BOUNDING COHOMOLOGY OF PROJECTIVE VARIETIES - A SURVEY

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1. The Historic Roots and ...

(1.1) In 1955, in his seminal work "Faisceaux algébriques cohérents (FAC)", Serre introduced the notion of "coherent sheaf over an algebraic variety" and developed a "cohomology theory for such varieties with respect to coherent sheaves of coefficients".

(1.2) This opened the door to the use of functorial methods in Algebraic Geometry and thus initiated a new aera in this theory. It lead Grothendieck to develop his revolutionary "scheme theoretic approach to Algebraic Geometry" presented in his series of monographs "Éléments de géométrie algébrique (EGA)".

(1.3) If X is a projective algebraic variety (over the algebraically closed field K) and F is a coherent sheaf over X then, for each non-negative integer i, the i-th cohomology group H^i(X,F) is a K-vector space of finite dimension, say

$$h^i(X,F) := \dim(H^i(X,F)).$$

(1.4) Moreover, for each integer n the so-called "n-th twist" F(n) of F is defined: another coherent sheaf over X.

… Cohomology Tables

(1.5) This provides us with an important "<u>system of (non-negative) numerical invariants"</u>

(*) h^i(X,F(n)) := dimH^i(X,F(n)) (i = 0,1,2,… ; n = … , -2,- 1,0,1, 2, …)

of the pair (X,F). We thus get a <u>"discrete skeleton of the continuous object (X,F)</u>": the family of non-negative integers

(**) h(X,F) := {h^i(X,F(n)) II i = 0,1,2,… ; n = … , -2, -1, 0, 1, 2, …}.

We call this family h(X,F) the <u>"cohomology table of the pair (X,F)"</u>.

(1.6) The study of these cohomology tables is a basic issue of Algebraic Geometry.

(1.7) Our aim is to present a few results on this subject. These are an outspring of the long-term research project:

<u>"What bounds cohomology of a projective scheme with coefficients in a coherent sheaf?"</u>

2. Reminders on Varieties and ...

(2.1) Let r a positive integer. Consider the projective r-space

$IP^r := \{(a_0 : a_1 : \dots : a_r) \mid$ all a_i in K, not all of them 0$\}$.

Let $K[x_0, x_1, \dots, x_r]$ denote the polynomial ring over the field K in the given r+1 indeterminates $x_0, x_1, \dots, x_r$.

(2.2) A projective algebraic variety X in IP^r is the common set of zeros of (finitely many) homogeneous polynomials $f_1, f_2, \dots, f_t$ in $K[x_1, x_2, \dots, x_r]$. We then write $X = V(f_1, f_2, \dots, f_t)$.

Keep in mind, that the point $p = (a_0: a_1: \dots : a_r)$ in IP^r belongs to the variety X if and only if

$$f_i(a_0, a_1, \dots, a_r) = 0 \text{ for } i=1, 2, \dots, t.$$

(2.3) Although projective algebraic varieties are the basic object of our investigation, our results can be understood without knowing the theory of these varieties, which is "Projective Algebraic Geometry".

… Sheaves

(2.4) A <u>coherent sheaf</u> F over the projective variety X is given by a map, which assigns to each (Zariski-) open set U of X an algebraic object F(U) such that certain properties hold … : *<u>We omit further details!</u>*

(2.5) A basic example of coherent sheaf over X is the so called <u>structure sheaf</u> F = O_X of X. It assigns to each open subset U of X the <u>ring of regular (or rational) functions</u> O_X(U) on U, hence the ring of all functions f: U –> K which are locally given by

f(a_0: a_1, … , a_r) = g(a_0, a_1, … , a_r)/h(a_0, a_1, … , a_n),

where g and h in K[x_1, x_2, … , x_r] are <u>homogeneous of the same degree</u> and the occurring denominator h(a_0, a_1, … , a_n) is not 0.

(2.6) There is a natural notion of <u>morphism of coherent sheaves</u> over X. The coherent sheaves and their morphisms form an <u>Abelian Category</u>. In particular we can speak of <u>short exact sequences</u> and <u>finite direct sums of coherent sheaves</u> over X. Moreover, the "<u>functor</u> F → F(X) <u>of global sections</u>" is <u>left-exact</u>. This makes coherent sheaves over X accessible to a <u>cohomology theory based on right derived functors</u>.

