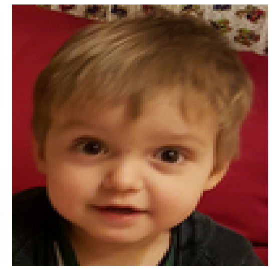
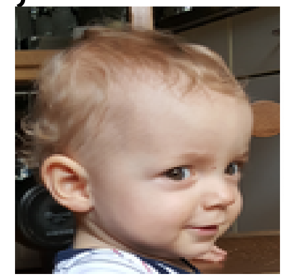
The Grasp for Infinity

With the welcome Assistance of Rona, Loris, Izaiah and Alec



Quotations from the New Jerusalem Bible

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Rona Counts as far as Eleven

Rona's Counting: **(performed at the Age of 2.5)**

1, 2, 3, ? , 6, 5, ? , 8, 10, 9, 11 !!

Children and Counting:

Children are fascinated by counting - by this mysterious „Poem of Numbers“ which finds no end – and hence offers a *First Glance at Infinity*.

The Juvenile Grasp for Infinity:

Which is the largest number?
Can one count on and on for ever?
Do all numbers have a name?
Can one write down all numbers?

A Word of Jesus:

Matthew 18, 10: ***See that you never despise any of these little ones, for I tell you that their angels in heaven are continually in the presence of my Father in heaven.***

The Attraction of Infinity

Man's Glance to Infinity:

During all Cultures and Ages, Humans were captured by thinking about the Infinite: In *Religion, Philosophy, Mathematics* and ... in *Childhood*. Does this express, that Humans are deeply aware of their determination for an ever-lasting Live? Does God put this interest for the infinite in our heart – to make us rise our eyes up toward Him, who created us to be with Him in Eternity? What does He tell us about our Immortality in the Holy Scripture?

Immortality of Human Beings in the Bible:

Wisdom 2, 23:

For God created human beings to be immortal, he made them as an image of his own nature;

1 Corinthians 15, 42-44:

42 *It is the same too with the resurrection of the dead: what is*
sown is perishable, but what is raised is imperishable;
43 *what is sown is contemptible but what is raised is glorious;*
what is sown is weak, but what is raised is powerful;
44 *what is sown is a natural body, and what is raised is a*
spiritual body. If there is a natural body, there is a spiritual
body too.

Revelation 22, 5:

And night will be abolished; they will not need lamplight or sunlight, because the Lord God will be shining on them. They will reign for ever and ever.

Uncountability: In the Bible and for a Child

Uncountability in the Bible – Literally and Spiritually:

Genesis 22, 15-17:

„The angel of Yahweh called Abraham a second time from heaven. 'I swear by my own self, Yahweh declares, that because you have done this, because you have not refused me your own beloved son, I will shower blessings on you and make your descendants as <u>numerous as the stars of heaven and the grains of sand on the seashore."</u>

Literally: ***No man can count the stars of heaven or the grains of sand of the seashore. But precise counting can be approximatively replaced by estimating:***

1) Supposing there are 100'000 km of sand covered seahores on earth, of an average width of 100 m, with an average sand layer of 1 m, and one grain of sand takes an average volume of 1 cube mm, we get an estimated number of 10'000'000'000'000'000'000 = 10^19 sand grains.

2) On use of radio-telescopic methods astronomers estimate the „total number of stars in the observable part of the universe" by 7 x 10^22.

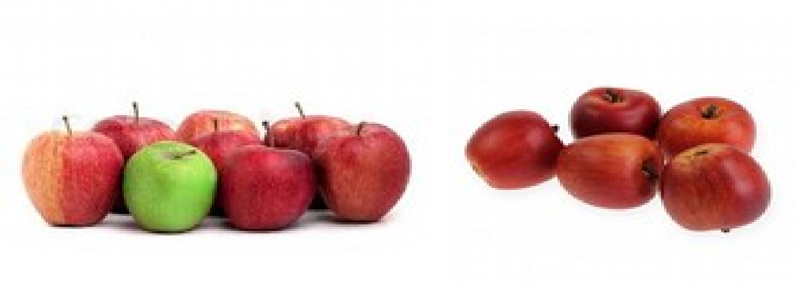
Spiritually: ***The descendants of Abraham are the children of promise, as we read in*** **Romans 9, 7-8:**

„and not all the descendants of Abraham count as his children, for Isaac is the one through whom your Name will be carried on. That is, it is not by being children through physical descent that people become children of God; it is the children of the promise that are counted as the heirs.“

Their number cannot be counted or estimated by man, but only by God, who knows the heart of each.

