TAME LOCI OF CERTAIN LOCAL COHOMOLOGY MODULES

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Abstract. Let M be a finitely generated graded module over a Noetherian homogeneous ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$. For each $i \in \mathbb{N}_0$ let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to the irrelevant ideal $R_+ = \bigoplus_{n>0} R_n$ of R, furnished with its natural grading. We study the tame loci $\mathfrak{T}^i(M)^{\leq 3}$ at level $i \in \mathbb{N}_0$ in codimension ≤ 3 of M, that is the sets of all primes $\mathfrak{p}_0 \subset R_0$ of height ≤ 3 such that the graded $R_{\mathfrak{p}_0}$-modules $H^i_{R_+}(M)_{\mathfrak{p}_0}$ are tame.

1. Introduction

Throughout this note let $R = \bigoplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring. So, R is an $\mathbb{N}_0$-graded R_0-algebra and $R = R_0[l_1, ..., l_r]$ with finitely many elements $l_1, ..., l_r \in R_1$. Moreover, let $R_+ := \bigoplus_{n>0} R_n$ denote the irrelevant ideal of R and let M be a finitely generated graded R-module. For each $i \in \mathbb{N}_0$ let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to R_+. It is well known, that the R-module $H^i_{R_+}(M)$ carries a natural grading and that the graded components $H^i_{R_+}(M)_n$ are finitely generated R_0-modules which vanish for all $n \gg 0$ (s. [11], §15 for example). So, the R_0-modules $H^i_{R_+}(M)_n$ are asymptotically trivial if $n \to +\infty$.

On the other hand a rich variety of phenomena occurs for the modules $H^i_{R_+}(M)_n$ if $i \in \mathbb{N}_0$ is fixed and $n \to -\infty$. So, it is quite natural to investigate the *asymptotic behaviour of cohomology*, e.g.the mentioned phenomena (s. [3]).

One basic question in this respect is to ask for the *asymptotic stability of associated primes*, more precisely the question, whether for given $i \in \mathbb{N}_0$ the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ (or some of its specified subsets) ultimately becomes independent of n, if $n \to -\infty$. In many particular cases this is indeed the case (s. [2], [5], [6], [7]), partly even in a more general setting (s. [16]). On the other hand it is known for quite a while, that the asymptotic stability of associated primes also may fail in many even surprisingly "nice" cases by various examples (s. [6], [8] and also [3]), which rely on the constructions given in [20] and [21].

Another related question is, whether for fixed $i \in \mathbb{N}_0$ certain numerical invariants of the R_0-modules $H^i_{R_+}(M)_n$ ultimately become constant if $n \to -\infty$. A number of such *asymptotic stability results for numerical invariants* are indeed known (s. [4], [9], [10] and also [14]).

April 12, 2012.

The second author was in part supported by a grant from IPM (No. 89130115).

The oldest - and most challenging - question around the asymptotic behaviour of cohomology was the so-called *tameness problem*, that is the question, whether for fixed $i \in \mathbb{N}_0$ the R_0-modules $H^i_{R_+}(M)_n$ are either always vanishing for all $n \ll 0$ or always non-vanishing for all $n \ll 0$. This question seems to have raised already in relation with Marley's paper [18]. In a number of cases, this tameness problem was shown to have an affirmative answer (s. [3], [7], [17], [19]).

Nevertheless by means of a duality result for bigraded modules given in [15], Cutkosky and Herzog [12] constructed an example which shows that the tameness-problem can have a negative answer also. In [13] an even more striking counter-example is given: a Rees-ring R of a three-dimensional local domain R_0 of dimension 4, which is essentially of finite type over a field such that the graded R-module $H^2_{R_+}(R)$ is not tame.

The present paper is devoted to the study of the tame loci $\mathfrak{T}^i(M)$ of M, that is the sets of all primes $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$ for which the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0} \cong H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0})$ is tame. These loci have been studied already in [19]. We restrict ourselves to the case in which the base ring R_0 is essentially of finite type over a field, as in this situation asymptotic stability of associated primes holds in codimension ≤ 2. As shown by Chardin-Jouanolou, this latter asymptotic stability result holds under the weaker assumption that R_0 is a homomorphic image of a Noetherian ring which is locally Gorenstein (oral communication by M. Chardin). So all results of our paper remain valid if R_0 is subject to this weaker condition.

One expects, that in such a specific situation the tame loci $\mathfrak{T}^i(M)$ show some "usual" well-behaviour, like being open for example. But as we shall see in Example 2.5 this is wrong in general. Namely, using the counter-example given in [13] we construct an example of graded R-module M of dimension 4 whose 2-nd tame locus $\mathfrak{T}^2(M)$ is not even stable under generalization. This shows in particular, that the tame loci $\mathfrak{T}^i(M)$ need not be open in codimension ≤ 4. The example of [13] also shows, that the tame loci $\mathfrak{T}^i(M)$ need not contain all primes $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$ of height 3. Therefore we shall focus to the "border line case" and investigate the sets $\mathfrak{T}^i(M)^{\leq 3}$ of all primes $\mathfrak{p}_0 \in \mathfrak{T}^i(M)$ of height ≤ 3.

In Section 2 of this paper we recall a few basic facts on the asymptotic stability of associated primes which shall be used constantly in our arguments. In this section we also introduce the so called *critical sets* $C^i(M) \subset \operatorname{Spec}(R_0)$ which consist of primes of height 3 and have the property that all primes $\mathfrak{p}_0 \notin C^i(M)$ of height ≤ 3 belong to the tame locus $\mathfrak{T}^i(M)$ (s. Proposition 2.8 (b)). Moreover the finiteness of the set $C^i(M)$ has the particularly nice consequence that M is *uniformly tame at level i in codimension* ≤ 3, e.g. there is an integer n_0 such that for each $\mathfrak{p}_0 \in \mathfrak{T}^i(M)^{\leq 3}$ the $(R_0)_{\mathfrak{p}_0}$-module $(H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$ is either vanishing for all $n \leq n_0$ or non-vanishing for all $n \leq n_0$ (s. Proposition 2.8 (c)).

In Section 3 we give some finiteness criteria for the critical sets $C^i(M)$. Here, we assume in addition that the base ring R_0 is a domain, so that the intersection $\mathfrak{a}^i(M)$ of all non-zero primes $\mathfrak{p}_0 \subset R_0$ which are associated to $H^i_{R_+}(M)$ is a non-zero ideal by a result of [5]. Our main result says, that the critical set $C^i(M)$ is finite, if $\mathfrak{a}^i(M)$ contains a *quasi-non-zero divisor* with respect to M (s. Theorem 3.4). This obviously applies in particular to the case in which M is torsion-free as an R_0-module in all large degrees or at all (s. Corollary 3.5 resp. Corollary 3.7). In order to force a situation as required in Theorem 3.4 one is tempted to replace M by $M/\Gamma_{(x)}(M)$ for some non-zero element $x \in R_0$. We therefore give a comparison result for the critical sets $C^i(M)$ and $C^i(M/\Gamma_{(x)}(M))$ (s. Proposition 3.7). As an application we prove that the critical sets $C^i(M)$ are finite if R_0 is a domain and the R_0-module M asymptotically satisfies some weak "unmixedness condition" (s. Corollary 3.8).

