On the Discrete Edwards Model in Three Dimensions

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Abstract: *We prove that the measures defining the three dimensional discrete Edwards models are weakly convergent in the continuum limit to the polymer measure (continuum Edwards model) in three dimensions.*

1. Introduction

Let us first introduce the concept of polymer measure (i.e. continuum Edwards model). Let $\{B_t\}_{t\geq 0}$ be the standard Brownian motion in $\mathbf{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mu)$. The so-called polymer measure ν_λ on $C_0([0,1] \to \mathbf{R}^d)$ is formally defined by

$$\nu_\lambda(\mathrm{d}\omega) = N_\lambda^{-1} \exp\Big(- \lambda \int_0^1 \int_s^1 \delta(B_s - B_t)\mathrm{d}s\mathrm{d}t \Big) \mu(\mathrm{d}\omega),$$

where $\lambda \geq 0$ is a coupling constant, μ is the Wiener measure and N_λ is the renormalization constant. The existence of the polymer measure ν_λ for $d = 2$ was first proved by Varadhan (see [15]). For $d = 4$, it was shown in [1] that a measure related to ν_λ exists if the coupling constant λ is non-positive and infinitesimal. For $d = 3$, the existence of the measure ν_λ was first proved by Westwater (see [16,17]), and an alternative construction of ν_λ was recently given with a simple proof, by one of the authors in [2], which is based on the approaches given in [3] and [4]. We let ν_λ resp. ν'_λ denote the polymer measure in three dimensions constructed by Westwater, and respectively by one of the authors in [2].

We now introduce the discrete Edwards models. Let $\{X_n\}_{n\geq 0}$ be the simple random walk in $\mathbf{Z}^d$ on a probability space $(\Omega', \mathcal{F}', \mathbf{P})$ such that $X_0 = 0$. Let

$$T_{m,n} = n^{(d-4)/2} \sum_{i=1}^{n} \sum_{j=i+m}^{n} \delta(X_i, X_j).$$

We first define a new probability measure $\mu_{m,n,\lambda}$ on $C_0([0,1] \to \mathbf{R}^d)$. Let $X^{(n)} \in C_0([0,1] \to \mathbf{R}^d)$ be defined as follows:

$$X^{(n)}(n^{-1}i) = n^{-1/2}X_i, \qquad i = 0, 1, \cdots, n,$$

and $X^{(n)}$ is linear on $[(i-1)n^{-1}, in^{-1}]$ for $i = 1, \cdots, n$. For $0 \le t_1 < t_2 < \cdots < t_k \le 1$, and $A_1, \cdots, A_k \in \mathcal{B}(\mathbf{R}^d)$, we define

$$\mu_{m,n,\lambda}(X(t_1) \in A_1, \cdots, X(t_k) \in A_k) = \mathbf{E}(\exp(-\lambda T_{m,n}))^{-1} \int_{\{X^{(n)}(t_1)\in A_1,\cdots,X^{(n)}(t_k)\in A_k\}} \exp(-\lambda T_{m,n})d\mathbf{P},$$

where $\mathbf{E}$ is the expectation with respect to $\mathbf{P}$ ($\widetilde{\mathbf{E}}$ in this paper will denote the expectation with respect to μ). Using this one can easily define a probability measure $\mu_{m,n,\lambda}$ on $C_0([0,1] \to \mathbf{R}^d)$, written by

$$\mu_{m,n,\lambda}(d\omega) = (\mathbf{E}\exp(-\lambda T_{m,n}))^{-1}\exp(-\lambda T_{m,n})\mathbf{P}(d\omega).$$

In general, the measure $\mu_{1,n,\lambda}$ is called the (discrete) Edwards model (see [9,11]). If $m > 1$, the measure $\mu_{m,n,\lambda}$ is called here a modified discrete Edwards model. It is believed that the polymer measure ν_λ (or ν'_λ) can be approximated by the measures $\mu_{1,n,\lambda}$, $n \ge 1$. As a fact, Stoll [14] already proved the above assertion for $d = 2$. The main aim of the present paper is to prove this assertion for $d = 3$. We will prove that the polymer measure ν'_λ can be approximated by the measures $\mu_{m_n,n,\lambda}$ for m_n, a suitable n-dependent sequence. The main result in this paper is as follows.

Theorem 1.1. *Assume that $m_n \in [1,n]$, $\forall n \ge 1$, and $\lim_{n\to\infty} n/m_n = \infty$. Then there is a constant $\lambda_0 > 0$ such that*

$$\lim_{n\to\infty} \mu_{m_n,n,\lambda} \stackrel{\mathcal{D}}{=} \nu'_\lambda, \quad \forall\lambda \in [0,\lambda_0],$$

where $\mathcal{D}$ represents the weak convergence. In particular

$$\lim_{n\to\infty} \mu_{1,n,\lambda} \stackrel{\mathcal{D}}{=} \nu'_\lambda, \quad \lambda \in [0,\lambda_0].$$

Using the invariance principle for intersection local time one can easily show that for any $\lambda \in [0,\lambda_0)$ there is a sequence $\{m_n\}$ with $\lim_{n\to\infty} n/m_n = \infty$ such that

$$\mu_{m_n,n,\lambda} \xrightarrow{\mathcal{D}} \nu'_\lambda, \qquad n \to \infty. \tag{1.1}$$

Hence, to prove Theorem 1.1 we need only to show that $\mu_{m_n,n,\lambda}$ and $\mu_{1,n,\lambda}$ have the same asymptotic behavior, if $\lim_{n\to\infty} n/m_n = \infty$. For this purpose we will first derive a reasonable estimate for the so-called two-point functions with small coupling constants (see Theorem 3.1 below). In Sect.4 we will derive an estimate for the two-point functions with large coupling constants. The main theorem (i.e. Theorem 1.1 above) in this paper will be proved in Sect.5. In the appendix we will prove several lemmas.

In a forthcoming paper we will use the approach given in [2] to construct the polymer measure ν'_λ for all $\lambda \in [0,\infty)$ and also prove that $\nu_\lambda = \nu'_\lambda$, $\forall \lambda \in [0,\infty)$. Thus, Theorem 1.1 can be extended as follows. If $\lim_{n\to\infty} n/m_n = \infty$, then

$$\lim_{n\to\infty} \mu_{m_n,n,\lambda} \stackrel{\mathcal{D}}{=} \nu_\lambda = \nu'_\lambda, \quad \forall \lambda \in [0,\infty).$$

At the end of this section we give a remark on our main result. It is easy to see that the intersections of the Brownian motion $\{B_t\}$ within short ranges (e.g. the set $B_t = B_s$ for $|t-s| \le 2^{-n}$) are ignored in the definition of $\nu'_{n,\lambda}$ (see Sect.2 below). Due to the relation (2.4) below we can say that the intersections of $\{B_t\}$ within short ranges are not so important in the construction of the polymer measure ν'_λ. However, the intersections of the random walk $\{X_n\}$ within short ranges are heuristically important in the definition of $\mu_{m,n,\lambda}$, since the random variable $n^{-1/2}\sum_{i=1}^{n-2} I_{\{X_i = X_{i+2m}\}}$ is not convergent to a constant if $m \ge 1$ is fixed. In this sense, it is not so simple to prove that $\{\mu_{m,n,\lambda}\}$ and $\{\nu'_{n,\lambda}\}$have the same asymptotic behavior as $n \to \infty$, if $m \ge 1$ is fixed. We will pay more attention to the discussions for the intersections of $\{X_n\}$ within short ranges when we consider the asymptotic behavior of $\{\mu_{m,n,\lambda}\}_{n\ge1}$ if $m \ge 1$ is fixed.

2. Notations and Preliminaries

In this section we first derive an embedding theorem in multi-dimensional case (i.e. $d \ge 2$), and then use this theorem to prove that (1.1) is true. We will also introduce some notations which will be used in the proof of Theorem 3.1 below.

We now derive an extension of the Skorokhod embedding theorem to the high dimensional case (For the other extensions, see e.g. [8] and references therein). Let $\xi_1, \xi_2, \cdots$ be i.i.d. random vectors in $\mathbf{Z}^d$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying:

$$\mathbf{P}(\xi_1 = (i_1, \cdots, i_d)) = (2d)^{-1}$$

for any $(i_1,\cdots,i_d) \in \mathbf{Z}^d$ with $|i_1| + \cdots + |i_d| = 1$. Set $X_n = \sum_{i=1}^n \xi_i$, $\forall n \ge 1$. For convenience, the distribution function of ξ_1 is denoted by F. Let $\{B(t) = (B_1(t),\cdots,B_d(t))\}_{t\ge0}$ be the standard Brownian motion in $\mathbf{R}^d$ on a probability space $(\Omega_1, \mathcal{F}_1, \mu_1)$, and $\eta_1, \eta_2, \cdots$ be i.i.d. random vectors in Z^d_+ on a probability space $(\Omega_2, \mathcal{F}_2, \mu_2)$ satisfying

$$\mu_2(\eta_1 = (i_1, \cdots, i_d)) = d^{-1}$$

for any $(i_1,\cdots,i_d) \in Z^d_+$ with $i_1 + \cdots + i_d = 1$. Let $\tau_k = (\tau_k^1, \cdots, \tau_k^d) \in R^d_+$, $\forall k \ge 1$, be given. For convenience, we set

$$B(\tau_k) = (B_1(\tau_k^1), \cdots, B_d(\tau_k^d)) \in \mathbf{R}^d, \qquad \tau_k - \tau_{k-1} = (\tau_k^1 - \tau_{k-1}^1, \cdots, \tau_k^d - \tau_{k-1}^d),$$

and write $\tau_{k-1} \le \tau_k$ if $\tau_{k-1}^i \le \tau_k^i$, for $i = 1, \cdots, d$. We say that $\tau \in \mathbf{R}^d$ is non-negative if $\tau^i \ge 0$ for $i = 1, \cdots, d$.

The embedding theorem is as follows.

Proposition 2.1. *There are non-negative random vectors $\tau_1 \le \tau_2 \le \cdots$ in $\mathbf{R}^d$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$ such that*

(i) $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \cdots$ are independent, identically distributed and finite,

(ii) $\widetilde{\mathbf{E}}\tau_1 = (1, 1, \cdots, 1) \in \mathbf{R}^d$, where $\widetilde{\mathbf{E}}$ is the expectation with respect to $\mu_1 \times \mu_2$.

(iii) $B(\tau_1), B(\tau_2) - B(\tau_1), \cdots$ are independent, identically distributed, and their distribution function is F.

Proof. Let $\tau_0 = (0, 0, \cdots, 0) \in \mathbf{R}^d$. Suppose that $\tau_n = (\tau_n^1, \cdots, \tau_n^d)$ has already been constructed. Let

$$\sigma_{n+1}^i = \inf\{t \ge \tau_n^i : |B_i(t) - B_i(\tau_n^i)| = 1\}, \quad i = 1, \cdots, d.$$

Then, $\tau_{n+1} = (\tau_{n+1}^1, \cdots, \tau_{n+1}^d)$ is defined by

$$\tau_{n+1}^i = \eta_{n+1}^i(\sigma_{n+1}^i - \tau_n^i) + \tau_n^i, \quad i = 1, \cdots, d,$$

where $\eta_{n+1} = (\eta_{n+1}^1, \cdots, \eta_{n+1}^d)$. By our hypotheses, one can show that $\tau_1, \tau_2 - \tau_1, \cdots$ are i.i.d. and finite, and $B(\tau_1), B(\tau_2) - B(\tau_1), \cdots$ are also i.i.d.. It is easy to check that the distribution function of $B(\tau_1)$ is F, and

$$\widetilde{\mathbf{E}}\tau_1 = (\widetilde{\mathbf{E}}(\eta_1^1\sigma_1^1), \cdots, \widetilde{\mathbf{E}}(\eta_1^d\sigma_1^d)) = (1, \cdots, 1),$$

which proves the desired result. □

Note. (i) It is easy to see from the above definition that $\widetilde{\mathbf{E}}(\sigma_1^i)^2 < \infty$, $i = 1, \cdots, d$. By the above construction we also know that $\widetilde{\mathbf{E}}(\tau_1^i)^2 < \infty$, $i = 1, \cdots, d$.

