

ON VARIETIES OF ALMOST MINIMAL DEGREE III : TANGENT SPACES AND EMBEDDING SCROLLS

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ABSTRACT. Let $X \subset \mathbb{P}^r$ be a variety of almost minimal degree which is the projected image of a rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}$ from a point p outside of \tilde{X} . In this paper we study the tangent spaces at singular points of X and the geometry of the embedding scrolls of X , i.e. the rational normal scrolls $Y \subset \mathbb{P}^r$ which contain X as a codimension one subvariety.

1. INTRODUCTION

Varieties of minimal degree, namely (irreducible non-degenerate) projective varieties $X \subseteq \mathbb{P}^r$ with $\deg(X) = \text{codim}(X) + 1$ have been studied and classified already in the 19th century by del Pezzo in the case of surfaces and by Bertini in the general case. Varieties of almost minimal degree, e.g. projective varieties $X \subseteq \mathbb{P}^r$ which satisfy the equality $\deg(X) = \text{codim}(X) + 2$ are still an active branch of projective algebraic geometry. These latter varieties were studied and classified by Fujita (see [Fu1] or [Fu3]). A purely algebraic approach to varieties of almost minimal degree was given by Hoa-Stückrad-Vogel [H-St-V] in 1991. In [B-S] it was shown that varieties $X \subseteq \mathbb{P}^r$ of almost minimal degree which are either non-linearly normal or non-normal are precisely the linear projections of varieties $\tilde{X} \subseteq \mathbb{P}^{r+1}$ of minimal degree from a point $p \in \mathbb{P}^{r+1} \setminus \tilde{X}$.

So, understanding varieties of almost minimal degree which are either non-linearly normal or else non-normal is equivalent to knowing the possible linear projections $\pi_p : \tilde{X} \rightarrow X_p := \pi_p(\tilde{X})$ of a variety of minimal degree $\tilde{X} \subseteq \mathbb{P}^{r+1}$ from points $p \in \mathbb{P}^{r+1} \setminus \tilde{X}$. If \tilde{X} is (a cone over) the Veronese surface in \mathbb{P}^5 , this is a task which can be solved easily. In the "general case", namely if the projecting variety $\tilde{X} \subseteq \mathbb{P}^{r+1}$ is a (cone over) a smooth rational normal scroll this same task turns out to be more demanding. The crucial point here consist in knowing, how the so called secant locus

$$\Sigma_p(\tilde{X}) := \{q \in \tilde{X} \mid \#(\langle p, q \rangle \cap \tilde{X}) > 1\}$$

of \tilde{X} with respect to the center of projection p depends on p . In [B-P] we have solved this problem, by making explicit the so called secant stratification of \tilde{X} . One application of this is an extension of Fujita's classification of

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normal del Pezzo varieties to possibly non-normal del Pezzo varieties (s.[B-P]).

In the present paper, we are concerned with local aspects of varieties of almost minimal degree.

Our first aim is to determine the embedding dimension $\dim(T_x X)$ and the multiplicity $m_x(X)$ of a closed singular point x of a variety $X \subseteq \mathbb{P}^r$ of almost minimal degree which is not normal. It turns out that for all such points x which are not vertex points of X we have

$$\dim(T_x X) = 2 \dim(X) + 2 - \text{depth}(X) \text{ and } m_x(X) = 2,$$

where $\text{depth}(X)$ denotes the arithmetic depth of X (s. Theorem 3.9). Clearly the behavior of the tangent spaces $T_x X$ of a variety $X \subseteq \mathbb{P}^r$ of almost minimal degree is closely related to the question how the tangent spaces $T_{q_1} \tilde{X}, T_{q_2} \tilde{X}$ of the projecting variety $\tilde{X} \subseteq \mathbb{P}^{r+1}$ in two distinct points $q_1, q_2 \in \tilde{X}$ intersect. Again, the case where $\tilde{X} \subseteq \mathbb{P}^{r+1}$ is a rational normal scroll is crucial here. We treat this problem completely in the case where \tilde{X} is smooth (s. Theorem 4.2) and a cone (s. Corollary 4.4). Once more, the secant stratification of \tilde{X} is the basic tool we need to do this.

The final sections 5 and 6 are devoted to the study of the so called embedding scrolls $Y \subseteq \mathbb{P}^r$ of a variety $X \subseteq \mathbb{P}^r$ of almost minimal degree ≥ 5 , that is of scrolls Y containing X and satisfying $\dim(Y) = \dim(X) + 1$. In [B-S] it is shown that these embedding scrolls always exist. They are a very useful tool for the study of Betti diagrams of varieties of almost minimal degree (s.[B-S],[N] and in particular [P1]). Our aim is to give an account on all possible embedding scrolls of a given variety $X \subseteq \mathbb{P}^r$ of almost minimal degree (which is not a cone). We show that the singular embedding scrolls of X are always of the shape $Y = \text{Join}(\text{Sing}(X), X)$ and hence unique, and that in the case where $2 \leq \text{depth}(X) \leq \dim(X)$ there are no smooth embedding scrolls (s. Theorems 5.5 and 5.8). We also describe the possible smooth embedding scrolls in the case $\text{depth}(X) = \dim(X) + 1$, that is if X is maximally del Pezzo. In this situation we have (s. Theorem 6.10)

- a one-dimensional family of smooth embedding scrolls if X is a curve
- a unique smooth embedding scroll if the projecting scroll \tilde{X} is a surface without line sections
- no smooth embedding scroll in the remaining cases.

In the case where X is smooth, we show that all its embedding scrolls are smooth (s. Theorem 5.5 (1)). But in this case, we are not able yet to describe all possible embedding scrolls (which indeed occur in families now).

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2. PRELIMINARIES

Notation and Remark 2.1. (A) Let K be an algebraically closed field, let r be an integer ≥ 2 and let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a variety of minimal degree with

$$n := \dim(\tilde{X}) \text{ and } e := \text{codim}(\tilde{X}) \geq 2.$$

So, $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is either a rational normal scroll or (a cone over) the Veronese surface in \mathbb{P}_K^5 . Keep in mind that \tilde{X} is integral, non-degenerate, arithmetically normal, arithmetically Cohen-Macaulay (CM) and of degree $e + 1$.

(B) Now, let $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ be a closed point. We fix a projective space \mathbb{P}_K^r and a linear projection

$$\pi_p : \tilde{X} \rightarrow X := \pi_p(\tilde{X}) \subseteq \mathbb{P}_K^r$$

of \tilde{X} from p . We may consider \mathbb{P}_K^r as a subspace of \mathbb{P}_K^{r+1} with $p \notin \mathbb{P}_K^r$ and π_p as given by the canonical projection of \mathbb{P}_K^{r+1} from p onto \mathbb{P}_K^r , so that $\pi_p^{-1}(x) = \tilde{X} \cap \langle x, p \rangle$ for all closed points $x \in X$. Keep in mind that π_p is finite and birational and that X is a *variety of almost minimal degree*, in the sense of [B-S], so that $\deg(X) = \text{codim}(X) + 2$.

(C) Keep the above notations and consider the *secant cone of \tilde{X}* with respect to p , defined by

$$\text{Sec}_p(\tilde{X}) := \bigcup_{q \in \tilde{X} : \text{length}(\tilde{X} \cap \langle p, q \rangle) \geq 2} \langle p, q \rangle$$

if \tilde{X} admits secant lines passing through p , and $\text{Sec}_p(\tilde{X}) = \{p\}$ else. We furnish $\text{Sec}_p(\tilde{X})$ with its reduced scheme structure. We also introduce the *secant locus of \tilde{X}* with respect to p , which is defined as the scheme theoretic intersection

$$\Sigma_p(\tilde{X}) := \text{Sec}_p(\tilde{X}) \cap \tilde{X}.$$

Let us also consider the *arithmetic depth* of X , which we denote by t , thus

$$t := \text{depth}(X).$$

In these notations we have (s. [B-S, Theorem 1.3]):

- (2.1) *If $t = 1$, then \tilde{X} and X are smooth, $\pi_p : \tilde{X} \rightarrow X$ is an isomorphism and X is not linearly normal.*
- (2.2) *If $t \geq 2$, then $\text{Sec}_p(\tilde{X}) = \mathbb{P}_K^{t-1} \subseteq \mathbb{P}_K^{r+1}$, the secant locus $\Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^{t-1}$ is a hyperquadric and $\pi_p(\Sigma_p(\tilde{X})) = \mathbb{P}_K^{t-2}$ is the non-normal locus of X .*

In addition, if X is not arithmetically CM, then the generic point of the non CM-locus of X is of Goto type. More precisely:

- (2.3) *If $1 \leq t \leq n$, then $\pi_p(\Sigma_p(\tilde{X}))$ is the non CM-locus of X and the generic point x of this locus satisfies*

$$H_{\mathfrak{m}_{X,x}}^i(\mathcal{O}_{X,x}) \cong \begin{cases} 0, & \text{if } i \neq 1, \dim(\mathcal{O}_{X,x}), \\ \kappa(x), & \text{if } i = 1. \end{cases}$$

(D) According to [B-S] a *maximal del Pezzo* variety $X \subseteq \mathbb{P}_K^r$ is a variety of almost minimal degree which is arithmetically CM. These are indeed the del Pezzo varieties in the sense of Fujita [Fu3][Fu2] which are in addition linearly normal. A *del Pezzo variety* is a projective variety which is the image of a maximal del Pezzo variety under a linear isomorphic projection. Using this terminology we can say (s. [B-S, Theorem 1.2], [Fu1] and [Fu3]):

(2.4) *A variety $X \subseteq \mathbb{P}_K^r$ of almost minimal degree is either a normal maximal del Pezzo variety, or obtained by a projection $\pi_p : \tilde{X} \rightarrow X$ of a variety $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ of minimal degree and codimension ≥ 2 from a closed point $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$.*

In the latter case we call $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ a projecting variety (of minimal degree) for X . •

We now recall a few facts on smooth rational normal scrolls and fix some further notation.

Notation and Reminder 2.2. (A) For a positive integer a , let $\sigma_a : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^a$ be the a -uple embedding and let $S(a) := \sigma_a(\mathbb{P}_K^1) \subset \mathbb{P}_K^a$ be the rational normal curve of degree a .

(B) Let a_1, \dots, a_n be a non-decreasing sequence of positive integers such that $r+1 = \sum a_i + n - 1$. Then we choose n complementary linear subspaces $\Lambda_i = \mathbb{P}_K^{a_i} \subset \mathbb{P}_K^{r+1}$ and rational normal curves $S(a_i) \subset \Lambda_i$ for all $i \in \{1, \dots, n\}$. We may define the smooth n -fold rational normal scroll $S(a_1, \dots, a_n)$ as

$$\tilde{X} = S(a_1, \dots, a_n) := \bigcup_{x \in \mathbb{P}^1} \langle \sigma_{a_1}(x), \dots, \sigma_{a_n}(x) \rangle \subset \mathbb{P}_K^{r+1}.$$

Note that $S(a_1, \dots, a_n)$ is determined up to projective equivalence by the integers a_i . Moreover, any smooth rational normal scroll $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ of dimension n is projectively equivalent to a scroll $S(a_1, \dots, a_n)$ with uniquely determined positive integers $a_1 \leq a_2 \leq \dots \leq a_n$.

(C) For each closed point $x \in \mathbb{P}_K^1$, we consider the linear $(n-1)$ -space

$$\mathbb{L}(x) := \langle \sigma_{a_1}(x), \dots, \sigma_{a_n}(x) \rangle,$$

called the ruling of \tilde{X} over x . Moreover, let

$$\varphi : S(a_1, \dots, a_n) \rightarrow \mathbb{P}_K^1$$

be the natural projection morphism, so that $\varphi^{-1}(x) = \mathbb{L}(x)$ for all $x \in \mathbb{P}_K^1$. •

Next, we recall a few facts on hyperplane sections of smooth rational normal scrolls and their linear projections.

Remark 2.3. (A) (cf. [Ha, Theorem 8.29]) Let $n > 1$ and let $\tilde{X} := S(a_1, \dots, a_n) \subset \mathbb{P}^{r+1}$ be a smooth rational normal n -fold scroll. For a hyperplane $\tilde{H} \subset \mathbb{P}^{r+1}$, the hyperplane section $\tilde{X} \cap \tilde{H} \subseteq \tilde{H}$ is a smooth rational

$(n - 1)$ -fold scroll if and only if $\mathbb{L}(x) \not\subseteq \tilde{H}$ for all $x \in \mathbb{P}^1$.

