FAMILIES OF BLOWUPS OF THE REAL AFFINE PLANE: CLASSIFICATION, ISOTOPIES AND VISUALIZATIONS

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ABSTRACT. We classify embedded blowups of the real affine plane up to oriented isomorphy. We show that two blowups in the same isomorphism class are isotopic, using a matrix deformation argument similar to an idea given in [12]. This answers two questions which were motivated by the interactive visualizations of such blowups (see [10], [13], [14]).

1. INTRODUCTION

The Visualization Project for Blowups of the Real Affine Plane. The present paper is primarily of theoretical nature. Nevertheless we begin with a "warm up" related to one aim of our whole Visualization Project, which is "to bring Algebraic Geometry to the Class Room" already on early undergraduate level. Namely, in Figure 1 we illustrate the *resolving effect of blowing up* – a basic issue in Algebraic Geometry (see [6] for the rôle of this effect in the resolution of singularities of algebraic varieties in characteristic zero). Our example, which will be explained later in detail, shows how a simple nodal singularity of a plane curve is resolved by blowing up.

Our paper is motivated by several investigations on the visualization of blowups of the real affine plane (see [1],[2],[3],[7],[8],[9]) in particular by the interactive visualizations suggested by the second named author and C. Stussak [10]. Our principal aim is to consolidate the theoretical background of our Visualization Project and focuses on the following two problems:

(1.0) (a) *Deformation Problem*: "Can one connect two arbitrary oriented isomorphic embedded blowups of the real affine plane by a continuous family within their isomorphism class?"

(b) *Classification Problem*: "Is there a simple criterion to detect whether two regular embedded blowups of the affine plane are oriented isomorphic?"

We shall see, that both of these problems find an affirmative answer (see Theorem 4.8 and Theorem 3.9). At first view, these are results of theoretical nature – but, indeed, they also are of considerable practical meaning: Namely, once having tested that two embedded blowups B and C of the real affine plane are oriented isomorphic, one can use the animated visualization procedure of [10] to produce a family or sequence of pictures which shows a deformation between the two blowups B and C within their common

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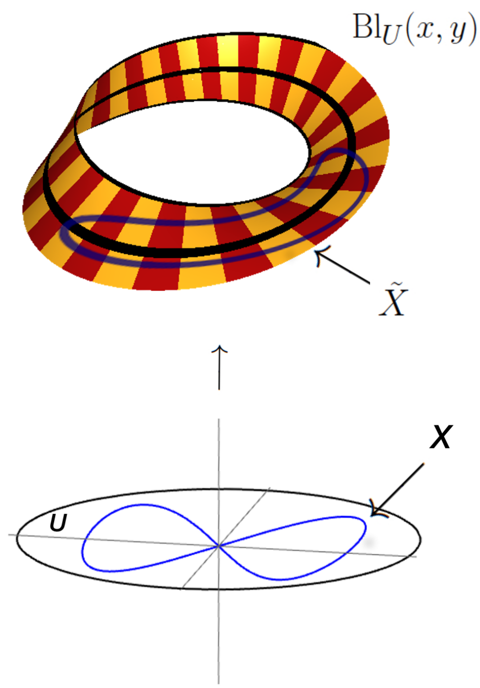


FIGURE 1. The resolving effect of blowing up

isomorphism class. Moreover, our answer to the classification problem gives an easy way to detect whether two regular embedded blowups are oriented isomorphic. We shall provide a number of examples of this, including illustrations based on the visualization program REALSURF developed by C. Stussak (see [13]).

Blowups of the Real Affine Plane. We now start to set the precise setting in which we shall work. Let $Z \subset \mathbb{R}^2$ be a finite set and let $U \subset \mathbb{R}^2$ be an open bounded and star-shaped set with closure $\overline{U}$ such that $Z \subset U$ – for example an open disk containing Z. We fix a pair of two-variate real polynomials.

(1.1) $\underline{f} := (f_0, f_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ such that $\mathrm{Z}_{\overline{U}}(\underline{f}) := \{p \in \overline{U} \mid f_0(p) = f_1(p) = 0\} = Z.$

We always shall denote by $\mathbb{P}^1 := \{(x_0 : x_1) = [(x_0, x_1)] \mid (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0,0)\}\}$ the *real projective line*, whereas the complex projective line will be denoted by $\mathbb{P}^1_{\mathbb{C}}$.
For any set $S \subset U \times \mathbb{P}^1$ we denote by $\overline{S}$ the *Zariski closure* of S in $U \times \mathbb{P}^1$, that is the restriction to $U \times \mathbb{P}^1$ of the closure of S with respect to the Zariski topology in the ambient complex algebraic variety $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Now, the *embedded blowup* $\mathrm{Bl}_U(\underline{f})$ of U with respect to the pair $\underline{f}$ is defined as the Zariski closure of the graph of the map

(1.2) $\varepsilon_{U,\underline{f}} : U \setminus Z \longrightarrow \mathbb{P}^1$, given by $p \mapsto [\underline{f}(p)] = (f_0(p) : f_1(p))$

in $U \times \mathbb{P}^1$. More precisely, our embedded blowup is given by

(1.3) (a) the set $\mathrm{Bl}_U(\underline{f}) := \overline{\{(p, [\underline{f}(p)]) \mid p \in U \setminus Z\}}$ and
(b) the *canonical projection* map $\pi_{U,\underline{f}} : \mathrm{Bl}_U(\underline{f}) \longrightarrow U$, given by $(p, (x_0 : x_1)) \mapsto p$ for all $(p, (x_0 : x_1)) \in \mathrm{Bl}_U(\underline{f}) \subset U \times \mathbb{P}^1$.

(1.4) (a) The set Z is called the *center* of the blowup $\mathrm{Bl}_U(\underline{f})$, whereas
(b) the graph $\mathrm{Bl}_U^\circ(\underline{f}) := \{(p, [\underline{f}(p)]) \mid p \in U \setminus Z\} = \mathrm{Bl}_U(\underline{f}) \setminus (Z \times \mathbb{P}^1)$ of $\varepsilon_{U,\underline{f}}$ is called the *open kernel* of our embedded blowup, and
(c) the set $\mathrm{E}_U(\underline{f}) := \pi_{U,\underline{f}}^{-1}(Z) = \mathrm{Bl}_U(\underline{f}) \cap (Z \times \mathbb{P}^1)$ is called the *exceptional set* of this embedded blowup.

Our basic aim is to study the class of embedded blowups

(1.5) $\mathfrak{Bl}_U(Z) := \{\mathrm{Bl}_U(\underline{f}) \mid \underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2 \text{ with } Z_{\overline{U}}(\underline{f}) = Z\}$,

up to (relative oriented embedded) isomorphisms – a concept which will be defined below. If we write $\mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ we tacitly mean that $(f_0, f_1) = \underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ satisfies the condition $Z_{\overline{U}}(\underline{f}) = Z$.

Isomorphisms of Embedded Blowups. A *(relative oriented) automorphism* (we often omit the wording in brackets from now on) of $U \times \mathbb{P}^1$ is a map

(1.6) (a) $\varphi = \varphi_M : U \times \mathbb{P}^1 \longrightarrow U \times \mathbb{P}^1$ given by $(p, [\underline{v}]) \mapsto (p, [\underline{v}M(p)])$ for all $p \in U$ and all $\underline{v} \in \mathbb{R}^2 \setminus \{\underline{0}\}$, where
(b) $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$.

It is indeed justified to call these maps automorphisms. Namely: If $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$, its inverse $M^{-1} \in \mathbb{R}(\mathbf{x}, \mathbf{y})^{2\times 2}$ may be written in the form $M^{-1} = \frac{1}{\det M} N$ with $N \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ and $\det\big(N(p)\big) = \det\big(M(p)\big) > 0$ for all $p \in U$. It is immediate, that the map φ_N is inverse to φ_M. Observe that a relative oriented automorphism of $U \times \mathbb{P}^1$ leaves fix the fiber $\{p\} \times \mathbb{P}^1 \cong \mathbb{P}^1$ of the canonical projection $\pi : U \times \mathbb{P}^1 \longrightarrow U$ over each point $p \in U$ and acts as an orientation preserving *Möbius-Transformation* on this fiber.

We say that two embedded blowups $B, C \in \mathfrak{Bl}_U(Z)$ are *(relatively oriented embedded) isomorphic* (we often omit the wording in brackets from now on) – and write $B \cong C$ – if there is an automorphism φ of $U \times \mathbb{P}^1$ such that $C = \varphi(B)$. This means in particular:

(1.7) If $B = \mathrm{Bl}_U(\underline{f}), C \in \mathfrak{Bl}_U(Z)$, then $B \cong C$ if and only if $C = \mathrm{Bl}_U(\underline{f}M)$ for some $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$.

Regular Embedded Blowups and their Classification. We say that the pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is *regular* with respect to Z on U if:

(1.8) (a) $\mathrm{Z}_{\overline{U}}(\underline{f}) = Z$.
(b) The Jacobian

$$\partial\underline{f} := \begin{pmatrix} \frac{\partial f_0}{\partial \mathbf{x}} & \frac{\partial f_1}{\partial \mathbf{x}} \\ \frac{\partial f_0}{\partial \mathbf{y}} & \frac{\partial f_1}{\partial \mathbf{y}} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2} \text{ of } \underline{f} \text{ is of rank 2 in all points } p \in Z.$$

If the pair $\underline{f} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ is regular with respect to Z on U, we call $\mathrm{Bl}_U(\underline{f})$ a *regular embedded blowup* of the set U along Z – and we define:

(1.9) $\mathfrak{Bl}_U^{\mathrm{reg}}(Z) := \{\mathrm{Bl}_U(\underline{f}) \mid \underline{f} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ is regular with respect to Z on $U\}$.

If we write $\mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, we tacitly mean that $\underline{f} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ is regular with respect to Z on U. If $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ with $Z \neq \emptyset$, there is a map, depending only on B,

(1.10) $\mathrm{sgn}_B : Z \longrightarrow \{\pm 1\}$ given by $p \mapsto \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big)$ for all $p \in Z$

(see Definition and Remark 3.4), called the *sign distribution* of B.
If $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ with $\#Z = n \in \mathbb{N}$, we call B a *regular (embedded) n-point blowup*. We shall present examples of such n-point blowups and families of them for $n = 1$ (see Example 2.1), for $n = 2$ (see Examples 5.2 (B) and (C)), for $n = 3$ (see Examples 5.2 (A) and (B)) and for $n = 4$ (see Example 2.2).

Our *Classification Problem* (1.0)(b) is answered as follows (see Theorem 3.9):

(1.11) ***Classification Theorem*:** Two embedded blowups $B, C \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, are relatively oriented embedded isomorphic if and only if they have the same sign distribution. Hence, for short: $B \cong C$ if and only if $\mathrm{sgn}_B = \mathrm{sgn}_C$.

Isotopies of Blowups and the Deformation Theorem. Now, we turn to the *Deformation Problem* (1.0)(a). Given $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_Z(U)$, we are interested in families $\big(B^{(t)}\big)_{t\in[0,1]} \subset \mathfrak{Bl}_Z(U)$, such that $B^{(0)} = B$ and $B^{(t)} \cong B$ for all $t \in [0,1]$. In view of (1.6) and (1.7) it is natural to consider such families which come from an *isotopy* of $U \times \mathbb{P}^1$-automorphisms. This means:

(1.12) There is a family of relative oriented $U \times \mathbb{P}^1$-automorphisms $\big(\varphi^{(t)} = \varphi_{M^{(t)}}\big)_{t\in[0,1]}$, given by a (2×2)-matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$, such that
  - (a) for all $t \in [0,1]$ and all $p \in U$, the matrix $M^{(t)} := \widetilde{M}(\mathbf{x},\mathbf{y},t) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ satisfies $\det\big(M^{(t)}(p)\big) > 0$;
  - (b) $M^{(0)} = \mathbf{1}^{2\times 2}$ and $B^{(t)} = \varphi^{(t)}(B) = \mathrm{Bl}_U(\underline{f}M^{(t)})$ for all $t \in [0,1]$.

In this context we shall solve the Deformation Problem (1.0)(a) (see Theorem 4.8):

(1.13) ***Deformation Theorem*:** Let $B, C \in \mathfrak{Bl}_U(Z)$ be relatively oriented embedded isomorphic. Then, B and C are connected by an isotopy of $U \times \mathbb{P}^1$-automorphisms. More precisely, there is an isotopy $\big(\varphi^{(t)} = \varphi_{M^{(t)}}\big)_{t\in[0,1]}$ as in (1.12) such that $\varphi^{(0)}(B) = B$ and $\varphi^{(1)}(B) = C$.

Deformation of Matrices. Our Deformation Theorem (1.13) is a consequence of the following deformation result for matrices (see Proposition 4.4 and Remark 4.6):

(1.14) ***Polynomial Deformations of Matrices*:** Let $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ for all $p \in \overline{U}$. Then M is connected to the unit matrix $\mathbf{1}^{2\times 2} \in \mathbb{R}^{2\times 2}$ by a polynomial family of (2×2)-matrices with positive determinants on $\overline{U}$. More precisely: There is a (2×2)-matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$, such that with $M^{(t)} := \widetilde{M}(\mathbf{x},\mathbf{y},t)$ we have

(a) $\det\big(M^{(t)}(p)\big) > 0$ for all $t \in [0,1]$ and all $p \in \overline{U}$.
(b) $M^{(0)} = \mathbf{1}^{2\times 2}$ and $M^{(1)} = M$.