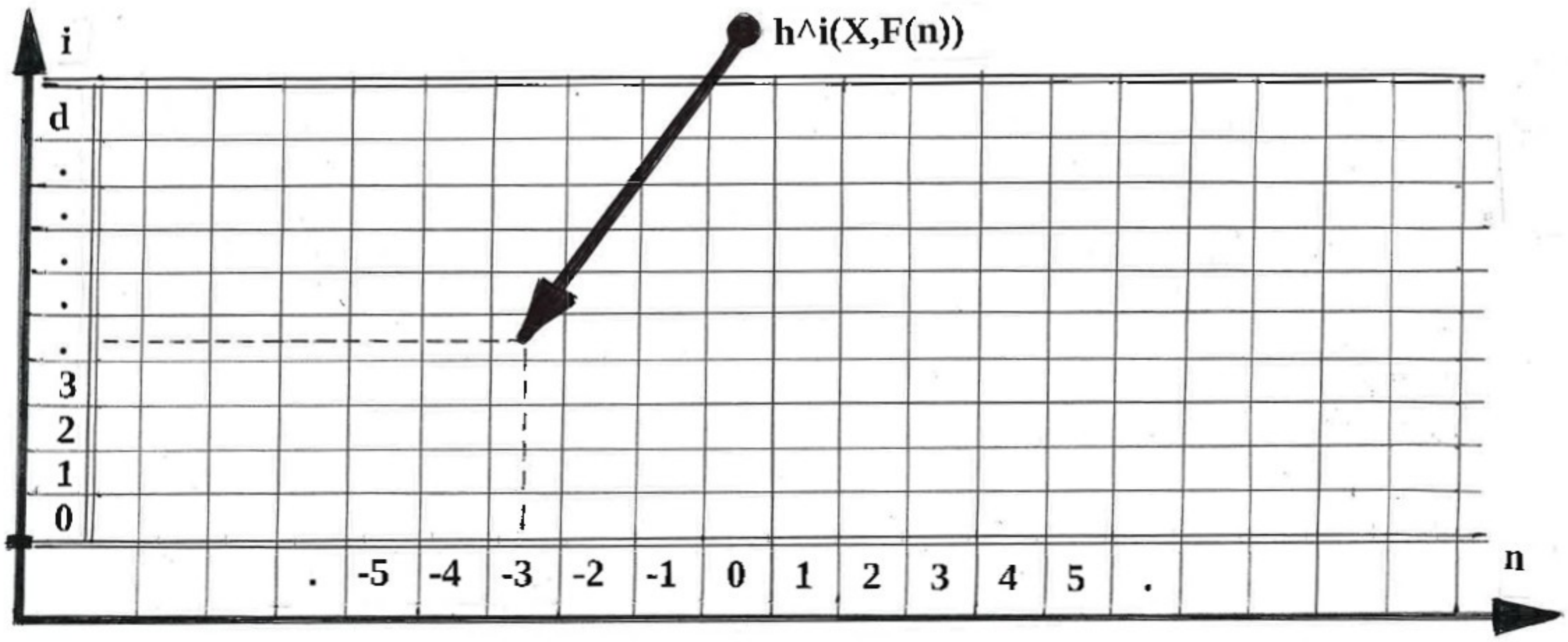
(2.7) *<u>In our talk we shall not introduce this cohomology theory. Instead we shall content ourselves to present results obtained by this theory.</u>*

3. Cohomology Tables and ...

(3.1) Let X be a projective variety over the algebraically closed field K and let F be a non-zero coherent sheaf over X. Then the <u>dimension</u> of (X,F) is a non-negative integer d, which can be characterized by the property

d = dim(X,F) := max{i II H^i(X,F(n)) does not vanish for some n}.

