NORI 1-MOTIVES

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ABSTRACT. Let EHM be Nori's category of effective homological mixed motives. In this paper, we consider the thick abelian subcategory $\mathsf{EHM}_1 \subset \mathsf{EHM}$ generated by the i-th relative homology of pairs of varieties for $i \in \{0, 1\}$. We show that EHM_1 is naturally equivalent to the abelian category ${}^t\mathcal{M}_1$ of 1-motives with torsion; this is our main theorem. Along the way, we obtain several interesting results. Firstly, we realize ${}^t\mathcal{M}_1$ as the universal abelian category obtained, using Nori's formalism, from the Betti representation of an explicit diagram of curves. Secondly, we obtain a conceptual proof of a theorem of Vologodsky on realizations of 1-motives. Thirdly, we verify a conjecture of Deligne on extensions of 1-motives in the category of mixed realizations for those extensions that are effective in Nori's sense.

CONTENTS

Introduction

Nori's construction of the category of effective homological mixed motives EHM is drafted in [11], [14], [15] and generalized in [1] (see also [8] for the abstract part of the construction). For details on EHM we here mainly refer to [15] (which is published and consistent with Nori's notation in [11]). The category EHM depends on a base field k together with a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. Given a k-scheme X and a closed subset $Y \subset X$, we have motives $\widetilde{H}_i(X, Y; \mathbb{Z}) \in \mathsf{EHM}$ for all $i \in \mathbb{Z}$. Their Betti realizations are the usual relative homology groups $H_i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$.

For $n \in \mathbb{N}$, we denote by $\mathsf{EHM}_n \subset \mathsf{EHM}$ the thick abelian subcategory generated by $\widetilde{H}_i(X, Y; \mathbb{Z})$ for all $i \leq n$ and all pairs (X, Y) consisting of a k-scheme X and a closed subset $Y \subset X$. Objects of EHM_n will be called Nori n-motives.

We can well describe EHM_n for $n \leq 1$. Quite straightforwardly, EHM_0 is equivalent to the category of 0-motives (see Theorem 3.4). The case $n = 1$ is much more difficult, but nevertheless we are able to show the expected property: EHM_1 is equivalent to the abelian category of 1-motives with torsion; this is our main result (see Theorem 5.1). Its proof relies on two preliminary results which are of independent interest. The first result (see Theorem 4.5) claims that the category of 1-motives with torsion is the universal category, in the sense of Nori, obtained from the Betti representation of an explicit diagram of curves. The second result (see Theorem 7.10) is a particular case of a conjecture of Deligne [10, 2.4]. Roughly speaking, we prove that a mixed realization which is an extension of 1-motives is itself a 1-motive provided that it is *effectively* coming from geometry, *i.e.*, is the mixed realization of a Nori *effective* motive. (The original conjecture of Deligne would predict that the same holds provided that the extension is the realization of a non-necessarily effective Nori motive.)

Also, in the course of proving our main theorem, we obtain a conceptual proof of a result of Vologodsky comparing two Hodge realizations on the category of 1-motives: the classical one constructed by Deligne, and the composition of Huber's Hodge realization with the embedding of 1-motives into Voevodsky's triangulated motives (see Remark 7.7). Finally, with rational coefficients, we are also able to construct a left adjoint to the inclusion $\mathsf{EHM}_1 \hookrightarrow \mathsf{EHM}$ (see Theorem 7.13).

In any case, a comparison between the abelian category of 2-motives proposed in [2] and EHM_2 seems far beyond the scope of existing techniques.

Notation and Conventions. We let k be our base field and tacitly fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$. We denote by Sch_k the category of k-schemes. (By k-scheme we always mean a finite type, separated and reduced k-scheme.) Given a k-scheme X, we denote by X^{an} the associated analytic space given by $X(\mathbb{C})$. We fix a Noetherian commutative ring R and denote by R-mod the category of finitely generated R-modules. (In practice, R will be $\mathbb{Z}$ or $\mathbb{Q}$.) If E is an R-algebra which is non-necessary commutative, we also denote by E-mod the category of left E-modules which are finitely generated over R. (This will not be confusing: we only use this when E is finitely generated as an R-module, in which case a left E-module is finitely generated if and only if it is so as an R-module.) We say that a functor $\mathcal{E} \to R$-mod from an R-linear abelian category $\mathcal{E}$ is *forgetful* if it is R-linear, faithful and exact. (For instance, the obvious functor E-mod $\to R$-mod is forgetful.) A *thick abelian* subcategory is a full subcategory, containing the zero object, and stable under sub-quotients and extensions. A *thick triangulated* subcategory is a full subcategory, containing the zero object, and stable under cones, desuspensions and direct summands.

1. NORI'S UNIVERSAL CATEGORY

We recall Nori's construction of a universal abelian category from a representation of a diagram. In some situation, we give a characterization of this universal category.

1.1. **Generalities.** Recall the following construction due to Nori (see [8] and [15, §5.3.3] for details). Given a representation $T : D \to R$-mod of a diagram D, there are an R-linear abelian category $\mathcal{C}(T)$, a forgetful functor $F_T : \mathcal{C}(T) \to R$-mod and a representation $\widetilde{T} : D \to \mathcal{C}(T)$ such that $F_T \circ \widetilde{T} = T$. Moreover, the triple $(\widetilde{T}, \mathcal{C}(T), F_T)$ is initial (up to isomorphisms of functors) among factorizations of the representation T as a representation to an R-linear abelian category followed by a forgetful functor. For the precise statement, we refer the reader to [15, Theorem 41].

1.2. When D is finite, one takes $\mathcal{C}(T) = \mathrm{End}(T)$-mod where $\mathrm{End}(T)$ is the R-algebra of endomorphisms of T. More precisely, an element of $\mathrm{End}(T)$ is a family

$$(a_p)_{p \in Ob(D)} \in \prod_{p \in Ob(D)} \mathrm{End}(T(p))$$

such that for every arrow $a : p \to q$, one has $T(a) \circ a_p = a_q \circ T(a)$. For $p \in Ob(D)$, the algebra $\mathrm{End}(T)$ acts on the left on $T(p)$. The

resulting left $\mathrm{End}(T)$-module will be denoted by $\widetilde{T}(p)$. This gives the representation $\widetilde{T} : D \to \mathrm{End}(T)$-mod. We clearly have $T = F_T \circ \widetilde{T}$ where F_T is the obvious forgetful functor.

1.3. When the diagram is no longer assumed to be finite, we set

$$\mathcal{C}(T) := 2 - \operatorname*{Colim}_{E \subset D} \ \mathrm{End}(T|_F)\text{-mod} \tag{1.1}$$

taking the colimit over all finite sub-diagrams E. The universality of $\mathcal{C}(T)$ is established in [8]. From this description, it follows that if D is a filtered union of a family of (non-necessarily finite) sub-diagrams $(D_\alpha)_{\alpha \in I}$, there is an equivalence of categories

$$2 - \operatorname*{Colim}_{\alpha \in I} \mathcal{C}(T|_{D_\alpha}) \simeq \mathcal{C}(T). \tag{1.2}$$

1.4. Assume that R is an integral domain with field of fractions K. Denote by $T_K : D \to K$-mod the representation defined by $T_K(-) = T(-) \otimes_R K$. Then there is a canonical equivalence of categories

$$\mathcal{C}(T) \otimes_R K \simeq \mathcal{C}(T_K). \tag{1.3}$$

(For an R-linear category $\mathcal{A}$, the category $\mathcal{B} = \mathcal{A} \otimes_R K$ is given by $Ob(\mathcal{B}) = Ob(\mathcal{A})$ and $\hom_\mathcal{B}(-, -) = \hom_\mathcal{A}(-, -) \otimes_R K$.) To check this, it is enough, by (1.1), to consider the case where D is finite. As K is a flat R-algebra, we have $\mathrm{End}(T_K) = \mathrm{End}(T) \otimes_R K$. This easily implies the equivalence (1.3).

1.5. Assume that the ring R is artinian (*e.g.*, a field). Then the pro-system $\{\mathrm{End}(T|_E)\}_{E \subset D}$, where E runs over the ordered set of finite sub-diagrams of D, satisfies the Mittag-Leffler condition. Thus, it is tempting to consider

$$\mathrm{End}(T) := \operatorname*{Lim}_{E \subset D} \mathrm{End}(T|_E) \tag{1.4}$$

endowed with the inverse limit topology. Then $0 \in \mathrm{End}(T)$ has a fundamental system of neighborhoods consisting of open and closed two-sided ideals I such that $\mathrm{End}(T)/I$ is a finite length R-module. We denote by $\mathrm{End}(T)$-mod the category of continuous $\mathrm{End}(T)$-modules which are of finite type over R and discrete, *i.e.*, annihilated by an open and closed 2-sided ideal of $\mathrm{End}(T)$. It follows immediately that there are equivalences of categories (*cf.* [11, §1.2.1])

$$\begin{aligned} \mathcal{C}(T) &\simeq 2 - \operatorname*{Colim}_I \mathrm{End}(T)/I\text{-mod} \\ &\simeq \mathrm{End}(T)\text{-mod}. \end{aligned} \tag{1.5}$$

(The colimit above is over the open and closed two sided ideals in $\mathrm{End}(T)$.)

1.6. **Criterion for an equivalence.** We keep the notation as in the previous paragraphs. Assume that we are given a representation $S : D \to \mathcal{E}$ into an R-linear abelian category $\mathcal{E}$ and a forgetful functor $G : \mathcal{E} \to R\text{-mod}$ such that $G \circ S = T$. By universality (*i.e.*, [15, Theorem 41]) there is an exact faithful R-linear functor

$$U : \mathcal{C}(T) \to \mathcal{E} \qquad (1.6)$$

such that $F_T = G \circ U$ and $S = U \circ \widetilde{T}$.

1.7. **Proposition.** *Assume the following conditions:*
  (a) *Given p, $p' \in Ob(D)$, there exist $p \sqcup p' \in Ob(D)$, and arrows $i : p \to p \sqcup p'$ and $i' : p' \to p \sqcup p'$ such that*

$$T(i) + T(i') : T(p) \oplus T(p') \to T(p \sqcup p')$$

  is an isomorphism.
  (b) *Every object of $\mathcal{E}$ is a quotient of an object of the form $S(p)$ with $p \in Ob(D)$.*
  (c) *For every map $S(p) \to A$ in $\mathcal{E}$ there exists a finite sub-diagram $E \subset D$ containing p such that*

$$\mathrm{Ker}\{T(p) = G \circ S(p) \to G(A)\}$$

  is a sub-$\mathrm{End}(T|_E)$-module of $T(p)$.
Then U is an equivalence of categories.

We first note the following lemma.

1.8. **Lemma.** *Assume that 1.7(a) is satisfied and that 1.7(c) holds when $A = S(q)$ with $q \in D$. For every map $f : S(p) \to S(p')$ in $\mathcal{E}$, there exists a finite sub-diagram $E \subset D$ containing p and p' such that $G(f) : G \circ S(p) \to G \circ S(p')$ is a morphism of $\mathrm{End}(T|_E)$-modules.*

Proof. Consider the morphism

$$g = f - \mathrm{id}_{S(p')} : S(p) \oplus S(p') \to S(p').$$

It suffices to show that $\mathrm{Ker}(G(g)) \subset T(p) \oplus T(p')$ is a sub-$\mathrm{End}(T|_E)$-module for some finite sub-diagram $E \subset D$ containing p and p'.

With the notation as in 1.7(a), the morphism

$$S(i) + S(i') : S(p) \oplus S(p') \to S(p \sqcup p')$$

is invertible as G is faithful and exact. Moreover, the isomorphism

$$G \circ S(i) + G \circ S(i') : G \circ S(p) \oplus G \circ S(p') \simeq G \circ S(p \sqcup p')$$

is a morphism of $\mathrm{End}(T|_E)$-modules for any finite diagram E containing the objects p, p' and $p \sqcup p'$, and the arrows i and i'. (Indeed, this coincides with the isomorphism $T(i) + T(i') : T(p) \oplus T(p') \simeq T(p \sqcup p')$.)

Hence, it suffices to show that the kernel of the map $G \circ S(p \sqcup p') \to G \circ S(p')$ is a sub-End$(T|_E)$-module for some finite sub-diagram E. This is granted by 1.7(c) for $A = S(p')$.   □

1.9. **Proof of Proposition 1.7.** We first construct a functor $V : \mathcal{E} \to \mathcal{C}(T)$. For $A \in \mathcal{E}$, we choose an epimorphism $\alpha : S(p) \twoheadrightarrow A$ in $\mathcal{E}$ with $p \in Ob(D)$. (For this, we use 1.7(b).) Let $E \subset D$ be a finite sub-diagram such that the kernel of $G(\alpha)$ is a sub-End$(T|_E)$-module, *i.e.*, E is as in 1.7(c). As $T(p) = G \circ S(p) \to G(A)$ is surjective, there is a unique structure of End$(T|_E)$-module on $G(A)$ such that $G(\alpha)$ is End$(T|_E)$-linear. This End$(T|_E)$-module defines an objet of $\mathcal{C}(T)$ which we denote by $V(A, \alpha)$. By construction, $F_T(V(A, \alpha)) = G(A)$.

Next consider a commutative square in $\mathcal{E}$:

$$
\begin{array}{ccc}
S(p) & \xrightarrow{\ \alpha\ } & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} \\
S(p') & \xrightarrow{\ \alpha'\ } & A'.
\end{array}
$$

Using Lemma 1.8, the map $G \circ S(p) \to G \circ S(p')$ is a morphism of End$(T|_E)$-modules for some finite diagram $E \subset T$. Enlarging E so that the kernels of $G(\alpha)$ and $G(\alpha')$ are sub-End$(T|_E)$-modules, we get that $G(e) : G(A) \to G(A')$ is End$(T|_E)$-linear. This gives a morphism

$$V(e, f, \alpha, \alpha') : V(A, \alpha) \to V(A', \alpha')$$

in $\mathcal{C}(T)$. By construction, $F_T(V(e, f, \alpha, \alpha')) = G(e)$.