The Visualization Procedure. We now present a visualization procedure for embedded blowups $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$. We use a method originally suggested in [1] and [2] – in the modified form given in [10]. So, let $\rho, r \in \mathbb{R}$ with $0 < \rho < r$ and consider

(1.15) (a) the open disk $\mathbb{D} := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < \rho^2\} \subset \mathbb{R}^2$, with $U \subsetneq \mathbb{D}$ and
(b) the open solid torus $\mathbb{T} := \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + \big(r - \sqrt{v^2+w^2}\big)^2 < \rho^2\} \subset \mathbb{R}^3$

together with the diffeomorphism

(1.16) $\iota : \mathbb{D} \times \mathbb{P}^1 \xrightarrow{\cong} \mathbb{T}$, given by

$$\big((x,y),(x_0 : x_1)\big) \mapsto \Big(x, (r-y)\frac{x_0^2 - x_1^2}{x_0^2 + x_1^2}, (r-y)\frac{2x_0x_1}{x_0^2+x_1^2}\Big), \text{for all } (x,y) \in U, (x_0 : x_1) \in \mathbb{P}^1.$$

(1.17) The blowup $B = \mathrm{Bl}_U(\underline{f})$ is visualized by its diffeomorphic image

$$\iota\big(\mathrm{Bl}_U(\underline{f})\big) = \iota\big(\mathrm{Bl}^\circ_U(\underline{f})\big) \ \dot\cup\ \iota\big(\mathrm{E}_U(\underline{f})\big) \subset \mathbb{T}, \text{ so that we have:}$$

(a) $\iota\big(\mathrm{Bl}^\circ_U(\underline{f})\big) = \{\big(x, (r-y)\frac{f_0(x,y)^2 - f_1(x,y)^2}{f_0(x,y)^2+f_1(x,y)^2}, (r-y)\frac{2f_0(x,y)f_1(x,y)}{f_0(x,y)^2+f_1(x,y)^2} \mid (x,y) \in U \setminus Z\}$.
(b) $\iota\big(\mathrm{E}_U(\underline{f})\big) \subseteq \iota(Z \times \mathbb{P}^1) = \dot{\bigcup}_{p \in Z} \iota(\{p\} \times \mathbb{P}^1)$.
(c) If $p = (x,y) \in Z$, then $\iota(\{p\} \times \mathbb{P}^1) \subset \mathbb{T}$ is the circle of radius $r - y$ given by:

$$\begin{aligned}\iota(\{p\} \times \mathbb{P}^1) &= \{\big(x, (r-y)\frac{x_0^2 - x_1^2}{x_0^2+x_1^2}, (r-y)\frac{2x_0x_1}{x_0^2+x_1^2}\big) \mid (x_0,x_1) \in \mathbb{R}^2 \setminus \{(0,0)\}\} \\ &= \{\big(x, (r-y)\cos(\beta), (r-y)\sin(\beta) \mid -\pi \le \beta \le \pi\}.\end{aligned}$$

Observe that $\iota\big(\mathrm{Bl}^\circ_U(\underline{f})\big) \subset \mathbb{T}$ is a surface without boundary and that $\iota(Z \times \mathbb{P}^1) \subset \mathbb{T}$ is a finite union of circles parallel to the central circle of $\mathbb{T}$ and centered at the rotation axis of $\mathbb{T}$.

For each point $p \in Z$, the accumulation points (with respect to the *strict* (or *metric*) *topology*) of the open kernel $B^\circ = \mathrm{Bl}^\circ_U(\underline{f})$ in the fiber $\pi^{-1}_{U,\underline{f}}(p)$ are called the *limit points* of B above p. We denote the set of these limit points by $\mathcal{L}_p(B)$, thus:

(1.18) $\mathcal{L}_p(B) := \{(p,s) \in \{p\} \times \mathbb{P}^1 \mid \exists \big(p_n\big)_{n\in\mathbb{N}} \subset U \setminus Z : \lim_{n\to\infty}(p_n, \varepsilon_{U,\underline{f}}(p_n)) = (p,s)\}$.

The sets $\mathcal{L}_p(B)$ are of particular importance for the shape of the embedded blowup B. Therefore, in some of our illustrations, their images $\iota\big(\mathcal{L}_p(B)\big)$ are colored in bold black and they usually appear as closed arcs on the circle $\iota\big(\{p\} \times \mathbb{P}^1\big)$.

The Technique of Visualization. For visualizations the parametric presentation given in (1.17) is used by Brandenberg (see [1]) and also by Brodmann and Prager (see [2] and [9]) for a very few examples. The difficulty of the parametrization for further examples is its instability in the neighborhood of Z (see also Prager in [9] for a further discussion). The new idea of C. Stussak (see [14] and [10]) – which will be applied in this paper – was to derive the implicit equation of the parametrized surface (based on the work of [2]) and to use the program RealSurf (see [13]) for its visualization. RealSurf is a graphic GPU-program for the visualization of algebraic surfaces. It allows an interactive view of

algebraic surfaces in $\mathbb{A}^3_{\mathbb{R}} = \mathbb{R}^3$ in real time.
In his PhD dissertation (see [14]) C. Stussak studied exact rasterization of algebraic curves and surfaces for the visualization on a personal computer with GPU-programming. As an application of his technique he and the second named author studied interactive visualizations of blowups of the real affine plane (see [14] and [10]). These interactive visualizations are based on RealSurf with several adaptations for the particular situation of our concrete examples (see [10] for the technical details). The modified program allows continuous parameter changes by mouse action. With the help of these modifications we produced the pictures of the present paper. We are grateful to C. Stussak for making the adaption of RealSurf available to us.

The pictures were produced on a PC with graphic cards nVidia GT 525 Windows 7.

A Few Preliminary Examples. Let us first recall the notion of *affine standard charts* of an embedded blowup $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{B}_U(Z)$, which are given by

(1.19) $\big(B\big)_i = \big(\mathrm{Bl}_U(\underline{f})\big)_i := \{(p, \frac{f_j(p)}{f_i(p)}) \in U \times \mathbb{R} \mid p \in U, f_i(p) \neq 0\} \quad (i, j \in \{0, 1\}, i \neq j).$

Keep in mind, that the blowup B is obtained by pasting together the two affine standard charts $\big(B\big)_i, (i = 0, 1)$ by identifying (for $w \neq 0$) the two points (p, w) and $(p, \frac{1}{w})$ of $U \times \mathbb{R}$. Moreover, we can say:

(1.20) If f_0 and f_1 have no common divisor, then $\big(\mathrm{Bl}_U(\underline{f})\big)_i = Z_{U\times\mathbb{R}}(f_i\mathbf{w} - f_j), (i, j \in \{0, 1\}, i \neq j\}$,

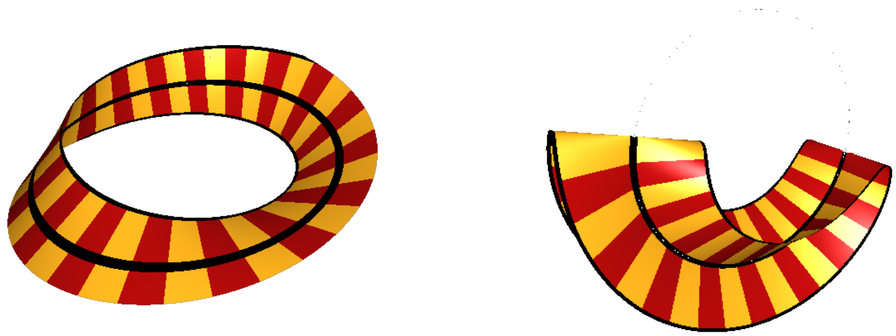
where $Z_{U\times\mathbb{R}}(h) := \{(x, y, w) \in U \times \mathbb{R} \mid h(x, y, z) = 0\}$ for $h \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{w}]$.
To present two basic examples of blowups, we choose $\rho = 2, r = 4, Z = \{(0, 0)\}, U = \mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$. Then, for the choice $f_0 = \mathbf{x}, f_1 = \mathbf{y}$, the blowup $\mathrm{Bl}_U(\underline{f})$ is regular and appears as a *Möbius Strip* under our visualization process (see Figure 2 (a); see also [5], pg. 29, Figure 3, and [11], pg. 100, Figure 6, which both present sketches of an affine standard chart of this blowup).

For the choice $f_0 = \mathbf{x}^2, f_1 = \mathbf{y}^2$, the blowup $\mathrm{Bl}_U(\underline{f})$ is not regular and appears as a *Double Whitney Umbrella* (see Figure 2 (b)). Indeed, according to (1.20) the two embedded standard affine charts of this blowup are given respectively by $Z_{U\times\mathbb{R}}(\mathbf{x}^2\mathbf{w} - \mathbf{y}^2), Z_{U\times\mathbb{R}}(\mathbf{y}^2\mathbf{w} - \mathbf{x}^2) \subset \mathbb{R}^3$ and hence appear as Whitney Umbrellas folded along the positive $\mathbf{w}$-axis and rotated around this axis with respect to each other by 90°.

We now explain in detail the example shown in Figure 1 which illustrates the resolving effect of blowing up. We choose ρ, r, Z, U as above, set $f_0 = \mathbf{x}, f_1 = \mathbf{y}$ and consider the lemniscate $X := \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - \frac{1}{2}x^4 = 0\} = Z_{\mathbb{A}^2_{\mathbb{C}}}(\mathbf{x}^2 - \mathbf{y}^2 - \frac{1}{2}\mathbf{x}^4) \cap \mathbb{R}^2 \subset U$, which has a nodal singularity of multiplicity 2 at the origin $\underline{0} := (0, 0)$ and is smooth elsewhere. Finally we consider the so called *strict transform*

$$\widetilde{X} := \overline{\pi^{-1}_{U,(\mathbf{x},\mathbf{y})}(X \setminus \{(0, 0)\})} = \overline{\pi^{-1}_{U,(\mathbf{x},\mathbf{y})}(X) \cap \mathrm{Bl}^{\circ}_U(\mathbf{x}, \mathbf{y})} \subset \mathrm{Bl}_U(\mathbf{x}, \mathbf{y})$$

of X, (with respect to the pair $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$) which is a non-singular curve contained in our embedded blowup $\mathrm{Bl}_U(\mathbf{x}, \mathbf{y})$ (see Example 4.9.1 in Chapter I of [5]) – and hence appears as a smooth simple closed curve on a Möbius strip – as illustrated in Figure 2. The resolving effect of the same blowup is also illustrated on an affine standard chart in



(a) (b)

FIGURE 2. (a) Möbius Strip (b) Double Whitney Umbrella

[5], pg. 29, Figure 3, but with a plane nodal quadric curve instead of a lemniscate. We did choose the lemniscate as its whole strict transform appears on the blowup $\mathrm{Bl}_U(\mathbf{x},\mathbf{y})$.

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2. First Examples of Families of Blowups

Examples and their Visualizations. We shall continue with a few examples of families of embedded blowups and their visualizations. Already now, we present three examples, which give a first flavor of the subject and illuminate some typical features. Again, as in the examples visualized by Figure 2, we choose $\rho = 2, r = 4$ and $U = \mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$.

Example 2.1. In our first example, we consider the most simple regular blowup of the real affine plane, namely the *regular one-point blowup* $B := \mathrm{Bl}_U(\mathbf{x},\mathbf{y})$, whose visualization shows up as a Möbius strip (see Figure 2(a)). We deform this blowup by means of the family of polynomial matrices

$$\Big(M^{(t)} := \begin{pmatrix} 1-t & \frac{t}{2} \\ -\frac{t}{2} & 1+t \end{pmatrix}\Big)_{t\in]-\frac{2}{3}\sqrt{3},\frac{2}{3}\sqrt{3}[} \text{ with } \det(M^{(t)}) = 1 - \frac{3}{4}t^2 > 0 \text{ for } t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[.$$

This leads us to the family of regular embedded blowups $\big(B^{(t)}\big)_{t\in]-\frac{2}{3}\sqrt{3},\frac{2}{3}\sqrt{3}[}$ with

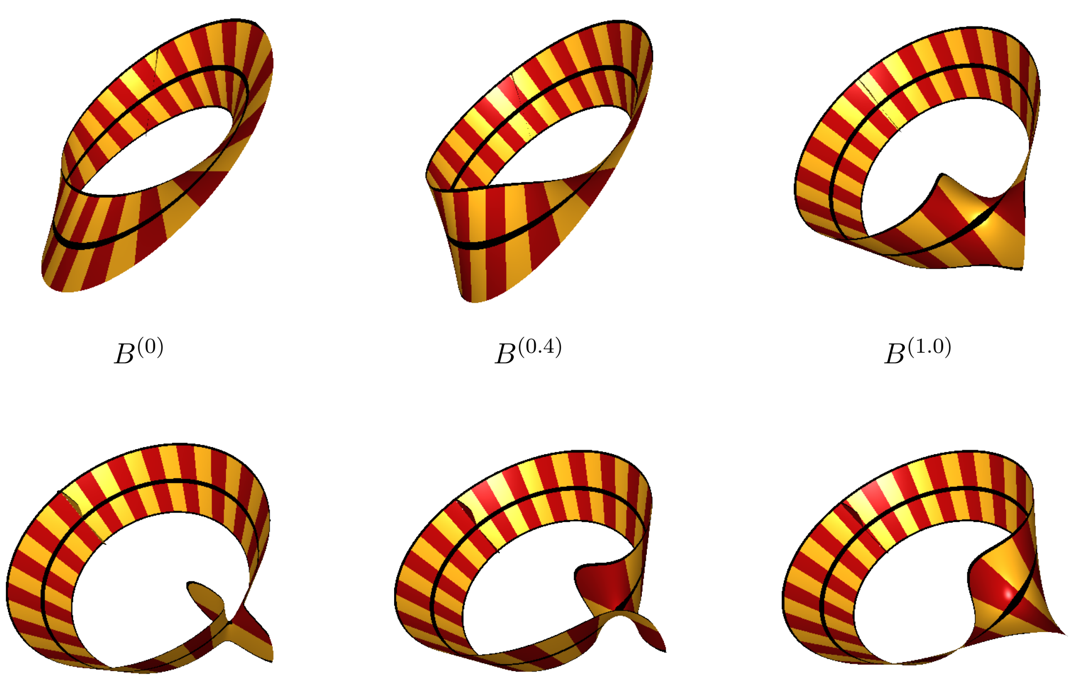
$$B^{(t)} := \mathrm{Bl}_U\big((\mathbf{x},\mathbf{y})M^{(t)}\big) = \mathrm{Bl}_U\big(f_0^{(t)} = (1-t)\mathbf{x} - \frac{t}{2}\mathbf{y}, f_1^{(t)} = \frac{t}{2}\mathbf{x} + (1+t)\mathbf{y}\big) \in \mathfrak{Bl}_U^{\mathrm{reg}}\big(\{\underline{0}\}\big)$$

and

$$Z = Z_U\big(\underline{f}^{(t)} := (f_0^{(t)}, f_1^{(t)})\big) = \{(0,0)\} \text{ for all } t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[.$$

