

HAZARD PROCESSES AND MARTINGALE HAZARD PROCESSES

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ABSTRACT. In this paper, we build a bridge between different reduced-form approaches to pricing defaultable claims. In particular, we show how the well known formulas by Duffie et al. [12] and by Elliott et al. [14] are related. Moreover, in the spirit of Collin Dufresne et al. [8], we propose a simple pricing formula under an equivalent change of measure.

Two processes will play a central role: the hazard process and the martingale hazard process attached to a default time. The crucial step is to understand the difference between them, which has been an open question in the literature so far. We show that pseudo-stopping times appear as the most general class of random times for which these two processes are equal. We also show that these two processes always differ when τ is an honest time, providing an explicit expression for the difference. Eventually we provide a solution to another open problem: we show that if τ is an arbitrary random (default) time such that its Azéma's supermartingale is continuous, then τ avoids stopping times.

1. INTRODUCTION

Defaults occur when some contractual cash flows are not met. Therefore for pricing or hedging a defaultable claim, a crucial step is to model the arrival of the default event and its effect on the prices of the different financial assets.

In this paper, our aim is to understand better the nature of the default times by analyzing their properties in a background filtration, which represents the information attached to those financial assets which survive (i.e., do not default) when a particular default event occurs. This approach was originally developed in [14]. We point out what we believe are the adequate mathematical tools to deal with the models of default risk. Using these tools, we build a bridge between different reduced-form approaches to pricing defaultable claims. In particular we show how the well known formulas by Duffie et al. [12] and by Elliott et al. [14] are related. In these formulas the behavior of two processes plays a central role: the hazard process and the martingale hazard process attached to a default time. The crucial step is to understand the difference between them.

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We provide a solution to two problems which have been open in default time modeling in credit risk. We first show that if τ is an arbitrary random (default) time such that its Azéma's supermartingale is continuous, then τ avoids stopping times. We then disprove a conjecture about the equality between the hazard process and the martingale hazard process, which first appeared in [19], and we show how it should be modified to become a theorem. The pseudo-stopping times introduced in [26] appear as the most general class of random times for which these two processes are equal. We also show that these two processes always differ when τ is an honest time and give a simple formula for the difference.

Finally, all the different theoretical results are used in an effective manner for pricing defaultable claims.

1.1. Random times as default times: the modeling framework.

Following [5], [14], [19], a reduced-form model for defaultable claims may be constructed in two steps. We begin with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbf{P})$ satisfying the usual assumptions. The default time τ is defined as a random time (i.e., a nonnegative \mathcal{F} -measurable random variable) which is not an \mathbb{F} -stopping time. Then, a second filtration $\mathbb{G} = (\mathcal{G}_t)$ is obtained by progressively enlarging the filtration \mathbb{F} with the random time τ : \mathbb{G} is the smallest filtration satisfying the usual assumptions, containing the original filtration \mathbb{F} , and for which τ is a stopping time, such as explained in [20], [22].

When the random time is not a stopping time, several quantities play an important role in the analysis of the model. The most fundamental object attached to an arbitrary random time τ is certainly the supermartingale $Z_t^\tau = \mathbf{P}(\tau > t | \mathcal{F}_t)$, chosen to be càdlàg, called the Azéma supermartingale associated with τ ([2]). Two more processes, closely related to the Azéma supermartingale Z^τ and the \mathbb{G} predictable compensator of $\mathbf{1}_{\{\tau \leq t\}}$, are often used in the evaluation of defaultable claims: the hazard process and the martingale hazard process, which we now define.

Definition 1.1. (1) Let τ be a random time such that $Z_t^\tau > 0$, for all $t \geq 0$ (in particular τ is not an \mathbb{F} -stopping time). The nonnegative stochastic process $(\Gamma_t)_{t \geq 0}$ defined by

$$\Gamma_t = -\ln Z_t^\tau,$$

is called the *hazard process*.

(2) An \mathbb{F} -predictable right-continuous increasing process Λ is called an *\mathbb{F} -martingale hazard process* of the random time τ if the process $\widetilde{M}_t = \mathbf{1}_{\{\tau \leq t\}} - \Lambda_{t \wedge \tau}$ is a \mathbb{G} martingale.

We see that a martingale hazard process is only defined up to time τ and that the stopped martingale hazard process is the \mathbb{G} -predictable compensator of the process $\mathbf{1}_{\{\tau \leq t\}}$. This has two implications. First, several martingale hazard processes might exist for a default time, even if the predictable compensator is unique. Unicity of Λ is a priori insured only on the

interval $[0, \tau]$. Secondly, this representation allows the martingale hazard process to be \mathbb{F} -adapted as stated in the definition even if, obviously, the compensator is only \mathbb{G} -adapted. In Section 3 we will characterize the situation where the martingale hazard process is unique, and give examples of constructions.

The filtration \mathbb{G} is usually considered as the relevant filtration in credit risk models: it represents the information available on the market, to be used for pricing or hedging defaultable claims. The enlargement of filtration appears to be a useful tool for several reasons. First, it provides a simple formula to compute the \mathbb{G} -predictable compensator of the process $\mathbf{1}_{\{\tau \leq t\}}$, using the \mathbb{F} -adapted process Λ . Secondly, prior to default, i.e., on the set $\{(t, \omega) : \tau(\omega) > t\}$, prices of defaultable claims can always be expressed using \mathbb{F} -adapted processes (here the process Γ comes into play, as shall be detailed below). From a purely economic point of view, modeling distinctly the two information sets forces the modeler to a reflection about the links between the defaultable asset prices (which are \mathbb{G} -adapted) and the non-defaultable ones (which can be used as hedging instruments and are \mathbb{F} -adapted), hence allows a finer construction of the default event. This approach has been proven particularly efficient in pricing (see [5], [14], [19]), hedging ([4]) and, more recently, in models with imperfect information (see ([11], [23], [14], [16], [6], [15])).

Note that an alternative and more direct reduced-form approach, which historically appeared first, consists in introducing one single global filtration \mathbb{G} from the start, where the default time is a totally inaccessible stopping time with a given intensity process λ (that is: $\Lambda_{t \wedge \tau} = \int_0^{t \wedge \tau} \lambda_s ds$). Some of the major papers using the intensity-based framework are [1], [18], [12], [17], [24], [25], [13].