(B) Keep the notations of part (A) and let $\mathbb{P}_K^s = \Lambda \subset \mathbb{P}_K^{r+1}$ be a linear subspace such that $\dim(\tilde{X} \cap \Lambda) \leq 0$. Let $\tilde{\mathcal{H}} := \{\tilde{H} \in (\mathbb{P}_K^{r+1})^* \mid \Lambda \subseteq \tilde{H}\}$ be the linear $(r - s)$ -subspace of $(\mathbb{P}_K^{r+1})^*$ which consists of all hyperplanes $\tilde{H} \subset \mathbb{P}_K^{r+1}$ containing Λ . Moreover let $\mathcal{G} = \{\tilde{H} \in \tilde{\mathcal{H}} \mid \exists x \in \mathbb{P}_K^1 : \mathbb{L}(x) \subseteq \tilde{H}\}$ be the family of all hyperplanes in \mathbb{P}_K^{r+1} which contain Λ and some ruling $\mathbb{L}(x)$ of \tilde{X} . Writing $\mathcal{G} = \bigcup_{x \in \mathbb{P}_K^1} \{\tilde{H} \in (\mathbb{P}_K^{r+1})^* \mid \langle \mathbb{L}(x), \Lambda \rangle \subseteq \tilde{H}\}$ and observing that $\langle \mathbb{L}(x), \Lambda \rangle = \mathbb{P}_K^{n+s}$ for generic $x \in \mathbb{P}_K^1$ and $\langle \mathbb{L}(x), \Lambda \rangle = \mathbb{P}_K^{n+s-1}$ if $\mathbb{L}(x) \cap \Lambda \neq \emptyset$ we see that the closed set $\mathcal{G} \subseteq (\mathbb{P}_K^{r+1})^*$ is of dimension $\leq (r+1) - (n+s+1) + 1 = r - n - s + 1 < r - s$, so that $\mathcal{G} \subsetneq \tilde{\mathcal{H}} = \mathbb{P}_K^{r-s}$. According to the observation made in part (A) we thus see that for a generic hyperplane $\tilde{H} \subset \mathbb{P}_K^{r+1}$ containing Λ , the intersection $\tilde{X} \cap \tilde{H} \subseteq \tilde{H} = \mathbb{P}_K^r$ is a smooth rational $(n - 1)$ -fold scroll.

(C) Keep the above notations and let $\mathbb{P}_K^u = L \subseteq \mathbb{P}_K^{r+1}$ be a linear subspace with $L \cap \tilde{X} = \emptyset$. Let $\pi_L : \tilde{X} \rightarrow X := \pi_L(\tilde{X}) \subseteq \mathbb{P}_K^{r-u}$ be a linear projection of \tilde{X} from L . We may assume that $\mathbb{P}_K^{r-u} \subseteq \mathbb{P}_K^{r+1}$ is disjoint to L and π_L is given by the canonical projection of \mathbb{P}_K^{r+1} from L onto \mathbb{P}_K^{r-u} . Now, let $x \in X$ be a closed point. Let $\Lambda := \langle x, L \rangle$. As the morphism $\pi_L : \tilde{X} \rightarrow X$ is finite, the fibre $\tilde{X} \cap \Lambda = \pi_L^{-1}(x)$ is of dimension 0. Moreover the assignment $H \mapsto \tilde{H} := \langle H, L \rangle$ yields an isomorphism between the space $\mathcal{H} \subseteq (\mathbb{P}_K^{r-u})^*$ of all hyperplanes $H \subseteq \mathbb{P}_K^r$ running through x and the space $\tilde{\mathcal{H}} \subseteq (\mathbb{P}_K^{r+1})^*$ of all hyperplanes $\tilde{H} \subseteq \mathbb{P}_K^{r+1}$ containing Λ . As $\pi_L^{-1}(X \cap H) = \tilde{X} \cap \tilde{H}$ for all $H \in \mathcal{H}$ we conclude from part (B) that $X \cap H$ is irreducible for a generic hyperplane $H \subseteq \mathbb{P}_K^{r-u}$ running through x .

(D) Keep the previous notations and assume in addition that $\pi_L : \tilde{X} \rightarrow X$ is birational. Then, there is a non-empty open set $U \subseteq X$ such that π_L induces an isomorphism from $\tilde{U} := \pi_L^{-1}(U)$ onto U . For generic $H \in \mathcal{H}$ we have $U \cap (X \cap H) = U \cap H \neq \emptyset$ and the induced isomorphism $\tilde{U} \cap (\tilde{X} \cap \tilde{H}) \cong U \cap (X \cap H)$ implies that $U \cap (X \cap H)$ is smooth. So, if $\pi_L : \tilde{X} \rightarrow X$ is in addition birational we conclude from part (C) that $X \cap H$ is not only irreducible but also generically reduced for a generic hyperplane $H \subseteq \mathbb{P}_K^{r-u}$ with $x \in H$. \bullet

Before resuming the above observations we prove the following lemma.

Lemma 2.4. *Let $X \subset \mathbb{P}_K^r$ be a nondegenerate irreducible projective variety of degree d and codimension $e \geq 2$. Let p be the minimal prime divisor of d . Then for any linear space $\mathbb{P}_K^t = \Lambda \subset \mathbb{P}_K^r$ with $t \leq \lceil e - 1 - \frac{d}{p} \rceil$ and $\Lambda \cap X = \emptyset$, the linear projection $\pi_\Lambda : X \rightarrow \mathbb{P}_K^{r-t-1}$ of X from Λ is birational onto $X_\Lambda := \pi_\Lambda(X)$.*

Proof. Suppose that $\pi_\Lambda : X \rightarrow X_\Lambda$ is not birational. Then $\deg(\pi_\Lambda) \geq p$ since it is a divisor of d . On the other hand, the codimension of X_Λ in \mathbb{P}_K^{r-t-1} is equal

to $e - t - 1$ and so we have $\deg(X_\Lambda) \geq e - t$. Therefore

$$d = \deg(\pi_\Lambda) \cdot \deg(X_\Lambda) \geq (e - t)p,$$

which contradicts the assumption that $t \leq \lceil e - 1 - \frac{d}{p} \rceil$. \square

Now we summarize the observations made in Remark 2.3 as follows:

Proposition 2.5. *Let $r > 2$ and let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a smooth rational normal scroll of degree d with $\dim(\tilde{X}) > 1$. Let $\mathbb{P}_K^u = L \subseteq \mathbb{P}_K^{r+1}$ be a linear subspace such that $L \cap \tilde{X} = \emptyset$, and let $\pi_L : \tilde{X} \rightarrow X := \pi_L(\tilde{X}) \subseteq \mathbb{P}_K^{r-u}$ be a linear projection of \tilde{X} from L . Let $x \in X$ be a closed point and let $\mathbb{P}_K^{r-u-1} = H \subset \mathbb{P}_K^{r-u}$ be a generic hyperplane running through x . Then*

- (a) $X \cap H$ is irreducible.
- (b) If $u \leq \lceil d - 2 - \frac{d}{p} \rceil$, where p is the minimal prime divisor of d , then $X \cap H$ is in addition generically reduced.

Proof. Clear from Remark 2.3 (C), (D) and Lemma 2.4. \square

3. TANGENT SPACES OF VARIETIES OF ALMOST MINIMAL DEGREE

The aim of this section is to calculate the dimension of the tangent space $T_x X$ and the multiplicity $m_x(X)$ of an n -dimensional projective variety $X \subseteq \mathbb{P}_K^r$ of almost minimal degree at a closed singular point x . We begin with a few preparations.

First of all, let us recall that a noetherian ring R is said to satisfy the *second Serre condition* S_2 , if for each $\mathfrak{p} \in \text{Spec}(R)$ we have

$$\text{depth}(R_{\mathfrak{p}}) \geq \min\{2, \dim(R_{\mathfrak{p}})\}.$$

Correspondingly, a locally noetherian scheme X is said to be S_2 at the point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ satisfies the property S_2 . Clearly, if X is CM at x , then it is S_2 at x .

Lemma 3.1. *Let X be a locally noetherian scheme, let $W \subset X$ be an effective Cartier divisor and let $x \in W$ be such that W is irreducible and generically reduced at x . Let $Z \subseteq X$ be a closed set such that X is S_2 at all points $w \in W \setminus Z$ and assume that either*

- (1) $\text{depth}(\mathcal{O}_{X,x}) \geq 2$ and $\dim_x(Z) \leq 0$

or

- (2) $\text{depth}(\mathcal{O}_{X,x}) > \dim_x(Z) > \dim_x(W \cap Z) \geq 0$.

Then W is integral at x .

Proof. The statement is of local nature. So, we may assume that $X = \text{Spec}(R)$ and $x = \mathfrak{m}$, where (R, \mathfrak{m}) is a local noetherian ring. We then find a non-zero divisor $h \in \mathfrak{m}$ of R such that $\mathcal{I}_{W,x} = hR$ and an ideal $\mathfrak{a} \subseteq R$ such that $Z = \text{Var}(\mathfrak{a})$. As W is irreducible (at x) we have $\mathfrak{p} := \sqrt{hR} \in \text{Spec}(R)$. As W is generically reduced (at x) we have $hR_{\mathfrak{p}} = \sqrt{hR_{\mathfrak{p}}} = \mathfrak{p}R_{\mathfrak{p}}$. It remains to show

that the set $\text{Ass}_R(R/hR)$ of primes associated to the R -module R/hR consists only of \mathfrak{p} . So, let $\mathfrak{q} \in \text{Ass}_R(R/hR)$. As $\mathfrak{q} \in \text{Var}(hR) = \text{Var}(\mathfrak{p})$ we have $\mathfrak{p} \subseteq \mathfrak{q}$. Assume now that $\mathfrak{q} \neq \mathfrak{p}$, so that $\mathfrak{p} \subsetneq \mathfrak{q}$.

Suppose first that $\mathfrak{q} \in \text{Var}(\mathfrak{a}) = Z$. If $\dim(R/\mathfrak{a}) = \dim_x(Z) \leq 0$, we have $\mathfrak{q} = \mathfrak{m}$ and condition (1) implies that $\text{depth}(R) \geq 2$. This leads to the contradiction that $\mathfrak{q} = \mathfrak{m} \notin \text{Ass}_R(R/hR)$. Therefore $\dim(R/\mathfrak{a}) = \dim_x(Z) > 0$ and so condition (2) yields that

$$\text{depth}(R) > \dim(R/\mathfrak{a}) > \dim(R/(\mathfrak{a} + Rh)) \geq \dim(R/\mathfrak{q}),$$

whence $\dim(R/\mathfrak{q}) < \text{depth}(R) - 1 = \text{depth}(R/hR)$. But this contradicts the fact that $\mathfrak{q} \in \text{Ass}_R(R/hR)$. Therefore $\mathfrak{q} \notin \text{Var}(\mathfrak{a})$. As $\mathfrak{p} \subsetneq \mathfrak{q}$ and $h \in \mathfrak{p}$ is a non-zero divisor in R we have $\dim(R_{\mathfrak{q}}) > \dim(R_{\mathfrak{p}}) > 0$. As $R_{\mathfrak{q}}$ is S_2 it follows $\text{depth}(R_{\mathfrak{q}}) \geq 2$, whence $\text{depth}(R_{\mathfrak{q}}/hR_{\mathfrak{q}}) \geq 1$, which contradicts the fact that $\mathfrak{q} \in \text{Ass}_R(R/hR)$. So, $\mathfrak{q} = \mathfrak{p}$ as requested. \square

Lemma 3.2. *Let $f : \tilde{X} \rightarrow X$ be a finite morphism of integral locally noetherian schemes and let $x \in X$ be with $n := \dim_x(X) \geq 1$ and such that*

- (1) $f^{-1}(x)$ consists of smooth points of \tilde{X} and
- (2) $(f_*\mathcal{O}_{\tilde{X}})_x/\mathcal{O}_{X,x}$ is a simple $\mathcal{O}_{X,x}$ -module.

Then

- (a) The Hilbert-Samuel function of X at x is given by

$$H_{X,x}(t) := \text{length}(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^{t+1}) = 2 \binom{n+t}{n} - 1 \text{ for all } t \in \mathbb{N}_0.$$

- (b) The embedding dimension of X at x and the multiplicity of X at x are given respectively by $\dim(T_x X) = 2n$ and $m_x(X) = 2$.

