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# **Around the Circle**

**Notes Based on a Course for High School  
Teachers in Mathematics**

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# Preface

**The Significance of the Circle.** The circle may be considered as one of the most basic geometric objects at all, whose omnipresence in every-days life made it to an object of human investigation since the ancient age. Therefore, the circle also plays a fundamental rôle in Mathematics, and we all know that a great part of Geometry taught at Secondary- and High Schools is more or less intimately related to the circle. This made us choose the circle as the central object of the present course.

Clearly, we all know the circle as an object of great practical meaning, as we meet it in form of wheels and related objects at many instances. But the circle is and always was also an object of great impact on Mathematics and Philosophy. In this course, we hope to shed some light to both aspects of the circle. It is obviously impossible to present a complete picture of the tremendous wealth of mathematical treasures related to the circle. We shall restrict ourselves to some mathematical aspects of the circle, which are accessible to elementary Geometry, Algebra and Calculus, as they are taught at High Schools.

**Brief Preview to the Content.** The present notes contain essentially more material, than treated in our course. Our idea is, to present a text, which encourages the reader to study the subjects treated in the course in more detail on his own. The course is rather understood to just give a guideline, how to work through the notes. We follow the idea, that the notes should be written in stringent mathematical style, with proper definitions, strictly formulated statements, and deductive proofs, as far as possible. So, part of our presentation will exceed what can be treated in the class room. But we hope that this offers a chance to the reader, to brush up what he has learned while attending University. It namely is a most important issue for mathematics teacher, that they keep staying familiar with professional mathematical thinking.

But clearly, we also keep in mind the point of view of teaching Mathematics in High School. As said already above, the choice of our theme was inspired by this “class-room perspective”. But we also include a number of examples and exercises, which can be transferred directly to the classroom. To provide some help for this transfer, we include a special section “Didactic Comments” to each Chapter. For some proofs, we also offer two variants, with the idea to increase the chance to find a presentation fitting to the class-room.

A further aim of this course is to do initiate some “eye-opening”, by introducing subjects which are natural and inspiring extensions of what we did while treating the circle. So, inspired by the classical definition of circle, we to introduce the concept of metric and present a few examples of metrics in the plane. We also refer to the Bible to discuss the statement, that the “circumference of a round vessel of diameter of 10 ells is 30 ells”. Moreover, we present an anecdote which claims that the circumference of a circle is three times its diameter – and turn this statement in challenging exercise. We also twice consider the cycloid, a curve intimately related to the circle. The problem of

defining the notion of circumference of a circle in a proper way will lead us to look at lengths of smooth parametrized curves. Here we use some arguments, which go beyond what can be treated at High School. Our aim is to “bring back to the the readers ear some sounds” they heard during their University studies. We do this, to give evidence, that the elementary subjects treated at High School usually are just the tip of the “iceberg” of some deeper mathematical considerations. This hopefully provokes some eye-opening (or rather ear-opening) in the readers. Generalizing the notion of length of a curve, we even just look at path lengths of sets. This subject is illustrated by a number of examples and exercises, which could be treated at High School – hopefully with an “eye-opening” effect. Finally, we take up the problem of squaring the circle. This will lead us to two different types of eye-opening: We namely first discuss how to perform the four basic arithmetic operations by straightedge and compasses. This clearly is eye-opening on the class-room level. Then we recall and make precise the notions of constructible, algebraic and transcendent numbers. This second eye-opening takes place on the teachers level and will remind the readers of some subjects they heard in their basic Algebra courses at University.

These notes are divided in five Chapters, namely:

Chapter 1: The Circle

Chapter 2: Approaching  $\pi$

Chapter 3: Defining  $\pi$  as a Limit

Chapter 4: What is the Circumference?

Chapter 5: Squaring the Circle.

**Prerequisites.** Clearly, we have to assume that the reader has a basic background in Set Theory, Elementary Logics, Elementary Arithmetics, Elementary and Linear Algebra, Plane Geometry, Vector Geometry, Basic Calculus and – above all – the motivation and practice to follow some basic abstract mathematical arguments. We now fix a few notations which we will keep throughout these notes.

**0.1 Notations and Conventions.** (A) (*Basic Sets of Numbers*) By  $\mathbb{R}$  we denote the set of *real numbers*, by  $\mathbb{Q}$  the subset of  $\mathbb{R}$  which consists of all *rational numbers* – hence of all *fractions* – and by  $\mathbb{Z}$  the set of all *integers*. The numbers which belong to the set  $\mathbb{R} \setminus \mathbb{Q}$ , hence the non-rational real numbers, are referred to as *irrational numbers*. Note that we have

$$\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

with

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\},$$

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}, \text{ and}$$

$$\mathbb{R} = \left\{ \lim_{n \rightarrow \infty} q_n \mid q_n \in \mathbb{Q}, q_n \leq q_{n+1} \leq Q \text{ for } n = 1, 2, \dots \text{ and some } Q \in \mathbb{Q} \right\}.$$

(B) (*Specified Sets of Numbers*) If  $\mathbb{S} \subseteq \mathbb{R}$  and  $a \in \mathbb{R}$  we write

$$\mathbb{S}_{\geq a} := \{s \in \mathbb{S} \mid s \geq a\} \text{ and } \mathbb{S}_{>a} := \{s \in \mathbb{S} \mid s > a\}.$$

Similarly, for each  $b \in \mathbb{R}$  we set

$$\mathbb{S}_{\leq b} := \{s \in \mathbb{S} \mid s \leq b\} \text{ and } \mathbb{S}_{<b} := \{s \in \mathbb{S} \mid s < b\}.$$

This allows to introduce the set of *non-negative integers* and the set of *natural numbers* respectively by

$$\mathbb{N}_0 := \mathbb{Z}_{\geq 0} \text{ and } \mathbb{N} := \mathbb{Z}_{\geq 1} = \mathbb{N}_0 \setminus \{0\}.$$