In our final Section 4 we give a few conditions for the *tameness at level i in codimension* ≤ 3 in terms of the "asymptotic smallness" of the graded R-modules $H^{i-1}_{R_+}(M)$ and $H^{i-2}_{R_+}(M)$. We first prove that all primes $\mathfrak{p}_0 \subset R_0$ of height ≤ 3 belong to the tame locus $\mathfrak{T}^i(M)$, provided that $\dim_{R_0}(H^{i-1}_{R_+}(M)_n) \leq 1$ and $\dim_{R_0}(H^{i-2}_{R_+}(M)_n) \leq 2$ for all $n \ll 0$ (s. Theorem 4.2). In addition we show that M is tame at almost all primes $\mathfrak{p}_0 \subset R_0$ of height ≤ 3 provided that R_0 is a domain and $\dim_{R_0}(H^{i-1}_{R_+}(M)_n) \leq 0$ for all $n \ll 0$ (s. Theorem 4.4). We actually prove in both cases slightly sharper statements namely: the corresponding graded $R_{\mathfrak{p}_0}$-modules $H^i_{R_+}(M)_{\mathfrak{p}_0}$ are not only tame, but even what we call *almost Artinian*. Using this terminology we get in particular the following conclusion. If R_0 is a domain and the graded R-module $H^{i-1}_{R_+}(M)$ is almost Artinian, then for almost all primes $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$ of height ≤ 3 either the $(R_0)_{\mathfrak{p}_0}$-module $(H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$ is of dimension > 0 for all $n \ll 0$ or else the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is almost Artinian (s. Corollary 4.5).

2. Tame Loci in Codimension ≤ 3

We keep the previously introduced notations.

Convention and Notation 2.1. (A) Throughout this section we convene that the base ring R_0 of our Noetherian homogeneous ring $R = R_0 \bigoplus R_1 \bigoplus ...$ is essentially of finite type over some field. So, $R_0 = S^{-1}A$, where $A = K[a_1, \ldots, a_s]$ is a finitely generated algebra over some field K, $S \subseteq A$ is multiplicatively closed and there are finitely many elements $l_1, \ldots, l_r \in R_1$ such that $R = R_0[l_1, \ldots, l_r]$.
(B) If $n \in \mathbb{N}_0$ and $\mathfrak{P} \subseteq \operatorname{Spec}(R_0)$ we write

$$\mathfrak{P}^{=n} := \{\mathfrak{p}_0 \in \mathfrak{P} \mid \operatorname{height}(\mathfrak{p}_0) = n\}$$
$$\mathfrak{P}^{\leq n} := \{\mathfrak{p}_0 \in \mathfrak{P} \mid \operatorname{height}(\mathfrak{p}_0) \leq n\}.$$

Reminder and Remark 2.2. (A) According to [1] for all $n \ll 0$ the set $\operatorname{Ass}_{R_0}(M_n)$ is equal to the set $\{\mathfrak{p} \cap R_0 \mid \mathfrak{p} \in \operatorname{Ass}_R \cap \operatorname{Proj}(R)\}$ and hence asymptotically stable for $n \to \infty$, thus:

There is a least integer $m(M) \geq 0$ and a finite set $\operatorname{Ass}^*_{R_0}(M) \subseteq \operatorname{Spec}(R_0)$ such that $\operatorname{Ass}_{R_0}(M_n) = \operatorname{Ass}^*_{R_0}(M)$ for all $n > m(M)$.

(B) Let $f(M)$ denote the finiteness dimension of M with respect to R_+, that is "the least integer" for which the R-module $H^i_{R_+}(M)$ is not finitely generated. Clearly we may write

$$f(M) = \inf\{i \in \mathbb{N}_0 \mid \sharp\{n \in \mathbb{Z} \mid H^i_{R_+}(M)_n \neq 0\} = \infty\}.$$

(C) Keep in mind that $f(M) > 0$. According to [BH, Theorem 5.6] we know that the set $\mathrm{Ass}_{R_0}(H^{f(M)}_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$:

There is a largest integer $n(M) \leq 0$ and a finite set $\mathfrak{U}(M) \subseteq \mathrm{Spec}(R_0)$ such that $\mathrm{Ass}_{R_0}(H^{f(M)}_{R_+}(M)_n) = \mathfrak{U}(M)$ for all $n \leq n(M)$.

In particular

$$\mathrm{Supp}_{R_0}(H^{f(M)}_{R_+}(M)_n) = \overline{\mathfrak{U}(M)}, \quad \forall n \leq n(M),$$

where $\overline{\bullet}$ denotes the formation of the topological closure in $\mathrm{Spec}(R_0)$.

(D) According to [B1, Theorem 4.1] we know that for each $i \in \mathbb{N}_0$ the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ is asymptotically stable in codimension ≤ 2 for $n \to -\infty$:

For each $i \in \mathbb{N}_0$ there is a largest integer $n^i(M) \leq 0$ and a finite set $\mathfrak{P}^i(M) \subseteq \mathrm{Spec}(R_0)^{\leq 2}$ such that $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2} = \mathfrak{P}^i(M)$ for all $n \leq n^i(M)$.

Now, combining this with the observations made in parts (B) and (C) we obtain:

(i) $i < f(M) \Rightarrow \forall n \leq n^i(M) : H^i_{R_+}(M)_n = 0;$

(ii) $\forall n \leq n(M) : \mathrm{Supp}_{R_0}(H^{f(M)}_{R_+}(M)_n) = \overline{\mathfrak{U}(M)};$

(iii) $i > f(M) \Rightarrow \forall n \leq n^i(M) : \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2} = \overline{\mathfrak{P}^i(M)}^{\leq 2}.$

Definition and Remark 2.3. (A) Let $i \in \mathbb{N}_0$. We say that the finitely generated graded R-module M is *(cohomologically) tame at level i* if the graded R-module $H^i_{R_+}(M)$ is tame, e.g.

$$\exists n_0 \in \mathbb{Z} : (\forall n \leq n_0 : H^i_{R_+}(M)_n = 0) \vee (\forall n \leq n_0 : H^i_{R_+}(M)_n \neq 0).$$

(B) Let $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$. We say that M is *(cohomologically) tame at level i in $\mathfrak{p}_0$* if the graded $R_{\mathfrak{p}_0}$-module $M_{\mathfrak{p}_0}$ is cohomologically tame at level i. In view of the graded flat base change property of local cohomology it is equivalent to say that the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is tame.

(C) We define the *i-th (cohomological) tame locus of M* as the set $\mathfrak{T}^i(M)$ of all primes $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$ such that M is (cohomologically) tame at level i in $\mathfrak{p}_0$. So, if $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$ we have

$$\mathfrak{p}_0 \in \mathfrak{T}^i(M) \Leftrightarrow \exists n_0 \in \mathbb{Z} : \begin{cases} \forall n \leq n_0 : \mathfrak{p}_0 \in \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n) \\ \qquad\qquad \text{or} \\ \forall n \leq n_0 : \mathfrak{p}_0 \notin \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n) \end{cases}$$

If $k \in \mathbb{N}_0$, the set $\mathfrak{T}^i(M)^{\leq k}$ is called the *i-th (cohomological) tame locus of M in codimension $\leq k$.*

(D) Let $\mathfrak{U} \subseteq \mathrm{Spec}(R_0)$. We say that M is *(cohomologically) tame at level i along* $\mathfrak{U}$, if $\mathfrak{U} \subseteq \mathfrak{T}^i(M)$. We say that M is *uniformly (cohomologically) tame at level i along* $\mathfrak{U}$ if there is an integer n_0 such that for all $\mathfrak{p}_0 \in \mathfrak{U}$

$$\big(\forall n \leq n_0 : \mathfrak{p}_0 \in \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n\big) \vee \big(\forall n \leq n_0 : \mathfrak{p}_0 \notin \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n\big).$$

(E) If M is uniformly tame at level i along the set $\mathfrak{U} \subseteq \mathrm{Spec}(R_0)$, then it is tame along $\mathfrak{U}$ at level i.