Proof. We write $R = \mathcal{O}_{X,x}$, $\mathfrak{m} = \mathfrak{m}_{X,x}$ and $\tilde{R} = (f_*\mathcal{O}_{\tilde{X}})_x$. According to hypothesis (2) the ring \tilde{R} is a finite integral extension domain of R , that is the semilocal ring of \tilde{X} at the finitely many points of $f^{-1}(x)$. Now, by hypotheses (2) we have $\mathfrak{m} = \mathfrak{m}\tilde{R}$ and $\text{length}_R(\tilde{R}/\mathfrak{m}\tilde{R}) = 2$. In particular $f^{-1}(x)$ contains at most two points. Moreover, by hypotheses (1) the ring \tilde{R} is regular and of dimension n with one or two maximal ideals, according to the number of points in the set $f^{-1}(x)$.

(a): We first treat the case in which the set $f^{-1}(x)$ consists of a single point, so that \tilde{R} has a unique maximal ideal $\tilde{\mathfrak{m}}$. Assume first that $\tilde{\mathfrak{m}} = \mathfrak{m}$. Then $(\tilde{R}, \mathfrak{m})$ is a regular local ring of dimension n , so that $\text{length}_{\tilde{R}}(\tilde{R}/\mathfrak{m}^{t+1}) = \binom{n+t}{n}$ for all $t \in \mathbb{N}_0$. As the field \tilde{R}/\mathfrak{m} is of degree 2 over the field R/\mathfrak{m} and as \tilde{R}/R is a simple R -module it follows that $\text{length}_R(R/\mathfrak{m}^{t+1}) = 2 \binom{n+t}{n} - 1$ for all $t \in \mathbb{N}_0$ and this is our claim.

Assume now that $\tilde{\mathfrak{m}} \neq \mathfrak{m}$, so that the residue fields $\tilde{R}/\tilde{\mathfrak{m}}$ and R/\mathfrak{m} are isomorphic and $\tilde{\mathfrak{m}}/\mathfrak{m}$ is a simple R -module. Choosing $b \in \tilde{\mathfrak{m}} \setminus (\tilde{\mathfrak{m}}^2 \cup \mathfrak{m})$ we thus get $\tilde{\mathfrak{m}} = \mathfrak{m} + bR = \mathfrak{m} + b\tilde{R}$. As $(\tilde{R}, \tilde{\mathfrak{m}})$ is regular of dimension n and

$b \notin \tilde{\mathfrak{m}}^2$ we find elements $a_1, \dots, a_{n-1} \in \mathfrak{m}$ such that $\tilde{\mathfrak{m}} = \sum_{i=1}^{n-1} a_i \tilde{R} + b \tilde{R}$. Now a_1, \dots, a_{n-1}, b is a regular system of parameters of \tilde{R} and so $\tilde{R}/\sum_{i=1}^{n-1} a_i \tilde{R}$ is a regular local ring of dimension 1. We thus find some element $a \in \mathfrak{m}$ such that $\mathfrak{m}/\sum_{i=1}^{n-1} a_i \tilde{R} \subseteq \tilde{R}/\sum_{i=1}^{n-1} a_i \tilde{R}$ is the principal ideal generated by $a + \sum_{i=1}^{n-1} a_i \tilde{R}$. Consequently $\mathfrak{m} = \sum_{i=1}^{n-1} a_i \tilde{R} + a \tilde{R}$. As a_1, \dots, a_{n-1}, a form an \tilde{R} -sequence we thus get the equality $\text{length}_{\tilde{R}}(\tilde{R}/\mathfrak{m}^{t+1}) = \text{length}_{\tilde{R}}(\tilde{R}/\mathfrak{m}) \binom{n+t}{n}$ for all $t \in \mathbb{N}_0$. As $\text{length}_R(\tilde{R}/\mathfrak{m}) = 2$, $\text{length}_R(\tilde{R}/R) = 1$ and $\tilde{R}/\tilde{\mathfrak{m}} \cong R/\mathfrak{m}$ we get our claim.

So, let us assume now that $f^{-1}(x)$ consists of two points. Then \tilde{R} has precisely two maximal ideals $\tilde{\mathfrak{m}}_1$ and $\tilde{\mathfrak{m}}_2$. As $\mathfrak{m} = \mathfrak{m}\tilde{R} \subseteq \tilde{\mathfrak{m}}_1 \cap \tilde{\mathfrak{m}}_2 \subsetneq \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2$ and $\dim_K(\tilde{R}/\mathfrak{m}) = 2$ we get $\mathfrak{m} = \tilde{\mathfrak{m}}_1 \cap \tilde{\mathfrak{m}}_2 = \tilde{\mathfrak{m}}_1 \tilde{\mathfrak{m}}_2$ and $\tilde{R}/\tilde{\mathfrak{m}}_1 \cong \tilde{R}/\tilde{\mathfrak{m}}_2 \cong K$. Now, for each $t \in \mathbb{N}_0$ the Chinese Remainder Theorem yields an isomorphism $\tilde{R}/\mathfrak{m}^{t+1} \cong \tilde{R}/\tilde{\mathfrak{m}}_1^{t+1} \times \tilde{R}/\tilde{\mathfrak{m}}_2^{t+1}$. As the rings $\tilde{R}_{\tilde{\mathfrak{m}}_i}$ are regular local of dimension n , we get $\text{length}_R(\tilde{R}/\tilde{\mathfrak{m}}_i^{t+1}) = \text{length}_{\tilde{R}}(\tilde{R}/\tilde{\mathfrak{m}}_i^{t+1}) = \text{length}(\tilde{R}_{\tilde{\mathfrak{m}}_i}/(\tilde{\mathfrak{m}}_i \tilde{R}_{\tilde{\mathfrak{m}}_i})^{t+1}) = \binom{n+t}{n}$ for all $t \in \mathbb{N}_0$ and for $i = 1, 2$. It follows that

$$\text{length}_R(\tilde{R}/\mathfrak{m}^{t+1}) = 2 \binom{n+t}{n} \text{ for all } t \in \mathbb{N}_0.$$

Now, we get our claim as in the previous case.

(b): This is clear from statement (a) as $\dim(T_x X) = H_{X,x}(1) - 1$ and $m_x(X) = n! \lim_{t \rightarrow \infty} t^{-n} H_{X,x}(t)$. \square

We now want to prove an intermediate result which concerns a more general setting than what we are basically heading for in this paper. To do so, we recall a few facts on projections of certain varieties.

Reminder 3.3. (A) Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a smooth rational normal scroll. Then according to Eisenbud-Green-Hulek-Popescu [E-G-H-P] the variety \tilde{X} has the syzygetic property $N_{2,2}$, and hence in particular the condition K_2 introduced by Vermeire [V]. Thus, a natural extension of the program performed in this paper would be to study simple exterior birational projections $X \subseteq \mathbb{P}_K^r$ of varieties $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ which satisfy the property $N_{2,2}$ or even just the condition K_2 . So, let $\pi_p : \tilde{X} \rightarrow X = \pi_p(\tilde{X})$ be a birational morphism induced by a projection from a point $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$. Then by Vermeire [V] one knows that the secant cone $\text{Sec}_p(\tilde{X}) \subseteq \mathbb{P}_K^{r+1}$ of \tilde{X} with respect to p is a linear subspace and the secant locus $\Sigma_p(\tilde{X}) = \text{Sec}_p(\tilde{X}) \cap \tilde{X}$ of \tilde{X} with respect to p is a hyperquadric in this subspace. This implies in particular, that the singular locus $\text{Sing}(\pi_p) = \pi_p(\Sigma_p(\tilde{X}))$ of the morphism π_p is a linear subspace of dimension $s := \dim(\text{Sing}(\pi_p)) - 1$ and the sheaf $(\pi_p)_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ has support $\text{Sing}(\pi_p) = \mathbb{P}_K^s$. In the particular case in which \tilde{X} is a rational normal scroll we know in addition that $\dim(\text{Sing}(\pi_p)) = \text{depth}(X) - 2$ (s. (2.1),(2.2)).

(B) Keep the hypotheses and notations of part (A). Let $x \in X$ be the generic point of $\text{Sing}(\pi_p)$. We are interested in the local behavior of X at x , notably in the dimension of the tangent space $T_x X$ of X at x . To this end, we aim to apply Lemma 3.2 and in order to do so, we should know that the sheaf $(\pi_p)_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X$ is simple at x . This latter requirement is satisfied if \tilde{X} satisfies the property $N_{2,2}$. In fact, in this case we can say even more. Namely, according to Ahn-Kwak [A-K] we have an isomorphism $(\pi_p)_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X \cong \mathcal{O}_{\text{Sing}(\pi_p)}(-1) \cong \mathcal{O}_{\mathbb{P}^s}(-1)$ so that indeed $((\pi_p)_* \mathcal{O}_{\tilde{X}})_x / \mathcal{O}_{X,x} \cong \kappa(x)$. \bullet

Proposition 3.4. *Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a smooth nondegenerate variety which satisfies the property $N_{2,2}$, let $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ and let $\pi_p : \tilde{X} \rightarrow X := \pi_p(\tilde{X}) \subseteq \mathbb{P}_K^r$ be a birational linear projection of \tilde{X} from p . Set $d := \dim(\text{Sing}(\pi_p))$, $n := \dim(X)$ and let $x \in X$ be the generic point of $\text{Sing}(\pi_p)$. Then the local ring $\mathcal{O}_{X,x}$ is of dimension $n - d$ and*

- (a) $H_{X,x}(t) := \text{length}(\mathcal{O}_{X,x} / \mathfrak{m}_{X,x}^{t+1}) = 2 \binom{n-d+t}{n-d} - 1$ for all $t \in \mathbb{N}_0$.
- (b) $\dim(T_x X) = 2(n - d)$ and $m_x(X) = 2$.

Proof. By Reminder 3.3 (A) the singular locus $\text{Sing}(\pi_p) \subseteq X \subseteq \mathbb{P}_K^r$ of π_p is a linear subspace of dimension d and so, the local local ring $\mathcal{O}_{X,x}$ has indeed dimension $n - d$. According to Reminder 3.3 (B) the $\mathcal{O}_{X,x}$ -module $((\pi_p)_* \mathcal{O}_{\tilde{X}})_x / \mathcal{O}_{X,x}$ is simple. Now, we may conclude by Lemma 3.2. \square

Corollary 3.5. *Let $X \subseteq \mathbb{P}_K^r$ be a variety of almost minimal degree with smooth projecting variety $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ of minimal degree. Let $x \in \text{Sing}(X)$ be a closed point and set $n = \dim(X)$. Assume that either $n = 1$ or $\text{depth}(X) = 2 \leq n$. Then*

- (a) $H_{X,x}(t) := \text{lenght}(\mathcal{O}_{X,x} / \mathfrak{m}_{X,x}^{t+1}) = 2 \binom{n+t}{n} - 1$ for all $t \in \mathbb{N}_0$.
- (b) $\dim(T_x X) = 2n$ and $m_x(X) = 2$.

Proof. By our hypothesis we have a finite birational linear projection morphism $\pi_p : \tilde{X} \rightarrow X$ such that the restriction $\pi_p | : \tilde{X} \setminus \Sigma_p(\tilde{X}) \rightarrow X \setminus \pi_p(\Sigma_p(\tilde{X}))$ is an isomorphism (s. Notation and Remark 2.1).

Assume first that $n = 1$. Then $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is a rational normal curve and $\text{Sing}(X) \neq \emptyset$ yields that π_p is not an isomorphism, whence X is arithmetically CM (s. Notion and Remark 2.1). So, by [B-S, Theorem 6.2] we have $h^1(X, \mathcal{O}_X) = 1$ and hence the Hilbert polynomial of X is given by $p_{\mathcal{O}_X}(t) = p_X(t) = (r + 1)t$, whereas for \tilde{X} we have $p_{(\pi_p)_* \mathcal{O}_{\tilde{X}}}(t) = p_{\tilde{X}}(t) = (r + 1)t + 1$. Therefore $p_{(\pi_p)_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X}(t) \equiv 1$, and as x is in the support of $(\pi_p)_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X$ it follows that $((\pi_p)_* \mathcal{O}_{\tilde{X}})_x / \mathcal{O}_{X,x}$ is a simple $\mathcal{O}_{X,x}$ -module. Now we conclude by Lemma 3.2.