Both hazard-rate approaches mentioned above, i.e., the direct approach or the one based on two different sets of filtrations, model the occurring of the default as a surprise for the market, that is, the default time is a totally inaccessible stopping time in the global market filtration \mathbb{G} . Another common feature of these two approaches is that under specific assumptions, the price of a defaultable claim is obtained as the one of a default free security, but using adjusted discount rates, i.e., rates which are modified to reflect the default risk. These simplified formulas represent a substantial computational interest, since they permit to the well developed default-free valuation techniques to be applied in the defaultable framework. However, the pricing formulas obtained in each approach (i.e., the direct approach or the one based on two different sets of filtrations) are not directly comparable, for several reasons: the adjusted discount rates are not the same in general; there are different specific assumptions under which these formulas can be used; and eventually prices are computed conditionally to different information sets. We are going to study the relation between the two approaches, compare the adjusted discount factors in general and clarify the assumptions under which these adjusted discount factors can be used. We

point out that there are important classes of models where these simplified formulas cannot be directly applied (e.g. the important case where the default time is modeled by a last passage time). In this situation, we find that it is sometimes possible to recover again the known formulas, under a suitable equivalent change of the probability measure, that we shall define.

1.2. Pricing formulas in the reduced-form approach. The defaultable claims we are going to analyze in this paper have the specific form

$$X = P\mathbf{1}_{\{\tau > T\}} + C_\tau\mathbf{1}_{\{\tau \leq T\}}, \quad (1.1)$$

where we assume that P is a square integrable, \mathcal{F}_T -measurable random variable which represents a single payment which occurs at time T and (C_t) is a bounded, \mathbb{F} -predictable process. P stands for the promised payment, while the process C models the recovery in case of default. Hence, it is assumed that a model for the recovery has already been selected and that the focus is now to model the occurrence of the default τ .

Let us also denote $R_t = \int_0^t r_u du$, where r_u is the locally risk-free interest rate, and let $\tilde{R}_t := R_t + \Lambda_t$. \tilde{R}_t will be called default-adjusted account, for reasons that will become obvious next.

We recall that the arbitrage-free price of a defaultable claim is given by the following conditional expectation:

$$S(X)_t := e^{R_t} \mathbf{E}[Pe^{-R_T}\mathbf{1}_{\{\tau > T\}} + C_\tau e^{-R_\tau}\mathbf{1}_{\{\tau \leq T\}} | \mathcal{G}_t] \mathbf{1}_{\{t \leq T\}}.$$

It follows that if the default has occurred before the maturity, i.e., on $\{\tau \leq t \leq T\}$ the price process is: $S(X)_t = C_\tau e^{R_t - R_\tau}$. Note other assumptions exist in the literature regarding the post-default evolution of the price: the recovery may be payed out at default and the price shrinks to zero in case of default, or it may be paid at the maturity date T . In any case, as the recovery process C is here given exogenously, the quantity of interest is now the pre-default price process of the claim, i.e., $S(X)_t \mathbf{1}_{\{t > \tau\}}$.

Using the enlargement of filtrations framework, pre-default prices can always be expressed in terms of an \mathbb{F} adapted process, via projections on the smaller filtration \mathbb{F} :

Proposition 1.2 ([5]). *The price of the defaultable claim X is given by*

$$S(X)_t = e^{R_t + \Gamma_t} \mathbf{E} \left[\int_t^T C_u e^{-R_u} dZ_u^\tau + P Z_T^\tau e^{-R_T} | \mathcal{F}_t \right] \quad \text{on } \{t < \tau\}, t \geq 0. \quad (1.2)$$

If moreover the process Z^τ is continuous decreasing (or alternatively, the hazard process Γ is continuous increasing), then

$$S(X)_t = e^{R_t + \Gamma_t} \mathbf{E} \left[\int_t^T C_u e^{-(R_u + \Gamma_u)} d\Gamma_u + P e^{-(R_T + \Gamma_T)} | \mathcal{F}_t \right] \quad \text{on } \{t < \tau\}, t \geq 0. \quad (1.3)$$

We note that when Γ is continuous and decreasing, pricing a defaultable claim is similar to pricing in the filtration \mathbb{F} a fictitious default-free claim

which pays out dividends at the rate $C_t d\Gamma_t$ and using an adjusted account $R_t + \Gamma_t$.

On the other hand, in the direct, intensity-based approach, pre-default prices are directly computed in the filtration \mathbb{G} :

Proposition 1.3 ([12]). *Let the RCLL processes V and J be defined respectively by:*

$$V_t = e^{\tilde{R}_t} \mathbf{E} \left[\int_t^T C_u e^{-\tilde{R}_u} d\Lambda_u + P e^{-\tilde{R}_T} | \mathcal{G}_t \right] \quad t < T, \quad (1.4)$$

with $V_t = 0$ for $t \geq T$, and:

$$J_t := e^{R_t} \mathbf{E} \left[e^{-R_\tau} \Delta V_\tau | \mathcal{G}_t \right]. \quad (1.5)$$

Then the claim's price process $S(X)$ satisfies:

$$S(X)_t = V_t - J_t \quad \text{on } \{t < \tau\}, t \geq 0.$$

If V is predictable, then $S(X)_t \mathbf{1}_{\{\tau > t\}} = V_t \mathbf{1}_{\{\tau > t\}}$.

The expression for V can be seen as the risk-neutral valuation formula for a fictitious security that pays out dividends at rate $C_t d\Lambda_t$ under a fictitious short rate $d\tilde{R}_t = d(R_t + \Lambda_t)$, i.e., the default-adjusted account. The term J adds however much complexity to the above formula, since it is difficult to compute in practice the expected jumps at default. For this reason, many models assume the so-called no-jump assumption, that is the assumption that the process V does not jump at time τ . No explicit expression for this term is known in general, hence the formula is difficult to implement in practice and this is still nowadays a major problem in the credit risk research.

In [8] it is shown that when the process $D_t := \mathbf{1}_{\{\tau > t\}} e^{\Lambda_t}$ is a martingale, then using the absolutely continuous change of measure: $d\mathbf{P}' = D_T \cdot d\mathbf{P}$ on \mathcal{G}_T , it is possible to express the pre-default price as:

$$S(X)_t = e^{\tilde{R}_t} \mathbf{E}^{\mathbf{P}'} \left[\int_t^T C_u e^{-\tilde{R}_u} d\Lambda_u + P e^{-\tilde{R}_T} | \mathcal{G}_t \right] \quad (1.6)$$

on $\{t < \tau\}$. In Section 4 we shall show that in most of the situations (i.e., whenever C or P are not constants), this formula does not circumvent the difficulty to make explicit the jumps of \mathbb{G} -adapted processes at the time τ . We therefore propose an alternative change of measure, but on the filtration \mathbb{F} and equivalent to \mathbf{P} , and show that it is efficient for pricing complex products.