Remark 2.4. (A) According to Reminder and Remark 2.2 (D) (i) and (ii) we have

M is uniformly tame along $\mathrm{Spec}(R_0)$ *at all levels* $i \leq f(M)$.

(B) Using the notation of Reminder and Remark 2.2 (A) we write $\mathrm{Supp}^*_{R_0}(M) := \overline{\mathrm{Ass}^*_{R_0}(M)}$ so that $\mathrm{Supp}_{R_0}(M_n) = \mathrm{Supp}^*_{R_0}(M)$ for all $n \geq m(M)$. Now, on use of Reminder and Remark 2.2 (D) it follows easily:

for all $i > f(M)$, the module M is uniformly tame at level i along the set $W^i(M) := (\mathrm{Spec}(R_0) \setminus \mathrm{Supp}^*_{R_0}(M)) \cup \overline{\mathfrak{P}^i(M)} \cup \mathrm{Spec}(R_0)^{\leq 2}$.

It follows in particular that $W^i(M) \subseteq \mathfrak{T}^i(M)$, and moreover, for all $i \in \mathbb{N}_0$:

(i) M is uniformly tame at level i along the set $\mathrm{Spec}(R_0)^{\leq 2}$.

(ii) $\mathfrak{T}^i(M)^{\leq 3}$ is stable under generalization.

If the graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is tame, and $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$, then the graded $R_{\mathfrak{p}_0}$-module $T_{\mathfrak{p}_0}$ need not to be tame any more. This hints that in general the loci $\mathfrak{T}^i(M)$ could be non-stable under generalization. We now present such an example.

Example 2.5. Let K be algebraically closed. Then according to [CCHS], there exists a normal homogeneous Noetherian domain $R' = \bigoplus_{n\geq 0} R'_n$ of dimension 4 such that $(R'_0, \mathfrak{m}'_0)$ is local, of dimension 3 with $R'_0/\mathfrak{m}'_0 = K$ and such that for all negative integers n we have $H^2_{R'_+}(R')_n = K^2$ if n is even and $H^2_{R'_+}(R')_n = 0$ if n is odd.

Now, let $l_1, ..., l_r \in R'_1$ be such that $R'_1 = \sum_{i=1}^r R'_0 l_i$. Let $x, x_1, ..., x_r$ be indeterminates, let R_0 denote the 4-dimensional local domain $R'_0[x]_{(\mathfrak{m}'_0, x)}$ with maximal ideal $\mathfrak{m}_0 := (\mathfrak{m}'_0, x)R'_0$, consider the homogeneous R_0-algebras $R := R_0[x_1, ..., x_r]$ and $\overline{R} := R_0 \otimes_{R'_0} R'$ together with the surjective graded homomorphism of R_0-algebras

$$\Phi : R = R_0[x_1, ..., x_r] \twoheadrightarrow \overline{R}; \quad x_i \mapsto 1_{R_0} \otimes l_i.$$

Now, let $\alpha \in \mathfrak{m}'_0 \setminus \{0\}$, let t be a further indeterminate, consider the Rees algebra

$$S = R_0[xt, (x+\alpha)t] = \bigoplus_{n\geq 0} ((x, x+\alpha)R_0)^n$$

and the surjective graded homomorphism of R_0-algebras

$$\Psi : R \twoheadrightarrow S, \quad x_1 \mapsto xt, \quad x_2 \mapsto (x+\alpha)t, \quad x_i \mapsto 0 \text{ if } i \geq 3.$$

We consider $\overline{R}$ and S as graded R-modules by means of Φ and Ψ respectively. Then $M := \overline{R} \oplus S$ is a finitely generated graded R-module which is, in addition, torsion-free over R_0.

By the graded base ring independence and flat base change properties of local cohomology we get isomorphisms of graded R-modules

$$H^2_{R_+}(\overline{R}) \cong R_0 \otimes_{R'_0} H^2_{R'_+}(R'), \quad H^2_{R_+}(S) \cong H^2_{S_+}(S).$$

As $\mathrm{cd}_{S_+}(S) = \dim(S/\mathfrak{m}_0 S) = 2$ we have $H^2_{S_+}(S)_n \neq 0$ for all $n \ll 0$. It follows that $H^2_{R_+}(M)_n \cong H^2_{R_+}(\overline{R})_n \oplus H^2_{S_+}(S)_n \neq 0$ for all $n \ll 0$ and so M is tame at level 2. In particular we have $\mathfrak{m}_0 \in \mathfrak{T}^2(M)$.

Now, consider the prime $\mathfrak{p}_0 := \mathfrak{m}_0' R_0 \in \operatorname{Spec}(R_0)^{=3}$. Then, for each $n < 0$ we have

$$(H^2_{R_+}(\overline{R})_n)_{\mathfrak{p}_0} \cong (R_0)_{\mathfrak{m}_0' R_0} \otimes_{R_0'} H^2_{R'_+}(R')_n \cong \begin{cases} K(x)^2, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Moreover $S_{\mathfrak{p}_0} = (R_0)_{\mathfrak{p}_0}[(x, x+\alpha)(R_0)_{\mathfrak{p}_0} t] = (R_0)_{\mathfrak{p}_0}[t]$ shows that $H^2_{S_+}(S)_{\mathfrak{p}_0} \cong H^2_{(S_{\mathfrak{p}_0})_+}(S_{\mathfrak{p}_0}) = 0$. It follows that $(H^2_{R_+}(M)_n)_{\mathfrak{p}_0}$ vanishes precisely for all odd negative integers n. So $H^2_{R_+}(M)_{\mathfrak{p}_0}$ is not tame and hence $\mathfrak{p}_0 \notin \mathfrak{T}^2(M)$.

Observe in particular that here $\mathfrak{T}^2(M) = \mathfrak{T}^2(M)^{\leq 4}$ is not stable under generalization, and that R_0 is a domain and the graded R-module M is torsion-free over R_0. On the other hand $\mathfrak{T}^i(M)^{\leq 3}$ is always stable under generalization, (cf. Remark 2.4 (B) (ii)).

One of our aims is to show that quite a lot can be said about the sets $\mathfrak{T}^i(M)^{\leq 3}$ if the base ring R_0 is a domain and M is torsion-free over R_0. Indeed, we shall attack the problem in a more general context, beginning with the following result, in which $\mathfrak{P}^i(M)$ is defined according to Definition and Remark 2.2 (D).

