_{j \in W} \partial z'_j} \left(\prod_{i<j} (z'_i - z'_j)^{-1} \right) \Bigg|_{z'_1 = z_1, \dots, z'_k = z_k} .$$

In order to compute the differential with respect to z'_j, $j \in V$ in this formula, we need, for each $j \in V$, to remove one of the factors $z_i - z'_j$ and to replace it by -1. The index i in the removed factor is free as soon as it is different from j and $\sigma^{-1}(j)$, now $j = \sigma^{-1}(j)$ since $j \in V \subset F(\sigma)$. Hence, we get:

$$\frac{\partial^{|V|}}{\prod_{j \in V} \partial z'_j} \left(\prod_{i,j \neq i, \sigma(i)} (z_i - z'_j) \right) \Bigg|_{z'_1 = z_1, \dots, z'_k = z_k}$$

$$= (-1)^{|V|} \sum_{\forall j \in V, w_j \in \{1, \dots, k\} \setminus \{j\}} \prod_{j \notin V, i \neq j, \sigma^{-1}(j)} (z_i - z_j) \prod_{j \in V, i \neq j, w_j} (z_i - z_j).$$

The computation of the derivation with respect to the indices in W is done by using the following lemma:

Lemma 5.8. *For $x_1 \neq x_2 \neq \cdots \neq x_k$, let*

$$\Delta^{-1}(x_1, \dots, x_k) := \prod_{1 \leq i < j \leq k} (x_j - x_i)^{-1}.$$

Then, for $1 \leq m \leq k$

$$\frac{\partial^m}{\prod_{j=1}^m \partial x_j} \Delta^{-1}(x_1, \dots, x_k) = \Delta^{-1}(x_1, \dots, x_k) \sum_{i_1 \neq 1, \dots, i_m \neq m} 2^{N(i_1, \dots, i_m)} \prod_{p=1}^m (x_{i_p} - x_p)^{-1},$$

where $N(i_1, \dots, i_m)$ denotes the number of indices $p \in \{1, \dots, m\}$ such that $i_p \in \{1, \dots, p\}$ and $i_{i_p} = p$.

Proof. For $m = 1$, we obtain $k - 1$ terms, obtained by differentiating each of the factors $(x_j - x_1)^{-1}$ with respect to x_1. This multiplies the factor by $(x_j - x_1)^{-1}$, since the derivative is $(x_j - x_1)^{-2}$. Hence,

$$\frac{\partial^m}{\partial x_1} \Delta^{-1}(x_1, \dots, x_k) = \Delta^{-1}(x_1, \dots, x_k) \sum_{i \neq 1} (x_i - x_1)^{-1},$$

which proves the formula for $m = 1$, since $N(i) = 0$ for all $i \neq 1$. Let us now deduce the formula for $m \in \{2, \dots, k\}$ from the formula for $m - 1$. If the claimed forumula is true for $m - 1$,

$$\frac{\partial^m}{\prod_{j=1}^m \partial x_j} \Delta^{-1}(x_1, \dots, x_k) = \sum_{i_1 \neq 1, \dots, i_{m-1} \neq m-1} 2^{N(i_1, \dots, i_{m-1})} \prod_{p=1}^{m-1} (x_{i_p} - x_p)^{-1} \frac{\partial}{\partial x_m} \Delta^{-1}(x_1, \dots, x_k)$$

$$+ \Delta^{-1}(x_1, \dots, x_k) \sum_{i_1 \neq 1, \dots, i_{m-1} \neq m-1} 2^{N(i_1, \dots, i_{m-1})} \frac{\partial}{\partial x_m} \left(\prod_{p=1}^{m-1} (x_{i_p} - x_p)^{-1} \right).$$

The derivative in the first term gives terms with an extra factor $(x_{i_m} - x_m)^{-1}$, for all $i_m \neq m$. The derivative in the second term gives terms with an extra factor $(x_p - x_m)^{-1}$, for all $p \in \{1, \ldots, m-1\}$ such that $i_p = m$. Hence

$$\frac{\partial^m}{\prod_{j=1}^m \partial x_j} \Delta^{-1}(x_1, \ldots, x_k) = \sum_{i_1 \neq 1, \ldots, i_m \neq m} 2^{N(i_1, \ldots, i_m)} \prod_{p=1}^{m-1} (x_{i_p} - x_p)^{-1} \Delta^{-1}(x_1, \ldots, x_k)$$

$$+ \Delta^{-1}(x_1, \ldots, x_k) \sum_{i_1 \neq 1, \ldots, i_{m-1} \neq m-1} 2^{N(i_1, \ldots, i_{m-1})} \sum_{p \in \{1, \ldots, m-1\}, i_p = m} (x_p - x_m)^{-1} \prod_{q=1}^{m-1} (x_{i_q} - x_q)^{-1}.$$

Now, if the index p in the last sum is denoted i_m, the constraint on i_m is that $i_m \in \{1, \ldots, m-1\}$ and $i_{i_m} = m$, or equivalently, $i_m \neq m$, $i_m \in \{1, \ldots, m\}$, $i_{i_m} = m$. Moreover, the factor $(x_p - x_m)^{-1}$ is equal to $(x_{i_m} - x_m)^{-1}$. Hence,

$$\frac{\partial^m}{\prod_{j=1}^m \partial x_j} \Delta^{-1}(x_1, \ldots, x_k) = \sum_{i_1 \neq 1, \ldots, i_m \neq m} 2^{N(i_1, \ldots, i_{m-1})} \prod_{p=1}^m (x_{i_p} - x_p)^{-1} \Delta^{-1}(x_1, \ldots, x_k)$$

$$+ \Delta^{-1}(x_1, \ldots, x_k) \sum_{i_1 \neq 1, \ldots, i_m \neq m} 2^{N(i_1, \ldots, i_{m-1})} \mathbb{1}_{i_m \in \{1, \ldots, m\}, i_{i_m} = m} \prod_{p=1}^m (x_{i_p} - x_p)^{-1}.$$

and then

$$\frac{\partial^m}{\prod_{j=1}^m \partial x_j} \Delta^{-1}(x_1, \ldots, x_k)$$

$$= \Delta^{-1}(x_1, \ldots, x_k) \sum_{i_1 \neq 1, \ldots, i_m \neq m} 2^{N(i_1, \ldots, i_{m-1})} (1 + \mathbb{1}_{i_m \in \{1, \ldots, m\}, i_{i_m} = m}) \prod_{p=1}^m (x_{i_p} - x_p)^{-1}$$

$$= \Delta^{-1}(x_1, \ldots, x_k) \sum_{i_1 \neq 1, \ldots, i_m \neq m} 2^{N(i_1, \ldots, i_{m-1}) + \mathbb{1}_{i_m \in \{1, \ldots, m\}, i_{i_m} = m}} \prod_{p=1}^m (x_{i_p} - x_p)^{-1}$$

$$= \Delta^{-1}(x_1, \ldots, x_k) \sum_{i_1 \neq 1, \ldots, i_m \neq m} 2^{N(i_1, \ldots, i_m)} \prod_{p=1}^m (x_{i_p} - x_p)^{-1}.$$

$\square$

From this lemma, we immediately get

$$\frac{\partial^{|W|}}{\prod_{j \in W} \partial z_j'} \left(\prod_{i<j} (z_i' - z_j')^{-1} \right)_{z_1' = z_1, \ldots, z_k' = z_k}$$

$$= \left(\prod_{i<j} (z_i - z_j)^{-1} \right) \sum_{\forall j \in W, w_j \in \{1, \ldots, k\} \setminus \{j\}} 2^{N(w_j, j \in W)} \prod_{j \in W} (z_{w_j} - z_j)^{-1},$$

where $N(w_j, j \in W)$ denotes the number of pairs $\{j, k\} \subset W$, such that $w_j = k$ and $w_k = j$, in other words $1/2$ of the number of $j \in W$ such that $w_j \in W$ and $w_{w_j} = j$. Hence, we deduce