We are now ready to define the functor V. First, we note that $V(A, \alpha)$ is independent of α. More precisely, given another epimorphism $\alpha' : S(p') \twoheadrightarrow A$, there is a unique isomorphism $v_{\alpha, \alpha'} : V(A, \alpha) \simeq V(A, \alpha')$ such that $F_T(v_{\alpha, \alpha'}) = \mathrm{id}_{G(A)}$. It is given by the composition of

$$V(A, \alpha) \xrightarrow{\sim} V(A, \alpha + \alpha') \xleftarrow{\sim} V(A, \alpha')$$

where the maps are $V(\mathrm{id}_A, S(i), \alpha, \alpha + \alpha')$ and $V(\mathrm{id}_A, S(i'), \alpha', \alpha + \alpha')$ with $i : p \to p \sqcup p'$ and $i' : p' \to p \sqcup p'$ as in 1.7(a).

Similarly, given a morphism $e : A \to A'$, there is a unique morphism $V(e) : V(A, \alpha) \to V(A, \alpha')$ such that $F_T(V(e)) = G(e)$. It is given by the composition of

$$V(A, \alpha) \to V(A', \alpha + \alpha') \xleftarrow{\sim} V(A', \alpha')$$

where the maps are $V(e, S(i), \alpha, \alpha + \alpha')$ and $V(\mathrm{id}_A, S(i'), \alpha', \alpha + \alpha')$. Hence, choosing for every A an epimorphism α_A yields a functor $V : \mathcal{E} \to \mathcal{C}(T)$ such that $F_T \circ V = G$.

Now, as G and F_T are forgetful functors, it is immediate that V is an R-linear faithful and exact functor. Moreover, choosing the epimorphism α_A to be the identity when $A = S(p)$ for $p \in Ob(D)$, we see that $V \circ S = \widetilde{T}$. Hence, from the universal property, we should get that $V \circ U \simeq \mathrm{id}_{\mathcal{C}(T)}$. Since $V : \mathcal{E} \to \mathcal{C}(T)$ is faithful, $U : \mathcal{C}(T) \to \mathcal{E}$ is fully faithful. Condition 1.7(b) implies now that U is also essentially surjective. $\square$

For later use, we need to refine the criterion given by Proposition 1.7. We keep the notation as in 1.6.

1.10. **Proposition.** *We assume that R is an integral domain with field of fractions K. We denote by $T_K : D \to K$-mod the representation defined by $T_K(-) = T(-) \otimes_R K$ and we consider the pro-finite dimensional K-algebra $\mathrm{End}(T_K)$ (see 1.5).*

We assume that the conditions 1.7(a) and 1.7(b) are satisfied and that $T(p)$ is a torsion-free R-module for every $p \in Ob(D)$. Also, we assume the following variant of 1.7(c):

(c$'$) *For every map $S(p) \to A$ in $\mathcal{E} \otimes_R K$,*

$$\mathrm{Ker}\{T_K(p) = G(S(p)) \otimes_R K \to G(A) \otimes_R K\}$$

is a sub-$\mathrm{End}(T_K)$-module of $T_K(p)$.

Then U is an equivalence of categories.

Proof. Using Proposition 1.7, it remains to show that 1.10(c$'$) implies 1.7(c). First, note that if $G(A)$ is a torsion-free R-module, then we have

$$\mathrm{Ker}\{T(p) \to G(A)\} = T(p) \cap \mathrm{Ker}\{T_K(p) \to G(A) \otimes_R K\} \qquad (1.7)$$

where the intersection is taken inside $T_K(p)$. Let $E \subset D$ be a finite sub-diagram containing p and such that

$$\mathrm{Im}\{\mathrm{End}(T_K) \to \mathrm{End}(T_K(p))\} = \mathrm{Im}\{\mathrm{End}(T_K|_E) \to \mathrm{End}(T_K(p))\}.$$

By 1.10(c$'$), $\mathrm{Ker}\{T_K(p) \to G(A) \otimes_R K\}$ is a sub-$\mathrm{End}(T_K|_E)$-module of $T_K(p)$. As $\mathrm{End}(T_K|_E) = \mathrm{End}(T|_E) \otimes_R K$, we see that the right hand side of (1.7) is a sub-$\mathrm{End}(T|_E)$-module of $T(p)$. This shows that 1.7(c) is true for $S(p) \to A$.

For the general case we argue as follows. Given A, we may find an epimorphism $e : A' \twoheadrightarrow A$ such that $G(A')$ is torsion-free (*e.g.*, use 1.7(b)). We then choose an epimorphism $S(p') \twoheadrightarrow S(p) \times_A A'$ and

consider the commutative square

$$
\begin{array}{ccc}
S(p') & \longrightarrow & A' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} \\
S(p) & \longrightarrow & A.
\end{array}
$$

By the previous discussion, we know 1.7(c) for $S(p') \to A'$; let $E' \subset D$ be a finite sub-diagram containing p' and such that $\mathrm{Ker}\{T(p') \to G(A')\}$ is a sub-$\mathrm{End}(T|_{E'})$-module of $T(p')$. On the other hand, by Lemma 1.8, $G(f) : G \circ S(p') \to G \circ S(p)$ is a morphism of $\mathrm{End}(T|_E)$-modules with E a finite sub-diagram of D containing p and p'. As

$$
G(f) \left(\mathrm{Ker}\{T(p') \to G(A')\} \right) = \mathrm{Ker}\{T(p) \to G(A)\}
$$

we see immediately that $\mathrm{Ker}\{T(p) \to G(A)\}$ is a sub-$\mathrm{End}(T|_{E \cup E'})$-module of $T(p)$. $\qquad\square$

2. Nori n-motives

We define here the categories EHM_n of homological mixed n-motives. We also introduce some related categories which we denote by EHM'_n. These are obtained by simply repeating Nori's construction while restricting ourself to homological degree less or equal to n.

2.1. **Nori's diagram.** Recall the definition of Nori's diagram $^D(Sch_k)$ from [15]. Objects are triples (X, Y, i) where X is a k-scheme, $Y \subset X$ is a closed subset and i is an integer. Arrows are of the following kinds:

  a) $f : (X, Y, i) \to (X', Y', i)$ for any morphism $f : X \to X'$ such that $f(Y) \subset Y'$ and
  b) $\delta : (X, Y, i) \to (Y, Z, i-1)$ for any $Z \subset Y \subset X$ closed in X.

We have a canonical representation (associated to the complex embedding $\sigma : k \hookrightarrow \mathbb{C}$)

$$
H_* : {}^D(Sch_k) \to R\text{-}\mathsf{mod} \tag{2.1}
$$

given by $(X, Y, i) \rightsquigarrow H_i(X, Y; R)$ the singular homology of the pair $(X^{\mathrm{an}}, Y^{\mathrm{an}})$ with R-coefficients. For f and δ as before, $H_*(f) := f_*$ is given by the functoriality of singular homology and $H_*(\delta) := \partial$ is the boundary map in the long exact sequence associated to the triple $(X^{\mathrm{an}}, Y^{\mathrm{an}}, Z^{\mathrm{an}})$.

2.2. **Definition.** With the above notation, we set

$$
\mathsf{EHM}^R := \mathcal{C}\left(H_* : {}^D(Sch_k) \to R\text{-}\mathsf{mod} \right). \tag{2.2}
$$

This is the R-linear category of *Nori effective motives*. When $R = \mathbb{Z}$, we simply write EHM. If we wish to stress the dependence on the field k, we write $\mathsf{EHM}^R(k)$ and $\mathsf{EHM}(k)$.

Given a triple $(X, Y, i) \in {}^D(Sch_k)$, we denote by $\widetilde{H}_i(X, Y; R)$ its image in EHM^R by the universal representation of ${}^D(Sch_k)$ associated to H_*.

2.3. **Definition.** We denote by EHM_n^R (or $\mathsf{EHM}_n^R(k)$ if we wish to stress the dependence on k) the thick abelian subcategory of EHM^R generated by $\widetilde{H}_i(X, Y; R)$ where X is a k-scheme, $Y \subset X$ is a closed subset and i is an integer such that $i \leq n$. Objets in EHM_n^R are called *Nori n-motives*. When $R = \mathbb{Z}$, we simply write EHM_n (or $\mathsf{EHM}_n(k)$).

2.4. **Proposition.** *In* Definition 2.3 *we can restrict to k-schemes of dimension at most n, i.e., EHM_n^R is the thick abelian subcategory of EHM^R generated by $\widetilde{H}_i(X, Y; R)$ where X is a k-scheme of dimension at most n, $Y \subset X$ is a closed subset and i is an integer such that $i \leq n$.*

Proof. Let $\mathcal{A} \subset \mathsf{EHM}_n^R$ be the thick abelian subcategory generated by the objects as in the statement. It is enough to show that $\widetilde{H}_i(X, Y; R) \in \mathcal{A}$ for X a k-scheme of arbitrary dimension, $Y \subset X$ a closed subset and $i \leq n$. We derive this by a standard argument relying on Lefschetz theorem (on hyperplane sections). We reproduce this argument for the sake of completeness.

We argue by induction on the dimension of X. We may assume that $\dim(X) \geq n + 1$; otherwise there is nothing to prove. We may also assume that $\dim(Y) < \dim(X)$. (Indeed, replacing X and Y by the closures of the complements of the common irreducible components does not change the relative homology.) If $Y' \subset X$ is a closed subset containing Y, we have a short exact sequence

$$\widetilde{H}_i(Y', Y; R) \to \widetilde{H}_i(X, Y; R) \to \widetilde{H}_i(X, Y'; R).$$

If $\dim(Y') < \dim(X)$, it suffices by induction to treat the case of $\widetilde{H}_i(X, Y'; R)$. In other words, we may enlarge Y and assume that $X \setminus Y$ is smooth and that there exists a blow-up with center contained in Y rendering X a quasi-projective scheme. (This is possible by Chow's lemma.)

Next, given a blow-up $X_1 \to X$ inducing an isomorphism $X_1 \setminus Y_1 \to X \setminus Y$, with $Y_1 = X_1 \times_X Y$, we have an isomorphism $\widetilde{H}_i(X_1, Y_1; R) \simeq \widetilde{H}_i(X, Y; R)$. Thus, using Chow's lemma and Hironoka's resolution of singularities, we may assume that X is smooth and quasi-projective.

Moreover, we may fix an embedding $X \hookrightarrow \overline{X}$ into a smooth and projective k-scheme $\overline{X}$ such that $X_\infty = \overline{X} \smallsetminus X$ is a simple normal crossing divisor.

Now, using the exact sequence

$$\widetilde{H}_i(X, \emptyset; R) \to \widetilde{H}_i(X, Y; R) \to \widetilde{H}_{i-1}(Y, \emptyset; R)$$

and induction, it is enough to show that $\widetilde{H}_i(X, \emptyset; R) \in \mathcal{A}$. Let $\overline{Z} \subset \overline{X}$ be a smooth ample divisor meeting transversally the divisor X_∞. Letting $Z = X \cap \overline{Z}$, we claim that the map

$$\widetilde{H}_i(Z, \emptyset; R) \to \widetilde{H}_i(X, \emptyset; R)$$

is surjective if $i = \dim(X) - 1$ (resp. bijective if $i < \dim(X) - 1$); this will finish the proof. It is enough to check this after applying the forgetful functor $\mathsf{EHM} \to R\text{-mod}$. We argue by induction on the dimension of X and the number of irreducible components in X_∞. If $X_\infty = \emptyset$, then the claim is simply Lefschetz hyperplane theorem. Otherwise, let $\overline{D} \subset X_\infty$ be an irreducible component, $E \subset X_\infty$ the union of the remaining irreducible components and $D = \overline{D} \smallsetminus E$. We then have a commutative diagram

$$
\begin{array}{ccccccc}
H_{i-1}(D; R) & \longrightarrow & H_i(X; R) & \longrightarrow & H_i(\overline{X} \smallsetminus E; R) & \longrightarrow & H_{i-2}(D; R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} \\
H_{i-1}(D \cap \overline{Z}; R) & \longrightarrow & H_i(Z; R) & \longrightarrow & H_i(\overline{Z} \smallsetminus E; R) & \longrightarrow & H_{i-2}(D \cap \overline{Z}; R)
\end{array}
$$

where the lines are part of Gysin long exact sequences. The first and third vertical arrows are surjective (resp. bijective) by induction. The surjectivity (resp. bijectivity) of the second vertical arrow follows by a simple diagram chasing. $\square$

2.5. Consider the full sub-diagram $^D(Sch_k)_{\leq n} \subset {}^D(Sch_k)$ consisting of triples (X, Y, i) with $i \leq n$. As in 2.2 we set

$$\mathsf{EHM}_n'^R := \mathcal{C}\left(H_* : {}^D(Sch_k)_{\leq n} \to R\text{-mod}\right). \tag{2.3}$$

(When $R = \mathbb{Z}$, we simply write EHM_n'. If we wish to stress the dependence on the field k, we write $\mathsf{EHM}_n'^R(k)$ and $\mathsf{EHM}_n'(k)$.) For $i \leq n$, we also denote by $\widetilde{H}_i'(X, Y; R)$ the object in $\mathsf{EHM}_n'^R$ associated to (X, Y, i) by the universal representation

$$\widetilde{H}_*' : {}^D(Sch_k)_{\leq n} \to \mathsf{EHM}_n'^R.$$

By universality, there is a faithful exact functor

$$\mathsf{EHM}_n'^R \to \mathsf{EHM}_n^R. \tag{2.4}$$

2.6. **Remark.** We conjecture that (2.4) is an equivalence of categories although we expect this to be a difficult problem. For $n \in \{0, 1\}$, this conjecture follows from Theorems 3.4 and 5.1 at least when $R = \mathbb{Z}$.

