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Families of Blowups of the Real Affine Plane: Classification, Isotopies and Visualizations

by

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FAMILIES OF BLOWUPS OF THE REAL AFFINE PLANE: CLASSIFICATION, ISOTOPIES AND VISUALIZATIONS

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ABSTRACT. We classify embedded blowups of the real affine plane up to oriented isomorphy. We show that two blowups in the same isomorphism class are isotopic, using a matrix deformation argument similar to an idea given in [14]. This answers two questions which were motivated by the interactive visualizations of such blowups (see [11], [12], [13]).

1. INTRODUCTION AND SURVEY

Motivating Background: The Visualization Project for Blowups of the Real Affine Plane. The present paper is motivated by several investigations on the visualization of blowups of the real affine plane (see [1],[2],[3],[9],[8],[10]) in particular by the interactive visualizations suggested by the second named author and Ch. Stussak [11]. Our investigation is driven by the following two problems

(1.0) (a) *Deformation Problem*: "Can one connect two arbitrary oriented isomorphic embedded blowups of the real affine plane by a continuous family within their isomorphism class?"

(b) *Classification Problem*: "Is there a simple criterion to detect whether two regular embedded blowups of the affine plane are oriented isomorphic?"

We shall see, that both of these problems find an affirmative answer. At first view, this is a result of theoretical nature – but it also is of considerable practical meaning: Namely, once having tested that two embedded blowups B and C of the real affine plane are oriented isomorphic, one can use the animated visualization procedure of [11] to "produce a sequence of pictures which shows a deformation of the two blowups B to C within their common isomorphism class." Moreover, our answer to the classification problem gives an easy way to detect whether two regular embedded blowups are oriented isomorphic.
We shall provide a few simple examples of this. Let us also note, that all illustrations in the present paper base on the visualization REALSURF as developed by C. Stussak (see [12]).

Blowups of the Real affine Plane. We now start to set the precise setting in which we shall work. So, let $Z \subset \mathbb{R}^2$ be a finite set and let $U \subset \mathbb{R}^2$ be an open bounded and

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star-shaped set with closure $\overline{U}$ such that $Z \subset U$ – for example an open disk containing Z. We fix a pair of two-variate real polynomials

(1.1) $\underline{f} := (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ such that $\mathrm{Z}_{\overline{U}}(\underline{f}) := \{p \in \overline{U} \mid f_0(p) = f_1(p) = 0\} = Z$.

Then, the *embedded blowup* $\mathrm{Bl}_U(\underline{f})$ of U with respect to the pair $\underline{f}$ is defined as the closure (with respect to the topology induced by the Zariski topology on the ambient complex algebraic variety $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$) of the graph of the map

(1.2) $\varepsilon_{U,\underline{f}} : U \setminus Z \longrightarrow \mathbb{P}^1$, given by $p \mapsto [\underline{f}(p)] = (f_0(p) : f_1(p))$

in $U \times \mathbb{P}^1$. More precisely, our blowup is the pair consisting of

(1.3) (a) the set $\mathrm{Bl}_U(\underline{f}) := \overline{\{(p, [\underline{f}(p)]) \mid p \in U \setminus Z\}}$ (where $\overline{\bullet}$ denotes the operation of taking real Zariski closure) and
(b) the *canonical projection* map $\pi_{U,\underline{f}} : \mathrm{Bl}_U(\underline{f}) \longrightarrow U$, given by $(p, (x_0 : x_1)) \mapsto p, \quad$ for all $(p, (x_0 : x_1)) \in \mathrm{Bl}_U(\underline{f}) \subset U \times \mathbb{P}^1$.

(1.4) The set Z is called the *center* of the blowup $\mathrm{Bl}_U(\underline{f})$, whereas
(a) the graph $\mathrm{Bl}^\circ_U(\underline{f}) := \mathrm{Graph}(\varepsilon_{U,\underline{f}}) = \{(p, [\underline{f}(p)]) \mid p \in U \setminus Z\}$ of $\varepsilon_{U,\underline{f}}$ is called the *open kernel* of our blowup, and
(b) the set $\pi^{-1}_{U,\underline{f}}(Z) \subseteq (Z \times \mathbb{P}^1)$ – hence the set of boundary points of $\mathrm{Bl}^\circ_U(\underline{f})$ with respect to the complex Zariski topology – is called the *exceptional locus* of this blowup.

Observe the following fact:

(1.5) The blowup $\mathrm{Bl}_U(\underline{f})$ is the disjoint union of its open kernel and its exceptional set, more precisely:
(a) $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}^\circ_U(\underline{f}) \dot{\cup} \pi^{-1}_{U,\underline{f}}(Z)$.
(b) The restriction $\pi_{U,\underline{f}} \restriction: \mathrm{Bl}^\circ_U(\underline{f}) \xrightarrow{\cong} U \setminus Z$ of the canonical projection map $\pi_{U,\underline{f}}$ of (1.3)(b) to the open kernel is an isomorphism, whose inverse is given by $p \mapsto (p, \varepsilon_{U,\underline{f}}(p))$, for all $p \in U \setminus Z$.

Thus, if $Z \neq \emptyset$, the blowup $B = Bl_U(\underline{f})$ is obtained by replacing each point $p \in Z$ by the so called *exceptional fiber* $\pi^{-1}_{U,\underline{f}}(p) \subset \{p\} \times \mathbb{P}^1$ of $\pi_{U,\underline{f}}$ (or of B) above p – inserted to U instead of p in a way controlled by the two polynomials $f_0, f_1 \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$. The accumulation points of the open kernel B° in the exceptional fiber $\pi^{-1}_{U,\underline{f}}(p)$ are called *limit points* of B above p. We denote the set of these limit points by $\mathcal{L}_p(B)$. The open kernel B° is pasted to the exceptional fiber $\pi^{-1}_{U,\underline{f}}(p)$ along the set $\mathcal{L}_p(B)$. In Section 2 we shall have a closer look at the sets $\mathcal{L}_p(B) \subset \pi^{-1}_{U\underline{f}}(p)$.
In the *degenerate case* $Z = \emptyset$ we have $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}^\circ_U(\underline{f})$

Our basic aim is to study the class of blowups

(1.6) $\mathfrak{Bl}_U(Z) := \{\mathrm{Bl}_U(\underline{f}) \mid \underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with $Z_{\overline{U}}(\underline{f}) = Z\}$.

We obviously focus on the non-degenerate case in which $Z \neq \emptyset$. If we write $\mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$, we tacitly mean that $\underline{f} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ satisfies the condition $Z_{\overline{U}}(\underline{f}) = Z$.

A Glance to Algebraic Geometry. Blowups are one of the basic tools in Algebraic Geometry. Therefore we now relate the previous setting to the Algebraic Geometric context. If $(X, \mathcal{O}_X)$ is a locally Noetherian scheme and $\mathcal{J} \subseteq \mathcal{O}_X$ is a coherent sheaf of ideals, then the *blowup of X with respect to $\mathcal{J}$* is defined as the projective X scheme

(1.7) $\quad \mathrm{Bl}_X(\mathcal{J}) := \mathrm{Proj}(\bigoplus_{n\geq 0} \mathcal{J}^n \mathbf{t}^n) \xrightarrow{\pi_{X,\mathcal{J}}} X$

induced by the sheaf of *Rees Algebras* $\bigoplus_{n\geq 0} \mathcal{J}^n \mathbf{t}^n \subset \mathcal{O}_X[\mathbf{t}]$ associated to $\mathcal{J}$ (see [5], Chapter II, Section 7).

Blowups are of great significance in Algebraic Geometry mainly by two of their basic properties: The first is the *resolving effect on singularities* which allows "to blow away singular points" and hence gives rise to one of the most powerful tools of Algebraic Geometry: The *Resolution of Singularities* (see [6]). Below we shall illustrate this resolving effect by means of a simple example.

The second basic property says, that blowups of quasi-projective varieties are nothing else than proper birational morphisms (see [5], Chapter II, Theorem 7.17). This turns blowups into an indispensable tool of *Birational Algebraic Geometry*.

We now formulate a restricted notion of embedded blowup of scheme, sufficiently general to cover our embedded blowups of the real affine plane. Namely, If we are in the particular situation that $\mathcal{J} = \sum_{i=0}^s f_i \mathcal{O}_X$ is generated by a finite family of global sections $\underline{f} = (f_0, \ldots, f_s) \in \mathcal{J}(X)^{s+1}$, then, the surjective homomorphism of sheaves of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X[\mathbf{z}_0, \ldots \mathbf{z}_s] \twoheadrightarrow \bigoplus_{n\geq 0} \mathcal{J}^n \mathbf{t}^n, \quad \big(\mathbf{z}_i \mapsto f_i \mathbf{t}, i = 0, \ldots, s\big)$$

gives rise to a closed immersion

(1.8) $e_{\underline{f}} : \mathrm{Bl}_X(\mathcal{J}) \longrightarrow \mathbb{P}^s_X = X \times \mathbb{P}^s_{\mathbb{Z}}$ such that $\pi_{X,\mathcal{J}} = \pi_X \circ e_{\underline{f}}$, where $\pi_X : \mathbb{P}^s_X \longrightarrow X$ is the canonical projection.

We call

(1.9) $\mathrm{Bl}_X(\underline{f}) := e_{\underline{f}}\big(\mathrm{Bl}_X(\mathcal{J})\big) \subset \mathbb{P}^s_X$

the *embedded blowup* of X with respect to the family $\underline{f}$.

To relate this general algebraic geometric concept to our original setting, we let X be the complex affine plane $\mathrm{Spec}(\mathbb{C}[\mathbf{x},\mathbf{y}]) = \mathbb{A}^2_{\mathbb{C}}$, fix a pair $(f_0, f_1) = \underline{f} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2 \subset \mathbb{C}[\mathbf{x},\mathbf{y}]^2 = \mathcal{O}_X(X)^2$ as in (1.1) and set $\mathcal{J} = f_0\mathcal{O}_X + f_1\mathcal{O}_X$. Then, the closed immersion

$$e_{\underline{f}} : \mathrm{Bl}_X(\mathcal{J}) = \mathrm{Bl}_{\mathbb{A}^2_{\mathbb{C}}}(\mathcal{J}) \longrightarrow \mathbb{P}^1_X = \mathbb{A}^2 \times \mathbb{P}^1_{\mathbb{C}}$$

of (1.8) is $\mathbb{R}$-rational and it holds

(1.10) $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_X(\underline{f})_{\mathbb{R}} \cap \big(U \times \mathbb{P}^1_{\mathbb{R}}\big)$.

So, the embedded blowup $\mathrm{Bl}_U(\underline{f})$ defined in (1.3) is nothing else than the real trace of that part of the embedded blowup $\mathrm{Bl}_X(\underline{f})$ which lies over the open set U under the canonical projection $\pi_{\mathbb{A}^2_{\mathbb{C}}} : \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \twoheadrightarrow \mathbb{A}^2_{\mathbb{C}}$.

The Visualization Procedure. We now aim to present a visualization procedure for the embedded blowup $\mathrm{Bl}_U(\underline{f})$ with respect to a pair $\underline{f}$ of two-variate polynomials which satisfies the requirement (1.1), as defined in (1.3)(a). We use the method originally suggested in [1] and [2] – but in the slightly modified form used in [11]. Let $\rho, r \in \mathbb{R}$ with $0 < \rho < r$ and consider

(1.11) (a) the open disk $\mathbb{D} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \rho\} \subset \mathbb{R}^2$, with $U \subseteq \mathbb{D}$ and
(b) the open solid torus $\mathbb{T} := \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + \big(r - \sqrt{v^2 + w^2}\big)^2 < \rho^2\} \subset \mathbb{R}^3$

together with the diffeomorphism

(1.12) $\iota : \mathbb{D} \times \mathbb{P}^1 \xrightarrow{\cong} \mathbb{T}$, given by

$$\big((x, y), (x_0 : x_1)\big) \mapsto \big(x, (r - y)\frac{x_0^2 - x_1^2}{x_0^2 + x_1^2}, (r - y)\frac{2x_0x_1}{x_0^2 + x_1^2}\big), \text{for all } (x, y) \in U, (x_0 : x_1) \in \mathbb{P}^1.$$

We convene

(1.13) The blowup $B = \mathrm{Bl}_U(\underline{f})$ is visualized by its diffeomorphic image

$$\iota\big(\mathrm{Bl}_U(\underline{f})\big) = \iota\big(\mathrm{Bl}^\circ_U(\underline{f})\big) \quad \dot{\cup} \quad \iota\big(\pi^{-1}_{U,\underline{f}}(Z)\big) \subset \mathbb{T}, \text{ so that we have:}$$

(a) $\iota\big(\mathrm{Bl}^\circ_U(\underline{f})\big) = \{\big(x, (r - y)\frac{f_0(x,y)^2 - f_1(x,y)^2}{f_0(x,y)^2 + f_1(x,y)^2}, (r - y)\frac{2f_0(x,y)f_1(x,y)}{f_0(x,y)^2 + f_1(x,y)^2} \mid (x, y) \in U \setminus Z\}$.
(b) $\iota\big(\pi^{-1}_{U,\underline{f}}(Z)\big) \subseteq \iota(Z \times \mathbb{P}^1) = \dot{\bigcup}_{p \in Z} \quad \iota(\{p\} \times \mathbb{P}^1)$.
(c) If $p = (x, y) \in Z$, then $\iota(\{p\} \times \mathbb{P}^1) \subset \mathbb{T}$ is the circle of radius $r - y$ given by:

$$\begin{aligned}\iota(\{p\} \times \mathbb{P}^1) &= \{\big(x, (r - y)\frac{x_0^2 - x_1^2}{x_0^2 + x_1^2}, (r - y)\frac{2x_0x_1}{x_0^2 + x_1^2}\big) \mid (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0, 0)\}\} \\ &= \{\big(x, (r - y)\cos(\beta), (r - y)\sin(\beta) \mid -\pi \leq \beta \leq \pi\}.\end{aligned}$$

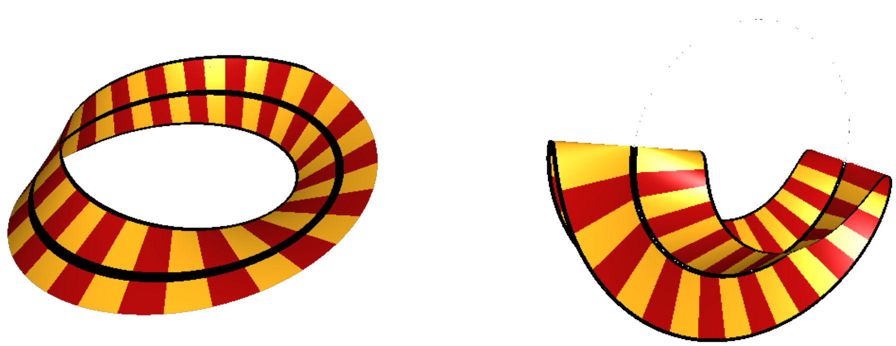
Observe that $\iota\big(\mathrm{Bl}^\circ_U(\underline{f})\big) \subset \mathbb{T}$ is a surface without boundary and that $\iota(Z \times \mathbb{P}^1)$ is a finite union of circles $\iota(\{p\} \times \mathbb{P}^1) \subset \mathbb{T}$ parallel to the central circle of $\mathbb{T}$, centered at the rotation axis of $\mathbb{T}$. Moreover, for each point $p \in Z$, the set of limit points and the exceptional fiber of B over p are visualized respectively by the two subsets $\iota\big(\mathcal{L}_p(B)\big) \subseteq \iota\big(\pi^{-1}_{U,\underline{f}}(p)\big)$ of the circle $\iota\big(\{p\} \times \mathbb{P}^1\big)$. The sets $\mathcal{L}_p(B)$ are of particular interest for the shape of the blowup B. Therefore, in some of our illustrations, their images $\iota\big(\mathcal{L}_p(B)\big)$ are colored in bold black and they usually appear (as arcs on) the circle $\iota\big(\{p\} \times \mathbb{P}^1\big)$.