If  $a, b \in \mathbb{R}$  we also may define the *closed interval* respectively the *open interval* between  $a$  and  $b$  respectively by:

$$[a, b] := \mathbb{R}_{\geq a} \cap \mathbb{R}_{\leq b} + \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

$$]a, b[ := \mathbb{R}_{<a} \cap \mathbb{R}_{>b} + \{x \in \mathbb{R} \mid a < x < b\}.$$

(C) (*Basic Notions of Plane Geometry*) Our basic geometric object is the (*standard affine*) *plane*, which we shall denote by  $\mathbb{E}$ . We endow  $\mathbb{E}$  with a (*standard normal coordinate system*) with an *x-axis* and a *y-axis*. So, each point  $P \in \mathbb{E}$  is determined by its *pair of coordinates*  $(x_P, y_P) \in \mathbb{R}^2$ . Without further mention, we use this to identify the point  $P \in \mathbb{E}$  with its pair of coordinates  $(x_P, y_P)$  and hence write

$$\mathbb{E} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

The line segment between two points  $P, Q \in \mathbb{E}$  will be denoted by  $|PQ|$ . So in the above notations we have

$$|PQ| = |QP| = \{(tx_P + (1-t)x_Q, ty_P + (1-t)y_Q) \mid 0 \leq t \leq 1\}.$$

The *distance* between two points  $P, Q \in \mathbb{E}$  coincides with the *length* of the line segment  $|PQ|$  and is given by

$$\text{dist}(P, Q) = \sqrt{(x_Q - x_P)^2 + (y_P - y_P)^2}.$$

(D) We also will sometimes consider  $\mathbb{R}^2$  as a *vector space* over  $\mathbb{R}$ . In this case we consider the pairs  $(x, y) \in \mathbb{R}^2$  as as vectors and we write them as columns, hence in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

Keep in mind, that in this context the *addition* of vectors respectively the *scalar multiplication* of vectors with real numbers are respectively defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} := \begin{pmatrix} x + x' \\ y + y' \end{pmatrix} \quad \text{and} \quad \lambda \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}, \quad \text{for all } x, x', y, y', \lambda \in \mathbb{R}.$$

The *length* or *norm* of the vector

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

is given by

$$\|\vec{v}\| = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := \sqrt{x^2 + y^2}.$$

If  $P, Q \in \mathbb{E} = \mathbb{R}^2$  are two points, the *vector pointing from P to Q* is given by

$$\overrightarrow{PQ} := \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix}.$$

Observe, that in this situation we have

$$\text{dist}(P, Q) = \|\overrightarrow{PQ}\|.$$

Often we write

$$O := (0, 0) \in \mathbb{E}$$

for the *origin* of our coordinate system, so that – for each point  $P \in \mathbb{E}$  – the vector

$$\overrightarrow{OP} = \begin{pmatrix} x_P \\ y_P \end{pmatrix} \in \mathbb{R}^2$$

is the *position vector* of  $P$ .

(E) (*Basic Notions from Calculus*) We assume that the reader has good knowledge on the basics of calculus, such as limits, suprema, infima, continuous functions, derivatives, Riemann integration.

(F) (*Basic Notions from Linear and Abstract Algebra*) In our last chapter, we assume that the reader knows certain basic notions from Linear Algebra and Abstract Algebra. This includes the notion of dimension of vector spaces (over the rational numbers) and the handling of polynomials with rational coefficients.

**Acknowledgement.** The author's thanks go to Professor Ernest Beyaraza, for having initiated the North-South Cooperation between the University of Zürich and Kabale University and Uganda Martyr's University in Kabale, which among other activities resulted in the present course. He also thanks very much to the great and generous hospitality and support offered to the Summer School by St. Mary's College at Rushoroza, Kabale. He also thanks to all the participants, who did sacrifice a holyday week for attending this course and for their attention and willingness to learn.

# 1. The Circle

**The Elementary Description of the Circle.** As usually in Mathematics, we need to define our objects of investigation, before we can start to study them. So, we now right away repeat the well-known definition of circle, which actually seems to go back to *Euclid* (Greek Mathematician, ca. 300 B.C.):

**1.1 Definition.** (A) (*Metric Definition of the Circle*) A circle  $\mathcal{C}$  in the plane  $\mathbb{E}$  is the locus (or – equivalently – the set) of all points  $P$  in  $\mathbb{E}$  which have the same distance (say  $r$ ) from a given point (say  $Z$ ) in  $\mathbb{E}$ . In this situation, the point  $Z$  is called the *center* of the circle  $\mathcal{C}$  and the common distance  $r$  of all points of  $\mathcal{C}$  from the center  $Z$  is called the *radius* of  $\mathcal{C}$ . In terms of Set Theory we thus could write

$$\mathcal{C} = \mathcal{C}_r(Z) := \{P \in \mathbb{E} \mid \text{dist}(P, Z) = r\}.$$

(B) (*Metric Definition of Closed and open Circular Disks*) Let  $\mathcal{C} = \mathcal{C}_r(Z) \subseteq \mathbb{E}$  be the circle of radius  $r$  and center  $Z$ . Then we introduce the *closed disk with boundary  $\mathcal{C}$* , thus the set

$$\mathcal{D} = \mathcal{D}_r(Z) := \{P \in \mathbb{E} \mid \text{dist}(P, Z) \leq r\},$$

which we call also the *closed disk of radius  $r$  and center  $Z$* . Sometimes, it will also be appropriate to consider the *interior* of the closed disk  $\mathcal{D}_r(Z)$ , hence the set

$$\mathcal{U} = \mathcal{U}_r(Z) := \mathcal{D}_r(Z) \setminus \mathcal{C}_r(Z) = \{P \in \mathbb{E} \mid \text{dist}(P, Z) < r\},$$

which is also called the *open disk of radius  $r$  and center  $Z$* .

Let us also recall that mainly in analysis the two disks  $\mathcal{D}_r(Z)$  and  $\mathcal{U}_r(Z)$  are called respectively the *closed* or *open  $r$ -neighborhood* of  $Z$ , with  $r$  replaced by the Greek letter  $\varepsilon > 0$ .