2.7. **Lemma.** *Let $n \leq n'$ be non-negative integers. There are (faithful, exact) functors* $\mathsf{EHM}_n'^R \to \mathsf{EHM}_{n'}'^R$ *inducing an equivalence*

$$2 - \operatorname*{Colim}_{n \geq 0} \mathsf{EHM}_n'^R \cong \mathsf{EHM}^R.$$

Proof. The first claim in clear and the second is a particular case of (1.2). $\square$

2.8. **Proposition.** *The category $\mathsf{EHM}_n'^R$ is generated, as a thick abelian subcategory of itself, by $\widetilde{H}_i'(X, Y; R)$ where X is a k-scheme of dimension at most n, $Y \subset X$ is a closed subset and i is an integer such that $i \leq n$.*

Proof. A proof is obtained by replacing $\widetilde{H}_i$ by $\widetilde{H}_i'$ everywhere in the proof of Proposition 2.4. $\square$

3. The case of Artin motives

In this section, we consider the case $n = 0$.

3.1. Let $\mathcal{M}_0^R = \mathcal{M}_0^R(k)$ be the category of 0-motives (with coefficients in R), *i.e.*, R-constructible étale sheaves on Et_k, the small étale site of k. (Recall that a sheaf on Et_k is R-constructible if and only if it is locally constant and its stalks are finitely generated R-modules.) Clearly, $\mathcal{M}_0^R$ is abelian. Moreover, the fiber functor associated to the geometric point given by the complex embedding $\iota : k \hookrightarrow \mathbb{C}$, yields a forgetful functor

$$\iota^* : \mathcal{M}_0^R \to R\text{-mod}. \tag{3.1}$$

Note the following observation.

3.2. **Lemma.** *There exist a representation $S : {}^D(Sch_k)_{\leq 0} \to \mathcal{M}_0^R$ and an isomorphism of representations $\iota^* \circ S \simeq H_*$.*

Proof. An objet of ${}^D(Sch_k)_{\leq 0}$ consists of a triple $(X, Y, 0)$. We define

$$S(X, Y, 0) := \frac{\pi_0(X) \otimes R}{\pi_0(Y) \otimes R}.$$

(In the above formula, $\pi_0(Z)$ denotes the étale k-scheme of geometric connected components of a k-scheme Z and, for $V \in Et_k$, $V \otimes R$ is the étale sheaf associated to the presheaf: $U \in Et_k \rightsquigarrow \bigoplus_{U \to V} R$.)

The verification that S gives a representation is easy and is left to the reader. Almost from the construction we have

$$\iota^* \circ S(X, Y, 0) = \frac{\bigoplus_{x \in \pi_0(X^{\mathrm{an}})} R}{\bigoplus_{y \in \pi_0(Y^{\mathrm{an}})} R}.$$

This gives the required identification $\iota^* \circ S = H_*$.   □

From Lemma 3.2 and universality (*i.e.*, [15, Theorem 41]) we get a canonical functor:

$$\mathsf{EHM}_0'^R \to \mathcal{M}_0^R. \tag{3.2}$$

3.3. **Proposition.** *The functor* (3.2) *is an equivalence of categories.*

Proof. We will use Proposition 1.7. Only the condition 1.7(c) needs to be discussed.

So let $\mathcal{F}$ be an R-constructible étale sheaf on Et_k and let

$$\alpha : \frac{\pi_0(X) \otimes R}{\pi_0(Y) \otimes R} \to \mathcal{F}$$

be a morphism. Let l/k be a finite Galois extension which trivializes $\pi_0(X)$, $\pi_0(Y)$ and $\mathcal{F}$. Let E be the full sub-diagram whose objects are $(\mathrm{Spec}(l), \emptyset, 0)$ and $(X, Y, 0)$. Then it is easy to see that $\mathrm{End}(T|_E) = R[\mathrm{Gal}(l/k)]$, the group algebra of the Galois group. The map $\iota^*(\alpha)$ being $\mathrm{Gal}(l/k)$-equivariant, it follows that $\mathrm{Ker}(\iota^*(\alpha))$ is a sub-$\mathrm{End}(T|_E)$-module. (A similar argument appears in [11, §6.1].)   □

Using Proposition 3.3, we define a functor from 0-motives to Nori 0-motives:

$$\nu_0 : \mathcal{M}_0^R \simeq \mathsf{EHM}_0'^R \to \mathsf{EHM}_0^R. \tag{3.3}$$

For simplicity, we state the following result only for $R = \mathbb{Z}$:

3.4. **Theorem.** *The functor* $\nu_0 : \mathcal{M}_0 \to \mathsf{EHM}_0$ *is an equivalence of categories.*

Proof. We will derive this result as a corollary of Theorem 5.1 which is the main theorem of this paper and the analogous result for 1-motives; as the conclusion of Theorem 3.4 is not used later in the paper, this won't result in a circular argument.

We claim that the following square is commutative up to a natural isomorphism

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{\nu_0} & \mathsf{EHM}_0 \\ \downarrow & & \downarrow \\ {}^t\mathcal{M}_1 & \xrightarrow{\nu_1} & \mathsf{EHM}_1. \end{array}$$

This will finish the proof. Indeed, modulo the equivalence ν_1 (from Theorem 5.1), the subcategory $\mathsf{EHM}_0 \subset \mathsf{EHM}_1$ corresponds to $\mathcal{M}_0 \subset {}^t\mathcal{M}_1$.

From the construction of the functors ν_0 and ν_1, we are left to check the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathsf{EHM}'_0 & \xrightarrow{\;\sim\;} & \mathcal{M}_0 \\
\downarrow & & \downarrow \\
\mathsf{EHM}'_1 \xleftarrow{\;\sim\;} \mathsf{EHM}''_1 & \xrightarrow{\;\sim\;} & {}^t\mathcal{M}_1.
\end{array}
$$

By the universality of Nori's construction, it is enough to construct a natural isomorphism between the two representations of ${}^D(Sch_k)_{\leq 0}$ into ${}^t\mathcal{M}_1$ deduced from the above diagram. Given a triple $(X, Y, 0) \in {}^D(Sch_k)_{\leq 0}$, there is a canonical isomorphism

$$S(X, Y, 0) \simeq \mathrm{Coker}\{A(\mathbb{A}^1_{\pi_0(Y)}, \{0,1\}, 1) \to A(\mathbb{A}^1_{\pi_0(X)}, \{0,1\}, 1)\}$$

(*cf.* §4.3 for the definition of the representation A). The left hand side is the image of $(X, Y, 0)$ by the composition of

$$
{}^D(Sch_k)_{\leq 0} \to \mathsf{EHM}'_0 \simeq \mathcal{M}_0 \hookrightarrow {}^t\mathcal{M}_1
$$

while the right hand side is the image of $(X, Y, 0)$ by the composition of

$$
{}^D(Sch_k)_{\leq 0} \hookrightarrow {}^D(Sch_k)_{\leq 1} \to \mathsf{EHM}'_1 \simeq \mathsf{EHM}''_1 \simeq {}^t\mathcal{M}_1.
$$

This finishes the proof. $\qquad\square$

3.5. **Remark.** It is certainly possible to give a direct (and more natural) proof of Theorem 3.4 which avoid the use of 1-motives. However, such a proof will necessarily follow the same strategy as the proof of Theorem 5.1 and will use 0-motivic versions of some of the intermediate results (such as Theorem 7.10). For the sake of not repeating some arguments twice, we decided not to include the "natural" proof of Theorem 3.4.

4. Deligne 1-motives as a universal category

In this section, we concentrate on the category EHM''_1 which we will define below. Our main goal is to identify it with the abelian category of 1-motives with torsion ${}^t\mathcal{M}_1$ (see [5, §1] and [4, Appendix C]).

4.1. **A smaller diagram of curves.** Let ${}^D(Crv_k)$ be the full sub-diagram of ${}^D(Sch_k)$ consisting of triples $(C, Z, 1)$ where C is a smooth affine curve and $Z \subset C$ a closed subset consisting of finitely many closed points.

By restriction, we get a representation $H_* : {}^D(Crv_k) \to R\text{-mod}$. We set

$$\mathsf{EHM}_1''^R = \mathcal{C}\left(H_* : {}^D(Crv_k) \to R\text{-mod}\right).$$

(As usual, when $R = \mathbb{Z}$, we simply write EHM_1''. If we want to stress the dependence on the field k, we write $\mathsf{EHM}_1''^R(k)$ and $\mathsf{EHM}_1''(k)$.) The image in $\mathsf{EHM}_1''^R$ of an object $(C, Z, 1) \in {}^D(Crv_k)$ is denoted by $\widetilde{H}_1''(C, Z; R)$. By universality, we get faithful and exact functors

$$\mathsf{EHM}_1''^R \to \mathsf{EHM}_1'^R \to \mathsf{EHM}_1^R. \qquad (4.1)$$

4.2.   Let ${}^t\mathcal{M}_1 = {}^t\mathcal{M}_1(k)$ be the abelian category of 1-motives with torsion [5, §1]. Recall that a 1-motive with torsion is a complex of commutative group schemes $[\mathcal{F} \to \mathcal{G}]$ where $\mathcal{G}$ is a semi-abelian variety and $\mathcal{F}$ is a lattice with torsion, *i.e.*, a $\mathbb{Z}$-constructible étale sheaf on Et_k (as in 3.1) considered as a group scheme in the obvious way.

Given an étale k-scheme V, we denote by $\mathbb{Z}_{tr}(V)$ the group scheme such that, for every k-scheme X, $\mathbb{Z}_{tr}(V)(X)$ is the free abelian group on the set of connected components of $X \times_k V$. This corresponds to the $\mathbb{Z}$-constructible étale sheaf $V \otimes \mathbb{Z}$ used in the proof of Lemma 3.2.

4.3.   We have a representation

$$A : {}^D(Crv_k) \to {}^t\mathcal{M}_1$$

given by

$$(C, Z, 1) \rightsquigarrow A(C, Z) := [\mathrm{Div}_Z^0(C) \to \mathrm{Alb}^0(C)]$$

where $\mathrm{Div}_Z^0(C)$ is the lattice $\mathrm{Ker}\{\mathbb{Z}_{tr}(Z) \to \mathbb{Z}_{tr}(\pi_0(C))\}$ and $\mathrm{Alb}^0(C)$ is the connected component of the identity of the Serre-Albanese scheme $\mathrm{Alb}(C)$ of C. (Note that $A(C, Z) = \mathrm{Alb}^-(C, Z)$ with the notation adopted in [6] and $A(C, Z) \cong \mathrm{L}_1\mathrm{Alb}(C, Z)$ according to [4].) On the other hand, we have a functor

$$T_{\mathbb{Z}} : {}^t\mathcal{M}_1 \to \mathbb{Z}\text{-mod} \qquad (4.2)$$

given as follows:

$$T_{\mathbb{Z}}([\mathcal{F} \to \mathcal{G}]) = \mathcal{F}(\mathbb{C}) \times_{\mathcal{G}(\mathbb{C})} \mathrm{Lie}\,\mathcal{G}(\mathbb{C})$$

This is a forgetful functor. Moreover, we have:

4.4. **Lemma.** *There is a canonical isomorphism $H_* \simeq T_{\mathbb{Z}} \circ A$ between representations of ${}^D(Crv_k)$.*

Proof. This is classical: see [6, Proposition 3.1.2 & §5.3]. For the sake of completeness, we recall the description of the isomorphism for $(C, Z, 1) \in {}^D(Crv_k)$ assuming $k = \mathbb{C}$, Z non-empty, and C connected and not isomorphic to the affine line. The last assumption is to insure

that there is an embedding $C \hookrightarrow \mathrm{Alb}^1(C)$ where $\mathrm{Alb}^1(C) \subset \mathrm{Alb}(C)$ is the connected component consisting of 0-cycles of degree 1. The group $H_1(C, Z; \mathbb{Z})$ is generated by classes of paths $\gamma : [0, 1] \to C^{\mathrm{an}}$ such that $\gamma(0), \gamma(1) \in Z$. Let $\gamma^1 : [0, 1] \to \mathrm{Alb}^1(C)^{\mathrm{an}}$ be the composition of γ with the canonical embedding. Fix a universal cover $U \to \mathrm{Alb}^1(C)^{\mathrm{an}}$; then U is naturally a torsor over the vector space $V = \mathrm{Lie}\,\mathrm{Alb}^0(C)$. If $\widetilde{\gamma} : [0, 1] \to U$ is any lift of γ^1, then $\widetilde{\gamma}(1) - \widetilde{\gamma}(0) \in V$ is well-defined. Moreover, it is an element of

$$\{\gamma^1(1) - \gamma^1(0)\} \times_{\mathrm{Alb}^0(C)^{\mathrm{an}}} V \subset \mathrm{Div}_Z^0(C) \times_{\mathrm{Alb}^0(C)^{\mathrm{an}}} V = T_{\mathbb{Z}}(A(C, Z)).$$

This yields the map $H_1(C, Z; \mathbb{Z}) \to T_{\mathbb{Z}}(A(C, Z))$ which is easily checked to be an isomorphism. $\qquad\square$

By Lemma 4.4 and universality (*i.e.*, [15, Theorem 41]) there is a canonical, faithful and exact functor

$$\mathsf{EHM}_1'' \to {}^t\mathcal{M}_1. \tag{4.3}$$

We can now state one of the key results of this paper.

4.5. **Theorem.** *The functor* (4.3) *is an equivalence of categories.*

From now on, we only use the coefficients rings $R = \mathbb{Z}$ or $R = \mathbb{Q}$. As a particular case of (1.3), we have $\mathsf{EHM}_n''^{\mathbb{Q}} = \mathsf{EHM}_n'' \otimes \mathbb{Q}$. We denote by $\mathcal{M}_1^{\mathbb{Q}} = {}^t\mathcal{M}_1 \otimes \mathbb{Q}$, the abelian category of 1-motives where isogenies are inverted; we have a forgetful functor $T_{\mathbb{Q}} : \mathcal{M}_1^{\mathbb{Q}} \to \mathbb{Q}\text{-}\mathsf{mod}$ induced from (4.2).

Let $\mathcal{R}_1$ be the pro-finite dimensional $\mathbb{Q}$-algebra of endomorphisms of the representation $H_* : {}^D(Crv_k) \to \mathbb{Q}\text{-}\mathsf{mod}$ (see 1.5).