Lemma 2.6. *Let $i \in \mathbb{N}_0$ and let $n^i(M)$ be defined as in Reminder and Remark 2.2 (D). Then for all $n \leq n^i(M)$ we have*

$$C^i_n(M) := \big(\operatorname{Supp}_{R_0}(H^i_{R_+}(M)_n) \backslash \overline{\mathfrak{P}^i(M)}\big)^{\leq 3} = \big(\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n) \backslash \overline{\mathfrak{P}^i(M)}\big)^{=3}.$$

Proof. Let $n \leq n^i(M)$ and $\mathfrak{p}_0 \in \big((\operatorname{Supp}_{R_0}(H^i_{R_+}(M)_n) \backslash \overline{\mathfrak{P}^i(M)}\big)^{\leq 3}$. Then, there is some $\mathfrak{q}_0 \in \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)$ with $\mathfrak{q}_0 \subseteq \mathfrak{p}_0$. As $\mathfrak{p}_0 \notin \overline{\mathfrak{P}^i(M)}$ we have $\mathfrak{q}_0 \notin \mathfrak{P}^i(M) = \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$. It follows that $\operatorname{height}(\mathfrak{q}_0) \geq 3$, hence $\mathfrak{q}_0 = \mathfrak{p}_0$ and therefore

$$\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{=3}.$$

This proves the inclusion " $\subseteq$ ". The converse inclusion is obvious. □

Definition 2.7. Let $i \in \mathbb{N}_0$ and let $n^i(M)$ and $C^i_n(M)$ be as in Lemma 2.6. Then the set

$$C^i(M) := \bigcup_{n \leq n^i(M)} C^i_n(M)$$

is called the *ith critical set of M*.

Proposition 2.8. *Let $i \in \mathbb{N}_0$. Then*
(a) *M is uniformly tame at level i along the set*

$$[\big(\operatorname{Spec}(R_0) \setminus \operatorname{Supp}^*_{R_0}(M)\big) \cup \overline{\mathfrak{P}^i(M)} \cup \operatorname{Spec}(R_0)^{\leq 3}] \setminus C^i(M).$$

(b) $\mathfrak{T}^i(M)^{\leq 3} \supseteq \operatorname{Spec}(R_0)^{\leq 3} \setminus C^i(M)$.
(c) *The following statements are equivalent:*
(i) *$C^i(M)$ is a finite set;*
(ii) *$\mathfrak{T}^i(M)^{\leq 3}$ is open in $\operatorname{Spec}(R_0)^{\leq 3}$ and M is uniformly tame at level i along $\mathfrak{T}^i(M)^{\leq 3}$.*
(iii) *$\operatorname{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)$ is finite and M is uniformly tame at level i along $\mathfrak{T}^i(M)^{\leq 3}$.*

Proof. (a): This follows from Remark 2.4 (B) and the fact that

$$\Big[\bigcup_{n\leq n^i(M)} \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)\Big]^{=3} \setminus \overline{\mathfrak{P}^i(M)} = C^i(M).$$

(b): This is immediate by statement (a).
(c): "(i) $\Rightarrow$ (ii)": This follows easily by statements (a) and (b) and the fact that M is uniformly tame at level i along each finite subset $V \subseteq \mathfrak{P}^i(M)$.
"(ii) $\Rightarrow$ (iii)": Assume that statement (ii) holds. As $\mathrm{Spec}(R_0)^{\leq 2} \subseteq \mathfrak{T}^i(M)^{\leq 3}$ (s. Remark 2.4 (B) (i)) and as $\mathfrak{T}^i(M)^{\leq 3}$ is open in $\mathrm{Spec}(R_0)^{\leq 3}$ it follows that $\mathrm{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)^{\leq 3}$ is a finite set, and this proves statement (iii).
"(iii) $\Rightarrow$ (i)": Assume that statement (iii) holds so that $\mathrm{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)$ is finite and M is uniformly tame along $\mathfrak{T}^i(M)^{\leq 3}$. By statement (b) we have $\mathrm{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)^{\leq 3} \subseteq C^i(M) \subseteq \mathrm{Spec}(R_0)^{=3}$. It thus suffices to show that the set $F := C^i(M) \cap \mathfrak{T}^i(M)$ is finite.

By uniform tameness there is some integer $n_0 \leq n^i(M)$ such that for each $\mathfrak{p}_0 \in F$ either

$$(I)\ \ \mathfrak{p}_0 \in \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)\ \text{ for all }\ n \leq n_0; \text{or}$$

$$(II)\ \ \mathfrak{p}_0 \notin \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)\ \text{ for all }\ n \leq n_0.$$

Let $F_I := \{\mathfrak{p}_0 \in F \mid \mathfrak{p}_0\,\text{satisfies}\,(I)\}$ and $F_{II} := \{\mathfrak{p}_0 \in F \mid \mathfrak{p}_0\,\text{satisfies}\,(II)\}$. As $F = F_I \cup F_{II}$ it suffices to show that F_I and F_{II} are finite.

If $\mathfrak{p}_0 \in F_I$, we have $\mathfrak{p}_0 \in \big(\mathrm{Supp}_{R_0}(H^i_{R_+}(M)_{n_0}) \setminus \overline{\mathfrak{P}^i(M)}\big)^{\leq 3}$. As $n_0 \leq n^i(M)$ statement (a) implies $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_{n_0})$. This proves that $F_I \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_{n_0})$ and thus F_I is finite.

Clearly $F_{II} \subseteq \big(\bigcup_{n_0\leq n\leq n^i(M)} \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n \setminus \overline{\mathfrak{P}^i(M)}\big)^{\leq 3}$. So, by statement (a) we see that F_{II} is contained in the finite set $\bigcup_{n_0\leq n\leq n^i(M)} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$. □

3. Finiteness of Critical sets

We keep all notations and hypotheses of the previous section. So $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a Noetherian homogeneous ring whose base ring R_0 is essentially of finite type over some field and M is a finitely generated graded R-module. By statement (c) of Proposition 2.8 it seems quite appealing to look for criteria which ensure that the critical sets $C^i(M)$ are finite. This is precisely the aim of the present section.

Reminder 3.1. (A) Assume that R_0 is a domain. Then, according to [BFL, Theorem 2.5] there is an element $s \in R_0\setminus\{0\}$ such that the $(R_0)_s$-module $(H^i_{R_+}(M))_s$ is torsion-free or 0 for all $i \in \mathbb{N}_0$. From this we conclude that (with the standard convention that $\bigcap_{\mathfrak{p}_0\in\emptyset} \mathfrak{p}_0 := R_0$):

If R_0 is a domain, the ideal

$$\mathfrak{a}^i(M) := \bigcap_{\mathfrak{p}_0\in \mathrm{Ass}_{R_0}(H^i_{R_+}(M))\setminus\{0\}} \mathfrak{p}_0$$

is $\neq 0$ for all $i \in \mathbb{N}_0$.

(B) Keep the notations and hypotheses of part (A). Then:

If $x \in \mathfrak{a}^i(M)$ and if N is a second finitely generated graded R-module such that the graded R_x-modules M_x and N_x are isomorphic, then $x \in \mathfrak{a}^i(N)$.

This follows immediately from the fact, that for all $n \in \mathbb{Z}$ there is an isomorphism of $(R_0)_x$-modules $(H^i_{R_+}(M)_n)_x \cong (H^i_{R_+}(N)_n)_x$. For our purposes the most significant application of this observation is:

If $x \in \mathfrak{a}^i(M)$ then $x \in \mathfrak{a}^i(M/\Gamma_{(x)}(M))$.