The Technique of Visualization. For visualizations the parametric presentation given in (1.12) is used by Brandenberg (see [1]) and also by Brodmann and Prager (see [2] and [10]) for a very few examples. The difficulty of the parametrization for further examples is its instability in the neighborhood of Z (see also Prager in [10] for a further discussion). The new idea of Stussak (see [13] and [11]) was to derive the implicit equation of the parametrized surface (based on the work of [2]) and to use the program RealSurf (see [12]) for its visualization.

As already announced previously, all single pictures and sequences of pictures illustrating deformations of blowups we present in this paper are build by the program RealSurf developed by C. Stussak (see [12]). RealSurf is a graphic GPU-program for the visualization of algebraic surfaces. It allows an interactive view of algebraic surfaces in $\mathbb{A}^3$ in real time.

In his PhD dissertation (see [13]) C. Stussak studied exact rasterization of algebraic curves and surfaces for the visualization on a personal computer with GPU-programming. As an application of his technique he and the second author studied interactive visualizations of blowups of the real affine plane (see [13] and [11]). These interactive visualizations are based on RealSurf with several adaptations for the particular situation of our concrete examples (see [11] for the technical details). The modified program allows continuous parameter changes by mouse action. With the help of these modifications we produced the pictures of the present paper. We are grateful to Christian Stussak for making the adaption of RealSurf available to us.
The pictures were produced on a PC with graphic cards nVidia GT 525 Windows 7.

A Few Preliminary Examples. To present two basic examples of blowups, we choose $\rho = 2, r = 4, Z = \{(0,0)\}, U = \mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$. Then, under our visualization process, and for the choice $f_0 = \mathbf{x}, f_1 = \mathbf{y}$ the blowup $\mathrm{Bl}_U(\underline{f})$ appears as a *Möbius Strip* (see Figure 1 (a)), whereas for the choice $f_0 = \mathbf{x}^2, f_1 = \mathbf{y}^2$ the blowup $\mathrm{Bl}_U(\underline{f})$ appears as a *Double Whitney Umbrella* (see Figure 1 (b)). The essential difference between these two blowups, which shows also in their visualizations, will be explained later: the first one is regular, whereas the second one is not.



(a) (b)

Figure 1. (a) Möbius Strip (b) Double Whitney Umbrella

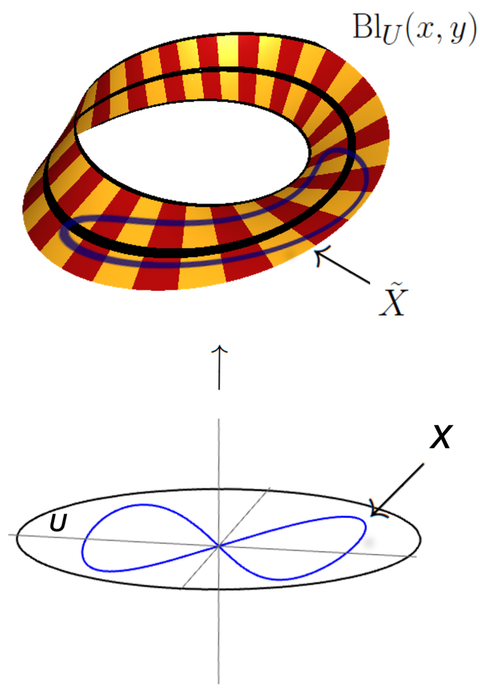


FIGURE 2. Blowing up the Lemniscate and Toroidal Embedding

We now present another example which illustrates the resolving effect of blowing up. We chose $\rho, r, Z, U = \mathbb{D}, f_0, f_1$ as in the first of the above examples. and consider the lemniscate, hence the plane quartic $X := \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 - \frac{1}{2}x^4 = 0\} \subset U$, which has a nodal singularity at the origin $\underline{0} := (0,0)$. Finally we consider the so called *strict transform or toroidal embedding*

$$\widetilde{X} := \overline{\pi^{-1}_{U,\underline{f}}(X) \cap \mathrm{Bl}^{\circ}_U(\underline{f})} = \overline{\pi^{-1}_{U,\underline{f}}(X \setminus Z)}$$

of X, which is a non-singular curve contained in our embedded blowup $\mathrm{Bl}_U(\underline{f})$ – and hence appears as a smooth simple closed curve on a Möbius strip – as illustrated in Figure 2.

Isomorphisms of Blowups. A *(relative oriented) automorphism* (we omit the wording in brackets from now on) of $U \times \mathbb{P}^1$ is a map

(1.14) (a) $\varphi = \varphi_M : U \times \mathbb{P}^1_{\mathbb{R}} \longrightarrow U \times \mathbb{P}^1_{\mathbb{R}}$ given by $(p, [\underline{v}]) \mapsto (p, [\underline{v}M(p)])$ for all $p \in U$ and all $\underline{v} \in \mathbb{R}^2 \setminus \{\underline{0}\}$, where
(b) $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$.

It is indeed justified to call these maps automorphisms. Namely: If $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$, its inverse $M^{-1} \in \mathbb{R}(\mathbf{x},\mathbf{y})^{2\times 2}$ may be written in the form $M^{-1} = \frac{1}{\det M}N$ with $N \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ and $\det\big(N(p)\big) > 0$ for all $p \in U$. It is immediate, that the map φ_N is inverse to φ_M. Observe that an automorphism of $U \times \mathbb{P}^1$ (in the above sense) leaves fix the fiber $\{p\} \times \mathbb{P}^1 \cong \mathbb{P}^1$ of the canonical projection

$\pi : U \times \mathbb{P}^1 \longrightarrow U$ over each point $p \in U$ and acts as a *Möbius-Transformation* on this fiber.

We say that two embedded blowups $B, C \in \mathfrak{Bl}_U(Z)$ are *(relative oriented embedded) isomorphic* (we omit the wording in brackets from now on) – and write $B \cong C$ – if there is an automorphism φ of $U \times \mathbb{P}^1$ such that $C = \varphi(B)$. This means in particular:

(1.15) If $B = \mathrm{Bl}_U(\underline{f}), C \in \mathfrak{Bl}_U(Z)$ then $B \cong C$ if and only if $C = \mathrm{Bl}_U(\underline{f}M)$ for some $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}])^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$.

Regular Embedded Blowups. We say that the pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is *regular* with respect to Z on U if the following requirements are satisfied:

(1.16) (a) $\mathrm{Z}_{\overline{U}}(\underline{f}) = Z$.
(b) The Jacobian

$$\partial \underline{f} := \begin{pmatrix} \frac{\partial f_0}{\partial \mathbf{x}} & \frac{\partial f_1}{\partial \mathbf{x}} \\ \\ \frac{\partial f_0}{\partial \mathbf{y}} & \frac{\partial f_1}{\partial \mathbf{y}} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$$

of the pair $\underline{f}$ is of rank 2 in all points $p \in Z$.

If the pair $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is regular with respect to Z on U, we call $\mathrm{Bl}_U(\underline{f})$ a *regular embedded blowup* of the set U along Z – and we aim to study the sub-class of $\mathfrak{Bl}_U(Z)$

(1.17) $\mathfrak{Bl}_U^{\mathrm{reg}}(Z) := \{\mathrm{Bl}_U(\underline{f}) \mid \underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is regular with respect to Z on $U\}$

consisting of all these regular blowups for fixed Z and U. From now on, if we write $\mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, we tacitly mean that $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is regular with respect to Z on U. Clearly, in the degenerate case $Z = \emptyset$ the blowup $\mathrm{Bl}_U(\underline{f})$ is regular, so that we have $\mathfrak{Bl}_U(\emptyset) = \mathfrak{Bl}_U^{\mathrm{reg}}(\emptyset)$.

If $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, there is a map

(1.18) $\mathrm{sgn}_B : Z \longrightarrow \{\pm 1\}$ given by $p \mapsto \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big)$ for all $p \in Z$

(which depends indeed only on B, see Definition and Remark 4.3, called the *sign distribution* of B.
If $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ with $\#Z = n \in \mathbb{N}$, we call B a *regular (embedded) n-point blowup*. We shall present examples of such n-point blowups and families of such for $n = 1$ (see Example 2.1), for $n = 2$ (see Examples 6.2 (B) and (C)), for $n = 3$ (see Examples 6.2 (A) and (B)) and for $n = 4$ (see Example 2.2).

In the framework of regular blowups we will give an affirmative answer to the *Classification Problem* (1.0)(b) by proving (see Theorem 4.8):

(1.19) ***Classification Theorem*:** If $B, C \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, then $B \cong C$ if and only if $\mathrm{sgn}_B = \mathrm{sgn}_C$.

Isotopies of Blowups. Now, we turn to the *Deformation Problem* (1.0)(a). Generally, one obtains families of embedded blowups of the real plane, if the coefficients of the defining polynomials f_0, f_1 vary. On application of the previously described visualization procedure, this leads to appealing "movies" showing the deformation of a blowup. Motivated by this, we aim to study families of the form:

(1.20) $\big(B^{(t)} = \mathrm{Bl}_U(\underline{f}^{(t)})\big)_{t\in[0,1]} \in \mathfrak{Bl}_Z(U)^{[0,1]}$, given by a pair of polynomials $(\widetilde{f_0}, \widetilde{f_1}) \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^2$, such that for all $t \in [0,1]$ the pair $\underline{f}^{(t)} := \big(\widetilde{f_0}(\mathbf{x},\mathbf{y},t), \widetilde{f_1}(\mathbf{x},\mathbf{y},t)\big) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ satisfies the the condition $Z_{\overline{U}}(\underline{f}^{(t)}) = Z$ of (1.1).

We are interested in such families for which the embedded blowups $B^{(t)} \in \mathfrak{Bl}_Z(U)$ are all isomorphic. It therefore is natural to study classes $\big(B^{(t)}\big)_{t\in[0,1]}$ of embedded blowups which come from an *isotopy* of $U \times \mathbb{P}^1$-automorphisms, hence from a family:

(1.21) $\big(\varphi^{(t)} = \varphi_{M^{(t)}}\big)_{t\in[0,1]}$, given by a (2×2)-matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$, such that for all $t \in [0,1]$ the matrix $M^{(t)} := \widetilde{M}(\mathbf{x},\mathbf{y},t) \in \mathbb{R}[\mathbf{x},\mathbf{y}])^{2\times 2}$ satisfies $\det\big(M(p)\big) > 0$ for all $p \in U$.

In this situation, the family of (1.20) takes the form (see (1.15)):

(1.22) $\big(B^{(t)} = \varphi^{(t)}(B) = \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]}$, for $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$.

In this context we will give an affirmative answer to the Deformation Problem (1.0)(a) by proving (see Theorem 5.9):

(1.23) ***Deformation Theorem*:** Let $B, C \in \mathfrak{Bl}_U(Z)$ with $B \cong C$. Then, B and C are connected by an isotopy of $U \times \mathbb{P}^1$-automorphisms. More precisely, there is an isotopy $\big(\varphi^{(t)} = \varphi_{M^{(t)}}\big)_{t\in[0,1]}$ as in (1.21) such that $\varphi^{(0)}(B) = B$ and $\varphi^{(1)}(B) = C$.

Deformation of Matrices. In view of (1.18) the Deformation Theorem (1.23) for blowups follows immediately from the following deformation result for matrices, (see Proposition 5.4 and Remark 5.6):

(1.24) *Polynomial Deformations of Matrices*: Let $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in \overline{U}$. Then M is connected to the unit matrix $\mathbf{1}^{2\times 2} \in \mathbb{R}^{2\times 2}$ by a polynomial family of (2×2)-matrices with positive determinants on $\overline{U}$. More precisely: There is a (2×2)-matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$, such that with $M^{(t)} := \widetilde{M}(\mathbf{x},\mathbf{y},t)$ for all $t \in \mathbb{R}$ we have
(a) $\det\big(M^{(t)}(p)\big) > 0$ for all $t \in [0,1]$ and all $p \in \overline{U}$.
(b) $M^{(0)} = \mathbf{1}^{2\times 2}$ and $M^{(1)} = M$.

We shall approach this deformation result in a more general context, which is appropriate for the study of blowups in the framework of Real Analytic Geometry, too. (See Proposition 5.2).

2. First Examples of Families of Blowups

Examples and their Visualizations. We shall conclude our paper with a few examples of families of embedded blowups and their visualizations. Already now, we present three examples, which give a first flavor of the subject and illuminate some typical features. Again, as in the examples visualized by Figure 1, we chose $\rho = 2, r = 4$ and

$U = \mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$.

Example 2.1. In our first example, we consider the most simple regular blowup of the real affine plane, namely the "regular one-point blowup" $B := \mathrm{Bl}_U(\mathbf{x}, \mathbf{y})$, whose visualization shows up as a Möbius strip (see Figure 1(a)). We deform this blowup by means of the family of polynomial matrices

$$\big(M^{(t)} := \begin{pmatrix} 1-t & \frac{t}{2} \\ -\frac{t}{2} & 1+t \end{pmatrix}\big)_{t\in[0,1]} \text{ with } \det(M^{(t)}) = 1 - \frac{3}{4}t^2 > 0 \quad \text{for all } t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[.$$

This leads us to the family of regular embedded of blowups $\big(B^{(t)}\big)_{t\in[0,1]}$ with

$$B^{(t)} := \mathrm{Bl}_U\big((\mathbf{x},\mathbf{y})M^{(t)}\big) = \mathrm{Bl}_U\big(f_0^{(t)} = (1-t)\mathbf{x} - \frac{t}{2}\mathbf{y}, f_1^{(t)} = \frac{t}{2}\mathbf{x} + (1+t)\mathbf{y}\big) \in \mathfrak{Bl}_U^{\mathrm{reg}}\big(\{\underline{0}\}\big)$$

and

$$Z = Z_U\big(\underline{f}^{(t)} := (f_0^{(t)}, f_1^{(t)})\big) = \{(0,0)\} \text{ for all } t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[.$$

In view of Figure 1(a) we expect that the visualization $\big(\iota(B^{(t)})\big)_{t\in[0,1]}$ of this family presents itself as a deformation of a Möbius strip. In Figure 3 we present this deformation for the values $t = 0, t = 0.4$ and $t = 1$. We also allow ourselves to leave the range $0 \leq t \leq 1$ and consider the three values $t = 1.15, t = 1.2$ and $t = 1.4$, which come close or lie beyond the critical value $t = \frac{2}{3}\sqrt{3} = 1.15470\ldots$.

These choices illustrate the following fact: If t takes its critical values $\pm\frac{2}{3}\sqrt{3}$, the two linear forms $f_0^{(t)}$ and $f_1^{(t)}$ are linearly dependent and hence do not define a blowup in our sense. If $t \notin [-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}]$ the blowup $B^{(t)}$ shows up again as a Möbius strip, but with reverse orientation.