4.6. **Proposition.** *Let* $(C, Z, 1)$ *be in* ${}^D(Crv_k)$ *and* $\alpha : A(C, Z) \to M$ *a morphism in* $\mathcal{M}_1^{\mathbb{Q}}$. *Then, the kernel of*

$$H_1(C, Z; \mathbb{Q}) \simeq T_{\mathbb{Q}} \circ A(C, Z) \xrightarrow{T_{\mathbb{Q}}(\alpha)} T_{\mathbb{Q}}(M)$$

is a sub-$\mathcal{R}_1$-module of $\widetilde{H}_1''(C, Z; \mathbb{Q})$.

The proof is contained in 4.15 below and it relies on Lemmas 4.8, 4.9, 4.11, 4.12, 4.13 and 4.14 below.

Once Proposition 4.6 is proven, Theorem 4.5 will follow from Proposition 1.10.

4.7. We break the proof of Proposition 4.6 into small steps. In what follows, the kernel in the statement of Proposition 4.6 will be denoted by $K(C, Z, \alpha)$; it is a subspace of $H_1(C, Z; \mathbb{Q})$.

4.8. Lemma. *Assume that $M = N \oplus N'$ (in $\mathcal{M}_1^{\mathbb{Q}}$) and let β and β' be the corresponding components of α. If* Proposition 4.6 *holds for β and β' then it holds for α.*

Proof. Indeed, we have $K(C, Z, \alpha) = K(C, Z, \beta) \cap K(C, Z, \beta')$. $\qquad\square$

4.9. Lemma. *Let $f : (D, T, 1) \to (C, Z, 1)$ be an arrow in ${}^D(Crv_k)$ such that $f : D \to C$ is dominant and the image of the induced morphism*

$$f_* : \mathrm{Div}_T^0(D) \to \mathrm{Div}_Z^0(C)$$

has a finite index. Then Proposition 4.6 *for $\alpha : A(C, Z) \to M$ follows from the case of $\alpha \circ A(f) : A(D, T) \to M$.*

Proof. Indeed, the hypothesis of the lemma imply that the homomorphism $H_1(D, T; \mathbb{Q}) \to H_1(C, Z, \mathbb{Q})$ is surjective. If follows that $K(C, Z, \alpha)$ is the image of $K(D, T, \alpha')$ with $\alpha' = \alpha \circ A(f)$. This finishes the proof. $\qquad\square$

4.10. Remark. Assume that $f : D \to C$ induces a bijection on the sets of connected components and that $f(T) = Z$. Then the hypothesis of Lemma 4.9 are satisfied.

4.11. Lemma. Proposition 4.6 *holds when M is a 0-motive, i.e., $M = [\mathcal{F} \to 0]$.*

Proof. Using Lemma 4.8, we may assume that the lattice $\mathcal{F}$ is simple (as an object of $\mathcal{M}_0^{\mathbb{Q}}$). Therefore, there exists a finite extension l/k such that $\mathcal{F}$ is a direct factor of $\mathbb{Z}_{tr}(l)$ (in $\mathcal{M}_0^{\mathbb{Q}}$). Thus, we may assume that $\mathcal{F} = \mathbb{Z}_{tr}(l)$. We may also enlarge l and assume that l is Galois and contains the residue field of every point in Z. Let $C' = C \otimes_k l$ and $Z' = Z \otimes_k l$. By Lemma 4.9, we may replace (C, Z) by (C', Z'). In other words, we may assume that C is defined over l and every point of Z is rational over l.

Now, $\mathrm{Div}_Z^0(C)$ is the kernel of $\mathbb{Z}_{tr}(Z) \to \mathbb{Z}_{tr}(\pi_0(C))$. Hence, it is a direct factor of $\mathbb{Q}_{tr}(Z)$. On the other hand, $\mathrm{Hom}(\mathbb{Z}_{tr}(Z), \mathbb{Z}_{tr}(l))$ has a basis which is indexed by (z, σ) where z is a point of Z and $\sigma : k(z) \simeq l$ is a k-isomorphism. A couple (z, σ) corresponds to the composition of

$$u(z, \sigma) : \mathbb{Z}_{tr}(Z) \to \mathbb{Z}_{tr}(k(z)) \overset{\sigma}{\simeq} \mathbb{Z}_{tr}(l).$$

It follows that $\alpha_0 : \mathrm{Div}_Z^0(C) \to \mathbb{Z}_{tr}(l)$ can be written as the composition of $\mathrm{Div}_Z^0(C) \hookrightarrow \mathbb{Z}_{tr}(Z)$ and a linear combination $\sum_{(z,\sigma)} a_{z,\sigma} \cdot u(z, \sigma)$ with $a_{z,\sigma} \in \mathbb{Q}$. (Recall that α is a morphism in $\mathcal{M}_1^{\mathbb{Q}}$.)

We claim that, for a fixed (z, σ), the composition of

$$H_1(C, Z; \mathbb{Q}) \simeq T_{\mathbb{Q}} A(C, Z) \overset{u(z,\sigma)}{\longrightarrow} T_{\mathbb{Q}}(\mathbb{Z}_{tr}(l)) \simeq H_0(\mathrm{Spec}(l); \mathbb{Q}) \qquad (4.4)$$

coincides with a morphism obtained from a zigzag in the diagram $^D(Crv_k)$ modulo the isomorphism $H_0(\mathrm{Spec}(l); \mathbb{Q}) \simeq H_1(\mathbb{A}_l^1, \{0,1\}; \mathbb{Q})$. To show this, we need a construction. Let $z_0 \in Z$ be a point different from z. (We may assume that $Z \neq \{z\}$ because otherwise, the morphism (4.4) is necessarily zero.) Consider a finite morphism $C \to \mathbb{A}_l^1$ which is injective on Z and sends z_0 to the zero section. Denote by $T \subset \mathbb{A}_l^1$ the image of Z and $t \in T$ the image of z. Then our zigzag is the following:

$$(C, Z, 1) \to (\mathbb{A}_l^1, T, 1) \leftarrow (\mathbb{A}_l^1, \{0,t\}, 1) \sqcup (\mathbb{A}_l^1, T \smallsetminus \{t\}, 1) \xrightarrow{(*)}$$

$$(\mathbb{A}_l^1, \{0,t\}, 1) \sqcup (\mathbb{A}_l^1, \{0\}, 1) \leftarrow (\mathbb{A}_l^1, \{0,t\}, 1) \xrightarrow{t^{-1}} (\mathbb{A}_l^1, \{0,1\}, 1)$$

where the arrow $(*)$ is given by the identity on the first factor and by the zero morphism on the second factor. It follows that (4.4) is a morphism of $\mathcal{R}_1$-modules. This proves that

$$H_1(C, Z; \mathbb{Q}) \simeq T_{\mathbb{Q}} A(C, Z) \xrightarrow{\alpha} T_{\mathbb{Q}}(\mathbb{Z}_{tr}(l)) \simeq H_0(\mathrm{Spec}(l); \mathbb{Q})$$

is also a morphism of $\mathcal{R}_1$-modules. Hence, its kernel is a sub-$\mathcal{R}_1$-module of $H_1(C, Z; \mathbb{Q})$. $\qquad\square$

4.12. Lemma. *Let $(C, Z, 1)$ be an object of $^D(Crv_k)$ and let*

$$\beta : [\mathcal{L} \to 0] \to A(C, Z)$$

be a morphism in $\mathcal{M}_1^{\mathbb{Q}}$ from a lattice $\mathcal{L}$. Then, the image of the composition

$$T_{\mathbb{Q}}\mathcal{L} \to T_{\mathbb{Q}}A(C, Z) \simeq H_1(C, Z; \mathbb{Q})$$

is a sub-$\mathcal{R}_1$-module of $H_1(C, Z; \mathbb{Q})$.

Proof. It suffices to consider the case where $\mathcal{L}$ is simple in $\mathcal{M}_0^{\mathbb{Q}}$. In this case, $\mathcal{L}$ is a direct factor of $\mathbb{Z}_{tr}(l)$ (in $\mathcal{M}_0^{\mathbb{Q}}$). Thus, we may assume that $\mathcal{L} = \mathbb{Z}_{tr}(l)$. Then a multiple of β corresponds to a morphism of lattices

$$\beta_0 : \mathbb{Z}_{tr}(l) \to \mathbb{Z}_{tr}(Z)$$

whose image is contained in $\mathrm{Div}_Z^0(C)$ and such that the composition

$$\mathbb{Z}_{tr}(l) \to \mathrm{Div}_Z^0(C) \to \mathrm{Alb}^0(C)$$

is zero. Therefore we can find a finite correspondence $\gamma : \mathbb{A}_l^1 \to C$ such that $\gamma \circ (i_1 - i_0) = j_Z \circ \beta_0$ where $j_Z : Z \hookrightarrow C$ is the inclusion.

Recall that we want to show that the image of

$$T_{\mathbb{Q}}(\mathbb{Z}_{tr}(l)) \xrightarrow{\beta} T_{\mathbb{Q}}A(C, Z) \simeq H_1(C, Z; \mathbb{Q})$$

is a sub-$\mathcal{R}_1$-module. For this, we are free to add to Z any closed subset $T \subset C \setminus Z$ of dimension 0. We may find a finite Galois cover $c : E \to \mathbb{A}_l^1$ such that $\gamma \circ c$ is a linear combination of maps from E to

C, i.e, $\gamma \circ c = \sum_{i=1}^{n} a_i f_i$ with $a_i \in \mathbb{Z}$. We take $T = \bigcup_{i=1}^{n} f_i(F) \setminus Z$ with $F = c^{-1}\{0,1\}$.

Now, let G be the Galois group of $c : E \to \mathbb{A}_l^1$. Then G acts on the $\mathcal{R}_1$-module $\widetilde{H}_1''(E, F; \mathbb{Q})$ and the canonical map

$$H_1(E, F; \mathbb{Q}) \to H_1(\mathbb{A}_l^1, \{0,1\}; \mathbb{Q})$$

identifies $H_1(\mathbb{A}_l^1, \{0,1\}; \mathbb{Q})$ with the sub-$\mathcal{R}_1$-modules of invariants. On the other hand, the maps

$$f_{i*} : H_1(E, F; \mathbb{Q}) \to H_1(C, Z \cup T; \mathbb{Q})$$

are $\mathcal{R}_1$-linear. Hence, the image of $H_1(E, F; \mathbb{Q})^G$ by $\sum_{i=1}^{n} a_i f_{i*}$ is a sub-$\mathcal{R}_1$-module of $\widetilde{H}_1''(C, Z \cup T; \mathbb{Q})$. By construction, it coincides with the image of the composition of

$$T_{\mathbb{Q}}(\mathbb{Z}_{tr}(l)) \xrightarrow{\beta} T_{\mathbb{Q}} A(C, Z \cup T) \simeq H_1(C, Z \cup T; \mathbb{Q}).$$

This finishes the proof of the lemma.   □

We still need some more lemmas.

4.13. **Lemma.** *It suffices to prove* Proposition 4.6 *when* $M = [\mathcal{F} \to \mathrm{Alb}^0(D)]$ *with* D *a smooth affine curve.*

Proof. Let $M = [\mathcal{F} \to \mathcal{G}]$ be a general 1-motive. It is enough to show that $\mathcal{G}$ embeds, up to isogeny, into $\mathrm{Alb}^0(D)$ with D a smooth and affine curve. Indeed, letting $N = [\mathcal{F} \to \mathrm{Alb}^0(D)]$, one gets a monomorphism (in $\mathcal{M}_1^{\mathbb{Q}}$) of 1-motives $u : M \to N$ and the equality $K(C, Z, \alpha) = K(C, Z, u \circ \alpha)$ holds.

We now construct a monomorphism, up to isogeny, of group schemes $\mathcal{G} \hookrightarrow \mathrm{Alb}^0(D)$. If there is an isogeny between $\mathcal{G}$ and a product $\mathcal{G}_1 \times \mathcal{G}_2$, it is enough to consider $\mathcal{G}_1$ and $\mathcal{G}_2$ separately. (If $\mathcal{G}_i \subset \mathrm{Alb}^0(D_i)$ for $i \in \{1, 2\}$, just take $D = D_1 \sqcup D_2$.) If $\mathcal{G}$ is a torus, which splits over a finite extension l/k, one can embed $\mathcal{G}$ into a product of tori of the form $\mathbb{G}_m \otimes \mathbb{Z}_{tr}(l) = \mathrm{Alb}^0(\mathbb{A}_l^1 \setminus \{0\})$. Therefore, it remains to treat the case where $\mathcal{G}$ is not isogenous to a product where one of the factor is a non-trivial torus. This is equivalent to say that $\mathcal{G}$ has no non-trivial map to a torus. Consider the dual 1-motive $\mathcal{G}^{\vee}$. It is of the form $[\mathcal{L} \to \mathcal{A}]$ with $\mathcal{A}$ an abelian variety and $\mathcal{L}$ a torsion-free lattice. Our condition on $\mathcal{G}$ implies that the map $\mathcal{L} \to \mathcal{G}$ is a monomorphism of schemes. By drawing a general smooth curve $\overline{D} \subset \mathcal{A}$ containing generators of $\mathcal{L}$, we obtain a pair $(\overline{D}, T)$ with a surjective morphism $A(\overline{D}, T) \twoheadrightarrow \mathcal{G}^{\vee}$. Dualizing back, we see that $\mathcal{G}$ injects inside $\mathrm{Alb}^0(D)$ with $D = \overline{D} - T$.   □

From now on, we assume that $M = [\mathcal{F} \to \mathrm{Alb}^0(D)]$ with D a smooth affine curve. A multiple of the morphism $\alpha : A(C, Z) \to M$ induces a morphism of semi-abelian varieties $\alpha_1 : \mathrm{Alb}^0(C) \to \mathrm{Alb}^0(D)$.