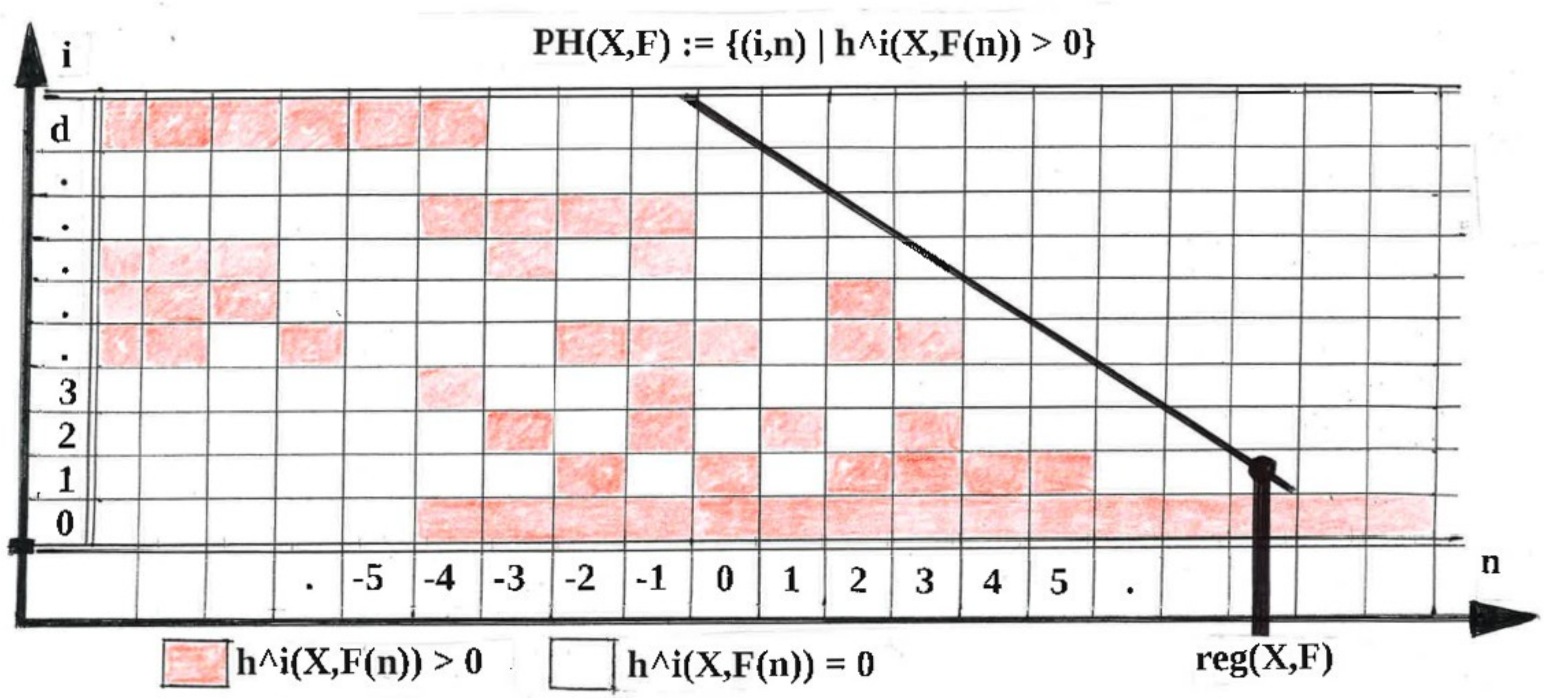
So, the cohomology table of (X,F) can be presented as follows:



h(X,F) = Cohomology table of (X,F)

… Cohomological Patterns

(3.2) Let dim(X,F) = d. Then, the <u>cohomological pattern</u> PH(X,F) of (X,F) is defined as the set of pairs (i,n) for which H^i(X,F(n)) does not vanish.



(3.3) For all i > 0 and all n >> 0 it holds H^i(X,F(n)) = 0. This allows to define the "<u>Castelnuovo-Mumford regularity</u>" of (X,F), a basic invariant of the pair (X,F):

reg(X,F) := min{n II H^i(X,F(n-i))=0 for all I > 0}.

4. Structure of Cohomological Patterns and...

(4.1) PROPOSITION (* - Hellus, 2002): *The cohomological pattern P := PH(X,F) of the pair (X,F) has the following properties:*

(a) There are integers m and n with (0,m), (d,n) in P.