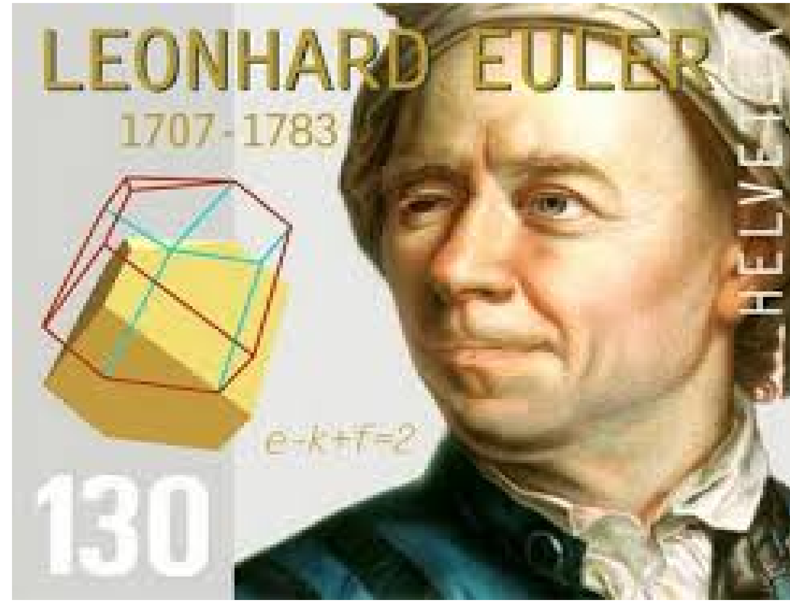
Uncountability for Rona (aged 2.5) :

12 Apples



Calculus: Speaking on Infinity by Avoiding it

In 1735 the Great Swiss-Russian Mathematician *Leonhard Euler* (1707-1783) gave the following formula:



Sum of Reciprocals of Squares:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The left hand side in this formula denotes the „infinite sum“ of all reciprocals of squares of natural numbers, a sum having infinitely many summands.

To recall the precise meaning of this, for each positive integer n consider the finite sum

$$s_n := \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$ **and set** $$s := \frac{\pi^2}{6}.$$

Then, the *Formula of Euler* says that: $\lim_{n\to\infty} s_n = s.$

By definition, and written down in predicate calculus, this precisely means:

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : |s_n - s| < \varepsilon.$$

Note: ***Infinity is not mentioned in the precise formulation of the fact expresssed by Eulers Formula !!***

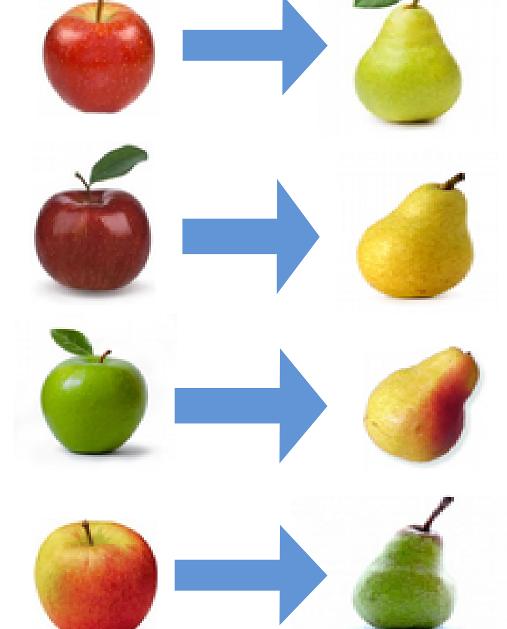
Infinity appears disguised in the „for all – exists – such that for all“ statement, which says that *„something is true for all $n > n_0$,“* hence *„for all large n“*. This is the way Mathematicians often speak about Infinity.

Their approach is a precise definition avoiding the notion of infinity, but guided by the heuristic idea, that „something holds on-and-on ad libidum.“ This is the point of view of *Potential Infinity*, contrasting the notion of *Actual Infinity.*

NB: Euler first used the common Symbol for Infinity !

Comparing: More Pears than Apples?

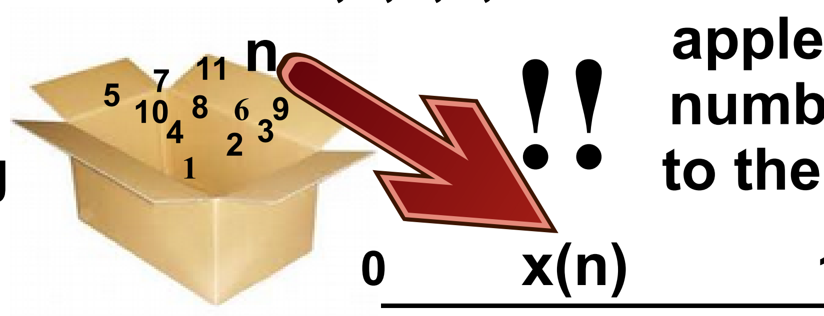
How can Rona decide ... with the help of Loris, Izaiah and Alec?



There are more fruits in each basket than Rona can count. But she takes one apple out of the first basket, a pear out of the second basket, and <u>assigns the pear to the apple</u>.

Can we Count the Points on a Line?

Question: ***„Are there less positive integers*** **1,2,3,4,5, …** ***than points on a line ?“*** **To get the answer, we imitate the „Apple-Pear“ Case: We imagine that all numbers 1,2,3,4,... are contained in a box (corresponding to the apples) and assign to each of them a real number x with 0 < x < 1 (corresponding to the pears).**



? ?

After some misunderstandings on the handling of numbers in a box we start – with the assistance of Rona, Loris, Izaiah and Alec: We take out of our box the number 1 and we assign to it a number x(1) between 0 and 1. Then we take out the number 2 and we assign to it a number x(2) between 0 and 1 ...