4.14. **Lemma.** *To prove* Proposition 4.6, *we may assume that α_1 is induced by a linear combination of maps from C to D. In other words, we may assume that $\alpha_1 = \sum_{i=1}^{n} a_i \mathrm{Alb}^0(f_i)$ for some maps $f_i : C \to D$ and integers $a_i \in \mathbb{Z}$.*

Proof. Let l/k be a finite Galois extension with Galois group G. Consider the morphism of 1-motives

$$\alpha' = \alpha \otimes \mathbb{Z}_{tr}(l) : A(C \otimes_k l, Z \otimes_k l) \simeq A(C, Z) \otimes \mathbb{Z}_{tr}(l) \to M \otimes \mathbb{Z}_{tr}(l).$$

Clearly, this is a G-equivariant morphism. It follows that G acts on $K(C \otimes_k l, Z \otimes_k l, \alpha')$ and that the space of G-invariants identifies with $K(C, Z, \alpha)$. Thus, it is enough to prove Proposition 4.6 for α'. Using this, we may assume that both projections $C \to \pi_0(C)$ and $D \to \pi_0(D)$ have sections.

The section of $C \to \pi_0(C)$ is used to construct a retraction $r : \mathrm{Alb}(C) \to \mathrm{Alb}^0(C)$ to the natural inclusion. (This is needed in (4.7) below.)

Let $h_0(D)$ the homotopy invariant presheaf with transfers (on the category Sm_k of smooth k-schemes) associated to $\mathbb{Z}_{tr}(D)$. There is an obvious morphism of homotopy invariant presheaves with transfers

$$h_0(D) \to \mathrm{Alb}(D). \tag{4.5}$$

The section $D \to \pi_0(D)$ is used to ensure that (4.5) induces an isomorphism on finitely generated extensions K/k. Indeed, to check this property, we may assume without loss of generality that $k = K$, *i.e.*, it is enough to check that $h_0(D)(k) \to \mathrm{Alb}(D)(k)$ is an isomorphism. The group $h_0(D)(k)$ is canonically isomorphic to the relative Picard group $Pic(\overline{D}, D_\infty)$ where $\overline{D}$ is a smooth compactification of D and $D_\infty = \overline{D} \smallsetminus D$. Using the exact sequences

$$0 \to \frac{\mathcal{O}^\times(D_\infty)}{\mathcal{O}^\times(\pi_0(D))} \to Pic(\overline{D}, D_\infty) \to Pic(\overline{D}) \to 0$$

and

$$0 \to \frac{\mathcal{O}^\times(D_\infty)}{\mathcal{O}^\times(\pi_0(D))} \to \mathrm{Alb}(D)(k) \to \mathrm{Alb}(\overline{D})(k)$$

it is enough to show that $Pic(\overline{D}) \to \mathrm{Alb}(\overline{D})(k)$ is surjective. Clearly, $\mathrm{Alb}(\overline{D})$ is equal to the Picard variety $\mathrm{Pic}(\overline{D})$ of $\overline{D}$. Moreover, we have the well-known exact sequence (see for example [7, p. 203]):

$$Pic(\overline{D}) \to \mathrm{Pic}(\overline{D})(k) \to H^2_{\text{ét}}(\pi_0(\overline{D}), \mathbb{G}_m) \to H^2_{\text{ét}}(\overline{D}, \mathbb{G}_m).$$

The existence of a section to $D \to \pi_0(D)$ implies that the last morphism is injective. Thus, the first morphism is surjective as needed.

Applying [16, Proposition 11.1] to the kernel and cokernel of (4.5) and using the previous discussion, we deduce an isomorphism

$$h_0(D)(C_Z) \simeq \mathrm{Alb}(D)(C_Z) \tag{4.6}$$

where C_Z is the spectrum of the semi-local ring of C at the points of Z. Now, the left hand side in (4.6) is the group of finite correspondences from C_Z to D up to homotopy. Taking the inverse image of the element $\psi \in \mathrm{Alb}(D)(C_Z)$ given by the composition

$$\psi : C_Z \hookrightarrow C \xrightarrow{r} \mathrm{Alb}^0(C) \xrightarrow{\alpha_1} \mathrm{Alb}^0(D) \hookrightarrow \mathrm{Alb}(D) \tag{4.7}$$

we arrive at the following conclusion. There exists a dense open neighborhood C' of Z in C and a finite correspondence $\gamma \in Cor(C', D)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Alb}^0(C') & \longrightarrow & \mathrm{Alb}^0(C) \\
& \underset{\mathrm{Alb}^0(\gamma)}{\searrow} & \downarrow{\scriptstyle \alpha_1} \\
& & \mathrm{Alb}^0(D).
\end{array}
$$

By Lemma 4.9, we may replace C by C'. In other words, we may assume that α_1 itself is induced by a correspondence $\gamma \in Cor(C, D)$.

To finish the proof, we choose a finite cover $r : C'' \to C$ such that $\gamma \circ r$ is a linear combination of morphisms. Using Lemma 4.9, we may replace C by C'' and Z by $r^{-1}(Z)$. In particular, we may indeed assume that $\gamma = \sum_{i=1}^n a_i f_i$ where $a_i \in \mathbb{Z}$ and $f_i : C \to D$. $\qquad\square$

4.15. **Proof of Proposition 4.6.** We are now ready to complete the proof of Proposition 4.6. First, remark that we may assume that $\mathcal{F} \to \mathrm{Alb}^0(D)$ is injective. Indeed, if $\mathcal{N}$ is the kernel of this morphism and $\mathcal{I}$ its image, there is a (non-canonical) decomposition

$$M = [\mathcal{N} \to 0] \oplus [\mathcal{I} \hookrightarrow \mathrm{Alb}^0(D)]$$

in $\mathcal{M}_1^{\mathbb{Q}}$. We then apply Lemmas 4.8 and 4.11 (for the 0-motive $\mathcal{N}$) to conclude.

Arguing as in the beginning of the proof of Lemma 4.13, we may replace $(C, Z, 1)$ by $(C \otimes_k l, Z \otimes_k l, 1)$ and M by $M \otimes \mathbb{Z}_{tr}(l)$ for any finite Galois extension l/k. Therefore, we may assume that there exists such l/k, with Galois group G, such that the following properties are satisfied:

- $\mathcal{F} = \bigoplus_{s=1}^r \mathbb{Z}_{tr}(l)e_s$ where $e_s \in \mathcal{F}(l)$ form a basis of the $\mathbb{Z}[G]$-module $\mathcal{F}(l)$.

- The image of e_s in $\mathrm{Alb}^0(D)(l)$ is represented by a 0-cycle g_s of $D \otimes_k l$.

Let $T \subset D$ be a finite set of closed points containing the supports of the 0-cycles g_s's and $\bigcup_{i=1}^n f_i(Z)$. There is a morphism of 1-motives

$$\delta : M \to A(D, T).$$

which is the identity on $\mathrm{Alb}^0(D)$. The induced morphism on lattices $\mathcal{F} = \bigoplus_{s=1}^r \mathbb{Z}_{tr}(l)e_s \to \mathrm{Div}_T^0(D)$ sends e_s to the 0-cycle g_s. Also, the arrows $f_i : (C, Z, 1) \to (D, T, 1)$ in $^D(Crv_k)$ induces a morphism of 1-motives

$$\gamma = \sum_{i=1}^n A(f_i) : A(C, Z) \to A(D, T).$$

However, the triangle

$$A(C, Z) \xrightarrow{\ \alpha\ } M$$
$$\ \ \ \searrow{\gamma} \qquad \downarrow{\delta}$$
$$\qquad\qquad A(D, T)$$

is not necessarily commutative. Let $\epsilon := \gamma - \delta \circ \alpha$. This is a morphism of 1-motives such that the component $\epsilon_1 : \mathrm{Alb}^0(C) \to \mathrm{Alb}^0(D)$ is zero.

In $\mathcal{M}_1^{\mathbb{Q}}$, we may decompose $\mathrm{Alb}(C, Z) = \mathcal{I} \oplus N$ where N is a 1-motive $[\mathcal{L} \hookrightarrow \mathrm{Alb}^0(C)]$ given by an injective morphism of group schemes. From our assumption on M, we have $\mathcal{I} \subset \mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\alpha|_N)$ is of the form $[\mathcal{L} \cap \mathcal{G} \hookrightarrow \mathcal{G}]$ where $\mathcal{G} = \mathrm{Ker}\{\alpha_1 : \mathrm{Alb}^0(C) \to \mathrm{Alb}^0(D)\}$. It follows that

$$\mathrm{Ker}(\gamma|_N) \subset \mathrm{Ker}(\alpha|_N). \tag{4.8}$$

Indeed, both 1-motives in (4.8) have the same semi-abelian part.

Now, consider the sub-1-motive $\mathcal{T} \subset A(D, T)$ given by

$$\gamma(\mathrm{Ker}(\alpha|_N)) = \epsilon(\mathrm{Ker}(\alpha|_N)).$$

As ϵ is zero on the semi-abelian part, we see that $\mathcal{T}$ is a lattice. Also, using (4.8), we get

$$\mathrm{Ker}(\alpha|_N) = (\gamma|_N)^{-1}(\mathcal{T}).$$

It follows that

$$\mathrm{Ker}(\alpha) = \mathcal{I} + \gamma^{-1}(\mathcal{T}). \tag{4.9}$$

Thus, we are left to show that $T_{\mathbb{Q}}(\mathcal{I})$ and $T_{\mathbb{Q}}(\gamma^{-1}(\mathcal{T}))$ are sub-$\mathcal{R}_1$-modules of $T_{\mathbb{Q}}(A(C, Z))$.

For $T_{\mathbb{Q}}(\mathcal{I})$, this follows from Lemma 4.12. For the second one, remark that $T_{\mathbb{Q}}(\gamma^{-1}(\mathcal{T}))$ is nothing but the inverse image of $T_{\mathbb{Q}}(\mathcal{T}) \subset$

$T_{\mathbb{Q}}(A(D,T))$ by the map

$$\sum_{i=1}^{n} a_i f_{i*} : H_1(C, Z; \mathbb{Q}) \to H_1(D, T; \mathbb{Q}).$$

The latter being a morphism of $\mathcal{R}_1$-modules, we may again apply Lemma 4.12 to conclude.

5. Deligne versus Nori 1-motives

Using (4.1) and Theorem 4.5, we define a functor from Deligne 1-motives to Nori 1-motives:

$$\nu_1 : {}^t\mathcal{M}_1 \simeq \mathsf{EHM}_1'' \to \mathsf{EHM}_1. \tag{5.1}$$

The main result of the paper is:

5.1. **Theorem.** *The functor* $\nu_1 : {}^t\mathcal{M}_1 \to \mathsf{EHM}_1$ *is an equivalence of categories.*

The goal of this section is to show that the functor ν_1 in (5.1) is fully faithful. In order to do this, we use that ${}^t\mathcal{M}_1$ embeds fully faithfully inside the category of mixed realizations. Then, we reduce Theorem 5.1 to showing that the essential image of ${}^t\mathcal{M}_1$ is thick in EHM (see the key Lemma 5.10 below). The proof of the latter property will be the subject of the next sections (completed in Section 7).

5.2. **Mixed realizations.** Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$. We consider a variant, which we denote by $\mathcal{MR}^{\sigma}(k)$, of the category of mixed realizations (see [10] and *cf.* [13]) where, roughly speaking, we only retain the Betti component corresponding to σ, the de Rham component, and the ℓ-adic components corresponding to the algebraic closure of $\sigma(k)$ in $\mathbb{C}$. More specifically, an objet of $\mathcal{MR}^{\sigma}(k)$ is a tuple $M := (M_{\mathrm{B}}, M_{\mathrm{dR}}, M_\ell, \dots)$ consisting of:

- a finitely generated abelian group M_{B} together with an increasing filtration $W_\bullet$ on $M_{\mathrm{B}} \otimes \mathbb{Q}$, called the weight filtration,
- a finitely generated k-vector space M_{dR} together with a decreasing filtration $F^\bullet$, called the Hodge filtration,
- for every prime ℓ, a finitely generated $\mathbb{Z}_\ell$-module M_ℓ together with a continuous action of the Galois group of $\overline{k}/k$, where $\overline{k} \subset \mathbb{C}$ is the algebraic closure of k in $\mathbb{C}$,
- a comparison isomorphism $M_{\mathrm{B}} \otimes \mathbb{C} \simeq M_{\mathrm{dR}} \otimes_k \mathbb{C}$ such that $(M_{\mathrm{B}}, M_{\mathrm{dR}} \otimes_k \mathbb{C}, W_\bullet, F^\bullet)$ is a polarizable mixed Hodge structure,
- for every prime ℓ, a comparison isomorphism $M_{\mathrm{B}} \otimes \mathbb{Z}_\ell \simeq M_\ell$.

It is known that $\mathcal{MR}^\sigma(k)$ is an abelian category (see [10] and *cf.* [13]); this is actually an easy consequence of the fact that mixed Hodge structures form an abelian category.

The following two simple remarks are useful.

5.3. Remark. Projections yield functors from $\mathcal{MR}^\sigma(k)$ to $\mathbb{Z}$-mod as well as $\mathsf{MHS} = \{\text{polarizable mixed Hodge structures}\}$ and $G_k - \mathsf{Rep}_\ell = \{\ell - \text{adic Galois representations}\}$ where $G_k = \mathrm{Gal}\,(\overline{k}/k)$. The first two functors are faithful. The third one is faithful up to ℓ'-torsion.

5.4. Remark. Given an extension k'/k and a complex embedding σ' : $k' \hookrightarrow k$ extending σ, one has a base-change functor

$$- \otimes_k k' : \mathcal{MR}^\sigma(k) \to \mathcal{MR}^{\sigma'}(k')$$

which is also faithful. If M is a mixed realization over k, then $M \otimes_k k'$ is simply given by

$$(M_{\mathrm{B}}, M_{\mathrm{dR}} \otimes_k k', M_\ell, \cdots)$$

where the action of $G_{k'}$ on M_ℓ is deduced from the action of G_k by restricting along the canonical morphism $G_{k'} \to G_k$.

The following is a variant of [10, 2.2 & 2.3] for 1-motives with torsion.