Notation 3.2. An element $x \in R_0$ is called a *quasi-non-zero divisor with respect to (the finitely generated graded R-module) M* if x is a non-zero divisor on M_n for all $n \gg 0$. We denote the set of these quasi-non-zero divisors by $\mathrm{NZD}^*_{R_0}(M)$. Thus in the notation of Reminder and Remark 2.2 (A) we may write

$$\mathrm{NZD}^*_{R_0}(M) = R_0 \setminus \bigcup_{\mathfrak{p}_0 \in \mathrm{Ass}^*_{R_0}(M)} \mathfrak{p}_0.$$

Lemma 3.3. *Let $i, k \in \mathbb{N}_0$ and assume that* $\mathrm{height}(\mathfrak{p}_0) \geq k$ *for all* $\mathfrak{p}_0 \in \mathrm{Ass}\,^*_{R_0}(M)$. *Then, the set* $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq k+2}$ *is asymptotically stable for $n \to -\infty$. In particular, if $k > 0$, then $C^i(M)$ is finite.*

Proof. There is some integer $n_0 \in \mathbb{Z}$ such that $(0 :_{R_0} M_{\geq n_0}) \subseteq R_0$ is of height $\geq k$, where we use the notation $M_{\geq n_0} := \bigoplus_{n \geq n_0} M_n$. As $H^i_{R_+}(M)$ and $H^i_{R_+}(M_{\geq n_0})$ differ only in finitely many degrees we may replace M by $M_{\geq n_0}$ and hence assume that $\mathfrak{a}_0 M = 0$ for some ideal $\mathfrak{a}_0 \subseteq R_0$ with $\mathrm{height}(\mathfrak{a}_0) \geq k$. As $\mathrm{height}(\mathfrak{p}_0/\mathfrak{a}_0) \leq \mathrm{height}(\mathfrak{p}_0) - k$ for all $\mathfrak{p}_0 \in \mathrm{Var}(\mathfrak{a}_0)$ and in view of the natural isomorphisms of R_0-modules $H^i_{R_+}(M)_n \cong H^i_{(R/\mathfrak{a}_0 R)_+}(M)_n$ we now get a canonical bijection

$$\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq k+2} \leftrightarrow \mathrm{Ass}_{R_0/\mathfrak{a}_0}(H^i_{R_+}(M)_n)^{\leq 2},$$

for all $n \in \mathbb{Z}$. So, by Reminder and Remark 2.2 (D) the left hand side set is asymptotically stable for $n \to -\infty$. If $k > 0$ the finiteness of $C^i(M)$ now follows easily from statement (a) of Lemma 2.6. □

Let $i \in \mathbb{N}_0$. According to Remark 2.4 (B) we know that M is uniformly tame at level i in codimension ≤ 2. we also know that M need not be tame at level i in codimension 3. It is natural to ask, whether there are only finitely many primes $\mathfrak{p}_0$ of height 3 in R_0 such that M is not tame at level i in $\mathfrak{p}_0$ and whether outside of these "bad" primes the module M is uniformly tame at level i in codimension ≤ 3. We aim to give a few sufficient criteria for this behaviour. The following theorem plays a crucial rôle in this respect.

Theorem 3.4. *Let $i \in \mathbb{N}_0$. Assume that R_0 is a domain and that* $\mathrm{NZD}^*_{R_0}(M) \cap \mathfrak{a}^i(M) \neq \emptyset$. *Then $C^i(M)$ is a finite set. In particular the set* $\mathrm{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)$ *consists of finitely many primes of height 3 and M is uniformly tame at level i along $\mathfrak{T}^i(M)^{\leq 3}$.*

Proof. If $i \leq f(M)$ our claim is clear by Remark 2.4 (A) and Proposition 2.8 (c). So, let $i > f(M)$. Then in particular $i > 1$.

Now, let $m(M) \in \mathbb{Z}$ be as in Reminder and Remark 2.2 (A) and set $N := M_{\geq m(M)} := \bigoplus_{n \geq m(M)} M_n$. Then $\mathrm{NZD}^*_{R_0}(M)$ equals the set $\mathrm{NZD}_{R_0}(N)$ of non-zero divisors in R_0 on

N. As $i > 1$ we have $H^i_{R_+}(N) = H^i_{R_+}(M)$ and hence $\mathfrak{a}^i(M) = \mathfrak{a}^i(N)$ and $C^i(M) = C^i(N)$. So, we may replace M by N and hence assume that $\mathrm{NZD}_{R_0}(M) \cap \mathfrak{a}^i(M) \neq \emptyset$.

Let $x \in \mathrm{NZD}_{R_0}(M) \cap \mathfrak{a}^i(M)$. Then, the short exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ implies exact sequences

$$H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_n \longrightarrow H^i_{R_+}(M/xM)_n$$

for all $n \in \mathbb{Z}$. Now, let $\mathfrak{p}_0 \in C^i(M)$ so that $\mathrm{height}(\mathfrak{p}_0) = 3$ (s. Lemma 2.6). Then, there is an integer $n \leq n^i(M)$ such that $\mathfrak{p}_0$ is a minimal associated prime of $H^i_{R_+}(M)_n$. We thus get an exact sequence of $(R_0)_{\mathfrak{p}_0}$-modules

$$(H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \xrightarrow{\frac{x}{1}} (H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \xrightarrow{\varrho} (H^i_{R_+}(M/xM)_n)_{\mathfrak{p}_0}$$

in which the middle module is of finite length $\neq 0$. As $x \in \mathfrak{a}^i(M) \subseteq \mathfrak{p}_0$ it follows by Nakayama that ϱ is not the zero map. Therefore $(H^i_{R_+}(M/xM)_n)_{\mathfrak{p}_0}$ contains a non-zero $(R_0)_{\mathfrak{p}_0}$-module of finite length. It follows that $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M/xM)_n)^{=3}$. This shows that $C^i(M) \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+}(M/xM)_n)^{=3}$. So, by Lemma 3.3 the set $C^i(M)$ is finite. □

Corollary 3.5. *Let $i \in \mathbb{N}_0$. Assume that R_0 is a domain and that M_n is a torsion-free R_0-module for all $n \gg 0$. Then the set $C^i(M)$ is finite. In particular, M is uniformly tame at level i along $\mathfrak{T}^i(M)^{\leq 3}$ and the set $\mathrm{Spec(R_0)}^{\leq 3} \setminus \mathfrak{T}^{\mathrm{i}}(\mathrm{M})$ is finite.*

Proof. By our hypotheses we have $\mathrm{NZD}^*_{R_0}(M) = R_0 \setminus \{0\}$. By Reminder 3.1 (A) we have $\mathfrak{a}^i(M) \neq 0$. Now we conclude by Theorem 3.4. □

Corollary 3.6. *Let $i \in \mathbb{N}_0$ and assume that R_0 is a domain and M is torsion-free over R_0. Then M is uniformly tame at level i along a set which is obtained by removing finitely many primes of height 3 from $\mathrm{Spec}(R_0)^{\leq 3}$.*

Proof. This is clear by Corollary 3.5. □

Our next aim is to replace the requirement that M_n is R_0 torsion-free for all $n \gg 0$, which was used in Corollary 3.5 by a weaker condition. We begin with the following finiteness result for certain subsets of critical sets:

Proposition 3.7. *Let R_0 be a domain, let $i \in \mathbb{N}$ and let $x \in R_0 \setminus \{0\}$ be such that $x\Gamma_{(x)}(M) = 0$. Then*
(a) $[C^i(M) \setminus [C^i(M/\Gamma_{(x)}(M)) \cup [\overline{\mathfrak{P}^{i-1}(M/xM)} \cap \overline{\mathfrak{P}^{i+1}(\Gamma_{(x)}(M))}]^{=3}]$ *is a finite set.*
(b) *If $x \in \mathfrak{a}^i(M)$, then the set $C^i(M/\Gamma_{(x)}(M))$ and hence also the set*

$$C^i(M) \setminus \left[[\overline{\mathfrak{P}^{i-1}(M/xM)} \cap \overline{\mathfrak{P}^{i+1}(\Gamma_{(x)}(M))}]^{=3} \setminus C^i(M/\Gamma_{(x)}(M))\right]$$

is finite.