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\xi_\infty'(z_j)}{\xi_\infty(z_j)} \right] = \prod_{i \neq j} (z_j - z_i)^{-1} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) (-1)^{k - |F(\sigma)|} \prod_{i \in A} e^{2i\pi(z_{\sigma(i)} - z_i)} \ldots$$

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$$\ldots \times \sum_{U \sqcup V \sqcup W = F(\sigma), U \subset A} (2i\pi)^{|U|} (-1)^{|V|} \sum_{\forall j \in V \cup W, w_j \in \{1, \ldots, k\} \setminus \{j\}} 2^{N(w_j, j \in W)} \ldots$$

$$\ldots \times \prod_{j \notin V, i \neq j, \sigma^{-1}(j)} (z_i - z_j) \prod_{j \in V, i \neq j, w_j} (z_i - z_j) \prod_{j \in W} (z_{w_j} - z_j)^{-1}$$

The sum indexed by U, V, W can be simplified by considering $X = V \cup W$, and by using the fact that U is the complement of X in $F(\sigma)$. We have $|U| = |F(\sigma)| - |X|$, and by splitting the following product in two factors corresponding to $j \in W \subset F(\sigma)$ and $j \notin V \cup W = X$,

$$\prod_{j \notin V, i \neq j, \sigma^{-1}(j)} (z_i - z_j) = \prod_{j \in W, i \neq j} (z_i - z_j) \prod_{j \notin X, i \neq j, \sigma^{-1}(j)} (z_i - z_j),$$

which implies

$$\prod_{j \notin V, i \neq j, \sigma^{-1}(j)} (z_i - z_j) \prod_{j \in V, i \neq j, w_j} (z_i - z_j) \prod_{j \in W} (z_{w_j} - z_j)^{-1}$$

$$= \prod_{j \in X, i \neq j, w_j} (z_i - z_j) \prod_{j \notin X, i \neq j, \sigma^{-1}(j)} (z_i - z_j).$$

We deduce

$$\mathbb{E} \left[\prod_{j=1}^{k} \frac{\xi'_\infty(z_j)}{\xi_\infty(z_j)} \right] = \prod_{i \neq j} (z_j - z_i)^{-1} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) (-1)^{k - |F(\sigma)|} \prod_{i \in A} e^{2i\pi(z_{\sigma(i)} - z_i)} \ldots$$

$$\ldots \times \sum_{F(\sigma) \setminus A \subset X \subset F(\sigma)} (2i\pi)^{|F(\sigma)| - |X|} \sum_{\forall j \in X, w_j \in \{1, \ldots, k\} \setminus \{j\}} \prod_{j \in X, i \neq j, w_j} (z_i - z_j) \ldots$$

$$\ldots \times \prod_{j \notin X, i \neq j, \sigma^{-1}(j)} (z_i - z_j) \sum_{W \subset X} (-1)^{|X| - |W|} 2^{N(w_j, j \in W)}.$$

Let us now compute the last sum indexed by W. If there exists $j \in X$ such that $w_j \notin X$ or $w_{w_j} \neq j$, then for all $W \subset X$ such that $j \notin W$, we have

$$2N(w_i, i \in W \cup \{j\}) = \sum_{i \in W} \mathbb{1}_{w_i \in W \cup \{j\}, w_{w_i} = i} + \mathbb{1}_{w_j \in W \cup \{j\}, w_{w_j} = j}.$$

The last indicator function is equal to zero since $w_j \notin X$ or $w_{w_j} \neq j$ by assumption. Hence,

$$2N(w_i, i \in W \cup \{j\}) = \sum_{i \in W} \mathbb{1}_{w_i \in W, w_{w_i} = i} + \sum_{i \in W} \mathbb{1}_{w_i = j, w_{w_i} = i}.$$

In the second sum, the indicator functions are also zero: otherwise we would have $w_j = w_{w_i} = i \in W$, and $w_{w_j} = w_i = j$. Hence

$$N(w_i, i \in W \cup \{j\}) = N(w_i, i \in W)$$

and

$$\sum_{W \subset X} (-1)^{|X| - |W|} 2^{N(w_i, i \in W)}$$

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$$= \sum_{W \subset X, j \notin W} \left((-1)^{|X|-|W|} 2^{N(w_i, i \in W)} + (-1)^{|X|-|W|-1} 2^{N(w_i, i \in W \cup \{j\})} \right)$$

$$= \sum_{W \subset X, j \notin W} \left((-1)^{|X|-|W|} 2^{N(w_i, i \in W)} + (-1)^{|X|-|W|-1} 2^{N(w_i, i \in W)} \right) = 0.$$

We can then restrict our computations to the case where $w_j \in X$ and $w_{w_j} = j$ for all $j \in X$: in other words $w_j = \tau(j)$, where τ is an involution of X without fixed point. If $\mathcal{I}_X$ denotes the set of all involutions of X without fixed point (in particular $\mathcal{I}_X$ is empty if $|X|$ is odd), we get

$$\mathbb{E}\left[\prod_{j=1}^{k} \frac{\xi'_\infty(z_j)}{\xi_\infty(z_j)} \right] = \prod_{i \neq j} (z_j - z_i)^{-1} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)(-1)^{k-|F(\sigma)|} \prod_{i \in A} e^{2i\pi(z_{\sigma(i)} - z_i)} ...$$

$$... \times \sum_{F(\sigma) \backslash A \subset X \subset F(\sigma)} (2i\pi)^{|F(\sigma)|-|X|} \sum_{\tau \in \mathcal{I}_X} \prod_{j \in X, i \neq j, \tau(j)} (z_i - z_j) ...$$

$$... \times \prod_{j \notin X, i \neq j, \sigma^{-1}(j)} (z_i - z_j) \sum_{W \subset X} (-1)^{|W|} 2^{N(\tau(j), j \in W)}.$$

In the sum in W, we have replaced $(-1)^{|X|-|W|}$ by $(-1)^{|W|}$, since τ can exist only for $|X|$ even. Now, if $X_1, \ldots, X_h$ are the supports of the cycles of τ ($h = |X|/2$), we have for all $W \subset X$,

$$(-1)^{|W|} = \prod_{p=1}^{h} (-1)^{|W \cap X_p|}, \quad 2^{N(\tau(j), j \in W)} = \prod_{p=1}^{h} 2^{\mathbb{1}_{W \cap X_p = X_p}}.$$

Hence,

$$\sum_{W \subset X} (-1)^{|W|} 2^{N(\tau(j), j \in W)} = \sum_{W_1 \subset X_1, \ldots, W_h \subset X_h} \prod_{p=1}^{h} (-1)^{|W_p|} 2^{\mathbb{1}_{W_p = X_p}}$$

$$= \prod_{p=1}^{h} \sum_{W_p \subset X_p} (-1)^{|W_p|} 2^{\mathbb{1}_{W_p = X_p}} = \prod_{p=1}^{h} (1 - 1 - 1 + 2) = 1.$$