So, let $\text{depth}(X) = 2 \leq n$. Observe that X is not arithmetically CM so that \tilde{X} cannot be the Veronese surface in \mathbb{P}^5 (s. [B-P, Remark 6.3]). Hence \tilde{X} must be a smooth rational normal scroll. So, according to the observations made in

Reminder 3.3 (A) the variety \tilde{X} satisfies $N_{2,2}$ and $\text{Sing}(\pi_p) = \{x\}$. Now, we may conclude by Proposition 3.4. \square

Lemma 3.6. *Let $X \subseteq \mathbb{P}_K^4$ be a surface of almost minimal degree, whose projecting variety $\tilde{X} \subseteq \mathbb{P}_K^5$ is the Veronese surface. Let $x \in \text{Sing}(X)$ be a closed point. Then*

- (a) $H_{X,x}(t) := \text{length}(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^{t+1}) = (t+1)^2$ for all $t \in \mathbb{N}_0$.
- (b) $\dim(T_x X) = 3$, $m_x(X) = 2$ and $\text{depth}(X) = 3$.

Proof. Observe that X is a non-normal del Pezzo surface whose vanishing ideal $I \subseteq K[X_0, \dots, X_4]$ is generated by two quadrics (s. [B-P, Remark 6.3]); in particular $\text{depth}(X) = 3$. After an appropriate linear coordinate transformation we may assume that x is the origin of an affine 4-space $\mathbb{A}_K^4 = \text{Spec}(K[\frac{X_1}{X_0} \dots \frac{X_4}{X_0}])$. So, with $y_i = \frac{X_i}{X_0}$ we may write $\mathcal{O}_{X,x} = (k[y_1, \dots, y_4]/(f_1, f_2))_{(y_1, \dots, y_4)}$ with two polynomials $f_1, f_2 \in k[y_1, \dots, y_4]$ of degree 2 vanishing at $x = (0, 0, 0, 0)$. We claim that one of the two polynomials f_i satisfies $\mu_x(f_i) = 1$. Otherwise f_1 and f_2 would be homogeneous and so infinitely many straight lines $L \subseteq X$ would run through x . If $L \subseteq X$ is a line not contained in $\pi_p(\Sigma_p(\tilde{X}))$, then $\pi_p^{-1}(L) = \langle L, p \rangle \cap \tilde{X} = \mathbb{P}_K^2 \cap \tilde{X} \not\subseteq \Sigma_p(\tilde{X})$ together with the fact that $\langle p, q \rangle \cap \tilde{X} = \{q\}$ for all $q \in \tilde{X} \setminus \Sigma_p(\tilde{X})$ shows that $\pi_p^{-1}(L)$ is a line. So there is a point $\tilde{x} \in \pi_p^{-1}(x)$ contained in infinitely many lines $\tilde{L} \subseteq \tilde{X}$, a contradiction. Therefore we may assume that $\mu_x(f_1) = 1$. As $x \in \text{Sing}(X)$ we then must have $\mu_x(f_2) = 2$ whence the completion of $\mathcal{O}_{X,x}$ is the formal local ring of the vertex of a quadratic cone in \mathbb{A}_K^3 . So, the Hilbert-Samuel function $H_{X,x}$ is of the requested type and our claim follows. \square

Proposition 3.7. *Let $X \subseteq \mathbb{P}_K^r$ be a variety of almost minimal degree of dimension n and arithmetic depth t with smooth projecting variety $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ of minimal degree. Let $x \in \text{Sing}(X)$. Then*

$$\dim(T_x X) = 2n + 2 - t \text{ and } m_x(X) = 2.$$

Proof. If $\tilde{X} \subseteq \mathbb{P}_K^5$ is the Veronese surface we conclude by Lemma 3.6. So, we may assume that $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is a smooth rational normal scroll of codimension ≥ 2 . We now proceed by induction on t . According to (2.1) we have $t \geq 2$. If $t = 2$, we conclude by Corollary 3.5.

So, let $t > 2$ and let $\pi_p : \tilde{X} \rightarrow X$ be as in Notation and Remark 2.1. Again we have $x \in Z := \pi_p(\Sigma_p(\tilde{X})) = \mathbb{P}_K^{t-2}$. Now, let $\mathbb{P}_K^{r-1} = H \subseteq \mathbb{P}_K^r$ be a generic hyperplane containing x . Then, by Proposition 2.5 the hyperplane section $W := X \cap H$ is irreducible and generically reduced. Observe that X is S_2 at all points $w \in W \setminus Z$ (cf (2.2)). Moreover, by the genericity of H we have $W \cap Z = H \cap Z = \mathbb{P}_K^{t-3}$. As $\text{depth}(X) = t > 2$ it follows $\text{depth}(\mathcal{O}_{X,w}) \geq t-1 \geq 2$

for all closed points $w \in W$. But now, Lemma 3.1 yields that $H \cap X = W$ is an integral scheme. In particular, we have

$$x \in \text{Sing}(H \cap X), \dim(X \cap H) = n - 1, \text{depth}(X \cap H) = t - 1$$

and the genericity of H also implies that $\dim(T_x(X \cap H)) = \dim(T_x X) - 1$ and $m_x(X \cap H) = m_x(X)$. Finally, in view of the isomorphism $\mathcal{H} \cong \tilde{\mathcal{H}}$ of Remark 2.3 (C) we may consider $\tilde{H} := \langle H, p \rangle$ as a generic hyperplane running through $\langle x, p \rangle = \Lambda$. So, by Remark 2.3 (B) the hyperplane section $\tilde{X} \cap \tilde{H} \subseteq \tilde{H} = \mathbb{P}_K^r$ is a smooth rational $(n - 1)$ -fold scroll. Restricting π_p we also have a projection $\tilde{\pi}_p : \tilde{X} \cap \tilde{H} \rightarrow \pi_p(\tilde{X} \cap \tilde{H}) = X \cap H$ of $\tilde{X} \cap \tilde{H} \subseteq \tilde{H}$ from $p \in \tilde{H}$. So, by induction $\dim_x(T_x(X \cap H)) = 2(n - 1) + 2 - (t - 1) = 2n + 1 - t$ and $m_x(X \cap H) = 2$. In view of the previous observations this proves our claim. \square

Notation and Remark 3.8. (A) Assume that our variety $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ of minimal degree is not necessarily smooth and let h denote the dimension of the vertex $\text{Vert}(\tilde{X})$ of \tilde{X} , so that $\text{Vert}(\tilde{X}) \cong \mathbb{P}_K^h$ with the convention that $\mathbb{P}_K^{-1} = \emptyset$. Thus \tilde{X} is a smooth rational normal scroll if and only if $h = -1$. If $h \geq 0$, choose a linear subspace $\Lambda = \mathbb{P}_K^{r-h} \subset \mathbb{P}_K^{r+1}$ which passes through p and satisfies $\Lambda \cap \text{Vert}(\tilde{X}) = \emptyset$. Then $\tilde{X}_0 := \tilde{X} \cap \Lambda \subset \mathbb{P}_K^{r-h}$ is an $(n - h - 1)$ -dimensional smooth rational normal scroll and $\tilde{X} = \text{Join}(\text{Vert}(\tilde{X}), \tilde{X}_0)$. As the finite birational morphism $\pi_p : \tilde{X} \rightarrow X$ is induced by a linear projection, X is a cone and the set of vertex points of X is given by $\text{Vert}(X) = \pi_p(\text{Vert}(\tilde{X}))$. Moreover $X_0 := \pi_p(\tilde{X}_0) \subseteq \langle \pi_p(\tilde{X}_0) \rangle = \mathbb{P}_K^{r-h-1}$ is a variety of almost minimal degree without vertex points and with projecting variety $\tilde{X}_0 \subseteq \mathbb{P}_K^{r-h}$ of minimal degree. In particular

$$X = \text{Join}(\text{Vert}(X), X_0).$$

(B) Now, let $\bullet_0 : \mathbb{P}_K^r \setminus \text{Vert}(X) \rightarrow \mathbb{P}_K^{n-h-1}$ be the canonical projection. Then, for each closed point $x \in X \setminus \text{Vert}(X)$ we have $x_0 \in X_0$ and moreover

$$T_x X = \langle \text{Vert}(X), T_{x_0} X_0 \rangle \text{ and } \mathcal{O}_{X,x} \cong \mathcal{O}_{X_0,x_0}[z_1, \dots, z_{h+1}]_{(\mathfrak{m}_{X_0,x_0}, z_1, \dots, z_{h+1})}$$

with indeterminates z_1, \dots, z_{h+1} . \bullet

Theorem 3.9. *Let $X \subseteq \mathbb{P}_K^r$ be a variety of almost minimal degree which is not a normal maximal del Pezzo variety. Let $x \in \text{Sing}(X) \setminus \text{Vert}(X)$ be a closed point. Then*

$$\dim(T_x X) = 2 \dim(X) + 2 - \text{depth}(X) \text{ and } m_x(X) = 2.$$

Proof. According to 2.4 and Notation and Remark 3.8 (A) we may consider X as a cone over a variety $X_0 \subseteq \mathbb{P}_K^{r-h-1}$ of almost minimal degree with vertex \mathbb{P}_K^h and smooth projecting variety $\tilde{X}_0 \subseteq \mathbb{P}_K^{r-h}$ of minimal degree. Now, according to Notation and Remark 3.8 (B) we have $\dim(T_{X_0,x_0}) = \dim(T_{X,x}) - h - 1$ and $m_{x_0}(X_0) = m_x(X)$. As $\dim(X_0) = \dim(X) - h - 1$ and $\text{depth}(X_0) = \text{depth}(X) - h - 1$ we get our claim by applying Proposition 3.7 to the closed point $x_0 \in X_0$. \square

4. INTERSECTION OF TANGENT SPACES OF A RATIONAL NORMAL SCROLL

We keep the notations and hypotheses of section 2.

Reminder 4.1. Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a smooth rational normal scroll of codimension ≥ 2 and let $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ be a closed point. Then according to [B-P, Theorem 3.2] (and in the notation introduced in [B-P, (4.1) - (4.6)]) we have the following six possibilities for the pairs $\Sigma_p(\tilde{X}) \subseteq \text{Sec}_p(\tilde{X})$:

$$(4.1) \quad p \in SL^0(\tilde{X}) : \text{Sec}_p(\tilde{X}) = \{p\} \text{ and } \Sigma_p(\tilde{X}) = \emptyset.$$

$$(4.2) \quad p \in SL^{q_1, q_2}(\tilde{X}) : \text{Sec}_p(\tilde{X}) = \mathbb{P}_K^1 \text{ and } \Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^1 \text{ consists of two simple points.}$$

$$(4.3) \quad p \in SL^{2q}(\tilde{X}) : \text{Sec}_p(\tilde{X}) = \mathbb{P}_K^1 \text{ and } \Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^1 \text{ consists of a double point.}$$

$$(4.4) \quad p \in SL^{L_1 \cup L_2}(\tilde{X}) : \text{Sec}_p(\tilde{X}) = \mathbb{P}_K^2 \text{ and } \Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^1 \text{ consists of two distinct lines, one of them being contained in a ruling } \mathbb{L}(x) \text{ for some } x \in \mathbb{P}_K^1, \text{ the other being a line section.}$$

$$(4.5) \quad p \in SL^C(\tilde{X}) : \text{Sec}_p(\tilde{X}) = \mathbb{P}_K^2 \text{ and } \Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^2 \text{ is a smooth conic curve.}$$

$$(4.6) \quad p \in SL^Q(\tilde{X}) : \text{Sec}_p(\tilde{X}) = \mathbb{P}_K^3 \text{ and } \Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^3 \text{ is a smooth quadric surface.}$$

Note also that the six strata $SL^*(\tilde{X}) \subseteq \mathbb{P}_K^{r+1} \setminus \tilde{X}$ (with $*$ running through the above six suffixes) are described in geometric terms by [B-P, Theorem 4.2]. •

Theorem 4.2. Let $\tilde{X} \subset \mathbb{P}^{r+1}$ be a smooth rational normal scroll of codimension ≥ 2 and dimension n . Let q_1, q_2 be two distinct closed points of \tilde{X} . Also let $x_i = \varphi(q_i)$ for $i = 1, 2$.

$$(a) \quad \text{If } x_1 = x_2, \text{ then } T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X} = \mathbb{L}(x_1).$$

$$(b) \quad \text{If } \langle q_1, q_2 \rangle \subset \tilde{X} \text{ and } x_1 \neq x_2, \text{ then } T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X} = \langle q_1, q_2 \rangle.$$

$$(c) \quad \text{If } \langle q_1, q_2 \rangle \not\subset \tilde{X}, \text{ then } \langle q_1, q_2 \rangle \cap \tilde{X} = \{q_1, q_2\} \text{ and for each closed point } p \text{ in } \langle q_1, q_2 \rangle \setminus \tilde{X}$$

$$\dim(T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X}) = \dim \Sigma_p(\tilde{X}) - 1.$$

Proof. For the sake of simplicity we denote $T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X}$ by Λ . As for statement (a) we refer the reader to Remark 7.4.(B) in [B-S]. Now, suppose that $x_1 \neq x_2$. Then $\mathbb{L}(x_1) \cap \mathbb{L}(x_2) = \emptyset$ and as $\mathbb{P}_K^{n-1} = \mathbb{L}(x_i) \subseteq T_{q_i} \tilde{X} = \mathbb{P}_K^n$ for $i = 1, 2$ we have $\dim \langle T_{q_1} \tilde{X}, T_{q_2} \tilde{X} \rangle \in \{2n - 1, 2n, 2n + 1\}$ whence $\dim(\Lambda) \in \{-1, 0, 1\}$. From this, statement (b) is obvious.