1.3. Objectives and outline of the paper. Due to the simplicity of the formula (1.3), very often the random time τ is given with extra regularity assumptions, such as continuity or monotonicity of its Azéma supermartingale Z^τ . However, these assumptions were not translated into properties of the random time τ . We shall try to clarify the link between the assumptions about the process Z_t^τ and the properties of the default time τ , since it is

crucial for the modeler to select the properties of the random time which appear to be the most sensible.

From the above pricing formulas it already appears that a main quantity to study in order to link the two approaches is the difference between the hazard process Γ and the martingale hazard process Λ . A first important problem is to clarify under which conditions they coincide: this was the object of a conjecture made in [19]:

Conjecture: Suppose that the process Z_t^τ is decreasing. If Λ is continuous, then $\Lambda = \Gamma$.

We shall show that the problem was not well posed and we shall see how it should be phrased in order to have the equality between the hazard process and the martingale hazard process under some general conditions.

More generally, the aim of this paper is to show that the general theory of stochastic processes provides a natural framework to pose and to study the modeling of default times, and that it helps solve in a simple way some of the problems raised there.

The paper is organized as follows:

In Section 2, we recall some basic facts from the general theory of stochastic processes that will be relevant for this paper.

In Section 3, we show that if Z_t^τ is continuous, then τ avoids stopping times. We also see under which conditions the martingale hazard process and the hazard process coincide: the pseudo-stopping times, introduced in [26], appear there as the most general class of random times for which these two processes are equal. Moreover, we prove that for honest times, which form another remarkable class of random times, the hazard process and the martingale hazard process always differ.

Eventually, in Section 4, we apply the results from Section 3 to analyse the pricing formulas introduced above. We shall show that, even when $\Gamma \neq \Lambda$, it is sometimes possible to find an equivalent martingale measure under which simple pricing formulas still apply: the pre-default price process can be computed as a default-free security paying a flow of negative dividends at the rate $Cd\Lambda$ and using the default adjusted account \tilde{R} . We shall see that for pricing, the intensity process is not sufficient in general, and that one also needs the hazard process Γ . We illustrate this fact with an example of last passage time.

2. BASIC FACTS

Throughout this paper, we assume we are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ satisfying the usual assumptions.

Definition 2.1. A random time τ is a nonnegative random variable $\tau : (\Omega, \mathcal{F}) \rightarrow [0, \infty]$.

When dealing with arbitrary random times, one often works under the following conditions:

- Assumption **(C)**: all (\mathcal{F}_t) -martingales are continuous (e.g: the Brownian filtration).
- Assumption **(A)**: the random time τ avoids every (\mathcal{F}_t) -stopping time T , i.e. $\mathbf{P}[\rho = T] = 0$.

When we refer to assumptions **(CA)**, this will mean that both the conditions **(C)** and **(A)** hold.

We also recall that the Azéma supermartingale is the (\mathcal{F}_t) supermartingale

$$Z_t^\tau = \mathbf{P}[\tau > t \mid \mathcal{F}_t] \quad (2.1)$$

chosen to be càdlàg, associated with τ by Azéma ([2]). We note that the supermartingale (Z_t^τ) is the \mathbb{F} -optional projection of $\mathbf{1}_{[0, \tau[}$.

There are some other processes related to Z^τ , which play an important role when characterizing the time τ . First, the \mathbb{F} dual optional and dual predictable projections of the process $\mathbf{1}_{\{\tau \leq t\}}$, denoted respectively by A_t^τ and a_t^τ .

Secondly, the càdlàg martingale:

$$\mu_t^\tau = \mathbf{E}[A_\infty^\tau \mid \mathcal{F}_t] = A_t^\tau + Z_t^\tau.$$

Finally, the Doob-Meyer decomposition of (2.1):

$$Z_t^\tau = m_t^\tau - a_t^\tau. \quad (2.2)$$

Let us now rigorously define the progressively enlarged filtration \mathbb{G} that was already mentioned in the introduction as being the market filtration in the default models.

We enlarge the initial filtration \mathbb{F} with the process $(\tau \wedge t)_{t \geq 0}$, so that the new enlarged filtration \mathbb{G} is the smallest filtration (satisfying the usual assumptions) containing \mathbb{F} and making τ a stopping time, that is

$$\mathcal{G}_t = \mathcal{K}_{t+},$$

where

$$\mathcal{K}_t = \mathcal{F}_t \bigvee \sigma(\tau \wedge t).$$

Modeling default times as \mathbb{F} random times forces one to a reflection on what should be the impact of default not only on the defaultable assets, but also on the ones which do not default at time τ . This type of modeling is therefore more elaborate than the one which directly assumes that the default is a \mathbb{G} stopping time with a given intensity, since we have here to deal with projections on the smaller filtration \mathbb{F} . The useful characteristics of τ are synthesized in the properties of the process Z^τ .

We can identify two types of random times that have been studied in the theory of the enlargements of filtrations and which have also been applied in default models: pseudo-stopping times and last passage times.

Definition 2.2 ([26]). We say that τ is a \mathbb{F} pseudo-stopping time if for every \mathbb{F} -martingale (M_t) in \mathcal{H}^1 , we have

$$\mathbf{E}M_\tau = \mathbf{E}M_0. \quad (2.3)$$

Remark. It is equivalent to assume that (2.3) holds for bounded martingales, since these are dense in \mathcal{H}^1 . It can also be proved that then (2.3) also holds for all uniformly integrable martingales (see [26]).

The most common situation encountered in the default risk literature is the (H) hypothesis (or immersion property): every \mathbb{F} -local martingale is also a \mathbb{G} -local martingale. For instance, this property is always satisfied when the default time is a Cox time. Also it has been shown that this property has links with no arbitrage and informational efficiency (see [5], [7]). The following characterization of pseudo-stopping times, which will be often used in the sequel, shows that pseudo-stopping times are the natural extension of the (H) hypothesis framework:

Theorem 2.3 ([26]). *The following four properties are equivalent:*

- (1) τ is a (\mathcal{F}_t) pseudo-stopping time, i.e (2.3) is satisfied;
- (2) $\mu_t^\tau \equiv 1$, a.s, that is, $(Z_t^\tau)_{t \geq 0}$ is decreasing.
If, furthermore, all (\mathcal{F}_t) martingales are continuous, then each of the preceding properties is equivalent to:
- (3) $(Z_t^\tau)_{t \geq 0}$ is a decreasing \mathbb{F} -predictable process

Remark. Of course, every stopping time is a pseudo-stopping time by the optional sampling theorem. But there are many examples or families of pseudo-stopping which are not stopping times (see [26]). Similarly, all random times which ensure that the (H) hypothesis holds are pseudo-stopping times. But there are pseudo-stopping times for which the (H) hypothesis does not hold (in particular those which are \mathcal{F}_∞ -measurable, see [26] for more details).