In view of Figure 2(a) we expect that the visualization $\big(\iota(B^{(t)})\big)_{t\in]-\frac{2}{3}\sqrt{3},\frac{2}{3}\sqrt{3}[}$ of this family presents itself as a deformation of a Möbius strip. In Figure 3 we present this deformation for the values $t = 0, t = 0.4$ and $t = 1$. We also allow ourselves to leave the range $0 \leq t \leq 1$ and consider the three values $t = 1.15, t = 1.2$ and $t = 1.4$, which come close or lie beyond the critical value $t = \frac{2}{3}\sqrt{3} = 1.15470\ldots$.
These choices illustrate the following fact: If t takes its critical values $\pm\frac{2}{3}\sqrt{3}$, the two linear forms $f_0^{(t)}$ and $f_1^{(t)}$ are linearly dependent and hence do not define a blowup in our sense. If $t \notin [-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}]$ the blowup $B^{(t)}$ shows up again as a Möbius strip, but reversely twisted along its central circle.

Example 2.2. As a second example, we consider a family of *regular four-point blowups* of the real affine plane, which is indeed a modification of the example shown in Figure 9 of [10]. To this end, we choose $a \in [0,1]$ and consider the two pairs of polynomials



$B^{(0)}$ $B^{(0.4)}$ $B^{(1.0)}$

$B^{(1.15)}$ $B^{(1.2)}$ $B^{(1.4)}$

FIGURE 3. Deformation of a Möbius Strip

$\underline{f} := (f_0, f_1)$ and $\underline{g} := (g_0, g_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ given by

$$f_0 = \mathbf{x}^2 - \frac{1}{2}\mathbf{y}^2 - \frac{1}{2}, \quad f_1 = -\frac{1}{2}\mathbf{x}^2 + \mathbf{y}^2 - \frac{1}{2} \text{ and}$$
$$g_0 = \mathbf{x}^2 + (a - \frac{1}{2})\mathbf{y}^2 - a - \frac{1}{2}, \quad g_1 = (a - \frac{1}{2})\mathbf{x}^2 + \mathbf{y}^2 - a - \frac{1}{2}.$$

Then $\det(\partial \underline{f}) = 3\mathbf{x}\mathbf{y}$ and $\det(\partial \underline{g}) = 4(1 - (a - \frac{1}{2})^2)\mathbf{x}\mathbf{y}$. Taking $\mathbf{x}$-resultants, we get $\mathrm{Res}_{\mathbf{x}}(g_0, g_1) = (((a - \frac{1}{2})^2 - 1)(1 - \mathbf{y}^2))^2$. As $(a - \frac{1}{2})^2 - 1 < 0$ for $a \in [0, 1]$ it follows that

$$Z = Z_U(\underline{f}) = Z_U(\underline{g}) = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.$$

In particular $\det(\partial \underline{f})(p)$ and $\det(\partial \underline{g})(p)$ are $\neq 0$ for all $p \in Z$, so that $\underline{f}$ and $\underline{g}$ are regular pairs with respect to Z on U, with $B := \mathrm{Bl}_U(\underline{f}), C := \mathrm{Bl}_U(\underline{g}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Moreover,

$$\underline{g} = \underline{f}M \text{ with } M = \begin{pmatrix} 1 + \frac{2}{3}a & \frac{4}{3}a \\ \frac{4}{3}a & 1 + \frac{2}{3}a \end{pmatrix}.$$

so that $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) = 1 + \frac{4}{3}a(1 - a) > 0$ for all $p \in U$. Setting

$$\widetilde{M} := \begin{pmatrix} 1 + \frac{2}{3}a\mathbf{t} & \frac{4}{3}a\mathbf{t} \\ \frac{4}{3}a\mathbf{t} & 1 + \frac{2}{3}a\mathbf{t} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^2 \text{ and } M^{(t)} := \begin{pmatrix} 1 + \frac{2}{3}at & \frac{4}{3}at \\ \frac{4}{3}at & 1 + \frac{2}{3}at \end{pmatrix} \text{ for all } t \in [0, 1]$$

we get $\det(M^{(t)}) = (1 + \frac{2}{3}at)^2 - \frac{16}{9}(at)^2 > 0$ for all $t \in [0, 1]$. Moreover, $M^{(0)} = \mathbf{1}^{2\times 2}$ and $M^{(1)} = M$. So, $\big(M^{(t)}\big)_{t\in[0,1]}$ is a family which connects $\mathbf{1}^{2\times 2}$ and M. Correspondingly $\big(\varphi^{(t)} := \varphi_{M^{(t)}}\big)_{t\in[0,1]}$ is an isotopy. As $\det(M^{(t)}) > 0$ for all $t \in [0, 1]$ and $\det\big(\partial(\underline{f}M^{(t)})\big) = \det(M^{(t)})\det(\partial \underline{f})$ it is clear that

$$\big(B^{(t)} = \varphi^{(t)}(B) = \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]}$$

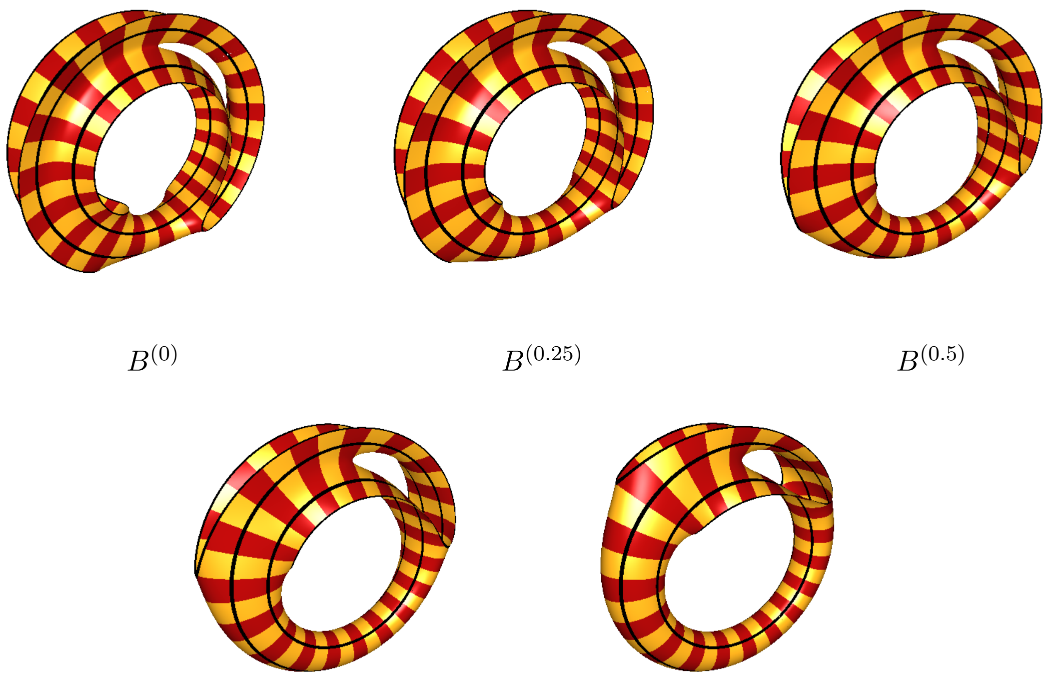
is a family of regular blowups $B^{(t)} \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ with $B^{(0)} = B$ and $B^{(1)} = C$.
We now choose $a = 1$. Then looking at the conics $f_0^{(t)} = 0$ and $f_1^{(t)} = 0$ defined by the two polynomials

$$f_0^{(t)}, f_1^{(t)} \in \mathbb{R}[\mathbf{x}, \mathbf{y}] \text{ with } \underline{f}^{(t)} := (f_0^{(t)}, f_1^{(t)}) = \underline{f}M^{(t)} \text{ for all } t \in [0, 1]$$

we have the following situation: Two hyperbolas ($t = 0$) are deformed to two ellipses ($t = 1$) via a degeneration to a pair of lines ($t = \frac{1}{2}$). A rough visualization of this family is shown in Figure 4.

Example 2.3. Up to now, we have considered two families of regular blowups of the real affine plane. Next, we aim to consider a family of blowups, which is obtained by deforming the singular blowup $B := \mathrm{Bl}_U(\mathbf{x^2}, \mathbf{y^2})$, whose visualization shows up as a Double Whitney Umbrella (see Figure 2(b)). We fix the matrix

$$\widetilde{M} = \widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{t}) := \begin{pmatrix} 1 - \mathbf{t} & \frac{1}{2}\mathbf{t} \\ -\frac{1}{2}\mathbf{t} & 1 + \mathbf{t} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2} \text{ with } \det(\widetilde{M}) = 1 - \frac{3}{4}\mathbf{t}^2.$$



$B^{(0)}$ $B^{(0.25)}$ $B^{(0.5)}$

$B^{(0.75)}$ $B^{(1)}$

FIGURE 4. Deformation of a regular four-point blowup

For all $t \in \mathbb{R}$ we set

$$M^{(t)} := \widetilde{M}(\mathbf{x}, \mathbf{y}, t) = \begin{pmatrix} 1-t & \frac{1}{2}t \\ -\frac{1}{2}t & 1+t \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}, \text{ so that } \det(M^{(t)}) = 1 - \frac{3}{4}t^2.$$

Clearly, $\det(M^{(t)}) > 0$ whenever $|t| < \frac{2}{3}\sqrt{3}$, so that $\big(\varphi_{M^{(t)}} = \varphi^{(t)}\big)_{t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[}$ is an isotopy of $U \times \mathbb{P}^1$-automorphisms. Thus for any blowup $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ we get a family $\big(B^{(t)} := \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[}$ *with* $B^{(t)} \in \mathfrak{Bl}_U(Z)$ and $B^{(t)} \cong B$ for all $t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[$.

With $f_0 = \mathbf{x}^2, f_1 = \mathbf{y}^2$ and $\underline{f}^{(t)} := \underline{f}M^{(t)}$ we then have

$$Z := Z_U(\underline{f}^{(t)}) = \{\underline{0}\} \text{ for all } t \neq \pm\frac{2}{3}\sqrt{3}.$$

In Figure 5, the blowups $B^{(t)}$ are visualized in $\mathbb{R}^3$ for $t = 0,\ 0.5,\ 1,\ 1.1,\ 1.25,\ 4$. Remember that $B = B^{(0)}$ is the so-called Double Whitney Umbrella.

Note that while passing from $t = 1.1$ to $t = 1.25$ (hence by passing through the critical value $t = \frac{2}{3}\sqrt{3}$) the orientation of embedded blowup $B^{(t)}$ swaps. Observe also, that the fiber $\pi^{-1}_{U,\underline{f}^{(t)}}(\underline{0}) = \{\underline{0}\} \times \mathbb{P}^1$ of $B^{(t)}$ over $\underline{0}$ is visualized by the same circle for all $t \neq \pm\frac{2}{3}\sqrt{3}$

and that the corresponding set of limit points $\mathcal{L}_{\underline{0}}(B^{(t)})$ is visualized by an arc on this circle, whose length depends on t. Near to the degeneration value $t = \frac{2}{3}\sqrt{3}$ we enlarged the scale of our visualization in order to improve the picture of the details. Therefore the coloring appears larger for the last three values of t.

3. Structure and Classification of Regular Embedded Blowups

Equality of Embedded Blowups. Let all notations be as in the Introduction. Our first aim is to make clear, when the embedded blowups of U with respect to two pairs of polynomials are equal.