Proof. (a): Fix an integer $n_0 \leq n^i(M/xM), n^i(\Gamma_{(x)}(M)), n^i(M), n^i(M/\Gamma_{(x)}(M))$ and let $\mathfrak{p}_0 \in C^i(M)$. Then $\mathfrak{p}_0 \in \min \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ for some $n \leq n^i(M)$. If $n_0 \leq n$, $\mathfrak{p}_0$ thus belongs to the finite set $\bigcup_{m \geq n_0} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_m)$. So, let $n < n_0$. The graded short exact sequences

$$0 \longrightarrow M/\Gamma_{(x)}(M) \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma_{(x)}(M) \longrightarrow M \longrightarrow M/\Gamma_{(x)}(M) \longrightarrow 0$$

imply exact sequences

$$(H^{i-1}_{R_+}(M/xM)_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M/xM)_n)_{\mathfrak{p}_0}$$

and

$$(H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M/\mathit{\Gamma}_{(x)}(M))_n)_{\mathfrak{p}_0} \longrightarrow (H^{i+1}_{R_+}(\mathit{\Gamma}_{(x)}(M))_n)_{\mathfrak{p}_0}.$$

Assume that $\mathfrak{p}_0 \notin C^i(M/\Gamma_{(x)}(M))$. Then $(H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}$ either vanishes or is an $(R_0)_{\mathfrak{p}_0}$-module of infinite length. In the first case we have $(H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \subseteq (H^i_{R_+}(M/xM)_n)_{\mathfrak{p}_0}$. As $(H^i_{R_+}(M)_n)_{\mathfrak{p}_0}$ is a non-zero $(R_0)_{\mathfrak{p}_0}$-module of finite length it follows $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M/xM)_n)$. So $\mathfrak{p}_0$ belongs to the finite set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M/xM))^{\leq 3}$ (s. Remark 3.3).

Assume now that $(H^i_{R_+}(M/\mathit{\Gamma}_{(x)}(M))_n)_{\mathfrak{p}_0}$ is not of finite length. Then, by the above sequences $(H^{i-1}_{R_+}(M/xM)_n)_{\mathfrak{p}_0}$ and $(H^{i+1}_{R_+}(\mathit{\Gamma}_{(x)}(M))_n)_{\mathfrak{p}_0}$ are both of infinite length, so that $\mathfrak{p}_0 \in \overline{\mathfrak{P}^{i-1}(M/xM)}$ and $\mathfrak{p}_0 \in \overline{\mathfrak{P}^{i+1}(\mathit{\Gamma}_{(x)}(M))}$.

(b): According to Reminder 3.1 (B) we have $x \in \mathfrak{a}^i(M/\Gamma_{(x)}(M))$. As moreover it holds $x \in \mathrm{NZD}_{R_0}(M/\Gamma_{(x)}(M))$ our claim follows be Theorem 3.4. $\square$

Corollary 3.8. *Let $i \in \mathbb{N}_0$, let R_0 be a domain and assume that* $\mathrm{height}(\mathfrak{p}_0) \geq 3$ *for all $\mathfrak{p}_0 \in \mathrm{Ass}^*_{R_0}(M)\setminus(\{0\} \cup \overline{\mathfrak{P}^i(M)})$. Then $C^i(M)$ is a finite set. In particular the set* $\mathrm{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)$ *is finite and M is uniformly tame at level i along the set $\mathfrak{T}^i(M)^{\leq 3}$.*

Proof. Let $m(M) \in \mathbb{Z}$ be as in Reminder and Remark 2.2 (A) so that $\mathrm{Ass}_{R_0}(M_n) = \mathrm{Ass}^*_{R_0}(M)$ for all $n \geq m(M)$. As $H^i_{R_+}(M)$ and $H^i_{R_+}(M_{\geq m(M)})$ differ only in finitely many degrees we may replace M by $M_{\geq m(M)}$ and hence assume that $\mathrm{Ass}^*_{R_0}(M) = \mathrm{Ass}_{R_0}(M)$. If $0 \notin \mathrm{Ass}_{R_0}(M)$ we get our claim by Lemma 3.3. So, let $0 \in \mathrm{Ass}_{R_0}(M)$ and consider the non-zero ideal $\mathfrak{b}_0 := \bigcap_{\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(M)\setminus\{0\}} \mathfrak{p}_0$. Then $\mathrm{Ass}_{R_0}(M/\Gamma_{\mathfrak{b}_0}(M)) = \{0\}$ so that $M/\Gamma_{\mathfrak{b}_0}(M)$ is torsion-free over R_0. Let $x \in \mathfrak{b}_0 \setminus \{0\}$ with $x\Gamma_{(x)}(M) = 0$. Then it follows that $\Gamma_{\mathfrak{b}_0}(M) = \Gamma_{(x)}(M)$. By Corollary 3.5 we therefore obtain that $C^i(M/\Gamma_{(x)}(M))$ is finite. According to Proposition 3.7 (a) it thus suffices to show that $C^i(M) \cap \overline{\mathfrak{P}^{i+1}(\Gamma_{\mathfrak{b}_0}(M))}^{=3}$ is finite. So, let $\mathfrak{q}_0$ be an element of this latter set. Then $\mathrm{height}(\mathfrak{q}_0) = 3$ and $\mathfrak{q}_0 \notin \overline{\mathfrak{P}^i(M)}$. Moreover, there is a minimal prime $\mathfrak{p}_0$ of $\mathfrak{b}_0$ with $\mathfrak{p}_0 \subseteq \mathfrak{q}_0$. In particular $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(M)\setminus\{0\}$ and $\mathfrak{p}_0 \notin \overline{\mathfrak{P}^i(M)}$. So, by our hypothesis $\mathrm{height}(\mathfrak{p}_0) \geq 3$, whence $\mathfrak{q}_0 = \mathfrak{p}_0 \in \mathrm{Ass}^*_{R_0}(M)\setminus\{0\}$. This shows that $\overline{C^i(M) \cap \mathfrak{P}^{i+1}(\Gamma_{\mathfrak{b}_0}(M))}^{=3} \subseteq \mathrm{Ass}^*_{R_0}(M)$ and hence proves our claim. $\square$

Remark 3.9. Clearly Corollary 3.6 applies to the domain R' constructed in [13] (s. Example 2.5), taken as a module over itself. In this example we have in particular $\mathfrak{T}^2(R')^{\leq 3} = \mathrm{Spec}(R'_0) \setminus \{\mathfrak{m}_0\}$. Moreover the uniform tameness of R' at level 2 along this set can be verified by a direct calculation.