5.5. Proposition. *Considering* $T := (T_\mathbb{Z}, T_{\mathrm{dR}}, T_\ell, \dots)$ *where* $T_\mathbb{Z}$ *denotes the Betti realization,* T_{dR} *the de Rham realization,* T_ℓ *the ℓ-adic realization, etc., of 1-motives with torsion, we obtain a functor*

$$T : {}^t\mathcal{M}_1(k) \to \mathcal{MR}^\sigma(k) \tag{5.2}$$

which is exact and fully faithful.

Proof. We split the proof in three steps.

Step 1: If $k = \mathbb{C}$, it is well-known [5, Proposition 1.5] that the composition

$${}^t\mathcal{M}_1(\mathbb{C}) \to \mathcal{MR}^{\mathrm{id}}(\mathbb{C}) \to \mathsf{MHS}$$

is fully faithful. From Remark 5.3, we know that the second functor is faithful. This implies that the first functor is fully faithful.

Step 2: If $k = \overline{k}$ is algebraically closed, the base change functor

$$- \otimes_k \mathbb{C} : {}^t\mathcal{M}_1(k) \to {}^t\mathcal{M}_1(\mathbb{C})$$

is fully faithful. (This easily follows from the case of lattices with torsion and semi-abelian varieties.) From Step 1, we deduce that the composition of

$${}^t\mathcal{M}_1(k) \to {}^t\mathcal{M}_1(\mathbb{C}) \to \mathcal{MR}^{\mathrm{id}}(\mathbb{C})$$

which is also the composition of

$${}^t\mathcal{M}_1(k) \to \mathcal{MR}^\sigma(k) \to \mathcal{MR}^{\mathrm{id}}(\mathbb{C})$$

is fully faithful. Now, by Remark 5.4, the functor

$$- \otimes_k \mathbb{C} : \mathcal{MR}^\sigma(k) \to \mathcal{MR}^{\mathrm{id}}(\mathbb{C})$$

is faithful (in fact, it is also full, but we don't need to know this). This implies that ${}^t\mathcal{M}_1(k) \to \mathcal{MR}^\sigma(k)$ is fully faithful.

Step 3: We now consider the general case. Let $\overline{k}$ be the algebraic closure of k in $\mathbb{C}$ and denote by $G_k = \mathrm{Gal}\,(\overline{k}/k)$ the absolute Galois group. Fix two 1-motives M and M' over k and denote by N and N' their mixed realizations. Consider the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{{}^t\mathcal{M}_1}(M, M') & \longrightarrow & \mathrm{Hom}_{{}^t\mathcal{M}_1}(M \otimes_k \overline{k}, M' \otimes_k \overline{k}) \\
\downarrow & & \downarrow{\scriptstyle\sim} \\
\mathrm{Hom}_{\mathcal{MR}^\sigma}(N, N') & \longrightarrow & \mathrm{Hom}_{\mathcal{MR}^\sigma}(N \otimes_k \overline{k}, N' \otimes_k \overline{k}) \\
\uparrow & & \uparrow \\
\prod_\ell \mathrm{Hom}_{G_k}(N_\ell, N'_\ell) & \longrightarrow & \prod_\ell \mathrm{Hom}(N_\ell, N'_\ell)
\end{array}
$$

where the two lower vertical arrows are injective and the first vertical arrow on the right is invertible. Using a diagram chasing, it is enough to show that the commutative square

$$
\begin{array}{ccc}
\mathrm{Hom}_{{}^t\mathcal{M}_1}(M, M') & \longrightarrow & \mathrm{Hom}_{{}^t\mathcal{M}_1}(M \otimes_k \overline{k}, M' \otimes_k \overline{k}) \\
\downarrow & & \uparrow \\
\prod_\ell \mathrm{Hom}_{G_k}(N_\ell, N'_\ell) & \longrightarrow & \prod_\ell \mathrm{Hom}(N_\ell, N'_\ell)
\end{array}
$$

is cartesian. The group G_k acts on $\mathrm{Hom}_{{}^t\mathcal{M}_1}(M \otimes_k \overline{k}, M' \otimes_k \overline{k})$ and $\mathrm{Hom}(N_\ell, N'_\ell)$ and the vertical arrow on the right is G_k-equivariant. Moreover, we have

$$\mathrm{Hom}_{G_k}(N_\ell, N'_\ell) = \mathrm{Hom}(N_\ell, N'_\ell)^{G_k}.$$

Thus, we are reduced to showing that the natural mapping

$$\mathrm{Hom}_{{}^t\mathcal{M}_1}(M, M') \to \mathrm{Hom}_{{}^t\mathcal{M}_1}(M \otimes_k \overline{k}, M' \otimes_k \overline{k})^{G_k} \qquad (5.3)$$

is a bijection. This follows from the Hoschschild-Serre spectral sequence. Indeed, from [4, Theorem 2.1.2], one has a fully faithful embedding

$$\mathrm{Tot} : \mathbf{D}^b({}^t\mathcal{M}_1(k)) \to \mathbf{DM}^{\mathrm{\acute{e}t}}_{\mathrm{eff}}(k)$$

where $\mathbf{DM}^{\mathrm{\acute{e}t}}_{\mathrm{eff}}(k)$ is the full subcategory of $\mathbf{D}(Shv^{\mathrm{\acute{e}t}}_{tr}(k))$ given by $\mathbb{A}^1$-local objects (*cf.* §7.1 & Proposition 7.2 below for a more detailed

discussion, but only with rational coefficients). Also, there is a similar functor for $\overline{k}$. Therefore

$$\mathrm{Hom}_{{}^{t}\mathcal{M}_1}(M, M') = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{eff}}^{\mathrm{\acute{e}t}}}(\mathrm{Tot}(M), \mathrm{Tot}(M'))$$

and similarly after applying $- \otimes_k \overline{k}$. Now in $\mathbf{D}(Shv_{tr}^{\mathrm{\acute{e}t}}(k))$ as well as in $\mathbf{DM}_{\mathrm{eff}}^{\mathrm{\acute{e}t}}(k)$ we have that

$$\mathrm{RHom}(C, C') = R\Gamma(G_k, \mathrm{RHom}(C \otimes_k \overline{k}, C' \otimes_k \overline{k}))$$

for objects $C, C' \in \mathbf{D}(Shv_{tr}^{\mathrm{\acute{e}t}}(k))$. For $C = \mathrm{Tot}(M)$ and $C' = \mathrm{Tot}(M')$, this gives (5.3). $\square$

5.6. Mixed realization of effective Nori motives. Considering

$$(X, Y, i) \rightsquigarrow R_i(X, Y) = (H_i^{\mathrm{B}}(X, Y), H_i^{\mathrm{dR}}(X, Y), H_i^{\ell}(X, Y), \dots)$$

given by singular homology, de Rham homology and ℓ-adic homology, we get a representation $R : {}^{D}(Sch_k) \to \mathcal{MR}^{\sigma}(k)$ which factors the representation (2.1). By universality (*i.e.*, [15, Theorem 41]), we obtain an exact faithful functor

$$\widetilde{R} : \mathsf{EHM}(k) \to \mathcal{MR}^{\sigma}(k). \tag{5.4}$$

5.7. Lemma. *For $(C, Z, 1) \in {}^{D}(Crv_k)$, there is a canonical isomorphism $T \circ A(C, Z) \simeq R(C, Z, 1)$. In other words, the following square is commutative:*

$$
\begin{array}{ccc}
{}^{D}(Crv_k) & \xrightarrow{\ A\ } & {}^{t}\mathcal{M}_1 \\
\downarrow & & \downarrow{\scriptstyle T} \\
{}^{D}(Sch_k) & \xrightarrow{\ R\ } & \mathcal{MR}^{\sigma}.
\end{array}
$$

Proof. Since we deal with curves this is essentially due to Deligne [9, §10.3]. (Deligne deals with the cohomology of curves: one needs to dualize to get the statement we need.) $\square$

5.8. Proposition. *The exact functor $\mathsf{EHM}_1'' \to \mathsf{EHM}$ is fully faithful.*

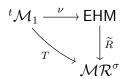*Proof.* Indeed, the commutative square in Lemma 5.7 gives a commutative square of exact faithful functors

$$
\begin{array}{ccc}
\mathsf{EHM}_1'' & \xrightarrow{\ \sim\ } & {}^{t}\mathcal{M}_1 \\
\downarrow & & \downarrow \\
\mathsf{EHM} & \longrightarrow & \mathcal{MR}^{\sigma}
\end{array}
$$

where the upper horizontal arrow is an equivalence by Theorem 4.5 and the right vertical arrow is fully faithful by Proposition 5.5. This proves the claim. $\square$

5.9. **Corollary.** *The functor*

$$\nu : {}^t\mathcal{M}_1 \simeq \mathsf{EHM}_1'' \to \mathsf{EHM} \tag{5.5}$$

is fully faithful and the triangle

$$
\begin{array}{ccc}
{}^t\mathcal{M}_1 & \xrightarrow{\ \nu\ } & \mathsf{EHM} \\
 & \searrow{\scriptstyle T} & \downarrow{\scriptstyle \widetilde{R}} \\
 & & \mathcal{M}\mathcal{R}^\sigma
\end{array}
$$

is commutative up to a canonical isomorphism. Moreover, EHM_1 *is the thick abelian subcategory generated by the image of* ν.

Proof. That ν is fully faithful follows from Proposition 5.8. Also, the equality $T = \widetilde{R} \circ \nu$ is clear from the construction. It remains to show that EHM_1 is the thick abelian subcategory of EHM generated by the image of ν. By Proposition 2.4, it is enough to show that the essential image of ν contains the motives $\widetilde{H}_i(C, Z, \mathbb{Z})$ for $i \leq 1$ and C a k-scheme with $\dim(C) \leq 1$.

By Proposition 3.3, this is clear for $i = 0$. Thus, we may assume that $i = 1$. If $(C, Z, 1) \in {}^D(Crv_k)$, then the property we need follows from the construction. We reduce the general case to the previous one as follows. We may assume that $\dim(Z) = 0$. If $Z' \subset C$ is a zero dimensional sub-scheme containing Z, then $\widetilde{H}_1(C, Z, \mathbb{Z}) \to \widetilde{H}_1(C, Z', \mathbb{Z})$ is injective. Therefore we may enlarge Z and assume that $C \smallsetminus Z$ is smooth. If C'' is the normalization of C and Z'' is the inverse image of Z by $C'' \to C$, we have $\widetilde{H}_1(C'', Z'', \mathbb{Z}) \simeq \widetilde{H}_1(C, Z, \mathbb{Z})$. Thus, we may assume that C is smooth. Finally if C is complete, and $c \in C \smallsetminus Z$ a closed point, the morphism $\widetilde{H}_1(C \smallsetminus \{c\}, Z; \mathbb{Z}) \to \widetilde{H}_1(C, Z; \mathbb{Z})$ is surjective. Therefore, we may assume that C is affine. This finishes the proof. $\qquad\square$

5.10. **Lemma.** *The following conditions are equivalent:*
   (a) ν_1 *is an equivalence of categories (cf.* (5.1)*),*
   (b) *the essential image of* ν *is a thick abelian subcategory of* EHM.

Proof. This follows from Corollary 5.9. $\qquad\square$

6. Some reductions

In this section we start the verification of 5.10(b). (By Lemma 5.10, this is what we still need to prove in order to complete the proof of Theorem 5.1.) We will see here that the essential image of the functor ν in (5.5) is stable under sub-quotients in EHM. Stability by extensions will be the subject of Section 7.

6.1. **Proposition.** *The essential image of the fully faithful exact functor $T : {}^t\mathcal{M}_1(k) \to \mathcal{MR}^\sigma(k)$ is stable under sub-quotients.*

Proof. It suffices to prove stability by sub-objects. Fix a 1-motive M and a sub-object $N' \subset N$ of its mixed realization $N = T(M)$. We need to construct a sub-1-motive $M' \subset M$ such that $T(M') = N'$. As for Proposition 5.5, we split the proof in three steps.

Step 1: If $k = \mathbb{C}$, we know that the composition

$$ {}^t\mathcal{M}_1(\mathbb{C}) \to \mathcal{MR}^{\mathrm{id}}(\mathbb{C}) \to \mathsf{MHS} $$

induces an equivalence between ${}^t\mathcal{M}_1(\mathbb{C})$ and the subcategory of MHS consisting of mixed hodge structures of type

$$ \{(0,0), (0,-1), (-1,0), (-1,-1)\}. $$

The latter is a thick abelian subcategory of MHS. Applying this to the mixed Hodge structure determined by N', we find a sub-1-motive $M' \subset M$ such that the sub-objects $T(M') \subset T(M)$ and $N' \subset T(M)$ determine the same sub-mixed Hodge structure. This implies that $T(M') = N'$.

Step 2: If $k = \overline{k}$ is algebraically closed, we know that the base change functor

$$ - \otimes_k \mathbb{C} : {}^t\mathcal{M}_1(k) \to {}^t\mathcal{M}_1(\mathbb{C}) $$

is fully faithful. Moreover, its essential image is stable under sub-objects. This follows from the fact that, for a lattice $\mathcal{L}$ (resp. a semi-abelian variety $\mathcal{G}$) defined over k, every sub-lattice of $\mathcal{L} \otimes_k \mathbb{C}$ (resp. sub-semi-abelian variety of $\mathcal{G} \otimes_k \mathbb{C}$) is defined over k. Therefore, to construct a sub-1-motive of M that realizes to N', it is enough to construct a sub-1-motive of $M \otimes_k \mathbb{C}$ that realizes to $N' \otimes_k \mathbb{C}$. We use the previous step to conclude.

Step 3: We now consider the general case. Let $\overline{k}$ be the algebraic closure of k in $\mathbb{C}$. By the previous step, we may find a sub-1-motive $M'_{\overline{k}} \subset M \otimes_k \overline{k}$ such that $T(M'_{\overline{k}}) = N' \otimes_k \overline{k}$.