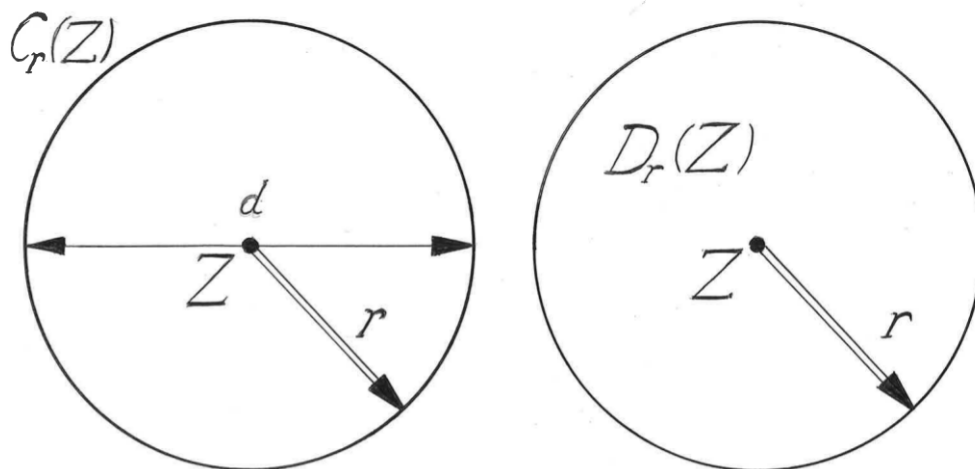


Figure 1.1: Circle and Disk with Center  $Z$  and Radius  $r$

**1.2 Remark.** (A) (*The Relative Nature of the Metric Definition of the Circle*) Clearly, the above definition is of *relative nature*. It namely supposes, that the notion of *plane* and the notion of *distance* between points in this plane are already available. So, if one would like to proceed in a completely rigorous way, one first would have to introduce these two notions in a consistent way. This actually is a task of *Axiomatic Geometry*, which goes beyond the framework of these lectures.

Let us just say here, that Euclids “Elements” are usually considered as the first attempt to build up a mathematical theory in an axiomatic manner. From the point of view of contemporary Mathematical Logics, the “Elements” do not yet quite satisfy the requirements of being really an axiom based theory. Nevertheless the impact of Euclid’s idea, to build up a mathematical theory only by logical deduction from a few axioms, was of basic significance for the development of Mathematics at all.

It was actually only towards the end of the 19th century, when a strictly axiomatic build-up of Geometry was formulated, for example by the German mathematician David Hilbert (1863-1943).

(B) (*The Premathematical Aspect of the Metric Definition of the Circle*) From a *heuristic* or *premathematical* point of view, the above definition of circle is quite satisfactory, and it indicates for example how a circle can be drawn by means of a nail (marking the center  $Z$ ) and a string of length  $r$  fixed to this nail.

A technically more advanced version of the same idea is the use of compasses (see Figure 1.2).

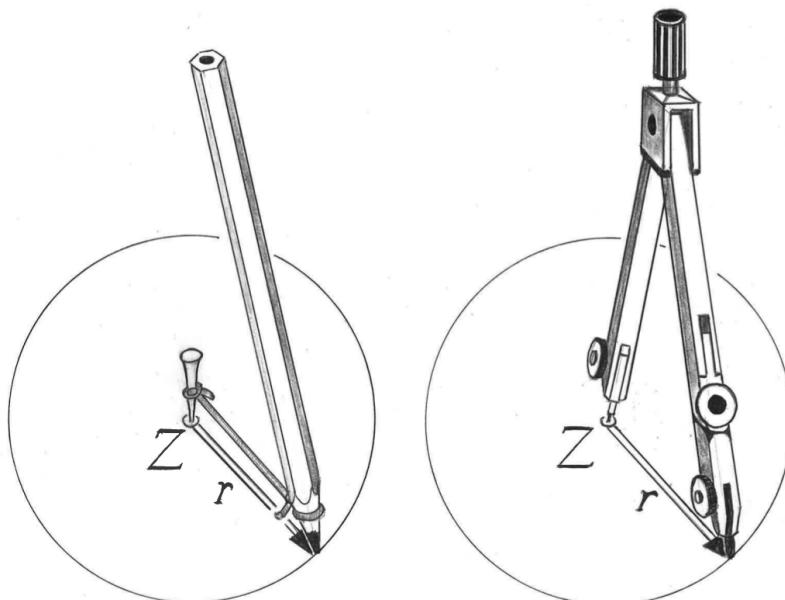


Figure 1.2: Drawing a Circle with Center  $Z$  and Radius  $r$

**Metrics and Generalized Circles.** The metric definition of the circle and of the circular disk make indeed sense in a very general setting - namely in the

situation in which the plane  $\mathbb{E}$  is replaced by an arbitrary metric space. We present this idea below. The first step consists in introducing the notion of metric space, or equivalently, the notion of metric on a set. We also will introduce a few metrics of the plane, among them the standard metric, hence the usual distance. A metric is a device mimicking – more precisely: generalizing – the notion of distance in the plane. Correspondingly, using a metric instead the usual distance to define a circle, will give us new types of “circles”, which sometimes are of unexpected shape.

**1.3 Definition, Remark and Exercise.** (A) (*The Concept of Metric Space*) Let  $\mathbb{M}$  be a non-empty set. A *metric* on  $\mathbb{M}$  is a map

$$d : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{R}, \quad (P, Q) \mapsto d(P, Q) \in \mathbb{R} \quad (\forall P, Q \in \mathbb{M})$$

which assigns to each pair  $(P, Q)$  of elements  $P, Q \in \mathbb{M}$  a real number  $d(P, Q)$  – a real number which is considered as the *distance* between  $P$  and  $Q$ . So, one subjects the map  $d : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{R}$ , to some additional conditions, which reflect the idea of distance in a very general manner. These requirements on the map  $d$  are:

- (a) (*Symmetry*)  $d(P, Q) = d(Q, P)$  for all  $P, Q \in \mathbb{M}$ .
- (b) (*Positivity*) For all  $P, Q \in \mathbb{M}$  it holds  $d(P, Q) \geq 0$  with equality if and only if  $P = Q$ .
- (c) (*The Triangle Inequality*) For all  $P, Q, R \in \mathbb{M}$  it holds

$$d(P, R) \leq d(P, Q) + d(Q, R).$$

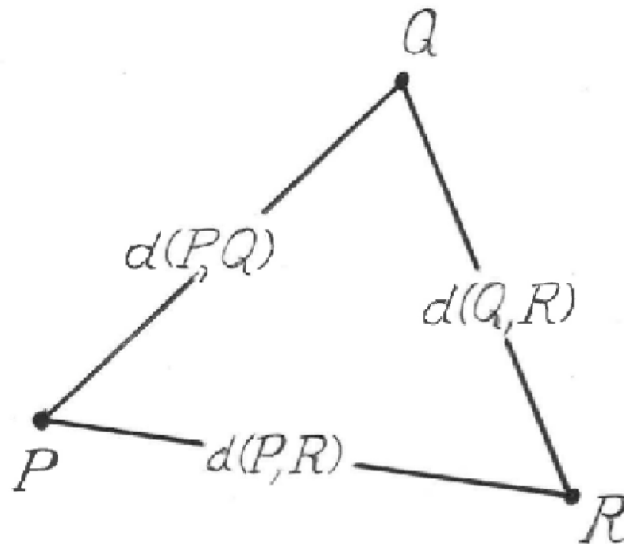


Figure 1.3: The Notion of Metric

By a *metric space* we mean a pair  $(\mathbb{M}, d)$  in which  $\mathbb{M} \neq \emptyset$  is a (non-empty) set and  $d : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$  is a metric on  $\mathbb{M}$ .



(B) (*The Standard Metric of the Plane*) Convince yourself – in the sense of an exercise – that the assignment

$$\text{dist} : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}, \quad (P, Q) \mapsto \text{dist}(P, Q) \quad (\forall P, Q \in \mathbb{E})$$

defines a metric on the plane  $\mathbb{E}$ . This metric is called the *standard metric* on the plane  $\mathbb{E}$ .

To describe this metric in a more formal way, we introduce a rectangular standard *coordinate system* in  $\mathbb{E}$ , consisting of an  $x$ - and a  $y$ -axis. Then consider the points

$$P = (x_P, y_P) \in \mathbb{E} \text{ and } Q = (x_Q, y_Q) \in \mathbb{E}$$

with coordinates  $x_P, y_P$  and  $x_Q, y_Q$  respectively. Check as an exercise, that

$$\text{dist}(P, Q) = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}.$$

Prove, exclusively on use of this algebraic description of distance, that the map  $\text{dist} : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$  satisfies indeed the three requirements of symmetry, positivity and of the triangle inequality as formulated in part (A).

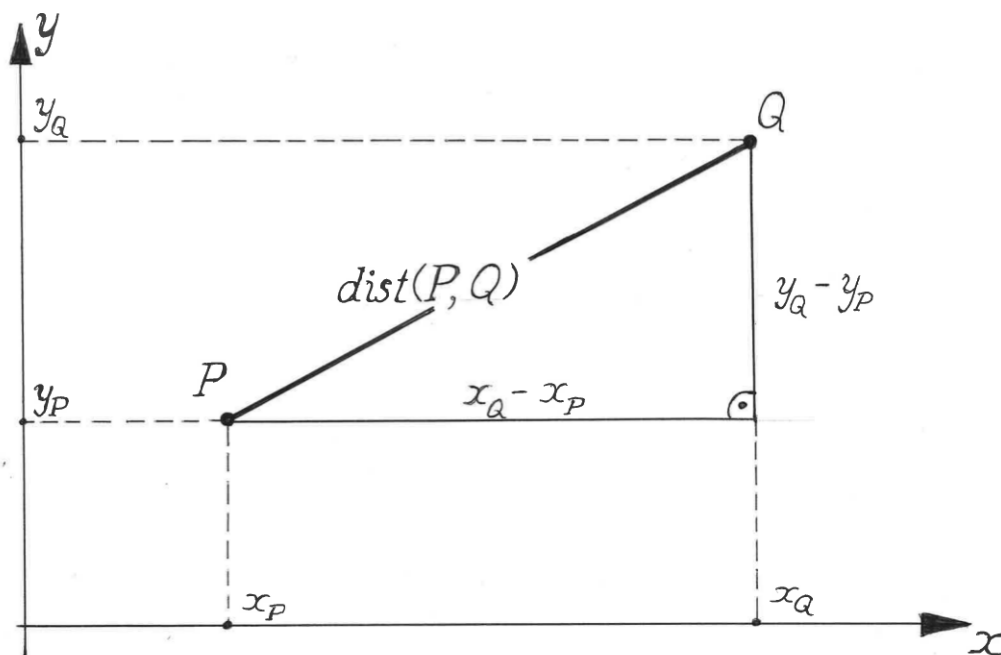


Figure 1.4: The Standard Metric

(C) (*The New York Taxi Driver Metric*) Let again  $P = (x_P, y_P) \in \mathbb{E}$  and  $Q = (x_Q, y_Q) \in \mathbb{E}$  be two points in the plane  $\mathbb{E}$  with coordinates  $x_P, y_P$  and  $x_Q, y_Q$  respectively. We set

$$d^{[sq]}(P, Q) := |x_Q - x_P| + |y_Q - y_P|.$$

Prove that the assignment

$$d^{[sq]} : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}, \quad (P, Q) \mapsto d^{[sq]}(P, Q) \in \mathbb{R} \quad (\forall P, Q \in \mathbb{E})$$

defines a metric on the plane  $\mathbb{E}$ . This metric is called the *square metric* or the *New York taxi driver metric*.