We know have

$$\mathbb{E}\left[\prod_{j=1}^{k} \frac{\xi'_\infty(z_j)}{\xi_\infty(z_j)} \right] = \prod_{i \neq j} (z_i - z_j)^{-1} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)(-1)^{k-|F(\sigma)|} \prod_{i \in A} e^{2i\pi(z_{\sigma(i)} - z_i)} ...$$

$$... \times \sum_{F(\sigma) \backslash A \subset X \subset F(\sigma), |X| \in 2\mathbb{Z}} (2i\pi)^{|F(\sigma)|-|X|} \sum_{\tau \in \mathcal{I}_X} \prod_{j \in X, i \neq j, \tau(j)} (z_i - z_j) \prod_{j \notin X, i \neq j, \sigma^{-1}(j)} (z_i - z_j),$$

which can be simplified in

$$\mathbb{E}\left[\prod_{j=1}^{k} \frac{\xi'_\infty(z_j)}{\xi_\infty(z_j)} \right] = \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)(-1)^{k-|F(\sigma)|} \prod_{j \notin F(\sigma)} (z_{\sigma^{-1}(j)} - z_j)^{-1} ...$$

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$$\ldots \times e^{2i\pi \sum_{j \in A}(z_{\sigma(j)} - z_j)} \sum_{F(\sigma)\backslash A \subset X \subset F(\sigma), |X| \in 2\mathbb{Z}} (2i\pi)^{|F(\sigma)| - |X|} \sum_{\tau \in \mathcal{I}_X} \prod_{j \in X} (z_{\tau(j)} - z_j)^{-1},$$

Let us now reorder this sum in function of the permutation $\rho = \sigma \circ \tau$. The condition $F(\sigma) \backslash A \subset X \subset F(\sigma)$ means that only the points in A can be fixed by ρ. Moreover, for a given ρ, σ can be any permutation obtained by removing some of the 2-cycles of ρ. If $S(\rho)$ denotes the set of such permutations σ, and if for $\sigma \in S(\rho)$, we denote by $N(\sigma, \rho)$ the number of 2-cycles which are removed, we get

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\zeta'_\infty(z_j)}{\zeta_\infty(z_j)}\right] = (-1)^k \sum_{\rho \in \mathfrak{S}_k, F(\rho) \subset A} \epsilon(\rho)(-2i\pi)^{|F(\rho)|} \ldots$$

$$\ldots \times \prod_{j \notin F(\rho)} (z_{\rho^{-1}(j)} - z_j)^{-1} \sum_{\sigma \in S(\rho)} (-1)^{N(\sigma,\rho)} e^{2i\pi \sum_{j \in A}(z_{\sigma(j)} - z_j)}.$$

Let us suppose that ρ has a 2-cycle completely outside or inside A. In this case, if $\sigma, \sigma' \in S(\rho)$ differ only by this cycle,

$$(-1)^{N(\sigma,\rho)} e^{2i\pi \sum_{j \in A}(z_{\sigma(j)} - z_j)} = -(-1)^{N(\sigma',\rho)} e^{2i\pi \sum_{j \in A}(z_{\sigma'(j)} - z_j)}$$

Hence, the sum for $\sigma \in S(\rho)$ vanishes. Otherwise, the 2-cycles of ρ are of the form (a, b) where $a \in A$ and $b \notin A$, and removing such a cycle removes a factor $-e^{2i\pi(z_b - z_a)}$ in the term

$$(-1)^{N(\sigma,\rho)} e^{2i\pi \sum_{j \in A}(z_{\sigma(j)} - z_j)}.$$

Hence, if $\mathfrak{S}_{k,A}$ denotes the set of permutations of order k such that all the fixed points are in A and all the 2-cycles have one element in A and one outside A, and if for $\rho \in \mathfrak{S}_{k,A}$, $G(\rho, A)$ is the set of elements in A in the 2-cycles of ρ, then we get

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\zeta'_\infty(z_j)}{\zeta_\infty(z_j)}\right] = (-1)^k \sum_{\rho \in \mathfrak{S}_{k,A}} \epsilon(\rho)(-2i\pi)^{|F(\rho)|} \ldots$$

$$\ldots \times \prod_{j \notin F(\rho)} (z_{\rho^{-1}(j)} - z_j)^{-1} e^{2i\pi \sum_{j \in A}(z_{\rho(j)} - z_j)} \prod_{j \in G(\rho,A)} \left(1 - e^{2i\pi(z_j - z_{\rho(j)})}\right).$$

We now summarize the above in the following Proposition:

Proposition 5.9. *Let $z_1, \cdots, z_n$ be distinct complex numbers in $\mathbb{C} \setminus \mathbb{R}$, and let A denote the set of indexes j such that z_j has negative real part. Let us also note $\mathfrak{S}_{k,A}$ the set of permutations of order k such that all the fixed points are in A and all the 2-cycles have one element in A and one outside A, and let $F(\sigma)$ be the set of fixed points of σ. For $\rho \in \mathfrak{S}_{k,A}$, we note $G(\rho, A)$ the set of elements in A in the 2-cycles of ρ. Then the following formula for the moments of the logarithmic derivative holds:*

$$\mathbb{E}\left[\prod_{j=1}^k \frac{\zeta'_\infty(z_j)}{\zeta_\infty(z_j)}\right] = (-1)^k \sum_{\rho \in \mathfrak{S}_{k,A}} \epsilon(\rho)(-2i\pi)^{|F(\rho)|} \ldots$$

$$\ldots \times \prod_{j \notin F(\rho)} (z_{\rho^{-1}(j)} - z_j)^{-1} e^{2i\pi \sum_{j \in A}(z_{\rho(j)} - z_j)} \prod_{j \in G(\rho,A)} \left(1 - e^{2i\pi(z_j - z_{\rho(j)})}\right).$$

Remark 5.10. *For $k = 1$ and $k = 2$, we recover what we obtained before.*

6 Mesoscopic fluctuations and blue noise

The function $\frac{\zeta'_\infty}{\zeta_\infty}(z) - i\pi$ studied in the pervious section was recently considered by Aizenman and Warzel in [AW13]. They prove that for any $z \in \mathbb{R}$, the value of this function follows the Cauchy distribution: in fact, their result applies to more general point processes than the sine kernel process. In the present paper, we deal with the same function but away from the real line. In this section we shall view this function in the framework of linear statistics and will study its fluctuations on a mesoscopic level. It is may be worth noting here that $\frac{\zeta'_\infty}{\zeta_\infty}(z) - i\pi$ also has a spectral interpretation: informally, it is the trace of the resolvent of the (unbounded) random Hermitian operator whose spectrum consists exactly of the points $(y_k)_{k\in\mathbb{Z}}$ that we constructed in [MNN13]. This interpretation is informal since the series corresponding to the resolvant is not absolutely convergent.

For $s \geq 0$, we consider the Sobolev space:

$$H^s := \left\{ f \in L^2(\mathbb{R}, \mathbb{C}) \mid \int_{\mathbb{R}} \left|\hat{f}(k)\right|^2 \left(1 + |k|^2\right)^s dk \right\},$$

We then call *blue noise* a Gaussian family of centered variables indexed by $H^{1/2}$, denoted $(\mathcal{B}(f))_{f\in H^{1/2}}$, such that $f \mapsto \mathcal{B}(f)$ is linear, $\mathcal{B}(f)$ is a.s. real if f is a real-valued function, and

$$\mathbb{E}[|\mathcal{B}(f)|^2] = \frac{1}{2\pi} \int_{\mathbb{R}} |k||\hat{f}(k)|^2 dk.$$

The covariance structure of $\mathcal{B}$ is then:

$$\mathbb{E}[\mathcal{B}(f)\mathcal{B}(g)] = \frac{1}{2\pi} \int_{\mathbb{R}} |k|\hat{f}(k)\hat{g}(-k)dk,$$

$$\mathbb{E}[\mathcal{B}(f)\overline{\mathcal{B}(g)}] = \frac{1}{2\pi} \int_{\mathbb{R}} |k|\hat{f}(k)\overline{\hat{g}(k)}dk.$$

Similarly as for the Brownian motion, we can take the notation:

$$\int_{\mathbb{R}} f(t)d\mathcal{B}_t := \mathcal{B}(f).$$

Now, for any function $f \in L^1(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$, we have

$$\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}} |f(y_k)|\right)^2\right] = \int_{\mathbb{R}} |f|^2 + \int_{\mathbb{R}^2} (1 - S^2(x-y))|f(x)||f(y)|dxdy$$

$$\leq \int_{\mathbb{R}} |f|^2 + \left(\int_{\mathbb{R}} |f|\right)^2 < \infty,$$

and then

$$X_f := \sum_{k\in\mathbb{Z}} f(y_k) - \int f$$

is well-defined as a square-integrable random variable. As we will see in Corollary 6.3, X_f can also be defined as a square-integrable random variable as soon as $f \in H^{1/2}$, even if f is not integrable.