The sub-1-motive $M'_{\overline{k}}$ can be defined over a finite Galois extension $l \subset \overline{k}$, *i.e.*, there exists a sub-1-motive $M'_l \subset M \otimes_k l$ such that

$$ T(M'_l \otimes_l \overline{k}) = N' \otimes_k \overline{k}. $$

This implies that, at least, $T_{\mathrm{dR}}(M'_l) = N'_{\mathrm{dR}} \otimes_k l$.

Let $\widetilde{M'} \in {}^t\mathcal{M}_1(k)$ be the Weil restriction of $M'_l \in {}^t\mathcal{M}_1(l)$. This is a sub-1-motive of $\widetilde{M} = M \otimes \mathbb{Z}_{tr}(l)$. It is characterized by the property that

$$ \widetilde{M'} \otimes_k l = \bigoplus_{\tau \in \hom_k(l,l)} M'_l \otimes_{l,\tau} l $$

as a sub-1-motive of

$$\widetilde{M} \otimes_k l = \bigoplus_{\tau \in \mathrm{hom}_k(l,l)} M \otimes_k l.$$

There is a canonical morphism of 1-motives $M \to \widetilde{M}$ and we define M' to be the intersection of M and $\widetilde{M'}$ inside $\widetilde{M}$. Then, by construction, we have $M' \otimes_k l \subset M'_l$. Therefore, we also have $T(M') \subset N'$.

Now, by construction

$$T_{\mathrm{dR}}(\widetilde{M}) = T_{\mathrm{dR}}(M \otimes_k l) = T_{\mathrm{dR}}(M) \otimes_k l,$$

viewed as a k-vector space. Moreover,

$$T_{\mathrm{dR}}(\widetilde{M'}) = \mathrm{T}_{\mathrm{dR}}(M'_l) = N'_{\mathrm{dR}} \otimes_k l,$$

viewed as a sub-k-vector space of $T_{\mathrm{dR}}(M \otimes_k l)$. As T_{dR} is an exact functor, it follows from the construction of M' that

$$T_{\mathrm{dR}}(M') = T_{\mathrm{dR}}(M) \cap (N'_{\mathrm{dR}} \otimes_k l) \qquad \text{inside} \quad T_{\mathrm{dR}}(M) \otimes_k l.$$

This shows that $T_{\mathrm{dR}}(M') = N'_{\mathrm{dR}}$. Therefore, $T(M') \subset N'$ has finite index.

Replacing M by M/M' and N' by $N'/T(M')$, we may assume that N' has zero de Rham component, *i.e.*, that N' is a torsion object of $\mathcal{MR}^\sigma$. In particular, N' lies in the essential image of $\mathcal{M}_0 \to \mathcal{MR}^\sigma$: let $\mathcal{L}$ be a torsion lattice (*i.e.*, a finite étale commutative group scheme over k) such that $N' = T([\mathcal{L} \to 0])$. We may use Proposition 5.5 to find a monomorphism $[\mathcal{L} \to 0] \hookrightarrow M$ that realizes to N' as a sub-object of N. This finishes the proof. $\qquad \square$

6.2. **Corollary.** *The essential image of the functor $\nu : {}^t\mathcal{M}_1 \to \mathsf{EHM}$ is an abelian subcategory that is stable under sub-quotients.*

Proof. Indeed, let M be a 1-motive and let $L' \subset L$ be a sub-motive of the effective Nori motive $L = \nu(M)$. Set $N = T(M) = \widetilde{R}(\nu(M))$ and $N' = \widetilde{R}(L')$. Then N' is a sub-object of N and, by Proposition 6.1, there exists a sub-1-motive $M' \subset M$ such that $T(M') = N'$. Then, necessarily $\nu(M') = L'$ as sub-objects of $\nu(M)$. Indeed, this can be checked after applying the forgetful functor $\mathsf{EHM} \to \mathbb{Z}\text{-mod}$. $\qquad \square$

6.3. **Lemma.** *To check 5.10(b), it is enough to check that the essential image of $\mathcal{M}_1^{\mathbb{Q}}$ by $\nu \otimes \mathbb{Q}$ is stable under extensions in $\mathsf{EHM}^{\mathbb{Q}}$.*

Proof. In view of Corollary 6.2, it remains to check that the essential image of ${}^t\mathcal{M}_1$ by ν is stable under extensions in EHM. We need to

prove this property assuming its rational analogue. So, consider an exact sequence in EHM:

$$0 \to \nu(M') \xrightarrow{r} N \xrightarrow{s} \nu(M'') \to 0 \qquad (6.1)$$

where M' and M'' are 1-motives. To show that N is in the essential image of ν, we consider several special cases.

Case 1: M' is torsion-free and M'' is torsion. Then $\nu(M') \to N$ is an isomorphism in $\mathsf{EHM}^{\mathbb{Q}}$. Therefore, there exists a morphism $t : N \to \nu(M')$ such that the composition $t \circ r$ is a multiplication by an non-zero integer. Consider the morphism $(t, s) : N \to \nu(M') \oplus \nu(M'')$. As M' is torsion-free, this morphism is injective. Hence, we have realized N as a sub-object of the image of a 1-motive by ν. By Corollary 6.2 we are done.

Case 2: both M' and M'' are torsion-free. In this case N is also torsion-free (*i.e.*, its Betti realization is a free $\mathbb{Z}$-module). Using the assumption in the statement, there exists a 1-motive M and an isomorphism $N \simeq \nu(M)$ in $\mathsf{EHM}^{\mathbb{Q}}$. Let $N \to \nu(Q)$ be a morphism in $\mathsf{EHM}^{\mathbb{Q}}$ inducing this isomorphism. As N is torsion-free, this morphism is injective. Again, we are done by Corollary 6.2.

Case 3: M' is torsion-free and M'' is general. For every finite extension l/k, the exact sequence (6.1) induces an exact sequence

$$0 \to \nu(M' \otimes \mathbb{Z}_{tr}(l)) \to N \otimes \nu(\mathbb{Z}_{tr}(l)) \to \nu(M'' \otimes \mathbb{Z}_{tr}(l)) \to 0$$

and, there is an injective morphism $N \hookrightarrow N \otimes \nu(\mathbb{Z}_{tr}(l))$. Using Corollary 6.2, it is enough to show that $N \otimes \nu(\mathbb{Z}_{tr}(l))$ is in the essential image of ν. Therefore, we may replace M' and M'' by $M' \otimes \mathbb{Z}_{tr}(l)$ and $M'' \otimes \mathbb{Z}_{tr}(l)$, for l/k large enough, and assume that the torsion part of the lattice $\mathcal{L}''$ of M'' is a direct summand. This implies that $M'' = M''_t \oplus M''_f$ where M''_t is a torsion 0-motive and M''_f is a torsion-free 1-motive. It follows that N embeds in a direct sum $N_t \oplus N_f$ where N_t and N_f are extensions of $\nu(M')$ by $\nu(M''_t)$ and $\nu(M''_f)$. By cases 1 and 2, N_t and N_f are in the essential image of ν. Again, we are done by Corollary 6.2.

Case 4: M' and M'' are general. Let $M'_t \subset M'$ be the torsion part of M'. By the previous case, $N/\nu(M'_t)$ is in the essential image of ν. In other words, we may change the exact sequence (6.1) and assume that M' is torsion.

It follows that $N \to \nu(M'')$ is an isomorphism in $\mathsf{EHM}^{\mathbb{Q}}$. Therefore, there exists a torsion-free 1-motive M''' and a morphism $t : \nu(M'') \to N$ such that the composition $s \circ t$ is given by an isogeny from M''' to M''. Using case 1, it is enough to show that $N/\nu(M''')$ is in the essential

image of ν. The latter being an extension of $\nu(M')$ by $\nu(M''/M''')$, we are reduced to the case where M'' is also torsion. Now, by a result of Nori [11], all torsion objets in EHM are 0-motives. This finishes the proof of the lemma. $\square$

7. ON DELIGNE'S CONJECTURE ON EXTENSIONS OF 1-MOTIVES

In this section, we prove a "piece" of Deligne's conjecture [10, 2.4] on extensions of 1-motives that is needed to complete the proof of Theorem 5.1. More precisely, we prove Deligne's conjecture under an effectivity condition coming from Nori's formalism of mixed motives. (This condition plays a crucial role in our argument and seems difficult to remove with the actual motivic technology.)

7.1. Voevodsky motives.
Let $\mathbf{DM}_{\mathrm{eff}}(k; R)$ be Voevodsky's category of motives with coefficients in R. There is a fully faithful embedding:

$$\mathbf{DM}_{\mathrm{eff}}(k; R) \hookrightarrow \mathbf{D}(Shv_{tr}^{\mathrm{Nis}}(k; R)).$$

Its image consists of those complexes of Nisnevich sheaves with transfers which are $\mathbb{A}^1$-local, *i.e.*, such that their Nisnevich hypercohomology presheaves are $\mathbb{A}^1$-invariant.

As usual, we denote by $\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k; R)$ the subcategory of $\mathbf{DM}_{\mathrm{eff}}(k; R)$ of geometric motives. It is the thick triangulated subcategory generated by motives of smooth k-schemes.

Given a pair (X, Y) where X is a k-scheme and $Y \subset X$ a closed subset, one has an object $\mathrm{M}(X, Y; R) \in \mathbf{DM}_{\mathrm{eff}}(k; R)$, called the motive of the pair. (As a presheaf with transfers, this is simply given by $R_{tr}(X)/R_{tr}(Y)$.) Since our base field has characteristic zero, we know, thanks to Hironaka's resolution of singularities, that $\mathrm{M}(X, Y; R)$ is a geometric motive.

Another source of examples is given by the following result (claimed by Voevodsky in [18, §3.4] and proved in [17]).

7.2. Proposition.
There exists a fully faithful embedding

$$\mathrm{Tot} : \mathbf{D}^b(\mathcal{M}_1^{\mathbb{Q}}(k)) \to \mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k; \mathbb{Q}). \tag{7.1}$$

It induces an equivalence of categories with the thick triangulated subcategory $\mathbf{DM}_{\leq 1}^{\mathrm{gm}}(k; \mathbb{Q})$ generated by motives of curves.

Proof. The functor is easily defined: it sends a 1-motive $[\mathcal{L} \to \mathcal{G}]$ to the complex of Nisnevich sheaves with transfers $[\mathcal{L} \otimes \mathbb{Q} \to \mathcal{G} \otimes \mathbb{Q}]$ placed in homological degrees 0 and -1. For details concerning the proof, we refer the reader to [17]. $\square$

7.3. **Voevodsky versus Nori.** Nori [11] constructed a triangulated functor

$$\Gamma : \mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k; R) \to \mathbf{D}^b(\mathsf{EHM}^R(k)). \tag{7.2}$$

This functor transforms $\mathrm{M}(X, Y; R)$ into a complex of effective Nori motives $\Gamma(X, Y; R)$ such that, canonically,

$$H_i(\Gamma(X, Y; R)) \simeq \widetilde{H}_i(X, Y; R). \tag{7.3}$$

(In the left hand side of the above formula, $H_i(-)$ is the homology functor with respect to the canonical t-structure on $\mathbf{D}^b(\mathsf{EHM}^R(k))$.)

Nori's functor (7.2) can be used to recover all the realization functors on $\mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k, \mathbb{Q})$ constructed by Huber [12]. For instance, one gets a mixed realization functor on $\mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k; R)$ by taking the composition

$$\mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k; \mathbb{Z}) \xrightarrow{\Gamma} \mathbf{D}^b(\mathsf{EHM}(k)) \xrightarrow{\widetilde{R}} \mathbf{D}^b(\mathcal{MR}^\sigma(k)).$$

7.4. **Remark.** The verification that the composition $\widetilde{R} \circ \Gamma$ is isomorphic to Huber's functor is tedious but routine; it will not be carried out in this paper. Happily, this is not needed for any of our main results.

7.5. **Proposition.** *The following square commutes up to a natural isomorphism*

$$\begin{array}{ccc} \mathcal{M}^{\mathbb{Q}}_1(k) & \xrightarrow{\ \nu\ } & \mathsf{EHM}^{\mathbb{Q}}(k) \\ \downarrow{\scriptstyle \mathrm{Tot}} & & \downarrow \\ \mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k; \mathbb{Q}) & \xrightarrow{\ \Gamma\ } & \mathbf{D}^b(\mathsf{EHM}^{\mathbb{Q}}(k)). \end{array}$$

Proof. The image of the composition of

$$\mathcal{M}^{\mathbb{Q}}_1(k) \to \mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k; \mathbb{Q}) \to \mathbf{D}^b(\mathsf{EHM}^{\mathbb{Q}}(k))$$

lies in the heart of the canonical t-structure on $\mathbf{D}^b(\mathsf{EHM}^{\mathbb{Q}}(k))$. To check this, it is enough to prove the same claim for the composition of

$$\mathcal{M}^{\mathbb{Q}}_1(k) \to \mathbf{DM}^{\mathrm{gm}}_{\mathrm{eff}}(k; \mathbb{Q}) \xrightarrow{R^{\mathrm{B}}} \mathbf{D}^b(\mathbb{Q})$$

where R^{B} is the Betti realization. Using the weight filtration on 1-motives, it is enough to consider separately the case of a lattice, of a torus, and of an abelian variety. Also, we may assume that $k = \mathbb{C}$. Then, the first two cases are obvious. For the third case, we use that for an abelian variety A, $\mathrm{Tot}([0 \to A]) = A \otimes \mathbb{Q}[-1]$ is a direct factor of a motive $\mathrm{M}(C \smallsetminus \{c_1\}, c_2; \mathbb{Q})[-1]$ where C is a complete smooth curve, and c_1 and c_2 are two distinct rational points. The Betti realization of such a motive is the complex $H^{\mathrm{B}}_1(C)[0]$.