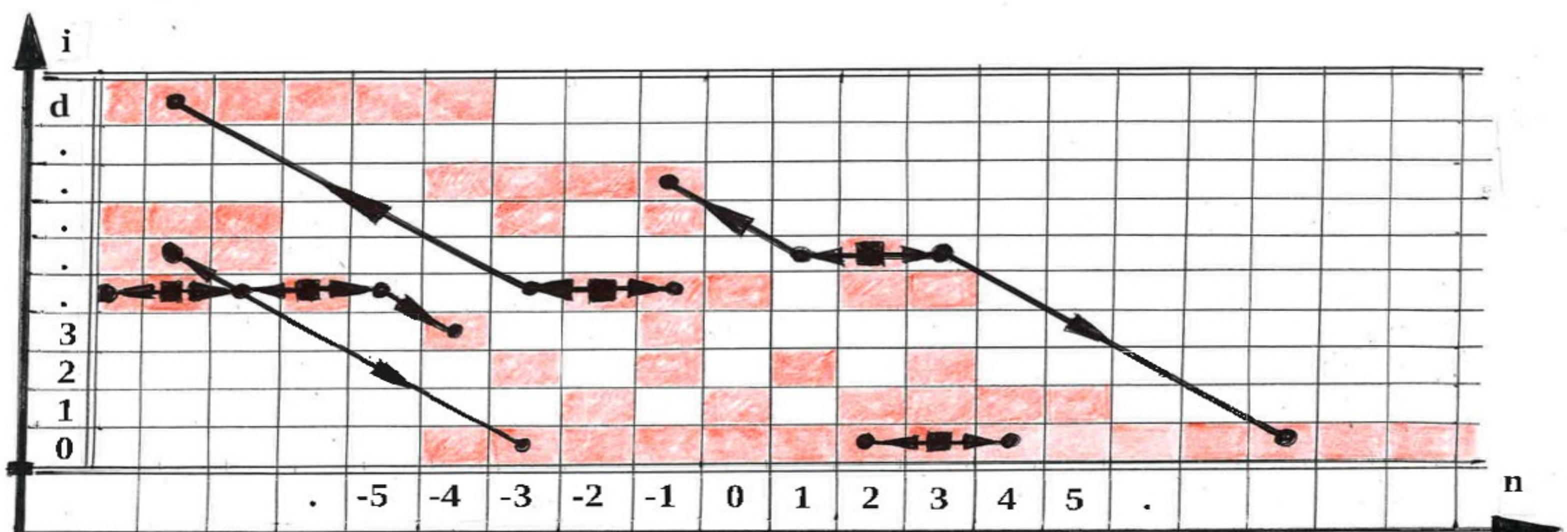
(b) If i>d, then (i,n) cannot be in P.

(c) If (i,n) in P, there is some j < i+1 such that (j,n+i-j+1) in P.

(d) If (i,n) in P, there is some k > i-1 such that (k,n+-k-1) in P.

(e) If i>0, then (i,n) is not in P for all n>>0.

(f) For each i either (i,n) in P for all n<<0 or else (i,n) not in P for all n<<0.



… their Realization

(4.2) From now on, we fix some non-negative integer d, and <u>let CI denote the class of all pairs (X,F) for which X is a projective variety over some algebraically closed field and F is a coherent sheaf over X such that dim(X,F) = d</u>.

(4.3) From now on, we write ID : = {(i,n) II i = 0, 1, ... , d; n an integer} for the <u>range of indices of all cohomology tables</u> h(X,F) of pairs (X,F) in CI.

(4.4) Now, a subset P of ID is called a <u>combinatorial pattern</u> if it satifies the properties (a) – (f) of Proposition 4.1. It is natural to ask, whether any such combinatorial pattern is realized as the cohomological pattern PH(X,F) of some pair (X,F) in CI. Indeed we have the following result:

(4.5) PROPOSITION (* - Hellus, 2002): ***For each combinatorial pattern P contained in ID and each algebraically closed field K, there existst a coherent sheaf over the projective d-space IP^d over the field K such that P = PH(IP^d,F).***

5. Cohomology Functions and…

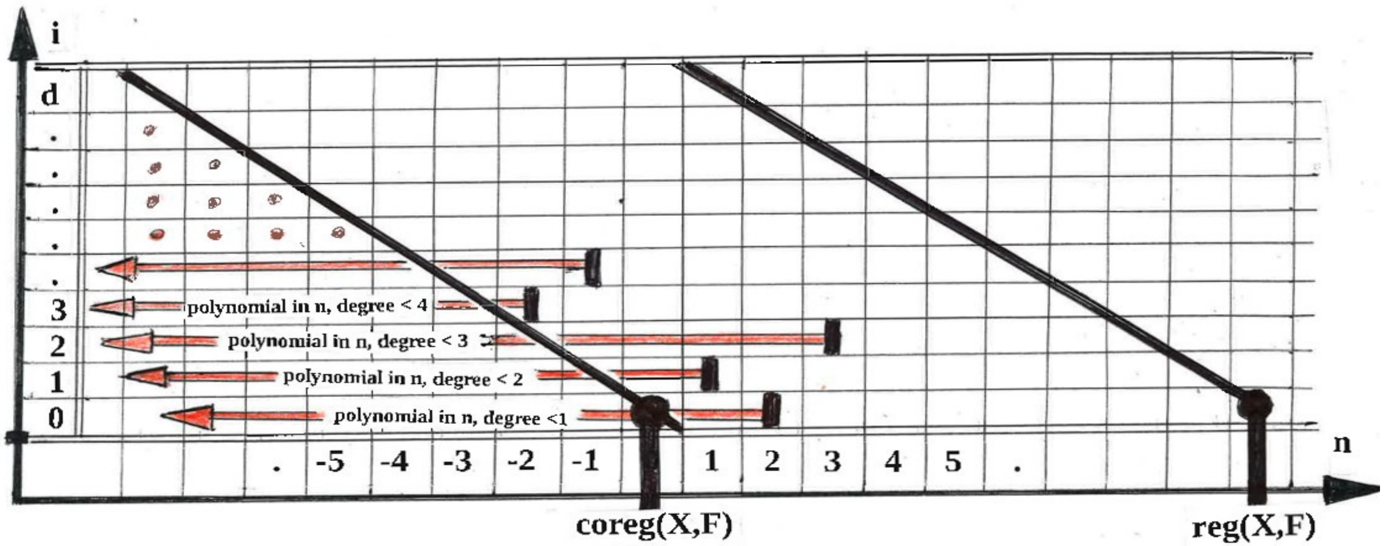
(5.1) Fix some non-negative integer i. Then, the function which maps n to h^i(X,F(n)) is called the <u>i-th cohomology function</u> of (X,F).