In this section, we examine the behavior of $\sum_{k\in\mathbb{Z}} f\left(\frac{y_k}{L}\right) - L \int f$ as $L \to \infty$ for suitable functions f:

Theorem 6.1. *If* $(y_k; k \in \mathbb{Z})$ *is a sine kernel point process, there is a blue noise $\mathscr{B}$ such that*

$$\left(X_{f\left(\frac{\cdot}{L}\right)} \right)_{f \in H^{1/2}} \overset{L \to \infty}{\longrightarrow} \left(\int f(t) d\mathscr{B}_t \right)_{f \in H^{1/2}},$$

the convergence holding in law for finite-dimensional marginals.

In Subsection 6.2, we analyse the asymptotic behavior of the Stieltjes transform of the sine kernel process. To that endeavor, we apply the result to the complex-valued functions $f_z(t) = \frac{1}{z-t}$.

6.1 The sine kernel from afar

We will need an intermediate proposition:

Proposition 6.2 (Adapted from Soshnikov [Sos00])**.** *If f is a smooth, real-valued function with compact support and if the p-th cumulant of X_f is denoted $C_p(f)$, then we have:*

$$C_1(f) = 0$$

$$\left| C_2(f) - \frac{1}{2\pi} \int \left| \hat{f}(k) \right|^2 |k| dk \right| \ll \int |k| \left| \hat{f}(k) \right|^2 \mathbb{1}_{|k| \geq \pi} dk$$

$$\forall p \geq 3, \left| C_p(f) \right| \ll_p \int_{k_1 + \cdots + k_p = 0} \mathbb{1}_{|k_1| + \cdots + |k_p| > 2\pi} |k_1| \left| \hat{f}(k_1) \dots \hat{f}(k_p) \right| dk$$

where in the previous equation, dk stands for the Lebesgue measure on the hyperplane $\{k_1 + \cdots + k_p = 0\}$.

Proof. The first equality is immediate. Now, since $y_k^{(n)} = \frac{n}{2\pi} \theta_k^{(n)}$ converges almost surely to y_k, X_f is the almost sure limit of:

$$\begin{aligned} X_{n,f} &:= \sum_{k \in \mathbb{Z}} f\left(\frac{n}{2\pi} \theta_k^{(n)} \right) - \int f \\ &= \sum_{k=1}^{n} \left(-\frac{1}{n} \int f + \sum_{l \in \mathbb{Z}} f\left(\frac{n}{2\pi} \theta_k^{(n)} + nl \right) \right) \\ &= \sum_{k=1}^{n} \psi_n \left(\theta_k^{(n)} \right), \end{aligned}$$

where ψ_n is the sequence of 2π-periodic functions with zero mean:

$$\psi_n(\theta) = -\frac{1}{n} \int f + \sum_{l \in \mathbb{Z}} f\left(\frac{n}{2\pi} \theta + nl \right)$$

If $\hat{f}$ is the Fourier transform of f, the Fourier coefficients

$$\left(c_k(\psi_n) := \frac{1}{2\pi} \int_0^{2\pi} \psi_n(\theta) e^{-ik\theta} d\theta; k \in \mathbb{Z} \right)$$

of ψ_n are given by:

$$c_0(\psi_n) = 0,$$

$$\forall k \in \mathbb{Z}^*, c_k(\psi_n) = \frac{\sqrt{2\pi}}{n} \hat{f}\left(\frac{2\pi k}{n}\right).$$

If $C_{p,n}(f)$ is the p-th cumulant of $X_{n,f}$, thanks to the main combinatorial lemma and Lemma 1 in [Sos00], we have:

$$C_{1,n}(f) = 0$$

$$\left| C_{2,n}(f) - \frac{2\pi}{n} \sum_{k \in \mathbb{Z}} \frac{|k|}{n} \hat{f}\left(\frac{2\pi k}{n}\right) \hat{f}\left(-\frac{2\pi k}{n}\right) \right| \ll \frac{1}{n} \sum_{|k| > \frac{1}{2}n} \frac{|k|}{n} \hat{f}\left(\frac{2\pi k}{n}\right) \hat{f}\left(-\frac{2\pi k}{n}\right)$$

$$\forall p \geq 3, \left| C_{p,n}(f) \right| \ll_p \frac{1}{n^{p-1}} \sum_{\substack{k_1 + \cdots + k_p = 0 \\ |k_1| + \cdots + |k_p| > n}} \frac{|k_1|}{n} \left| \hat{f}\left(\frac{2\pi k_1}{n}\right) \ldots \hat{f}\left(\frac{2\pi k_p}{n}\right) \right|$$

As $\hat{f}$ decays at infinity faster than any power, we recognize three converging Riemann sums. The first one is:

$$\frac{2\pi}{n} \sum_{k \in \mathbb{Z}} \frac{|k|}{n} \hat{f}\left(\frac{2\pi k}{n}\right) \hat{f}\left(-\frac{2\pi k}{n}\right) \xrightarrow{n \to \infty} 2\pi \int |k| \left| \hat{f}(2\pi k) \right|^2 dk = \frac{1}{2\pi} \int |k| \left| \hat{f}(k) \right|^2 dk.$$

The others appear as error terms and are Riemann sums converging to integrals on the hyperplane $\{k_1 + \cdots + k_p = 0\} \subset \mathbb{R}^p$.

$$\forall p \geq 2, \frac{1}{n^{p-1}} \sum_{\substack{k_1 + \cdots + k_p = 0 \\ |k_1| + \cdots + |k_p| > n}} \frac{|k_1|}{n} \left| \hat{f}\left(\frac{2\pi k_1}{n}\right) \ldots \hat{f}\left(\frac{2\pi k_p}{n}\right) \right|$$

$$\xrightarrow{n \to \infty} \int_{k_1 + \cdots + k_p = 0} \mathbb{1}_{|k_1| + \cdots + |k_p| > 1} |k_1| \left| \hat{f}(2\pi k_1) \ldots \hat{f}(2\pi k_p) \right| dk.$$

Therefore, for every $p \geq 1$, the p-th cumulant of $X_{n,f}$ is bounded independently of n and the sequence $|X_{n,f}|^p$ is uniformly integrable. Thus, the convergence of $X_{n,f}$ to X_f is not only almost sure but also in every $L^p(\Omega)$, Ω being the underlying probability space.