4. Conditions on Neighbouring Cohomologies for Tameness in Codimensions ≤ 3

We keep the hypotheses and notations of the previous sections. So $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a homogeneous Noetherian ring whose base ring R_0 is essentially of finite type over a field and M is a finitely generated graded R-module.

Our first result says that M is tame in codimension ≤ 3 at a given level $i \in \mathbb{N}$, if the two neigbouring local cohomology modules $H^{i-1}_{R_+}(M)$ and $H^{i-2}_{R_+}(M)$ are "asymptotically sufficiently small". (We set $H^k_{R_+}(\bullet) := 0$ for $k < 0$). We actually shall prove a more specific statement. To formulate it, we first introduce an appropriate notion.

Definition and Remark 4.1. (A) We say that a graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is *almost Artinian* if there is some graded submodule $N = \bigoplus_{n\in\mathbb{Z}} N_n \subseteq T$ such that $N_n = 0$ for all $n \ll 0$ and such that the graded R-module T/N is Artinian.

(B) A graded R-module T which is the sum of an Artinian graded submodule and a Noetherian graded submodule clearly is almost Artinian. Moreover, the property of being almost Artinian passes over to graded subquotients.

(C) As R_0 is Noetherian and R is homogeneous each graded almost Artinian R-module T has the property that $\dim_{R_0}(T_n) \leq 0$ for all $n \ll 0$.

(D) Clearly an almost Artinian graded R-module is tame.

Now, we are ready to formulate and to prove the announced result.

Theorem 4.2. *Let $i \in \mathbb{N}$ such that $\dim_{R_0}(H^{i-1}_{R_+}(M)_n) \leq 1$ and $dim_{R_0}(H^{i-2}_{R_+}(M)_n) \leq 2$ for all $n \ll 0$. Then the following statements hold.*
(a) *The graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is almost Artinian for all $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)^{=3}\setminus\overline{\mathfrak{P}^i(M)}$.*
(b) *$\mathfrak{T}^i(M)^{\leq 3} = \mathrm{Spec}(R_0)^{\leq 3}$ and hence M is tame at level i in codimension ≤ 3.*

Proof. (a): Let $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)^{=3}\setminus\overline{\mathfrak{P}^i(M)}$. We consider the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{p}_0}(H^q_{R_+}(M))_{\mathfrak{p}_0} \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{p}_0+R_+}(M)_{\mathfrak{p}_0}.$$

By our assumption on the dimension of the R_0-modules $H^{i-1}_{R_+}(M)_n$ and $H^{i-2}_{R_+}(M)_n$, the n-th graded component $(E_2^{p,q})_n$ of the graded $R_{\mathfrak{p}_0}$-module $E_2^{p,q}$ vanishes for all $n \ll 0$ if $(p,q) = (2, i-1)$ or $(p,q) = (3, i-2)$. Therefore

$$(E_2^{0,i})_n \cong (E_\infty^{0,i})_n, \quad \forall n \ll 0.$$

As the graded $R_{\mathfrak{p}_0}$-module $E_\infty^{0,i}$ is a subquotient of the Artinian $R_{\mathfrak{p}_0}$-module $H^i_{\mathfrak{p}_0+R_+}(M)_{\mathfrak{p}_0}$, it follows by Definition and Remark 4.1 (B) that the graded $R_{\mathfrak{p}_0}$-module

$$H^0_{\mathfrak{p}_0 R_{\mathfrak{p}_0}}\big(H^i_{R_+}(M)_{\mathfrak{p}_0}\big) \cong H^0_{\mathfrak{p}_0}(H^i_{R_+}(M))_{\mathfrak{p}_0} = E_2^{0,i}$$

is almost Artinian. Now, since $\mathfrak{p}_0 \notin \overline{\mathfrak{P}^i(M)}$ and $\mathfrak{p}_0$ is of height 3 we must have

$$\dim_{R_{0\mathfrak{p}_0}}\big((H^i_{R_+}(M)_{\mathfrak{p}_0})_n\big) \leq 0, \quad \forall n \ll 0.$$

and hence $H^0_{\mathfrak{p}_0 R_{\mathfrak{p}_0}}\big(H^i_{R_+}(M)_{\mathfrak{p}_0}\big)$ and $H^i_{R_+}(M)_{\mathfrak{p}_0}$ coincide in all degrees $n \ll 0$. Therefore $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is indeed almost Artinian.

(b): This follows immediately from statement (a), as $\overline{\mathfrak{P}^i(M)} \subseteq \mathfrak{T}^i(M)$ (s. Remark 2.4 (B)). $\square$

Remark 4.3. The domain R' constructed in [13] (s. Example 2.5), taken as a module over itself, clearly cannot satisfy the hypotheses of Theorem 4.1 with $i = 2$ as it does not fulfill the corresponding conclusion of this theorem. Indeed a direct calculation shows that $\dim_{R'_0}(H^1_{R'_+}(R')_n) = 3$ for all $n < 0$.

Our next result says that the module M is tame at level i almost everywhere in codimension ≤ 3 provided that R_0 is a domain and the local cohomology module $H^{i-1}_{R_+}(M)$ is "asymptotically very small". Again, we aim to prove a more specific result.

Theorem 4.4. *Let R_0 be a domain and $i \in \mathbb{N}$ such that $\dim_{R_0}(H^{i-1}_{R_+}(M)) \leq 0$ for all $n \ll 0$. Then the following statements hold.*
(a) *There is a finite set $Z \subset \operatorname{Spec}(R_0)^{=3}$ such that the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is almost Artinian for all $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)^{=3} \setminus \big(Z \cup \overline{\mathfrak{P}^i(M)}\big)$.*
(b) $\operatorname{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)$ *is a finite subset of* $\operatorname{Spec}(R_0)^{=3}$.

Proof. (a): According to Reminder 3.1 (A) there is an element $x \in \mathfrak{a}^i(M) \setminus \{0\}$ such that $x\Gamma_{(x)}(M) = 0$. If we apply Lemma 3.3 with $k = 1$ to the the R-module M/xM (also with $i-1$ instead of i) and to the R-module $\Gamma_{(x)}(M)$ (with $i+1$ instead of i) we see that the three sets

$$\operatorname{Ass}_{R_0}(H^{i-1}_{R_+}(M/xM)_n)^{\leq 3}, \quad \operatorname{Ass}_{R_0}(H^{i}_{R_+}(M/xM)_n)^{\leq 3}, \quad \operatorname{Ass}_{R_0}(H^{i}_{R_+}(\Gamma_{(x)}(M)_n)^{\leq 3}$$

are asymptotically stable for $n \to -\infty$. So, there is a finite set $Z \subset \operatorname{Spec}(R_0)^{=3}$ such that

$$\operatorname{Ass}_{R_0}(H^{i-1}_{R_+}(M/xM)_n)^{=3} \cup \operatorname{Ass}_{R_0}(H^{i}_{R_+}(M/xM)_n)^{=3} \cup \operatorname{Ass}_{R_0}(H^{i+1}(\Gamma_{(x)}(M)_n)^{=3} = Z$$

for all $n \ll 0$. Let

$$\mathfrak{p}_0 \in \operatorname{Spec}(R_0)^{=3} \setminus \big(Z \cup \overline{\mathfrak{P}^i(M)}\big).$$