(ii) Since $B(\tau_1)$ and ξ_1 have the same distribution function F, $\{B(\tau_n)\}_{n\ge 0}$ is a simple random walk in $\mathbf{Z}^d$ on the probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$.

Let

$$B_n(t) = n^{-1/2}B\Big(nt\Big), \qquad \forall t \in [0, \infty), \ \forall n \ge 1.$$

Let $\widehat{S}_0 = 0$, and $\widehat{S}_i = B(\tau_i)$, $i \ge 1$, where $\{\tau_i\}_{i\ge 1}$ has been defined in Proposition 2.1. Let $\widehat{S}^{(n)} \in C_0([0, 1] \to \mathbf{R}^d)$ be defined as follows:

$$\widehat{S}^{(n)}(in^{-1}) = n^{-1/2}\widehat{S}_i, \qquad i = 0, 1, \cdots, n,$$

and $\widehat{S}^{(n)}$ is linear on $[(i-1)n^{-1}, in^{-1}]$ for $i = 1, \cdots, n$. By definition, one can check that $\omega \to \widehat{S}^{(n)}(., \omega)$ is a measurable map from $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ to $C_0([0, 1] \to \mathbf{R}^d)$. Let $\mu = \mu_1 \times \mu_2$, and

$$||\widehat{S}^{(n)} - B_n|| = \max_{0\le t\le 1} \{|\widehat{S}^{(n)}(t) - B_n(t)|\}.$$

Since $\lim_{i\to\infty} \tau_i/i = (1, \cdots, 1) \in \mathbf{R}^d$, we can easily prove that

$$\lim_{n\to\infty} \widetilde{\mathbf{E}}||\widehat{S}^{(n)} - B_n|| = 0. \tag{2.1}$$

We now turn to the discussion of the so-called invariance principle for intersection local time.

For any given $l \in [0, [\log_2 n]]$ and $k = 1, \cdots, 2^{l-1}$, we define

$$I(k,l,n) = n^{(d-4)/2} \sum_{i=(2k-2)2^{[\log_2 n]-l}+1}^{(2k-1)2^{[\log_2 n]-l}} \sum_{j=(2k-1)2^{[\log_2 n]-l}+1}^{(2k)2^{[\log_2 n]-l}} \delta(\widehat{S}_i, \widehat{S}_j)$$

$$\alpha_n(x, A) = \int_A \delta_x(B_n(t) - B_n(s))dtds,$$

where δ_x is the δ-function, and $A \subset [2k-2, 2k-1] \times [2k-1, 2k]$. In general, $\alpha_n(x, A)$ is called the intersection local time of $\{B_n(t)\}_{t\geq 0}$ on the set A. Let

$$\alpha_n(x, k, l) = \alpha_n(x, [(2k-2)2^{-l}, (2k-1)2^{-l}] \times [(2k-1)2^{-l}, (2k)2^{-l}]).$$

There has been a lot of investigations about the intersection local time $\alpha_n(x, A)$ (e.g. see [6, 10, 12, 13, 18] and references therein).

Let

$$I_n(k,l) = n^{(d-4)/2} \sum_{i=k2^{[\log_2 n]-l}+1}^{(k+1)2^{[\log_2 n]-l}} \sum_{j=(k+1)2^{[\log_2 n]-l}+1}^{i+2^{[\log_2 n]-l}} \delta(\widehat{S}_i, \widehat{S}_j),$$

$$\alpha_n(k,l) = \alpha_n(0, \{(s,t) : \ k2^{-l} < s \leq (k+1)2^{-l} < t \leq s + 2^{-l}\}).$$

By a similar argument as in [13] (see also [6]) we can use (2.1) to prove that the following holds

$$\lim_{n\to\infty} \widetilde{\mathbf{E}}|I(k,l,n) - \alpha_n(0;k,l)| = 0, \tag{2.2}$$

$$\lim_{n\to\infty} \widetilde{\mathbf{E}}|I_n(k,l) - \alpha_n(k,l)| = 0. \tag{2.3}$$

For $k \geq 1$ we set $Q_k = \{(x,y) \in [0,1]^2 : \ y - x \geq 2^{-k}\}$. Let $\nu'_{k,\lambda}(d\omega)$be a new probability measure on $C_0([0,1] \to \mathbf{R}^3)$ defined by

$$\nu'_{k,\lambda}(d\omega) = \widehat{N}_{k,\lambda}^{-1} \exp(-\lambda(\alpha_1(0, Q_k) - \widetilde{\mathbf{E}}\alpha_1(0, Q_k) + \lambda\chi(k)))\mu(d\omega),$$

where $\chi(k) = (2\pi)^{-2}|\log 2^{-k}|$, $\lambda \in [0, \infty)$ and $\widehat{N}_{k,\lambda}$ is the normalization constant. It was shown in [2] that there is a constant $\lambda_0 > 0$ such that the following holds in the weak sense

$$\lim_{n\to\infty} \nu'_{n,\lambda} = \nu'_\lambda, \qquad \forall \lambda \in [0, \lambda_0). \tag{2.4}$$

Moreover, there is a constant $C \in (0,1]$ such that

$$C \le \widehat{N}_{k,\lambda} \le C^{-1}, \qquad \forall k \ge 1,\ \lambda \in [0,\lambda_0).$$

Let

$$\widehat{T}_{m,n} = n^{-1/2} \sum_{i=1}^{n} \sum_{j=i+m}^{n} \delta(\widehat{S}_i, \widehat{S}_j),$$

It is clear that $T_{m,n}$ and $\widehat{T}_{m,n}$ with $d=3$ have the same distribution.

Let $\psi : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ be bounded and infinitely often differentiable with all bounded derivatives, Ψ_n and $\widehat{\Psi}_n : \Omega' \to (0,\infty)$ be defined by

$$\Psi_n = \exp\Big(\int_0^1 \psi(u, B_n(u))du\Big), \qquad \widehat{\Psi}_n = \exp\Big(\int_0^1 \psi(u, \widehat{S}^{(n)}(u))du\Big).$$

Using (2.2), (2.3) and the Hölder inequality we can show that for any given $\lambda \in [0,\lambda_0)$ there is a sequence $\{m_n\}$ with $\lim_{n\to\infty} n/m_n = \infty$ such that

$$\lim_{n\to\infty} \widetilde{\mathbf{E}}\Big|\Psi_n \exp(-\lambda\alpha_n(0,Q_{n-m_n}) + \lambda\widetilde{\mathbf{E}}\alpha_n(0,Q_{n-m_n}) - \lambda^2\chi(n-m_n)) \\ - \widehat{\Psi}_n \exp(-\lambda\widehat{T}_{m_n,n} + \lambda\widehat{T}_{m_n,n} - \lambda^2\chi(n-m_n))\Big| = 0,$$

It follows by (2.4) that

$$\lim_{n\to\infty} \mu_{m_n,n,\lambda} \xrightarrow{\mathcal{D}} \nu'_\lambda, \qquad n \to \infty.$$

This then completes the proof of (1.1).

To estimate the two point functions, let us first introduce some notations. Let

$$p_n(x)\ (:= p(n,x)) = \mathbf{P}(X_n = x),$$
$$\bar{p}_n(x)\ (:= \bar{p}(n,x)) = 2\Big(\frac{d}{2\pi n}\Big)^{d/2} \exp\Big(-\frac{d|x|^2}{2n}\Big).$$

As in [9], we write $x \leftrightarrow n$ if $p_n(x) > 0$. For later use, we first recall some basic results about $p_n(x)$ and $\bar{p}_n(x)$.

(i) There is a constant $C_1 \in (0,\infty)$ such that for $|x|^2 > n$ (see [7])

$$p_n(x) \le C_1 \bar{p}_{nd}(x).$$

(ii) There is a constant $C_2 \in (0,\infty)$ such that (see [9; Theorem 1.2.1])

$$|p_n(x) - \bar{p}_n(x)| \le C_2 n^{-5/2}, \qquad \text{if } x \leftrightarrow n.$$

(iii) For any given $\alpha \in (1/2, 2/3)$ there is a constant $C_3 \in (0,\infty)$ such that (see [9; Proposition 1.2.5])

$$|p_n(x) - \bar{p}_n(x)| \leq C_3 n^{3\alpha-2}\bar{p}_n(x),$$

if $|x| \leq n^\alpha$ and $x \leftrightarrow n$.

(iv) There is a constant $C_4 \in (0,\infty)$ such that

$$\bar{p}_n * \bar{p}_m(x) \leq C_4 \bar{p}_{n+m}(x), \qquad \forall x \in \mathbf{Z}^3.$$

For any given $n \geq m \geq 1$, set

$$\gamma_j\ (=: \gamma(m,n)) = \sum_{k=m+1}^{n} \sum_{j_1=m}^{k-1} \sum_{i=1}^{j_1\wedge(k-m)} \sum_{y\in\mathbf{Z}^3} p_i(y)p_{j_1-i}(y)p_{k-j_1}(y),$$

$$\beta_k\ (=: \beta(m,n)) = \sum_{j=m}^{\infty} p_j(0).$$

It is easy to show that

$$\gamma(m,n) = \sum_{i=1}^{n} \sum_{j_1=m\vee i}^{n} \sum_{k=j_1\vee(m+i)}^{n} \sum_{y\in\mathbf{Z}^3} p_i(y)p_{j_1-i}(y)p_{k-j_1}(y) + O(1).$$

Remark. One can show that

$$n^{-1/2}\sum_{k=1}^{n}\beta_k = \mathbf{E}T_{m,n} + O(1); \qquad \gamma(m,n) = \frac{1}{2}\,\mathrm{Var}\,(T_{m,n}) + O(1).$$

For convenience, we assume $p_j(x) = 0$ if $j < 0$. Having the above sequences, we can introduce the following quantities. Let

$$J^{m,\lambda}_{i_1,j_1;i_2,j_2} = \lambda n^{-1/2}\sum_{i=i_1}^{j_1}\ \sum_{j=(i_1+m)\vee i_2}^{j_2} \delta(X_i,X_j),$$

$$J^{m,\lambda}_{k,l} = \lambda n^{-1/2}\sum_{i=k}^{l}\sum_{j=i+m}^{l}\delta(X_i,X_j),$$

$$\bar{J}^{m,\lambda}_{k,l} = J^{m,\lambda}_{k,l} - \lambda n^{-1/2}\sum_{i=k}^{l}\beta_i + \lambda^2 n^{-1}\sum_{i=k}^{l}\gamma_i,$$

$$g^{m,\lambda}_k(x) = \mathbf{E}(\exp(-\bar{J}^{m,\lambda}_{1,k})\delta(X_k,x)).$$