Now since

$$\forall p \geq 1, C_p(f) \xrightarrow{n \to \infty} C_p(f),$$

we have

$$\left| C_2(f) - \frac{1}{2\pi} \int \left| \hat{f}(k) \right|^2 |k| dk \right| \ll \int |k| \, |f(2\pi k)|^2 \, \mathbb{1}_{|k| \geq \frac{1}{2}} dk$$

$$\forall p \geq 3, \left| C_p(f) \right| \ll_p \int_{k_1 + \cdots + k_p = 0} \mathbb{1}_{|k_1| + \cdots + |k_p| > 1} |k_1| \left| \hat{f}(2\pi k_1) \ldots \hat{f}(2\pi k_p) \right| dk$$

After an obvious change of variables, we recover the claimed estimates. $\square$

Corollary 6.3. *The map*

$$f \mapsto X_f$$

from $L^1(\mathbb{R}, \mathbb{C}) \cap H^{1/2}$ to $L^2(\Omega)$ admits a linear extension to $H^{1/2}$, which satisfies the following property of continuity:

$$\mathbb{E}\left(\left| X_f \right|^2 \right)^{\frac{1}{2}} \ll \|f\|_{H^{\frac{1}{2}}},$$

uniformly, for all $f \in H^{1/2}$. This extension is unique up to almost sure equality.

Proof. The estimate on the second cumulant, given by Proposition 6.2, implies

$$\mathbb{E}\left(\left|X_f\right|^2\right)^{\frac{1}{2}} \ll \|f\|_{H^{\frac{1}{2}}}$$

for every smooth, real-valued function f with compact support. By linearity, this estimate remains true without the assumption that f is real-valued. We deduce the existence of a family $(Y_f)_{f \in H^{1/2}}$ of random variables such that $Y_f = X_f$ a.s. if f is smooth with compact support, and

$$\mathbb{E}\left(\left|Y_f\right|^2\right)^{\frac{1}{2}} \ll \|f\|_{H^{\frac{1}{2}}}$$

This family is unique up to almost sure equality. Then, we are done if we show that $X_f = Y_f$ almost surely as soon as $f \in L^1 \cap H^{1/2}$. Now, the map $f \mapsto X_f - Y_f$ from $f \in L^1 \cap H^{1/2}$ to $L^2(\Omega)$ is a.s. equal to zero on $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$. Moreover, we have seen above, by using the two first correlation functions of the sine kernel process, that

$$\mathbb{E}\left(\left|X_f\right|^2\right)^{\frac{1}{2}} \leq ||f||_{L^1} + ||f||_{L^2}$$

which implies:

$$\mathbb{E}\left(\left|X_f - Y_f\right|^2\right)^{\frac{1}{2}} \ll ||f||_{L^1} + ||f||_{L^2} + ||f||_{H^{1/2}} \ll ||f||_{L^1} + ||f||_{H^{1/2}}.$$

Hence, the map $f \mapsto X_f - Y_f$ from $f \in L^1 \cap H^{1/2}$ is continuous, and since it vanishes on $\mathcal{C}_c^\infty$, which is dense in $L^1 \cap H^{1/2}$, it vanishes everywhere.

$\square$

Proof of Theorem 6.1. It is sufficient to prove convergence in law of the one-dimensional marginals, for real-valued functions f. Indeed, if we have this convergence, if $f_1, \ldots, f_m$ are real-valued functions in $H^{1/2}$, and if $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, then we have the convergence in law

$$X_{f\left(\frac{\cdot}{L}\right)} = \sum_{j=1}^m \lambda_j X_{f_j\left(\frac{\cdot}{L}\right)} \xrightarrow{L \to \infty} \mathcal{B}(f) = \sum_{j=1}^m \lambda_j \mathcal{B}(f_j),$$

for

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m.$$

Applying the bounded, continuous function $x \mapsto e^{ix}$ gives the convergence of the Fourier transform of $(X_{f_j\left(\frac{\cdot}{L}\right)})_{1 \leq j \leq m}$ towards the Fourier transform of $(\mathcal{B}(f_j))_{1 \leq j \leq m}$, and then the convergence of the finite-dimensional marginals claimed in Theorem 6.1, for real-valued functions. The case of complex-valued functions is then deduced by linearity.

If remains to prove that for all $f \in H^{1/2}$, real-valued,

$$X_{f\left(\frac{\cdot}{L}\right)} \xrightarrow{L \to \infty} \mathcal{B}(f).$$

Let us first assume that f is smooth function with compact support. If $C_p^{(L)}(f)$ is the p-th cumulant of $X_{f\left(\frac{\cdot}{L}\right)}$, then by rescaling the space variable:

$$\forall k \in \mathbb{R}, \widehat{f\left(\frac{\cdot}{L}\right)}(k) = L\hat{f}(Lk)$$

and

$$C_1^{(L)}(f) = 0$$

$$\left| C_2^{(L)}(f) - \frac{1}{2\pi} \int \left|\hat{f}(k)\right|^2 |k| dk \right| \ll \int |k| \left|\hat{f}(k)\right|^2 \mathbb{1}_{\{|k| \geq L\pi\}} dk$$

$$\forall p \geq 3, \left| C_p^{(L)}(f) \right| \ll_p \int_{k_1 + \cdots + k_p = 0} \mathbb{1}_{\{|k_1| + \cdots + |k_p| > 2\pi L\}} |k_1| \left|\hat{f}(k_1) \ldots \hat{f}(k_p)\right| dk$$

Therefore, as $L \to \infty$, $X_{f\left(\frac{\cdot}{L}\right)}$ converges in law to a centered Gaussian random variable with variance $\frac{1}{2\pi} \int |k| |f(k)|^2 dk$, i.e. to $\mathcal{B}(f)$.

Now, if f is only supposed to be in $H^{\frac{1}{2}}$, let us consider a sequence of smooth compactly supported functions $(f_n)_{n \in \mathbb{N}}$ such that:

$$\|f - f_n\|_{H^{\frac{1}{2}}} \overset{n \to \infty}{\longrightarrow} 0$$

We will be done after proving that for any t in a compact set:

$$\mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right) \overset{n \to \infty}{\longrightarrow} \exp\left(-\frac{t^2}{4\pi} \int |k||\hat{f}(k)|^2 dk\right)$$

We have because of the triangular inequality, for fixed n:

$$\left| \mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right) - \exp\left(-\frac{t^2}{4\pi} \int |k||\hat{f}(k)|^2 dk\right) \right|$$

$$\leq \left| \mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right) - \mathbb{E}\left(e^{itX_{f_n\left(\frac{\cdot}{L}\right)}}\right) \right|$$

$$+ \left| \mathbb{E}\left(e^{itX_{f_n\left(\frac{\cdot}{L}\right)}}\right) - \exp\left(-\frac{t^2}{4\pi} \int |k||\hat{f_n}(k)|^2 dk\right) \right|$$

$$+ \left| \exp\left(-\frac{t^2}{4\pi} \int |k||\hat{f}(k)|^2 dk\right) - \exp\left(-\frac{t^2}{4\pi} \int |k||\hat{f_n}(k)|^2 dk\right) \right|$$

The third term is a $\mathcal{O}\left(t^2\|f - f_n\|_{H^{\frac{1}{2}}}^2\right)$. The second disappears when we take the $\limsup_{L \to \infty}$. As for the first term, we have for any $\varepsilon > 0$:

$$\left| \mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right) - \mathbb{E}\left(e^{itX_{f_n\left(\frac{\cdot}{L}\right)}}\right) \right|$$

$$\leq \mathbb{E}\left(\left| e^{itX_{f\left(\frac{\cdot}{L}\right)} - itX_{f_n\left(\frac{\cdot}{L}\right)}} - 1 \right|\right)$$

$$\leq 2\mathbb{P}\left(\left| X_{f\left(\frac{\cdot}{L}\right)} - X_{f_n\left(\frac{\cdot}{L}\right)} \right| \geq \varepsilon\right) + \varepsilon|t|$$

$$\leq 2\frac{\mathbb{E}\left(\left|X_{f(\frac{\cdot}{L})} - X_{f_n(\frac{\cdot}{L})}\right|^2\right)}{\varepsilon^2} + \varepsilon|t|$$