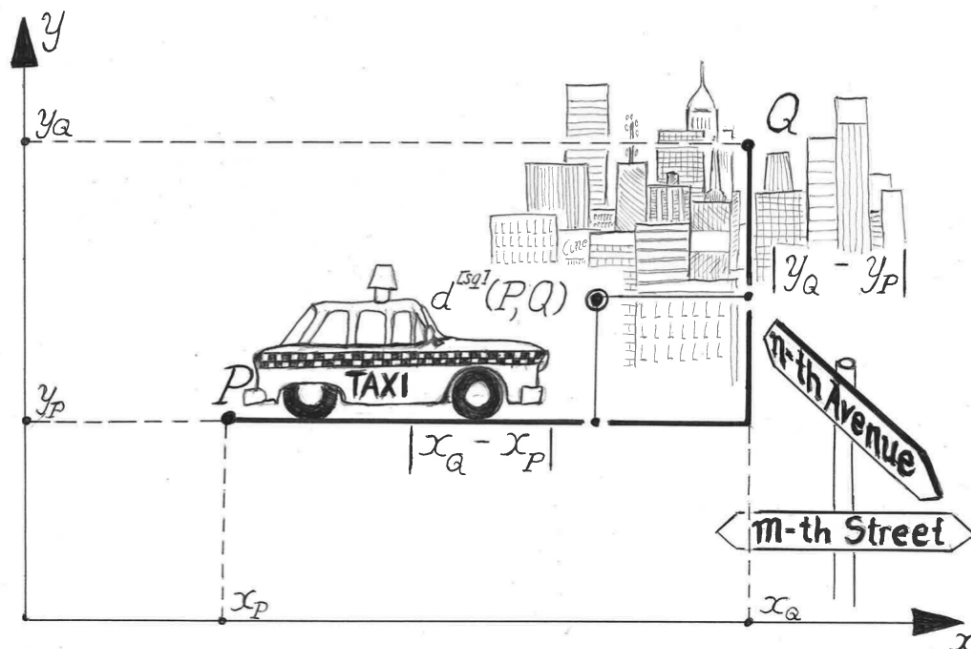


Figure 1.5: The New York Taxi Driver Metric

Explain the name of New York taxi driver metric. Show also, that

$$\text{dist}(P, Q) \leq d^{[sq]}(P, Q) \quad \forall P, Q \in \mathbb{E}.$$

Fix some point  $O \in \mathbb{E}$  and sketch the set

$$\{Q \in \mathbb{E} \mid \text{dist}(P, O) = d^{[sq]}(P, O)\}.$$

(D) (*The French Railway Metric*) We fix a point  $O \in \mathbb{E}$  and set

$$d^{[O]}(P, Q) := \begin{cases} \text{dist}(P, Q), & \text{if } O, P \text{ and } Q \text{ are collinear,} \\ \text{dist}(P, O) + \text{dist}(O, Q), & \text{else} \end{cases}$$

for all  $P, Q \in \mathbb{E}$ . Prove, that the assignment

$$d^{[O]} : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}, \quad (P, Q) \mapsto d^{[O]}(P, Q) \in \mathbb{R} \quad (\forall P, Q \in \mathbb{E})$$

defines a metric on the plane  $\mathbb{E}$ . This metric is called the *central point metric* or *French railway metric* on  $\mathbb{E}$  with respect to  $O$ .

Explain the name of French railway metric. Show that

$$\text{dist}(P, Q) \leq d^{[O]}(P, Q) \text{ for all } P, Q \in \mathbb{E}.$$

Fix  $P \in \mathbb{E} \setminus \{O\}$  and sketch the sets

$$\mathbb{U} := \{Q \in \mathbb{E} \mid \text{dist}(P, Q) = d^{[O]}(P, Q)\},$$

$$\mathbb{V} := \{Q \in \mathbb{E} \mid d^{[O]}(P, Q) \leq 3\text{dist}(P, Q)\},$$

Let  $O := (0, 0)$ ,  $P := (0, 2)$  compute the function

$$f : \mathbb{R} \longrightarrow \mathbb{R} \text{ given by } t \mapsto f(t) := d^{[O]}((t, 1), P) \text{ for all } t \in \mathbb{R}$$

and sketch the graph of this function.

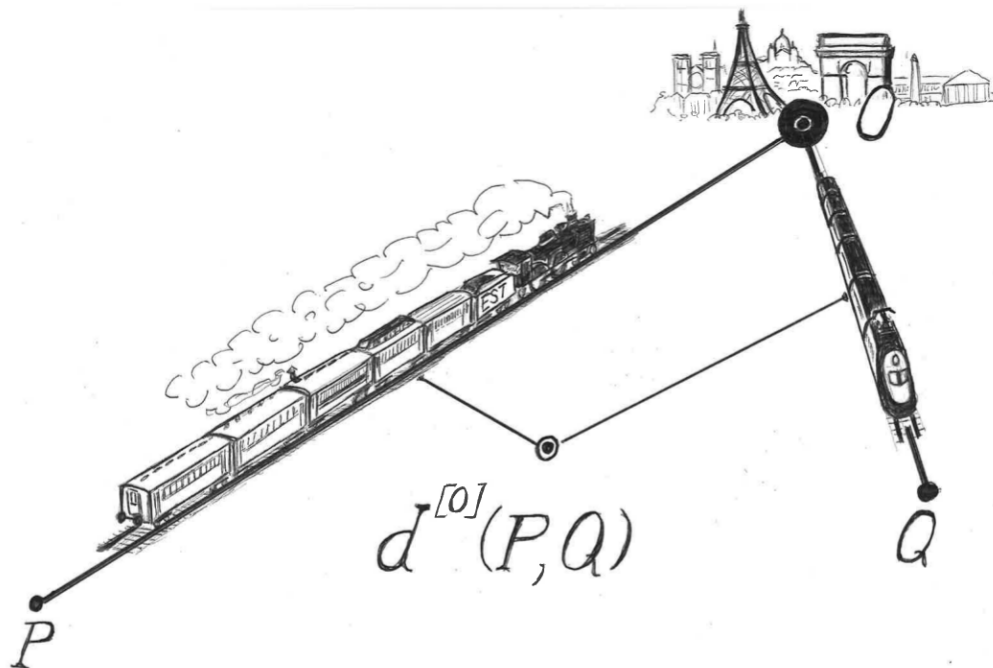


Figure 1.6: The French Railway Metric

Now, we will define the notion of circle and circular disk with respect to arbitrary metrics, and consider a few examples.