The following classical lemma will be very helpful: it indicates the properties of the above processes under the assumptions **(A)** or **(C)** (for more details or references, see [9] or [28]).

Lemma 2.4. *Under condition **(A)**, $A_t^\tau = a_t^\tau$ is continuous.*

*Under condition **(C)**, A^τ is predictable (recall that under **(C)** the predictable and optional sigma fields are equal) and consequently $A^\tau = a^\tau$.*

*Under conditions **(CA)**, Z^τ is continuous.*

We give a first application of theorem 2.3 and lemma 2.4 to illustrate how the general theory of stochastic processes shed a new light on default time modeling. It is very often assumed in the literature on default times that τ is a random time whose associated Azéma supermartingale is continuous and decreasing (as seen in the introduction, this simplifies the pricing procedure).

Proposition 2.5. *Let τ be a random time that avoids stopping times, that is condition **(A)** holds. Then (Z_t^τ) is continuous and decreasing if and only if τ is a pseudo-stopping time.*

Proof. From lemma 2.4, when **(A)** holds, A^τ is continuous and consequently $Z^\tau = \mu^\tau - A^\tau$ is the Doob-Meyer decomposition of Z^τ . If τ is a pseudo-stopping, then from theorem 2.3, $\mu^\tau \equiv 1$ a.s. hence $Z^\tau = 1 - A_t^\tau$ is continuous decreasing.

Conversely, if Z^τ is continuous decreasing, then from the uniqueness of the Doob-Meyer decomposition, $Z^\tau = 1 - a^\tau$. But since τ avoids stopping times, we have $a_t^\tau = A_t^\tau$ (lemma 2.4) and hence $Z^\tau = 1 - A^\tau$. Consequently, from theorem 2.3, τ is a pseudo-stopping time. \square

Remark. We shall see a slight reinforcement of this proposition in the next section: indeed, we shall prove that if Z^τ is continuous, then τ avoids stopping times.

3. MAIN THEOREMS

First, we clarify a situation concerning the hazard process. Indeed, in the credit risk literature, the \mathbb{G} martingale $L_t \equiv \mathbf{1}_{\tau > t} e^{\Gamma_t}$ plays an important role (see [19] or [5]). But from definition 1.1, the hazard process is defined only when $Z_t^\tau > 0$ for all $t \geq 0$. We wish to show that nevertheless, the martingale (L_t) is always well defined. For this, it is enough to show that on the set $\{\tau > t\}$, $\Gamma_t = -\log Z_t^\tau$ is always well defined. This is the case thanks to the following result from the general theory of stochastic processes:

Proposition 3.1 ([20], [9], p.134). *Let τ be an arbitrary random time. The sets $\{Z^\tau = 0\}$ and $\{Z_-^\tau = 0\}$ are both disjoint from the stochastic interval $[0, \tau[$, and have the same lower bound T , which is the smallest stopping time larger than τ .*

The next proposition gives general conditions under which Γ is continuous, which is generally taken as an assumption in the literature on default times: indeed, when computing prices or hedging, one often has to integrate with respect to Γ (see [5], [14] or [19]).

Proposition 3.2. *Let τ be a random time.*

- (i) *Under **(CA)**, (Γ_t) is continuous and $\Gamma_0 = 0$.*
- (ii) *If τ is a pseudo-stopping time and if **(A)** holds, then (Γ_t) is a continuous increasing process, with $\Gamma_0 = 0$.*

Proof. This is a consequence of Lemma 2.4 and theorem 2.3. \square

Now, what can one say about the random time τ if one assumes that its associated Azéma's supermartingale is continuous? It seems to have been an open question in the literature on credit risk modeling. The next proposition answers this question:

Proposition 3.3. *Let τ be a finite random time such that its associated Azéma's supermartingale Z_t^τ is continuous. Then τ avoids stopping times.*

Proof. It is known that

$$Z_t^\tau = {}^o(\mathbf{1}_{[0,\tau)}),$$

that is Z_t^τ is the optional projection of the stochastic interval $[0, \tau)$. Now, following Jeulin-Yor [22], define \tilde{Z}_t as the optional projection of the stochastic interval $[0, \tau]$:

$$\tilde{Z}_t = {}^o(\mathbf{1}_{[0,\tau]}).$$

It can be shown (see [22]) that

$$\tilde{Z}_+ = Z^\tau \quad \text{and} \quad \tilde{Z}_- = Z_-^\tau.$$

Since Z^τ is continuous, we have

$$\tilde{Z}_+ = \tilde{Z}_- = Z^\tau,$$

and consequently, for any stopping time T :

$$\mathbf{E}[\mathbf{1}_{\tau \geq T}] - \mathbf{E}[\mathbf{1}_{\tau > T}] = 0,$$

which means that $\mathbf{P}[\tau = T] = 0$ for all stopping times T . \square

As an application, we can state the following enforcement of proposition 2.5:

Corollary 3.4. *Let τ be a random time. Then (Z_t^τ) is a continuous and decreasing process if and only if τ is a pseudo-stopping time that avoids stopping times.*

The above corollary says that pseudo-stopping times which avoid stopping times represent the more general class of default times such that the simple pricing formula (1.3) applies.

Now we recall a theorem which is useful in constructing the martingale hazard process.

Theorem 3.5 ([21]). *Let H be a bounded (\mathcal{G}_t) predictable process. Then*

$$H_\tau \mathbf{1}_{\tau \leq t} - \int_0^{t \wedge \tau} \frac{H_s}{Z_{s-}^\tau} da_s^\tau$$

is a (\mathcal{G}_t) martingale.

Corollary 3.6. *Let τ be a pseudo-stopping time that avoids \mathbb{F} stopping times. Then the \mathbb{G} dual predictable projection of $\mathbf{1}_{\tau \leq t}$ is $\log\left(\frac{1}{Z_{t \wedge \tau}^\tau}\right)$.*

Let g be an honest time (that means that g is the end of an \mathbb{F} optional set) that avoids \mathbb{F} stopping times. Then the \mathbb{G} dual predictable projection of $\mathbf{1}_{g \leq t}$ is A_t^g .