By linearity and the second cumulant estimate:

$$\mathbb{E}\left(\left|X_{f(\frac{\cdot}{L})} - X_{f_n(\frac{\cdot}{L})}\right|^2\right)^{\frac{1}{2}} = \mathbb{E}\left(\left|X_{(f-f_n)(\frac{\cdot}{L})}\right|^2\right)^{\frac{1}{2}} \ll \|f - f_n\|_{H^{\frac{1}{2}}}$$

Hence for any fixed n and $\varepsilon > 0$:

$$\limsup_{L\to\infty}\left|\mathbb{E}\left(e^{itX_{f(\frac{\cdot}{L})}}\right) - \exp\left(-\frac{t^2}{2}\int|k||\hat{f}(k)|^2 dk\right)\right|$$
$$\ll \|f - f_n\|_{H^{\frac{1}{2}}}^2\left(\frac{1}{\varepsilon^2} + t^2\right) + \varepsilon|t|$$

Taking $n \to \infty$, then $\varepsilon \to 0$ concludes the proof. $\qquad\square$

6.2 Application to the Stieltjes transform of the sine kernel

For $z \in \mathbb{C}\backslash\mathbb{R}$, $f_z : t \mapsto 1/(z - t)$ is in $H^{1/2}$. Indeed, one can check (by using the inverse Fourier transform for example) that

$$\hat{f}_z(k) = -i\sqrt{2\pi}\,\mathrm{sgn}\,\Im(z)e^{-izk}\mathbb{1}_{k\Im(z)<0},$$

and then $\hat{f}_z$ decays exponentially at infinity. Moreover, X_{f_z} can be related to the logarithmic derivative of ξ_∞:

Proposition 6.4. *For all $z \notin \mathbb{R}$, we have almost surely,*

$$X_{f_z} = \frac{\xi'_\infty(z)}{\xi_\infty(z)} - 2i\pi\mathbb{1}_{\Im z<0} = i\pi\,\mathrm{sgn}\,\Im z + \frac{1}{z - y_0} + \sum_{k=1}^\infty\left(\frac{1}{z - y_k} + \frac{1}{z - y_{-k}}\right).$$

Proof. Let φ be a smooth even function from $\mathbb{R}$ to $[0,1]$, nonincreasing on $\mathbb{R}_+$, equal to 1 on $[-1,1]$ and to 0 on $\mathbb{R}\backslash[-2,2]$. If for $A > 0$, $f_z^{(A)}(t) = f_z(t)\varphi(t/A)$, we have

$$|f_z^{(A)}(t) - f_z(t)| \leq |f_z(t)|\mathbb{1}_{|t|\geq A}$$

and

$$|(f_z^{(A)})'(t) - f_z'(t)| = \left|f_z'(t)\varphi(t/A) + \frac{1}{A}\varphi'(t/A)f_z(t) - f_z'(t)\right| \ll |f_z'(t)|\mathbb{1}_{|t|\geq A} + \frac{|f_z(t)|}{A}.$$

For z fixed, $|f_z(t)|$ is dominated by $1/(1 + |t|)$, $|f_z'(t)|$ is dominated by $1/(1 + |t|)^2$, and then

$$|f_z^{(A)}(t) - f_z(t)|^2 + |(f_z^{(A)})'(t) - f_z'(t)|^2 \ll \frac{\mathbb{1}_{|t|\geq A}}{(1 + |t|)^2} + \frac{1}{A^2(1 + |t|)^2}.$$

We deduce that $f_z^{(A)}$ converges to f_z in H^1, and a fortiori in $H^{1/2}$. Hence, in $L^2(\Omega)$,

$$
X_{f_z} = \lim_{A \to \infty} X_{f_z^{(A)}} = \lim_{A \to \infty} \left(\sum_{k \in \mathbb{Z}} \frac{\varphi(y_k/A)}{z - y_k} - \int_{\mathbb{R}} \frac{\varphi(y/A)}{z - y} dy \right)
$$

$$
= \lim_{A \to \infty} \int_1^2 (-\varphi'(u)) \left(\sum_{k \in \mathbb{Z}} \frac{\mathbb{1}_{|y_k| \le Au}}{z - y_k} - \int_{-Au}^{Au} \frac{dy}{z - y} \right) du
$$

From Proposition 4.6, one easily deduces that

$$
\sum_{k \in \mathbb{Z}} \frac{\mathbb{1}_{|y_k| \le B}}{z - y_k} - \int_{-B}^{B} \frac{dy}{z - y} \xrightarrow[B \to \infty]{} \frac{\xi_\infty'(z)}{\xi_\infty(z)} - 2i\pi \mathbb{1}_{\Im z < 0}
$$

in $L^p(\Omega)$ for all $p \ge 1$, and in particular in $L^2(\Omega)$. Now, since $-\varphi'$ is nonnegative in $[1, 2]$ and has integral 1, one has

$$
\left\| \int_1^2 (-\varphi'(u)) \left(\sum_{k \in \mathbb{Z}} \frac{\mathbb{1}_{|y_k| \le Au}}{z - y_k} - \int_{-Au}^{Au} \frac{dy}{z - y} \right) du - \frac{\xi_\infty'(z)}{\xi_\infty(z)} + 2i\pi \mathbb{1}_{\Im z < 0} \right\|_{L^2(\Omega)}
$$

$$
\le \int_1^2 (-\varphi'(u)) du \left\| \sum_{k \in \mathbb{Z}} \frac{\mathbb{1}_{|y_k| \le Au}}{z - y_k} - \int_{-Au}^{Au} \frac{dy}{z - y} - \frac{\xi_\infty'(z)}{\xi_\infty(z)} + 2i\pi \mathbb{1}_{\Im z < 0} \right\|_{L^2(\Omega)}
$$

$$
\le \sup_{B \in [A, 2A]} \left\| \sum_{k \in \mathbb{Z}} \frac{\mathbb{1}_{|y_k| \le B}}{z - y_k} - \int_{-B}^{B} \frac{dy}{z - y} - \frac{\xi_\infty'(z)}{\xi_\infty(z)} + 2i\pi \mathbb{1}_{\Im z < 0} \right\|_{L^2(\Omega)},
$$

which tends to zero when A goes to infinity. Hence, in $L^2(\Omega)$,

$$
X_{f_z} = \lim_{A \to \infty} \int_1^2 (-\varphi'(u)) \left(\sum_{k \in \mathbb{Z}} \frac{\mathbb{1}_{|y_k| \le Au}}{z - y_k} - \int_{-Au}^{Au} \frac{dy}{z - y} \right) du = \frac{\xi_\infty'(z)}{\xi_\infty(z)} - 2i\pi \mathbb{1}_{\Im z < 0}.
$$

$\square$

A consequence of the previous proposition is the following result:

Proposition 6.5. *For $z \in \mathbb{C} \backslash \mathbb{R}$, let*

$$
F(z) := X_{f_z} = \frac{\xi_\infty'(z)}{\xi_\infty(z)} - 2i\pi \mathbb{1}_{\Im z < 0}.
$$

Then, one has the convergence in law:

$$
(LF(Lz))_{z \in \mathbb{C} \backslash \mathbb{R}} \xrightarrow[L \to \infty]{} (G(z))_{z \in \mathbb{C} \backslash \mathbb{R}},
$$

where $G(z) = \mathcal{B}(f_z)$ for all $z \in \mathbb{C} \backslash \mathbb{R}$. The centered gaussian process $(G(z))_{z \in \mathbb{C} \backslash \mathbb{R}}$ has the covariance structure given, for all $z_1, z_2 \notin \mathbb{R}$, by

$$
\mathbb{E}[G(z_1)G(z_2)] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2) < 0}}{(z_2 - z_1)^2},
$$

$$\mathbb{E}[G(z_1)\overline{G(z_2)}] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)>0}}{(\overline{z_2}-z_1)^2},$$

and in particular

$$\mathbb{E}[|G(z_1)|^2] = \frac{1}{4\Im^2(z_1)}.$$

Proof. We have, for $L > 0$,

$$f_z(t/L) = \frac{1}{z-(t/L)} = \frac{L}{Lz-t} = Lf_{Lz}(t),$$

and then

$$X_{f_z(\cdot/L)} = LX_{f_{Lz}} = LF(Lz).$$

The convergence in law given in this proposition is then a consequence of Theorem 6.1. It remains to compute the covariance structure. For $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$,