Example 2.2. As a second example, we consider a family of "regular four-point blowups" of the real affine plane, which is indeed a modification of the example shown in Figure 9 of [11]. To this end, we chose $a \in [0,1]$ and consider the two pairs of polynomials $\underline{f} := (f_0, f_1)$ and $\underline{g} := (g_0, g_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ given by

$$f_0 = \mathbf{x}^2 - \frac{1}{2}\mathbf{y}^2 - \frac{1}{2}, \quad f_1 = -\frac{1}{2}\mathbf{x}^2 + \mathbf{y}^2 - \frac{1}{2} \textit{ and}$$

$$g_0 = \mathbf{x}^2 + (a - \frac{1}{2})\mathbf{y}^2 - a - \frac{1}{2}, \quad g_1 = (a - \frac{1}{2})\mathbf{x}^2 + \mathbf{y}^2 - a - \frac{1}{2}.$$

Then $\det(\partial\underline{f}) = 3\mathbf{xy}$ and $\det(\partial\underline{g}) = 4(1 - (a - \frac{1}{2})^2)\mathbf{xy}$. Taking $\mathbf{x}$-resultants, we get $\mathrm{Res}_{\mathbf{x}}(g_0, g_1) = (((a - \frac{1}{2})^2 - 1)(1 - \mathbf{y}^2))^2$. As $(a - \frac{1}{2})^2 - 1 < 0$ for $a \in [0,1]$ it follows that

$$Z = Z_U(\underline{f}) = Z_U(\underline{g}) = \{\pm 1, \pm 1\}.$$

This shows that $\underline{f}$ and $\underline{g}$ are regular pairs with respect to Z on U, so that $B := \mathrm{Bl}_U(\underline{f}), C := \mathrm{Bl}_U(\underline{g}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ and $\mathrm{sgn}_B = \mathrm{sgn}_C$. Moreover

$$\underline{g} = \underline{f}M \text{ with } M = \begin{pmatrix} 1 + \frac{2}{3}a & \frac{4}{3}a \\ \frac{4}{3}a & 1 + \frac{2}{3}a \end{pmatrix}.$$

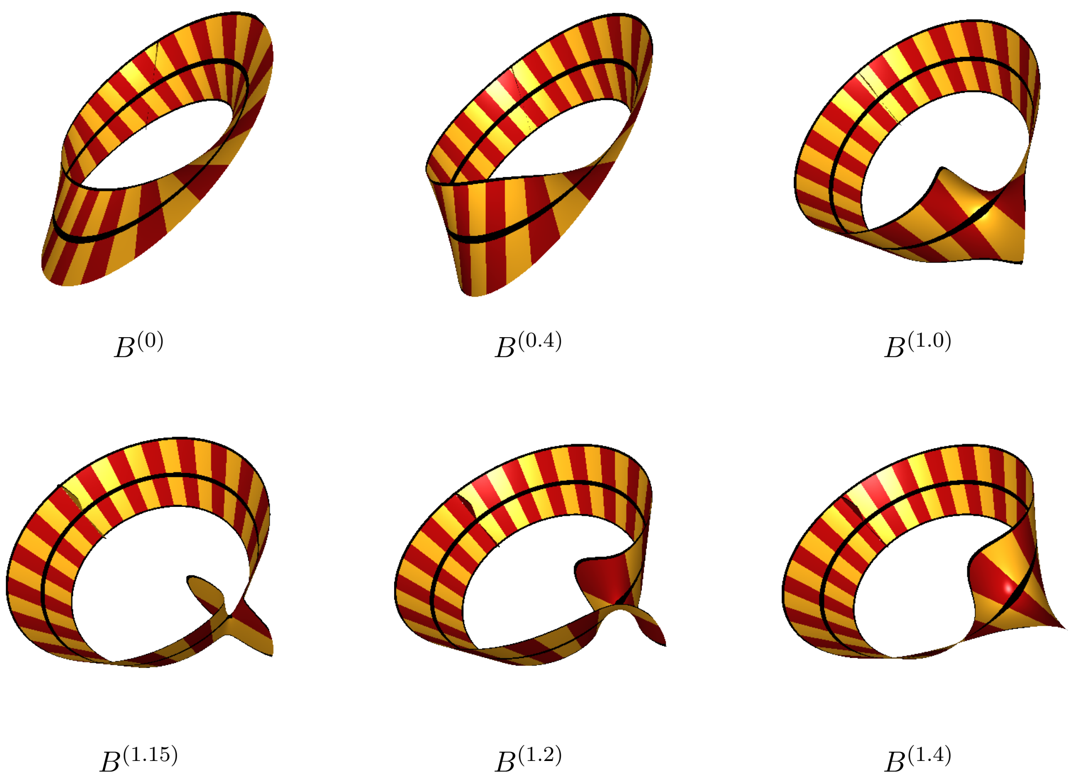


FIGURE 3. Deformation of a Möbius Strip

so that $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big) > 0$ for all $p \in U$. Setting

$$\widetilde{M} := \begin{pmatrix} 1+\frac{2}{3}a\mathbf{t} & \frac{4}{3}a\mathbf{t} \\ \frac{4}{3}a\mathbf{t} & 1+\frac{2}{3}a\mathbf{t} \end{pmatrix} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^2 \text{ and } M^{(t)} := \begin{pmatrix} 1+\frac{2}{3}at & \frac{4}{3}at \\ \frac{4}{3}at & 1+\frac{2}{3}at \end{pmatrix} \text{ for all } t \in \mathbb{R}$$

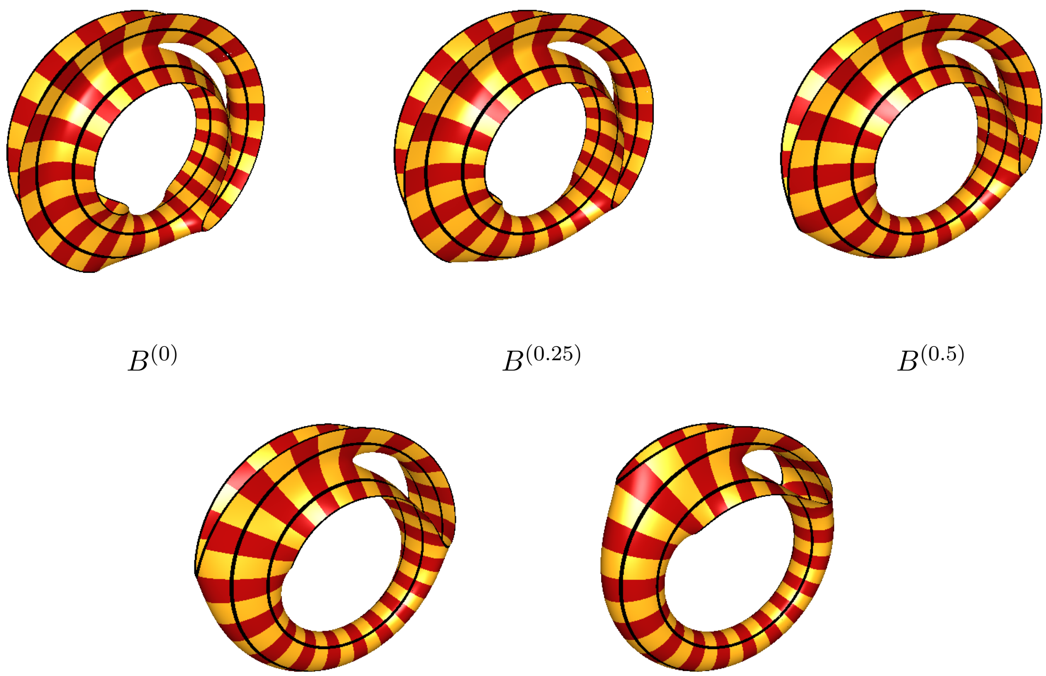
we get $\det(M^{(t)}) = (1+\frac{2}{3}at)^2 - \frac{16}{9}(at)^2 > 0$ and hence $\det(M^{(t)}) > 0$ for all $t \in [0,1]$. Moreover $M^{(0)} = \mathbf{1}^{2\times 2}$ and $M^{(1)} = M$. So, $\big(M^{(t)}\big)_{t\in[0,1]}$ is a family which connects $\mathbf{1}^{2\times 2}$ and M. Correspondingly $\big(\varphi^{(t)} := \varphi_{M^{(t)}}\big)_{t\in[0,1]}$ is an isotopy and

$$\big(B^{(t)} = \varphi^{(t)}(B) = \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]}$$

is a family of regular blowups $B^{(t)} \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ with $B^{(0)} = B$ and $B^{(1)} = C$.
We now choose $a = 1$. Then looking at the conics $f_0^{(t)} = 0$ and $f_1^{(t)} = 0$ defined by the two polynomials

$$f_0^{(t)}, f_1^{(t)} \in \mathbb{R}[\mathbf{x},\mathbf{y}] \text{ with } \underline{f}^{(t)} := (f_0^{(t)}, f_1^{(t)}) = \underline{f}M^{(t)} \text{ for all } t \in [0,1]$$

we have the following situation: Two hyperbolas ($t = 0$) are deformed to two ellipses ($t = 1$) via a degeneration to a pair of lines ($t = \frac{1}{2}$). A rough visualization of this family presents itself as shown in Figure 4.



$B^{(0)}$ $B^{(0.25)}$ $B^{(0.5)}$

$B^{(0.75)}$ $B^{(1)}$

FIGURE 4. Deformation of a Regular Four-Point Blowup

Example 2.3. Up to now, we have considered two families of regular blowups of the real affine plane. Now, we aim to consider a family of blowups, which is obtained by deforming the singular blowup $B := \mathrm{Bl}_U(\mathbf{x^2}, \mathbf{y^2})$, whose visualization shows up as a Double Whitney Umbrella (see Figure 1(b)). To this end, we fix the matrix

$$\widetilde{M} = \widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{t}) := \begin{pmatrix} 1-\mathbf{t} & \frac{1}{2}\mathbf{t} \\ -\frac{1}{2}\mathbf{t} & 1+\mathbf{t} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2} \text{ with } \det(\widetilde{M}) = 1 - \frac{3}{4}\mathbf{t}^2.$$

For all $t \in \mathbb{R}$ we set

$$M^{(t)} := \widetilde{M}(\mathbf{x}, \mathbf{y}, t) = \begin{pmatrix} 1-t & \frac{1}{2}t \\ -\frac{1}{2}t & 1+t \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}, \text{ so that } \det(M^{(t)}) = 1 - \frac{3}{4}t^2.$$

Clearly, $\det(M^{(t)}) > 0$ whenever $|t| < \frac{2}{3}\sqrt{3}$, so that $\left(\varphi_{M^{(t)}} = \varphi^{(t)}\right)_{t\in[0,\frac{2}{3}\sqrt{3}[}$ is an isotopy of $U \times \mathbb{P}^1$-automorphisms. Thus for any blowup $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ we get a family

$$\left(B^{(t)} := \mathrm{Bl}_U(\underline{f}M^{(t)})\right)_{t\in]-\frac{2}{3}\sqrt{3},\frac{2}{3}\sqrt{3}[}$$

with $B^{(t)} \in \mathfrak{Bl}_U(Z)$ and $B^{(t)} \cong B$ for all $t \in]-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}[$.

With $f_0 = \mathbf{x}^2, f_1 = \mathbf{y}^2$ and $\underline{f}^{(t)} := \underline{f}M^{(t)}$ we then have

$$Z := Z_U(\underline{f}^{(t)}) = \{\underline{0}\} \text{ for all } t \neq \pm\frac{2}{3}\sqrt{3}.$$

In Figure 5, the blowups $B^{(t)}$ are visualized by their images $\iota\big(B^{(t)}\big) \subset \mathbb{R}^3$ for $t = 0,\ 0.5,\ 1,\ 1.1,\ 1.25,\ 4$. Remember that $B = B^{(0)}$ is the so-called Double Whitney Umbrella.

Note that while passing from $t = 1.1$ to $t = 1.25$ (hence by passing through the critical value $t = \frac{2}{3}\sqrt{3}$) the embedded isomorphism type of $B^{(t)}$ swaps. Observe also, that the exceptional fiber $\pi^{(-1)}_{U,\underline{f}^{(t)})}(\underline{0}) = \{\underline{0}\} \times \mathbb{P}^1$ of $B^{(t)}$ over $\underline{0}$ is visualized by the same circle for all $t \neq \pm\frac{2}{3}\sqrt{3}$ and that the corresponding set of limit points $\mathcal{L}_{\underline{0}}(B^{(t)})$ is visualized by an arc on this circle (compare Example 3.9 (B)). For $t = 0$ this arc is a half circle, whereas the length of this arc converges to 0 if $t \to \pm\frac{2}{3}\sqrt{3}$ - hence if the degenerate case is approached. Near to the degeneration value $t = \frac{2}{3}\sqrt{3}$ we enlarged the scale of our visualization in order to improve the picture of the details. For that reason the coloring appears larger for the last three values of t.

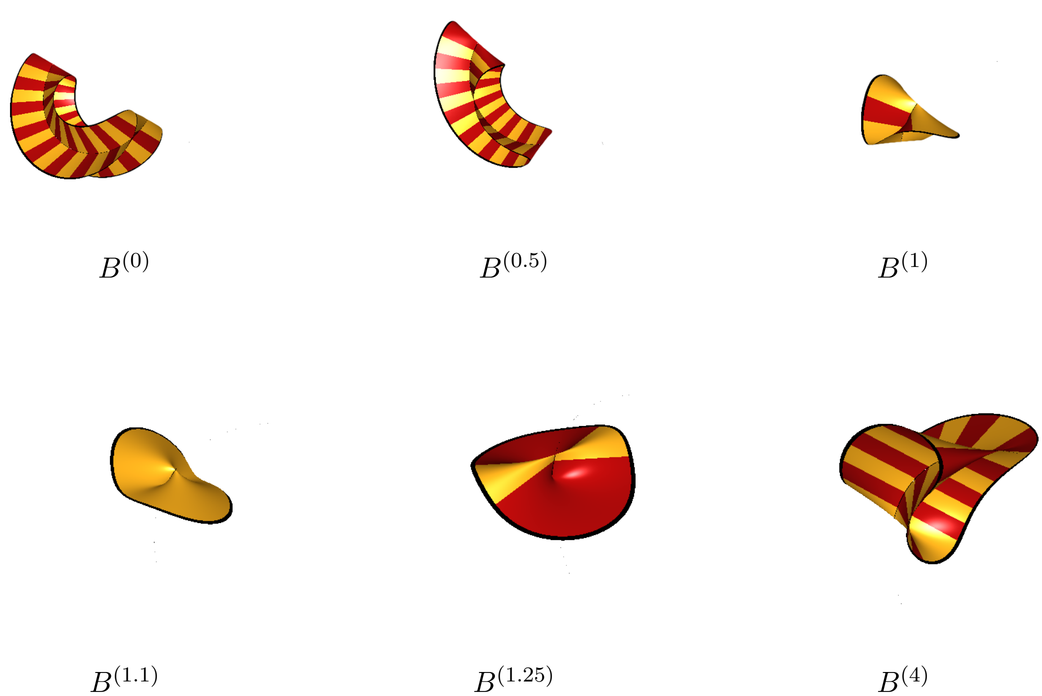


FIGURE 5. Deformation of a Double Whitney Umbrella

3. Exceptional Fibers and Limit Points

Exceptional Fibers. In this section, we will have a closer look at the exceptional set (1.4)(b) of a blowup. We keep the previous notations and hypotheses.

Definition 3.1. Let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be a pair which satisfies the requirement (1.1) of the introduction. A point $p \in Z$ is called *superfluous* with respect to $\underline{f}$, if there are polynomials $g_0, g_1, h \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ with $\underline{f} = h\underline{g}$ and $\underline{g}(p) \neq 0$. Observe, that in this situation we may assume that h is a greatest common divisor of f_0 and f_1.