Proof. Let τ be a random time; taking $H \equiv 1$, in Theorem 3.5 we find that $\int_0^{t \wedge \tau} \frac{1}{Z_{s-}^\tau} dA_s^\tau$ is the \mathbb{G} dual predictable projection of $\mathbf{1}_{\tau \leq t}$.

When τ is a pseudo-stopping time that avoids \mathbb{F} stopping times, we have from Theorem 2.3 that the \mathbb{G} dual predictable projection of $\mathbf{1}_{\tau \leq t}$ is $-\log(Z_{t \wedge \tau}^\tau)$ since in this case $A_t^\tau = 1 - Z_t^\tau$ is continuous.

The second fact is an easy consequence of the well known fact that the measure dA_t^g is carried by $\{t : Z_t^g = 1\}$ (see [2]). \square

As a consequence, we have the following characterization of the martingale hazard process:

Proposition 3.7. *Let τ be a random time. Suppose that $Z_t^\tau > 0, \forall t$. Then, there exists a unique martingale hazard process Λ_t , given by*

$$\Lambda_t = \int_0^t \frac{da_u^\tau}{Z_{u-}}, \quad (3.1)$$

where a_t^τ is the dual predictable projection of $\mathbf{1}_{\tau \leq t}$.

Proof. We suppose there exist two different martingale hazard processes Λ^1 and Λ^2 and denote

$$T(\omega) = \inf \{t : \Lambda_t^1(\omega) \neq \Lambda_t^2(\omega)\}.$$

T is an (\mathcal{F}_t) -stopping time hence a \mathbb{G} stopping time. Due to the uniqueness of the predictable compensator we must have for all $t \geq 0$:

$$\Lambda_{t \wedge \tau}^1 = \Lambda_{t \wedge \tau}^2 \text{ a.s.}$$

Hence, $T > \tau$ a.s. and hence $Z_t^\tau = 0, \forall t \geq T$. By assumption, this is impossible, hence $\Lambda^1 = \Lambda^2$ a.s. \square

It is conjectured in [19] that if τ is any random time (possibly a stopping time) such that $\mathbf{P}(\tau \leq t | \mathcal{F}_t)$ is an increasing process, and if the martingale hazard process Λ is continuous, then $\Lambda = \Gamma$, where Γ is the hazard process. We now provide a counterexample to this conjecture. Indeed, let τ be a totally inaccessible stopping time of the filtration \mathbb{F} . Then of course $\mathbf{P}(\tau \leq t | \mathcal{F}_t) = \mathbf{1}_{\tau \leq t}$ is an increasing process. Let now (A_t) be the predictable compensator of $\mathbf{1}_{\tau \leq t}$. It is well known (see [2] or [20] for example) that A_t is a continuous process (that satisfies $A_t = A_{t \wedge \tau}$) and hence $\Lambda_t = A_t$ is continuous. But clearly $\Gamma_t \neq \Lambda_t$.

We propose the following theorem instead of the above conjecture (recall the fact that the Azéma supermartingale is continuous and decreasing means that τ is a pseudo-stopping time):

Theorem 3.8. *Let τ be a pseudo-stopping time. Assume further that $Z_t^\tau > 0$ for all t .*

- (i) Under **(A)**, Γ is continuous and $\Gamma_t = \Lambda_t = -\ln Z_t$.
- (ii) Under **(C)**, if Λ is continuous, then $\Gamma_t = \Lambda_t = -\ln Z_t$.

Proof. (i) follows from lemma 2.4, Theorem 2.3 and proposition 3.7.

(ii) Assume **(C)** holds. Since Λ is assumed to be continuous, it follows from proposition 3.7 (2) that a_t^τ is continuous. Hence τ avoids all predictable stopping times. But under **(C)**, all stopping times are predictable. Consequently τ avoids all stopping times and we apply part (i). \square

Remark. In the next section, the above theorem will be used to compare the two different pricing formulas mentioned in the introduction.

It has been proved in [19] that in general, even under the assumptions **(CA)**, the hazard process and the martingale hazard process may differ. The example they used was $g \equiv \sup\{t \leq 1 : W_t = 0\}$, where W denotes as usual the standard Brownian Motion. This time is a typical example of an honest time (i.e. the end of an optional set). We shall now show that this result actually holds for any honest time g and compute explicitly the difference in this case. We shall need for this the following characterisation of honest times given in [27]:

Theorem 3.9 ([27]). *Let g be an honest time. Then, under the conditions **(CA)**, there exists a unique continuous and nonnegative local martingale $(N_t)_{t \geq 0}$, with $N_0 = 1$ and $\lim_{t \rightarrow \infty} N_t = 0$, such that:*

$$Z_t^g = \mathbf{P}(g > t \mid \mathcal{F}_t) = \frac{N_t}{\Sigma_t},$$

where $\Sigma_t = \sup_{s \leq t} N_s$. The honest time g is also given by:

$$\begin{aligned} g &= \sup \{t \geq 0 : N_t = \Sigma_\infty\} \\ &= \sup \{t \geq 0 : \Sigma_t - N_t = 0\}. \end{aligned} \quad (3.2)$$

Proposition 3.10. *Let g be an honest time. Under **(CA)**, assume that $\mathbf{P}(g > t \mid \mathcal{F}_t) > 0$. Then there exists a unique strictly positive and continuous local martingale N , with $N_0 = 1$ and $\lim_{t \rightarrow \infty} N_t = 0$, such that:*

$$\Gamma_t = \ln \Sigma_t - \ln N_t \text{ whilst } \Lambda_t = \ln \Sigma_t,$$

where $\Sigma_t = \sup_{s \leq t} N_s$. Consequently,

$$\Lambda_t - \Gamma_t = \ln N_t, \quad (3.3)$$

and $\Gamma \neq \Lambda$.