Going on like this, to each positive integer n we assign a number $x(n)$ with $0 < x(n) < 1$.
Each of the numbers $x(n)$ has a *decimal expansion*:

$$x(n) = 0,a_{(n,1)}a_{(n,2)}a_{(n,3)} \dots a_{(n,k)} \dots a_{(n,n-1)}a_{(n,n)}a_{(n,n+1)} \dots$$

with digits $a_{(n,k)}$ in $\{0,1,...,9\}$ (for $k,n = 1,2,...$).

We set:

$$y := 0,b_1b_2b_3 \dots b_{n-1}b_nb_{n+1} \dots$$

with b_n in $\{1,2,...,8\}$ and different from $a_{(n,n)}$ for all $n = 1,2,3...$

It follows:

$0 < y < 1$ *and* y *is different from all the numbers* $x(n)$, ($n =1,2,3,...$)!

Conclusion: ***There are less positive integers than points on a line !*** **Hence:** ***The points on a line are not countable :***

The Continuum (= the Line) is Uncountable !!

Can we Count the Fractions?

Question: ***Are there more positive integers or more (positive) fractions?***

We write each (positive) fraction in the form q = n/d, with positive integers d and n which have no common divisor: the *denominator* d and the *numerator* n. The *height* of the fraction q = n/d is defined as h(q) := n+d. Clearly h(q) > 1.
For each integer h > 1 there are at most h-1 different fractions which have height h. So we can count the fractions of height h by increasing denominators:

h = 2: 1/1; h = 3: 2/1,1/2; h = 4: 3/1,1/3; h = 5: 4/1,3/2,2/3,1/4;
h = 6: 5/1,1/5; h = 7: 6/1,5/2,4/3,3/4,2/5,1/6; h = 8: 7/1,5/3,3/5,1/7;
h = 9: 8/1,7/2,5/4,4/5,2/7,1/8; …

Enumeration of Fractions: **As each fraction has a uniquely determined height h, this furnishes a *complete enumeration of all fractions*, namely: 1/1,2/1,1/2,3/1,1/3,4/1,3/2,2/3,1/4,5/1,1/5,6/1, 5/2,4/3,3/4,2/5,1/6,7/1,5/3,3/5,1/7,8/1,7/2,5/4,4/5,2/7,1/8, …**

Successful Comparison: **So, we have assigned to each positive integer n a fraction q(n). Our assignement n |–> q(n) is *one-to-one*: For each (positive) fraction q there is a unique positive integer n such that q = q(n).**

Comparing with the Apple-Pear Experiment we can say that there are equally many (postive) fractions as positive integers. This means:

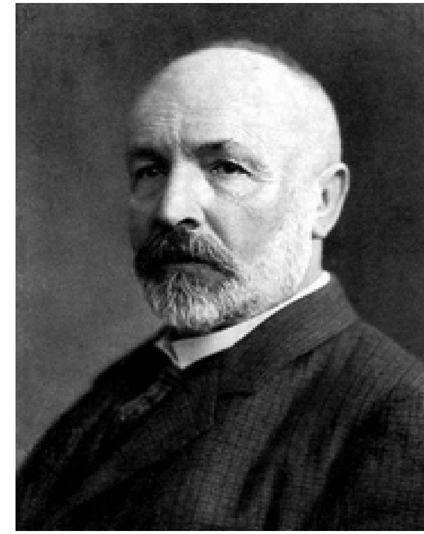
Conclusion: ***<u>We can count the (positive) fractions!</u>***

Hence:

The Fractions are Countable !!

Paradox of Infinity: **By Countability:** ***<u>There are equally many positive integers as (positive) fractions!</u>*** **But:** ***<u>All positive integers n = n/1 are fractions, and there are infinitely many fractions which are not positive integers!</u>***

About Sets and Maps



Set Theory was introduced by the German Mathematician and Philosopher *Georg Cantor* (1845-1918).

Definition: (A) A *Set* M is a collection of objects, which latter are called *Elements* of the set M.
(B) If m is an element of M, we say „m in M“.
(C) If a collection $m_1,m_2,m_3,...$ of objects is given, we denote the set which consists of these objects by $\{m_1,m_2,m_3,...\}$.
(D) If *P* is a property, we write {m: m satisfies *P*} for the set of all objects m which have this property.

Examples: The set N := {1,2,3,...} of *Natural Numbers.*
The set Z := { … -3,-2,-,1,0,1,2,3,...} of *Integers*.
The set Q := {n/d: n in Z, d in N} of *Rational Numbers.*
The set Q_+ := {q in Q: q > 0} of *Positive Fractions.*
The set R := {$x = a_{m+1}a_{m+2}..a_0,a_1a_2a_3...$: m<0, a_n in {0,1,2,...,9} for all n>m} of *Real Numbers* – the *Continuum.*
The set]0,1[:= {x in R: 0<x<1} – the *Open Interval* between 0 and 1.

Definition: **(A) A *Map* f:M –> K from the set M to the set K (given by m|–> f(m)) assigns to each element m in M an element f(m) in K.**

(B) The map f is called *injective* if f(m) and f(m') are different whenever m and m' are different from each other.