**1.4 Definition and Exercise.** (A) (*Metric Circles and Disks*) Let

$$d : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}, \quad (P, Q) \mapsto d(P, Q) \in \mathbb{R} \quad (\forall P, Q \in \mathbb{E})$$

be a metric on  $\mathbb{E}$ . Let  $Z \in \mathbb{E}$  and  $r \in \mathbb{R}_{>0}$ . Then, the *circle of radius  $r$  and center  $Z$  with respect to  $d$*  is defined as

$$\mathcal{C}_r^{[d]}(Z) := \{P \in \mathbb{E} \mid d(P, Z) = r\}.$$

Moreover the *closed* and *open disk of radius  $r$  and center  $Z$  with respect to  $d$*  are respectively defined by

$$\mathcal{D}_r^{[d]}(Z) := \{P \in \mathbb{E} \mid d(P, Z) \leq r\},$$

$$\mathcal{U}_r^{[d]}(Z) := \{P \in \mathbb{E} \mid d(P, Z) < r\},$$

(B) (*Strange Circles*) Sketch the following disks and circles:

- $\mathcal{D}_2^{[d^{[O]}]}(Z)$  and  $\mathcal{C}_{\frac{1}{2}}^{[d^{[O]}]}(Z)$  for  $O, Z \in \mathbb{E}$  with  $\text{dist}(O, Z) = 1$ .
- $\mathcal{D}_2^{[d^{[sq]}]}(Z)$  and  $\mathcal{D}_{\frac{1}{2}}^{[d^{[O]}]}(Z)$  for  $O, Z \in \mathbb{E}$  with  $\text{dist}(O, Z) = 1$ .
- $\mathcal{U}_{\frac{1}{2}}^{[d^{[O]}]}(Z)$  and  $\mathcal{C}_{\frac{1}{2}}^{[d^{[O]}]}(Z)$  for  $O, Z \in \mathbb{E}$  with  $\text{dist}(O, Z) = 1$ .

**Parametrizing the Circle.** After this look at strange circles and disks, we return to our standard circle, which we aim to describe now by means of a *parametrization*. This analytic description is obviously a most powerful tool to describe and to study the circle. But parametrizations are indeed a much more general tool, which allows to describe and to study arbitrary plane curves. The basic idea of the *parametric description* is to study geometric properties of plane curves by means of analytic tools, usually from bi-variate calculus. We will make use of this later in our course at several instances.

Another analytic or algebraic approach to plane curves is to describe them as sets of points whose coordinates satisfy a given bi-variate equation. This way of looking at plane curves is often called the *implicit description*. This way to look at curves is inherent to *Algebraic Geometry* or *Analytic Geometry*. Nevertheless, in this course, we will not treat the implicit approach to the circle or other curves.

Both – the parametric and the implicit approach to plane curves rely fundamentally on the concept of *coordinatization* of the plane, hence the use of a (standard) coordinate system. The idea of coordinatization of the plane (or of the space) was around in Mathematics already very early. But it was only the French mathematician and philosopher R. Descartes (1596-1650, latinized name: Cartesius) who introduced and used this concept in a systematic way and thus revolutionized Mathematics fundamentally. Therefore, standard coordinate systems in the plane or in space are often called *Cartesian coordinate systems*. The notion of *Cartesian product* used for example in *Set Theory* also refers to this.

**1.5 Exercise and Definition.** (A) (*The Standard Parametrization of the Circle*) Let  $Z = (x_Z, y_Z) \in \mathbb{E} = \mathbb{R}^2$  and let  $r \in \mathbb{R}_{>0}$ . We consider the map

$$\varphi = \varphi_r^{[Z]} : [0, 2\pi] \longrightarrow \mathbb{R}^2 = \mathbb{E}, \quad t \mapsto \varphi(t) = \varphi_r^{[Z]}(t) := (x(t), y(t))$$

with *coordinate functions*  $x, y : [0, 2\pi] \longrightarrow \mathbb{R}$  given by

$$x(t) := x_Z + r \cos(t) \quad \text{and} \quad y(t) := y_Z + r \sin(t) \quad \text{for all } t \in [0, 2\pi],$$

where  $\cos(t)$  and  $\sin(t)$  are understood to be formed in the "rad"-mode, so that  $t$  is an *angle expressed in its arc-length*, as usually in Calculus. Show that:

- (a)  $\text{Im}(\varphi_r^{[Z]}) := \varphi_r^{[Z]}([0, 2\pi]) = \mathcal{C}_r(Z)$ .
- (b)  $\varphi_r^{[Z]}(0) = \varphi_r^{[Z]}(2\pi)$ .
- (c) If  $0 \leq s < t \leq 2\pi$  and  $s \neq 0$  or  $t \neq 2\pi$ , then  $\varphi_r^{[Z]}(s) \neq \varphi_r^{[Z]}(t)$ . (Hint: Use that  $\cos(s) = \sin(t)$  implies  $t = 2\pi - s$ , and that  $t = 2\pi - s$  implies  $\sin(s) = -\sin(t)$ .)

The map  $\varphi_r^{[Z]} : [0, 2\pi] \longrightarrow \mathbb{R}^2 = \mathbb{E}$  is called the *standard parametrization* of the circle  $\mathcal{C}_r(Z)$ . Often, one also refers to the induced surjective map

$$\varphi_r^{[Z]} : [0, 2\pi] \twoheadrightarrow \mathcal{C}_r(Z)$$

as the *standard parametrization* of  $\mathcal{C}_r(Z)$ .