For short, we drop m and λ from the above notations, and set $\bar{J}_{l+1,l} = 0$ and

$$g_0(x) = p_0(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Let

$$\mathcal{A}_l^{(1)} = \sum_{k=1}^{l} \mathbf{E}\Big[\exp(-\bar{J}_{k+1,l})\Big(\lambda n^{-1/2} \sum_{i=k+m}^{l} \delta(X_k, X_i) - \lambda n^{-1/2}\beta_k + \lambda^2 n^{-1}\gamma_k\Big)\delta(X_l, x)\Big],$$

$$\mathcal{A}_l^{(2)} = \sum_{k=1}^{l} \mathbf{E}\Big[\exp(-\bar{J}_{k+1,l})(1 - \exp(-\bar{J}_{k,l} + \bar{J}_{k+1,l}) - (\bar{J}_{k,l} - \bar{J}_{k+1,l}))\delta(X_l, x)\Big].$$

Then we have

$$p(l, x) - g_l(x) = \mathcal{A}_l^{(1)} + \mathcal{A}_l^{(2)}. \tag{2.5}$$

Let

$$A_l^{(1)} = \lambda n^{-1/2} \sum_{k=1}^{l} \sum_{j=k+m}^{l} (g_{j-k}(0) p_k * g_{l-j}(x) - \beta_k p_k * g_{l-k}(x)),$$

$$A_l^{(2)} = \lambda^2 n^{-1} \sum_{k=1}^{l} \Big(\gamma_k p_k * g_{l-k}(x) - \sum_{j_1=k+m}^{l} \sum_{i=k+1}^{j_1} \sum_{j=(j_1+1)\vee(i+m)}^{l} p_k$$
$$* (g_{i-k} g_{j_1-i} g_{j-j_1}) * g_{l-j}(x)\Big),$$

$$A_l^{(3)} = \lambda n^{-1/2} \sum_{k=1}^{l} \sum_{j=k+m}^{l} \mathbf{E}\exp(-\bar{J}_{k+1,j} - \bar{J}_{j+1,l})\delta(X_k, X_j)(J_{k+1,j;j+1,l})^2$$
$$+ \lambda^2 n^{-1} \sum_{k=1}^{l} \sum_{j_1=k+m}^{l} \sum_{i=k+1}^{j_1} \sum_{j=(j_1+1)\vee(i+m)}^{l} \mathbf{E}\delta(X_k, X_{j_1})$$
$$\times \exp(-\bar{J}_{k+1,i} - \bar{J}_{i+1,j_1} - \bar{J}_{j_1+1,j} - \bar{J}_{j,l})\delta(X_i, X_j)$$
$$\times \delta(X_l, x)(J_{k+1,i;i+1,j_1} + J_{i+1,j_1;j_1+1,j} + J_{j_1+1,j;j+1,l})^2.$$

Using the fact:

$$1 - x \le e^{-x} \le 1 - x + x^2, \qquad x > 0, \tag{2.6}$$

as in [2] we can show that

$$\lambda A_l^{(1)} - \lambda^2 A_l^{(2)} \le \mathcal{A}_l^{(1)} \le \lambda A_l^{(1)} - \lambda^2 A_l^{(2)} + \lambda^3 A_l^{(3)}. \tag{2.7}$$

We introduce the following quantity.

$$K(m, n) = \sup_{1 \le k \le n} \sup_{x \in \mathbf{Z}^3} \frac{|g_k(x) - p(k, x)|}{(k/n)^{1/2} \bar{p}(2dk, x)} \vee \sup_{m+1 \le i \le n} n^{1/2} \Big| \sum_{j=1}^{i} (g_j(0) - p_j(0)) \Big|.$$

For notational convenience, we set $K = K(m, n)$. As in [2], we let $\phi(x)$ be a generic polynomial in x with nonnegative coefficients, which might be different from line to line.

3. An Estimate for Two Point Function

We use the notations introduced in Sect.2. The main aim of this section is to prove the following theorem.

Theorem 3.1. *There are constants $\lambda_0 > 0$ and $C \in (0, \infty)$ such that*

$$|g_k^{m,\lambda}(x) - p_k(x)| \le C \cdot \left(\frac{k}{n}\right)^{1/2} \bar{p}_{6k}(x), \qquad \forall x \in \mathbf{Z}^3,\ 1 \le k,\ m \le n.$$

We begin with several lemmas.

Lemma 3.2. *The following holds for any $x \in \mathbf{Z}^3$.*

$$A_l^{(1)} \le \lambda\phi(K)(l/n)^{1/2}\bar{p}(2dl, x).$$

Proof. Let

$$I_1 = \lambda n^{-1/2} \sum_{k=1}^{l} \sum_{j=k+m}^{l} p_j * g_{l-j}(x)(g_{j-k}(0) - p_{j-k}(0)),$$

$$I_2 = \lambda n^{-1/2} \sum_{k=1}^{l} \sum_{j=k+m}^{l} g_{j-k}(0)(p_k - p_j) * g_{l-j}(x),$$

$$I_3 = \lambda n^{-1/2} \sum_{k=1}^{l} \left(\sum_{j=k+m}^{l} p_{j-k}(0) p_j * g_{l-j}(x) - \beta_k p_k * g_{l-k}(x) \right).$$

Clearly, we have $A_l^{(1)} = I_1 + I_2 + I_3$, and $I_1 \le \lambda\phi(K)(l/n)^{3/4}\bar{p}(2dl, x)$. From the definition of β_j we can see that I_3 is less than

$$\lambda n^{-1/2} \left(\phi(K) \sum_{j=1}^{l} j^{-1/2} + \sum_{j=1}^{m \wedge l} \beta_j \right) \bar{p}(2dl, x) \le \lambda\phi(K)(l/n)^{1/2}\bar{p}(2dl, x).$$

To consider I_2 we let

$$L_1 = \sum_{k=1}^{l} \sum_{j=k+m}^{l} I_{\{k<j/2\}} \bar{p}(2d(j-k), 0)(p_k + p_j) * \bar{p}(2d(l-j), x),$$

$$L_2 = \sum_{k=1}^{l} \sum_{j=k+m}^{l} I_{\{k \ge j/2\}} g_{j-k}(0) |p_k - p_j| * \bar{p}(2d(l-j), x).$$

Then, $I_2 \le \lambda n^{-1/2}\phi(K)L_1 + \lambda n^{-1/2}L_2$. It is clear that

$$L_1 \le \phi(K)\sum_{k=1}^{l}\sum_{j=k+m}^{l} I_{\{k<j/2\}}\Big(j^{-3/2}+j^{-3/2}I_{\{j<l/2\}}+I_{\{j\ge l/2\}}(l-j+k)^{-3/2}\Big)\bar{p}(2dl,x)$$
$$\le \phi(K)l^{1/2}\bar{p}(2dl,x).$$

To consider L_2, we let $\alpha \in (1/2, 2/3)$ be a fixed constant. We note that $g_{j-k}(0) = 0$ if $x \leftrightarrow k$ and $p_j(x) = 0$, or $x \leftrightarrow j$ and $p_k(x) = 0$. By (iii) we know that if $k \ge j/2$ and $|x| \le j^\alpha$

$$g_{j-k}(0)|p_k(x) - p_j(x)| \le g_{j-k}(0)|\bar{p}_k(x) - \bar{p}_j(x)|(1 + O(j^{3\alpha-2})).$$

It is easy to check that $|\bar{p}_k(x) - \bar{p}_j(x)| \le \phi(K)|j-k|j^{-1}\bar{p}(2j,x)$, if $k \ge j/2$. Therefore,

$$L_2 \le \phi(K)\sum_{k=1}^{l}\sum_{j=k+m}^{l} I_{\{k\ge j/2\}}\bar{p}(2d(j-k),0)\sum_{|y|\le j^\alpha}|j-k|j^{-1}\bar{p}(2j,y)\bar{p}(2d(l-j),x-y)$$
$$+\phi(K)\sum_{k=1}^{l}\sum_{j=k+m}^{l} I_{\{k\ge j/2\}}\bar{p}(2d(j-k),0)j^{-2}\sum_{|y|>j^\alpha}\bar{p}(2dj,y)\bar{p}(2d(l-j),x-y)$$
$$\le \phi(K)l^{1/2}\bar{p}(2dl,x).$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *The following holds for any $x \in \mathbf{Z}^3$.*

$$A_l^{(2)} \le \lambda^2\phi(K)(l/n)^{3/2}\bar{p}(2dl,x).$$

Proof. Let

$$B_1 = \lambda^2 n^{-1}\sum_{k=1}^{l}\sum_{j_1=k+m}^{l}\sum_{i=k+1}^{j_1}\sum_{j=(j_1+1)\vee(i+m)}^{l} p_k * (p_{i-k}p_{j_1-i}p_{j-j_1}$$
$$-g_{i-k}g_{j_1-i}g_{j-j_1}) * g_{l-j}(x),$$

$$B_2 = \lambda^2 n^{-1}\sum_{k=1}^{l}\Big[\gamma_k p_k * g_{l-k}(x) - \sum_{j_1=k+m}^{l}\sum_{i=k+1}^{j_1}\sum_{j=(j_1+1)\vee(i+m)}^{l} p_k$$
$$*(p_{i-k}p_{j_1-i}p_{j-j_1}) * g_{l-j}(x)\Big].$$

It is clear that $A_l^{(2)} = B_1 + B_2$, and

$$B_1 \le \lambda^2 n^{-1}\phi(K)\sum_{k=1}^{l}\sum_{j_1=k+m}^{l}\sum_{i=k+1}^{j_1}\sum_{j=(j_1+1)\vee(i+m)}^{l}(i-k)^a(j_1-i)^b$$
$$\times(j-j_1)^c n^{-(a+b+c)}p_k * \bar{p}_{2d(i-k)}\bar{p}_{2d(j_1-i)}\bar{p}_{2d(j-j_1)} * \bar{p}(2d(l-j),x),$$

where $a,b,c \ge 0$ and $a\vee b\vee c \ge 1/2$. Using these we can show that (see e.g. [2; page 89]) $B_1 \le \lambda^2(l/n)^{3/2}\bar{p}(2dl,x)$.