It remains to prove statement (c). As $\langle q_1, q_2 \rangle \not\subset \tilde{X}$ and as \tilde{X} is defined by quadrics, we have $\langle q_1, q_2 \rangle \cap \tilde{X} = \{q_1, q_2\}$. Moreover $q \in \langle q_1, q_2 \rangle \cap \Lambda \setminus \{q_1, q_2\}$ would imply the contradiction that $\langle q_1, q_2 \rangle$ is a tangent line at \tilde{X} in q_1 and q_2 . Therefore $\langle q_1, q_2 \rangle \cap \Lambda \setminus \{q_1, q_2\} = \emptyset$. Now, let $p \in \langle q_1, q_2 \rangle \setminus \{q_1, q_2\}$. Then $\{q_1, q_2\} \subseteq \Sigma_p(\tilde{X})$ and so p must belong to one of the four strata $SL^*(\tilde{X})$ listed

under (4.3) - (4.6).

Case (4.3). Suppose that $p \in SL^{q_1, q_2}(\tilde{X})$. Then $\Sigma_p(\tilde{X}) = \{q_1, q_2\}$. If there exists a point $q \in \Lambda$ then either $\langle q_1, q_2, q \rangle$ is a 4-secant 2-plane to \tilde{X} or else it meets \tilde{X} along a curve D . Obviously the first case does not occur as \tilde{X} satisfies N_2 (s. [E-G-H-P]). In the second case, D must be a line since otherwise $D \subset \Sigma_p(\tilde{X})$. Moreover D passes through exactly one of the points q_1 and q_2 , say through q_1 . So for any $r \in D \setminus \{q_1\}$, the line $\langle q_2, r \rangle$ is trisecant to \tilde{X} since it is tangential to \tilde{X} at q_2 . This is impossible since \tilde{X} is cut out by quadrics. Therefore $\Lambda = \emptyset$.

Case (4.4). Suppose that $p \in SL^C(\tilde{X})$ so that $C := \Sigma_p(\tilde{X})$ is a smooth plane conic. Obviously $T_{q_1}C$ and $T_{q_2}C$ meet at a point $q \in \langle C \rangle$ and so Λ is non-empty. Assume that Λ is a line and let $r \in \Lambda$ be a general point. If $\Lambda \subset \langle C \rangle$, then the line $\langle q_1, r \rangle$ is tri-secant to \tilde{X} , a contradiction. So Λ and $\langle C \rangle$ meet transversally. Note that either $\langle q_1, q_2, r \rangle$ is a 4-secant 2-plane to \tilde{X} or else meets \tilde{X} along a curve D . By the same argument as in the Case (4.3), we see that these two situations cannot occur. Therefore Λ is a single point.

Case (4.5). Suppose that $p \in SL^{L_1 \cup L_2}(\tilde{X})$ so that $\Sigma_p(\tilde{X})$ is the union of a line L which is contained in a ruling $\mathbb{L}(x) \subset \tilde{X}$ and a line section L' of \tilde{X} . We may assume that $q_1 \in L$ (and hence $x = x_1$) and $q_2 \in L'$. Let q be the intersection point of $\mathbb{L}(x_1)$ and L' . Obviously $q \in \Lambda$. Assume that Λ is a line. As Λ and $\mathbb{L}(x_2) = \mathbb{P}_K^{n-1}$ are contained in $T_{q_2}\tilde{X}$ clearly Λ and $\mathbb{L}(x_2)$ meet at a point q' . Note that $\langle q_1, q' \rangle$ is a line section of \tilde{X} since otherwise it would be a trisecant line to \tilde{X} . As $p \in \langle q_1, q_2 \rangle \subseteq \langle \langle q_1, q' \rangle, \langle q_2, q' \rangle \rangle$ and $\langle q_2, q' \rangle \subseteq \mathbb{L}(x_2) \subseteq \tilde{X}$ it follows that $\langle q_1, q' \rangle \subseteq \Sigma_p(\tilde{X})$. As $\langle q_1, q_2 \rangle \not\subseteq \tilde{X}$ we have $q_2 \notin \langle q_1, q' \rangle$, whence $\langle q_1, q' \rangle \neq L'$. So $\Sigma_p(\tilde{X})$ contains two line sections – a contradiction. Therefore Λ cannot be a line, whence $\Lambda = \{q\}$.

Case (4.6). Suppose that $p \in SL^Q(\tilde{X})$. So $Q := \Sigma_p(\tilde{X})$ is a smooth quadric surface. Obviously $T_{q_1}Q$ and $T_{q_2}Q$ meet along a line and so Λ contains a line. Since $\dim \Lambda \leq 1$, this proves that Λ is a line. \square

Notation and Remark 4.3. (A) Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a rational normal scroll of dimension n and codimension ≥ 2 which is not necessarily smooth. We write $\tilde{X} = \text{Join}(\text{Vert}(\tilde{X}), \tilde{X}_0)$ with a smooth rational normal scroll $\tilde{X}_0 \subseteq \mathbb{P}_K^{r-h} = \langle \tilde{X}_0 \rangle$, where $\mathbb{P}_K^{r-h} \subseteq \mathbb{P}_K^{r+1}$ is a linear subspace disjoint to the vertex $\mathbb{P}_K^h = \text{Vert}(\tilde{X})$ of \tilde{X} (s. Notation and Remark 3.8 (A)). If $\varphi_0 : \tilde{X}_0 \rightarrow \mathbb{P}_K^1$ is defined according to Notation and Remark 2.2 (C) we now write

$$\mathbb{L}(x) := \langle \text{Vert}(\tilde{X}), \varphi_0^{-1}(x) \rangle$$

for all closed points $x \in \mathbb{P}_K^1$. So, $\mathbb{L}(x) = \mathbb{P}_K^{n-1}$ and $\tilde{X} = \bigcup_{x \in \mathbb{P}_K^1} \mathbb{L}(x)$ with $\mathbb{L}(x_1) \cap \mathbb{L}(x_2) = \text{Vert}(\tilde{X})$ for all $x_1, x_2 \in \mathbb{P}_K^1$ with $x_1 \neq x_2$.

(B) Let the notations and hypotheses be as in part (A) and let

$$\bullet_0 : \mathbb{P}_K^{r+1} \setminus \text{Vert}(\tilde{X}) \rightarrow \langle \tilde{X}_0 \rangle = \mathbb{P}_K^{r-h}$$

be the natural projection map. Observe that $\tilde{X}_0 = (\tilde{X} \setminus \text{Vert}(\tilde{X}))_0$ and that

$$(4.7) \quad T_q \tilde{X} = \langle \text{Vert}(\tilde{X}), T_{q_0} \tilde{X}_0 \rangle \text{ for all closed points } q \in \tilde{X} \setminus \text{Vert}(\tilde{X}).$$

Moreover (s. [B-P, Remark 5.4]):

$$(4.8) \quad \text{Sec}_p(\tilde{X}) = \text{Join}(\text{Vert}(\tilde{X}), \text{Sec}_{p_0}(\tilde{X}_0)),$$

$$(4.9) \quad \Sigma_p(\tilde{X}) = \text{Join}(\text{Vert}(\tilde{X}), \Sigma_{p_0}(\tilde{X}_0)) \text{ if } p_0 \notin SL^{2q}(\tilde{X}_0),$$

and

$$(4.10) \quad \Sigma_p(\tilde{X}) = 2\langle \text{Vert}(\tilde{X}), p_0 \rangle \subseteq \langle \text{Vert}(\tilde{X}), p_0 \rangle \text{ if } p_0 \in SL^{2q}(\tilde{X}_0).$$

•

Corollary 4.4. *Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a rational normal scroll of codimension ≥ 2 and dimension n . Let $q_1, q_2 \in \tilde{X}$ be two distinct closed points such that $\langle q_1, q_2 \rangle \cap \text{Vert}(\tilde{X}) = \emptyset$. Let $x_1, x_2 \in \mathbb{P}_K^1$ be such that $q_i \in \mathbb{L}(x_i)$ for $i = 1, 2$.*

(1) *Suppose that $\langle q_1, q_2 \rangle \subseteq \tilde{X}$.*

(a) *If $x_1 = x_2$, then $T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X} = \mathbb{L}(x_1)$.*

(b) *If $x_1 \neq x_2$, then $T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X} = \langle \text{Vert}(\tilde{X}), q_1, q_2 \rangle$.*

(2) *Suppose that $\langle q_1, q_2 \rangle \not\subseteq \tilde{X}$. Then*

$$\langle \text{Vert}(\tilde{X}), q_1, q_2 \rangle \cap \tilde{X} = \langle \text{Vert}(\tilde{X}), q_1 \rangle \cup \langle \text{Vert}(\tilde{X}), q_2 \rangle$$

and for each closed point $p \in \langle \text{Vert}(\tilde{X}), q_1, q_2 \rangle \setminus \tilde{X}$ we have

$$\dim(T_{q_1} \tilde{X} \cap T_{q_2} \tilde{X}) = \dim \Sigma_p(\tilde{X}) - 1.$$

Proof. Observe (4.7) and (4.9) and apply Theorem 4.2 to the smooth rational normal scroll \tilde{X}_0 and the points $(q_1)_0$ and $(q_2)_0$ of \tilde{X}_0 . \square

Theorem 4.5. *Let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a rational normal scroll of codimension ≥ 2 and let $X = \pi_p(\tilde{X}) \subseteq \mathbb{P}_K^r$, where $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$. If x is a closed singular point of X with $x \notin \text{Vert}(X)$ such that $\pi_p^{-1}(x)$ consist of two distinct points $x_1, x_2 \in \tilde{X}$, then*

$$T_x X = \langle \pi_p(T_{x_1} \tilde{X}), \pi_p(T_{x_2} \tilde{X}) \rangle.$$

Proof. Let $n = \dim(\tilde{X})$ and $t = \text{depth}(X)$. By Theorem 3.9 we have $\dim(T_x X) = 2n + 2 - t$. As $p \in \langle x_1, x_2 \rangle$ and $x \notin \text{Vert}(X)$ we have $\langle x_1, x_2 \rangle \cap \text{Vert}(\tilde{X}) = \emptyset$. So Corollary 4.4 (2) implies that $\dim(T_{x_1} \tilde{X} \cap T_{x_2} \tilde{X}) = t - 3$ and hence

$$\dim \langle T_{x_1} \tilde{X}, T_{x_2} \tilde{X} \rangle = 2n + 3 - t.$$

As $\pi_p^{-1}(x)$ consists of the two distinct points x_1 and x_2 we have $p \notin T_{x_1}\tilde{X} \cup T_{x_2}\tilde{X}$ and this implies that $\pi_p(T_{x_i}\tilde{X})$ are defined for $i = 1, 2$ and

$$\pi_p \left(\langle T_{x_1}\tilde{X}, T_{x_2}\tilde{X} \rangle \setminus \{p\} \right) \subseteq \langle \pi_p(T_{x_1}\tilde{X}), \pi_p(T_{x_2}\tilde{X}) \rangle.$$

The left-hand side space now has dimension $2n + 3 - t - 1 = \dim(T_x X)$. As $\pi_p(T_{x_i}\tilde{X}) \subseteq T_x X$ for $i = 1, 2$ we get our claim. \square

5. SINGULAR EMBEDDING SCROLLS

Notation and Reminder 5.1. (A) Throughout this section let $X \subseteq \mathbb{P}_K^r$ be a variety of almost minimal degree, of dimension n and codimension $e \geq 2$. Assume that X admits a projecting rational normal scroll $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ and let $\pi_p : \tilde{X} \rightarrow X = \pi_p(\tilde{X})$ be the corresponding projection morphism.