(C) The map f is called *surjective* if for each k in K there is some m in M such that f(m) = k.

(D) The map f is called *bijective* if it is injective and surjective.

Examples: **(A) There is no surjective map f:N –>]0,1[and no surjective map N –> R.**

(B) The previous enumeration of positive fractions defines a bijective map

$q:\underline{N} \to \underline{Q}_+$ (given by n|–> q(n)).

Transfinite Numbers

Cardinal Numbers **were introduced by Cantor as „Numbers of Elements“ of arbitrary sets. These numbers need not be finite any more and hence they are called *Transfinite Numbers.* To each set M there is assigned a cardinal number card(M) – the *Cardinality* of M.**

Properties: (A) If M is finite, then card(M) is the usual number of elements of M.

(B) card(M) = card(K) if and only if there is a bijective map f:M –> K.

Comparability: (A) Cardinal numbers may be *compared*: If $\aleph$:= card(M) and $\aleph'$:= card(M'), then it holds $\aleph < \aleph'$ if and only if there is no surjective map f:M –> M'.

(B) If $\aleph$ and $\aleph'$ are cardinals, then either:

$$\aleph < \aleph', \quad \aleph = \aleph' \text{ or else } \aleph' < \aleph.$$

Remarks: (A) One also may introduce certain *Arithmeitic Operations* between Cardinal Numbers (*Transfinite Arithmetics*). (B) There is another kind of Transfinite Numbers – the *Ordinal Numbers.*

Examples: (A) $\aleph_0$:= card(N) = card(Z) = card(Q) = card(Q_+) = the *Least Transfinite Cardinal Number*, the *Cardinality of Countable Sets.*

(B) $\aleph_0$ < card(]0,1[) = card(R) = the *Cardinality of the Continuum.*

NB: Comparability of Cardinal Numbers is based on the *Well Ordering Principle,* which claims:

Each set M *admits a Well Ordering* < .

Definition: A *Well Ordering* of the set M is a relation < such that for all elements m,m',m'' in M and each non-empty subset S of M (i.e. the set S contains elements and all of these are elements of M) it holds:

(A) *Transivity:* If m < m' and m' < m'', then m < m''.
(B) *Totality:* Either m < m', m' < m or m = m'.
(C) *Existence of Least Elements:* There is an element s_0 in S such that for each element s in S it holds either s_0 < s or s_0 = s. The element s_0 then is unique and called the *Least Element* of S.

Example: *The standard ordering < is a well ordering of the set*

N = {1,2,3,...}.

The Character Aleph: א

Psalm 119, א (1-8):

1 *How blessed are those whose way is blameless, who walk in the Law of Yahweh!*
2 *Blessed are those who observe his instructions, who seek him with all their hearts,*
3 *and, doing no evil, who walk in his ways.*
4 *You lay down your precepts to be carefully kept.*
5 *May my ways be steady in doing your will.*
6 *Then I shall not be shamed, if my gaze is fixed on your commandments.*
7 *I thank you with a sincere heart for teaching me your upright judgements.*
8 *I shall do your will; do not ever abandon me wholly*.

Psalm 119 **– the longest of all Psalms – consists of 22** ***Pericopes*** **composed of 8** ***Verses*****, which gives it a total number of 22 x 8 = 176 Verses. Each of the 22 Pericopes carries one of the 22 Characters of the Hebrew „Alphabet“. The complete list of Hebrew letters counts a total of 27; as there are 5 characters which can be written in two ways there are indeed 22 genuine characters.**

Psalm 119 is a great praise of the Law of Yahweh, His Commandements and His Everlasting Word. The use of the 22 Letters of the Hebrew Alphabet expresses the Universal Presence and Power of the Word of Yahweh – and yields that God gave His Commandments to Mose in stone carved letters (see Exodus 24, 12 / 32, 15-19).

Observe also, that the Hebrew Alphabet is written from the right to the left: ***Aleph***, the initial letter and sign of *Beginning* is in the upper right corner of our table.

The last letter ***Tav*** is in the lower left corner. Tav is the sign of *Fulfillment*, of *Final Judgement* and the sign of the *Righteous* who find the *Mercy of God* (see Ezechiel 9, 2- 6), the *Seal* of the *True Servants of God* (compare Revelation 7, 1-8).

Cantor's Inequality

Definition: **(A) Let <u>M</u> be a set. A set <u>S</u> is called a *Subset* of <u>M</u>, if each element of <u>S</u> is an element of <u>M</u>.**
(B) The set <u>M</u> and the *Empty Set* {} (which contains no element) are subsets of <u>M</u>.
(C) The *Power Set* <u>*IP*</u>(<u>M</u>) is the set of all subsets of <u>M</u>, thus <u>*IP*</u>(<u>M</u>) := {<u>S</u>: <u>S</u> is a subset of <u>M</u>}.

Remark and Definition: **(A) If <u>M</u> is a finite set, it holds:**

card(<u>*IP*</u>(<u>M</u>)) = 2^card(<u>M</u>).