We now consider B_2. Let

$$D_1 = \lambda^2 n^{-1}\sum_{k=1}^{l}\Bigg[\gamma_k p_k * g_{l-k}(x) - \sum_{j_1=k+m}^{l}\sum_{i=k+1}^{j_1}\sum_{j=(j_1+1)\vee(i+m)}^{l}\sum_{y,z\in\mathbf{Z}^3}p_j(z)$$
$$\times p_{i-k}(z-y)p_{j_1-i}(z-y)p_{j-j_1}(z-y)g_{l-j}(x-z)\Bigg],$$

$$D_2 = \lambda^2 n^{-1}\sum_{k=1}^{l}\sum_{j_1=k+m}^{l}\sum_{i=k+1}^{j_1}\sum_{j=(j_1+1)\vee(i+m)}^{l}\sum_{y,z\in\mathbf{Z}^3}(p_j(z)-p_k(y))$$
$$\times p_{i-k}(z-y)p_{j_1-i}(z-y)p_{j-j_1}(z-y)g_{l-j}(x-z).$$

It is obvious that $B_2 = D_1 + D_2$. For later use, we let

$$\tau = ((i-k)(j_1-i)+(i-k)(j-j_1)+(j_1-i)(j-j_1))^{-3/2},$$
$$\sigma = (i-k)(j_1-i)(j-j_1)\tau^{2/3}.$$

It is easy to check that $\sigma \le j-k$. We remark that $\gamma_k \le O(1)(\log n/m+1)$, and

$$\sum_{k=j}^{n}\sum_{j_1=m}^{k-1}\sum_{i=1}^{j_1\wedge(k-m)}|p_ip_{j_1-i}p_{k-j_1}|_1 \le O(1)(\log n/j+1),$$

where $|p_ap_bp_c|_1 = \sum_{y\in\mathbf{Z}^3}p_a(y)p_b(y)p_c(y)$. However, D_1 is less than

$$\lambda^2 n^{-1}\left|\sum_{k=1}^{l}\left(\gamma_k p_k * g_{l-k}(x) - \sum_{j=k+m+1}^{l}\sum_{j_1=k+m}^{j-1}\sum_{i=k+1}^{j_1\wedge(j-m)}|p_{i-k}p_{j_1-i}p_{j-j_1}|_1 p_j * g_{l-j}(x)\right)\right|$$
$$\le \lambda^2 n^{-1}\sum_{k=1}^{(m+1)\wedge l}\gamma_k p_k * g_{l-k}(x) + \sum_{j=m+2}^{l}\left|\gamma_j - \sum_{k=m+1}^{j-1}\sum_{j_1=m}^{k-1}\sum_{i=1}^{j_1\wedge(k-m)}|p_ip_{j_1-i}p_{k-j_1}|_1\right|$$
$$\times p_j * g_{l-j}(x)$$
$$\le \lambda^2\phi(K)\left((l/n)^{3/4} + \sum_{j=m+2}^{l}\log n/j\right)\bar{p}(2dl,x)$$
$$\le \lambda^2(l/n)^{3/4}\phi(K)\bar{p}(2dl,x).$$

We now consider D_2. In fact, we can use the estimate (iii) to show (see e.g. [2; page 90]) that $|D_2|$ is less than

$$\lambda^2 n^{-1}\phi(K)\sum_{k=1}^{l}\sum_{j_1=k+m}^{l}\sum_{i=k+1}^{j_1}\sum_{j=(j_1+1)\vee(i+m)}^{l}\tau|p_{k+\sigma}-p_j|*g_{l-j}(x)$$

which is bounded from above by $\lambda^2\phi(K)(l/n)\bar{p}(2dl,x)$. This implies the desired result. □

Lemma 3.4. *The following holds for any* $x\in\mathbf{Z}^3$.

$$A_l^{(3)}\le\phi(K)(l/n)^{3/2}\bar{p}(2dl,x).$$

Proof. It is easy to show that

$$\begin{aligned}
&\lambda n^{-1/2}\sum_{k=1}^{l}\sum_{j=k+m}^{l}\mathbf{E}\exp(-\bar{J}_{k+1,j}-\bar{J}_{j+1,l})\delta(X_k,X_j)J^2_{k+1,j;j+1,l}\\
&\le\lambda^3\phi(K)n^{-3/2}\sum_{k=1}^{l}\sum_{j=k+m}^{l}\sum_{t_1=k+1}^{j}\sum_{s_1=t_1}^{j}\sum_{t_2=j+1}^{l}\sum_{s_2=t_2}^{l}\Big((p_k\\
&\quad*((g_{t_1-k}g_{t_2-j})*(g_{s_1-t_1}g_{s_2-t_2}))g_{j-s_1})*g_{l-s_2}(x)\\
&\quad+(p_k*((g_{s_1-t_1}g_{s_2-t_2})*(g_{j-s_1}g_{t_2-j}))g_{t_1-k})*g_{l-s_2}(x)\Big).
\end{aligned}\tag{3.1}$$

Thus we can prove that (see e.g. the argument for $A_T^{(3)}$ given in [2; page 91]) the right hand side of (3.1) is less than $\lambda^3\phi(K)(l/n)^{3/2}\bar{p}(2dl,x)$. Similarly, we can show that the last term in the expression of $A_l^{(3)}$ is also less than $\lambda^3\phi(K)(l/n)^{3/2}\bar{p}(2dl,x)$, which implies the desired result. □

Lemma 3.5. *There is a constant* $\lambda_0>0$ *such that the following holds for any* $x\in\mathbf{Z}^3$ *and* $\lambda\in[0,\lambda_0]$

$$|A_l^{(2)}|\le\lambda^2\phi(K)l/n\bar{p}(2dl,x).$$

Proof. We use (2.6) and the following facts:

$$\gamma_k\le O(1)(1+\log n/m);\qquad e^x=1+x+O(x^2),$$

if x is bounded, to show, as in the proofs of Lemma 3.2 and Lemma 3.3 that, for $\lambda<\lambda_0$

$$A_l^{(2)}\le\sum_{k=1}^{l}\mathbf{E}\Big\{\exp(-\bar{J}_{k+1,l})\Big[\lambda n^{-1/2}\beta_k-\lambda^2n^{-1}\gamma_k+1-\exp(\lambda n^{-1/2}\beta_k-\lambda^2n^{-1}\gamma_k)$$

$$\times\left(1-\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)\right)-\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)\Bigg]\delta(X_l,x)\Bigg\}$$

$$=\sum_{k=1}^{l}\mathbf{E}\Big\{\exp(-\bar{J}_{k+1,l})\Big[O((\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k)^2)+(\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k$$

$$+O((\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k)^2))\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)\Big]\delta(X_l,x)\Big\}$$

$$\le \lambda^2\phi(K)\frac{l}{n}\bar{p}(2dl,x).$$

On the other hand, we have

$$\mathcal{A}_l^{(2)}\ge\sum_{k=1}^{l}\mathbf{E}\Big\{\exp(-\bar{J}_{k+1,l})\Big[1-\exp(\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k)$$

$$\times\left(1-\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)+\lambda^2 n^{-1}\Big(\sum_{j=k+m}^{l}\delta(X_j,X_k)\Big)^2\right)$$

$$-\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)+\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k\Big]\cdot\delta(X_l,x)\Big\}$$

$$\ge\sum_{k=1}^{l}\mathbf{E}\Big\{\exp(-\bar{J}_{k+1,l})\Big[-O((\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k)^2)$$

$$\times\left(1-\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)+\lambda^2 n^{-1}\Big(\sum_{j=k+m}^{l}\delta(X_j,X_k)\Big)^2\right)$$

$$+(\lambda n^{-1/2}\beta_k-\lambda^2 n^{-1}\gamma_k)\Big(\lambda n^{-1/2}\sum_{j=k+m}^{l}\delta(X_j,X_k)-\lambda^2 n^{-1}$$

$$\times\Big(\sum_{j=k+m}^{l}\delta(X_j,X_k)\Big)^2\Big)-\lambda^2 n^{-1}\Big(\sum_{j=k+m}^{l}\delta(X_j,X_k)\Big)^2\Big]\cdot\delta(X_l,x)\Big\}.$$

By the arguments as for $A_l^{(3)}$ given before we can show that

$$n^{-1}\sum_{k=1}^{l}\mathbf{E}\Big\{\exp(-\bar{J}_{k+1,l})\Big(\sum_{j=k+m}^{l}\delta(X_j,X_k)\Big)^2\delta(X_l,x)\Big\}\le\phi(K)\frac{l}{n}\bar{p}(2dl,x);$$

$$n^{-1}\sum_{k=1}^{l}\mathbf{E}\Big\{\exp(-\bar{J}_{k+1,l})\sum_{j=k+m}^{l}\delta(X_j,X_k)\delta(X_l,x)\Big\}\le\phi(K)\frac{l}{n}\bar{p}(2dl,x).$$

Hence, we obtain that for some $\lambda_0 > 0$

$$\mathcal{A}_l^{(2)} \geq -\lambda^2 \phi(K) \frac{l}{n} \bar{p}(2dl, x), \qquad \lambda \in [0, \lambda_0],$$

which implies the desired result. □

We are now in a position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. By (2.5) and (2.7) we know that

$$\lambda A_l^{(1)} - \lambda^2 A_l^{(2)} + \mathcal{A}_l^{(2)} \leq p_l(x) - g_l(x) \leq \lambda A_l^{(1)} - \lambda^2 A_l^{(2)} + \lambda^3 A_l^{(3)} + \mathcal{A}_l^{(2)}.$$

By Lemma 3.2 – Lemma 3.5 we know that $|p_l(x) - g_l(x)| \leq \lambda \phi(K)(l/n)^{1/2}\ \bar{p}(2dl, x)$. We now show that for any $i \in [1, n]$

$$\left| \sum_{j=1}^{i} (g_{i-j}(0) - p_{i-j}(0)) \right| \leq \lambda \phi(K) n^{-1/2}. \tag{3.2}$$

If $\lambda \in [0, \lambda_0]$, $x = 0$ and $i \leq n$, then l.h.s. of (3.2) is less than

$$\begin{aligned}
&\leq \lambda \left| \sum_{j=1}^{i} A_{i-j}^{(1)} \right| + \lambda^2 \left| \sum_{j=1}^{i} A_{i-j}^{(2)} \right| + \lambda^3 \left| \sum_{j=1}^{i} A_{i-j}^{(3)} \right| + \left| \sum_{j=1}^{i} \mathcal{A}_{i-j}^{(2)} \right| \\
&\leq \lambda \left| \sum_{j=1}^{i} A_{i-j}^{(1)} \right| + \lambda^2 \phi(K) \left(n^{-1} \sum_{j=1}^{i} (i-j)^{-1/2} + n^{-3/4} \sum_{j=1}^{i} (i-j)^{-3/4} \right) \\
&\leq \lambda \left| \sum_{j=1}^{i} A_{i-j}^{(1)} \right| + \lambda^2 \phi(K) n^{-1/2}.
\end{aligned}$$

Thus, it suffices to prove the following for $x = 0$

$$\left| \sum_{j=1}^{i} A_{i-j}^{(1)} \right| \leq \Phi(K) \lambda n^{-1/2}. \tag{3.3}$$

By the argument as for I_1 given in the proof of Lemma 3.2 we can show that for $i \geq m+1$

$$\left| \sum_{l=1}^{i} \sum_{k=1}^{l} \sum_{j=k+m}^{l} p_k * g_{l-j}(0) p_{j-k}(0) - \sum_{l=1}^{i} \sum_{k=1}^{l} \sum_{j=m}^{\infty} p_k * g_{l-j}(0) g_{j-k}(0) \right| \leq \phi(K).$$

By definition we know that for $x = 0$

$$A_l^{(1)} = \lambda n^{-1/2} \sum_{k=1}^{l} \left(\sum_{j=k+m}^{l} p_k * g_{l-j}(0) g_{j-k}(0) - \beta_k p_k * g_{l-k}(0) \right).$$

Thus, by computation we can show that the left hand side of (3.3) is less than

$$\sum_{k=1}^{i-m}\sum_{l=0}^{i-k-m} p_k * g_l(0) \sum_{j=i-l-k+1}^{\infty} p_j(0) + \sum_{k=i-m+1}^{i}\sum_{l=0}^{i-k} p_k * g_l(0) \sum_{j=m}^{n} p_j(0)$$
$$+\sum_{k=1}^{i}\sum_{l=(i-k-m)\vee 1}^{i-k} p_k * g_l(0) \sum_{j=m}^{\infty} p_j(0) \le \phi(K),$$

which proves (3.3). Thus, the proof of (3.2) is complete.