Lemma and Definition 3.2. *Let $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ and let $p \in Z$. Then $\pi^{-1}_{U,\underline{f}}(p)$ is called the exceptional fiber of B over p and it holds*

(a) $\pi^{-1}_{U,\underline{f}}(p) = \{p\} \times \mathbb{P}^1$ *if p is not a superfluous point with respect to $\underline{f}$.*
(b) $\pi^{-1}_{U,\underline{f}}(p) = \{\big(p, (g_0(p) : g_1(p))\big)\}$, *if p is a superfluous point with respect to $\underline{f}$ and $g_0, g_1 \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ are as in Definition 3.1.*

Proof. The homogeneous coordinate ring of $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ takes the form $\mathbb{C}[\mathbf{x}, \mathbf{y}, \mathbf{t}_0, \mathbf{t}_1]$ with $\deg(\mathbf{x}) = \deg(\mathbf{y}) = 0$ and $\deg(\mathbf{t}_0) = \deg(\mathbf{t}_1) = 1$. Let $g_0, g_0, h \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ with $f_0 = hg_0$ and $f_1 = hg_1$, where h is a greatest common divisor of f_0 and f_1 in $\mathbb{R}[\mathbf{x}, \mathbf{y}]$ – and hence in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. For all $q \in U$ it holds $\big(f_1(q)g_0(q) - f_0(q)g_1(q)\big)h(q) = 0$. Clearly, there is an open neighborhood $W \subset U$ of p, such that $h(q) \neq 0$ for all $q \in W \setminus \{p\}$. It follows that $f_1(q)g_0(q) - f_0(q)g_1(q) = 0$ for all $q \in \mathbb{R}^2$. This means, that the Zariski closure $\overline{B}$ of the open kernel $B^\circ := \mathrm{Bl}^\circ_U(\underline{f})$ in $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is contained in the irreducible surface $\mathbb{S} \subset \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ defined by the equation $\mathbf{t}_1 g_0 - \mathbf{t}_1 g_1 = 0$. As the image of B° under the canonical projection $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^2_{\mathbb{C}}$ covers the set $U \setminus Z$, we have $\dim(\overline{B}) \geq 2$ and it follows that $\overline{B} = \mathbb{S}$. This implies that

$$\pi^{-1}_{U,\underline{f}}(p) = \big(\{p\} \times \mathbb{P}^1\big) \cap \widetilde{B} = \big(\{p\} \times \mathbb{P}^1\big) \cap \mathbb{S} = \{p\} \times \{(t_0 : t_1) \in \mathbb{P}^1 \mid t_1 g_0(p) - t_0 g_1(p) = 0\}.$$

As p is superfluous with respect to $\underline{f}$ if and only if $\underline{g}(p) \neq 0$, we get our claim. □

As an immediate application we get the following result, which justifies to speak of "superfluous points."

Proposition 3.3. *Let $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ and let S be the set of superfluous points $p \in Z$ with respect $\underline{f}$. Then*

(a) $B \in \mathfrak{Bl}_U(Z \setminus S)$.
(b) *If $S = \emptyset$, then* $\pi^{-1}_{U,\underline{f}}(Z) = Z \times \mathbb{P}^1$.

Remark 3.4. Proposition 3.3 recommends to consider only blowups $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ without superfluous points with respect to $\underline{f}$. All our examples will satisfy this requirement, as we shall consider only pairs $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ whose greatest common divisor has no zero in Z. In this situation we may always write (see (1.5)(a))

$$\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}^\circ_U(\underline{f}) \quad \dot\cup \quad (Z \times \mathbb{P}^1).$$

Limit Points. Let $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$ and let B° be the open kernel of B. In some sense, it is more natural to consider instead of B the closure $\overline{B^\circ}$ of the open kernel with respect of the standard topology. As the standard topology is finer than the Zariski topology, this leads to the problem to determine the points in the exceptional set of B which are accumulation points of B°, hence to determine the set $\mathcal{L}_p(B)$ of limit points of B above each $p \in Z$. We have mentioned these sets of limit points already in the introduction (see (1.13)). We now will have a closer look at them.

Definition and Remark 3.5. (A) Let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be a pair which satisfies the requirement (1.1) of the introduction and consider the blowup $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$, its open kernel $B^\circ = \mathrm{Bl}^\circ_U(\underline{f})$ and fix a point $p \in Z$. A point

$$q = (p, s) \in \pi^{-1}_{U,\underline{f}}(p) \in \{p\} \times \mathbb{P}^1$$

is called a *limit point* of B above p, if it is a point of accumulation of B°. As in the introduction, we write $\mathcal{L}_p(\underline{f})$ or $\mathcal{L}_p(B)$ for the set of these points, hence:

$$\mathcal{L}_p(B) := \{(p, s) \mid \exists \big(p_n\big)_{n\in\mathbb{N}} \subset U \setminus Z \text{ with } \lim_{n\to\infty} p_n = p, \lim_{n\to\infty}\big(f_0(p_n) : f_1(p_n)\big) = s\}.$$

(B) Observe that the closure of B° with respect to the standard topology can be written in the form

$$\overline{B^\circ} = B^\circ \quad \dot\cup \quad \bigcup_{p\in Z} \mathcal{L}_p(B).$$

In the sequel, we restrict ourselves to treat a particular case, which is sufficient to understand our examples.

Notation 3.6. (A) Let $f \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and let $p = (x, y) \in \mathbb{R}$. For each $i \in \mathbb{N}_0$ we consider the *i-th term in the Taylor expansion*

$$f^{[i,p]} = f^{[i,p]}(\mathbf{x}, \mathbf{y}) := \sum_{j=0}^{i} \frac{\partial^i f}{\partial \mathbf{x}^j \partial \mathbf{y}^{i-j}}(x, y)(\mathbf{x} - x)^j (\mathbf{y} - y)^{i-j}$$

of f around p and the *multiplicity*

$$\mathrm{mult}_p(f) := \min\{m \in \mathbb{N}_0 \mid f^{[m,p]} \neq 0\}$$

of f in p.

(B) Let $\varrho \in \mathbb{Q}(\mathbf{w})$ be a real rational function. We write $\overline{\mathrm{Im}}(\varrho)$ for the closure of the set $\varrho\big(\mathbb{R} \setminus \mathrm{Pole}(\varrho)\big)$ in $\mathbb{R} \cup \{\pm\infty\}$ where $\mathrm{Pole}(\varrho)$ denotes the set of poles of ϱ.

Proposition 3.7. *Let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be as in (1.1) of the introduction and consider the blowup $B = \mathrm{Bl}_U(\underline{f}) \in \mathfrak{Bl}_U(Z)$. Let $p = (x, y) \in Z$, and assume that $\mathrm{mult}_p(f_0) = \mathrm{mult}_p(f_1) =: m$ and that $f_0^{[m,p]}$ and $f_1^{[m,p]}$ have no common linear factor.*

Consider the rational function $\varrho(\mathbf{w}) := \frac{f_0^{[m,p]}(1+x,\mathbf{w}+y)}{f_1^{[m,p]}(1+x,\mathbf{w}+y)} \in \mathbb{Q}(\mathbf{w})$. *Then*

$$\mathcal{L}_p(B) = \{p\} \times \{\big(\frac{1}{\sqrt{1+\tau^2}} : \frac{\tau}{\sqrt{1+\tau^2}}\big) \mid \tau \in \overline{\operatorname{Im}}(\varrho)\} =$$
$$= \{p\} \times \{\big(\cos(\alpha) : \sin(\alpha)\big) \mid -\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} \text{ and } \tan(\alpha) \in \overline{\operatorname{Im}}(\varrho)\}.$$

In particular, $\mathcal{L}_p(B)$ *is a closed segment of the projective line* $\{p\} \times \mathbb{P}^1$, *visualized under the map* ι *of (1.12) by the closed arc*

$$\iota\big(\mathcal{L}_p(B)\big) = \{\big(x, (r-y)\cos(\beta), (r-y)\sin(\beta) \mid -\pi \leq \beta \leq \pi \text{ and } \tan(\frac{\beta}{2}) \in \overline{\operatorname{Im}}(\varrho)\}$$

on the circle

$$\iota(\{p\} \times \mathbb{P}^1) = \{(x, (r-y)\cos(\beta), (r-y)\sin(\beta) \mid -\pi \leq \beta \leq \pi\}.$$

Proof. We may assume that $p = \underline{0} := (0,0)$ so that $f_0^{[i,\underline{0}]}, f_1^{[i,\underline{0}]} \in \mathbb{R}[\mathbf{x},\mathbf{y}]_i$ for all $i \in \mathbb{N}$. It suffices to prove the first equality (see also (1.13) (c)). We set

$$\mathcal{S} := \{\big(\frac{1}{\sqrt{1+\tau^2}} : \frac{\tau}{\sqrt{1+\tau^2}}\big) \mid \tau \in \overline{\operatorname{Im}}(\varrho)\}.$$

It remains to show that $\mathcal{L}_{\underline{0}}(B) = \{\underline{0}\} \times \mathcal{S}$. Let $\mathcal{C} := \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 = 1\}$. As $f_0^{[m,\underline{0}]}$ and $f_1^{[m,\underline{0}]}$ have no common linear factor it holds

$$\{\big(f_0^{[m,\underline{1}]}(q) : f_1^{[m,\underline{0}]}(q)\big) \mid q \in \mathcal{C}\} = \mathcal{S}.$$

Let $s \in \mathbb{P}^1$ with $(\underline{0}, s) \in \mathcal{L}_{\underline{0}}(B)$ and let $\big(p_n\big)_{n\in\mathbb{N}}$ be a sequence in $U \setminus Z$ with $\lim_{n\to\infty} p_n = \underline{0}$ and $\lim_{n\to\infty}\big(f_0(p_n) : f_1(p_n)\big) = s$. For all $n \in \mathbb{N}$ we may write $p_n = r_n q_n$ with $r_n \in \mathbb{R}_{>0}$ and $q_n \in \mathcal{C}$. Clearly $\lim_{n\to\infty} r_n = 0$. As $\mathcal{C}$ is compact, we may replace $\big(p_n\big)_{n\in\mathbb{N}}$ by an appropriate subsequence and hence assume that $\lim_{n\to\infty} q_n = q$ for some $q \in \mathcal{C}$. Keep in mind, that for all $n \in \mathbb{N}$ we have

$$\big(f_0(p_n) : f_1(p_n)\big) =$$
$$= \big(f_0^{[m,\underline{0}]}(q_n) + r_n \sum_{i>m_0} r_n^{i-m-1} f_0^{[i,\underline{0}]}(q_n) : f_1^{[m,\underline{0}]}(q_n) + r_n \sum_{i>m_1} r_n^{i-m-1} f_1^{[i,\underline{0}]}(q_n)\big).$$

This yields that $s = \big(f_0^{[m,\underline{0}]}(q) : f_1^{[m,\underline{0}]}(q)\big) \in \mathcal{S}$ and hence proves that $\mathcal{L}_p(B) \subseteq \{p\} \times \mathcal{S}$. The converse inclusion is immediate. □

Remark 3.8. If $\operatorname{mult}_p(f_0) \neq \operatorname{mult}_p(f_1)$, or if $\operatorname{mult}_p(f_0) = \operatorname{mult}_p(f_1) =: m$ and $f_0^{[m,p]}$ and $f_1^{[m,p]}$ have a common linear factor, the set $\mathcal{L}_p(B)$ may behave more complicated. But in the present paper, we shall not consider such examples.

Example 3.9. (A) Keep the notations and hypotheses of Proposition 3.7. It holds:

$$\text{If } \operatorname{rank}(\partial \underline{f})(p) = 2, \text{ then } \mathcal{L}_p(B) = \pi_{U,\underline{f}}^{-1}(p) = \{p\} \times \mathbb{P}^1.$$

Indeed, if $\operatorname{rank}(\partial \underline{f})(p) = 2$ we have $\operatorname{mult}_p(f_0) = \operatorname{mult}_p(f_1) = 1$ and $f_0^{[1,p]}$ and $f_1^{[1,p]}$ have no common linear factor and hence the rational function $\varrho(\mathbf{w}) := \frac{f_0^{[m,p]}(1+x,\mathbf{w}+y)}{f_1^{[m,p]}(1+x,\mathbf{w}+y)}$ is fractional linear, so that $\overline{\operatorname{Im}}(\varrho) = \mathbb{R} \cup \{\pm\infty\}$ and hence $\{\big(\frac{1}{\sqrt{1+\tau^2}} : \frac{\tau}{\sqrt{1+\tau^2}}\big) \mid \tau \in \overline{\operatorname{Im}}(\varrho)\} = \mathbb{P}^1$.

(B) We consider the blowup of Example 2.3 visualized in Figure 5 as a deformed double Whitney Umbrella:

$$B^{(t)} = \operatorname{Bl}_U\big(f_0^{(t)} = (1-t)\mathbf{x}^2 - \frac{t}{2}\mathbf{y}^2, f_1^{(t)} = \frac{t}{2}\mathbf{x}^2 + (1+t)\mathbf{y}^2\big) \in \mathfrak{Bl}_U\big(\{\underline{0}\}\big) \text{ with } t \in \mathbb{R}\setminus\{\pm\frac{2}{3}\sqrt{3}\}$$

and the corresponding rational function

$$\varrho(\mathbf{w}) := \frac{f_0^{[m,\underline{0}]}(1,\mathbf{w})}{f_1^{[m,\underline{0}]}(1,\mathbf{w})} = \frac{2(1-t) - t\mathbf{w}^2}{t + (2+2t)\mathbf{w}^2} \in \mathbb{Q}(\mathbf{w}).$$

In this case on use of Proposition 3.7 and Proposition 3.3 we obtain that

$$\iota\big(\mathcal{L}_p(B)\big) = \{\big(0, r\cos(\beta), r\sin(\beta) \mid -\pi \leq \beta \leq \pi \text{ and } \tan(\frac{\beta}{2}) \in \overline{\operatorname{Im}}(\varrho)\}$$

is a closed arc of variable length $\lambda(t) \leq \frac{\pi}{2}$ on the circle

$$\iota\big(\pi_{U,\underline{f}}^{-1}(\underline{0})\big) = \iota\big(\underline{0} \times \mathbb{P}^1\big) = \{\big(0, r\cos(\beta), r\sin(\beta) \mid -\pi \leq \beta \leq \pi\}$$

with $\lim_{t\to\pm\frac{2}{3}\sqrt{3}} \quad \lambda(t) = 0$.

4. Structure and Classification of Regular Embedded Blowups

Structure of Regular Embedded Blowups. We begin this section with the following structure result for regular blowups.

Proposition 4.1. *Let* $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. *Then*

(a) *For all* $p \in Z$ *the set of limit points of* B *above* p *coincides with the exceptional fiber of* B *above* p, *hence*

$$\mathcal{L}_p(B) = \pi_{U,\underline{f}}^{-1}(p) = \{p\} \times \mathbb{P}^1 \text{ and } B = B^\circ \dot{\cup} (Z \times \mathbb{P}^1).$$

(b) B *is a smooth real algebraic hyper-surface in* $U \times \mathbb{P}^1$.

Proof. Statement (a) is clear by Example 3.9 (A). To prove statement (b), let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ be a regular pair on U with respect to Z, such that $B = \operatorname{Bl}_U(\underline{f})$ and consider the polynomial

$$h := \mathbf{v}f_0(\mathbf{x},\mathbf{y}) - \mathbf{u}f_1(\mathbf{x},\mathbf{y}) \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}].$$

If $(x,y) \in U \setminus Z$ and $(u:v) \in \mathbb{P}^1$ it holds $h(x,y,u,v) = 0$ if and only if $\big((x,y),(u:v)\big) \in B^\circ$. By statement (a) it follows that

$$\{((x,y),(u:v)) \in U \times \mathbb{P}^1 \mid h(x,y,u,v) = 0\} = B.$$

It remains to show, that

$$\big(\frac{\partial h}{\partial \mathbf{x}}(x,y,u,v), \frac{\partial h}{\partial \mathbf{y}}(x,y,u,v), \frac{\partial h}{\partial \mathbf{u}}(x,y,u,v), \frac{\partial h}{\partial \mathbf{v}}(x,y,u,v)\big) \neq 0,$$

whenever $\big((x,y),(u:v)\big) \in B$. As $\frac{\partial h}{\partial \mathbf{u}} = -f_1$ and $\frac{\partial h}{\partial \mathbf{v}} = f_0$, this is clear if $p := (x,y) \notin Z$. If $p \in U$, we have $\mathrm{rank}\big((\partial \underline{f})(p)\big) = 2$ and $(u,v) \neq (0,0)$ shows that

$$\big(\frac{\partial h}{\partial \mathbf{x}}(x,y,u,v), \frac{\partial h}{\partial \mathbf{y}}(x,y,u,v)\big) = \big(v\frac{\partial f_0}{\partial \mathbf{x}}(p) - u\frac{\partial f_1}{\partial \mathbf{x}}(p), v\frac{\partial f_0}{\partial \mathbf{y}}(p) - u\frac{\partial f_1}{\partial \mathbf{y}}(p)\big) \neq 0.$$

□

Reduced and Strongly Regular Pairs and Application to Sign Distributions. The remaining part of this section is devoted to the Isomorphy Criterion mentioned in (1.19) and hence to the solution of the Classification Problem (1.0)(b) for regular embedded blowups. We first will introduce two special types of regular pairs of polynomials and relate these to the sign distribution map which was mentioned already in (1.16).