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})]$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} |k|(-i\sqrt{2\pi}\operatorname{sgn}\Im(z_1)e^{-iz_1k}\mathbb{1}_{k\Im(z_1)<0})(-i\sqrt{2\pi}\operatorname{sgn}\Im(z_2)e^{iz_2k}\mathbb{1}_{-k\Im(z_2)<0})dk.$$

If $\Im(z_1)$ and $\Im(z_2)$ have the same sign, the product of the indicator functions vanishes for all $k \in \mathbb{R}$, so

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = 0.$$

If $\Im(z_1)$ and $\Im(z_2)$ have not the same sign, we get

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = \int_{\mathbb{R}} |k|e^{ik(z_2-z_1)}\mathbb{1}_{k\Im(z_2)>0}.$$

By doing the change of variable $k' = k\operatorname{sgn}\Im(z_2)$, we get

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = \int_0^\infty ke^{ik(z_2-z_1)\operatorname{sgn}\Im(z_2)}dk$$

Now, for all $y > 0$,

$$\int_0^\infty ke^{-yk}dk = \int_0^\infty (u/y)e^{-u}d(u/y) = 1/y^2,$$

and by analytic continuation, this formula is true for all y with strictly positive real part. Applying this to $y = -i(z_2-z_1)\operatorname{sgn}\Im(z_2)$, we have

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = -1/(z_2-z_1)^2$$

for $\Im(z_1)\Im(z_2) < 0$, and then in any case,

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)<0}}{(z_2-z_1)^2}.$$

Since the blue noise here is real-valued for real functions, $\mathscr{B}(f_{\overline{z_2}}) = \overline{\mathscr{B}(f_{z_2})}$, and then

$$\mathbb{E}[\mathscr{B}(f_{z_1})\overline{\mathscr{B}(f_{z_2})}] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)>0}}{(\overline{z_2}-z_1)^2}.$$

$\square$

Remark 6.6. *The covariance structure of F has been computed above in this paper. We have*

$$\mathbb{E}[F(z_1)F(z_2)] = -\frac{1 - e^{2i\pi(z_1 - z_2)\operatorname{sgn}\Im(z_1 - z_2)}}{(z_1 - z_2)^2}\mathbb{1}_{\Im(z_1)\Im(z_2)<0},$$

and then

$$\mathbb{E}[(LF(Lz_1))(LF(Lz_2))] \xrightarrow[L\to\infty]{} -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)<0}}{(z_2 - z_1)^2} = \mathbb{E}[G(z_1)G(z_2)].$$

Similarly,

$$\mathbb{E}[(LF(Lz_1))(\overline{LF(Lz_2)})] \xrightarrow[L\to\infty]{} \mathbb{E}[G(z_1)\overline{G(z_2)}].$$

This convergence is naturally expected once the previous proposition is proven.

The stochastic process $z \mapsto X_{f_z}$ admits the version

$$z \mapsto F(z) = \frac{\zeta_\infty'(z)}{\zeta_\infty(z)} - 2i\pi\mathbb{1}_{\Im z<0},$$

which is holomorphic on $\mathbb{C}\backslash\mathbb{R}$. One can ask if the situation is similar for G. The answer is positive:

Proposition 6.7. *The random function G admits a version which is holomorphic on $\mathbb{C}\backslash\mathbb{R}$. Moreover, $z \mapsto LF(Lz)$ converges in law to an holomorphic version of G when L goes to infinity, in the sense of the uniform convergence on compact sets of $\mathbb{C}\backslash\mathbb{R}$.*

Proof. We first compute the L^2 norm of $G(z_1) - G(z_2)$ when $z_1, z_2 \notin \mathbb{R}$:

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = \mathbb{E}[|G(z_1)|^2] + \mathbb{E}[|G(z_2)|^2] - \mathbb{E}[G(z_1)\overline{G(z_2)}] - \mathbb{E}[G(z_2)\overline{G(z_1)}]$$

$$= -\frac{1}{(z_1 - \overline{z_1})^2} - \frac{1}{(z_2 - \overline{z_2})^2} + \mathbb{1}_{\Im(z_1)\Im(z_2)>0}\left(\frac{1}{(z_1 - \overline{z_2})^2} + \frac{1}{(z_2 - \overline{z_1})^2}\right).$$

Let us now assume that z_1 and z_2 are in a given compact set K of $\mathbb{C}\backslash\mathbb{R}$. Let us denote:

$$c_K := \inf\{|\Im(z)|, z \in K\} > 0.$$

If $z_1, z_2 \in K$ have imaginary parts of different signs, necessarily $|z_1 - z_2| \geq 2c_K$ and from the computations above,

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = \frac{1}{4\Im^2(z_1)} + \frac{1}{4\Im^2(z_2)} \leq \frac{1}{2c_K^2}.$$

One deduces

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] \leq \frac{1}{8c_K^4}|z_1 - z_2|^2.$$

If $z_1, z_2 \in K$ have imaginary parts with the same sign,

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = A(z_1, \overline{z_1}) + A(z_2, \overline{z_2}) - A(z_1, \overline{z_2}) - A(z_2, \overline{z_1})),$$

where

$$A(u,v) := -\frac{1}{(u-v)^2}.$$

The function A of two variables is holomorphic in the open set of $(a,b) \in \mathbb{C}^2$ such that $\Im(a)\Im(z_1) > 0$ and $\Im(b)\Im(z_1) < 0$. Since the set $[z_1, z_2] \times [\overline{z_1}, \overline{z_2}]$ is included in this domain (recall that $\Im(z_1)$ and $\Im(z_2)$ have the same sign), we have

$$A(z_1, \overline{z_1}) + A(z_2, \overline{z_2}) - A(z_1, \overline{z_2}) - A(z_2, \overline{z_1})) = \int_{z_1}^{z_2} \int_{\overline{z_1}}^{\overline{z_2}} A_{1,2}''(u,v) du dv,$$

where $A_{1,2}''$ is the second derivative of A with respect to the two variables. Hence,

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = 6 \int_{z_1}^{z_2} \int_{\overline{z_1}}^{\overline{z_2}} \frac{du dv}{(u-v)^4}.$$

Now, for $u \in [z_1, z_2]$, $v \in [\overline{z_1}, \overline{z_2}]$, we have $|\Im(u) - \Im(v)| \geq 2c_K$, since $z_1, z_2 \in K$. Hence, $|u - v|^4 \geq 16c_K^4$, and

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] \leq \frac{3}{8c_K^4} \int_{z_1}^{z_2} \int_{\overline{z_1}}^{\overline{z_2}} |du||dv|.$$

Hence, similarly as in the case $\Im(z_1)\Im(z_2) < 0$, we have

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] \leq \frac{3}{8c_K^4} |z_1 - z_2|^2.$$

By Kolmogorov's criterion, G admits a continuous version on $\mathbb{C}\backslash\mathbb{R}$. We now assume that G itself is continuous.