Combining (2.5), (2.7) with Lemma 3.2 – Lemma 3.5, we can see that there are a constant $\lambda_0 > 0$ and a generic polynomial with nonnegative coefficients ϕ such that

$$K(m,n) \le \lambda\phi(K(m,n)), \qquad \lambda \in [0,\lambda_0],\ \forall m \le n.$$

Note that $K(m,n) = 0$ if $\lambda = 0$. It is clear that $K(m,n)$ is continuous with respect to $\lambda \in [0,\infty)$. Thus, we can see that

$$K(m,n) \le 1, \qquad \forall\lambda \le (\phi(1)\vee 1)^{-1},\ \forall m \le n,$$

which proves the desired result. The proof of Theorem 3.1 is complete. □

By Theorem 3.1 we know that for any given $\varepsilon \in (0,1)$ there is a constant $\lambda(\varepsilon) > 0$ such that

$$K(m,n) \le \varepsilon, \qquad \forall\lambda \le \lambda(\varepsilon),\ \forall m \le n,$$

and so

$$\sum_{x\in\mathbf{Z}^3} p_n(x) - \varepsilon \sum_{x\in\mathbf{Z}^3, x\leftrightarrow n} \bar{p}(2dn,x) \le \sum_{x\in\mathbf{Z}^3} g_n(x) \le \sum_{x\in\mathbf{Z}^3} p_n(x) + \varepsilon \sum_{x\in\mathbf{Z}^3, x\leftrightarrow n} \bar{p}(2dn,x),$$

if $\lambda < \lambda(\varepsilon)$. Remark that

$$\bar{J}_{k,l}^{m,\lambda} = \lambda T_{m,n} - \lambda n^{-1/2}\sum_{k=1}^{n}\beta_k + \lambda^2 n^{-1}\sum_{k=1}^{n}\gamma_k. \tag{3.4}$$

Thus we can find $\lambda_1 > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\varepsilon_0 \le \mathbf{E}\exp\left(- \lambda T_{m,n} + \lambda n^{-1/2}\sum_{k=1}^{n}\beta_k - \lambda^2 n^{-1}\sum_{k=1}^{n}\gamma_k\right) \le \varepsilon_0^{-1}$$

for all $\lambda \in [0,\lambda_1)$, $m \le n$ and $n \ge 1$.

4. An Extention of Theorem 3.1

In this section we first prove a lemma (see Lemma 4.1 below) which concerns the estimates of $g_k^{m_n,\lambda}(x)$ for any given $\lambda \in [0,\infty)$. Using this lemma we can derive an estimate for the normalization constant with all coupling constants $\lambda \in [0,\infty)$ (see Proposition 4.2 below). Let $\mu_{k,l}$ be the measure on $(\Omega', \mathcal{F}', \mathbf{P})$ defined by

$$\mu_{k,l}(A) = \int_A \exp(-\bar{J}_{k,l}^{m_n,\lambda})\mathrm{d}\mathbf{P}.$$

In particular we denote $\mu_{1,n}$ by μ_n. Lemma 4.1 (ii) can be thought of as an extension of Theorem 3.1.

Lemma 4.1. *(i) For any given $\lambda \in [0,\infty)$ there are constants $C_1, C_2 \in (0,\infty)$ such that*

$$\mu_n(X_l = x) \le C_1 \bar{p}_{C_2 l}(x), \qquad \forall x \in \mathbf{Z}^3,\ l \le n, s \in [0,1],$$

where $\bar{p}_u(x)$ was defined in Sect.2.

(ii) For any given $\lambda \in [0,\infty)$ there is a constant $C_3 \in (0,\infty)$ such that

$$|\mu_{1,l}(X_l = x) - p_l(x)| \le C_3 (l/n)^{1/2} \bar{p}_{C_2 l}(x)$$

for $x \in \mathbf{Z}^3$, $s \in [0,1]$ and $l \le n$.

Proof. (i) By Theorem 3.1 we know that there is a constant $\lambda_0 > 0$ such that

$$|\mu_n(X_n = x) - p_n(x)| \le O(1)\bar{p}_{6n}(x), \qquad \forall x \in \mathbf{Z}^3, \lambda \in [0,\lambda_0).$$

Let

$$\hat{J}_1(n) = \lambda n^{-1/2} \sum_{i=1}^{[n/2]} \sum_{j=i+m_n}^{[n/2]} \delta(X_i, X_j) - \lambda n^{-1/2} \sum_{k=1}^{[n/2]} \beta_k + \lambda^2 n^{-1} \sum_{k=1}^{[n/2]} \gamma_k;$$

$$\hat{J}_2(n) = \lambda n^{-1/2} \sum_{i=[n/2]+1}^{n} \sum_{j=([n/2]+1)\vee(i+m_n)}^{n} \delta(X_i, X_j) - \lambda n^{-1/2} \sum_{k=[n/2]+1}^{n} \beta_k$$
$$+ \lambda^2 n^{-1} \sum_{k=[n/2]+1}^{n} \gamma_k.$$

If $\lambda \in [0, 2^{1/2}\lambda_0)$ and $n \ge 1$ is large enough, then

$$\begin{aligned}
\mu_n(X_n = x) &\le \mathbf{E}\exp(-\hat{J}_1(n) - \hat{J}_2(n))\delta(X_n, x) \\
&= \sum_{y\in\mathbf{Z}^3} g_{[n/2]}^{m_n,([n/2]/n)^{1/2}\lambda}(y) g_{n-[n/2]}^{m_n,((n-[n/2])/n)^{1/2}\lambda}(x-y) \\
&\le O(1) \sum_{y\in\mathbf{Z}^3} \bar{p}_{6[n/2]}(y)\bar{p}_{6(n-[n/2])}(x-y) \\
&\le O(1)\bar{p}_{12n}(x),
\end{aligned}$$

where $g_k^{m,\lambda}(x)$ was defined in Sect.2. It follows that $\mu_n(X_n = x) \le O(1)\,\bar{p}_{12n}(x)$, $\lambda \in [0, 2^{1/2}\lambda_0)$. By induction on $k \ge 1$ we can show that $\mu_n(X_n = x) \le O(1)\bar{p}_{2^k 3n}(x)$, $\lambda \in [0, 2^{k/2}\lambda_0)$. Thus we have actually proved that for any given $\lambda \in [0, \infty)$ there is a constant $C_2 = C_2(\lambda) \in (0, \infty)$ such that

$$\mu_n(X_n = x) \le O(1)\bar{p}_{C_2 n}(x), \qquad \forall x \in \mathbf{Z}^3, \forall\, n \ge 1.$$

From this we can see that there is a $C_4(\lambda) \in (0, \infty)$ such that

$$\mathbf{E}\exp(-\bar{J}_{1,n}^{m,\lambda}) = \mu_n(\widetilde{\Omega}) \le C_4(\lambda), \qquad \forall n \ge 1.$$

Therefore,

$$\begin{aligned}\mu_n(X_l = x) &\le \int_{\{X_l = x\}} \exp(-\bar{J}_{1,l}^{m_n,\lambda}) \int_{\Omega'} \exp(-\bar{J}_{l+1,n}^{m_n,\lambda})\mathrm{d}\mathbf{P} \\ &\le C_4(\lambda)\mu_l(X_l = x) \le O(1)\bar{p}_{C_2 l}(x), \qquad \forall x \in \mathbf{Z}^3,\ l \le n,\end{aligned}$$

which proves (i).

(ii) From our definitions we see that $\mu_{1,l}(X_l = x) = g_l^{m_n,\lambda}(x)$. By Theorem 3.1 we know that if $\lambda \in [0, \lambda_0)$

$$|\mu_{1,l}(X_l = x) - p_l(x)| \le O(1)(l/n)^{1/2}\bar{p}_{6l}(x), \qquad 1 \le l \le n.$$

We remark that if $l \le 2^{-1/2}n$

$$\begin{aligned}\mu_{1,l}(X_l = x) = \mathbf{E}\Bigg(\exp\Bigg(&-\lambda(n/2)^{-1/2}\sum_{i=1}^{l}\sum_{j=i+m_n}^{l}\delta(X_i, X_j) \\ &+\lambda(n/2)^{-1/2}\sum_{i=1}^{l}\beta_i - \lambda^2\frac{2}{n}\sum_{i=1}^{l}\gamma_i\Bigg)\delta(X_l, x)\Bigg).\end{aligned}$$

By Theorem 3.1 again we know that if $\lambda \in [0, 2^{1/2}\lambda_0)$

$$|\mu_{1,l}(X_l = x) - p_l(x)| \le O(1)(l/n)^{1/2}\bar{p}_{6l}(x), \qquad 1 \le l \le 2^{-1/2}n.$$

If $l \ge 2^{-1/2}n$, by (i) we know that

$$|\mu_{1,l}(X_l = x) - p_l(x)| \le O(1)(l/n)^{1/2}\bar{p}_{C_2 l}(x), \qquad \forall x \in \mathbf{Z}^3.$$

Therefore, if $\lambda \in [0, 2^{1/2}\lambda_0)$,

$$|\mu_{1,l}(X_l = x) - p_l(x)| \le C_3(l/n)^{1/2}\bar{p}_{C_2 l}(x), \qquad \forall x \in \mathbf{Z}^3.$$

By induction on $k \ge 1$ we can show that the above estimate also holds for $\lambda \in [0, 2^{k/2}\lambda_0)$ (of course, the constants C_2 and C_3 depend on $k \ge 1$). This then ends the proof. □

Proposition 4.2. *For any given $\lambda \in [0,\infty)$ there are $C_5, C_6 \in (0,\infty)$ such that*

$$C_5 \le \mathbf{E}\exp(-\bar{J}_{1,n}^{m_n,\lambda}) \le C_6, \quad m_n \in [1,n], \forall n \ge 1.$$

Proof. The upper bound is an immediate consequence of Lemma 4.1 (i). We need only to show the lower bound. Let $\{X_i'\}$ be an independent copy of $\{X_i\}$ under a finite measure $\bar{\mu}_n$, and $\bar{\mu}_n(X_{i_1} = x_1, \cdots, X_{i_k} = x_k) = \bar{\mu}_n(X'_{i_1} = x_1, \cdots, X'_{i_k} = x_k)$ $= \mu_n(X_{i_1} = x_1, \cdots, X_{i_k} = x_k)$. By Lemma 4.1 (i) we know that there is a constant $C_7(\lambda) \in (0,\infty)$ such that

$$\begin{aligned} & n^{-1/2}\mathbf{E}_{\bar{\mu}_n} \sum_{i=1}^{n}\sum_{j=1}^{n} \delta(X_i, X_j') \\ & = n^{-1/2} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{x\in\mathbf{Z}^3} \mu_n(X_i = x)\mu_n(X_j = x) \le C_7(\lambda), \quad \forall n \ge 1. \end{aligned}$$

For convenience we may assume that n is even. By Jensen's inequality we have

$$\begin{aligned} & \mathbf{E}\exp(-\bar{J}_{1,n}^{m_n,2^{1/2}\lambda}) \ge \mathbf{E}_{\bar{\mu}_{n/2}} \exp\Big(-2^{1/2}\lambda n^{-1/2} \sum_{i=1}^{n/2}\sum_{j=1}^{n/2} \delta(X_i, X_j') \Big) \\ & \ge \mathbf{E}_{\bar{\mu}_{n/2}} \cdot \exp\Big(-(\mathbf{E}_{\bar{\mu}_{n/2}}1)^{-1} 2^{1/2}\lambda n^{-1/2} \mathbf{E}_{\bar{\mu}_{n/2}} \sum_{i=1}^{n/2}\sum_{j=1}^{n/2} \delta(X_i, X_j') \Big) \\ & \ge \mathbf{E}_{\bar{\mu}_{n/2}}1 \exp(-2^{1+1/2}\lambda C_7(\lambda)(\mathbf{E}_{\bar{\mu}_n}1)^{-1}). \end{aligned}$$