(B) An *embedding scroll* of X is a rational normal scroll $Y \subseteq \mathbb{P}_K^r$ such that $X \subseteq Y$ and $\dim(Y) = n + 1$. According to [B-S, Theorem 7.3] such embedding scrolls of X exist. \bullet

In this section we mainly focus on singular embedding scrolls.

Lemma 5.2. *Let x be a closed point in $X \setminus \text{Vert}(X)$. Then*

$$Y_x := \text{Join}(x, X) \subset \mathbb{P}_K^r$$

is a non-degenerate irreducible subvariety of dimension $n + 1$. Furthermore,

- (a) *If x is a smooth point of X , then $\deg(Y_x) = e + 1$.*
- (b) *If x is a singular point of X , then $\deg(Y_x) = e$ and hence Y_x is a variety of minimal degree.*

Proof. It is clear that $Y_x = \overline{\bigcup_{x' \in X \setminus \{x\}} \langle x, x' \rangle}$ is a non-degenerate irreducible variety of dimension $n + 1$ in \mathbb{P}_K^r . Therefore $\deg(Y_x) \geq e$ and the inner projection of X from x to a generic hyperplane section of Y_x is birational. So $\deg(Y_x) = e + 2 - m_x(X)$ where $m_x(X)$ denotes the multiplicity of X at x (cf. [Ha, Chapter 20]). Now, we conclude by Theorem 3.9. \square

Lemma 5.3. *Let Y be an embedding scroll of X . Then*

$$\text{Vert}(X) \subseteq \text{Vert}(Y) \subseteq \text{Sing}(X).$$

Proof. For any $x \in \text{Vert}(X)$, we have $\mathbb{P}_K^r = T_x X \subset T_x Y$. Therefore x is a singular point of Y . Since $\text{Sing}(Y) = \text{Vert}(Y)$, it follows that $\text{Vert}(X) \subseteq \text{Vert}(Y)$.

Let $y \in \text{Vert}(Y) \setminus \text{Vert}(X)$. As $Y \supseteq \text{Join}(y, X) \not\supseteq X$ and X is a codimension one subvariety of Y we have $Y = \text{Join}(y, X)$. If $y \notin X$, then Lemma 2.4 guarantees that the linear projection of X from y is a birational morphism from X onto a general hyperplane section of Y . Thus $\deg(Y) = \deg(X) = e + 2$, which is impossible since $\deg(Y) = e$. Therefore $y \in X$. But now Lemma 5.2 says that y must be a singular point of X . \square

Lemma 5.4. *Let X_1 be a general hyperplane section of X . If Y is an embedding scroll of X , then $Y_1 := Y \cap \langle X_1 \rangle$ is an embedding scroll of X_1 and*

$$Y = \text{Join}(\text{Vert}(X), Y_1).$$

Conversely, if Y_1 is an embedding scroll of X_1 then $Y := \text{Join}(\text{Vert}(X), Y_1)$ is an embedding scroll of X .

Proof. Suppose that Y is an embedding scroll of X . Since $\langle X_1 \rangle$ is a general hyperplane, Y_1 is an embedding scroll of X_1 . By Lemma 5.3, we have $\text{Vert}(X) \subseteq \text{Vert}(Y)$ and so $Y_1 := Y \cap \langle X_1 \rangle$ is an embedding scroll of X_1 . Since Y contains $\text{Join}(\text{Vert}(X), Y_1)$ and has the same dimension as this latter, it follows that $Y = \text{Join}(\text{Vert}(X), Y_1)$. The second part of our statement is obvious. \square

Lemma 5.4 says that the problem of classifying all embedding scrolls of X is reduced to that classifying the embedding scrolls of X_1 . So we concentrate to the case where X is not a cone, or equivalently, where the projecting rational normal scroll is smooth.

We first consider the cases in which $X \subseteq \mathbb{P}_K^r$ is not arithmetically CM.

Theorem 5.5. *Let $X \subseteq \mathbb{P}_K^{r+1}$ be as in Notation and Reminder 5.1. Suppose that X is not a cone. Let Y be an embedding scroll of X .*

- (a) *If $\text{depth}(X) = 1$ (and hence X is smooth), then Y is smooth.*
- (b) *If $2 \leq \text{depth}(X) \leq n$ (and hence X is singular but not arithmetically Cohen-Macaulay), then $\text{Vert}(Y) = \text{Sing}(X)$ and*

$$Y = \text{Join}(\text{Sing}(X), X).$$

In particular, X has a unique embedding scroll, which always is singular.

Proof. (a) : Since X is smooth, Lemma 5.3 implies that $\text{Vert}(Y) = \emptyset$. Therefore Y is a smooth rational normal scroll.

(b) : Let x be a singular point of X . Note that $\text{Sing}(X) = \text{NCM}(X)$ since $\text{depth}(X) \leq n$. Therefore the local ring $\mathcal{O}_{X,x}$ is not Cohen-Macaulay. In particular, $\dim_K T_x X \geq n+2$. Since $T_x X$ is a subspace of $T_x Y$, it follows that x is a singular point of Y and hence $x \in \text{Vert}(Y)$. By combining this fact with Lemma 5.3, we conclude that $\text{Vert}(Y) = \text{Sing}(X)$. Thus we have

$$Y = \text{Join}(\text{Vert}(Y), Y) \supseteq \text{Join}(\text{Sing}(X), X) \supseteq \text{Join}(x, X)$$

where Y and $\text{Join}(x, X)$ have the same dimension $n+1$. This shows that $Y = \text{Join}(\text{Sing}(X), X)$. \square

Remark 5.6. Theorem 5.5.(a) says that if $\text{depth}(X) = 1$ (and hence X is smooth), then any embedding scroll Y of X is smooth. This was first proved in [P1, Theorem1.2] by using [N, Theorem 5.10]. Our present proof is direct and elementary. \bullet

We now consider embedding scrolls Y of X in the case where X is arithmetically CM. We begin with the case in which Y is singular. As previously we restrict ourselves to non-conic varieties X . Moreover, we shall only consider the case in which $e := \text{codim}(X) > 2$. If $\text{codim}(X) = 2$, the embedding scrolls of X are precisely the quadrics of rank 3 or 4 which contain X , and so they are covered by the investigation [L-P-S].

Reminder 5.7. Let $X \subseteq \mathbb{P}_K^r$ be as in Notation and Reminder 5.1. If $\text{depth}(X) = n + 1$, then X is maximally del Pezzo and non-normal. Assume in addition that X is not a cone and that $e := \text{codim}(X) \geq 3$ so that $\text{deg}(X) = \text{deg}(\tilde{X}) \geq 5$. Then, the possible pairs (\tilde{X}, p) with $\text{deg}(\tilde{X}) \geq 5$ are completely classified in [B-P, Theorem 6.2] as follows:

- (5.1) $\tilde{X} = S(a)$ for some integer $a > 4$ and $p \in \text{Sec}(\tilde{X}) \setminus \tilde{X}$.
- (5.2) (i) $\tilde{X} = S(1, b)$ for some integer $b > 3$ and $p \in \text{Join}(S(1), \tilde{X}) \setminus \tilde{X}$;
(ii) $\tilde{X} = S(2, b)$ for some integer $b > 2$ and $p \in \langle S(2) \rangle \setminus \tilde{X}$;
- (5.3) $\tilde{X} = S(1, 1, c)$ for some integer $c > 2$ and $p \in \langle S(1, 1) \rangle \setminus \tilde{X}$. •

Theorem 5.8. Let $X \subseteq \mathbb{P}_K^r$ be as in Notation and Reminder 5.1. Suppose that X is not a cone. If X is arithmetically Cohen-Macaulay and Y is a singular embedding scroll of X , then $\text{Vert}(Y) = \text{Sing}(X)$ and

$$Y = \text{Join}(\text{Sing}(X), X).$$

In particular, X has a unique singular embedding scroll.

Proof. For each closed point $x \in \text{Vert}(Y)$, the embedding scroll Y contains $\text{Join}(x, X)$ and both are of dimension $n + 1$. Therefore $Y = \text{Join}(x, X)$ and hence $Y \subset \text{Join}(\text{Sing}(X), X)$. So we need to verify that $\text{Join}(\text{Sing}(X), X)$ has dimension $n + 1$. To this end, we use the classification given in Reminder 5.7.

Case (5.1). Suppose that $\tilde{X} = S(a)$ for some integer $a > 3$ and that $p \in \text{Sec}(\tilde{X}) \setminus \tilde{X}$. Since $\text{Vert}(Y)$ is non-empty and $\text{Sing}(X) := \{x\}$ is a point, Lemma 5.3 says that $\text{Vert}(Y) = \text{Sing}(X) = \{x\}$. Therefore we have $Y = \text{Join}(\text{Sing}(X), X)$.

Case (5.2) (i). Suppose that $\tilde{X} = S(1, b)$ for some integer $b > 3$ and that $p \in \text{Join}(S(1), \tilde{X}) \setminus \tilde{X}$. Thus (cf Notation and Reminder 2.2 (A))

$$\tilde{X} = \bigcup_{\lambda \in \mathbb{P}^1} \langle \sigma_1(\lambda), \sigma_b(\lambda) \rangle$$

and $\Sigma_p(\tilde{X}) = S(1) \cup \mathbb{L}(\mu)$ for some $\mu \in \mathbb{P}^1$. In particular $S(1) \cap \mathbb{L}(\mu) = \{\sigma_1(\mu)\}$. So, for each $\lambda \neq \mu$, the line $\mathbb{L}(\lambda) = \langle \sigma_1(\lambda), \sigma_b(\lambda) \rangle$ maps to $\langle \pi_p(\sigma_1(\lambda)), \pi_p(\sigma_b(\lambda)) \rangle$ where $\pi_p(\sigma_1(\lambda))$ is contained in the line $\text{Sing}(X)$. Therefore

$$X = \overline{\bigcup_{\lambda \in \mathbb{P}^1 \setminus \{\mu\}} \langle \pi_p(\sigma_1(\lambda)), \pi_p(\sigma_b(\lambda)) \rangle}.$$

Let v be a general closed point in $\text{Join}(\text{Sing}(X), X)$. Then there exist closed points $\lambda \in \mathbb{P}^1 \setminus \{\mu\}$ and $z \in \langle \pi_p(\sigma_1(\lambda)), \pi_p(\sigma_b(\lambda)) \rangle$ such that $v \in \langle \text{Sing}(X), z \rangle$. Since $\pi_p(\sigma_1(\lambda))$ is contained in $\text{Sing}(X)$, it holds that

$$\langle \text{Sing}(X), z \rangle = \langle \text{Sing}(X), \pi_p(\sigma_b(\lambda)) \rangle.$$

In particular, v is contained in $\text{Join}(\text{Sing}(X), S(b))$ and hence

$$\text{Join}(\text{Sing}(X), X) = \text{Join}(\text{Sing}(X), S(b)).$$

This shows that $\text{Join}(\text{Sing}(X), X)$ has dimension 3.

Case (5.2) (ii). Suppose that $\tilde{X} = S(2, b)$ for some integer $b > 2$ and $p \in \langle S(2) \rangle \setminus \tilde{X}$. Now,

$$\tilde{X} = \bigcup_{\lambda \in \mathbb{P}^1} \langle \sigma_2(\lambda), \sigma_b(\lambda) \rangle$$

and $\Sigma_p(\tilde{X}) = S(2)$. Moreover,

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \langle \pi_p(\sigma_2(\lambda)), \pi_p(\sigma_b(\lambda)) \rangle.$$

Let v be a general closed point in $\text{Join}(\text{Sing}(X), X)$. Then there exist closed points $\lambda \in \mathbb{P}^1$ and $z \in \langle \pi_p(\sigma_2(\lambda)), \pi_p(\sigma_b(\lambda)) \rangle$ such that $v \in \langle \text{Sing}(X), z \rangle$. Since $\pi_p(\sigma_2(\lambda))$ is contained in $\text{Sing}(X)$, we have $\langle \text{Sing}(X), z \rangle = \langle \text{Sing}(X), \pi_p(\sigma_b(\lambda)) \rangle$. In particular, v is contained in $\text{Join}(\text{Sing}(X), S(b))$ and hence

$$\text{Join}(\text{Sing}(X), X) = \text{Join}(\text{Sing}(X), S(b)).$$

This proves that $\text{Join}(\text{Sing}(X), X)$ has dimension 3.