Let $\Gamma : [0,1] \mapsto \mathbb{C}$ be a closed, piecewise smooth contour in $\mathbb{C}\backslash\mathbb{R}$. Since G is continuous, the integral of G along Γ is well-defined, and one has

$$\left|\int_\Gamma G(z)dz\right|^2 = \int_0^1 \int_0^1 G(\Gamma(t))\overline{G(\Gamma(u))}\Gamma'(t)\overline{\Gamma'(u)}dtdu.$$

If we denote $\overline{\Gamma}$ the contour given by $\overline{\Gamma}(t) = \overline{\Gamma(t)}$, we can write

$$\left|\int_\Gamma G(z)dz\right|^2 = \int_0^1 \int_0^1 G(\Gamma(t))\widetilde{G}(\overline{\Gamma}(u))\Gamma'(t)\overline{\Gamma}'(u)dtdu,$$

where $\widetilde{G}$ is the function from $\mathbb{C}\backslash\mathbb{R}$, given by

$$\widetilde{G}(z) = \overline{G(\overline{z})}.$$

Hence,

$$\left|\int_\Gamma G(z)dz\right|^2 = \int_\Gamma \int_{\overline{\Gamma}} G(z_1)\widetilde{G}(z_2)dz_1 dz_2.$$

Now, for $z_1 \in \Gamma$, $z_2 \in \overline{\Gamma}$

$$\mathbb{E}[|G(z_1)||\widetilde{G}(z_2)|] \leq \left(\mathbb{E}[|G(z_1)|^2]\right)^{1/2} \left(\mathbb{E}[|\overline{G(z_2)}|^2]\right)^{1/2} = \frac{1}{4|\Im(z_1)||\Im(z_2)|},$$

which implies

$$\int_{\Gamma}\int_{\overline{\Gamma}}\mathbb{E}[|G(z_1)\widetilde{G}(z_2)|]|dz_1|\,|dz_2| \leq \frac{(\ell(\Gamma))^2}{4c_{\Gamma}^2} < \infty,$$

where $\ell(\Gamma)$ is the length of Γ and c_{Γ} the infimum of $|\Im(z)|$ for $z \in \Gamma$. This bound allows to write

$$\mathbb{E}\left[\left|\int_{\Gamma} G(z)dz\right|^2\right] = \int_{\Gamma}\int_{\overline{\Gamma}}\mathbb{E}[G(z_1)\widetilde{G}(z_2)]dz_1dz_2.$$

Now, for $z_1 \in \Gamma$ and $z_2 \in \overline{\Gamma}$, $\Im(z_1)$ and $\Im(\overline{z_2})$ have the same sign, which implies

$$\mathbb{E}[G(z_1)\widetilde{G}(z_2)] = \mathbb{E}[G(z_1)\overline{G(\overline{z_2})}] = -\frac{1}{(z_2 - z_1)^2},$$

and then

$$\mathbb{E}\left[\left|\int_{\Gamma} G(z)dz\right|^2\right] = -\int_{\Gamma}\int_{\overline{\Gamma}}\frac{dz_1dz_2}{(z_2 - z_1)^2},$$

which is equal to zero, since the function $(z_1, z_2) \mapsto 1/(z_2 - z_1)^2$ is holomorphic and the contours Γ and $\overline{\Gamma}$ are closed. Hence, for all closed, piecewise smooth contours Γ on $\mathbb{C}\backslash\mathbb{R}$, one has almost surely

$$\int_{\Gamma} G(z)dz = 0.$$

One deduces that almost surely, this equality holds simultaneously for all polygonal closed contours whose vertices have rational real and imaginary parts. Then, by continity of G, one can remove the condition of rationality, and deduces that almost surely, G is holomorphic on $\mathbb{C}\backslash\mathbb{R}$.

We know $z \mapsto LF(Lz)$ converges in law to G in the sense of the finite-dimensional marginals: it remains to prove that this convergence occurs in the space of continuous functions, i.e. that the family of laws of $(LF(Lz))_{z\in\mathbb{C}}$ is tight in this space. For a compact set K of $\mathbb{C}\backslash\mathbb{R}$, and for $z_1, z_2 \in K$, one has

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] = \frac{1 - e^{-4L\pi|\Im(z_1)|}}{4\Im^2(z_1)} + \frac{1 - e^{-4L\pi|\Im(z_2)|}}{4\Im^2(z_2)}$$

if $\Im(z_1)\Im(z_2) < 0$, and

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] = A_L(z_1, \overline{z_1}) + A_L(z_2, \overline{z_2}) - A_L(z_1, \overline{z_2}) - A_L(z_2, \overline{z_1})$$
$$= \int_{z_1}^{z_2}\int_{\overline{z_1}}^{\overline{z_2}}(A_L'')_{1,2}(u, v)dudv$$

if $\Im(z_1)\Im(z_2) > 0$, for

$$A_L(u, v) = -\frac{1 - e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)}}{(u - v)^2}.$$

In the first case, we get

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] \leq \frac{|z_1 - z_2|^2}{8c_K^4}$$

and in the second case,

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] \leq |z_2 - z_1|^2 \sup_{|\Im(u)|,|\Im(v)|>c_K} |(A_L'')_{1,2}(u,v)|.$$

Note that A_L is holomorphic in $\{(u,v) \in \mathbb{C}^2, \Im(u)\Im(v) < 0\}$, since $\text{sgn}\,\Im(u-v)$ is locally constant on this set. Now,

$$(A_L')_1(u,v) = \frac{2(1 - e^{2i\pi L(u-v)\,\text{sgn}\,\Im(u-v)})}{(u-v)^3} + \frac{2i\pi L\,\text{sgn}\,\Im(u-v)e^{2i\pi L(u-v)\,\text{sgn}\,\Im(u-v)}}{(u-v)^2},$$

$$\begin{aligned}(A_L'')_{1,2}(u,v) &= \frac{6(1 - e^{2i\pi L(u-v)\,\text{sgn}\,\Im(u-v)})}{(u-v)^4} + \frac{8i\pi L\,\text{sgn}\,\Im(u-v)e^{2i\pi L(u-v)\,\text{sgn}\,\Im(u-v)}}{(u-v)^3} \\ &\quad + \frac{4\pi^2 L^2 e^{2i\pi L(u-v)\,\text{sgn}\,\Im(u-v)}}{(u-v)^2},\end{aligned}$$

$$\begin{aligned}|(A_L'')_{1,2}(u,v)| &\leq \frac{6(1 + e^{-2\pi L|\Im(u-v)|})}{|u-v|^4} + \frac{8\pi L e^{-2\pi L|\Im(u-v)|}}{|u-v|^3} + \frac{4\pi^2 L^2 e^{-2\pi L|\Im(u-v)|}}{|u-v|^2}. \\ &\leq \frac{12}{|\Im(u-v)|^4} + \frac{8\pi L e^{-2\pi L|\Im(u-v)|}}{|\Im(u-v)|^3} + \frac{4\pi^2 L^2 e^{-2\pi L|\Im(u-v)|}}{|\Im(u-v)|^2} \\ &\leq \frac{4\pi^2}{|\Im(u-v)|^4}\left(1 + (L|\Im(u-v)| + L^2(\Im(u-v))^2)e^{-2\pi L|\Im(u-v)|}\right) \\ &\leq \frac{\pi^2}{4c_K^4}\left(1 + \sup_{x\geq 0}(x+x^2)e^{-2\pi x}\right).\end{aligned}$$

Hence,

$$\sup_{L>0}\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] \leq \tilde{c}_K|z_2 - z_1|^2,$$

where $\tilde{c}_K > 0$ depends only on K. By Kolmogorov's criterion, the laws of $(LF(Lz))_{z\in\mathbb{C}\backslash\mathbb{R}}$ form a tight family for the uniform convergence on compact sets of $\mathbb{C}\backslash\mathbb{R}$.

$\square$

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