Proposition 3.1. *Let $Z, W \subset U$ be two finite sets and let $\underline{f} = (f_0, f_1), \underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ such that $Z_{\overline{U}}(\underline{f}) = Z$ and $Z_{\overline{U}}(\underline{g}) = W$.*

(a) *Then $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(\underline{g})$ if and only if $f_0 g_1 = f_1 g_0$.*
(b) *If f_0 and f_1 have no common divisor, the pair $\underline{f}$ is uniquely determined by $\mathrm{Bl}_U(\underline{f})$ up to multiplication with a non-zero constant.*

Proof. (a): As $U \setminus (Z \cup W) \neq \emptyset$, we have $\underline{f}, \underline{g} \neq (0,0)$. Assume first, that $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(\underline{g})$. Then clearly $\mathrm{Bl}_U(\underline{f}) \setminus \big((Z \cup W) \times \mathbb{P}^1\big) = \mathrm{Bl}_U(\underline{g}) \setminus \big((Z \cup W) \times \mathbb{P}^1\big)$. As $\mathrm{E}_U(\underline{f}), \mathrm{E}_U(\underline{g}) \subseteq (Z \cup W) \times \mathbb{P}^1$ (see (1.4)(c)), $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}^{\circ}_U(\underline{f}) \,\dot{\cup}\, \mathrm{E}_U(\underline{f})$ and $\mathrm{Bl}_U(\underline{g}) = \mathrm{Bl}^{\circ}_U(\underline{g}) \,\dot{\cup}\, \mathrm{E}_U(\underline{g})$, it follows that $\mathrm{Bl}^{\circ}_U(\underline{f}) \setminus \big((Z \cup W) \times \mathbb{P}^1\big) = \mathrm{Bl}^{\circ}_U(\underline{g}) \setminus \big((Z \cup W) \times \mathbb{P}^1\big)$. But according to

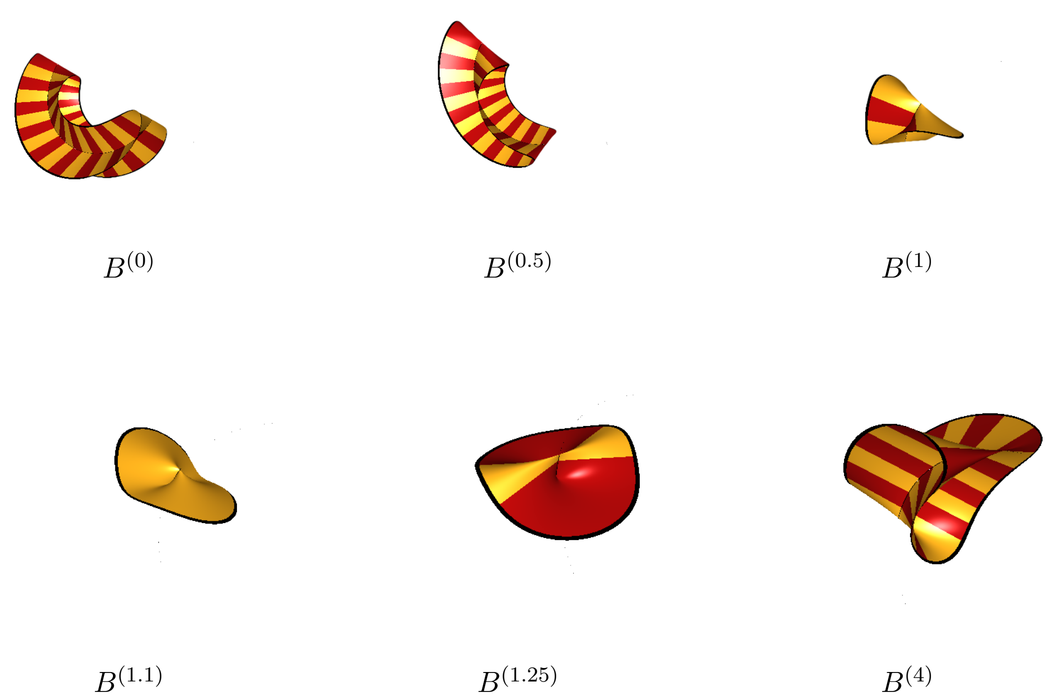


Figure 5. Deformation of a Double Whitney Umbrella

the definition of the operation $\mathrm{Bl}_U^\circ(\bullet)$ of taking open kernels (see (1.4)(b)), this means that the graphs of the two restricted maps $\varepsilon_{U,\underline{f}} \restriction, \varepsilon_{U,\underline{g}} \restriction : U \setminus (Z \cup W) \longrightarrow \mathbb{P}^1$ (and thus these restricted maps themselves) coincide. So, for all $p \in U \setminus (Z \cup W)$ it holds $(f_0(p) : f_1(p)) = (g_0(p) : g_1(p))$. Assume now, that $f_0 \neq 0$. Then, there is a dense open subset $V \subseteq U \setminus (Z \cup W)$ such that $f_0(p) \neq 0$ and $(f_0(p) : f_1(p)) = (g_0(p) : g_1(p))$ for all $p \in V$. As $V \neq \emptyset$ is open in $\mathbb{R}^2$, it follows that the two rational functions $\frac{f_1}{f_0}, \frac{g_1}{g_0} \in \mathbb{R}(\mathbf{x}, \mathbf{y})$ are defined and coincide and hence that $f_0 g_1 = f_1 g_0$. If $f_0 = 0$ we have $f_1 \neq 0$ and hence may conclude similarly.
Assume now that $f_0 g_1 = f_1 g_0$. Suppose first, that $f_0 \neq 0$. Then $g_0 = 0$ would imply $g_1 = 0$ and hence the contradiction that $\underline{g} = (0, 0)$. So, we have $g_0 \neq 0$. Therefore we find a dense open subset $V \subseteq U \setminus (Z \cup W)$ such that for all $p \in V$ we have $f_0(p), g_0(p) \neq 0$ and $(f_0(p) : f_1(p)) = (g_0(p) : g_1(p))$. This means, that the two restricted maps $\varepsilon_{U,\underline{f}} \restriction, \varepsilon_{U,\underline{g}} \restriction : V \longrightarrow \mathbb{P}^1$ coincide and hence have the same graph

$$S := \{\big(p, (f_0(p) : f_1(p)\big) = \big(p, (g_0(p) : g_1(p))\big) \mid p \in V\} \subseteq \mathrm{Bl}_U^\circ(\underline{f}) \cap \mathrm{Bl}_U^\circ(\underline{g}).$$

As V is open and dense in $U \setminus Z$, the isomorphism $\pi_{U,\underline{f}} \restriction : \mathrm{Bl}_U^\circ(\underline{f}) \xrightarrow{\cong} U \setminus Z$ yields that $S := \big(\pi_{U,\underline{f}} \restriction\big)^{-1}(V)$ is open and dense in $\mathrm{Bl}_U^\circ(\underline{f})$ with respect to the strict (e.g. metric) topology of $U \times \mathbb{P}^1$, so that S and $\mathrm{Bl}_U^\circ(\underline{f})$ have the same strict closure. The same applies for S and $\mathrm{Bl}_U^\circ(\underline{g})$. Therefore, $\mathrm{Bl}_U^\circ(\underline{f})$ and $\mathrm{Bl}_U^\circ(\underline{g})$ have the same strict closure. As the Zariski topology is coarser than the strict topology, it follows, that these two sets also have the same Zariski closure, and hence that $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(\underline{g})$ (see (1.3)(a)).
(b): Assume neither f_0 and f_1 nor g_0 and g_1 have a common divisor and that $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(\underline{g})$. By statement (a) we get $f_0 g_1 = f_1 g_0$. As $\mathbb{R}[\mathbf{x}, \mathbf{y}]$ is factorial we find some $c \in \mathbb{R} \setminus \{0\}$ such that $\underline{g} = c\underline{f}$. □

Structure of Regular Embedded Blowups. We next prove a structure result for regular blowups.

Proposition 3.2. *Let $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Then B is a smooth real algebraic hyper-surface in $U \times \mathbb{P}^1$.*

Proof. Let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be a regular pair on U with respect to Z, such that $B = \mathrm{Bl}_U(\underline{f})$. Let $h := \mathbf{z}_0 f_1(\mathbf{x}, \mathbf{y}) - \mathbf{z}_1 f_0(\mathbf{x}, \mathbf{y}) \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{z}_0, \mathbf{z}_1]$. If $(x, y) \in U \setminus Z$ and $(u : v) \in \mathbb{P}^1$ we have $h(x, y, u, v) = 0$ if and only if $\big((x, y), (u : v)\big) \in B^\circ$, so that $B^\circ = \{\big((x, y), (u : v)\big) \in (U \setminus Z) \times \mathbb{P}^1 \mid h(x, y, u, v) = 0\}$. Passing to Zariski closures we get (see (1.3)(a)) $B = \{((x, y), (u : v)) \in U \times \mathbb{P}^1 \mid h(x, y, u, v) = 0\}$. It remains to show, that

$$\big(\frac{\partial h}{\partial \mathbf{x}}(x, y, u, v), \frac{\partial h}{\partial \mathbf{y}}(x, y, u, v), \frac{\partial h}{\partial \mathbf{z}_0}(x, y, u, v), \frac{\partial h}{\partial \mathbf{z}_1}(x, y, u, v)\big) \neq \underline{0},$$

whenever $\big((x, y), (u : v)\big) \in B$. As $\frac{\partial h}{\partial \mathbf{z}_0} = f_1$ and $\frac{\partial h}{\partial \mathbf{z}_1} = -f_0$, this is clear if $p := (x, y) \notin Z$. If $p = (x, y) \in Z$, we have $\mathrm{rank}\big((\partial \underline{f})(p)\big) = 2$, and $(u, v) \neq (0, 0)$ shows that

$$\big(\frac{\partial h}{\partial \mathbf{x}}(x, y, u, v), \frac{\partial h}{\partial \mathbf{y}}(x, y, u, v)\big) = \big(u \frac{\partial f_1}{\partial \mathbf{x}}(p) - v \frac{\partial f_0}{\partial \mathbf{x}}(p), u \frac{\partial f_1}{\partial \mathbf{y}}(p) - v \frac{\partial f_0}{\partial \mathbf{y}}(p)\big) \neq \underline{0}.$$

□

Reduced and Strongly Regular Pairs and Application to Sign Distributions. The remaining part of this section is devoted to the Classification Theorem mentioned in (1.11) and hence to the solution of the Classification Problem (1.0)(b) for regular embedded blowups. We first will introduce two special types of regular pairs of polynomials.

Lemma and Definition 3.3. *Let* $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$*. Then, there is a regular pair* $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$*, with respect to* Z *on* U*, unique up to multiplication with a non-zero constant – and called a reduced regular pair for* B *– such that*

(a) f_0 *and* f_1 *have no common divisor.*
(b) $\mathrm{Bl}_U(\underline{f}) = B$.
(c) *If* $\underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ *is a regular pair with respect to* Z *on* U *with* $B = \mathrm{Bl}_U(\underline{g})$*, then there is a unique polynomial* $h \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ *such that* $\underline{g} = h\underline{f}$*. Moreover, in this situation*
  (1) $h(p) \neq 0$ *for all* $p \in U$*.*
  (2) $\mathrm{sgn}\big(\det(\partial \underline{g}(p))\big) = \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big)$ *for all* $p \in Z$*.*

Proof. By our definition (1.9) of $\mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ we may write $B = \mathrm{Bl}_U(\underline{g})$ where $(g_0, g_1) = \underline{g} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is a regular pair with respect to Z on U. Now, choose any such pair $\underline{g}$. Let $h \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ be a greatest common divisor of g_0 and g_1 and let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be such that $\underline{g} = h\underline{f}$. By Proposition 3.1 (a) we have $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(\underline{g}) = B$. The Leibniz product rule for derivatives gives

$$(@)\quad \partial \underline{g} = \partial(h\underline{f}) = h\partial \underline{f} + \begin{pmatrix} f_0 \frac{\partial h}{\partial \mathbf{x}} & f_1 \frac{\partial h}{\partial \mathbf{x}} \\ f_0 \frac{\partial h}{\partial \mathbf{y}} & f_1 \frac{\partial h}{\partial \mathbf{y}} \end{pmatrix}.$$

We claim that $h(p) \neq 0$ for all $p \in \overline{U}$. If we assume to the contrary that $h(p) = 0$ for some $p \in \overline{U}$, it would follow – by $\underline{g} = h\underline{f}$ – that $p \in Z_{\overline{U}}(\underline{g}) = Z$. But then by (@) the matrix $\partial \underline{g}(p)$ would be of rank at most 1, which contradicts the fact that $\underline{g}$ is regular with respect to Z on U. In particular we now get that $Z_{\overline{U}}(\underline{f}) = Z$. Now, another use of (@) gives that for all $p \in Z$ we have $h(p)\big(\partial \underline{f}\big)(p) = \partial \underline{g}(p)$ and hence $h(p)^2 \det\big(\partial \underline{f}\big)(p) = \det\big(\partial \underline{g}\big)(p) \neq 0$, thus $\big(\partial \underline{f}\big)(p) \neq 0$. This shows that $\underline{f}$ is a regular pair with respect to Z on U by definition. Finally, a further use of (@) shows that $\mathrm{sgn}\big(\det(\partial \underline{g}(p))\big) = \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big)$ for all $p \in Z$. As f_0 and f_1 have no common divisor, the stated uniqueness of the pair $\underline{f}$ follows by Proposition 3.1 (b). □

Definition and Remark 3.4. Let $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ and let $p \in Z$. We write $B = \mathrm{Bl}_U(\underline{g})$, where $\underline{g} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is a regular pair with respect to Z on U. Then, by Lemma and Definition 3.3 (c) (2) it is immediate, that $\mathrm{sgn}\big(\det(\partial \underline{g}(p))\big)$ depends only on the blowup B and not on the chosen defining pair $\underline{g}$. This allows to define a map (see (1.10))

$$\mathrm{sgn}_B = \mathrm{sgn}_{\underline{g}} : Z \longrightarrow \{\pm 1\} \text{ given by } p \mapsto \mathrm{sgn}_B(p) := \mathrm{sgn}\big(\det(\partial \underline{g}(p))\big) \text{ for all } p \in Z.$$

We call this map the *sign distribution* of B.

We now define a notion related to the complex affine plane. We like to do this, as in this way we get a stronger result (see Lemma 3.6).