Case (5.3). Suppose that $\tilde{X} = S(1, 1, c)$ for some integer $c > 2$ and $p \in \langle S(1, 1) \rangle \setminus \tilde{X}$. Then for the canonical ruling $(L_\lambda := \langle \sigma_1(\lambda), \sigma_1(\lambda) \rangle \mid \lambda \in \mathbb{P}^1)$ of $S(1, 1)$, we have

$$\tilde{X} = \bigcup_{\lambda \in \mathbb{P}^1} \langle L_\lambda, \pi_p(\sigma_c(\lambda)) \rangle$$

and $\Sigma_p(\tilde{X}) = S(1, 1)$. Moreover,

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \langle \pi_p(L_\lambda), \pi_p(\sigma_c(\lambda)) \rangle.$$

Let v be a general closed point in $\text{Join}(\text{Sing}(X), X)$. Then there exist closed points $\lambda \in \mathbb{P}^1$ and $z \in \langle \pi_p(L_\lambda), \pi_p(\sigma_c(\lambda)) \rangle$ such that $v \in \langle \text{Sing}(X), z \rangle$. Since $\pi_p(L_\lambda)$ is a subset of $\text{Sing}(X)$, it holds that $\langle \text{Sing}(X), z \rangle = \langle \text{Sing}(X), \pi_p(\sigma_c(\lambda)) \rangle$. In particular, v is contained in $\text{Join}(\text{Sing}(X), S(c))$ and hence

$$\text{Join}(\text{Sing}(X), X) = \text{Join}(\text{Sing}(X), S(c)).$$

This shows that $\text{Join}(\text{Sing}(X), X)$ has dimension 4. □

6. SMOOTH EMBEDDING SCROLLS

Finally, we consider the case where X is arithmetically CM and the embedding scroll Y of X is smooth.

Notation and Remark 6.1. (A) Let $X \subseteq \mathbb{P}_K^r$ be as in Notation and Remark 5.1 and assume, that X is arithmetically CM of degree ≥ 5 but not a cone. So, we are in of the three cases (5.1), (5.2) or (5.3) listed in Remark 5.7. Keep in mind, that then X is even arithmetically Gorenstein (see [B-S, Theorem 6.2]). So the Betti diagram of X tells us that the homogeneous vanishing ideal of X is generated by quadrics.

(B) Let $Y \subseteq \mathbb{P}_K^r$ be a smooth embedding scroll of X . Let H be a hyperplane divisor of Y and let F be a fibre divisor of the natural projection map $\tau : Y \rightarrow \mathbb{P}_K^1$. Keep in mind that the divisors of Y are all linearly equivalent to divisors of the form $aH + bF$ with $a, b \in \mathbb{Z}$. •

Lemma 6.2. X is linearly equivalent to $2H + (e - 3)F$ as a divisor of Y .

Proof. Let $aH + bF, a, b \in \mathbb{Z}$, be the divisor class of X in Y . Obviously $a \geq 1$. Suppose that $a = 1$. Then X is smooth since it is irreducible. So $a \geq 2$ since X is singular. For general $\eta \in \mathbb{P}^1$ the intersection $X \cap \tau^{-1}(\eta) \subset \tau^{-1}(\eta) = \mathbb{P}_K^n$ is a hypersurface of degree a . In particular, a general line in $\tau^{-1}(\eta)$ is a -secant to X . Since the homogeneous vanishing ideal of X is generated by quadrics, we conclude that $a = 2$. Finally we obtain the value of b from the equality $\deg(X) = 2 \deg(Y) + b$. □

Notation and Remark 6.3. Let \mathfrak{h} be a hyperplane divisor of the smooth projecting rational normal scroll $\tilde{X} \subset \mathbb{P}_K^{r+1}$ of X and let \mathfrak{f} be a fibre of the projection map $\varphi : \tilde{X} \rightarrow \mathbb{P}_K^1$. Consider the surjective morphism $f = \tau \circ \pi_p : \tilde{X} \rightarrow \mathbb{P}_K^1$ where $\tau : Y \rightarrow \mathbb{P}_K^1$ is defined as in Notation and Remark 6.1.(B), and the line bundle $\mathcal{L} := f^* \mathcal{O}_{\mathbb{P}_K^1}(1)$ on \tilde{X} . •

Lemma 6.4. $\mathcal{L} = \mathcal{O}_{\tilde{X}}(2\mathfrak{f})$.

Proof. We may write $\mathcal{L} = \mathcal{O}_{\tilde{X}}(a\mathfrak{h} + b\mathfrak{f})$ with appropriate integers $a, b \in \mathbb{Z}$. If $n = 1$, the isomorphism $\varphi : \tilde{X} \rightarrow \mathbb{P}_K^1$ yields that $\mathfrak{h} \sim \deg(X)\mathfrak{f}$ so that $\mathcal{L} = \mathcal{O}_{\tilde{X}}((a \deg(X) + b)\mathfrak{f})$. Assume now that $n > 1$. As \mathcal{L} defines a map $X \rightarrow \mathbb{P}_K^1$ it follows that $\deg(\mathcal{L}) = 0$. As $\mathfrak{h}^{n-i} \cdot \mathfrak{f}^i$ is equal to $\deg(\tilde{X})$ for $i = 0, 1$, and 0 otherwise, we get $0 = a^{n-1}(a \deg(\tilde{X}) + nb)$. Moreover by Lemma 6.2 the fibre $f^{-1}(x) \subset \tilde{X}$ is mapped by $\pi_p : \tilde{X} \rightarrow X \subset Y$ to a quadratic hypersurface in $\tau^{-1}(x) = \mathbb{P}_K^n$ for a generic point $x \in \mathbb{P}_K^1$. Therefore

$$2 = \mathcal{L} \cdot \mathfrak{f}^{n-1} = a \deg(\tilde{X}) + b.$$

As $\deg(\tilde{X}) \geq 5$ (s. Notation and Remark 6.1) we now get our claim. □

Notation and Remark 6.5. (A) Let X and Y be algebraic varieties. If $f \in \text{Mor}(X, Y)$ we write $[f] := \text{Aut}(Y) \circ f$ for the orbit of f under the natural action of the automorphism group of Y . Correspondingly, for $F \subset \text{Mor}(X, Y)$

we set $[F] := \{[f] \mid f \in F\}$.

(B) For two irreducible algebraic curves \mathcal{C} and \mathcal{C}' and a positive integer d , let $\text{Mor}_d(\mathcal{C}, \mathcal{C}')$ denote the set of all morphism $f : \mathcal{C} \rightarrow \mathcal{C}'$ of degree d and keep in mind that $[f] \subset \text{Mor}_d(\mathcal{C}, \mathcal{C}')$ for all $f \in \text{Mor}_d(\mathcal{C}, \mathcal{C}')$.

(C) Let $\nu : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^2$, $((s : t) \mapsto (s^2 : st : t^2))$ be the Veronese embedding. Then, for each $f \in \text{Mor}_d(\mathbb{P}_K^1, \mathbb{P}_K^1)$ there is a unique linear projection $\pi_f : \mathbb{P}_K^2 \setminus \{q_f\} \rightarrow \mathbb{P}_K^1$ from a point $q_f \in \mathbb{P}_K^2 \setminus \text{Im}(\nu)$ such that $f = \pi_f \circ \nu$. In particular, we have a natural bijection

$$i : [\text{Mor}_2(\mathbb{P}_K^1, \mathbb{P}_K^1)] \rightarrow \mathbb{P}_K^2 \setminus \text{Im}(\nu) \text{ given by } [f] \mapsto q_f.$$

•

Notation and Remark 6.6. : (A) Let $\mathcal{C} \subset \mathbb{P}_K^r$ be a nondegenerate irreducible projective curve, let $f \in \text{Mor}_2(\mathcal{C}, \mathbb{P}_K^1)$ and let $U \subset \mathbb{P}_K^1 \setminus f(\text{Sing}(\mathcal{C}))$ be a nonempty open set. We write

$$W := \overline{\bigcup_{t \in U} \langle f^{-1}(t) \rangle},$$

where $\langle f^{-1}(t) \rangle$ denotes the tangent line to \mathcal{C} in $f^{-1}(t)$, provided this fibre is a double point.

(B) Keep the notations of part (A). It is easy to see that $\mathcal{C} \subset W$, and that W does not depend on the chosen nonempty open set $U \subset \mathbb{P}_K^1 \setminus f(\text{Sing}(\mathcal{C}))$. So we may write $W =: W_f$. But now it is straight forward to check that for $f, g \in \text{Mor}_2(\mathcal{C}, \mathbb{P}_K^1)$ we have

$$[f] = [g] \iff W_f = W_g.$$

(C) We write $\mathfrak{S}(\mathcal{C})$ for the set of all smooth rational surface scrolls $S \subset \mathbb{P}_K^r$ with $\mathcal{C} \subset S$.

•

Lemma 6.7. *Let $r \geq 3$ and let $\mathcal{C} \subset \mathbb{P}_K^r$ be either*

- (1) *a rational normal curve or*
- (2) *a singular curve of almost minimal degree with $r \geq 4$.*

Let $f \in \text{Mor}_2(\mathcal{C}, \mathbb{P}_K^1)$. Then $W_f \in \mathfrak{S}(\mathcal{C})$.

Proof. Consider on \mathcal{C} the line bundles $\mathcal{L} := \mathcal{O}_{\mathcal{C}}(1)$, $\mathcal{L}_1 := f^* \mathcal{O}_{\mathbb{P}_K^1}(1)$ and $\mathcal{L}_2 := \mathcal{L} \otimes \mathcal{L}_1^{-1}$ so that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$. Our first aim is to show that $h^0(\mathcal{C}, \mathcal{L}_2) = r - 1$. To do so, assume that $\mathcal{C} \subset \mathbb{P}_K^r$ is a rational normal curve. Then \mathcal{L}_1 and \mathcal{L}_2 are line bundles on \mathcal{C} of degree 2 and $r - 2$, respectively. Therefore $h^0(\mathcal{C}, \mathcal{L}_2) = h^0(\mathbb{P}_K^1, \mathcal{O}_{\mathbb{P}_K^1}(r - 2)) = r - 1$. Next assume that $\mathcal{C} \subset \mathbb{P}_K^r$ is a singular curve of almost minimal degree. Then \mathcal{L}_1 and \mathcal{L}_2 are line bundles on \mathcal{C} of degree 2 and $r - 1 > 1$, respectively. In particular, $h^1(\mathcal{C}, \mathcal{L}_2) = 0$ since $\rho_a(\mathcal{C}) = 1$. Hence by Riemann-Roch we have $h^0(\mathcal{C}, \mathcal{L}_2) = r - 1$.

Now fix a basis b_1, \dots, b_{r-1} of the K -space $H^0(\mathcal{C}, \mathcal{L}_2)$ and fix a basis a_1, a_2 of $V := f^*H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \subset H^0(\mathcal{C}, \mathcal{L}_1)$. Then the bilinear map

$$V \times H^0(\mathcal{C}, \mathcal{L}_2) \rightarrow H^0(\mathcal{C}, \mathcal{L}) \cong H^0(\mathbb{P}_K^r, \mathcal{O}_{\mathbb{P}_K^r}(1))$$

induces the 1-generic $2 \times (r-1)$ matrix $[a_i b_j \mid 1 \leq i \leq 2, 1 \leq j \leq r-1]$ of linear forms on \mathbb{P}_K^r whose 2×2 minors define a rational normal surface scroll $S \subset \mathbb{P}_K^r$ which contains \mathcal{C} . Moreover the general ruling of S is spanned by a fiber of f and hence $S = W_f$. Also Castelnuovo's base point pencil trick enables us to show that the above bilinear map is surjective. This implies that S is smooth since its vertex is defined as the common zero set of the entries of the previous matrix. Consequently, $S = W_f \in \mathfrak{S}(\mathcal{C})$. \square

Lemma 6.8. *Let $r \geq 4$ and let $X \subset \mathbb{P}_K^r$ be a non-normal curve of almost minimal degree. Then, we have a bijection*

$$\varepsilon_X : [\text{Mor}_2(X, \mathbb{P}_K^1)] \rightarrow \mathfrak{S}(X); ([f] \mapsto W_f).$$

Proof. According to Lemma 6.7 and Notation and Remark 6.6 the map ε_X is indeed defined and injective. So, let $S \in \mathfrak{S}(X)$ with projection map $\varphi : S \rightarrow \mathbb{P}_K^1$. We consider the restricted morphism $f := \varphi|_X : X \rightarrow \mathbb{P}_K^1$, which must be of degree > 1 , as X is singular. On the other hand X satisfies the condition N_2 and thus admits no 4-secant plane (s. [E-G-H-P]) and hence no 3-secant line. Therefore $\deg(f) < 3$, whence $f \in \text{Mor}_2(X, \mathbb{P}_K^1)$. Now, clearly $S = W_f$. \square