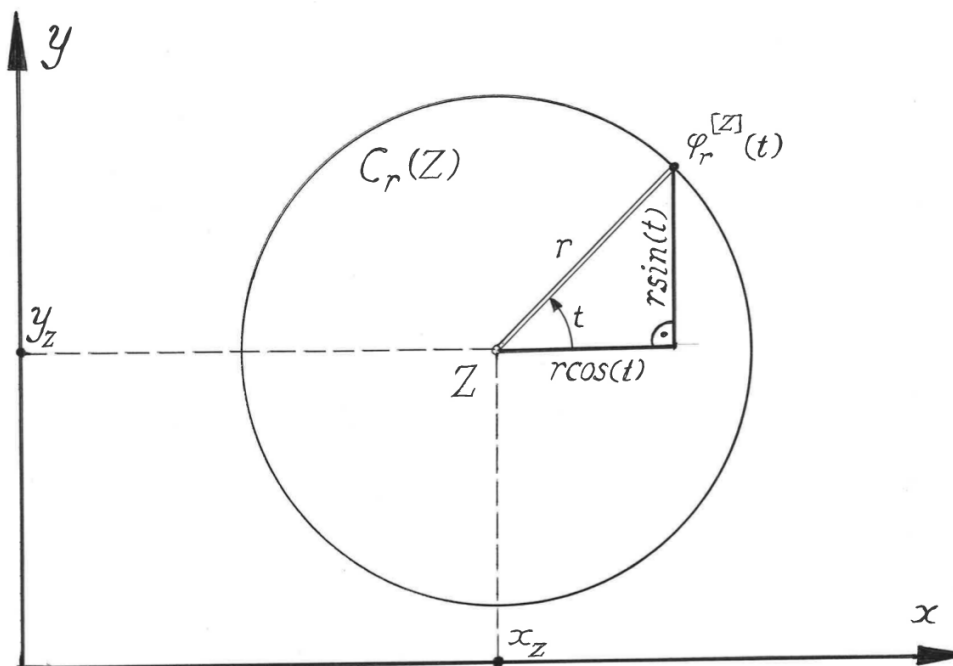


Figure 1.7: Standard Parametrization of the Circle  $\mathcal{C}_r(Z)$

(B) (*The Vectorial Derivative*) We consider the standard parametrization

$$\varphi = \varphi_r^{[Z]} : [0, 2\pi] \longrightarrow \mathbb{R}^2 = \mathbb{E}, \quad t \mapsto \varphi_r^{[Z]}(t) := (x(t), y(t))$$

and its (*vectorial*) derivative

$$\dot{\varphi} = (\varphi_r^{[Z]})' : [0, 2\pi] \longrightarrow \mathbb{R}^2, \quad t \mapsto \dot{\varphi}(t) = (\varphi_r^{[Z]})'(t) := (\dot{x}(t), \dot{y}(t)), \text{ where}$$

$$\dot{x}(t) := \frac{dx}{dt}(t) \text{ and } \dot{y}(t) := \frac{dy}{dt}(t) \text{ for all } t \in [0, 2\pi]$$

are the *derivatives* of the two coordinate functions  $x, y : [0, 2\pi] \longrightarrow \mathbb{R}$ . Verify that

- $\dot{\varphi}(t) = (-r\sin(t), r\cos(t))$  for all  $t \in [0, 2\pi]$ .
- $\|\dot{\varphi}(t)\| = r$  for all  $t \in [0, 2\pi]$ .
- For all  $t \in [0, 2\pi]$ , the two vectors  $\dot{\varphi}(t)$  and  $\overrightarrow{Z\varphi(t)}$  are perpendicular to each other.
- For all  $t \in [0, 2\pi]$ , line through  $\varphi(t)$  with directional vector  $\dot{\varphi}(t)$  meets the circle  $\mathcal{C}_r(Z)$  precisely in the point  $\varphi(t)$  – and hence is the tangent line to  $\mathcal{C}_r(Z)$  at this point.

What is shown in exercise (d) allows us to say that for all  $t \in [0, 2\pi]$  the vector  $\dot{\varphi}(t) = (\varphi_r^{[Z]})'(t)$  is a *tangent vector* to the circle  $\mathcal{C}_r(Z)$  at the point  $\varphi(t) = \varphi_r^{[Z]}(t) \in \mathcal{C}_r(Z)$ .

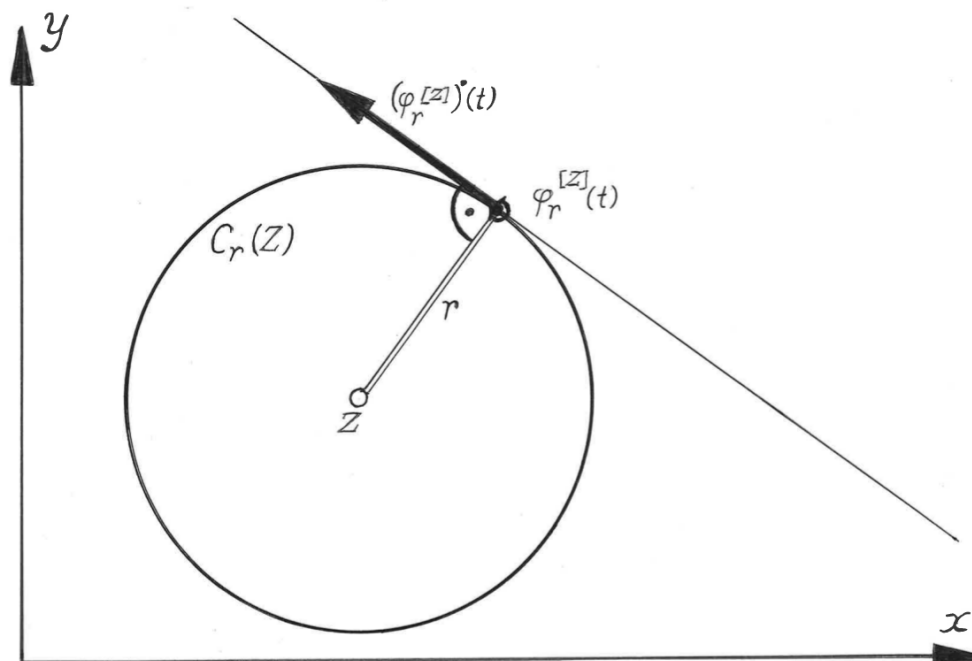


Figure 1.8: Tangent Vector and Tangent Line to  $C_r(Z)$  at  $\varphi_r^{[Z]}(t)$

**Prepaving a Path to Approach the Number  $\pi$ .** In the next Chapter we will approach the number  $\pi$  by computing circumferences of certain regular polygons on a circle. To pave the way to this proceeding, we therefore will introduce below the subject of polygons on a circle by means of our standard parametrization. To solve the exercises one needs basic trigonometry.