(B) Therefore we define:

$2^{\aleph}$ **:= card(<u>*IP*</u>(<u>M</u>)) for arbitrary cardinals** $\aleph$ **= card(<u>M</u>).**

Example: $2^{\aleph_0}$ **= card(]0,1[) = card(<u>R</u>) = *Cardinality of the Continuum.***

Cantor's Inequality: $\aleph < 2^{\aleph}$ **for all cardinal numbers** $\aleph$**.**

Corollary of Cantor's Inequality: ***For each cardinal number*** $\aleph$ ***there is a cardinal number*** $\aleph'$ ***such that*** $\aleph < \aleph'$ **:**

<u>There is no largest Transfinite Number</u> !!

Proof of Cantor's Inequality

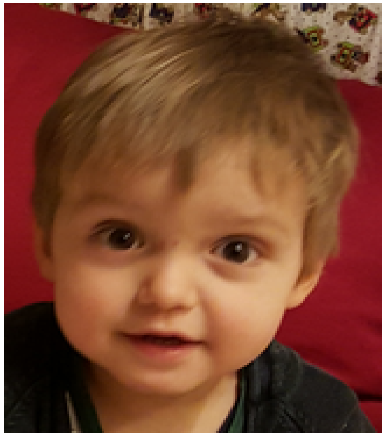
To show: There is no surjective map f: M –> *IP*(M).
Indeed, for any map f: M –> *IP*(M) let

T = T(f) := { m in M: m is not in f(m)}. (Observe that T in *IP*(M).)

It suffices to show that f(m) is different from T for all m in M. Assume to the contrary that T = f(m) for some m in M. If m in T, we have m in f(m), hence m not in T – a contradiction. If m is not in T, we have m not in f(m), thus m in T – a contradiction. So, assuming T = f(m), we always get a contradiction. Thus f(m) is indeed different from T for all m in M. *qued.*

Remark: The above proof uses an argument similar to the idea supposedly used by Rona, Loris, Izaiah and Alec to show that the continuum is uncountable...






The Controversy Around Infinity

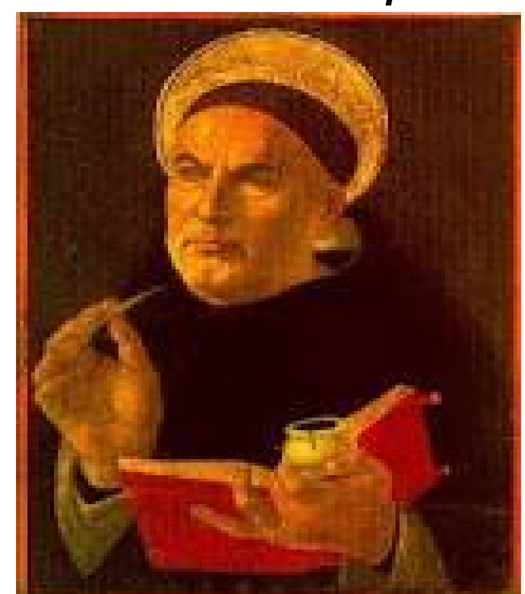
Cantor was blamed by:

– the *Mathematicians* for his *Unconstructible* handling of *Functions* and his use of the *Principle of Well Ordering,* issues which were not accepted by many Mathematicians at that time,

– the *Philosophers* who could not accept Cantors view of *Actual Infinity* and

– the *Theologists* (mainly the *Neo-Scholasticians*, who refered to the Teachings of *St. Thomas of Aquino* (1225-1274)), as the conclusion of the *„Ever On-going Infinities“* conflicts the *„One and Unique Infinity above All Infinities, Presented by God“.*

St. Thomas of Aquino



Involment of the Vatican: The controversy between *Cantor and the Theologists* gave rise to an extended *Correspondence*, reaching even the *Vatican.*

An Order of the Pope: **Finally *Pope Leo XIII* (1810-1903, Papacy 1878-1903) ordered to the Jesuit Theologist Cardinal *Johann Baptist Franzelin* (1816-1886) to examine the question whether Cantor's ideas conflict Christian Faith.**

Pope Leo XIII

Franzelin came to the conclusion that this is not the case: *Cantor's ideas about Infinity base on Human Intellect. They can at most give some image of the Infinity inherent to God. They cannot describe or reach the Infinity of God, which transcends Human Understanding*.

Pope Leo later published a Statement on the relation between the Church and Science, in which he wrote: „*The Church should not be opposed to new Scientific Results. The Church should rather examine these Findings and ask where and in which way new Scientific Ideas and Discoveries can contribute to the Welfare of Humanity and to a Deeper Understanding of Faith. (Free Quotation)*

J.B.Franzelin

Isaiah 55,9: ***For the heavens are as high above earth as my ways aree above your ways, my thoughts are above your thoughts.***

The Continuum Hypothesis

Remark: **Cantor's Inequality, applied with $\aleph = \aleph_0$, yields:**

$$\aleph_0 < 2^{\aleph_0} .$$

Question: ***Is there a cardinal number $\aleph$ with $\aleph_0 < \aleph < 2^{\aleph_0}$?***

The Continuum Hypothesis: **(Formulated by Cantor in 1878):**
There is no Cardinal Number $\aleph$ such that

$$\aleph_0 < \aleph < 2^{\aleph_0} \;!!$$

In other words: ***An Infinite Subset of the Continuum is either Countable or has the Same Cardinality as the Continuum !!***