(5.2) For each non-negative integer i, there is a polynomial p^i_(X,F) of degree at most i, such that

h^i(X,F(n)) = p^i_(X,F)(n) for all n << 0,

the <u>i-th Hilbert-Serre polynomial</u> of (X,F). Moreover, the <u>Hilbert-Serre coregularity</u> of (X,F) is defined by

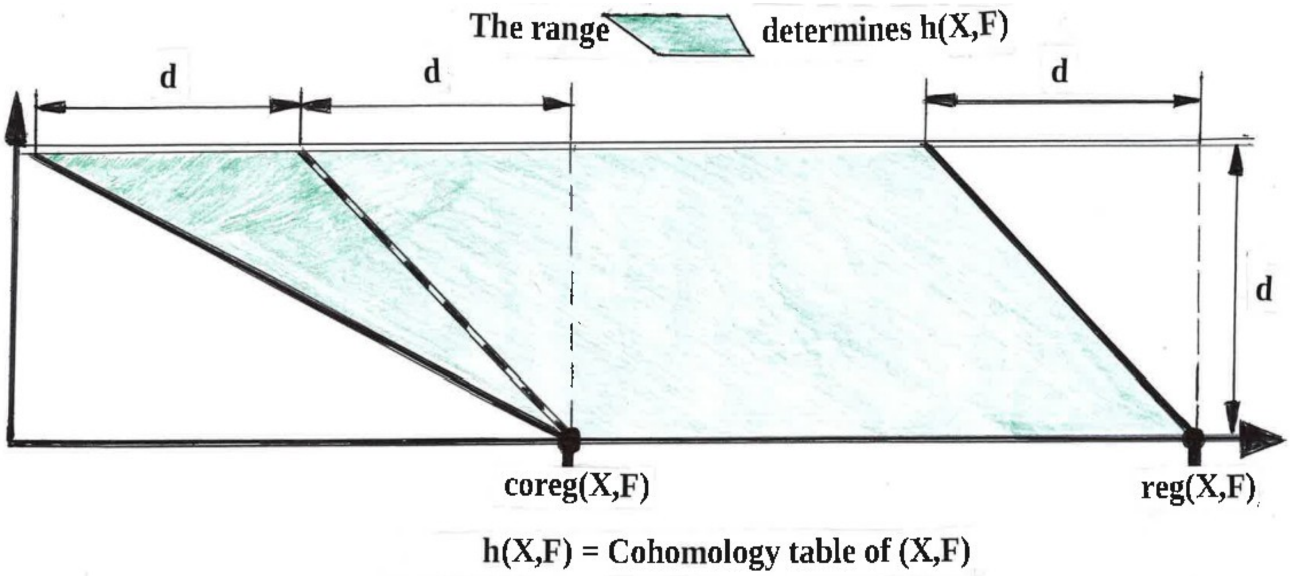
coreg(X,F) := max{c ll h^i(X,F(n)) = p^i_(X,F)(n) for all n << 0.



... Determinating Ranges

(5.3) We call a subset R of ID a <u>determinating range</u>, if for each pair (X,F) in Cl the values h^i(X,F(n)) with (i,n) in R determine the full cohomology table h(X,F) of this pair.

(5.4) An example of a determinating range is sketched here.