Definition and Remark 3.5. (A) Let $Z = \{p_i = (x_i, y_i) \mid i = 1, 2, \dots, n\} \subset U$, ($p_i \neq p_j$ for all $i \neq j$). A pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is called *strongly regular with respect to Z (on U)*, if it satisfies the following equivalent requirements:

(i) $\mathbb{C}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{C}[\mathbf{x}, \mathbf{y}]f_1 = \bigcap_{i=1}^n \big(\mathbb{C}[\mathbf{x}, \mathbf{y}](\mathbf{x} - x_i) + \mathbb{C}[\mathbf{x}, \mathbf{y}](\mathbf{y} - y_i)\big)$.
(ii) $\mathbb{C}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{C}[\mathbf{x}, \mathbf{y}]f_1 = I_{\mathbb{A}^2_{\mathbb{C}}}(Z) := \{f \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \mid f(p) = 0, \forall p \in Z\}$.

(B) Assume that $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is a strongly regular pair with respect to Z. Then, it is easy to see:

(a) $\underline{f}$ is a regular pair with respect to Z on U in the sense of (1.14).
(b) $\underline{f}$ is a reduced regular pair for $B := \mathrm{Bl}_U(\underline{f})$ in the sense of Lemma and Definition 3.3.
(c) $\mathbb{R}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{R}[\mathbf{x}, \mathbf{y}]f_1 = I_{\mathbb{A}^2_{\mathbb{R}}}(Z) := \{g \in \mathbb{R}[\mathbf{x}, \mathbf{y}] \mid g(p) = 0, \forall p \in Z\}$.

Lemma 3.6. *Let $n > 0$, let $Z := \{p_1, p_2, \dots, p_n\}$ a set of pairwise different points with $p_i := (x_i, y_i) \in U$ for $i = 1, 2, \dots, n$. Let $\chi : Z \longrightarrow \mathbb{R} \setminus \{0\}$ be a map. Then, there is a strongly regular pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z such that $\det\big(\partial \underline{f}(p)\big) = \chi(p)$ for all $p \in Z$.*

Proof. We shall give the proof under the assumption that $x_i \neq x_j$ for all $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$. Should this requirement not be satisfied, we first subject $\mathbb{R}^2$ to a general transformation $T \in \mathrm{SL}_2(\mathbb{R})$, such that our requirement is satisfied – and keep in mind that this does not affect our Jacobian determinants. Then, we perform our proof as below and finally apply the transformation T^{-1}. We set

$$f_0 := \prod_{i=1}^n (\mathbf{x} - x_i) \in \mathbb{R}[\mathbf{x}] \text{ and } f_1 := h(\mathbf{x})\big(\mathbf{y} - g(\mathbf{x})\big),$$

where $g(\mathbf{x}), h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ are the uniquely determined polynomials of degree $\leq n-1$ which respectively satisfy

$$g(x_i) = y_i \text{ and } h(x_i) = \frac{\chi(p_i)}{\prod_{j \in \{1,\dots,n\} \setminus \{i\}} (x_i - x_j)} \text{ for all } i = 1, 2, \dots, n.$$

Observe also, that

$$\frac{\partial f_0}{\partial \mathbf{x}}(p_i) = \prod_{j \in \{1,\dots,n\} \setminus \{i\}} (x_i - x_j) \text{ and } \frac{\partial f_1}{\partial \mathbf{y}}(p_i) = h(x_i) \text{ for all } i = 1, 2, \dots, n.$$

Now, for all $i = 1, 2, \dots, n$ we obtain:

$$\det\big(\partial \underline{f}(p_i)\big) = \det\begin{pmatrix} \frac{\partial f_0}{\partial \mathbf{x}}(p_i) & \frac{\partial f_1}{\partial \mathbf{x}}(p_i) \\ \\ \frac{\partial f_0}{\partial \mathbf{y}}(p_i) & \frac{\partial f_1}{\partial \mathbf{y}}(p_i) \end{pmatrix} =$$

$$= \det\begin{pmatrix} \prod_{j\neq i}(x_i - x_j) & \frac{\partial\big(h(\mathbf{x})(\mathbf{y}-g(\mathbf{x}))\big)}{\partial \mathbf{x}}(p_i) \\ \\ 0 & h(x_i) \end{pmatrix} = \chi(p_i).$$

Therefore $\det\big(\partial \underline{f}(p_i)\big) = \chi(p_i)$ for all $i = 1, 2, \dots, n$.
It is immediate to see, that $Z = \{p_1, p_2, \dots, p_n\}$ is precisely the set $Z_{\mathbb{C}^2}(\underline{f})$ of common zeros of the two polynomials $f_0, f_1 \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ in $\mathbb{C}^2$. As $\det\big(\partial \underline{f}(p_i)\big) = \chi(p_i) \neq 0$ for all $i \in \{1, 2 \dots, n\}$ it follows by the Jacobian Criterion, that $\mathbb{C}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{C}[\mathbf{x}, \mathbf{y}]f_1$ is reduced and hence is the vanishing ideal $I_{\mathbb{A}^2_{\mathbb{C}}}(Z)$ of Z in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. So $\underline{f}$ is strongly regular with respect to Z on U. □

The Classification Result. Now we will establish the Isomorphy Criterion we are heading for in this section, and hence solve the Classification Problem mentioned under (1.0) (b). We first shall prove two auxiliary results whose proofs are straight forward. As both of them are crucial for the proof of our Classification Theorem, we include their proofs for the reader's convenience.

Lemma 3.7. *Let $\underline{f} = (f_0, f_1), \underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be two pairs such that $\mathrm{Z}_U(\underline{f}) = \mathrm{Z}_U(\underline{g}) = Z$. Assume that there exists a matrix $N \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ such that $\underline{g} = \underline{f}N$. Moreover, for each $\gamma \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ we set*

$$N_\gamma := N + \gamma\begin{pmatrix} g_1 f_1 & -g_0 f_1 \\ -g_1 f_0 & g_0 f_0 \end{pmatrix}.$$

Then, it holds

(a) *$N_\gamma(p) = N(p)$ for all $p \in Z$.*
(b) *$\underline{g} = \underline{f}N_\gamma$.*
(c) *$\det(N_\gamma) = \det(N) + \gamma\big(g_0^2 + g_1^2\big)$.*
(d) *If $\det\big(N(p)\big) > 0$ for all $p \in Z$, then, there is some $b \in \mathbb{R}_{>0}$ such that $\det\big(N_\gamma(p)\big) > 0$ for all $p \in U$ and all $\gamma \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ with $\inf\{\gamma(p) \mid p \in U\} > b$.*

Proof. Statements (a) and (b) are immediate. To prove statement (c) we write

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

On use of the column bi-linearity of the determinant and as

$$\det\begin{pmatrix} f_1 & N_{12} \\ -f_0 & N_{22} \end{pmatrix} = g_1 \text{ and } \det\begin{pmatrix} N_{11} & -f_1 \\ N_{21} & f_0 \end{pmatrix} = g_0,$$

we get indeed

$$\begin{aligned}\det(N_\gamma) &= \det\bigl(N+\gamma\begin{pmatrix} g_1f_1 & -g_0f_1\\ -g_1f_0 & g_0f_0\end{pmatrix}\bigr) = \det\begin{pmatrix} N_{11}+\gamma g_1f_1 & N_{12}-\gamma g_0f_1\\ N_{21}-\gamma g_1f_0 & N_{22}+\gamma g_0f_0\end{pmatrix} =\\ &= \det\begin{pmatrix} N_{11} & N_{12}\\ N_{21} & N_{22}\end{pmatrix} + \det\begin{pmatrix} \gamma g_1f_1 & N_{12}\\ -\gamma g_1f_0 & N_{22}\end{pmatrix} +\\ &+ \det\begin{pmatrix} N_{11} & -\gamma g_0f_1\\ N_{21} & \gamma g_0f_0\end{pmatrix} + \det\begin{pmatrix} \gamma g_1f_1 & -\gamma g_0f_1\\ -\gamma g_1f_0 & \gamma g_0f_0\end{pmatrix} =\\ &= \det(N) + \gamma g_1\det\begin{pmatrix} f_1 & N_{12}\\ -f_0 & N_{22}\end{pmatrix} + \gamma g_0\det\begin{pmatrix} N_{11} & -f_1\\ N_{21} & f_0\end{pmatrix} + 0 =\\ &= \det(N) + \gamma g_1^2 + \gamma g_0^2 = \det(N) + \gamma\bigl(g_0^2+g_1^2\bigr).\end{aligned}$$

It remains to show statement (d). So, assume that $\det\bigl(N(p)\bigr) > 0$ for all $p \in Z$. We have to show that there is some constant $b \in \mathbb{R}_{>0}$ such that $\det\bigl(N_\gamma(p)\bigr) > 0$ for all $p \in U$ and all constants $\gamma > b$. As $\det\bigl(N(p)\bigr) > 0$ for all $p \in Z$, there is some open set $W \subset U$ such that $Z \subset W$ and $\det\bigl(N(p)\bigr) > 0$ for all $p \in W$. It follows by statement (a) and (c) that

$$\det\bigl(N_\gamma(p)\bigr) > 0 \text{ for all } p \in W \text{ and all } \gamma > 0.$$

As U is bounded and $\mathrm{Z}_{\mathbb{R}^2}(\underline{g})$ does not contain any points of the boundary of U it follows that there is some $c > 0$ such that $g_0(p)^2 + g_1(p)^2 > c$ for all $p \in U \setminus W$. As U is bounded, there is some $C > 0$ such that $\det\bigl(N(p)\bigr) \geq -C$ for all $p \in U$. If $\gamma > b := \frac{C}{c}$ it follows that

$$\det\bigl(N_\gamma(p)\bigr) \geq \det\bigl(N(p)\bigr) + b\bigl(g_0(p)^2 + g_1(p)^2\bigr) > 0 \text{ for all } p \in U \setminus W,$$

and hence $\det\bigl(N_\gamma(p)\bigr) > 0$ for all $p \in U$. □

Lemma 3.8. *Let* $\underline{f} = (f_0, f_1), \underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ *be two pairs of polynomials such that* $\underline{f}$ *is strongly regular with respect* Z *and* $\underline{g}$ *is regular with respect to* Z *on* U *and consider the two blowups* $B := \mathrm{Bl}_U(\underline{f}), C := \mathrm{Bl}_U(\underline{g}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. *Then, the following statements are equivalent:*

(i) $\mathrm{sgn}_C = \mathrm{sgn}_B$.
(ii) *There is a matrix* $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ *such that* $\det\bigl(M(p)\bigr) > 0$ *for all* $p \in U$ *and* $\underline{g} = \underline{f}M$.

Proof. (i) ⇒ (ii): Assume that statement (i) holds. As $g_0, g_1 \in I_{\mathbb{A}^2_{\mathbb{R}}}(Z)$, it follows by Definition and Remark 3.5(B)(c), that there is a matrix

$$N = \begin{pmatrix} N_{11} & N_{12}\\ N_{21} & N_{22}\end{pmatrix} = \begin{pmatrix} N_{1\bullet}\\ N_{2\bullet}\end{pmatrix} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2} \text{ with } \underline{g} = \underline{f}N.$$

By our assumption we have $\mathrm{sgn}\bigl(\det(\partial\underline{g}(p))\bigr) = \mathrm{sgn}_C(p) = \mathrm{sgn}_B(p) = \mathrm{sgn}\bigl(\det(\partial\underline{f}(p))\bigr)$ for all $p \in Z$. Moreover, by the Leibniz product rule for derivatives we have

(@@) $$\partial\underline{g} = \partial(\underline{f}N) = \partial\underline{f}\cdot N + f_0\cdot\partial N_{1\bullet} + f_1\cdot\partial N_{2\bullet}.$$

As $\underline{f}(Z) = 0$ it follows that $\det(\partial \underline{g}(p)) = \det(\partial \underline{f}(p)) \cdot \det\big(N(p)\big)$ and hence $\det\big(N(p)\big) > 0$ for all $p \in Z$. Now, by Lemma 3.7 (c), there is some $\gamma \in \mathbb{R}_{>0}$ such that the matrix $M := N_\gamma \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ satisfies $\det\big(M(p)\big) > 0$ for all $p \in U$. Moreover, Lemma 3.7 (b) yields that $\underline{g} = \underline{f}M$.
(ii) $\Rightarrow$ (i): Assume that statement (ii) holds. The Leibniz product rule for derivatives (see formula (@@) above) and the fact that $f_0(p) = f_1(p) = 0$ for all $p \in Z$ give

$$\partial \underline{g}(p) = \partial(\underline{f}M)(p) = \partial \underline{f}(p) \cdot M(p) \text{ for all } p \in Z.$$

Taking determinants and observing that $\det\big(M(p)\big) > 0$ for all $p \in Z$ we get statement (i). □

Now, we are ready to prove the main result of this section (cf. (1.11)).

Theorem 3.9. *(Classification of Regular Embedded Blowups)*

(a) *For each function $\sigma : Z \longrightarrow \{+1, -1\}$ there is a regular embedded blowup $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ such that $\mathrm{sgn}_B = \sigma$.*
(b) *Let $B, C \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Then B and C are relatively oriented embedded isomorphic if and only if they have the same sign distribution. Hence, for short: $B \cong C$ if and only if $\mathrm{sgn}_B = \mathrm{sgn}_C$.*
(c) *There are precisely $2^{\#Z}$ isomorphism types of regular embedded blowups of U along Z.*

Proof. (a): By Lemma 3.6 there is a strongly regular pair $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z such that $\det\big(\partial \underline{f}(p)\big) = \sigma(p)$ for all $p \in Z$. It suffices to choose $B = \mathrm{Bl}_U(\underline{f})$.