Lemma and Definition 4.2. *Let $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Then, there is a regular pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$, with respect to Z on U, unique up to multiplication with a non-zero constant – and called a reduced regular pair for B – such that*

(a) *f_0 and f_1 have no common divisor.*
(b) $\mathrm{Bl}_U(\underline{f}) = B$.
(c) *If $\underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ is a regular pair with respect to Z on U with $B = \mathrm{Bl}_U(\underline{g})$, then there is a unique polynomial $h \in \mathbb{R}[\mathbf{x},\mathbf{y}]$ such that $\underline{g} = h\underline{f}$. Moreover, in this situation*
   (1) *$h(p) \neq 0$ for all $p \in U$.*
   (2) *$\mathrm{sgn}\big(\det(\partial \underline{g}(p))\big) = \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big)$ for all $p \in Z$.*

Proof. We write $B = \mathrm{Bl}_U(\underline{g})$ where $\underline{g} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ is a regular pair with respect to Z on U. Let $h \in \mathbb{R}[\mathbf{x},\mathbf{y}]$ be a greatest common divisor of g_0 and g_1 and let $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ be such that $g_i = hf_i$ for $i = 0,1$. The Leibniz product rule for derivatives gives

$$(@)\quad \partial \underline{g} = \partial(h\underline{f}) = h\partial \underline{f} + \begin{pmatrix} f_0\frac{\partial h}{\partial \mathbf{x}} & f_1\frac{\partial h}{\partial \mathbf{x}} \\ f_0\frac{\partial h}{\partial \mathbf{y}} & f_1\frac{\partial h}{\partial \mathbf{y}} \end{pmatrix}.$$

Our immediate aim is to show that $h(p) \neq 0$ for all $p \in U$. If we assume to the contrary that $h(p) = 0$ for some $p \in U$, by $\underline{g} = h\underline{f}$, it would follow that $p \in Z$. But then by (@) the matrix $\partial g(p)$ would be of rank 1, which contradicts the fact that $\underline{g}$ is regular with respect to Z on U.

Now, another use of (@) gives that $\underline{f}$ is a regular pair with respect to Z on U. Moreover, it follows that the two maps $\varepsilon_{U,\underline{f}}$ and $\varepsilon_{U,\underline{g}}$ of (1.2) from $U \setminus Z$ to $\mathbb{P}^1$ coincide, so that $\mathrm{Bl}_U^\circ(\underline{f}) = \mathrm{Bl}_U^\circ(\underline{g})$ (see (1.5)(a)), and hence (see (1.6)(a)) $\mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(\underline{g}) = B$. Clearly f_0 and f_1 have no common divisor. Finally, a further use of (@) shows that $\mathrm{sgn}\big(\det(\partial \underline{g}(p))\big) = \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big)$ for all $p \in Z$, and this completes our proof. □

Definition and Remark 4.3. Let $B \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ and let $p \in Z$. We write $B = \mathrm{Bl}_U(\underline{g})$, where $\underline{g} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is a regular pair with respect to Z on U. Then, by Lemma 4.2 (c) it is immediate, that $\mathrm{sgn}\big(\det(\partial \underline{g}(p))\big)$ depends only on the blowup B and not on the chosen defining pair $\underline{g}$. This allows to define a map (see (1.16)

$$\mathrm{sgn}_B : Z \longrightarrow \{\pm 1\} \text{ given by } p \mapsto \mathrm{sgn}\big(\det(\partial \underline{f}(p))\big) \text{ for all } p \in Z.$$

We call this map the *sign distribution* of B.

Definition and Remark 4.4. (A) Let $Z = \{p_i = (x_i, y_i) \mid i = 1, 2, \ldots, n\} \subset U, (p_i \neq p_j$ for all $i \neq j)$. A pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is called *strongly regular with respect to Z (on U)*, if it satisfies the following equivalent requirements:

(i) $\mathbb{C}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{C}[\mathbf{x}, \mathbf{y}]f_1 = \bigcap_{i=1}^n \big(\mathbb{C}[\mathbf{x}, \mathbf{y}](\mathbf{x} - x_i) + \mathbb{C}[\mathbf{x}, \mathbf{y}](\mathbf{y} - y_i)\big)$.
(ii) $\mathbb{C}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{C}[\mathbf{x}, \mathbf{y}]f_1 = I_{\mathbb{A}^2}(Z) := \{f \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \mid f(Z) = 0\}$.

(B) Assume that $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ is a strongly regular pair with respect to Z. Then, it is easy to see:

(a) $\underline{f}$ is a regular pair with respect to Z on U in the sense of (1.14).
(b) $\underline{f}$ is a reduced regular pair for $B := \mathrm{Bl}_U(\underline{f})$ in the sense of Lemma and Definition 4.2.
(c) $\mathbb{R}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{R}[\mathbf{x}, \mathbf{y}]f_1 = I_{\mathbb{R}^2}(Z) := \{g \in \mathbb{R}[\mathbf{x}, \mathbf{y}] \mid g(Z) = 0\}$.

Lemma 4.5. *Let $n > 0$, let $Z := \{p_1, p_2, \ldots, p_n\}$ a set of pairwise different points with $p_i := (x_i, y_i) \in U$ for $i = 1, 2, \ldots, n$. Let $\chi : Z \longrightarrow \mathbb{R} \setminus \{0\}$ be a map. Then, there is a strongly regular pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z such that $\det\big(\partial \underline{f}(p)\big) = \chi(p)$ for all $p \in Z$.*

Proof. After a linear change of coordinates, we may assume that $x_i \neq x_j$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. We set

$$f_0 =: \prod_{i=1}^n (\mathbf{x} - x_i) \in \mathbb{R}[\mathbf{x}] \text{ and } f_1 = h(\mathbf{x})\big(\mathbf{y} - g(\mathbf{x})\big),$$

where $g(\mathbf{x}), h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ are the uniquely determined polynomials of degree $\leq n-1$ which respectively satisfy

$$g(x_i) = y_i \text{ and } h(x_i) = \frac{\chi(p_i)}{\prod_{j \neq i}(x_i - x_j)} \text{ for all } i = 1, 2, \ldots, n.$$

Observe also, that

$$\frac{\partial f_0}{\partial \mathbf{x}}(p_i) = \prod_{j \neq i}(x_i - x_j) \text{ and } \frac{\partial f_1}{\partial \mathbf{y}}(p_i) = h(x_i) \text{ for all } i = 1, 2, \ldots, n.$$

Now, for all $i = 1, 2, \ldots, n$ we obtain:

$$\det\big(\partial \underline{f}(p_i)\big) = \det\begin{pmatrix} \frac{\partial f_0}{\partial \mathbf{x}}(p_i) & \frac{\partial f_1}{\partial \mathbf{x}}(p_i) \\ \frac{\partial f_0}{\partial \mathbf{y}}(p_i) & \frac{\partial f_1}{\partial \mathbf{y}}(p_i) \end{pmatrix} =$$

$$= \det\begin{pmatrix} \prod_{j\neq i}(x_i - x_j) & \frac{\partial\big(h(\mathbf{x})(\mathbf{y}-g(\mathbf{x}))\big)}{\partial \mathbf{x}}(p_i) \\ 0 & h(x_i) \end{pmatrix} = \chi(p_i).$$

Therefore $\det\big(\partial \underline{f}(p_i)\big) = \chi(p_i)$ for all $i = 1, 2, \ldots, n$.
It is immediate to see, that $Z = \{p_1, p_2, \ldots, p_n\}$ is precisely the set of common zeros of the two polynomials $f_0, f_1 \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ in $\mathbb{C}^2$. As $\det\big(\partial \underline{f}(p_i)\big) = \chi(p_i) \neq 0$ for all $i \in \{1, 2 \ldots, n\}$ it follows, that $\mathbb{C}[\mathbf{x}, \mathbf{y}]f_0 + \mathbb{C}[\mathbf{x}, \mathbf{y}]f_1$ is the vanishing ideal $I_{\mathbb{A}^2_{\mathbb{C}}}(Z)$ of Z in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. So $\underline{f}$ is strongly regular with respect to Z on U. □

The Classification Result. Now we will establish the Isomorphy Criterion we are heading for in this section, and hence solve the Classification Problem mentioned under (1.0) (b). We first shall prove two auxiliary results.

Lemma 4.6. *Let $\underline{f} = (f_0, f_1), \underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ be two pairs such that $\mathrm{Z}_U(\underline{f}) = \mathrm{Z}_U(\underline{g}) = Z$. Let $N \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ such that $\underline{g} = \underline{f}N$. Moreover, for each $\gamma \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ we set*

$$N_\gamma := N + \gamma\begin{pmatrix} g_1 f_1 & -g_0 f_1 \\ -g_1 f_0 & g_0 f_0 \end{pmatrix}.$$

Then, it holds

(a) $N_\gamma(p) = N(p)$ *for all* $p \in Z$.
(b) $\underline{g} = \underline{f}N_\gamma$.
(c) $\det(N_\gamma) = \det(N) + \gamma\big(g_0^2 + g_1^2\big)$.
(d) *If* $\det\big(N(p)\big) > 0$ *for all* $p \in Z$*, then, there is some* $b \in \mathbb{R}_{>0}$ *such that* $\det\big(N_\gamma(p)\big) > 0$ *for all* $p \in U$ *and all* $\gamma \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ *with* $\inf\{\gamma(p) \mid p \in U\} > b$.

Proof. Statements (a) and (b) are immediate. To prove statement (c) we write

$$N = \begin{pmatrix} N_{11} & N_{21} \\ N_{12} & N_{22} \end{pmatrix}$$

On use of the column bi-linearity of the determinant and as

$$\det\begin{pmatrix} f_1 & N_{21} \\ -f_0 & N_{22} \end{pmatrix} = g_1 \text{ and } \det\begin{pmatrix} N_{11} & -f_1 \\ N_{12} & f_0 \end{pmatrix} = g_0,$$

we get indeed

$$\begin{aligned}\det(N_\gamma) &= \det\big(N+\gamma\begin{pmatrix} g_1f_1 & -g_0f_1\\ -g_1f_0 & g_0f_0\end{pmatrix}\big) = \det\begin{pmatrix} N_{11}+\gamma g_1f_1 & N_{21}-\gamma g_0f_1\\ N_{12}-\gamma g_1f_0 & N_{22}+\gamma g_0f_0\end{pmatrix} =\\ &= \det\begin{pmatrix} N_{11} & N_{21}\\ N_{12} & N_{22}\end{pmatrix} + \det\begin{pmatrix} \gamma g_1f_1 & N_{21}\\ -\gamma g_1f_0 & N_{22}\end{pmatrix} +\\ &+ \det\begin{pmatrix} N_{11} & -\gamma g_0f_1\\ N_{12} & \gamma g_0f_0\end{pmatrix} + \det\begin{pmatrix} \gamma g_1f_1 & -\gamma g_0f_1\\ -\gamma g_1f_0 & \gamma g_0f_0\end{pmatrix} =\\ &= \det(N) + \gamma g_1\det\begin{pmatrix} f_1 & N_{21}\\ -f_0 & N_{22}\end{pmatrix} + \gamma g_0\det\begin{pmatrix} N_{11} & -f_1\\ N_{12} & f_0\end{pmatrix} + 0 =\\ &= \det(N) + \gamma g_1^2 + \gamma g_0^2 = \det(N) + \gamma\big(g_0^2+g_1^2\big).\end{aligned}$$

It remains to show statement (d). So, assume that $\det\big(N(p)\big) > 0$ for all $p \in Z$. We have to show that there is some constant $b \in \mathbb{R}_{>0}$ such that $\det\big(N_\gamma(p)\big) > 0$ for all $p \in U$ and all constants $\gamma > b$. As $\det\big(N(p)\big) > 0$ for all $p \in Z$, there is some open set $W \subset U$ such that $Z \subset W$ and $\det\big(N(p)\big) > 0$ for all $p \in W$. It follows by statement (a) and (c) that

$$\det\big(N_\gamma(p)\big) > 0 \text{ for all } p \in W \text{ and all } \gamma > 0.$$

As U is bounded and $\mathrm{Z}_{\mathbb{R}^2}(\underline{g})$ does not contain any points of the boundary of U it follows that there is some $c > 0$ such that $g_0(p)^2 + g_1(p)^2 > c$ for all $p \in U \setminus W$. As U is bounded, there is some $C > 0$ such that $\det\big(N(p)\big) \geq -C$ for all $p \in U$. If $\gamma > b := \frac{C}{c}$ it follows that

$$\det\big(N_\gamma(p)\big) \geq \det\big(N(p)\big) + B\big(g_0(p)^2 + g_1(p)^2\big) > 0 \text{ for all } p \in U \setminus W,$$

and hence $\det\big(N_\gamma(p)\big) > 0$ for all $p \in U$. □

Lemma 4.7. *Let $\underline{f} = (f_0, f_1), \underline{g} = (g_0, g_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ be two pairs of polynomials such that $\underline{f}$ is strongly regular with respect Z and $\underline{g}$ is regular with respect to Z on U and consider the two blowups $B := \mathrm{Bl}_U(\underline{f}), C := \mathrm{Bl}_U(\underline{g}) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Then, the following statements are equivalent:*

(i) $\mathrm{sgn}_C = \mathrm{sgn}_B$.
(ii) *There is a matrix $M \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ such that $\det\big(M(p)\big) >$ for all $p \in U$ and $\underline{g} = \underline{f}M$.*

Proof. : (ii) ⇒ (i): This is immediate.
(i) ⇒ (ii): Assume that statement (i) holds. As $g_0, g_1 \in I_{\mathbb{R}^2}(Z)$, it follows by Definition and Remark 4.4(B)(c), that there is a matrix

$$N = \begin{pmatrix} N_{11} & N_{21}\\ N_{12} & N_{22}\end{pmatrix} = \begin{pmatrix} N_{\bullet 1}\\ N_{\bullet 2}\end{pmatrix} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2} \text{ with } \underline{g} = \underline{f}N.$$

By our assumption we have $\mathrm{sgn}\big(\det(\partial\underline{g}(p))\big) = \mathrm{sgn}_C(p) = \mathrm{sgn}_B(p) = \mathrm{sgn}\big(\det(\partial\underline{f}(p))\big)$ for all $p \in Z$. Moreover, by the Leibniz product rule for derivatives we have

$$\text{(@@)} \qquad \partial\underline{g} = \partial(\underline{f}N) = \partial\underline{f}\cdot N + f_0\cdot\partial N_{\bullet 1} + f_1\cdot\partial N_{\bullet 2}.$$

As $\underline{f}(Z) = 0$ it follows that $\det(\partial \underline{g}(p)) = \det(\partial \underline{f}(p)) \cdot \det\big(N(p)\big)$ and hence $\det\big(N(p)\big) > 0$ for all $p \in Z$. Now, by Lemma 4.6 (c), there is some $\gamma \in \mathbb{R}_{>0}$ such that the matrix $M := N_\gamma \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ satisfies $\det\big(M(p)\big) > 0$ for all $p \in U$. Moreover, Lemma 4.6 (b) yields that $\underline{g} = \underline{f}M$. □

Now, we are ready to formulate and to prove the main result of this section.