<u>Observe, that this determinating range if finite. So, a finite part of the cohomology table h(X,F) determines that table completely!</u>

6. Bounding Ranges, ...

(6.1) Let (X,F) be in Cl. The precise computation of the single entries h^i(X,F) of the cohomolgy table h(X,F) is usually very difficult. So, determinating ranges are not of great use in most cases. Instead we head for another concept, motivated by te following result:

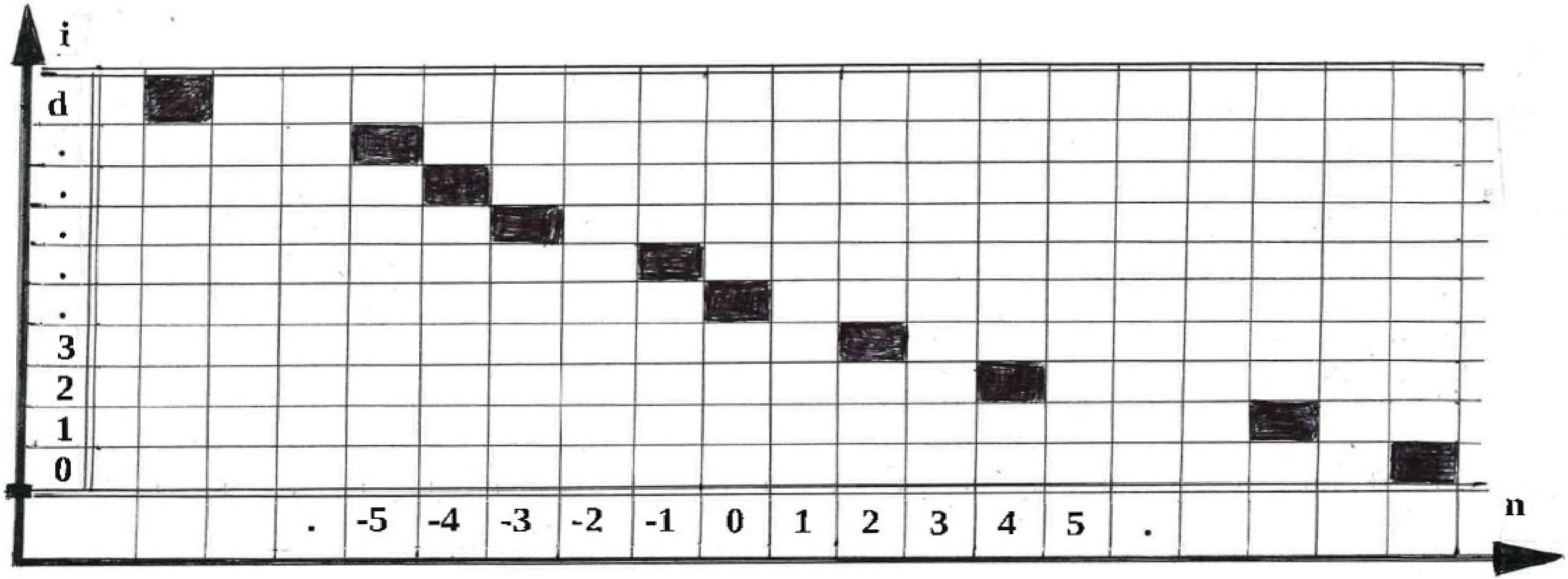
(6.2) PROPOSITION (* - Jahangiri - Linh, 2010) ***Let R be a subset of ID and, for each (X,F) in Cl let h(X,F)_R : = {h^i(X,F(n)) II (i,n) in R} denote the <u>restriction of the cohomology table</u> h(X,F) to the set of indices R. Then, the following statements are equivalent:***
(i) For each subclass S of Cl such that the set {h^i(X,F(n)) II (X,F) in S} is finite for all (i,n) in R, the set {h^i(X,F(n)) II (X,F) in S} is finite for all (i,n) in ID.
(ii) For each subclass S of Cl such that the set {h(X,F)_R II (X,F) in S} of restricted cohomoogy tables is finte, the set {h(X,F) II (X,F) in S} of unrestricted cohomology tables is finite, too.

(6.3) A subset R of ID which satisfies the equivalent conditions (i) and (ii) of Propsition (6.2) is called a <u>bounding range</u>.

... Quasi – Diagonals, ...

(6.4) We call a subset Q of ID a <u>Quasi-Diagonal</u> if is of the of the form

Q = {(i,n_i) II i = 0, 1, 2, ... , d and d_0 > d_1 > d_2 > ... > d_n}.