We remark that $\mathbf{E}_{\bar{\mu}_{n/2}} = (\mathbf{E}_{\mu_{n/2}}1)^2 = (\mathbf{E}\exp(-\bar{J}_{1,n/2}^{m_n,\lambda}))^2$. Then, there is a constant $C_8 \in (0,\infty)$ such that $\mathbf{E}_{\bar{\mu}_{n/2}}1 \ge C_8(\lambda)$, $n \ge 1$, $\lambda \in [0, \lambda_0)$, which implies

$$\mathbf{E}\exp(-\bar{J}_{1,n}^{m_n,2^{1/2}\lambda}) \ge C_8(\lambda)^2 \exp(-2^{1/2}\lambda C_7(\lambda) C_8(\lambda)^{-1}).$$

By induction on $k \ge 1$ we can show that $\mathbf{E}\exp(-\bar{J}_{1,n}^{m_n,2^{k/2}\lambda}) \ge C(k)$, $\forall n \ge 1$ for some constant $C(k) \in (0,\infty)$. This proves the desired result. □

5. Proof of Theorem 1.1

Let $M \ge 1$ be a given integer. For $k \in [1,n]$ we let $\beta(k,n) = \sum_{j=k}^{\infty} p_j(0)$. For $k \ge M$ we let

$$\gamma(k,n) = \sum_{i=1}^{n}\sum_{j=k\vee i}^{n}\sum_{l=j\vee(k+i)}^{n}\sum_{y\in\mathbf{Z}^3} p_i(y)p_{j-i}(y)p_{l-j}(y).$$

To define $\gamma(k,n)$ for $k \le M-1$, let us first introduce some notations. Let $J^{k+1}_{i,j;u,v}(x) = n^{-1/2}\sum_{s=i+1}^{j}\sum_{t=(u+1)\vee(s+k+1)}^{v}\delta(X_t - X_s, x)$, and

$$\alpha_1(k) = n^{-1}\sum_{i=1}^{n-k}\Big(\mathbf{E}\delta(X_i, X_{i+k})\sum_{j=i-k+1}^{i+k}\delta(X_j, X_{j+k}) - 2kp_k^2(0)\Big),$$

$$\alpha_2(k) = n^{-1/2}\sum_{i=1}^{n-k}\mathbf{E}\delta(X_i, X_{i+k})(J^{k+1}_{1,i;i,k+i} + J^{k+1}_{i,i+k;i+k,n}),$$

where $J^{k+1}_{i,j;u,v} = J^{k+1}_{i,j;u,v}(0)$. In general, $\alpha_1(k) \neq 0$, if $k \le M-1$. It is easy to show that $\alpha_2(k)$ is bounded uniformly on $n \ge 1$. Let y_k be the positive solution of the following equation:

$$\lambda^2 y^2 - \lambda^2 y\alpha_2(k) - \alpha_1(k) = 0. \tag{5.1}$$

For short, we let $\bar\gamma(k) = \gamma(k,n) - \gamma(k+1,n)$. We now let $\bar\gamma(k) = y_k$ for $1 \le k \le M-1$. Using this we can define $\gamma(k,n)$ for $1 \le k \le M-1$. Having $\beta(k,n)$ and $\gamma(k,n)$, as in Sect.2 we can define $\bar J^{k,\lambda}_{l,m}$. Let

$$\Psi_{s,t} = \exp\Big(\int_s^t \psi(u, X^{(n)}(u))\mathrm{d}u\Big),$$

where $X^{(n)}$ was defined in Sect.1, and ψ was defined in Sect.4. For short, we let $\Psi = \Psi_{0,1}$. For $1 \le k \le n$, we let $\rho(k) = \rho(k,\psi)$, where $\rho(k,\psi) = \mathbf{E}\Psi\exp(-\bar J^{k,\lambda}_{1,n})$. Further we let $\rho'(M,\psi) = \rho(M,\psi)$ and

$$\rho'(k,\psi) = \Big(\frac{\alpha_1(M-1)}{\bar\gamma(k)} + 1\Big)\cdots\Big(\frac{\alpha_1(k)}{\bar\gamma(k)} + 1\Big)\rho(k,\psi), \quad 1 \le k \le M-1.$$

For short, we let $\rho'(k) = \rho'(k,\psi)$. In this section we will mainly prove that there are constants $\delta \in (0,1)$ and $C, \delta' \in (0,\infty)$, independent of M, such that

$$|\rho(k) - \rho(k+1)| \le Cn^{-1}\Big(\frac{n}{k}\Big)^{\delta} + Ck^{-2}, \quad M \le k \le n, \forall n \ge 1, \tag{5.2}$$

$$|\rho'(k) - \rho'(k-1)| \le Cn^{-\delta'}, \qquad k \le M, \forall n \ge 1. \tag{5.3}$$

By (5.2) and (5.3) we know that there is a constant $C' \in (0,\infty)$, independent of M, such that

$$|\rho(m_n) - \rho'(1)| \le O(1)n^{-\delta'} + C'M^{-1} + O(1)\int_{M/n}^{m_n/n} x^{-\delta}\mathrm{d}x,$$

where $\{m_n\}$ satisfies (1.1). Since $\lim_{n\to\infty} m_n/n = 0$ and $M \ge 1$ is arbitrary, we get that $\lim_{n\to\infty}|\rho(m_n) - \rho'(1)| = 0$ for all ψ. Remark that

$$\mathbf{E}_{\mu_{m_n,n,\lambda}}\Psi = \frac{\rho(m_n,\psi)}{\rho(m_n,0)}, \qquad \mathbf{E}_{\mu_{1,n,\lambda}}\Psi = \frac{\rho'(1,\psi)}{\rho'(1,0)}.$$

From these expressions we see that the assertion of Theorem 1.1 is correct, provided (5.2) and (5.3) have been proved. The remainder of this section is devoted to the proofs of (5.2) and (5.3).

Let us introduce some other notations.

$$\chi^k_{l,m} = \lambda n^{-1/2} \sum_{i=l+1}^{m} \delta(X_i, X_{i+k}) - \lambda n^{-1/2}(m-l)p_k(0) + \lambda^2 n^{-1}(m-l)\bar{\gamma}(k),$$

$$\bar{J}^i_{1,n} = \bar{J}^{k+1,\lambda}_{1,n} + \bar{J}^{k+1,\lambda}_{k+i,n} + \lambda J^{k+1}_{1,i;i+k,n},$$

$$\chi_k(i) = \lambda n^{-1/2} \sum_{j=1}^{i-k} \delta(X_j, X_{j+k}) + \lambda n^{-1/2} \sum_{j=i+k+1}^{n-k} \delta(X_j, X_{j+k})$$
$$-\lambda n^{-1/2}(n-k)p_k(0) + \lambda^2 n^{-1}(n-k)\bar{\gamma}(k).$$

For short, we let $\chi^k_l = \chi^k_{1,l}$. It is obvious that $\bar{J}^{k+1,\lambda}_{1,n}$ is equal to

$$\bar{J}^i_{1,n} + \lambda J^{k+1}_{1,i;i,k+i} + \lambda J^{k+1}_{i,i+k;i+k,n} - \lambda n^{-1/2} k\beta(k+1,n) + \lambda^2 n^{-1} k\gamma(k+1,n),$$

and χ^k_n is equal to

$$\chi_k(i) + \lambda \sum_{j=i-k+1}^{i+k} \delta(X_j, X_{j+k}) - \lambda n^{-1/2} k p_k(0) + \lambda^2 n^{-1} k\bar{\gamma}(k).$$

We remark that for any given $m_0 < \infty$ there are constants $C_1, C_2 \in \mathbf{R}^1$ such that

$$1 - x + y + C_1|x-y|^2 \le e^{-x+y} \le 1 - x + y + C_2|x-y|^2, \quad x > 0,\ y \le m_0, \tag{5.4}$$

which will be denoted by $e^{-x+y} = 1 - x + y + O(1)|x-y|^2$. This kind of estimates will be used many times in the following discussions. Let us first prove a lemma.

Lemma 5.1. *Let f be a measurable function and $1 \le j_1 < \cdots < j_m \le n$. Then*

$$\mathbf{E}\int_0^1 \exp(-\bar{J}^i_{1,n} - s\chi_k(i))|f(X_{j_1}, \cdots, X_{j_m})|ds \le O(1)\mathbf{E}|f(X_{6j_1}, \cdots, X_{6j_m})|.$$

Proof. Using (5.4) we can show that the left hand side of the above estimate is less than $O(1)\mathbf{E}\exp(-\bar{J}^{k+1,\lambda}_{1,i} - \bar{J}^{k+1,\lambda}_{i+k,n})|f(X_{j_1}, \cdots, X_{j_m})|$. Using Lemma 4.1 we can show that the above quantity is then less than $O(1)\mathbf{E}|f(X_{6j_1}, \cdots, X_{6j_m})|$. This proves the desired result. □

Since $\gamma(k+1,n) \le O(1)\log n/k$, we can use Lemma 5.1 to show that there is a constant $\delta \in (0,1)$ such that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^i_{1,n} - s\chi_k(i))\delta(X_i, X_{i+k})\Big(n^{-1}k\gamma(k+1,n) + n^{-1}ks|\bar{\gamma}(k)|$$
$$+|\bar{J}^{k+1,\lambda}_{1,n} - \bar{J}^i_{1,n} - s\chi^k_n + s\chi_k(i)|^2\Big)ds \le O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^\delta.$$

From now on, δ is always assumed to be a constant in $(0,1)$. By (5.4) we know that

$$n^{-1/2}\sum_{i=1}^{n-k}\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}_{1,n}^{k+1,\lambda}-s\chi_n^k)\delta(X_i,X_{i+k})\mathrm{d}s$$

$$= n^{-1/2}\sum_{i=1}^{n-k}\int_0^1 \mathbf{E}\Psi\exp(-\tilde{J}_{1,n}^i-s\chi_k(i))\delta(X_i,X_{i+k})\Big[1-\lambda J_{1,i;i,k+i}^{k+1}-\lambda J_{i,i+k;i+k,n}^{k+1}$$

$$+\lambda n^{-1/2}k\beta(k+1,n)-\lambda^2 n^{-1}k\gamma(k+1,n)-\lambda n^{-1/2}s\sum_{j=i-k+1}^{i+k}\delta(X_j,X_{j+k})$$

$$+\lambda s n^{-1/2}kp_k(0)-s\lambda^2 n^{-1}k\bar{\gamma}(k)+O(1)(\bar{J}_{1,n}^{k+1,\lambda}-\tilde{J}_{1,n}^i-s\chi_n^k+s\chi_k(i))^2\Big]\mathrm{d}s$$

$$= O(1)n^{-1}\Big(\frac{n}{k}\Big)^{\delta}+n^{-1/2}\sum_{i=1}^{n-k}\int_0^1 \mathbf{E}\Psi\exp(-\tilde{J}_{1,n}^i-s\chi_k(i))\delta(X_i,X_{i+k})$$

$$\times\Big[1-\lambda J_{1,i;i,k+i}^{k+1}-\lambda J_{i,i+k;i+k,n}^{k+1}+\lambda n^{-1/2}k\beta(k+1,n)$$

$$-\lambda n^{-1/2}s\sum_{j=i-k+1}^{i+k}\delta(X_j,X_{j+k})+\lambda s n^{-1/2}kp_k(0)\Big]\mathrm{d}s. \qquad (5.5)$$

Using this we can show that $\rho(k+1)-\rho(k)$ is equal to

$$\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}_{1,n}^{k+1,\lambda}-s\chi_n^k)\chi_n^k\mathrm{d}s$$

$$= O(1)n^{-1}\Big(\frac{n}{k}\Big)^{\delta}+\lambda n^{-1/2}\sum_{i=1}^{n-k}\int_0^1 \mathbf{E}\Psi\exp(-\tilde{J}_{1,n}^i-s\chi_k(i))\delta(X_i,X_{i+k})\Big[1-\lambda J_{1,i;i,i+k}^{k+1}$$

$$-\lambda J_{i,i+k;i+k,n}^{k+1}+\lambda n^{-1/2}k\beta(k+1,n)-\lambda n^{-1/2}s\sum_{j=i-k+1}^{i+k}\delta(X_j,X_{j+k})$$

$$+\lambda s n^{-1/2}kp_k(0)\Big]\mathrm{d}s-(\lambda n^{1/2}p_k(0)-\lambda^2\bar{\gamma}(k))\bar{\rho}(k), \qquad (5.6)$$

where $\bar{\rho}(k)=\int_0^1\mathbf{E}\Psi\exp(-\bar{J}_{1,n}^{k+1,\lambda}-s\chi_n^k)\mathrm{d}s$.