The Generalized Continuum Hypothesis: ***There is no Pair of Transfinite Cardinal Numbers $\aleph$ and $\aleph'$, such that:***

$$\aleph < \aleph' < 2^{\aleph} \;!$$

The Great Controversies in Set Theory

Historical Fact: Around 1900, concerning Set Theory, there were two main Controversies among Mathematicians: The *Controversy around the Principle of Well Ordering* and the *Controversy around the Continuum Hypothesis.*

David Hilberts Position: The Great German Mathematician *David Hilbert* (1862-1943) was in favour of Cantor's Claims in both cases. Hilbert and others took a number of attempts to prove the Continuum Hypothesis, without succeeding.
Hilbert was convinced of Cantor's Non-Constructivistic View of Mathematics and expressed his conviction as follows:
„We will not allow to be expulsed from the Paradise which was opened to us by Cantor."

A New View: Finally, the Unsolvability of the two Great Controversies led to a *New View of Set Theory*, and of *Mathematics* at all...

Building on Axiomatic Footings

A commonly used approach do develop a Theory is to depart from a number of basic truths – the so called *Axioms* – and to build up the whole Theory by purely *logical deduction* from them. Clearly, a *System of Axioms* should satisfy two natural requirements:

1) Each of the single Axioms should be *Evidently True*.

2) The whole System of Axioms should be *Consistent*: It should be *impossible* to derive from the System of Axioms a *Conclusion* and the *Negation* of this same Conclusion.

Example: The *„Elements“* (of Plane Geometry) written by the Greek Mathematician *Euclid of Alexandria* (~ 300 BC). They had a tremendous influence on Medieval and even on Modern Science.

As early as in the 5th century, a *Controversy Around Euclid's Axiom on Parallel Lines* grew. This Axiom claims:

„If there is a line L in a plane E and a point p in that same plane, which avoids the line L, then there is precisely one line L' in E passing through p and having no point in common with L.“

Reconciliation in Geometry

The Points of View: Some Geometers thought the Axiom of Parallel Lines was wrong, others tried to prove it from the remaining axioms.

Lobatschewski's Independence Result: In 1826 the Russian Mathematician *Nikolai Lobatschewski* (1792-1856) proved that there are Geometries which do not satisfy the Axiom of Parallel Lines, but which satisfy all other Axioms of Euclid. This discovery was of great impact on the development of Mathematics.

Geometries not satisfying the Axiom of Parallel Lines, are called *Non-Euclidean Geometries* and are an important subject of modern Mathematics.

Hilberts Axioms: In 1899 *Hilber*t gave a modern approach to *Euclidian Geometry*, based on 20 Axioms.

At the same time Evidence grew:

Set Theory should be Axiomatized ...

From Choice Functions to Well-Orderings

Definition: Let M be a set with power set *IP*(M). A *Choice Function for* M is a map g: *IP*(M) –> M such that g(S) in S for each non-empty S in *IP*(M): g *chooses an element* g(S) *out of each non-empty subset* S *of* M.

In 1905 the German Mathematician *Ernst Zermelo* (1871-1953) proved:

Zermelo's Theorem: *If a non-empty set* M *admits a Choice Function* g: *IP*(M) –> M *, then it admits a Well Ordering* < *.*

A First Reconciliation: *Zermelo's Theorem brought a certain reconciliation in the Controversy around Cantor's Well Ordering Principle: Most Mathematicians found it easier to accept the existence of choice functions than of well orderings.*

The Axioms of Zermelo and Fraenkel

The System of Axioms of Zermelo and Fraenkel (ZF): In 1907 Zermelo proposed a System of Axioms for Set Theory. One of his axioms was the Axiom of Choice (AC), which claims: *Each non-empty Set has a Choice Function*. In 1921 the German-Israelite Mathematician *Abraham Halevi Fraenkel* (1891-1965) added some additional axioms to the system proposed by Zermelo. In 1930 Zermelo added a last extension. In 1929 the Norwegian Mathematician *Thoralf Albert Skolem* (1887-1963) formulated the ZF Axioms in the Language of *Predicate Calculus.*

A.H. Fraenkel

T.A. Skolem

Zermelo-Fraenkel axioms

(1) *Axiom of extension.* If A and B are sets and if, for all x, $x \in A$ if and only if $x \in B$, then $A = B$.

(2) *Axiom of the empty set.* There exists a set A such that, for all x, it is false that $x \in A$.

(3) *Axiom schema of separation.* If A is a set, there exists a set B such that, for all x, $x \in B$ if and only if $x \in A$ and $S(x)$. Here, $S(x)$ is any condition on x in which B is not free (it must be bound by a quantifier such as "all" or "some").

(4) *Axiom of pairing.* If A and B are sets, there exists a set (symbolized $\{A, B\}$ and called the unordered pair of A and B) having A and B as its sole members.

(5) *Axiom of union.* If C is a set, there exists a set A such that $x \in A$ if and only if $x \in B$ for some member B of C.

(6) *Axiom of power set.* If A is a set, there exists a set B, called its power set, such that $x \in B$ if and only if $x \subseteq A$.