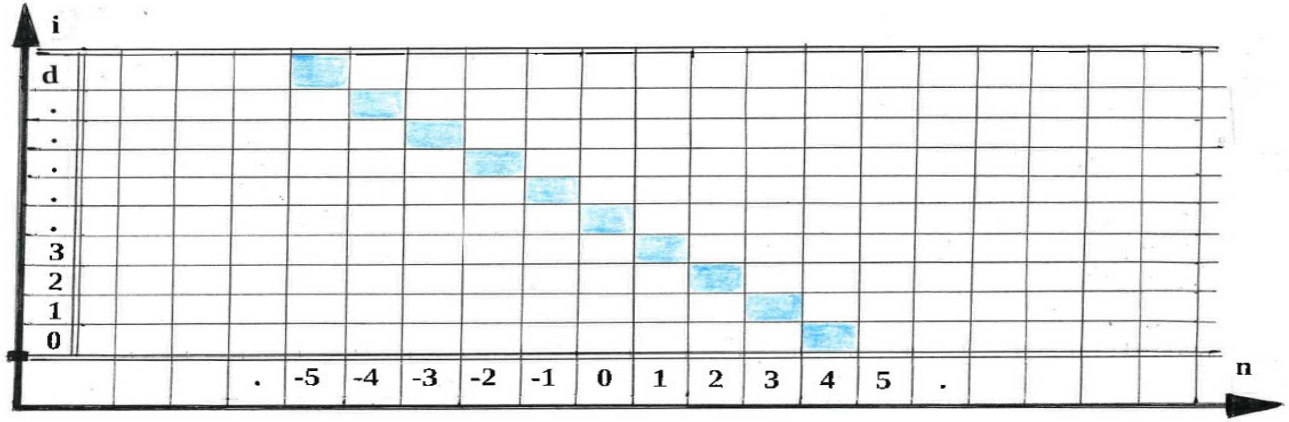


A Quasi-Diagonal

(6.5) THEOREM: (* - Jahangiri - Linh, 2010) [<u>Structure of Bounding Ranges</u>] ***A subset R of ID is a bounding range, if and only if it contains a quasi-diagonal.***

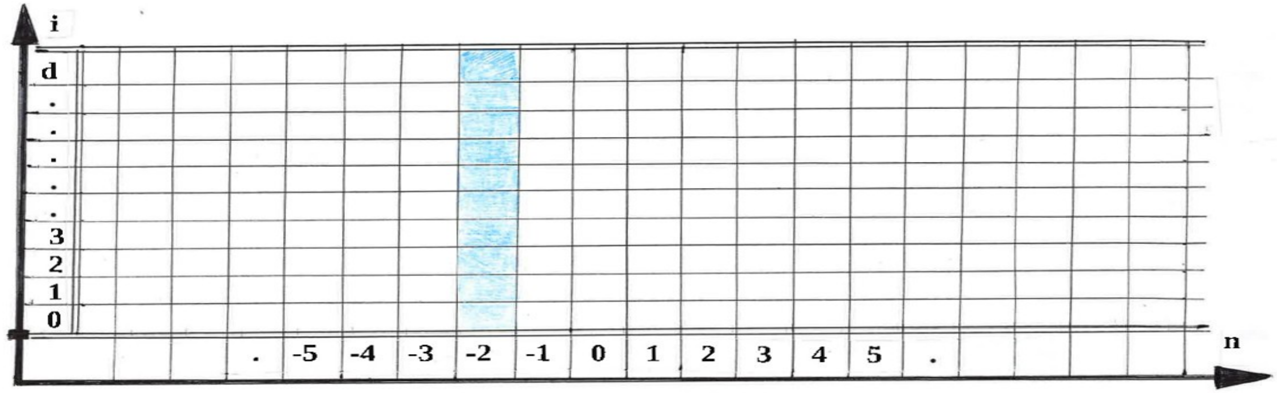
... Examples and ...

(6.6) The simplest type of bounding range is shown below: A diagonal.



A Bounding Range

Notably, a vertical in ID cannot be a bounding range if d > 0:



Not a Bounding Range

... Bounding Properties

(6.7) Unlike to determinating ranges, bounding ranges my be fairly small sets, namely just quasi-diagonals. *Nevertheless, quasi-diagonals enjoy a number of interesting properties which are indeed quite useful in the search of bounds for the numbers $h^i(X,F(n))$ for all (i,n) in ID!* We list a few of these Bounding Properties of Quasi-Diagonals:

(6.8) THEOREM: **(* - Jahangiri - Linh, 2010)** ***Fix an arbitrary quasi-diagonal $Q = \{(i,n_i) \;||\; i = 0, 1, ..., d\}$. For each sequence of positive integers $\underline{b} := b_0, b_1, ... , b_d$ we introduce the notation***

$$C(Q,\underline{b}) := \{ (X,F) \text{ in } Cl \;||\; h^i(X,F(n_i)) < b_i,\ i = 0, 1, ...\, d\}.$$

Then, there are algorithms which, for each sequence $\underline{b}$, allow to compute:

(a) For all (i,n) in ID a positive integer $b_{(i,n)}$ such that $h^i(X,F(n)) < b_{(i,n)}$ for all (X,F) in $C(Q,\underline{b})$.