Let θ_s be the shift operator, and

$$Y_i(k)=\lambda n^{-1/2}\sum_{u=i-k+1}^{i}\sum_{v=i+1}^{u+k}\delta(X_v,X_u),$$

$$\widetilde{Y}_s(i,k)=s\lambda n^{-1/2}\sum_{u=i-k+1}^{i}\delta(X_u,X_{u+k})-Y_i(k).$$

Using Lemma 5.1 we can show that

$$\int_0^1 p_k(0)\mathbf{E}\Psi\exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k})\widetilde{Y}^2_s(i,k)\mathrm{d}s \le O(1)n^{-1}k^{1/2}.$$

Since $\widetilde{Y}_s(i,k) < 0$, we can use (5.4) and Lemma A.1 below to show that

$$\begin{aligned}
&\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}^i_{1,n} - s\chi_k(i))\delta(X_i, X_{i+k})\mathrm{d}s\\
&= O(1)n^{-1}k^{-1/2} + \int_0^1 \mathbf{E}\Psi_{0,i/n}\Psi_{(i+k)/n,1}\circ\theta^{-1}_{k/n}p_k(0)\\
&\qquad \times\exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k} + \widetilde{Y}_s(i,k))\mathrm{d}s\\
&= O(1)n^{-1}k^{-1/2} + \int_0^1 p_k(0)\mathbf{E}\Psi\exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k})\\
&\qquad \times(1 + \widetilde{Y}_s(i,k) + O(1)\widetilde{Y}^2_s(i,k))\mathrm{d}s\\
&= O(1)n^{-1}k^{-1/2} + p_k(0)\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k})(1 + \widetilde{Y}_s(i,k))\mathrm{d}s\\
&= O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^{\delta} + p_k(0)\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)(1 + \mathbf{E}\widetilde{Y}_s(i,k))\mathrm{d}s \qquad (5.7)
\end{aligned}$$

for some $\delta \in (0,1)$. By computation we can show that there is $\delta \in (0,1)$ such that

$$\mathbf{E}Y_i(k) = \lambda n^{-1/2}k\beta(k+1,n) + O(1)n^{-3/2}k^{3/2}\Big(\frac{n}{k}\Big)^{\delta}.$$

Thus, the left hand side of (5.7) is equal to

$$\begin{aligned}
&O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^{\delta} + p_k(0)\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)\\
&\qquad \times(1 + s\lambda n^{-1/2}kp_k(0) - \lambda n^{-1/2}k\beta(k+1,n))\mathrm{d}s.
\end{aligned}$$

Combining the above estimate with (5.6) and Lemma A.2 (i) below we see that $\rho(k+1) - \rho(k)$ is equal to

$$\begin{aligned}
&O(1)n^{-1}\Big(\frac{n}{k}\Big)^{\delta} + \lambda n^{-1/2}\sum_{i=1}^{n-k}\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}^i_{1,n} - s\chi_k(i))\delta(X_i, X_{i+k})\Big(2\lambda s n^{-1/2}kp_k(0)\\
&-\lambda n^{-1/2}s\sum_{j=i-k+1}^{i+k}\delta(X_j, X_{j+k})\Big)\mathrm{d}s - (\lambda^2\alpha_2(k) - \lambda^2\bar{\gamma}(k))\bar{\rho}(k). \qquad (5.8)
\end{aligned}$$

As the argument given for (5.5) we can show that

$$\int_0^1 \mathbf{E}\Psi \exp(-\tilde{J}^i_{1,n} - s\chi_k(i))\delta(X_i, X_{i+k})ds = O(1)n^{-3/2}\left(\frac{n}{k}\right)^\delta + p_k(0)\bar{\rho}(k). \qquad (5.9)$$

Thus, by Lemma A.3 below, Lemma A.2 (ii) and (5.8) we know that $\rho(k+1) - \rho(k)$ is equal to

$$O(1)n^{-1}\left(\frac{n}{k}\right)^\delta + \left(2\lambda^2 k p_k^2(0) - \lambda^2 \mathbf{E}\delta(X_k, X_{2k})\sum_{j=1}^{2k}\delta(X_j, X_{j+k})\right)$$
$$\times \int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)sds = O(1)n^{-1}\left(\frac{n}{k}\right)^\delta + O(1)k^{-2},$$

if $k \geq M$, which proves (5.2).

We now prove (5.3). By (5.8), (5.9) and Lemma A.3 below we know that $\rho(k+1) - \rho(k)$ is equal to

$$O(1)n^{-1}\left(\frac{n}{k}\right)^\delta - \lambda^2\alpha_1(k)\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)sds - \lambda^2(\alpha_2(k) - \bar{\gamma}(k))\bar{\rho}(k),$$

if $k \leq M - 1$. We remark that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)s\chi^k_n ds = -\mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - \chi^k_n) + \bar{\rho}(k) = -\rho(k) + \bar{\rho}(k).$$

Using the similar argument as in the derivation of (5.5) we can show that there is $\delta \in (0,1)$ such that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)\delta(X_i, X_{i+k})sds = O(1)n^{-3/2}k^{3/2}\left(\frac{n}{k}\right)^\delta + p_k(0)\bar{\rho}(k).$$

It follows that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)s\chi^k_n ds$$
$$= O(1)n^{-1}\left(\frac{n}{k}\right)^\delta + \lambda^2\bar{\gamma}(k)\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)sds,$$

which implies that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)sds = \lambda^{-2}\bar{\gamma}(k)^{-1}(-\rho(k) + \bar{\rho}(k)) + O(1)n^{-1}\left(\frac{n}{k}\right)^\delta.$$

Therefore, by (5.1) we know that $\rho(k+1) - \rho(k)$ is equal to

$$O(1)n^{-1}(\frac{n}{k})^\delta + \frac{\alpha_1(k)}{\bar{\gamma}(k)}\rho(k) - \Big(\frac{\alpha_1(k)}{\bar{\gamma}(k)} + \lambda^2\alpha_2(k) - \lambda^2\bar{\gamma}(k)\Big)\bar{\rho}(k)$$
$$= O(1)n^{-1}\Big(\frac{n}{k}\Big)^\delta + \frac{\alpha_1(k)}{\bar{\gamma}(k)}\rho(k),$$

if $k \leq M-1$. Using this we obtain that

$$\Big|\rho(k+1) - \Big(1 + \frac{\alpha_1(k)}{\bar{\gamma}(k)}\Big)\rho(k)\Big| \leq O(1)n^{-1+\delta}, \qquad k \leq M-1$$

for some constant $\delta \in (0,1)$. In other words,

$$|\rho'(k+1) - \rho'(k)| \leq O(1)n^{-1+\delta}, \qquad k \leq M-1.$$

This proves (5.3).

Thus we complete the proofs of (5.2) and (5.3), provided Lemma A.1 – Lemma A.3 below have been proved.

Appendix

We now state and prove Lemma A.1 – Lemma A.3.

Lemma A.1. *There is a constant $\delta \in (0,1)$ such that*

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k})(1 + \widetilde{Y}_s(i,k))ds$$
$$= O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta} + \int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n)(1 + \mathbf{E}\widetilde{Y}_s(i,k))ds.$$

Proof. We only prove the following:

$$\int_0^1 \mathbf{E}\Psi(\exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k}) - \exp(-\bar{J}^{k+1,\lambda}_{1,n} - s\chi^k_n))ds = O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta}. \quad \text{(A.1)}$$

If (A.1) is indeed true, then we can repeat the argument given in (5.5) and get that

$$\Big|\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}^{k+1,\lambda}_{1,n-k} - s\chi^k_{n-k})|\widetilde{Y}_s(i,k) - \mathbf{E}\widetilde{Y}_s(i,k))ds\Big| \leq O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta}.$$

Combining this with (A.1) we get the desired result.

We now prove (A.1). As in the proof of Lemma 5.1 we can show that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}_{1,n-k}^{k+1,\lambda} - s\chi_{n-k}^k)(\bar{J}_{1,n-k}^{k+1,\lambda} - \bar{J}_{1,n}^{k+1,\lambda} + s\chi_{n-k}^k - s\chi_n^k)^2 \mathrm{d}s \leq O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta}.$$

Thus, by means of (5.4) we can show that l.h.s. of (A.1) is equal to

$$O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta} + \int_0^1 \mathbf{E}\Psi \exp(-\bar{J}_{1,n-k}^{k+1,\lambda} - s\chi_{n-k}^k)(\bar{J}_{1,n}^{k+1,\lambda} - \bar{J}_{1,n-k}^{k+1,\lambda} + s\chi_n^k - s\chi_{n-k}^k)\mathrm{d}s$$

$$= O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta} + \int_0^1 \mathbf{E}\Psi \exp(-\bar{J}_{1,n-k}^{k+1,\lambda} - s\chi_{n-k}^k)\Big[\lambda n^{-1/2}\Big(\sum_{v=n-k+1}^{n}\sum_{u=1}^{v-k-1}\delta(X_u, X_v)$$

$$k\beta(k+1,n)\Big) + s\lambda n^{-1/2}\sum_{v=n-k+1}^{n}\delta(X_{v-k}, X_v) - s\lambda n^{-1/2} k p_k(0)\Big]\mathrm{d}s. \qquad (A.2)$$

We let $t_n = n - n^{2/3} \vee (k(k/n)^{\delta'})$ for some $\delta' \in (0, 1/2)$. Then, as in the proof of Lemma 5.1 we can show that

$$n^{-1/2}\mathbf{E}\Psi \exp(-\bar{J}_{1,n-k}^{k+1,\lambda} - s\chi_{n-k}^k)\sum_{v=n-k+1}^{n}\sum_{u=1}^{v-t_n}\delta(X_u, X_v)$$

$$\leq O(1)n^{-1/2}\sum_{v=n-k+1}^{n}\sum_{u=1}^{v-t_n} p_{v-u}(0) \leq O(1)\Big(\frac{k}{n}\Big)^{3/2-\delta}.$$