(7) *Axiom of infinity.* There exists a set A such that $\varnothing \in A$ and, if $x \in A$, then $(x \cup \{x\}) \in A$, in which $x \cup \{x\}$ is the set x with x adjoined as a further member.

(8) *Axiom of choice.* If A is a set the elements of which are nonempty sets, then there exists a function f with domain A such that, for member B of A, $f(B) \in B$.

(9) *Axiom schema of replacement.* If A is a set and $f(x, y)$ a formula (in which x and y are free) such that for $x \in A$ there is exactly one y such that $f(x, y)$, then there exists a set B the members of which are the y's determined by $f(x, y)$ as x ranges over A.

(10) *Axiom of restriction (foundation axiom).* Every nonempty set A contains an element B such that $A \cap B = \varnothing$; i.e., A and B have no elements in common.

Partial Reconciliation: Relative Consistency

Notations: **(A) By ZF we denote the System of Axioms of Zermelo and Fraenkel.**
(B) By ZF-AC we denote the System ZF with the Axiom of Choice AC removed from it.
(C) By CH and GCM we respectively denote the Continuum Hypothesis and the Generalized Continuum Hypothesis.
(D) By ZF+CH and by ZF+GCH we denote the System ZF with CH respectively GCH added to it.

In 1940 the Cech-Austrian-American Mathematician *Kurt Gödel* (1906-1978) proved:

Gödel's Consistency Theorem: ***If the System* ZF-AC *is Consistent, then the Systems* ZF, ZF+CH *and* ZF+GCH *are Consistent, too.***

Consequences: **1) One cannot disprove AC from ZF-AC, and hence one can add AC to ZF-AC without getting a contradiction !!**
2) One cannot disprove CH and GCH from ZF and hence one can add CF and GCF to ZF without getting a contradiction !!

This means, that those Mathematicians who want to use* AC, CH *and* GCH *are justified and hence free to do so.

***The free use of* AC *is particularly interesting, as this allows to use <u>Zorn's Lemma</u>, which relies on* AC.**

This Lemma was proved in 1935 by the German-American Mathematician *Max August Zorn* (1906-1993) and is an important tool to Build-Up most Mathematical Theories.

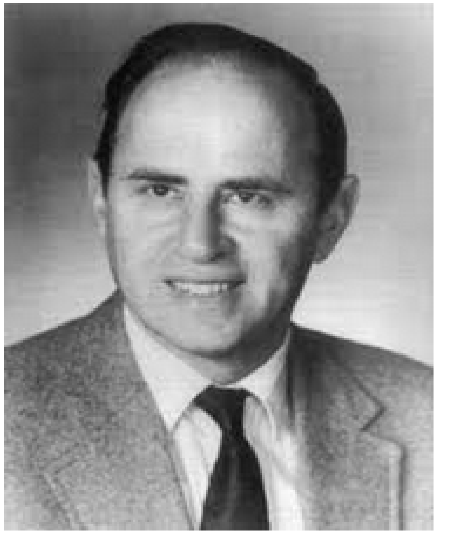
M.A.Zorn

Remark: ***It still could happen after Gödels Theorem, that one day a proof of* AC *out of* ZF-AC, *or a proof of* CH *(even of* GCH*) out of* ZF *would be given!***

The Final Reconciliation: Independence

Notation: If A is a Proposition, we write ¬A for the *Negation* „not A“ *of* A. If S is a System of Propositions, we write S+(¬A) for the System obtained adding ¬A to S.

In 1963 the American Mathematician *Paul Joseph Cohen* (1934-2007) proved:



Cohen's Independence Theorem: *If the System* ZF-AC *is Consistent, then the Systems* ZF-AC+(¬AC), ZF+(¬CH) *and* ZF+(¬GCH) *are Consistent, too.*

Consequences: 1) One neither can prove nor disprove AC from ZF-AC, hence one can add AC or else ¬AC to ZF-AC without getting a contradiction !!

2) One neither can prove nor disprove CH (respectively GCH) from ZF, hence one can add CF (respectively GCH) or else ¬CF (respectively ¬GCF) to ZF without getting a contradiction !!

Freedom of Choice: ***The Theorems of Gödel and Cohen say, that Mathematicians have a <u>Choice</u>:***
***They can decide to use once for all either* AC *or* ¬AC, *either* CH *or* ¬CH (*respectively either* GCH *or* ¬GCH).**

A New Method in Formal Logics: ***To give his Proof, Cohen developed a new method of formal Logic, called <u>Forcing</u>. This Method meanwhile is one of the fundamental tools of Logic.***
For his achievement, Cohen was given the <u>Fields Medal</u>, the most prestigeous award in Mathematics.

Fields Medal

Cohen's Result brought the Final Reconciliation in the Great Controversies Around Infinity among Mathematicians !!

But ... What is True Now?