(b): We may write $B = \mathrm{Bl}_U(\underline{g})$, where $\underline{g} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ is a regular pair of polynomials with respect to Z on U.
Assume first that B and C are oriented embedded isomorphic, more precisely, that $C = \varphi_M(B)$ for some automorphism $\varphi_M : U \times \mathbb{P}^1 \longrightarrow U \times \mathbb{P}^1$ with $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ and $\det\big(M(p)\big) > 0$ for all $p \in U$. Then we may write $C = \mathrm{Bl}_U(\underline{g}M)$. By the product rule for derivatives (see (@@), Proof of Lemma 3.8), as $\underline{g}(Z) = 0$ and as $\det(M(p)) > 0$ for all $p \in U$, we now obtain

$$\begin{aligned}\mathrm{sgn}_C(p) &= \mathrm{sgn}\big(\det[\partial(\underline{g}M)(p)]\big) = \mathrm{sgn}\big(\det[(\partial \underline{g})(p)M(p)]\big) = \\ &= \mathrm{sgn}\big(\det[\partial \underline{g}(p)]\det[M(p)]\big) = \mathrm{sgn}\big(\det[\partial \underline{g}(p)]\big) = \\ &= \mathrm{sgn}_B(p) \text{ for all } p \in Z.\end{aligned}$$

It follows that indeed $\mathrm{sgn}_C = \mathrm{sgn}_B$.
Assume conversely, that $\mathrm{sgn}_C = \mathrm{sgn}_B$. By Lemma 3.6 there is a strongly regular pair $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z on U such that $\det\big(\partial \underline{f}(p)\big) = \mathrm{sgn}_B(p) = \mathrm{sgn}_C(p)$ for all $p \in Z$. By Lemma 3.8 there is a matrix $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ for all $p \in U$ and $\underline{g} = \underline{f}M$. But this means, that $D := \mathrm{Bl}_U(\underline{f}) \cong B$. Similarly we see, that $D \cong C$. So B and C are embedded isomorphic.

(c): This is clear by statements (a) and (b). □

Remark 3.10. A (more complicated) proof of the Classification Theorem 3.9 – based on the ideas of the first named author – has been worked out in the Master thesis of S. Koller [7], but remained unpublished yet.

4. Deformation of Matrices and Isotopies of Embedded Blowups

Analytic Matrix Deformations. In this section, we approach the Deformation Problem (1.0)(a) mentioned in the introduction. We shall prove the Deformation Result (1.13). As already mentioned in the introduction, this means that we have to prove the result on polynomial deformations of matrices mentioned in (1.14). We first prove a result on real analytic deformation of matrices.

Notation and Remark 4.1. (A) Let $\mathcal{C}^\omega(U)$ denote the ring of real analytic functions on U. We choose a matrix

$$M = (M_{\bullet 1}\ M_{\bullet 2}) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathcal{C}^\omega(U)^{2\times 2} \text{ with } \det\big(M(p)\big) > 0 \text{ for all } p \in U,$$

where

$$M_{\bullet 1} := \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \text{ and } M_{\bullet 2} := \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix}$$

denote the column vectors of M.

Let $p, q \in U$ and let $\sigma : [0,1] \longrightarrow U$ be a smooth path with $\sigma(0) = p$ and $\sigma(1) = q$. As U is pathwise simply connected – by monodromy – the two values below (which are the total angles the vectors $\frac{M_{\bullet 1}}{\|M_{\bullet 1}\|}\big(\sigma(t)\big)$ and $\frac{M_{\bullet 2}}{\|M_{\bullet 2}\|}\big(\sigma(t)\big)$ respectively wander through if t runs from 0 to 1)

$$\alpha_M(p,q) = \int_0^1 \frac{M_{\bullet 1}}{\|M_{\bullet 1}\|}\big(\sigma(t)\big) \wedge \frac{d}{dt}\Big[\frac{M_{\bullet 1}}{\|M_{\bullet 1}\|}\big(\sigma(t)\big)\Big]dt,$$

$$\beta_M(p,q) = \int_0^1 \frac{M_{\bullet 2}}{\|M_{\bullet 2}\|}\big(\sigma(t)\big) \wedge \frac{d}{dt}\Big[\frac{M_{\bullet 2}}{\|M_{\bullet 2}\|}\big(\sigma(t)\big)\Big]dt$$

depend only (analytically) on p and q and not on their connecting path σ. Now, we fix a point $p_0 \in U$. Then, there are uniquely determined functions $\alpha_M, \beta_M : U \longrightarrow \mathbb{R}$ such that

$$0 \le \alpha_M(p_0), \beta_M(p_0) \le 2\pi,$$

$$\alpha_M(p) = \alpha_M(p_0) + \alpha_M(p_0,p) \text{ and } \beta_M(p) = \beta_M(p_0) + \beta_M(p_0,p) \text{ for all } p \in U$$

and

$$M_{\bullet 1}(p) = \|M_{\bullet 1}(p)\| \begin{pmatrix} \cos(\alpha_M(p)) \\ \sin(\alpha_M(p)) \end{pmatrix},\ M_{\bullet 2}(p) = \|M_{\bullet 2}(p)\| \begin{pmatrix} \cos(\beta_M(p)) \\ \sin(\beta_M(p)) \end{pmatrix}, \text{ for all } p \in U.$$

Observe, that in particular

$$\det\big(M(p)\big) = \|M_{\bullet 1}(p)\| \cdot \|M_{\bullet 2}(p)\| \cdot \sin\big(\beta_M(p) - \alpha_M(p)\big) > 0 \text{ for all } p \in U.$$

Now, by continuity, and as $\alpha_M(p,q)$ and $\beta_M(p,q)$ depend analytically on p and q, it follows that

(a) $0 < \beta_M(p) - \alpha_M(p) < \pi$ for all $p \in U$;

(b) $\alpha_M, \beta_M \in \mathcal{C}^\omega(U)$.

(B) Keep the notations and hypotheses of part (A). For each $t \in [0,1]$ and each $p \in U$ we set

$$\begin{aligned}
M_{11}^{(t)}(p) &:= \big[(1-t) + t\|M_{\bullet 1}\|\big] \cdot \cos\big(t\alpha_M(p)\big),\\
M_{21}^{(t)}(p) &:= \big[(1-t) + t\|M_{\bullet 1}\|\big] \cdot \sin\big(t\alpha_M(p)\big),\\
M_{12}^{(t)}(p) &:= \big[(1-t) + t\|M_{\bullet 2}\|\big] \cdot \cos\big((1-t)\frac{\pi}{2} + t\beta_M(p)\big),\\
M_{22}^{(t)}(p) &:= \big[(1-t) + t\|M_{\bullet 2}\|\big] \cdot \sin\big((1-t)\frac{\pi}{2} + t\beta_M(p)\big),
\end{aligned}$$

and consider the matrices

$$M^{(t)} := \begin{pmatrix} M_{11}^{(t)} & M_{12}^{(t)} \\ M_{21}^{(t)} & M_{22}^{(t)} \end{pmatrix} \in \mathcal{C}(U)^{2\times 2},\ \big(t \in [0,1]\big).$$

For all $t \in [0,1]$ and all $p \in U$ we obtain:

$$\det\big(M^{(t)}(p)\big) =$$
$$= \big[(1-t) + t\|M_{\bullet 1}(p)\|\big] \cdot \big[(1-t) + t\|M_{\bullet 2}(p)\|\big] \cdot \sin\big((1-t)\frac{\pi}{2} + t[\beta_M(p) - \alpha_M(p)]\big).$$

Moreover, $0 < \beta_M(p) - \alpha_M(p) < \pi$ (see statement (a) of Part (A)) implies

$$0 < (1-t)\frac{\pi}{2} + t[\beta_M(p) - \alpha_M(p)] < (1-t)\frac{\pi}{2} + t\pi = \frac{\pi}{2} + t\frac{\pi}{2} \le \pi.$$

So, in view of statement (b) of part (A) we can say:

(a) $M^{(t)} \in \mathcal{C}^\omega(U)^{2\times 2}$ and $\det\big(M^{(t)}(p)\big) > 0$ for all $t \in [0,1]$ and all $p \in U$.

Now, we solve our deformation problem for matrices with analytic entries.

Proposition 4.2. *Let $M \in \mathcal{C}^\omega(U)^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ for all $p \in U$. Then the family $\big(M^{(t)}\big)_{0\le t\le 1}$ of Notation and Remark 4.1 is an analytic family of matrices in $\mathcal{C}^\omega(U)^{2\times 2}$, with positive determinant on U, which connects the unit matrix $\mathbf{1}^{2\times 2}$ with the matrix M. More precisely,*

(a) $M^{(t)} \in \mathcal{C}^\omega(U)^{2\times 2}$ and $\det\big(M^{(t)}(p)\big) > 0$ for all $t \in [0,1]$ and all $p \in U$.

(b) $M^{(0)} = \mathbf{1}^{2\times 2}$ *and* $M^{(1)} = M$.

(c) *The map $\widetilde{M} : U \times [0,1] \longrightarrow \mathbb{R}^{2\times 2}$, given by $(p,t) \mapsto M^{(t)}(p)$, is continuous and analytic on the open set $U \times]0,1[$.*

Proof. (a): This is immediate by Notation and Remark 4.1 (B)(a).

(b): This is obvious by the definition of the Matrices $M^{(t)}$.

(c): This follows easily from the definition of the functions $p \mapsto M_{ij}^{(t)}(p)$ (see Notation and Remark 4.1 (B)) and statement (b) of Notation and Remark 4.1 (A). □

Polynomial and Rational Matrix Deformations. We now attack the case of polynomial or rational matrix deformations. We begin with the following auxiliary result.

Lemma 4.3. *Let $K \subset \mathbb{R}^2$ be a non-empty compact set. Let $P, Q \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ be two polynomials and let $F : K \times [0,1] \longrightarrow \mathbb{R}$ be a continuous function such that $F(p,0) = P(p)$ and $F(p,1) = Q(p)$ for all $p \in K$. Let $\varepsilon > 0$. Then, there is a polynomial $\widetilde{P} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]$ such that*

(a) $|F(p,t) - \widetilde{P}(p,t)| < \varepsilon$ *for all* $p \in K$ *and all* $t \in [0,1]$.
(b) $P(p) = \widetilde{P}(p,0)$ *and* $Q(p) = \widetilde{P}(p,1)$ *for all* $p \in K$.

Proof. By the Theorem of Stone-Weierstrass (see [4] (7.4.1)) there is a polynomial $\overline{P} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]$ such that

$$|F(p,t) - \overline{P}(p,t)| < \frac{\varepsilon}{2} \text{ for all } p \in K \text{ and all } t \in [0,1].$$

Now, set

$$\widetilde{P}(\mathbf{x}, \mathbf{y}, \mathbf{t}) := \overline{P}(\mathbf{x}, \mathbf{y}, \mathbf{t}) + (1 - \mathbf{t})\big(P(\mathbf{x}, \mathbf{y}) - \overline{P}(\mathbf{x}, \mathbf{y}, 0)\big) + \mathbf{t}\big(Q(\mathbf{x}, \mathbf{y}) - \overline{P}(\mathbf{x}, \mathbf{y}, 1)\big).$$

It is easy to see that $\widetilde{P}$ has the requested properties. □

Proposition 4.4. *Let $M, N \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ and $\det\big(N(p)\big) > 0$ for all $p \in \overline{U}$. Then, the matrix N is connected on $\overline{U}$ to M by a polynomial family of polynomial 2×2-matrices with positive determinant on $\overline{U}$. More precisely:*
There is a matrix

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2}$$

such that with $P^{(t)}(\mathbf{x}, \mathbf{y}) := \widetilde{P}(\mathbf{x}, \mathbf{y}, t)$ (for $t \in \mathbb{R}$) we have:

(a) $P^{(0)}(p) = N(p)$ *for all* $p \in \overline{U}$.
(b) $P^{(1)}(p) = M(p)$ *for all* $p \in \overline{U}$.
(c) $\det\big(P^{(t)}(p)\big) > 0$ *for all* $p \in \overline{U}$ *and all* $t \in [0,1]$.

Proof. Observe that the closed set

$$\mathbb{S} := \{p \in \mathbb{R}^2 \mid \det\big(M(p)\big) \leq 0 \ \text{ or } \det\big(N(p)\big) \leq 0\}$$

is disjoint to $\overline{U}$. We thus find a bounded open star-shaped set W such that $\overline{U} \subset W$ and $W \cap \mathbb{S} = \emptyset$. Now, clearly $M, N \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big), \det\big(N(p)\big) > 0$ for all $p \in W$. According to Proposition 4.2 we have two continuous maps

$$\widetilde{M} = \begin{pmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} : W \times [0,1] \longrightarrow \mathbb{R}^{2\times 2} \text{ with } \det\big(\widetilde{M}(p,t)\big) > 0, \text{ for all } (p,t) \in W \times [0,1],$$

$$\widetilde{N} = \begin{pmatrix} \widetilde{N}_{11} & \widetilde{N}_{12} \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{pmatrix} : W \times [0,1] \longrightarrow \mathbb{R}^{2\times 2} \text{ with } \det\big(\widetilde{N}(p,t)\big) > 0, \text{ for all } (p,t) \in W \times [0,1],$$

such that

$$\widetilde{M}(p,0) = \mathbf{1}^{2\times 2}, \text{ and } \widetilde{M}(p,1) = M(p), \text{ for all } p \in W,$$
$$\widetilde{N}(p,0) = \mathbf{1}^{2\times 2}, \text{ and } \widetilde{N}(p,1) = N(p), \text{ for all } p \in W.$$

Now, for all $i,j \in \{1,2\}$ we consider the continuous functions

$$\widetilde{F}_{ij} : W \times [0,1] \longrightarrow \mathbb{R}; \quad \widetilde{F}_{i,j}(p,t) := \begin{cases} \widetilde{N}_{ij}(p, 1-2t) & \text{if } t \in [0, \frac{1}{2}] \\ \widetilde{M}_{ij}(p, 2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and the matrix (here $\mathcal{C}(\bullet)$ denotes the ring of continuous functions)

$$\widetilde{F} := \begin{pmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ \widetilde{F}_{21} & \widetilde{F}_{22} \end{pmatrix} \in \mathcal{C}(W \times [0,1])^{2\times 2}.$$

Then $\widetilde{F}(p,0) = N(p), \widetilde{F}(p,1) = M(p)$ and $\det\big(\widetilde{F}(p,t)\big) > 0$ for all $p \in W$ and all $t \in [0,1]$. As $\overline{U} \subset W$ is compact, there are $c, \delta > 0$ such that for all $i,j \in \{1,2\}$, all $p \in \overline{U}$ and all $t \in [0,1]$ it holds

$$-c \leq \widetilde{F}_{ij}(p,t) \leq c \ \text{ and } \ \det\big(\widetilde{F}(p,t)\big) > \delta.$$

As the map $\det : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}$ is uniformly continuous on any compact subset of $\mathbb{R}^4$ we find some $\varepsilon > 0$ such that:

(1) $|\det\big(\widetilde{F}(p,t)\big) - \det\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}| < \frac{\delta}{2}$ for all $p \in \overline{U}$, all $t \in [0,1]$ and all $m_{ij} \in \mathbb{R}$ with $|m_{ij} - \widetilde{F}_{ij}(p,t)| < \varepsilon \quad (i,j \in \{1,2\})$.