If $0 \leq v - u \leq t_n$ and $v \geq n - k + 1$, we can use Taylor's expansion to show that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}_{1,n-k}^{k+1,\lambda} - s\chi_{n-k}^k)\delta(X_u, X_v)\mathrm{d}s$$

$$= \int_0^1 \mathbf{E}\Psi \exp(-\bar{J}_{1,u}^{k+1,\lambda} - s\chi_u^k)\delta(X_u, X_v)\mathrm{d}s$$

$$\times[1 + O(1)(\bar{J}_{1,n-k}^{k+1,\lambda} - \bar{J}_{1,u}^{k+1,\lambda} + s\chi_{n-k}^k - s\chi_u^k)]. \qquad (A.3)$$

As in the proof of Lemma 5.1 we can show that

$$\int_0^1 \mathbf{E}\Psi \exp(-\bar{J}_{1,u}^{k+1,\lambda} - s\chi_u^k)\delta(X_u, X_v)(|\bar{J}_{1,n-k}^{k+1,\lambda} - \bar{J}_{1,u}^{k+1,\lambda}| + s|\chi_{n-k}^k - \chi_u^k|)\mathrm{d}s$$

$$\leq O(1)\Big(\frac{k}{n}\Big)^{1-\delta} p_{v-u}(0).$$

Using $\Psi = \Psi_{0,u/n} + O(k/n)^{1-\delta}$, we then show that l.h.s. of (A.3) is equal to ($u \geq v - t_n$)

$$O(1)\Big(\frac{k}{n}\Big)^{1-\delta} p_{v-u}(0) + p_{v-u}(0)\int_0^1 \mathbf{E}\Psi_{0,u/n} \exp(-\bar{J}_{1,u}^{k+1,\lambda} - s\chi_u^k)\mathrm{d}s.$$

By Taylor's expansion we can show that

$$\int_0^1 \mathbf{E}\Psi_{0,u/n}\exp(-\bar{J}_{1,u}^{k+1,\lambda} - s\chi_u^k)\mathrm{d}s = \bar{\rho}(k) + O(1)\left(\frac{k}{n}\right)^{1-\delta},$$

if $u \geq v - t_n$. This proves that l.h.s. of (A.3) is equal to

$$O(1)\left(\frac{k}{n}\right)^{1-\delta} p_{v-u}(0) + p_{v-u}(0)\bar{\rho}(k).$$

This proves that r.h.s of (A.2) is of order $O(1)(k/n)^{3/2-\delta}$, which implies the desired result (A.1). □

Lemma A.2. *(i) There is a constant $\delta \in (0,1)$ such that*

$$\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}_{1,n}^i - s\chi_k(i))\delta(X_i, X_{i+k})(J_{1,i;i,i+k}^{k+1} + J_{i,i+k;i+k,n}^{k+1})\mathrm{d}s$$
$$= O(1)n^{-3/2}\left(\frac{n}{k}\right)^{\delta} + \bar{\rho}(k)\mathbf{E}\delta(X_i, X_{i+k})(J_{1,i;i,i+k}^{k+1} + J_{i,i+k;i+k,n}^{k+1}).$$

(ii) If $k \geq M$, then

$$n^{1/2}p_k^{-1}(0)\sum_{i=1}^{n-k}\mathbf{E}\delta(X_i, X_{i+k})(J_{1,i;i,i+k}^{k+1} + J_{i,i+k;i+k,n}^{k+1}) = O(1)n^{-1/2}\left(\frac{n}{k}\right)^{\delta} + \bar{\gamma}(k).$$

Proof. (i) We will mainly prove the following:

$$\int_0^1 \mathbf{E}\Psi\exp(-\bar{J}_{1,n}^i - s\chi_n(i))\delta(X_i, X_{i+k})J_{1,i;i,i+k}^{k+1}\mathrm{d}s$$
$$= O(1)n^{-3/2}\left(\frac{n}{k}\right)^{\delta} + \bar{\rho}(k)\mathbf{E}\delta(X_i, X_{i+k})J_{1,i;i,i+k}^{k+1}. \qquad \text{(A.4)}$$

For this purpose we choose three sequences $\{r_n\}$, $\{q_n\}$ and $\{l_n\}$ such that

(i) $n > r_n > q_n > l_n \geq 1$, $n \geq 1$,
(ii) $r_n k^{-1} = (n/k)^{\delta_1}$,
(iii) $q_n/l_n = (n/k)^{1-\delta_2}$,

and

(iv) $\sum_{u=1}^{i-l_n}\sum_{v=i+1}^{i+k} k^{-3/2}\mathbf{E}\delta(X_u, X_v) = (n/k)^{\delta_2}$, for $i \in [r_n, n - r_n]$,

where $\delta_1, \delta_2, \delta_3 \in (0,1)$ are constants. Let

$$L(s,i,n) = \bar{J}_{1,i-q_n}^{k+1} + \bar{J}_{i+q_n,n}^{k+1} + \lambda J_{1,i-q_n;i+q_n,n} + s\chi_{i-q_n}^k + s\chi_{i+q_n,n}^k.$$

Then $\tilde{J}^i_{1,n} + s\chi_n(i) = L(s,i,n) + O(k/n)^{1-\delta}$. Thus, l.h.s. of (A.4) is equal to

$$n^{-1/2} \sum_{u=i-l_n+1}^{i} \sum_{v=i+1}^{i+k} \int_0^1 \mathbf{E}\Psi \exp(-\tilde{J}^i_{1,n} - s\chi_n(i))ds + O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^\delta$$

$$= O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^\delta + n^{-1/2} \sum_{x,y\in\mathbf{Z}^3} \int_0^1 \mathbf{E}\Psi_{0,(i-q_n)/n}\Psi_{(i+q_n)/n,1}$$

$$\times \exp(-L(s,i,n)) I_{\{X_{i-q_n}=x, X_{i+q_n}=y\}} \sum_{u=i-l_n+1}^{i} \sum_{v=i+1}^{i+k} q_{u,i,v}(y-x)ds,$$

where $q_{u,i,v} = p_{2q_n-k-(i-u)} * (p_{i-u}p_{v-i}p_{i+k-v})$. Let $\tilde{q}_{u,i,v}(x)$ be equal to

$$\sum_{z\in\mathbf{Z}^3} [p_{2q_n-k-(i-u)}(z) - p_{2q_n}(x)] p_{i-u}(x-z) p_{v-i}(x-z) p_{i+k-v}(x-z)$$

$$= \tau(p_{2q_n-k-(i-u)+\sigma}(x) - p_{2q_n}(x)),$$

where $\tau = c((i-u)(v-i) + (i-u)(i+k-v) + (v-i)(i+k-v))^{-3/2}$, and $\sigma = (i-u)(v-i)(i+k-v)\tau^{3/2}$. Then

$$q_{u,i,v}(x) = p_{2q_n}(x) \sum_{y\in\mathbf{Z}^3} p_{i-u}(y)p_{v-i}(y)p_{i+k-v}(y) + \tilde{q}_{u,i,v}(x).$$

Therefore, l.h.s. of (A.4) is equal to

$$O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^\delta + n^{-1/2}\int_0^1 \mathbf{E}\Psi_{0,(i-q_n)/n}\Psi_{(i+q_n)/n,1}\exp(-L(s,i,n))ds$$

$$\times \sum_{u=i-l_n+1}^{i} \sum_{v=i+1}^{i+k} \mathbf{E}\delta(X_i, X_{i+k})\delta(X_u, X_v) + n^{-1/2} \sum_{x,y\in\mathbf{Z}^3} \int_0^1 \mathbf{E}\Psi_{0,(i-q_n)/n}$$

$$\times \Psi_{(i+q_n)/n,1} \exp(-L(s,i,n)) \sum_{u=i-l_n+1}^{i} \sum_{v=i+1}^{i+k} I_{\{X_{i-q_n}=x, X_{i+q_n}=y\}} \tilde{q}_{u,i,v}(y-x)ds.$$

The last term in r.h.s. of the above estimate is denoted by R. As in the proof of Lemma 3.2, we can show that

$$|p_{2q_n-m}(x) - p_{2q_n}(x)| \le O(1)\frac{m}{2q_n}\bar{p}_{4q_n}(x),$$

if $m \ge 0$ is even. Thus, by Lemma 4.1 we can show that $|R|$ is less than

$$O(1)n^{-1/2} \sum_{u=i-l_n+1}^{i} \sum_{v=i+1}^{i+k} \tau\frac{k+i-u}{2q_n} \le O(1)\frac{l_n}{n^{1/2}q_n}k^{-1} \le O(1)n^{-3/2}\Big(\frac{n}{k}\Big)^\delta.$$

Using this estimate we show that l.h.s. of (A.4) is equal to

$$O(1)n^{-3/2}\left(\frac{n}{k}\right)^{\delta} + n^{-1/2}\int_0^1 \mathbf{E}\exp(-L(s,i,n))\Psi_{0,(i-q_n)/n}\Psi_{(i+q_n)/n,1}\mathrm{d}s$$

$$\times \sum_{u=i-l_n+1}^{i}\sum_{v=i+1}^{i+k}\mathbf{E}\delta(X_i,X_{i+k})\delta(X_u,X_v)$$

$$= O(1)n^{-3/2}\left(\frac{n}{k}\right)^{\delta} + \mathbf{E}\delta(X_i,X_{i+k})J^{k+1}_{1,i;i,i+k}$$

$$\times \int_0^1 \mathbf{E}\exp(-L(s,i,n))\Psi_{0,(i-q_n)/n}\Psi_{(i+q_n)/n,1}\mathrm{d}s.$$

As in the proof of Lemma A.1 we can show that

$$\int_0^1 \mathbf{E}\exp(-L(s,i,n))\Psi_{0,(i-q_n)/n}\Psi_{(i+q_n)/n,1}\mathrm{d}s = O(1)\left(\frac{k}{n}\right)^{3/2-\delta} + \bar{\rho}(k).$$

Thus we finally obtain that l.h.s. of (A.4) is equal to

$$O(1)n^{-3/2}\left(\frac{n}{k}\right)^{\delta} + n^{-1/2}\bar{\rho}(k)\mathbf{E}\delta(X_i,X_{i+k})J^{k+1}_{1,i;i,i+k},$$

which proves (A.4).

Similarly we can show that

$$\int_0^1 \mathbf{E}\Psi\exp(-\tilde{J}^i_{1,n} - s\chi_k(i))\delta(X_i,X_{i+k})J^{k+1}_{i,i+k;i+k,n}\mathrm{d}s$$

$$= O(1)n^{-3/2}\left(\frac{n}{k}\right)^{\delta} + \bar{\rho}(k)\mathbf{E}\delta(X_i,X_{i+k})J^{k+1}_{i,i+k;i+k,n}.$$

Thus we complete the proof of (i).

(ii) By computation we can easily show the desired result. □

Lemma A.3. *There is a constant $\delta \in (0,1)$ such that*

$$\mathbf{E}\int_0^1 \Psi\exp(-\tilde{J}^i_{1,n} - s\chi_k(i))\delta(X_i,X_{i+k})\delta(X_j,X_{j+k})\mathrm{d}s$$

$$= O(1)k^{2-\delta}n^{-1+\delta} + \mathbf{E}\delta(X_i,X_{i+k})\delta(X_j,X_{j+k})\bar{\rho}(k).$$

The proof of Lemma A.3 is quite similar to that of Lemma A.2 (i) given before, so is omitted here.

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