Proof. From theorem 3.9, there exists a unique strictly positive continuous local martingale N , such that $N_0 = 1$ and $\lim_{t \rightarrow \infty} N_t = 0$, such that:

$$Z_t^g = \mathbf{P}(g > t \mid \mathcal{F}_t) = \frac{N_t}{\Sigma_t}.$$

Now an application of Itô's formula yields:

$$\mathbf{P}(g > t \mid \mathcal{F}_t) = 1 + \int_0^t \frac{dN_s}{\Sigma_s} - \int_0^t \frac{N_s}{\Sigma_s^2} d\Sigma_s.$$

But on the support of $(d\Sigma_s)$, we have $\Sigma_t = N_t$ and hence:

$$\mathbf{P}(g > t \mid \mathcal{F}_t) = 1 + \int_0^t \frac{dN_s}{\Sigma_s} - \ln \Sigma_t.$$

From the uniqueness of the Doob-Meyer decomposition, we deduce that the dual predictable projection of $\mathbf{1}_{g \leq t}$ is $\ln \Sigma_t$. Now, applying proposition 3.7, we have:

$$\Lambda_t = \int_0^t \frac{d(\ln \Sigma_s)}{\mathbf{P}(g > s \mid \mathcal{F}_s)} = \int_0^t \frac{\Sigma_s}{\Sigma_s N_s} d\Sigma_s = \ln \Sigma_t,$$

where we have again used the fact that the support of $(d\Sigma_s)$, we have $\Sigma_t = N_t$. The result of the proposition now follows easily. \square

The following theorem gathers properties of the difference between the hazard and the martingale hazard processes, which will reveal to be very important in pricing. We only concentrate on the situation where Λ is continuous, because in the reduced-form models the process Λ is continuous (it is usually supposed to be absolutely continuous with respect to the Lebesgue measure, i.e., an intensity process exists).

Theorem 3.11. *Suppose that $Z_t^\tau > 0, \forall t$ and denote $\Theta_t := \Gamma_t - \Lambda_t$. If Λ is continuous, then the process $N_t := e^{-\Theta_t}, t \geq 0$ is a strictly positive \mathbb{F} -local martingale. Moreover:*

- (i) *If τ is a pseudo stopping time, then $e^{-\Theta_t}$ is a finite variation local martingale. It is constant if one of the conditions **(C)** or **(A)** is satisfied.*
- (ii) *If τ is a honest time, then $e^{-\Theta_t}$ is never constant. It is a continuous local martingale if condition **(CA)** is satisfied.*

Proof. Using integration by parts and the expression of Λ (see equation (3.1)) we obtain

$$N_t = \exp\{-\Theta_t\} = Z_t^\tau e^{\Lambda_t} = 1 + \int_0^t Z_{s-}^\tau e^{\Lambda_s} d\Lambda_s + \int_0^t e^{\Lambda_s} dZ_s^\tau = 1 + \int_0^t e^{\Lambda_s} dm_s^\tau,$$

which is indeed a local martingale. When τ is a pseudo-stopping time, we have from Theorem 2.2 that $m_t^\tau = 1 - A_t^\tau + a_t^\tau$, which is a martingale of finite variation. Moreover, when **(C)** or **(A)** holds, $A^\tau = a^\tau$ as proved in Theorem 3.8 and consequently $m^\tau \equiv 1$.

When τ is a honest time, the result follows immediately from Proposition 3.10, formula (3.3). \square

4. APPLICATIONS TO THE PRICING OF DEFAULTABLE CLAIMS

In this section we always assume that $Z_t^\tau > 0, \forall t$, so that Γ is well defined and Λ is unique. We moreover assume that the process Λ is continuous, which is the usual assumption appearing in the reduced-form models. It is known that, when unique, Λ is a continuous process if and only if τ is a totally inaccessible \mathbb{G} -stopping time. We now apply the results of the last section in order to price a defaultable claim X of the type introduced in equation (1.1).

We shall denote by $\tilde{S}(X)$ the \mathbb{F} -adapted process which equals the pre-default price process of X , that is: $\tilde{S}(X)_t \mathbf{1}_{\{\tau > t\}} = S(X)_t \mathbf{1}_{\{\tau > t\}}, \forall t \geq 0$. $\tilde{S}(X)_t$ is given by the right-hand side of equation (1.2).

We now rewrite the general pricing result which was stated in equation (1.2) in order to emphasize the role of the process $N = e^{-\Theta}$:

$$\begin{aligned} \tilde{S}_t(X) &= e^{R_t + \Gamma_t} \mathbf{E} \left[\int_t^T C_s e^{-(\tilde{R}_s - \Lambda_s)} da_s^\tau + e^{-(\tilde{R}_T + \Theta_T)} P | \mathcal{F}_t \right] \\ &= e^{\tilde{R}_t + \Theta_t} \mathbf{E} \left[\int_t^T C_s e^{-(\tilde{R}_s - \Lambda_s)} Z_{s-}^\tau d\Lambda_s + e^{-(\tilde{R}_T + \Theta_T)} P | \mathcal{F}_t \right] \\ &= e^{\tilde{R}_t + \Theta_t} \mathbf{E} \left[\int_t^T C_s e^{-(\tilde{R}_s + \Theta_{s-})} d\Lambda_s + e^{-(\tilde{R}_T + \Theta_T)} P | \mathcal{F}_t \right] \end{aligned} \quad (4.1)$$

since Λ is continuous (recall that $\tilde{R}_t = R_t + \Lambda_t$ is the default-adjusted account).

Let us notice that the above formula is very general, but we cannot in this generality give the interpretation of a default-free pricing using an "adjusted" discount factor. Indeed, the "discounting rate": $\tilde{R} + \Theta = R + \Gamma$ is not in general an increasing process, as cumulated interests are. In some situations, it can be discontinuous, or have infinite variation. In what follows, we shall write the pricing formula under other, more useful forms.

We first state a lemma.

Lemma 4.1. *Assume that $(N_t)_{0 \leq t \leq T}$ is a square integrable martingale. Then we have*

$$\tilde{S}_t(X) = e^{\tilde{R}_t} N_t^{-1} \mathbf{E} \left[N_T \left(\int_t^T e^{-\tilde{R}_u} C_u d\Lambda_u + e^{-\tilde{R}_T} P \right) | \mathcal{F}_t \right] \quad (4.2)$$

Proof. Let us denote: $\tilde{C}_t := \int_0^t e^{-(R_u + \Lambda_u)} C_u d\Lambda_u$. An integration by parts (recall that Λ is supposed continuous) yields

$$N_T \tilde{C}_T = N_t \tilde{C}_t + \int_t^T N_u d\tilde{C}_u + \int_t^T \tilde{C}_u dN_u.$$

Consequently

$$\mathbf{E} \left[\int_t^T N_u d\tilde{C}_u | \mathcal{F}_t \right] = \mathbf{E} \left[N_T \tilde{C}_T - \int_t^T \tilde{C}_u dN_u | \mathcal{F}_t \right] - N_t \tilde{C}_t$$