Due to the previous discussion, it is enough to show that the following diagram

$$
\begin{array}{ccc}
\mathcal{M}_1^{\mathbb{Q}}(k) & \xrightarrow{\ \nu\ } & \mathsf{EHM}^{\mathbb{Q}}(k) \\
\downarrow & & \uparrow {\scriptstyle H_0} \\
\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k;\mathbb{Q}) & \xrightarrow{\ \Gamma\ } & \mathbf{D}^b(\mathsf{EHM}^{\mathbb{Q}}(k))
\end{array}
$$

commutes. Now we have to deal with two functors from $\mathcal{M}_1^{\mathbb{Q}}$ to $\mathsf{EHM}^{\mathbb{Q}}$, which are $\mathbb{Q}$-linear and exact. Thus, by Theorem 4.5 and universality (*i.e.*, [15, Theorem 41]), it will be enough to check that

$$
\begin{array}{ccc}
{}^D(Crv_k) & \xrightarrow{\ \widetilde{H}\ } & \mathsf{EHM}^{\mathbb{Q}} \\
\downarrow {\scriptstyle \mathrm{Tot}\,\circ\, A} & & \uparrow {\scriptstyle H_0} \\
\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k;\mathbb{Q}) & \xrightarrow{\ \Gamma\ } & \mathbf{D}^b(\mathsf{EHM}^{\mathbb{Q}})
\end{array}
$$

commutes. Now, for $(C, Z, 1) \in {}^D(Crv_k)$, it follows from [18, Theorem 3.4.2] that there exists an exact triangle in $\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k;\mathbb{Q})$:

$$
\mathrm{Tot}(A(C, Z)) \to \mathrm{M}(C, Z; \mathbb{Q})[-1] \to \mathcal{L}[-1] \to
$$

where $\mathcal{L} = \mathrm{Coker}\{\mathbb{Q}_{tr}(Z) \to \mathbb{Q}_{tr}(\pi_1(C))\}$. Moreover, this triangle splits (non-canonically). As $H_0(\Gamma(\mathcal{L}[-1])) = 0$, we get an isomorphism

$$
H_0(\Gamma(\mathrm{Tot}(A(C, Z)))) \simeq H_0(\Gamma(\mathrm{M}(C, Z; \mathbb{Q}))).
$$

Therefore, in the last square above, we may replace $\mathrm{Tot} \circ A$ by M. The commutativity is then a direct consequence of (7.3). $\square$

Although not needed for our main objective, we note the following concrete consequence of Proposition 7.5.

7.6. Corollary. *The following square commutes up to a natural isomorphism*

$$
\begin{array}{ccc}
\mathcal{M}_1^{\mathbb{Q}}(k) & \xrightarrow{\ T\ } & \mathcal{MR}^{\sigma}(k;\mathbb{Q}) \\
\downarrow {\scriptstyle \mathrm{Tot}} & & \downarrow \\
\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k;\mathbb{Q}) & \xrightarrow{\ \widetilde{R}\circ\Gamma\ } & \mathbf{D}^b(\mathcal{MR}^{\sigma}(k;\mathbb{Q}))
\end{array}
$$

(where we set $\mathcal{MR}^{\sigma}(k;\mathbb{Q}) = \mathcal{MR}^{\sigma}(k) \otimes \mathbb{Q}$).

Proof. This is a direct consequence of Proposition 7.5 and the equality $T = \widetilde{R} \circ \nu$ (see Corollary 5.9). $\square$

7.7. **Remark.** As a consequence (see however Remark 7.4) we obtain that Deligne's Hodge realization of 1-motives is isomorphic to the composition of Huber's Hodge realization with the embedding $\mathcal{M}_1^{\mathbb{Q}}(k) \hookrightarrow \mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k; \mathbb{Q})$. This was first proved by Vologodsky [19]. Our proof is arguably more conceptual.

7.8. Let $\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}} \subset \mathsf{MHS}^{\mathbb{Q}}$ denote the full subcategory of (homologically) effective mixed Hodge structures. Also, let $\mathsf{MHS}_{\leq 1}^{\mathbb{Q}}$ be the thick abelian subcategory of $\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}}$ consisting of mixed Hodge structures of type

$$\{(0,0), (-1,0), (0,-1), (-1,-1)\}.$$

We have a fully faithful embedding $\mathbf{D}^b(\mathsf{MHS}_{\leq 1}^{\mathbb{Q}}) \to \mathbf{D}^b(\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}})$ and a commutative square

$$
\begin{array}{ccc}
\mathbf{DM}_{\leq 1}^{\mathrm{gm}}(k; \mathbb{Q}) & \xrightarrow{R^{\mathrm{Hdg}}|_{\leq 1}} & \mathbf{D}^b(\mathsf{MHS}_{\leq 1}^{\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k; \mathbb{Q}) & \xrightarrow{R^{\mathrm{Hdg}}} & \mathbf{D}^b(\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}}).
\end{array}
\qquad (7.4)
$$

(For the sake of precision, we note that R^{Hdg} is taken to be the composition of

$$\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k; \mathbb{Q}) \xrightarrow{\Gamma} \mathbf{D}^b(\mathsf{EHM}^{\mathbb{Q}}(k)) \xrightarrow{R^{\mathrm{Hdg}}} \mathbf{D}^b(\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}})$$

where the second functor is derived from $R^{\mathrm{Hdg}} : \mathsf{EHM}^{\mathbb{Q}} \to \mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}}$ given by the universal property.)

Both vertical inclusions in (7.4) admit left adjoints (see [3, Theorem 2.4.1] or [4, Corollary 6.2.2] for the first one and [4, Proposition 17.1.1] for the second one); they are denoted respectively by LAlb and $(-)_{\leq 1}$. From the commutativity of (7.4), we get a natural transformation

$$(R^{\mathrm{Hdg}}(M))_{\leq 1} \to (R^{\mathrm{Hdg}}|_{\leq 1})(\mathrm{LAlb}(M)) = R^{\mathrm{Hdg}}(\mathrm{LAlb}(M)). \qquad (7.5)$$

7.9. **Proposition.** *The natural transformation* (7.5) *is invertible.*

Proof. This is essentially [4, Theorem 17.3.1]. For completeness, we give a sketch of the argument (with a slight modification). By the proof of [3, Corollary 2.4.6], it is enough to show that (7.5) is invertible after evaluating on motives $\mathrm{M}(X)$, where X is a smooth k-scheme which is NS^1-local (in the sense of [3, Definition 2.4.2]). Using [3, Proposition 2.4.4], we have $\mathrm{LAlb}(X) = \mathrm{Alb}(X)$, where the semi-abelian variety $\mathrm{Alb}(X)$ is considered as a Nisnevich sheaf with transfers.

Similarly, writing $H_i^{\mathrm{Hdg}}(X)$ for the mixed Hodge structure on the i-th homology of X, we have:

$$(H_i^{\mathrm{Hdg}}(X))_{\leq 1} = \begin{cases} H_i^{\mathrm{Hdg}}(X) & \text{if } i \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we have

$$(R^{\mathrm{Hdg}}(X))_{\leq 1} = \tau_{\leq 1} R^{\mathrm{Hdg}}(X)$$

where $\tau_{\leq 1}$ is the good truncation with respect to the canonical t-structure on $\mathbf{D}^b(\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}})$.

Thus, to finish the proof, we are left to show that the Betti realization of $\mathrm{M}(X) \to \mathrm{Alb}(X)$ is isomorphic to $R^{\mathrm{B}}(X) \to \tau_{\leq 1} R^{\mathrm{B}}(X)$. But, the complex $R^{\mathrm{B}}(\mathrm{Alb}(X))$ has homology in degree 0 and 1 (see the first part of the proof of Proposition 7.5). Moreover, we have canonically:

$$H_0(R^{\mathrm{B}}(\mathrm{Alb}(X))) = H_0^{\mathrm{B}}(X) \quad \text{and} \quad H_1(R^{\mathrm{B}}(\mathrm{Alb}(X))) = H_1^{\mathrm{B}}(X).$$

This finishes the proof. □

7.10. Theorem. *Let $M \in \mathsf{EHM}^{\mathbb{Q}}$ be an effective Nori motive whose Hodge realization is in $\mathsf{MHS}_{\leq 1}^{\mathbb{Q}}$. Then, M is in the essential image of the functor $\nu : \mathcal{M}_1^{\mathbb{Q}} \to \mathsf{EHM}^{\mathbb{Q}}$.*

Proof. We can realize M as a sub-quotient of a Nori motive of the form $\widetilde{H}_i(X, Y; \mathbb{Q})$ with X a k-scheme and $Y \subset X$ a closed subset. Consider the motive $\mathrm{M}(X, Y; \mathbb{Q}) \in \mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k; \mathbb{Q})$ and set

$$A = H_i(\mathrm{LAlb}(\mathrm{M}(X, Y; \mathbb{Q}))).$$

(In the above formula, H_i is with respect to the motivic t-structure on $\mathbf{DM}_{\leq 1}^{\mathrm{gm}}(k; \mathbb{Q})$ deduced from the canonical t-structure on $\mathbf{D}^b(\mathcal{M}_1^{\mathbb{Q}})$ via the equivalence in Proposition 7.2.)

Set $N = \Gamma(A)$. By construction, we have a map

$$\widetilde{H}_i(X, Y; \mathbb{Q}) \to N. \tag{7.6}$$

It is obtained by applying H_i to the obvious morphism

$$\Gamma(\mathrm{M}(X, Y; \mathbb{Q})) \to \Gamma(\mathrm{LAlb}(M(X, Y; \mathbb{Q}))).$$

In particular, Proposition 7.9 implies that (7.6) induces on the associated mixed Hodge structures, the obvious projection:

$$H_i^{\mathrm{Hdg}}(X, Y; \mathbb{Q}) \to (H_i^{\mathrm{Hdg}}(X, Y; \mathbb{Q}))_{\leq 1}. \tag{7.7}$$

(As before, we denote by $H_i^{\mathrm{Hdg}}(X, Y; \mathbb{Q})$ the mixed Hodge structure on the rational i-th homology of the pair (X, Y).)

Now, write $M = M'/M''$ where M' and M'' are sub-motives of $\widetilde{H}_i(X, Y; \mathbb{Q})$. Let N' and N'' be the images of M' and M'' in N.

It follows, by looking at the associated mixed Hodge structures, that $M \simeq N'/N''$. This proves that M is a sub-quotient of $\Gamma(A)$. Let $A' \in \mathcal{M}_1^{\mathbb{Q}}$ be a 1-motive such that $A = \mathrm{Tot}(A')$. By Proposition 7.5, we have

$$\Gamma(A) = \Gamma(\mathrm{Tot}(A')) = \nu(A').$$

Thus, we have realized M as a sub-quotient of an object in the image of ν. Now use Corollary 6.2. (In fact, by the commutativity of the triangle in Corollary 5.9 and Proposition 6.1 we are done.) $\qquad\square$

7.11. **Remark.** As $\mathsf{MHS}_{\leq 1}^{\mathbb{Q}}$ is a thick abelian subcategory of $\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}}$, Theorem 7.10 gives a positive answer to Deligne's conjecture [10, 2.4] in the case where the "geometric" extension of 1-motives is *effectively geometric*, *i.e.*, lying in the image of the functor $\widetilde{R} : \mathsf{EHM} \to \mathcal{MR}^{\sigma}$ from the category of *effective* Nori motives.

Let NHM be the category of (non-necessarily effective) Nori motives. It is natural to expect that Theorem 7.10 holds more generally for $M \in \mathsf{NHM}$. This would give a positive answer to Deligne's conjecture [10, 2.4] in full generality. On the other hand, it is reasonable to expect that $\mathsf{EHM} \subset \mathsf{NHM}$ is a thick abelian category. However, such a statement is completely out of reach and goes far beyond Deligne's conjecture.

7.12. **Corollary.** *The essential image of $\nu : \mathcal{M}_1^{\mathbb{Q}} \to \mathsf{EHM}^{\mathbb{Q}}$ is stable under extensions. Thus, the proof of* Theorem 5.1 *is complete.*

Proof. The first claim is a consequence of Theorem 7.10 and the fact that $\mathsf{MHS}_{\leq 1}^{\mathbb{Q}}$ is a thick abelian subcategory of $\mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}}$. The second claim follows from Lemmas 5.10 and 6.3. $\qquad\square$

We close the paper with the following result.

7.13. **Theorem.** *The inclusion $\mathsf{EHM}_1^{\mathbb{Q}} \hookrightarrow \mathsf{EHM}^{\mathbb{Q}}$ has a left adjoint, denoted by $(-)_{\leq 1}$. Moreover, the following square is commutative*

$$
\begin{array}{ccc}
\mathsf{EHM}^{\mathbb{Q}} & \longrightarrow & \mathsf{MHS}_{\mathrm{eff}}^{\mathbb{Q}} \\
\downarrow {\scriptstyle (-)_{\leq 1}} & & \downarrow {\scriptstyle (-)_{\leq 1}} \\
\mathsf{EHM}_1^{\mathbb{Q}} & \longrightarrow & \mathsf{MHS}_{\leq 1}^{\mathbb{Q}}.
\end{array}
$$

Proof. We will show that for every effective Nori motive M, there exists a map $M \to (M)_{\leq 1}$ to an object in $\mathsf{EHM}_1^{\mathbb{Q}}$ which realizes to the analogous map for effective mixed Hodge structures. This will implies that $M \to (M)_{\leq 1}$ is also universal and the theorem will follows. There will be a considerable overlap with the proof of Theorem 7.10.

First, we consider the case of $\widetilde{H}_i(X, Y; \mathbb{Q})$ for $(X, Y, i) \in {}^D(Sch_k)$. We use the map (7.6) constructed in the proof of Theorem 7.10. Its Hodge realization is given by (7.7): so we are done.

Now, let M be a general effective Nori motive and write $M = M'/M''$ with $M' \subset M'' \subset \widetilde{H}_i(X, Y; \mathbb{Q})$. As in the proof of Theorem 7.10, we consider $(M)_{\leq 1} = (M')_{\leq 1}/(M'')_{\leq 1}$ where $(M')_{\leq 1}$ and $(M'')_{\leq 1}$ are the images of M' and M'' in $(\widetilde{H}_i(X, Y; \mathbb{Q}))_{\leq 1}$. By construction, this map realizes to the projection $R^{\mathrm{Hdg}}(M) \to R^{\mathrm{Hgd}}(M)_{\leq 1}$. This finishes the proof of the theorem.   □

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