We aim to show that the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is almost Artinian. As $\mathfrak{p}_0 \notin \overline{\mathfrak{P}^i(M)}$ and $\operatorname{height}(\mathfrak{p}_0) = 3$ it follows

$$\operatorname{lenght}_{(R_0)_{\mathfrak{p}_0}}(H^i_{R_+}(M)_n)_{\mathfrak{p}_0}) < \infty$$

for all $n \ll 0$. As $\dim_{R_0}(H^{i-1}_{R_+}(M)_n) \leq 0$ for all $n \ll 0$ we also have

$$\operatorname{length}_{(R_0)_{\mathfrak{p}_0}}(H^{i-1}_{R_+}(M)_n)_{\mathfrak{p}_0} < \infty$$

for all $n \ll 0$. As $\mathfrak{p}_0 \notin Z$ and $\operatorname{height}(\mathfrak{p}_0) = 3$, we also can say

$$\Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}\big((H^{i-1}_{R_+}(M/xM)_n)_{\mathfrak{p}_0}\big) = \Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}\big((H^{i}_{R_+}(M/xM)_n)_{\mathfrak{p}_0}\big) =$$
$$= \Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}\big((H^{i+1}_{R_+}(\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}\big) = 0, \quad \forall n \ll 0.$$

Now, as in the proof of Proposition 3.8 (a), the canonical graded short exact sequences

$$0 \longrightarrow M/\Gamma(x)(M) \xrightarrow{\phi} M \longrightarrow M/xM \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma_{(x)}(M) \longrightarrow M \xrightarrow{\pi} M/\Gamma_{(x)}(M) \longrightarrow 0$$

respectively imply exact sequences of $(R_0)_{\mathfrak{p}_0}$-modules

$$(H^{i-1}_{R_+}(M)_n)_{\mathfrak{p}_0} \longrightarrow (H^{i-1}_{R_+}(M/xM)_n)_{\mathfrak{p}_0} \longrightarrow$$

$$\longrightarrow (H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0} \xrightarrow{(H^i_{R_+}(\phi)_n)_{\mathfrak{p}_0}} (H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M/xM)_n)_{\mathfrak{p}_0}$$

and

$$(H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \xrightarrow{(H^i_{R_+}(\pi)_n)_{\mathfrak{p}_0}} (H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0} \longrightarrow (H^{i+1}_{R_+}(\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}$$

for all $n \ll 0$. Keep in mind, that in the first of these sequences the first and the second but last module are of finite length for all $n \ll 0$, whereas the second and the last module are $\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}$-torsion-free for all $n \ll 0$. Observe further, that in the second sequence the first module is of finite length and the last module is $\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}$-torsion-free for all $n \ll 0$. So there is an integer $n(x)$ such that for each $n \leq n(x)$ we have the exact sequence

$$0 \longrightarrow (H^{i-1}_{R_+}(M/xM)_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0} \xrightarrow{(H^i_{R_+}(\phi)_n)_{\mathfrak{p}_0}} (H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \longrightarrow 0$$

and the relation

$$\mathrm{Im}(H^i_{R_+}(\pi)_n)_{\mathfrak{p}_0}) = \Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}(H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}).$$

Thus, for all $n \leq n(x)$ the image of the composite map

$$(H^i_{R_+}(\pi)_n)_{\mathfrak{p}_0} \circ (H^i_{R_+}(\phi)_n)_{\mathfrak{p}_0} : (H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0} \longrightarrow (H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}$$

is the torsion module $\Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}((H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0})$. As the composite map $\pi \circ \phi : M/\Gamma_{(x)}(M) \longrightarrow M/\Gamma_{(x)}(M))$ coincides with the multiplication map $x = x\mathrm{Id}_{M/\Gamma_{(x)}(M)}$ on $M/\Gamma_{(x)}(M)$ we end up with

$$\Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}((H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}) = x(H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}, \quad \forall n \leq n(x).$$

Now, without affecting $\Gamma_{(x)}(M)$ we may replace x by x^2 and thus get the equalities

$$x(H^i_{R_+}(M/\Gamma_{(x)}(M)_n)_{\mathfrak{p}_0} = x^2(H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}$$

for all $n \leq m(x) := \min\{n(x), n(x^2)\}$. Consequently, as $x \in \mathfrak{p}_0$ and as the $(R_0)_{\mathfrak{p}_0}$-modules $(H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}$ are finitely generated, it follows by Nakayama that

$$\Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}((H^i_{R_+}(M/\Gamma_{(x)}(M))_n)_{\mathfrak{p}_0}) = 0, \quad \forall n \ll 0.$$

Applying the functor $\Gamma_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}(\bullet)$ to the above short exact sequences and keeping in mind that the right hand side module in these sequences is of finite length, we get the natural monomorphisms

$$0 \longrightarrow (H^i_{R_+}(M)_n)_{\mathfrak{p}_0} \longrightarrow H^1_{\mathfrak{p}_0(R_0)_{\mathfrak{p}_0}}(H^{i-1}_{R_+}(M/xM)_n)_{\mathfrak{p}_0}, \quad \forall n \leq m(x).$$

It is easy to see, that these monomorphisms are the graded parts of a homomorphism of graded $R_{\mathfrak{p}_0}$-modules. Moreover, as $\dim((R_0/xR_0)_{\mathfrak{p}_0}) \leq 2$ the graded $R_{\mathfrak{p}_0}$-module

$$H^1_{\mathfrak{p}(R_0)_{\mathfrak{p}_0}}(H^{i-1}_{R_+}(M/xM)_{\mathfrak{p}_0}) \cong H^1_{\mathfrak{p}_0(R_0/xR_0)_{\mathfrak{p}_0}}(H^{i-1}_{(R/xR)_{\mathfrak{p}_0+}}((M/xM)_{\mathfrak{p}_0}))$$

is Artinian (s. [10] Theorem 5.10). In view of the observed monomorphisms and by Definition and Remark 4.1 (B), this implies immediately, that the graded $R_{\mathfrak{p}_0}$-module $(H^i_{R_+}(M))_{\mathfrak{p}_0}$ is almost Artinian.

(b): This follows immediately from statement (a), Reminder and Remark 4.1 (D) and Remark 2.4 (B). $\square$

This leads us immediately to the following observation.

Corollary 4.5. *If R_0 is a domain and $i \in \mathbb{N}$ is such that the R-module $H^{i-1}_{R_+}(M)$ is almost Artinian, then the set of all primes $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)^{\leq 3} \setminus \overline{\mathfrak{P}^i(M)}$ for which the graded $R_{\mathfrak{p}_0}$-module $H^i_{R_+}(M)_{\mathfrak{p}_0}$ is not almost almost Artinian as well as the set $\mathrm{Spec}(R_0)^{\leq 3} \setminus \mathfrak{T}^i(M)$ are both finite subsets of $\mathrm{Spec}(R_0)^{=3}$.*

Proof. This is immediate by Theorem 4.4 and Definition and Remark 4.1 (C). $\square$

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