The pre-default price of the defaultable claim from equation (4.1) can be re-written:

$$\begin{aligned} \tilde{S}_t(X) &= e^{\tilde{R}_t} N_t^{-1} \mathbf{E} \left[\int_t^T N_{s-} d\tilde{C}_s + e^{-\tilde{R}_T} N_T P | \mathcal{F}_t \right] \\ &= e^{\tilde{R}_t} N_t^{-1} \left(\mathbf{E} \left[N_T \tilde{C}_T - \int_t^T \tilde{C}_u dN_u + e^{-\tilde{R}_T} N_T P | \mathcal{F}_t \right] - N_t \tilde{C}_t \right) \end{aligned}$$

Since C is bounded and N is a square integrable martingale, \tilde{C} is also bounded and hence $\int \tilde{C}dN$ is also a square integrable martingale. Thus the pricing formula can be further simplified:

$$\tilde{S}_t(X) = e^{\tilde{R}_t} N_t^{-1} \left(\mathbf{E} \left[N_T \left(\tilde{C}_T + e^{-\tilde{R}_T} P \right) \middle| \mathcal{F}_t \right] - N_t \tilde{C}_t \right) \quad (4.3)$$

$$= e^{\tilde{R}_t} N_t^{-1} \left(\mathbf{E} \left[N_T \left(\int_t^T d\tilde{C}_u + e^{-\tilde{R}_T} P \right) \middle| \mathcal{F}_t \right] + \tilde{C}_t \mathbf{E}[N_T | \mathcal{F}_t] - N_t \tilde{C}_t \right) \quad (4.4)$$

$$= e^{\tilde{R}_t} N_t^{-1} \mathbf{E} \left[N_T \left(\int_t^T e^{-\tilde{R}_u} C_u d\Lambda_u + e^{-\tilde{R}_T} P \right) \middle| \mathcal{F}_t \right]. \quad (4.5)$$

□

Now we emphasize the link with the formula of Duffie et al. [12] in the case when the \mathbb{F} -martingales are \mathbb{G} -martingales, that is when the (H) hypothesis is valid. As explained in Section 2, in this case τ is a pseudo-stopping time.

Proposition 4.2. *Suppose that the (H) hypothesis holds and that $(N_t)_{0 \leq t \leq T}$ is a square integrable martingale. Define V as in equation (1.4) and also define the following \mathbb{F} -martingale:*

$$\tilde{V}_t := \mathbf{E} \left[\int_0^T C_u e^{-(R_u + \Lambda_u)} d\Lambda_u + P e^{-(R_T + \Lambda_T)} \middle| \mathcal{F}_t \right]$$

Then the claim's price process S satisfies for $t \geq 0$ and on $\{t < \tau\}$:

$$\tilde{S}_t(X) = V_t(X) - e^{\tilde{R}_t + \Theta_t} \mathbf{E} \left[[\tilde{V}, N]_T - [\tilde{V}, N]_t \middle| \mathcal{F}_t \right] = V_t(X) - J_t.$$

Therefore, if $[\tilde{V}, N] \equiv 0$ then $J \equiv 0$. In particular:

- (i) If one of the conditions (C) or (A) holds, then $J \equiv 0$.
- (ii) If V is predictable then $J \equiv 0$.

Proof. It is assumed that the (H) hypothesis holds. In this case, it is known (see [10]) that for every \mathcal{F}_∞ -measurable random variable F :

$$\mathbf{E}[F | \mathcal{G}_t] = \mathbf{E}[F | \mathcal{F}_t]$$

Hence, the process V , which was defined in equation (1.4) may be written as an \mathbb{F} -adapted process, that is:

$$V_t = e^{\tilde{R}_t} \mathbf{E} \left[\int_t^T C_u e^{-\tilde{R}_u} d\Lambda_u + P e^{-\tilde{R}_T} \middle| \mathcal{F}_t \right] \quad t < T,$$

and $V_t = 0$ for $t \geq T$. Hence, when (H) holds, we have:

$$\tilde{V}_t = V_t e^{-(R_t + \Lambda_t)} + \int_0^t e^{-(R_u + \Lambda_u)} C_u d\Lambda_u, \quad (4.6)$$

i.e., $\tilde{V}_t = V_t e^{-\tilde{R}t} + \tilde{C}_t$. Replacing this in equation (4.3) and using the fact that $V_T = P$, we obtain

$$\begin{aligned}\tilde{S}_t(X) &= e^{\tilde{R}t} N_t^{-1} \left(\mathbf{E} \left[N_T \tilde{V}_T \middle| \mathcal{F}_t \right] - N_t \tilde{C}_t \right) \\ &= e^{\tilde{R}t} N_t^{-1} \left(\mathbf{E} \left[N_T \tilde{V}_T \middle| \mathcal{F}_t \right] - N_t (\tilde{V}_t - V_t e^{-\tilde{R}t}) \right) \\ &= V_t + e^{\tilde{R}t} N_t^{-1} E \left[[\tilde{V}, N]_T - [\tilde{V}, N]_t \middle| \mathcal{F}_t \right].\end{aligned}$$

The statements about the term J are obtained from Theorem 3.11. \square

The proposition above gives another expression for the jump term J (see equation (1.5)), when the (H) hypothesis holds, and thus it allows to see better in which situations $J \equiv 0$.

Now, let us return to the general situation, when (H) does not necessarily hold. Whenever the local martingale N is a true martingale, it can serve as a change of measure which permits to recover again a simple pricing formula. Indeed, a simple use of the Girsanov's theorem gives:

Proposition 4.3. *Suppose that $(N_t)_{0 \leq t \leq T}$ is a square integrable martingale, and define the default-adjusted measure as:*

$$d\mathbf{Q}^\tau := N_T \cdot d\mathbf{P} \quad \text{on } \mathcal{F}_T.$$

Then, it follows that the pre-default price of the defaultable claims is:

$$\tilde{S}_t(X) = e^{\tilde{R}t} \mathbf{E}^{\mathbf{Q}^\tau} \left[\int_t^T C_u e^{-\tilde{R}u} d\Lambda_u + X e^{-\tilde{R}T} \middle| \mathcal{F}_t \right] \quad t < T;$$

Proof. The proof follows directly by application of the definition of the new measure \mathbf{Q}^τ and equation (4.2). \square

Remark. Note that $\left(\tilde{S}_t(X) e^{-\tilde{R}t} - \int_0^t C_u e^{-\tilde{R}u} d\Lambda_u \right)$ is a \mathbf{Q}^τ -martingale.