Conclusion: **The results of Gödel and Cohen show that for Set Theory we are in a situation similar to that in Geometry:** ***There is not a unique and self evident Set Theory based on a footing of undisputable truths:*** **Up to a certain extend, Mathematicians are free to use Systems of Axioms for Set Theory at their deliberate convenience. This means:** ***There are different Set Theories based on different Systems of Axioms.***

As Set Theory is the „Natural Biotope“ of all Mathematical Theories, this ***„Ambiguity of Foundations“*** **is inherited by most Mathematical Theories in which infinite sets play a role.**

However, the differences resulting from different underlying Set Theories show up only on „higher floors“ of „mathematical buildings“, and usually are without impact on practical applications of Mathematics.

Examples: **If one does accept ¬AC (which is allowed according to Cohen) then:**

1) There are vector spaces which have no basis.

2) Each bounded set of real numbers is measurable.

The Classical Point of View: **According to Leonhard Euler *„Mathemathics is Pure Nature“.* This expresses the *Classical Point of View*, that Mathematics studies and discovers basic laws and rules which reign Nature and exploits them for Applications. Accordingly, *„Mathematics is based on the Objective Footing given by Nature and its Undisputable Laws.“***

The New Point of View: **(Imposed by the results of Gödel and Cohen) *„Mathematics is not based on an a Footing of Undisputable Laws“*. It rests on Axioms which can be chosen at convenience. Instead of *„expressing undisputable truths“* these Axioms just must be *consistent* among each other: *„Mathematics must not be true; it must be correct .“* (David Hilbert).**

Conclusion: The *„Beautiful Creature of Mathematics“* (named so by *Ines,* former PhD Student) cannot fulfill the dream to find *Absolute Truth*, a dream which was a driving force of Philosophers and Scientists through all Ages.

But, if Mathematics cannot fulfill this dream, what else could do so?

Human Knowledge – Wisdom of God

So, we are taught by Mathematics, that Human Mind and Human Knowledge are of impressive size and can achieve great things – but on the other hand they cannot explain the deepest wisdom all men are in search of in their heart: the Wisdom of God. The full understanding of this Wisdom is beyond the reach of human mind. The Bible praises the glorious splendour of the Wisdom of God in the following verses:

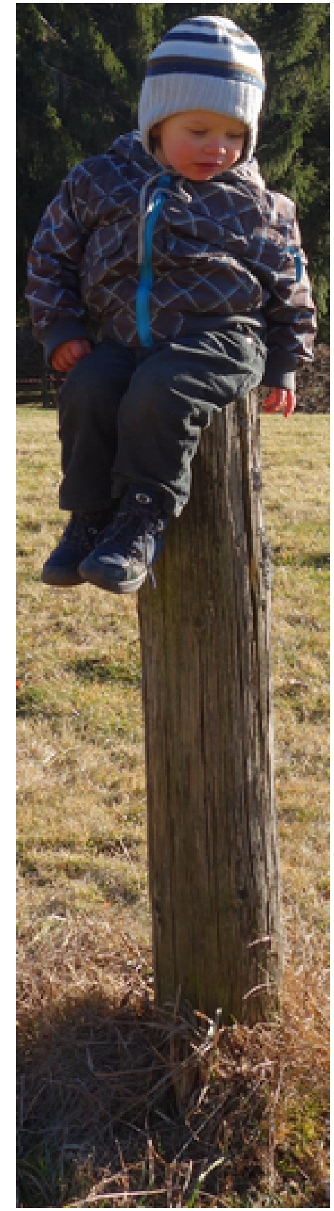
Wisdom 7, 22 – 8, 1:

22 For within her is a spirit intelligent, holy, unique, manifold, subtle, mobile, incisive, unsullied, lucid, invulnerable, benevolent, shrewd,

23 irresistible, beneficent, friendly to human beings, steadfast, dependable, unperturbed, almighty, all-surveying, penetrating all intelligent, pure and most subtle spirits.

24 ***For Wisdom is quicker to move than any motion; she is so pure, she pervades and permeates all things.***
25 ***She is a breath of the power of God, pure emanation of the glory of the Almighty; so nothing impure can find its way into her.***
26 ***For she is a reflection of the eternal light, untarnished mirror of God's active power, and image of his goodness.***
27 ***Although she is alone, she can do everything; herself unchanging, she renews the world, and, generation after generation, passing into holy souls, she makes them into God's friends and prophets;***
28 ***for God loves only those who dwell with Wisdom.***
29 ***She is indeed more splendid than the sun, she outshines all the constellations; compared with light, she takes first place,***
30 ***for light must yield to night, but against Wisdom evil cannot prevail.***
8,1 ***Strongly she reaches from one end of the world to the other and she governs the whole world for its good.***

Hallelujah !!



Psalm 131, 2-3:

Yahweh, my heart is not haughty, I do not set my sights too high. I have taken no part in great affairs, in wonders beyond my scope. No, I hold myself in quiet and silence, like a little child in its mother's arms, like a little child, so I keep myself.

12, 13, 14, … 117, 118, 119, 120, … 7893, 7894, 7895, …

$$\infty \quad \infty \quad \infty \quad \dots \quad 2^{2^{2^{2^{\aleph_0}}}} \quad 2^{2^{2^{\aleph_0}}} \quad 2^{2^{\aleph_0}} \quad 2^{\aleph_0} \quad \aleph_0$$

Ewa-Leonie (born 24. June 2020) trying to count the grains of sand at the „North Pole of Lucerne“, Switzerland