Now, we apply Lemma 4.3 to the four continuous functions $\widetilde{F}_{ij} : \overline{U} \times [0,1] \longrightarrow \mathbb{R}$ and obtain four polynomials $\widetilde{P}_{ij} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]$, such that for all $i,j \in \{1,2\}$ we have:

(2) $|\widetilde{F}_{ij}(p,t) - \widetilde{P}_{ij}(p,t)| < \varepsilon$ for all $p \in \overline{U}$ and all $t \in [0,1]$,
(3) $N_{ij}(p) = \widetilde{F}_{ij}(p,0) = \widetilde{P}_{ij}(p,0)$ for all $p \in \overline{U}$ and
(4) $M_{ij}(p) = \widetilde{F}_{ij}(p,1) = \widetilde{P}_{ij}(p,1)$ for all $p \in \overline{U}$.

We set

$$\widetilde{P} := \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{pmatrix}.$$

Then, the above statements (1) and (2) yield that

$$|\det\big(\widetilde{F}(p,t)\big) - \det\big(\widetilde{P}(p,t)\big)| < \frac{\delta}{2} \text{ for all } p \in \overline{U} \text{ and all } t \in [0,1],$$

so that with $P^{(t)}(p) := \widetilde{P}(p,t)$ and (because also $\det(\tilde{F}(p,t)) > \delta$) we get

$$\det\big(P^{(t)}(p)\big) = \det\big(\widetilde{P}(p,t)\big) > \frac{\delta}{2} > 0 \text{ for all } p \in \overline{U} \text{ and all } t \in [0,1].$$

By the above statements (3) and (4) we obtain

$$P^{(0)}(p) = \widetilde{P}(p,0) = N(p) \text{ and } P^{(1)}(p) = \widetilde{P}(p,1) = M(p) \text{ for all } p \in \overline{U}.$$

Altogether, this proves our claim. □

Remark 4.5. As an immediate consequence we now get the result announced in the introduction under (1.14).

Remark 4.6. As early as 2002, the first named author did ask for the existence of a connecting family $(M^{(t)})_{t\in[0,1]}$ as in Proposition 4.4 – but only continuous, not polynomial – at the occasion of a talk he gave at the IIT Bombay. A few weeks after this, A.R. Shastri [12] suggested a proof for the existence of a piecewise linear connecting family $(M^{(t)})_{t\in[0,1]}$. The authors are grateful to him for his hint. Clearly, instead of Proposition 4.2 one also could use Shastri's result to prove Proposition 4.4.

As an easy consequence of the above proposition we now get:

Corollary 4.7. *Let* $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{R}(\mathbf{x},\mathbf{y})^{2\times 2}$ *be such that none of its entries* $M_{ij}, (i,j \in \{1,2\})$ *has a pole in* $\overline{U}$, *and such that* $\det\big(M(p)\big) > 0$ *for all* $p \in \overline{U}$.
Then, the unit matrix $\mathbf{1}^{2\times 2}$ *is connected over* $\overline{U}$ *to* M *by a rational family of* 2×2*-matrices which are defined and of positive determinant on* $\overline{U}$. *More precisely:*
There is a matrix

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & \widetilde{Q}_{22} \end{pmatrix} \in \mathbb{R}(\mathbf{x},\mathbf{y},\mathbf{t})^{2\times 2}$$

such that no $\widetilde{Q}_{ij}$ *has a pole on* $\overline{U}$ *and such that, with* $Q^{(t)}(\mathbf{x},\mathbf{y}) := \widetilde{Q}(\mathbf{x},\mathbf{y},t)$ *(for* $t \in \mathbb{R}$*):*

(a) $Q^{(0)} = \mathbf{1}^{2\times 2}$.
(b) $Q^{(1)}(p) = M(p)$ *for all* $p \in \overline{U}$.
(c) $\det\big(Q^{(t)}(p)\big) > 0$ *for all* $p \in \overline{U}$ *and all* $t \in [0,1]$.

Proof. The closed set

$$\mathcal{P} := \bigcup_{1\le i,j\le 2} \mathrm{Pole}(M_{ij}) \cup \{p \in \mathbb{R}^2 \mid \det\big(M(p)\big) \le 0\}$$

is disjoint to $\overline{U}$. We thus find a bounded open star-shaped set W such that $\overline{U} \subset W$ and $W \cap \mathcal{P} = \emptyset$. So, none of the four entries M_{ij} of M has a pole in W and moreover $\det\big(M(p)\big) > 0$ for all $p \in W$. As W is path-wise connected and by taking common denominators we find

$$H \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2} \text{ and } \ G \in \mathbb{R}[\mathbf{x},\mathbf{y}] \text{ with } G(p) > 0 \text{ and } M(p) = \frac{H(p)}{G(p)} \text{ for all } p \in W.$$

In particular we have $\det\big(G(p)\mathbf{1}^{2\times 2}\big) > 0$ and $\det\big(H(p)\big) > 0$ for all $p \in W$, hence for all $p \in \overline{U}$. By Proposition 4.4 there is a matrix $\widetilde{P} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$ such that

(1) $\widetilde{P}(p,0) = G(p)\mathbf{1}^{2\times 2}$ for all $p \in \overline{U}$;
(2) $\widetilde{P}(p,1) = H(p)$ for all $p \in \overline{U}$;
(3) $\det\big(\widetilde{P}(p,t)\big) > 0$ for all $p \in \overline{U}$ and all $t \in [0,1]$.

Now, with $\widetilde{Q} := \frac{\widetilde{P}}{G}$ we get our claim. □

Isotopies of Embedded Blowups. As an application of Proposition 4.4 we now prove the result on the deformation of regular embedded blowups by means of isotopies mentioned in (1.13).

Theorem 4.8. *Let $B, C \in \mathfrak{Bl}_U(Z)$ be such that B and C are relatively oriented embedded isomorphic. Then, B and C are connected by an isotopy of $U \times \mathbb{P}^1$-automorphisms. More precisely, there is a matrix*

$$\widetilde{M} = \begin{pmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2}$$

such that with $M^{(t)}(\mathbf{x}, \mathbf{y}) := \widetilde{M}(\mathbf{x}, \mathbf{y}, t)\big($ for $t \in \mathbb{R})$ we have:

(a) $\det\big(M^{(t)}(p)\big) > 0$ *for all $p \in U$ and all $t \in [0,1]$ – and hence $\varphi^{(t)} := \varphi_{M^{(t)}}$ is a relative oriented automorphism of $U \times \mathbb{P}^1$ for all $t \in [0,1]$.*

(b) $\varphi^{(0)}(B) = B$ *and* $\varphi^{(1)}(B) = C$.

Proof. Let $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be such that $Z_{\overline{U}}(\underline{f}) = Z$. As $B \cong C$, we find some matrix $N \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ with $\det\big(N(p)\big) > 0$ for all $p \in U$ and such that, with $(g_0, g_1) = \underline{g} := \underline{f}N$, it holds $C = \mathrm{Bl}_U(\underline{g})$ (see (1.7)). Now, we choose $\gamma \in \mathbb{R}_{>0}$ and consider the matrix

$$M := N_\gamma = N + \gamma \begin{pmatrix} g_1 f_1 & -g_0 f_1 \\ -g_1 f_0 & g_0 f_0 \end{pmatrix}$$

of Lemma 3.7. Then, by statements (b), (c) and (d) of that Lemma and as g_0 and g_1 have no common zero on the boundary of U, it follows that for γ large enough we have $\det\big(M(p)\big) > 0$ for all $p \in \overline{U}$ and $\underline{g} = \underline{f}M$.

But now Proposition 4.4 yields that there is a matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2}$ such that, with $M^{(t)}(\mathbf{x}, \mathbf{y}) := \widetilde{M}(\mathbf{x}, \mathbf{y}, t)$, it holds

(1) $M^{(0)}(p) = \mathbf{1}^{2\times 2}$ for all $p \in \overline{U}$;

(2) $M^{(1)}(p) = M(p)$ for all $p \in \overline{U}$;

(3) $\det\big(M^{(t)}(p)\big) > 0$ for all $p \in \overline{U}$ and all $t \in [0,1]$.

In particular, we get the stated existence of the matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2}$ such that statement (a) holds.

As $\varphi^{(0)}(B) = \varphi_{M^{(0)}}(B) = \varphi_{\mathbf{1}^{2\times 2}}(B) = \mathrm{id}_{U\times\mathbb{P}^1}(B) = B$ and $C = \mathrm{Bl}_U(\underline{f}M) = \mathrm{Bl}_U(\underline{f}M^{(1)}) = \varphi_{M^{(1)}}\big(\mathrm{Bl}_U(\underline{f})\big) = \varphi_{M^{(1)}}(B) = \varphi^{(1)}(B)$, we get statement (b). □

5. Further Examples of Families of Blowups

Two Families of Regular Two-point Blowups. Already in Example 2.1 and Example 2.2 we have presented deformations of regular blowups by means of a particularly simple matrix deformation. We begin the present section with slightly more involved matrix deformations and we shall illustrate their effect on two non-isomorphic regular embedded two-point blowups. We fix our settings as in the examples given in the introduction and in Section 2 by choosing $\rho = 2, r = 4, U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$.

Example 5.1. (A) We fix a polynomial $a = a(\mathbf{x}, \mathbf{y}) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and consider the matrix

$$\widetilde{M} = \widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{t}) := \begin{pmatrix} 1 - a(\mathbf{x}, \mathbf{y})\mathbf{t} & a(\mathbf{x}, \mathbf{y})\mathbf{t} \\ -a(\mathbf{x}, \mathbf{y})\mathbf{t} & 1 + a(\mathbf{x}, \mathbf{y})\mathbf{t} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2} \text{ with } \det(\widetilde{M}) = 1$$

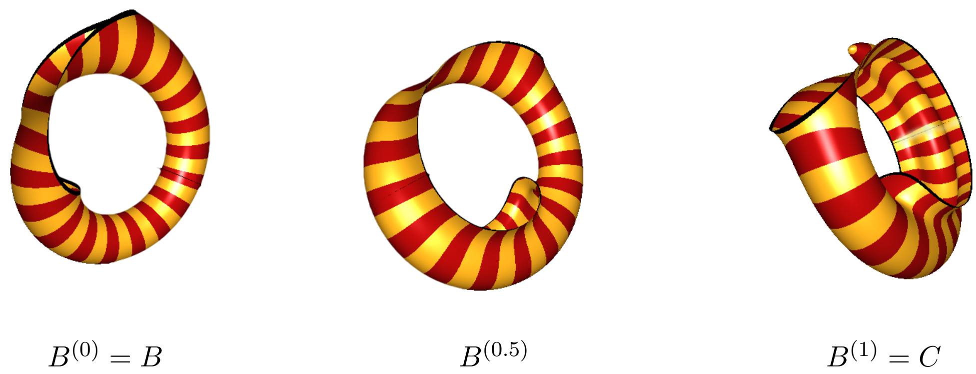


FIGURE 6. Deformation of a regular two-point blowup with non-constant sign distribution

and the matrices

$$M^{(t)} = M^{(t)}(\mathbf{x},\mathbf{y}) := \widetilde{M}(\mathbf{x},\mathbf{y},t) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2} \text{ with } \det(M^{(t)}) = 1 \text{ for all } t \in \mathbb{R}.$$

So, for any regular blowup $B = \mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(f_0, f_1) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ we get an isotopic family

$$\big(B^{(t)} = \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]} \text{ such that for all } t \in [0,1] \text{ it holds:}$$

$$B^{(t)} = \mathrm{Bl}_U\big(f_0 - t\cdot a(\mathbf{x},\mathbf{y})(f_0+f_1), f_1 + t\cdot a(\mathbf{x},\mathbf{y})(f_0+f_1)\big) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z) \text{ and } B^{(t)} \cong B.$$

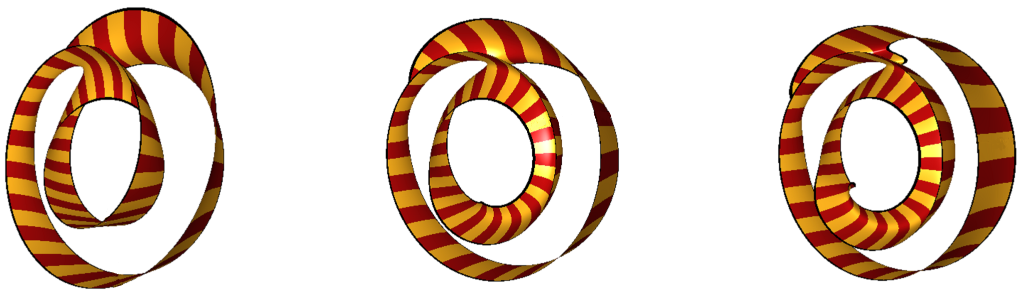
We thus get a family $\big(B^{(t)}\big)_{t\in[0,1]}$ of isotopic blowups $B^{(t)} \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, which connects $B = B^{(0)}$ with

$$C := B^{(1)} = \mathrm{Bl}_U(\underline{f}M^{(1)}) = \mathrm{Bl}_U\big(f_0 - a(\mathbf{x},\mathbf{y})(f_0+f_1), f_1 + a(\mathbf{x},\mathbf{y})(f_0+f_1)\big).$$

As announced, we aim to illustrate the situation by means of two regular two-point blowups, which are of are of different (relative oriented embedded) isomorphism type, a situation which can indeed only occur for regular blowups with respect to more than one point. More precisely, we shall blow up U with respect to two different pairs $\underline{f}$ of regular polynomials which both satisfy $Z_U(\underline{f}) = \{(\pm 1, 0)\}$, but such that the sign distribution $\mathrm{sgn}_{\underline{f}}$ (see Definition and Remark 3.4) is non-constant in the first case and constant in the second case.