(b) An upper bound for the set $\{reg(X,F) \;||\; (X,F) \text{ in } C(Q,\underline{b})\}$.

(c) A lower bound for the set $\{coreg(X,F) \;||\; (X,F) \text{ in } C(Q,\underline{b})\}$.

(d) An upper bound for the (finite!) cardinality of the set $\{h(X,F) \;||\; (X,F) \text{ in } C(Q,\underline{b})\}$.

7. Vector Bundles over IP^d

(7.1) A particularly important subclass of Cl is the subclass Vect of all vector bundles over a projective d-space. Vect is the class of all pairs (X,F) in Cl for which X is a projective d-space IP^d over an algebraically closed field K and the coherent sheaf F is locally free, hence "locally a direct sum of structure sheaves". *We do not explain more details!*

(7.2) On use of the so-called Vanishing Theorem of Severi-Enriques-Zariski-Serre, the pairs (X,F) in Cl which belong to Vect may be characterized cohomologically by saying that X must be some IP^d and H^i(X,F(n)) = 0 whenever i < d and n << 0. This means, that the cohomological patterns of pairs (X,F) in Vect are particularly simple. So, the question arises, whether the bounding ranges in Vect have to satisfy the same condition as the bounding ranges in the full class Cl. Indeed - surprisingly - this is the case:

(7.3) THEOREM: (* - Cathomen-Keller, 2014) *A subset R of ID is a bounding range in the class Vect of all vector bundles over some projective d-spcac if and only if it contains a quasi-diagonal*.

8. Hilbert Schemes and ...

(8.1) We recall that the notion of (projective) <u>scheme</u> was introduced by Grothendieck. It is a generalization of the notion of (projective) algebraic variety. <u>*We do not explain more details!*</u>

(8.2) Let (X,F) in Cl. Then, there is a polynomial p_(X,F) of degree d such that for all integers n it holds
$p_{(X,F)}(n) = h^0(X,F(n)) - h^1(X,F(n)) + h^2(X,F(n)) + (-1)^d h^d(X,F(n))$,
the <u>Hilbert-Serre Polynomial</u> of (X,F).

(8.3) A <u>coherent sheaf J of ideals</u> over a projective algebraic variety is given by the condition that J(U) is an <u>ideal</u> of the ring $O_X(U)$ for each open subset U of X.

(8.4) A particularly important projective scheme was introduced by Grothendieck in 1962 - the so-called <u>Hilbert Scheme</u>. Namely:
Let p be a polynomial of degree d. Then, there is is a projective scheme $Hilb_p$ (called the <u>Hilbert Scheme of</u> p), which parametrizes all coherent sheaves J of ideals over IP^d which satisfy $p_{(IP^d,J)} = p$.

... their Cohomological Strata

(8.5) In 1966, Hartshorne proved his fundamental <u>Connectivity Theorem for Hilbert Schemes</u>, which says that *the Hilbert scheme Hilb_p is (rationally) connected*.

(8.6) In 1988, Gotzmann improved on this by his <u>Connectivity Theorem for 0-th Cohomolgical Hilbert Scheme Strata</u>, which says: *Let f be a function which assigns to each integer n an integer f(n). Then, the subset*

Hilb^0_(p,f) := {(IP^d,J) in Hilb_p II h^0(IP^d,J(n)) > f(n) for all n}

of Hilb_p is (locally closed) and connected.

(8.7) Using techniques developed by Mall (2000) and Sbarra (2001), a PhD student of mine could improve the Connectivity Theorem of Gotzmann to the maximally possible extend:

(8.8) THEOREM (Fumasoli, 2007) *Let f^i (i = 0, 1, ..., d) be functions which assign to each integer n an integer f^i(n). Then each of the subsets*

Hilb_(p,f)^i := {(IP^d,J) in Hilb_p II h^i(IP^d,J(n)) > f^i(n) for all integers n}

of Hilb_p is (locally closed) and (rationally) connected.