Theorem 4.8. *(Classification of Regular Embedded Blowups)*

(a) *For each function $\sigma : Z \longrightarrow \{+1, -1\}$ there is a regular embedded blowup $B \in \mathfrak{Bl}_U^{\text{reg}}(Z)$ such that $\text{sgn}_B = \sigma$.*
(b) *Let $B, C \in \mathfrak{Bl}_U^{\text{reg}}(Z)$. Then $B \cong C$ if and only if $\text{sgn}_B = \text{sgn}_C$.*
(c) *There are precisely $2^{\#Z}$ isomorphism types of regular embedded blowups of U along Z.*

Proof. (a): By Lemme 4.5 there is a strongly regular pair $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z such that $\det\big(\partial \underline{f}(p)\big) = \sigma(p)$ for all $p \in Z$. It suffices to chose $B = \text{Bl}_U(\underline{f})$.

(b): We may write $B = \text{Bl}_U(\underline{g})$, where $\underline{g} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ is a regular pair of polynomials with respect to Z on U.
Assume first that B and C are oriented embedded isomorphic, more precisely, that $C = \varphi(B)$ for some automorphism $\varphi_M : U \times \mathbb{P}^1 \longrightarrow U \times \mathbb{P}^1$ with $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ and $\det\big(M(p)\big) > 0$ for all $p \in U$. Then we may write $C = \text{Bl}_U(\underline{g}M)$. By the product rule for derivatives (see (@@), Proof of Lemma 4.7), as $\underline{g}(Z) = 0$ and as $\det(M(p)) > 0$ for all $p \in U$, we now obtain

$$\begin{aligned}\text{sgn}_C(p) &= \text{sgn}\big(\det[\partial(\underline{g}M)(p)]\big) = \text{sgn}\big(\det[(\partial \underline{g})(p)M(p)]\big) = \\ &= \text{sgn}\big(\det[\partial \underline{g}(p)]\det[M(p)]\big) = \text{sgn}\big(\det[\partial \underline{g}(p)]\big) = \\ &= \text{sgn}_B(p) \text{ for all } p \in Z.\end{aligned}$$

It follows that indeed $\text{sgn}_C = \text{sgn}_B$.
Assume conversely, that $\text{sgn}_C = \text{sgn}_B$. By Lemma 4.5 there is a strongly regular pair $\underline{f} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$ with respect to Z on U such that $\det\big(\partial \underline{f}(p)\big) = \text{sgn}_B(p) = \text{sgn}_C(p)$ for all $p \in Z$. By Lemma 4.7 there is a matrix $M \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ for all $p \in U$ and $\underline{g} = \underline{f}M$. But this means, that $D := \text{Bl}_U(\underline{f}) \cong B$. Similarly we see, that $D \cong C$. So B and C are embedded isomorphic.

(c): This is clear by statements (a) and (b). □

Remark 4.9. (A) The classification result Theorem 4.8 has been shown in the Master thesis [8], but remained unpublished yet.

(B) It should be observed, that Theorem 4.8 applies also in the degenerate case $Z = \emptyset$. It says that for each $B \in \mathfrak{Bl}_U^{\text{reg}}(\emptyset)$ it holds $B \cong \text{Bl}_U(\underline{1})$, with $\underline{1} := (1, 1) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^2$.

5. Deformation of Matrices and Isotopies of Embedded Blowups

Analytic Matrix Deformations. In this section, we approach the deformation Problem (1.0)(a) mentioned in the introduction We shall prove the Deformation Result (1.23). As already mentioned in the introduction, this means that we have to prove the result on polynomial deformations of matrices mentioned in (1.24). We first prove a result on real analytic deformation of matrices.

Notation and Remark 5.1. (A) Let $\mathcal{C}^\omega(U)$ denote the ring of real analytic functions on U. We chose a matrix

$$M = (M_{\bullet 1} \quad M_{\bullet 2}) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathcal{C}^\omega(U)^{2\times 2} \text{ with } \det\bigl(M(p)\bigr) > 0 \text{ for all } p \in U,$$

where

$$M_{\bullet 1} := \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \text{ and } M_{\bullet 2} := \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix}$$

denote the column vectors of M. We fix a point p_0 in U. As U is path-wise simply connected, there are uniquely determined continuous functions $\alpha_M, \beta_M \in \mathcal{C}(U)$ such that (see [7])

$$0 \leq \alpha_M(p_0), \beta_M(p_0) \leq 2\pi$$

and

$$M_{\bullet 1}(p) = \|M_{\bullet 1}(p)\| \begin{pmatrix} \cos(\alpha_M(p)) \\ \sin(\alpha_M(p)) \end{pmatrix}, \quad M_{\bullet 2}(p) = \|M_{\bullet 2}(p)\| \begin{pmatrix} \cos(\beta_M(p)) \\ \sin(\beta_M(p)) \end{pmatrix}, \text{ for all } p \in U.$$

Observe, that in particular we have

$$\det\bigl(M(p)\bigr) = \|M_{\bullet 1}(p)\| \cdot \|M_{\bullet 2}(p)\| \cdot \sin\bigl(\beta_M(p) - \alpha_M(p)\bigr) \neq 0 \text{ for all } p \in U.$$

Now, by continuity it follows that

(a) $0 < \beta_M(p) - \alpha_M(p) < \pi$ for all $p \in U$.

For all $p, q \in U$ and each smooth path $\sigma : [0,1] \longrightarrow U$ with $\sigma(0) = p$ and $\sigma(1) = q$ we have

$$\alpha_M(q) - \alpha_M(p) = \int_0^1 \frac{M_{\bullet 1}}{\|M_{\bullet 1}\|}\bigl(\sigma(t)\bigr) \wedge \frac{d}{dt}\Bigl[\frac{M_{\bullet 1}}{\|M_{\bullet 1}\|}\bigl(\sigma(t)\bigr)\Bigr] dt,$$
$$\beta_M(q) - \beta_M(p) = \int_0^1 \frac{M_{\bullet 2}}{\|M_{\bullet 2}\|}\bigl(\sigma(t)\bigr) \wedge \frac{d}{dt}\Bigl[\frac{M_{\bullet 2}}{\|M_{\bullet 2}\|}\bigl(\sigma(t)\bigr)\Bigr] dt.$$

This allows to conclude:

(b) $\alpha_M, \beta_M \in \mathcal{C}^\omega(U)$.

(B) Keep the notations and hypotheses of part (A). For each $t \in [0,1]$ and each $p \in U$ we set

$$M_{11}^{(t)}(p) := \bigl[(1-t) + t\|M_{\bullet 1}\|\bigr] \cdot \cos\bigl(t\alpha_M(p)\bigr),$$
$$M_{21}^{(t)}(p) := \bigl[(1-t) + t\|M_{\bullet 1}\|\bigr] \cdot \sin\bigl(t\alpha_M(p)\bigr),$$
$$M_{12}^{(t)}(p) := \bigl[(1-t) + t\|M_{\bullet 2}\|\bigr] \cdot \cos\bigl((1-t)\frac{\pi}{2} + t\beta_M(p)\bigr),$$
$$M_{22}^{(t)}(p) := \bigl[(1-t) + t\|M_{\bullet 2}\|\bigr] \cdot \sin\bigl((1-t)\frac{\pi}{2} + t\beta_M(p)\bigr),$$

and consider the matrices

$$M^{(t)} := \begin{pmatrix} M_{11}^{(t)} & M_{12}^{(t)} \\ M_{21}^{(t)} & M_{22}^{(t)} \end{pmatrix} \in \mathcal{C}(U)^{2\times 2}, \quad \big(t \in [0,1]\big).$$

For all $t \in [0,1]$ and all $p \in U$ we obtain:

$$\det\big(M^{(t)}(p)\big) =$$
$$= \big[(1-t) + t\|M_{\bullet 1}(p)\|\big] \cdot \big[(1-t) + t\|M_{\bullet 2}(p)\|\big] \cdot \sin\Big((1-t)\frac{\pi}{2} + t[\beta_M(p) - \alpha_M(p)]\Big).$$

Moreover $0 < \beta_M(p) - \alpha_M(p) < \pi$ (see statement (a) of Part (A)) implies

$$0 < (1-t)\frac{\pi}{2} + t[\beta_M(p) - \alpha_M(p)] < (1-t)\frac{\pi}{2} + t\pi = \frac{\pi}{2} + t\frac{\pi}{2} \le \pi.$$

So, in view of statement (b) of part (A) we can say:

(a) $M^{(t)} \in \mathcal{C}^\omega(U)^{2\times 2}$ and $\det\big(M^{(t)}(p)\big) > 0$ for all $t \in [0,1]$ and all $p \in U$.

Now, we solve our deformation problem for matrices with analytic entries.

Proposition 5.2. *Let $M \in \mathcal{C}^\omega(U)^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ for all $p \in U$. Then the family $\big(M^{(t)}\big)_{0\le t\le 1}$ of Notation and Remark 5.1 is an analytic family of matrices in $\mathcal{C}^\omega(U)^{2\times 2}$, with positive determinant on U, which connects the unit matrix $\mathbf{1}^{2\times 2}$ with the matrix M. More precisely*

(a) $M^{(t)} \in \mathcal{C}^\omega(U)^{2\times 2}$ and $\det\big(M^{(t)}(p)\big) > 0$ for all $t \in [0,1]$ and all $p \in U$.

(b) $M^{(0)} = \mathbf{1}^{2\times 2}$ *and* $M^{(1)} = M$.

(c) *The map* $\widetilde{M} : U \times [0,1] \longrightarrow \mathbb{R}^{2\times 2}$, *given by* $(p,t) \mapsto M^{(t)}(p)$, *is continuous and analytic on the open set* $U \times]0,1[$.

Proof. (a): This is immediate by Notation and Remark 5.1 (B)(a).

(b): This is obvious by the definition of the Matrices $M^{(t)}$.

(c): This follows easily from the definition of the functions $p \mapsto M_{ij}^{(t)}(p)$ (see Notation and Remark 5.1 (B)) and statement (b) of Notation and Remark 5.1 (A). □

Polynomial and Rational Matrix Deformations. We now attack the case of polynomial or rational matrix deformations. We begin with the following auxiliary result.

Lemma 5.3. *Let $K \subset \mathbb{R}^2$ be a non-empty compact set. Let $P, Q \in \mathbb{R}[\mathbf{x},\mathbf{y}]$ be two polynomials and let $F : K \times [0,1] \longrightarrow \mathbb{R}$ be a continuous function such that $F(p,0) = P(p)$ and $F(p,1) = Q(p)$ for all $p \in K$. Let $\varepsilon > 0$. Then, there is a polynomial $\widetilde{P} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]$ such that*

(a) $|F(p,t) - \widetilde{P}(p,t)| < \varepsilon$ *for all* $p \in K$ *and all* $t \in [0,1]$.

(b) $P(p) = \widetilde{P}(p,0)$ *and* $Q(p) = \widetilde{P}(p,1)$ *for all* $p \in K$.

Proof. By the Theorem of Stone-Weierstrass (see [4] (7.4.1)) there is a polynomial $\overline{P} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]$ such that

$$|F(p,t) - \overline{P}(p,t)| < \frac{\varepsilon}{2} \text{ for all } p \in K \text{ and all } t \in [0,1].$$

Now, set

$$\widetilde{P}(\mathbf{x},\mathbf{y},\mathbf{t}) := \overline{P}(\mathbf{x},\mathbf{y},\mathbf{t}) + (1-\mathbf{t})\big(P(\mathbf{x},\mathbf{y}) - \overline{P}(\mathbf{x},\mathbf{y},0)\big) + \mathbf{t}\big(Q(\mathbf{x},\mathbf{y}) - \overline{P}(\mathbf{x},\mathbf{y},1)\big).$$

It is easy to see that $\widetilde{P}$ has the requested properties. □

Proposition 5.4. *Let $M, N \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ such that $\det\big(M(p)\big) > 0$ and $\det\big(N(p)\big) > 0$ for all $p \in \overline{U}$. Then, the matrix N is connected on $\overline{U}$ to M by a polynomial family of polynomial 2×2-matrices with positive determinant on $\overline{U}$. More precisely: There is a matrix*

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{pmatrix} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$$

such that with $P^{(t)}(\mathbf{x},\mathbf{y}) := \widetilde{P}(\mathbf{x},\mathbf{y},t)$ for all $t \in \mathbb{R}$ we have:

(a) $P^{(0)}(p) = N(p)$ *for all* $p \in \overline{U}$.
(b) $P^{(1)}(p) = M(p)$ *for all* $p \in \overline{U}$.
(c) $\det\big(P^{(t)}(p)\big) > 0$ *for all* $p \in \overline{U}$ *and all* $t \in [0,1]$.

Proof. Observe that the closed set

$$\mathbb{S} := \{p \in \mathbb{R}^2 \mid \det\big((M(p)\big) \leq 0 \text{ or } \det\big(N(p)\big) \leq 0\}$$

is disjoint to $\overline{U}$. We thus find a bounded open star-shaped set W such that $\overline{U} \subset W$ and $W \cap \mathbb{S} = \emptyset$. Now, clearly $M, N \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ with $\det\big(M(p)\big), \det\big(N(p)\big) > 0$ for all $p \in W$. According to Proposition 5.2 we have two continuous maps

$$\widetilde{M} = \begin{pmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} : W \times [0,1] \longrightarrow \mathbb{R}^{2\times 2} \text{ with } \det\big(\widetilde{M}(p,t)\big) > 0, \text{ for all } (p,t) \in W \times [0,1],$$

$$\widetilde{N} = \begin{pmatrix} \widetilde{N}_{11} & \widetilde{N}_{12} \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{pmatrix} : W \times [0,1] \longrightarrow \mathbb{R}^{2\times 2} \text{ with } \det\big(\widetilde{N}(p,t)\big) > 0, \text{ for all } (p,t) \in W \times [0,1],$$

such that

$$\widetilde{M}(p,0) = \mathbf{1}^{2\times 2}, \text{ and } \widetilde{M}(p,1) = M(p), \text{ for all } p \in W,$$
$$\widetilde{N}(p,0) = \mathbf{1}^{2\times 2}, \text{ and } \widetilde{N}(p,1) = N(p), \text{ for all } p \in W.$$

Now, for all $i,j \in \{1,2\}$ we consider the continuous functions

$$\widetilde{F}_{ij} : W \times [0,1] \longrightarrow \mathbb{R} \quad \widetilde{F}_{i,j}(p,t) := \begin{cases} \widetilde{N}_{ij}(p, 1-2t) & \text{if } t \in [0,\frac{1}{2}] \\ \widetilde{M}_{ij}(p, 2t-1) & \text{if } t \in [\frac{1}{2},1] \end{cases}$$

and the matrix

$$\widetilde{F} := \begin{pmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ \widetilde{F}_{21} & \widetilde{F}_{22} \end{pmatrix} \in \mathcal{C}(W \times [0,1])^{2\times 2}.$$

Then $\widetilde{F}(p,0) = N(p)$, $\widetilde{F}(p,1) = M(p)$ and $\det\big(\widetilde{F}(p,t)\big) > 0$ for all $p \in W$ and all $t \in [0,1]$. As $\overline{U} \subset W$ is compact, there are $c, \delta > 0$ such that for all $i,j \in \{1,2\}$, all $p \in \overline{U}$ and all $t \in [0,1]$ it holds

$$-c \leq \widetilde{F}_{ij}(p,t) \leq c \quad \text{and} \quad \det\big(\widetilde{F}(p,t)\big) > \delta.$$

As the map $\det : \mathbb{R}^4 \longrightarrow \mathbb{R}$ is uniformly continuous on any compact subset of $\mathbb{R}^4$ we find some $\varepsilon > 0$ such that:

(1) $|\det\big(\widetilde{F}(p,t)\big) - \det\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}| < \frac{\delta}{2}$ for all $p \in \overline{U}$, all $t \in [0,1]$ and all $m_{ij} \in \mathbb{R}$ with $|m_{ij} - \widetilde{F}_{ij}(p,t)| < \varepsilon \quad (i,j \in \{1,2\})$.