(B) We keep the general settings of part (A), set $a(\mathbf{x},\mathbf{y}) := \mathbf{xy}$ and consider the regular two-point blowup $B := \mathrm{Bl}_U(\underline{f})$ of U with respect to $Z := \{(\pm 1, 0)\}$ given by $f_0 := \mathbf{x}^2 + \mathbf{y}^2 - 1$ and $f_1 := \mathbf{y}$. We then have $\mathrm{sgn}_B\big((\pm 1, 0)\big) = \pm 1$, so that the sign distribution $\mathrm{sgn}_B = \mathrm{sgn}_{\underline{f}}$ is non-constant. The visualization of the resulting family of two-point blowups $B^{(t)} \cong B^{(0)} = B$ is presented in Figure 6 for $t = 0, 0.5, 1$.

(C) We now choose $a(\mathbf{x},\mathbf{y}) := \mathbf{y}$ and consider the the regular two-point blowup $B := \mathrm{Bl}_U(\underline{f})$ of U with respect to $Z := \{(\pm 1, 0)\}$ given by $f_0 := \mathbf{x}^2 - 1$ and $f_1 := \mathbf{xy}$. This time, it holds $\mathrm{sgn}_B\big((\pm 1, 0)\big) = 1$, so that the sign distribution $\mathrm{sgn}_B = \mathrm{sgn}_{\underline{f}}$ is constant. This means, that we get a two-point blowup whose embedded isomorphism type differs



$B^{(0)}$ $B^{(0.5)}$ $B^{(1)} = C$

FIGURE 7. Deformation of a regular two-point blowup with constant sign distribution

from the isomorphism type of the blowup of part (B). The visualization of the resulting family of two-point blowups $B^{(t)} \cong B^{(0)} = B$ is presented in Figure 7 for $t = 0, 0.5, 1$.

Two Families of Regular Three-point Blowups. Up to now, we have seen examples of families of regular n-point blowups for $n = 1, 2$ and $n = 4$ (see Figure 3, Figures 6 and 7 and Figure 4 respectively). We now aim to present two families of regular 3-point blowups. As above we choose $\rho = 2, r = 4, U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$ for our visualization.

Example 5.2. (A) We consider the following example of [8] given by:

$$B := \mathrm{Bl}_U(\underline{f}), \text{ with } f_0 := \frac{1}{2}(\mathbf{x} - 1) + \mathbf{y}^2 \text{ and } f_1 := (\mathbf{x} + \frac{1}{2})\mathbf{y}.$$

We have

$$Z = Z_U(\underline{f}) = \{p_1, p_2, p_3\} \text{ with } p_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), p_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}), p_3 = (1, 0)$$

and hence Z is the set of vertices of an equilateral triangle centered at the origin $\underline{0} \in \mathbb{R}^2$. Moreover, it holds

$$\det\big(\partial \underline{f}\big)(p_1) = \det\big(\partial \underline{f}\big)(p_2) = -\frac{3}{2} \text{ and } \det\big(\partial \underline{f}\big)(p_3) = \frac{3}{2}.$$

So B is a regular three-point blowup. The sign distribution and hence the embedded isomorphism type of B is given by

$$\mathrm{sgn}_B(p_i) = \begin{cases} -1, & \text{for } i = 1, 2 \\ 1, & \text{for } i = 3. \end{cases}$$

So, in this case we have a *regular three-point blowup with non-constant sign distribution.* We consider the family of matrices

$$\big(M^{(t)}\big)_{t \in [0,1]} \text{ with } M^{(t)} := \begin{pmatrix} t\mathbf{x} + (1-t) & -2t \\ 3t & t\mathbf{y} + (1-t) \end{pmatrix}.$$

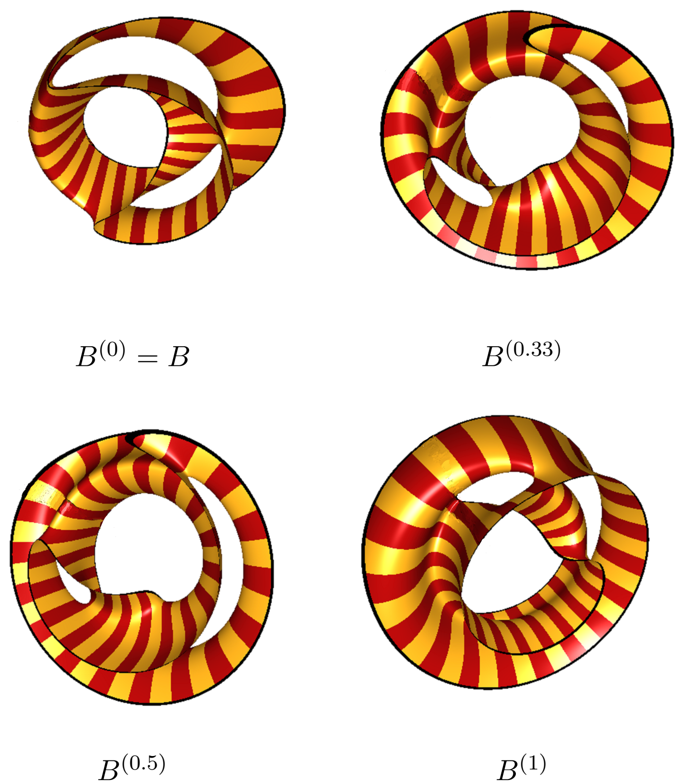


FIGURE 8. Deformation of a regular three-point blowup with non-constant sign distribution

As $\det(M^{(t)}) = (7 + \mathbf{xy} - \mathbf{x} - \mathbf{y})t^2 + (\mathbf{x} + \mathbf{y} - 2)t + 1$, and hence $\det(M^{(t)})(p) > 0$ for all $(x, y) = p \in U$ and all $t \in [0, 1]$, it follows that $\big(B^{(t)} := \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]}$ is an isotopic family of regular three-point blowups which non-constant sign distribution, whose visualization is presented in Figure 8 for $t = 0, 0.33, 0.5, 1$.

(B) We now aim to present a family of regular three-point blowups with constant sign distribution. We choose

$$Z := \{p_1 = (x_1, y_1) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), p_2 = (x_2, y_2) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}), p_3 = (x_3, y_3) = (1, 0)\}$$

as in part (A). Our first aim is to find a strongly regular pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z on U (see Definition 3.5) such that $\det\big(\partial \underline{f}\big)(p_i) = 1$ for $i = 1, 2, 3$. We do this according to the procedure suggested in the proof of Lemma 3.6, but with the rôles of $\mathbf{x}$, $\mathbf{y}$ and of f_0, f_1 exchanged respectively. We thus set

$$f_1 = \prod_{i=1}^{3}(\mathbf{y} - y_i) = \mathbf{y}^3 - \frac{3}{4}\mathbf{y} = \mathbf{y}(\mathbf{y}^2 - \frac{3}{4}) \text{ and } f_0 = h(\mathbf{y})\big(\mathbf{x} - g(\mathbf{y})\big) \text{ with}$$

$$\deg(h), \deg(g) \leq 2, \text{ and } g(y_i) = x_i, \quad h(y_i) = \frac{1}{\prod_{j\neq i}(y_i - y_j)} \text{ for } i = 1, 2, 3.$$

So

$$g(\mathbf{y}) = -2\mathbf{y}^2 + 1 \text{ and } h(\mathbf{y}) = \frac{4}{3}(2\mathbf{y}^2 - 1), \text{ thus}$$

$$f_0 = \frac{4}{3}(2\mathbf{y}^2 - 1)(\mathbf{x} + 2\mathbf{y}^2 - 1) = \frac{4}{3}(4\mathbf{y}^4 + 2\mathbf{y}^2\mathbf{x} - 4\mathbf{y}^2 - \mathbf{x} + 1).$$

Now, we have $Z_{\mathbb{R}^2}(\underline{f}) = \{p_1, p_2, p_3\}$ and $\det(\partial \underline{f}) = 4(2\mathbf{y}^2 - 1)(\mathbf{y}^2 - \frac{1}{4})$, so that $\det(\partial \underline{f})(p_i) = 1$ for $i = 1, 2, 3$. Therefore $B := \mathrm{Bl}_U(\underline{f})$ is a *regular three-point blowup with constant sign distribution* $\mathrm{sgn}_B(p_i) = 1$ for $i = 1, 2, 3$.

Our present example illustrates at this point, that the method suggested in the proof of Lemma 3.6 tends to furnish pairs of polynomials which may be simplified without changing the sign distribution (and hence the isomorphism type) in $\mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Namely, by setting

$$h_0 := \frac{3}{4}f_0 - 4\mathbf{y}, f_1 = 2\mathbf{x}\mathbf{y}^2 - \mathbf{y}^2 - \mathbf{x} + 1 \text{ and } h_1 := 4f_1 = 4\mathbf{y}^3 - 3\mathbf{y}$$

we get indeed $Z_{\mathbb{R}^2}(\underline{h}) = \{p_1, p_2, p_3\}$ and $\det(\partial \underline{h}) = 3(2\mathbf{y}^2 - 1)(4\mathbf{y}^2 - 1)$ so that $\det\big(\partial \underline{h}(p_i)\big) > 0$ and hence $\mathrm{sgn}_{\underline{h}}(p_i) = \mathrm{sgn}_{\underline{f}}(p_i) = 1$ for $i = 1, 2, 3$.

For a better visualization of the blowup $\mathrm{Bl}_U(\underline{h})$ we modify it slightly by interchanging the two indeterminates $\mathbf{x}$, $\mathbf{y}$ and the two polynomials h_0 and h_1 (which interchanges the coordinates of the common zeros of the two polynomials, and does not affect their Jacobian determinant – and hence preserves the (constant) sign distribution of the corresponding blowup), and by multiplying the first of them by $\frac{1}{3}$ (which gives a dilatation of the blowup in the "direction of the fibers"). So, we shall consider the blowup $B = \mathrm{Bl}_U(\underline{g})$ with $\underline{g} = (g_0, g_1)$,

$$g_0 = \mathbf{x}(\mathbf{x}^2 - \frac{3}{4}) \text{ and } g_1 = 2\mathbf{x}^2\mathbf{y} - \mathbf{x}^2 - \mathbf{y} + 1$$

under the deformation given by the family of matrices $M^{(t)}$ of part (A). This time, for the sake of virtual simplicity, we present with our method of visualization only the single blowup $B^{(t)} = \mathrm{Bl}_U(\underline{g}M^{(t)})$ for $t = 0.5$ (see Figure 9) and the two affine charts of the blowup $B^{(0)}$ given respectively by $g_1(\mathbf{x}, \mathbf{y}) - \mathbf{z}g_0(\mathbf{x}, \mathbf{y}) = 0$ and $g_0(\mathbf{x}, \mathbf{y}) - \mathbf{z}g_1(\mathbf{x}, \mathbf{y}) = 0$ (see Figure 10). The two charts were visualized by means of MATHEMATICA.

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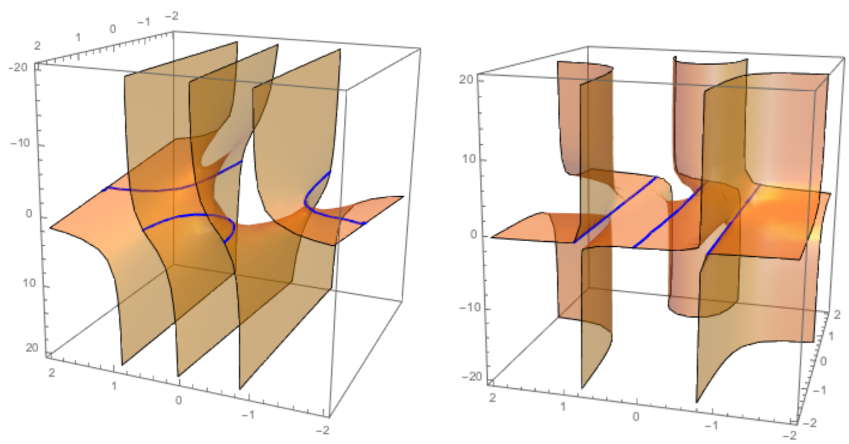
Figure 9. Deformation of a regular three-point blowup with constant sign distribution for $t = 0.5$



1. Chart 2. Chart

Figure 10. Two charts of a regular three-point blowup with constant sign Distribution

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