We now compare this result with the one which appears in Collin Dufresne et al. [8]. Indeed, as we already mentioned in the introduction, the following change of measure was used in [8]: $d\mathbf{P}' = D_t := \mathbf{1}_{\{\tau > t\}} e^{\Lambda t} \cdot d\mathbf{P}$. Now we show that in general, to use effectively pricing under the measure \mathbf{P}' , one needs to be able to compute the compensators of the jumps of martingales at τ . This problem is as complicated as the original problem of computing the jump term J .

To understand this, assume that $C \equiv 0$ and $P = Y_T$, where (Y_t) is a \mathbb{G} adapted process (let us recall that in this approach the smaller filtration \mathbb{F} does not play a role). Moreover take $\Lambda_t = \lambda t$, $t \geq 0$, where λ is a positive constant. Y can be a primary asset price (i.e. a traded asset, for instance a stock), in which case $\tilde{Y} := Y e^{-Rt}$ is a \mathbf{P} -martingale in the filtration \mathbb{G} , by the usual no-arbitrage arguments. The evaluation formula for X is before default (see 1.6):

$$S(\mathbf{1}_{\{\tau > T\}} Y_T)_t = e^{\tilde{R}t - \lambda T} \mathbf{E}^{\mathbf{P}'} [\tilde{Y}_T | \mathcal{G}_t].$$

Hence, the only needed thing is the dynamics of the \mathbf{P} -martingale \tilde{Y} under the measure \mathbf{P}' . Using the Lenglart-Girsanov theorem when the change of measure is only absolutely continuous (see [29] page 135), and noticing that the density process $D_t := \mathbf{1}_{\{\tau > t\}} e^{\Lambda t}$ is a finite variation martingale with a unique jump in τ , we get that the process:

$$\tilde{Y}'_t := \tilde{Y}_t - \int_0^t \frac{d\langle \tilde{Y}, D \rangle_s}{D_{s-}} = \tilde{Y}_t + \left(\mathbf{1}_{\{\tau \leq t\}} \Delta \tilde{Y}_\tau \right)^p$$

is a \mathbf{P}' -martingale, where $(\cdot)^p$ stands for the predictable compensator, in the filtration \mathbb{G} , under the measure \mathbf{P} . We find that before default:

$$S(\mathbf{1}_{\{\tau > T\}} Y_T)_t = e^{\tilde{R}_t - \Lambda T} \left(\tilde{Y}_t - J'_t \right).$$

where $J'_t := \mathbf{E}^{\mathbf{P}'} \left[\left(\mathbf{1}_{\{\tau \leq T\}} \Delta \tilde{Y}_\tau \right)^p \mid \mathcal{G}_t \right]$. We see that finding an explicit expression for J' has the same degree of difficulty as the original problem.

In what follows, in order to illustrate the method introduced in Proposition 4.3, we give an example of pricing when the default time is a last passage time for which neither Γ or Λ are absolutely continuous, and such that $(N_t)_{0 \leq t \leq T}$ is a square integrable and non constant martingale.

More precisely, consider the framework of Proposition 3.10 and assume that \mathbb{F} is generated by a standard Brownian motion $(W_t)_{t \geq 0}$. Let

$$N_t = \exp(2aW_t - 2a^2t),$$

where $a > 0$, and

$$\Sigma_t = \exp \left(\sup_{s \leq t} 2a(W_s - as) \right).$$

We assume that the default time is:

$$\tau := \sup \left\{ t : W_t - at = \sup_{s \geq 0} (W_s - as) \right\}$$

In this case, the default-adjusted measure is

$$d\mathbf{Q}^\tau := \exp(2aW_T - 2a^2T) \cdot d\mathbf{P} \quad \text{on } \mathcal{F}_T,$$

and the adjusted discount factor is:

$$\tilde{R}_t = R_t + \sup_{s \leq t} 2a(W_s - as) = R_t + \sup_{s \leq t} 2a(W_s^\tau + as),$$

where $W^\tau = W - 2at$ is a \mathbf{Q}^τ Brownian motion. The price of a defaultable zero-bond with maturity T and zero recovery in case of default is before default:

$$\tilde{B}(t, T) := \tilde{S}(\mathbf{1}_{\{\tau > T\}})_t = \mathbf{E}^{\mathbf{Q}^\tau} [e^{-(\tilde{R}_T - \tilde{R}_t)} \mid \mathcal{F}_t].$$

Therefore, there exists some \mathbb{F} -predictable volatility process $(\sigma(t, T), t \geq 0)$ such that the dynamics of the pre-default price of the bond is:

$$d\tilde{B}(t, T) = \tilde{B}(t, T) \left(d\tilde{R}_t + \sigma(t, T) dW_t^\tau \right) \quad (4.7)$$

$$= \tilde{B}(t, T) \{ (dR_t + d\Lambda_t - 2a\sigma(t, T)dt) + \sigma(t, T) dW_t \}. \quad (4.8)$$

We see that under the risk-neutral measure \mathbf{P} , the drift of the process is affected by two terms, which are specific to a defaultable bond: $d\Lambda$ and $-2a\sigma(t, T)dt$.

Let us now analyze the pricing of vulnerable options written on a stock. Let (Y_t) be the price process of a default-free stock. We may assume:

$$dY_t = Y_t(r_t dt + \sigma_t dW_t).$$

It follows that under \mathbf{Q}^τ the stock behaves as a negative-dividends paying stock:

$$dY_t = Y_t((r_t + \sigma a)dt + \sigma_t dW_t^\tau)$$

A vulnerable option with exercise date T and zero recovery in case of default is defined as:

$$X = \mathbf{1}_{\{\tau > T\}} f(Y_u, u \in [0, T]),$$

for some functional f of the paths of the process Y (this definition applies for European, Asian or lookback options). Because the adjusted account \tilde{R}_t is stochastic, it is convenient to price by using as numéraire the pre-default defaultable T -bond, i.e., $B(t, T)$. Let us introduce the default-adjusted forward measure as:

$$d\mathbf{Q}^T = \frac{e^{-\tilde{R}_T}}{\tilde{B}(0, T)} \cdot d\mathbf{Q}^\tau \text{ on } \mathcal{F}_T$$

Using the standard change of numéraire techniques we obtain:

$$S(X)_t = e^{\tilde{R}_t} \mathbf{E}^{\mathbf{Q}^\tau} \left[e^{-\tilde{R}_T} f(S_u, u \leq T) | \mathcal{F}_t \right] = \tilde{B}(t, T) \mathbf{E}^{\mathbf{Q}^T} [f(S_u, u \leq T) | \mathcal{F}_t]$$

Hence one can price a vulnerable option using the standard (i.e., default-free) evaluation techniques, but under the measure \mathbf{Q}^T .

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