Now, we apply Lemma 5.3 to the four continuous functions $\widetilde{F}_{ij} : \overline{U} \times [0,1] \longrightarrow \mathbb{R}$ and obtain four polynomials $\widetilde{P}_{ij} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]$, such that for all $i,j \in \{1,2\}$ we have:

(2) $|\widetilde{F}_{ij}(p,t) - \widetilde{P}_{ij}(p,t)| < \varepsilon$ for all $p \in \overline{U}$ and all $t \in [0,1]$,
(3) $N_{ij}(p) = \widetilde{F}_{ij}(p,0) = \widetilde{P}_{ij}(p,0)$ for all $p \in \overline{U}$ and
(4) $M_{ij}(p) = \widetilde{F}_{ij}(p,1) = \widetilde{P}_{ij}(p,1)$ for all $p \in \overline{U}$.

We set

$$\widetilde{P} := \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22}\end{pmatrix}.$$

Then, the above statements (1) and (2) yield that

$$|\det\big(\widetilde{F}(p,t)\big) - \det\big(\widetilde{P}(p,t)\big)| < \frac{\delta}{2} \text{ for all } p \in \overline{U} \text{ and all } t \in [0,1],$$

so that

$$\det\big(P^{(t)}(p)\big) = \det\big(\widetilde{P}(p,t)\big) > \frac{\delta}{2} > 0 \text{ for all } p \in \overline{U} \text{ and all } t \in [0,1].$$

By the above statements (3) and (4) we obtain

$$P^{(0)}(p) = \widetilde{P}(p,0) = N(p) \text{ and } P^{(1)}(p) = \widetilde{P}(p,1) = M(p) \text{ for all } p \in \overline{U}.$$

Altogether, this proves our claim. □

Remark 5.5. As an immediate consequence we now get the result announced in the introduction under (1.24).

Remark 5.6. As early as 2002, the first named author did ask for the existence of a connecting family $(M^{(t)})_{t\in[0,1]}$ as in Proposition 5.4 – but only continuous, not polynomial – at the occasion of a talk he gave at the IIT Bombay. A few weeks after this, A.R. Shastri [14] suggested a proof for the existence of a piecewise linear connecting family $(M^{(t)})_{t\in[0,1]}$. The authors are grateful to him for his hint. Clearly, instead of Proposition 5.2 one also could use Shastri's result to prove Proposition 5.4.

As an easy consequence of the above proposition we now get:

Corollary 5.7. *Let* $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix} \in \mathbb{R}(\mathbf{x},\mathbf{y})^{2\times 2}$ *be such that none of its entries* $M_{ij}, (i,j \in \{1,2\})$ *has a pole in* $\overline{U}$, *and such that* $\det\big(M(p)\big) > 0$ *for all* $p \in \overline{U}$.
Then, the unit matrix $\mathbf{1}^{2\times 2}$ *is connected over* $\overline{U}$ *to* M *by a rational family of* 2×2*-matrices*

which are defined and of positive determinant on $\overline{U}$. More precisely:
There is a matrix

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & \widetilde{Q}_{22} \end{pmatrix} \in \mathbb{R}(\mathbf{x}, \mathbf{y}, \mathbf{t})^{2\times 2}$$

such that no $\widetilde{Q}_{ij}$ has a pole on $\overline{U}$ and such that, with $Q^{(t)}(\mathbf{x}, \mathbf{y}) := \widetilde{Q}(\mathbf{x}, \mathbf{y}, t)$, (for all $t \in \mathbb{R}$) :

(a) $Q^{(0)} = \mathbf{1}^{2\times 2}$.
(b) $Q^{(1)}(p) = M(p)$ *or all* $p \in \overline{U}$.
(c) $\det\big(Q^{(t)}(p)\big) > 0$ *for all* $p \in \overline{U}$ *and all* $t \in [0, 1]$.

Proof. The closed set

$$\mathcal{P} := \bigcup_{1\leq i,j \leq 2)} \text{Pole}(M_{ij}) \cup \{p \in \mathbb{R}^2 \mid \det\big((M(p)\big) \leq 0\}$$

is disjoint to $\overline{U}$. We thus find a bounded open star-shaped set W such that $\overline{U} \subset W$ and $W \cap \mathcal{P} = \emptyset$. So, none of the four entries M_{ij} of M has a pole in W and moreover $\det\big(M(p)\big) > 0$ for all $p \in W$. As W is path-wise connected and by taking common denominators we find

$$H \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2} \text{ and } G \in \mathbb{R}[\mathbf{x}, \mathbf{y}] \text{ with } G(p) > 0 \text{ and } M(p) = \frac{H(p)}{G(p)} \text{ for all } p \in W.$$

In particular we have $\det\big(G(p)\mathbf{1}^{2\times 2}\big) > 0$ and $\det\big(H(p)\big) > 0$ for all $p \in W$, hence for all $p \in \overline{U}$. By Proposition 5.4 there is a matrix $\widetilde{P} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2}$ such that

(1) $\widetilde{P}(p, 0) = G(p)\mathbf{1}^{2\times 2}$ for all $p \in \overline{U}$;
(2) $\widetilde{P}(p, 1) = H(p)$ for all $p \in \overline{U}$;
(3) $\det\big(\widetilde{P}(p, t)\big) > 0$ for all $p \in \overline{U}$ and all $t \in [0, 1]$.

Now, with $\widetilde{Q} := \frac{\widetilde{P}}{G}$ we get our claim. □

Matrix Deformations Linear in Time. A particular simple case occurs if one can deform the unit matrix $\mathbf{1}^{2\times 2}$ to the matrix M of Corollary 5.7 by a family $\big(\widetilde{Q}_{(t)}\big)_{t\in[0,1]}$ which is linear t. The following Remark is devoted to this situation.

Remark 5.8. Let $M(\mathbf{x}, \mathbf{y}) = M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{R}(\mathbf{x}, \mathbf{y})^{2\times 2}$ be such that none of its entries $M_{ij}, (i, j \in \{1, 2\})$ has a pole in $\overline{U}$ and such that $\det\big(M(p)\big) > 0$ for all $p \in \overline{U}$. Then the unit matrix $\mathbf{1}^{2\times 2}$ can be deformed to $M = M^{(1)}$ by a family

$$\big(M^{(t)} = M^{(t)}(\mathbf{x}, \mathbf{y}) := \widetilde{M}(\mathbf{x}, \mathbf{y}, t)\big)_{t\in[0,1]}$$

which is linear in t if and only if the matrix

$$\widetilde{M} = \widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{t}) := \mathbf{t}M(\mathbf{x}, \mathbf{y}) + (1 - \mathbf{t})\mathbf{1}^{2\times 2} \in \mathbb{Q}(\mathbf{x}, \mathbf{y}, \mathbf{t})^{2\times 2}$$

satisfies $\det\big(M(x,y)^{(t)}\big) = \det\big(\widetilde{M}(x,y,t)\big) > 0$ for all $p = (x,y) \in \overline{U}$, and for all $t \in [0,1]$, hence if and only if

$$\big(\det(M(p)) - \mathrm{tr}(M(p)) + 1\big)t^2 + \big(\mathrm{tr}(M(p)) - 2\big)t + 1 > 0, \text{ for all } p \in \overline{U} \text{ and all } t \in [0,1].$$

This holds in particular, if the occurring quadratic polynomial in t has no real zero, hence if its discriminant $\mathrm{D}(p)$ satisfies

$$\mathrm{D}(p) = \mathrm{tr}\big(M(p)\big)^2 - 4\det\big(M(p)\big) < 0 \text{ for all } p \in \overline{U}.$$

Isotopies of Embedded Blowups. As an application of Proposition 5.4 we now prove the result on the deformation of regular embedded blowups by means of isotopies mentioned in (1.23).

Theorem 5.9. *Let $B, C \in \mathfrak{Bl}_U(Z)$ be such that $B \cong C$. Then, B and C are connected by an isotopy of $U \times \mathbb{P}^1$-automorphisms. More precisely, there is a matrix*

$$\widetilde{M} = \begin{pmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$$

such that with $M^{(t)}(\mathbf{x},\mathbf{y}) := \widetilde{M}(\mathbf{x},\mathbf{y},t)\big($ for all $t \in \mathbb{R})$ it holds

(a) *$\det\big(M^{(t)}\big) > 0$ for all $p \in U$ (and hence $\varphi^{(t)} := \varphi_{M^{(t)}}$ is an automorphism of $U \times \mathbb{P}^1$) for all $t \in [0,1]$.*
(b) *$\varphi^{(0)}(B) = B$ and $\varphi^{(1)}(B) = C$.*

Proof. Let $\underline{f} \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ be such that $Z_{\overline{U}}(\underline{f}) = Z$. As $B \cong C$ we find some matrix $N \in \mathbb{R}[\mathbf{x},\mathbf{y}]^{2\times 2}$ with $\det\big(N(p)\big) > 0$ for all $p \in U$ and such that, with $(g_0, g_1) = \underline{g} := \underline{f}N$, it holds $C = \mathrm{Bl}_U(\underline{g})$ (see (1.15)). Now, we chose $\gamma \in \mathbb{R}_{>0}$ and consider the matrix

$$M := N_\gamma = N + \gamma \begin{pmatrix} g_1 f_1 & -g_0 f_1 \\ -g_1 f_0 & g_0 f_0 \end{pmatrix}$$

of Lemma 4.6. Then, by statements (b), (c) and (d) of that Lemma and as g_0 and g_1 have no common zero on the boundary of U, it follows that for γ large enough we have $\det\big(M(p)\big) > 0$ for all $p \in \overline{U}$ and $\underline{g} = \underline{f}M$.

But now Proposition 5.4 yields that there is a matrix $\widetilde{M} \in \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$ such that, with $M^{(t)}(\mathbf{x},\mathbf{y}) := \widetilde{M}(\mathbf{x},\mathbf{y},t)$, it holds

(1) $M^{(0)}(p) = \mathbf{1}^{2\times 2}$ for all $p \in \overline{U}$;
(2) $M^{(1)}(p) = M(p)$ for all $p \in \overline{U}$;
(3) $\det\big(M^{(t)}(p)\big) > 0$ for all $p \in \overline{U}$ and all $t \in [0,1]$.

In particular we get the stated existence of the matrix $\widetilde{M} = \mathbb{R}[\mathbf{x},\mathbf{y},\mathbf{t}]^{2\times 2}$ and hence also statement (a).

As $\varphi^{(0)}(B) = \varphi_{M^{(0)}}(B) = \varphi_{\mathbf{1}^{2\times 2}}(B) = \mathrm{id}_{U\times\mathbb{P}^1}(B) = B$ and $C = \mathrm{Bl}_U(\underline{f}M) = \mathrm{Bl}_U(\underline{f}M^{(1)}) = \varphi_{M^{(1)}}\big(\mathrm{Bl}_U(\underline{f})\big) = \varphi_{M^{(1)}}(B) = \varphi^{(1)}(B)$ we get statement (b).

□

6. Further Examples of Families of Blowups

Two Families of Regular Two-point Blowups. Already in Example 2.1 and Example 2.2 we have presented deformations of regular blowups by means of a particularly simple matrix deformation. We begin the present section with slightly more involved matrix deformations and we shall illustrate their effect on two non-isomorphic regular embedded two-point blowups. We fix our settings as in the examples given in the introduction and in Section 2 by choosing $\rho = 2, r = 4, U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$.

Example 6.1. (A) We fix a polynomial $a = a(\mathbf{x}, \mathbf{y}) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and consider the matrix

$$\widetilde{M} = \widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{t}) := \begin{pmatrix} 1 - a(\mathbf{x}, \mathbf{y})\mathbf{t} & a(\mathbf{x}, \mathbf{y})\mathbf{t} \\ -a(\mathbf{x}, \mathbf{y})\mathbf{t} & 1 + a(\mathbf{x}, \mathbf{y})\mathbf{t} \end{pmatrix} \in \mathbb{R}[\mathbf{x}, \mathbf{y}, \mathbf{t}]^{2\times 2} \text{ with } \det(\widetilde{M}) = 1$$

and the matrices

$$M^{(t)} = M^{(t)}(\mathbf{x}, \mathbf{y}) := \widetilde{M}(\mathbf{x}, \mathbf{y}, t) \in \mathbb{R}[\mathbf{x}, \mathbf{y}]^{2\times 2} \text{ with } \det(M^{(t)}) = 1 \text{ for all } t \in \mathbb{R}.$$

So, for any regular blowup $B = \mathrm{Bl}_U(\underline{f}) = \mathrm{Bl}_U(f_0, f_1) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$ we get an isotopic family

$$\big(B^{(t)} = \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]} \text{ such that for all } t \in [0, 1] \text{ it holds:}$$

$$B^{(t)} = \mathrm{Bl}_U\big(f_0 - t \cdot a(\mathbf{x}, \mathbf{y})(f_0 + f_1), f_1 + t \cdot a(\mathbf{x}, \mathbf{y})(f_0 + f_1)\big) \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z) \text{ and } B^{(t)} \cong B.$$

We thus get a family $\big(B^{(t)}\big)_{t\in[0,1]}$ of isotopic blowups $B^{(t)} \in \mathfrak{Bl}_U^{\mathrm{reg}}(Z)$, which connects $B = B^{(0)}$ with

$$C := B^{(1)} = \mathrm{Bl}_U(\underline{f}M^{(1)}) = \mathrm{Bl}_U\big(f_0 - a(\mathbf{x}, \mathbf{y})(f_0 + f_1), f_1 + a(\mathbf{x}, \mathbf{y})(f_0 + f_1)\big).$$

As announced, we aim to illustrate the situation by means of two regular two-point blowups, which are of essentially different embedded isomorphism type, a situation which can indeed only occur for regular blowups with respect to more than one point. More precisely, we shall blow up U with respect to two different pairs $\underline{f}$ of regular polynomials which both satisfy $Z_U(\underline{f}) = \{(\pm 1, 0)\}$, but such that $\mathrm{sgn}_{\underline{f}}$ is non-constant in the first case and constant in the second case.

(B) We keep the general settings of part (A), set $a(\mathbf{x}, \mathbf{y}) := \mathbf{xy}$ and consider the regular two-point blowup $B := \mathrm{Bl}_U(\underline{f})$ of U with respect to $Z := \{(\pm 1, 0)\}$ given by $f_0 := \mathbf{x}^2 + \mathbf{y}^2 - 1$ and $f_1 := \mathbf{y}$. We then have $\mathrm{sgn}_B\big((\pm 1, 0)\big) = \pm 1$, so that the sign distribution $\mathrm{sgn}_B = \mathrm{sgn}_{\underline{f}}$ is non-constant. The visualization of the resulting family of two-point blowups $B^{(t)} \cong B^{(0)} = B$ is presented in Figure 6 for $t = 0, \quad 0.5, \quad 1$.