Notation and Remark 6.9. Let $r \geq 4$, let $X \subset \mathbb{P}_K^r$ be a singular curve of almost minimal degree with canonical projection $\pi_p : \tilde{X} = S(r+1) \rightarrow X$ and let $\sigma : \mathbb{P}_K^1 \rightarrow \tilde{X}$ be the Veronese map. Then, the assignment $[f] \mapsto [f \circ \pi_p \circ \sigma]$ defines a bijection

$$\kappa_p : [\text{Mor}_2(X, \mathbb{P}_K^1)] \rightarrow S_p := [\{f \in \text{Mor}_2(\mathbb{P}_K^1, \mathbb{P}_K^1) \mid \#\sigma^{-1}(\Sigma_p(\tilde{X})) = 1\}].$$

Moreover, by Notation and Remark 6.5, the assignment $[f] \mapsto q_{\tilde{f}}$ defines a bijection

$$i_p : S_p \rightarrow U_p := \langle \sigma^{-1}(\Sigma_p(\tilde{X})) \rangle \setminus \sigma^{-1}(\Sigma_p(\tilde{X})).$$

Choosing an isomorphism

$$\lambda_p : U_p \rightarrow \begin{cases} \mathbb{A}^1 \setminus \{0\}, & \text{if } \#\Sigma_p(\tilde{X}) = 2 \\ \mathbb{A}^1, & \text{if } \#\Sigma_p(\tilde{X}) = 1 \end{cases}$$

we finally get the bijection

$$\delta_p = \lambda_p \circ i_p \circ \kappa_p : [\text{Mor}_2(X, \mathbb{P}_K^1)] \rightarrow \begin{cases} \mathbb{A}^1 \setminus \{0\}, & \text{if } \#\Sigma_p(\tilde{X}) = 2 \\ \mathbb{A}^1, & \text{if } \#\Sigma_p(\tilde{X}) = 1 \end{cases}$$

•

Now, we are ready to formulate and to prove the conclusive result of this section.

Theorem 6.10. *Let $X \subset \mathbb{P}_K^r$, $\tilde{X} \subset \mathbb{P}_K^{r+1}$ and $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ be as in Reminder 5.7.*

- (a) *In the case (5.1), that is if $\tilde{X} = S(a)$ for some integer $a > 5$ and $p \in \text{Sec}(\tilde{X})$, then X admits a one dimensional family of smooth embedding scrolls. More precisely, we have the bijection*

$$\delta_p \circ \varepsilon_X^{-1} : \mathfrak{S}(X) \rightarrow \begin{cases} \mathbb{A}^1 \setminus \{0\}, & \text{if } p \in \text{Sec}(\tilde{X}) \setminus \text{Tan}(\tilde{X}) \\ \mathbb{A}^1, & \text{if } p \in \text{Tan}(\tilde{X}) \setminus \tilde{X} \end{cases}.$$

- (b) *In the case (5.2)(ii), that is if $\tilde{X} = S(2, b)$ for some $b > 2$ and $p \in \langle S(2) \rangle$, then X admits a unique smooth embedding scroll Y . More precisely $Y = S(1, \alpha, \beta)$ with $S(1) = \pi_p(\langle S(2) \rangle)$, $S(\alpha, \beta) \in \mathfrak{S}(\pi_p(S(b)))$ and*

$$(\alpha, \beta) = \begin{cases} \left(\frac{b-1}{2}, \frac{b-1}{2}\right) & \text{if } b \text{ is odd} \\ \left(\frac{b-2}{2}, \frac{b}{2}\right) & \text{if } b \text{ is even} \end{cases}.$$

- (c) *In the remaining cases (5.2)(i) and (5.3), X has no smooth embedding scroll.*

Proof. (a) : This is clear by Lemma 6.8, Notation and Remark 6.9 and by the fact $\Sigma_p(\tilde{X})$ consists of a single point if and only if $p \in \text{Tan}(\tilde{X}) \setminus \tilde{X}$.

Before proving the remaining statements (b) and (c), we make a preliminary remark. Let $Y \subset \mathbb{P}_K^r$ be a smooth embedding scroll of X , let $\varphi : \tilde{X} \rightarrow \mathbb{P}_K^1$ and $\tau : Y \rightarrow \mathbb{P}_K^1$ be the canonical projection maps and consider the line bundle $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}_K^1}(1)$ of Notation and Remark 6.3 (with $f := \tau \circ \pi_p : \tilde{X} \rightarrow \mathbb{P}_K^1$). Then by Lemma 6.4 we may write $\mathcal{L} = \varphi^* \mathcal{O}_{\mathbb{P}_K^1}(2)$. Therefore we have a commutative diagram

$$(6.1) \quad \begin{array}{ccccc} \tilde{X} & \longrightarrow & X & \hookrightarrow & Y \\ \varphi \downarrow & & & & \downarrow \pi \\ \mathbb{P}_K^1 & \xrightarrow{\nu} & \mathbb{P}_K^2 \setminus \{q\} & \xrightarrow{\pi_q} & \mathbb{P}_K^1 \end{array}$$

in which $\nu : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^2$ is the Veronese map and π_q is a linear projection from a point $q \in \mathbb{P}_K^2 \setminus \text{Im}(\nu)$. So, there is a nonempty open set $U \subset \mathbb{P}_K^1$ such that $(\pi_q \circ \nu)^{-1}(t) \subset \mathbb{P}_K^1$ consists of the two distinct points t_1, t_2 , whence

$$f^{-1}(t) = \varphi^{-1}(\pi_q \circ \nu)^{-1}(t) = \mathbb{L}(t_1) \cup \mathbb{L}(t_2)$$

for all closed points $t \in U$. As $\mathbb{P}_K^{n-1} = \pi_p(\mathbb{L}(t_i)) \subset \tau^{-1}(t) = \mathbb{P}_K^n$ it thus follows

$$(6.2) \quad \dim(\pi_p(\mathbb{L}(t_1)) \cap \pi_p(\mathbb{L}(t_2))) \geq n - 2, \quad \text{for all } t \in U.$$

(c) : First we assume that we are in the case (5.2)(i). Then $\tilde{X} = S(1, b)$ for some integer $b > 3$ and $p \in \text{Join}(S(1), \tilde{X}) \setminus \tilde{X}$. We then find some $s \in \mathbb{P}_K^1$ such

that $p \in \langle S(1), \mathbb{L}(s) \rangle$. Choosing $t \in U \setminus \pi_q \circ \nu(s)$ we get that t_1, t_2 and s are all different. As $\mathbb{L}(s) \not\subset \langle \mathbb{L}(t_1), \mathbb{L}(t_2) \rangle$ it follows

$$\langle \mathbb{L}(t_1), \mathbb{L}(t_2), p \rangle = \langle \mathbb{L}(t_1), \mathbb{L}(t_2), \mathbb{L}(s) \rangle \supsetneq \langle \mathbb{L}(t_1), \mathbb{L}(t_2) \rangle,$$

so that $p \notin \langle \mathbb{L}(t_1), \mathbb{L}(t_2) \rangle$, whence $\pi_p(\mathbb{L}(t_1)) \cap \pi_p(\mathbb{L}(t_2)) = \emptyset$. As $n = 2$ this contradicts the above inequality and so Y cannot exist.

Assume now that we are in the case (5.3). Then $\tilde{X} = S(1, 1, c)$ for some integer $c > 2$ and $p \in \langle S(1, 1) \rangle \setminus \tilde{X}$. As $\mathbb{L}(t_i) \cap \langle S(1, 1) \rangle$ ($i = 1, 2$) are two disjoint lines and $\Sigma_p(\tilde{X}) = S(1, 1)$, there is a unique line meeting p , $\mathbb{L}(t_1)$ and $\mathbb{L}(t_2)$. This implies that the two planes $\pi_p(\mathbb{L}(t_i))$ ($i = 1, 2$) intersect each other, which again contradicts our previous observation.

(b) : Assume that we are in the case (5.2)(ii), so that $\tilde{X} = S(2, b)$ for some integer $b > 2$ and let $p \in \langle S(2) \rangle \setminus \tilde{X}$. Consider the map

$$\varrho := \pi_p|_{S(2)} : S(2) \rightarrow \pi_p(\Sigma_p(\tilde{X})) = \mathbb{P}_K^1$$

and choose a non-empty open set $V \subset \pi_p(\langle \Sigma_p(\tilde{X}) \rangle)$ such that $\varrho^{-1}(s)$ consists of two distinct points for each closed point $s \in V$. So, for each $s \in V$ the set $\varphi^{-1}(\varrho^{-1}(s)) \subset \tilde{X}$ is the union of two distinct rulings of \tilde{X} and hence $Z_p(s) := \pi_p(\varphi^{-1}(\varrho^{-1}(s)))$ is the union of two distinct lines. In view of the diagram (6.1) we have $Z_p(s) \subset \tau^{-1}(s) = \mathbb{P}_K^2$, whence $\langle Z_p(s) \rangle = \tau^{-1}(s)$ for all $s \in V$. It follows that $Y = \overline{\bigcup_{s \in V} \langle Z_p(s) \rangle}$, and this proves the uniqueness of Y .

Our next aim is to show that there exists indeed a smooth embedding scroll Y of X . As $p \notin \langle S(b) \rangle$ the map $\eta := \pi_p|_{S(b)} : S(b) \rightarrow \pi_p(S(b)) =: \mathcal{C}$ is an isomorphism and $\mathcal{C} \subset \langle \pi_p(S(b)) \rangle = \mathbb{P}_K^b$ is a rational normal curve. Let $\sigma_1 : \mathbb{P}_K^1 \rightarrow \langle S(2) \rangle = \mathbb{P}_K^2$ be the Veronese embedding with $\text{Im}(\sigma_1) = S(2)$. Then

$$f := \varrho \circ \sigma_1 \circ \varphi \circ \eta^{-1} : \mathcal{C} \rightarrow \pi_p(\Sigma_p(\tilde{X})) = \mathbb{P}_K^1$$

is of degree 2. So, by Lemma 6.7 we get the smooth surface scroll $S(\alpha, \beta) := W_f \in \mathfrak{S}(\mathcal{C}) = \mathfrak{S}(\pi_p(S(b)))$. Now, for a general $t \in \mathbb{P}_K^1$ the span $\langle f^{-1}(t) \rangle \subset \mathbb{P}_K^b$ defines a ruling of W_f . From this it follows easily that for an appropriate non-empty open set $U \subset \mathbb{P}_K^1 = \pi_p(\Sigma_p(\tilde{X}))$ the union $\overline{\bigcup_{t \in U} \langle t, f^{-1}(t) \rangle}$ defines a 3-dimensional scroll Y which has $\pi_p(\Sigma_p(\tilde{X}))$ as a line section and thus is of type $S(1, \alpha, \beta)$ with $S(1) = \pi(\Sigma_p(\tilde{X}))$.

It remains to show that α and β are as stated in (b). We do this in the following lemma. \square

Lemma 6.11. *Let $S \subset \mathbb{P}_K^r$ be a smooth rational normal surface scroll which contains a rational normal curve $C = S(r) \subset \mathbb{P}_K^r$. Suppose that C is not a section of the projection $\varphi : S \rightarrow \mathbb{P}_K^1$. Then $S = S(\alpha, \beta)$ where*

$$(\alpha, \beta) = \begin{cases} \left(\frac{r-1}{2}, \frac{r-1}{2} \right) & \text{if } r \text{ is odd} \\ \left(\frac{r-2}{2}, \frac{r}{2} \right) & \text{if } r \text{ is even.} \end{cases}$$

Proof. Without loss of generality we may assume that $\alpha \leq \beta$. Let $C \sim uC_0 + vF$ for some $u \geq 1$ and $v \in \mathbb{Z}$. Then $u = 2$ since C is not a section of φ and admits no trisecant lines. Now the adjunction formula for C on S gives us $v = r - 2\alpha$. Thus we get the inequality

$$C.C_0 = r - 2\beta = \alpha - \beta + 1 \geq 0$$

which means that $\beta = \alpha$ or $\alpha + 1$. Therefore either $\beta = \alpha = \frac{r-1}{2}$ or else $\beta = \alpha + 1 = \frac{r}{2}$. \square

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