(C) We now chose $a(\mathbf{x}, \mathbf{y}) := \mathbf{y}$ and consider the the regular two-point blowup $B := \mathrm{Bl}_U(\underline{f})$ of U with respect to $Z := \{(\pm 1, 0)\}$ given by $f_0 := \mathbf{x}^2 - 1$ and $f_1 := \mathbf{xy}$. This time, it holds $\mathrm{sgn}_B\big((\pm 1, 0)\big) = 1$, so that the sign distribution $\mathrm{sgn}_B = \mathrm{sgn}_{\underline{f}}$ is constant. This means, that we get a two-point blowup whose embedded isomorphism type differs essentially from the isomorphism type of the blowup of part (B). The visualization of the resulting family of two-point blowups $B^{(t)} \cong B^{(0)} = B$ is presented in Figure 7 for $t = 0, \quad 0.5, \quad 1$.

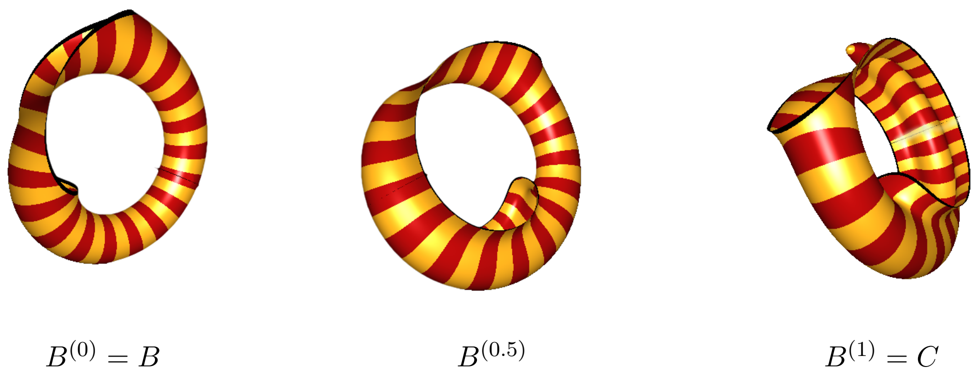


FIGURE 6. Deformation of a Regular Two-Point Blowup with Non-Constant Sign Distribution

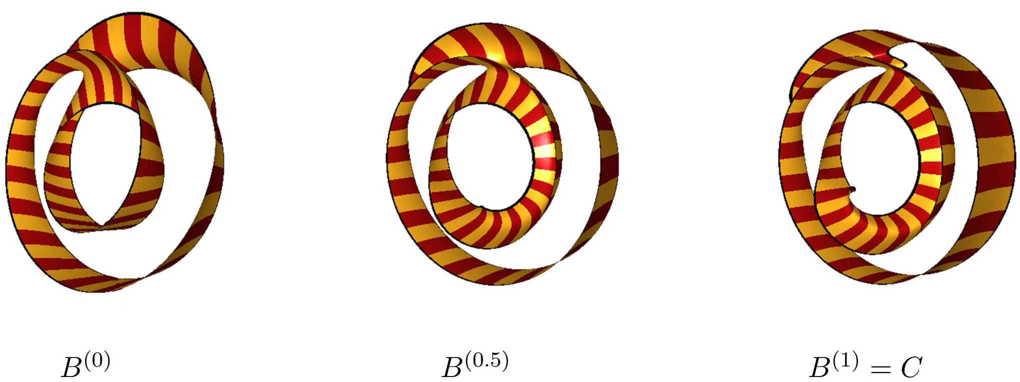


FIGURE 7. Deformation of a Regular Two-Point Blowup with Constant Sign Distribution

Two Families of Regular Three-point Blowups. Up to now, we have seen examples of families of regular n-point blowups for $n = 1, 2$ and $n = 4$ (see Figure 3, Figures 6 and 7 and Figure 4 respectively). We now aim to present two families of regular 3-point blowups. As above we chose $\rho = 2, r = 4, U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$ for our visualization.

Example 6.2. (A) We consider the following example of [9] given by:

$$B := \mathrm{Bl}_U(\underline{f}), \text{ with } f_0 := \frac{1}{2}(\mathbf{x} - 1) + \mathbf{y}^2 \text{ and } f_1 := (\mathbf{x} + \frac{1}{2})\mathbf{y}.$$

We have

$$Z = Z_U(\underline{f}) = \{p_1, p_2, p_3\} \text{ with } p_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), p_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}), p_3 = (1, 0)$$

and hence the set Z is an equilateral triangle centered at the origin $\underline{0} \in \mathbb{R}^2$. Moreover, it holds

$$\det\big(\partial \underline{f}\big)(p_1) = \det\big(\partial \underline{f}\big)(p_2) = -\frac{3}{2} \text{ and } \det\big(\partial \underline{f}\big)(p_3) = \frac{3}{2}.$$

So B is a regular three-point blowup. The sign distribution and hence the embedded isomorphism type of B is given by

$$\mathrm{sgn}_B(p_i) = \begin{cases} -1, & \text{for } i = 1,2 \\ 1, & \text{for } i = 3. \end{cases}$$

So, in this case we have a *regular three-point blowup with non-constant sign distribution* Inspired by Remark 5.8 we consider the matrix

$$M^{(1)} = M = M(\mathbf{x},\mathbf{y}) := \begin{pmatrix} \mathbf{x} & -2 \\ 3 & \mathbf{y} \end{pmatrix} \text{ with } \mathrm{tr}\big(M(p)\big)^2 - 4\det\big(M(p)\big) < 0 \text{ for all } p \in \overline{U}.$$

Then according to the quoted remark

$$\big(B^{(t)} := \mathrm{Bl}_U(\underline{f}M^{(t)})\big)_{t\in[0,1]} \text{ with } M^{(t)} = \begin{pmatrix} t\mathbf{x} + (1-t) & -2t \\ 3t & t\mathbf{y} + (1-t) \end{pmatrix}$$

is an isotopic family of regular three-point blowups which non-constant sign distribution, whose visualization is presented in Figure 8 for $t = 0, \quad 0.33, \quad 0.5, \quad 1.$

(B) We now aim to present a family of regular three-point blowups with constant sign distribution. We chose

$$Z = \{p_1 = (x_1,y_1) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), p_2 = (x_2,y_2) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}), p_3 = (x_3,y_3) = (1,0)\}$$

as in part (A). Our first aim is to find a strongly regular pair $\underline{f} = (f_0, f_1) \in \mathbb{R}[\mathbf{x},\mathbf{y}]^2$ with respect to Z on U (see Definition 4.4) such that $\det\big(\partial \underline{f}\big)(p_i) = 1$ for $i = 1,2,3$. We do this according to the procedure suggested in the proof of Lemma 4.5, but with the rôles of $\mathbf{x}$, $\mathbf{y}$ and of f_0, f_1 exchanged respectively. We thus set

$$f_1 = \prod_{i=1}^{3}(\mathbf{y} - y_i) = \mathbf{y}^3 - \frac{3}{4}\mathbf{y} = \mathbf{y}(\mathbf{y}^2 - \frac{3}{4}) \text{ and } f_0 = h(\mathbf{y})\big(\mathbf{x} - g(\mathbf{y})\big) \text{ with}$$

$$\deg(h), \deg(g) \leq 2, \text{ and } g(y_i) = x_i, \quad h(y_i) = \frac{1}{\prod_{j\neq i}(y_i - y_j)} \text{ for } i = 1,2,3.$$

So

$$g(\mathbf{y}) = -2\mathbf{y}^2 + 1 \text{ and } h(\mathbf{y}) = \frac{4}{3}(2\mathbf{y}^2 - 1), \text{ thus}$$

$$f_0 = \frac{4}{3}(2\mathbf{y}^2 - 1)(\mathbf{x} + 2\mathbf{y}^2 - 1) = \frac{4}{3}(4\mathbf{y}^4 + 2\mathbf{y}^2\mathbf{x} - 4\mathbf{y}^2 - \mathbf{x} + 1).$$

Now $B := \mathrm{Bl}_U(\underline{f})$ is a *regular three-point blowup with constant sign distribution* $\mathrm{sgn}_B(p_i) = 1$ for $i = 1,2,3$.

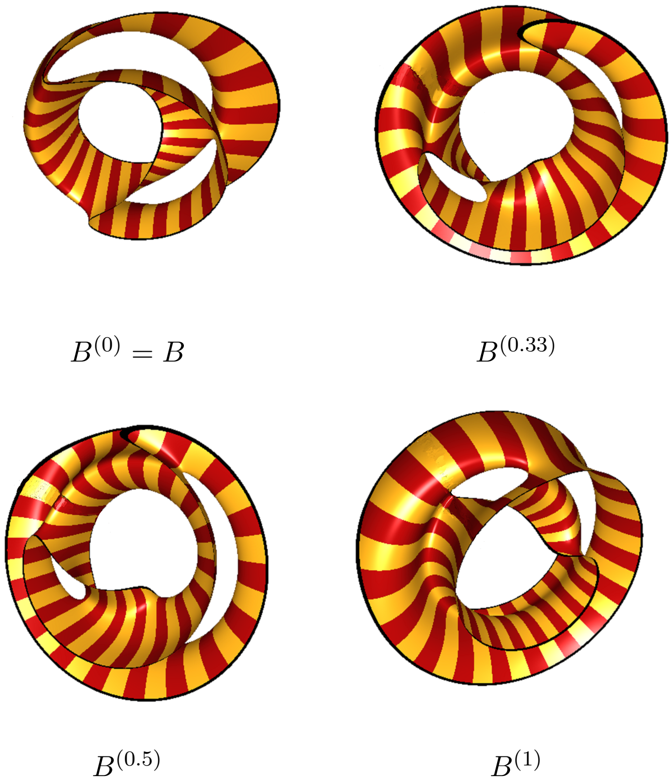


FIGURE 8. Deformation of a Regular Three-Point Blowup with Non-Constant Sign Distribution

Our present example illustrates at this point, that the method suggested in the proof of Lemma 4.5 tends to furnish pairs of polynomials which may be simplified without changing the sign distribution (and hence the isomorphism type) in $\mathfrak{Bl}_U^{\mathrm{reg}}(Z)$. Namely, by setting

$$h_0 := \frac{3}{4} f_0 - 4\mathbf{y} f_1 = 2\mathbf{x}\mathbf{y}^2 - \mathbf{y}^2 - \mathbf{x} + 1 \text{ and } h_1 := 4f_1 = 4\mathbf{y}^3 - 3\mathbf{y}$$

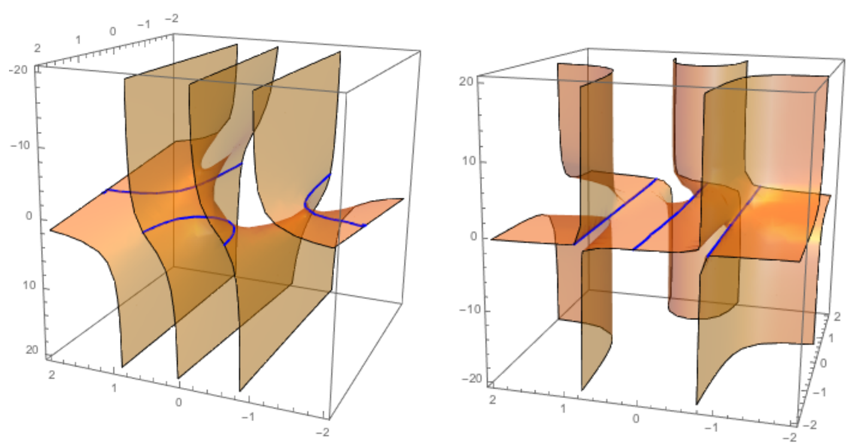
we get indeed $Z_{\mathbb{R}^2}(\underline{h}) = \{p_1, p_2, p_3\}$ and $\det(\partial \underline{h}) = 3(2\mathbf{y}^2 - 1)(4\mathbf{y}^2 - 1)$ so that $\det\big(\partial \underline{h}(p_i)\big) > 0$ and hence $\mathrm{sgn}_{\underline{h}}(p_i) = \mathrm{sgn}_{\underline{f}}(p_i) = 1$ for $i = 1, 2, 3$.

For a better visualization of the blowup $\mathrm{Bl}_U(\underline{h})$ we modify it slightly by interchanging the two indeterminates $\mathbf{x}$, $\mathbf{y}$ and the two polynomials h_0 and h_1 and by multiplying the first of them by $\frac{1}{3}$. So, we shall consider the blowup $B = \mathrm{Bl}_U(\underline{g})$ with $\underline{g} = (g_0, g_1)$,

$$g_0 = \mathbf{x}(\mathbf{x}^2 - \frac{3}{4}) \text{ and } g_1 = 2\mathbf{x}^2\mathbf{y} - \mathbf{x}^2 - \mathbf{y} + 1$$

under the deformation given by the family of matrices $M^{(t)}$ of part (A). This time, for the sake of virtual simplicity, we present with our method of visualization only the single blowup $B^{(t)} = \mathrm{Bl}_U(\underline{g}M^{(t)})$ for $t = 0.5$ and the two affine charts of the blowup $B^{(0)}$ given

Figure 9. Deformation of a Regular Three-Point Blowup with Constant Sign Distribution for $t = 0.5$



1. Chart 2. Chart

Figure 10. Two Charts of a Regular Three-Point Blowup with Constant Sign Distribution

respectively by $g_1(\mathbf{x}, \mathbf{y}) - \mathbf{z}g_0(\mathbf{x}, \mathbf{y}) = 0$ and $g_0(\mathbf{x}, \mathbf{y}) - \mathbf{z}g_1(\mathbf{x}, \mathbf{y}) = 0$. The two charts were visualized by means of Mathematica.

References

1. Brandenberg, M.: *Aufblasungen affiner Varietäten* (in German). Diploma Thesis, Institut für Mathematik, Universität Zürich (1992) [available on request as PDF].
2. Brodmann, M.: *Computerbilder von Aufblasungen* (in German). Elemente der Mathematik 50 (1995) 149-163.
3. Brodmann, M.: *Blowing-Up!* KIAS Newsletters, Korea Institute for Advanced Study, March 1st (2009), 40-43.

4. Dieudonné, J.: *Foundations of Modern Analysis*. Academic Press, New York and London, 1960.
5. Hartshorne, R.: *Algebraic Geometry*. Graduate Texts in Mathematics 52, Springer-Verlag New York, Heidelberg, Berlin, 1977.
6. Hironaka, H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero*. Annals of Mathematics 79 (1964). I 109-203; II: 205-226.
7. Knörrer, H.: *Geometrie*. Vieweg Studium, Aufbaukurs Mathematik, 2. Auflage, Springer Vieweg 2006.
8. Koller, S.: *Eingebettete Isomorphie zwischen Punktaufblasungen* (in German). Diploma Thesis, Institut für Mathematik, Universität Zürich (2001) [available on request as PDF].
9. Korolnik, N.: *Reelle Teile von Punktaufblasungen der affinen Ebene* (in German). Diploma Thesis, Institut für Mathematik, Universität Zürich (1999) [available on request as PDF].
10. Prager, A.: *Visualisierung von Mehrpunktaufblasungen affiner Varietäten* (in German). Master Thesis, Institut für Informatik, Martin Luther Universität, Halle-Wittenberg (2011) [available on request as PDF].
11. Schenzel, P. and Stussak, C.: *Interactive Visualizations of Blowups of the Plane*. IEEE Transactions on Visualization and Computer Graphics 19 (2013) 978-990.
12. Stussak, C.: *Echtzeit-Raytracing algebraischer Flächen auf der GPU. Diploma thesis, Martin Luther University Halle-Wittenberg, 2007.* URL `http://realsurf.informatik.uni-halle.de`
13. Stussak, C.: *On reliable visualization algorithms for real algebraic curves and surfaces*, PhD dissertation, Martin-Luther-Universität Halle-Wittenberg, Halle (Saale), 2013, `http://digital.bibliothek.uni-halle.de/ulbhalhs/urn/urn:nbn:de:gbv:3:4-10854`
14. Shastri, A.R.: *Personal Communication*. Department of Mathematics, Indian Institute of Technology Bombay (2002).

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