**1.6 Definition and Exercise.** (A) (*Polygons on the Circle*) Throughout we fix  $r \in \mathbb{R}_{>0}$  and write  $O$  for the origin  $(0, 0)$  of our coordinatized plane  $\mathbb{E} = \mathbb{R}^2$ . Then, we consider the circle

$$\mathcal{C} := C_r(O) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$$

of radius  $r$  and centered at the origin  $O$ . Moreover, we consider the standard parametrization

$$\varphi = \varphi_r^{[O]} : [0, 2\pi] \longrightarrow \mathcal{C}, \quad t \mapsto \varphi(t) = \varphi_r^{[O]}(t) = (r\cos(t), r\sin(t)).$$

Then, we fix  $n \in \mathbb{N}$  and consider a finite sequence

$$t_\bullet := (t_i)_{i=0}^n \in [0, 2\pi]^{n+1} \text{ with } 0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n.$$

Such a sequence is called an  $n$ -partition or just a *partition* of the interval  $[0, 2\pi]$ . The sequence

$$P_\bullet = (P_i)_{i=0}^n \text{ with } P_i := \varphi(t_i) = \varphi_r^{[O]}(t_i) \in \mathcal{C}, \quad (i = 0, 1, 2, \dots, n-1, n)$$

is called the  $n$ -gon with vertices  $P_i$  in  $\mathcal{C}$  induced by the partition  $t_\bullet$ . We also write this  $n$ -gon in the form

$$\varphi(t_\bullet) = \varphi_r^{[O]}(t_\bullet) = (\varphi(t_i))_{i=0}^n = (\varphi_r^{[O]}(t_i))_{i=0}^n.$$

The points  $P_i = \varphi(t_i) \in \mathcal{C}$  are called the vertices of the  $n$ -gon  $P_\bullet$ . Observe in particular that (see Exercise and Definition 1.5 (A)(c))

$$P_0 = P_n \text{ and } P_i \neq P_j \text{ for } i, j \in \{0, \dots, n\} \text{ with } i \neq j \text{ and } \{i, j\} \neq \{0, n\}.$$

Arbitrary  $n$ -gons of this type are called *polygons on the circle* or *polygons with vertices on the circle  $\mathcal{C}$* .

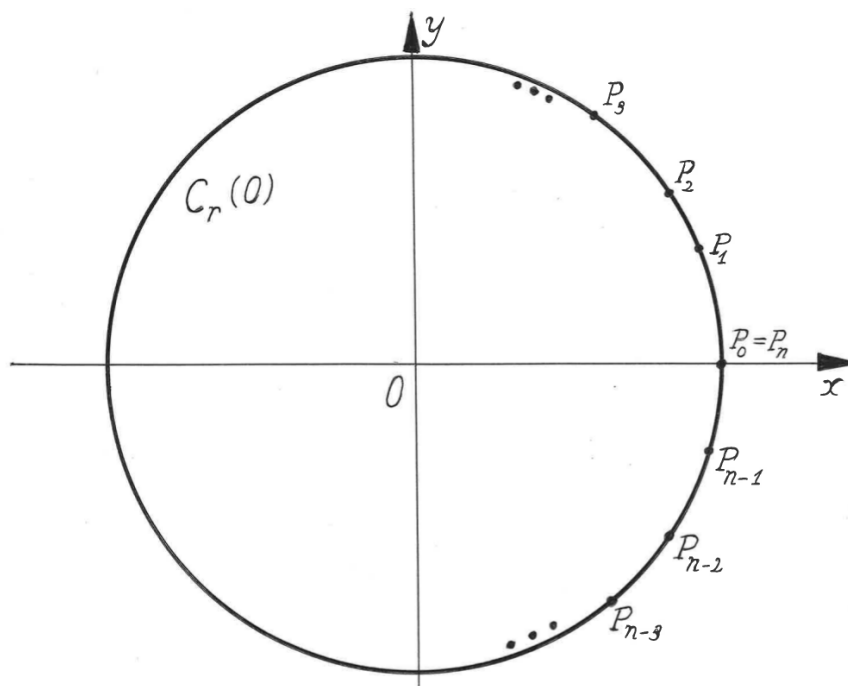


Figure 1.9:  $n$ -Gon with Vertices on  $\mathcal{C}$

(B) (*The Edges*) Let  $n \in \mathbb{N}$  and let  $t_\bullet := (t_i)_{i=0}^n$  be an  $n$ -partition of the interval  $[0, 2\pi]$ . We consider the induced  $n$ -gon  $P_\bullet = \varphi(t_\bullet)$  with vertices on  $\mathcal{C}$ . For each  $i \in \{1, 2, \dots, n\}$ , the  $i$ -th *edge* of the polygon  $P_\bullet$  is defined as the line section  $|P_{i-1}P_i|$  between the two vertices  $P_{i-1}$  and  $P_i$ . Correspondingly,  $i$ -th *central angle* of the polygon  $P_\bullet$  is the angle of the triangle  $P_{i-1}OP_i$  at  $O$  and hence given in the “rad-mode” by

$$\alpha_i = \alpha_i^{[P_\bullet]} := \angle P_{i-1}OP_i = t_i - t_{i-1}, \quad i = 1, 2, \dots, n$$

or in the “degree-mode” by

$$\frac{t_i - t_{i-1}}{2\pi} 360^\circ.$$

Prove the following relations, in which angles are understood to be taken in the “rad-mode”:

- (a) For each  $i \in \{1, 2, \dots, n\}$ , the length of the  $i$ -th edge of the polygon  $P_\bullet$  is given by

$$f_{P_\bullet}^{[i]} := \text{dist}(P_i, P_{i-1}) = 2r \sin\left(\frac{t_i - t_{i-1}}{2}\right).$$

- (b) The *circumference* or *perimeter* of the polygon  $P_\bullet$ , hence the sum of the lengths of all its edges, is given by

$$c^{[P_\bullet]} := \sum_{i=1}^n f_{P_\bullet}^{[i]} = 2r \sum_{i=1}^n \sin\left(\frac{t_i - t_{i-1}}{2}\right).$$

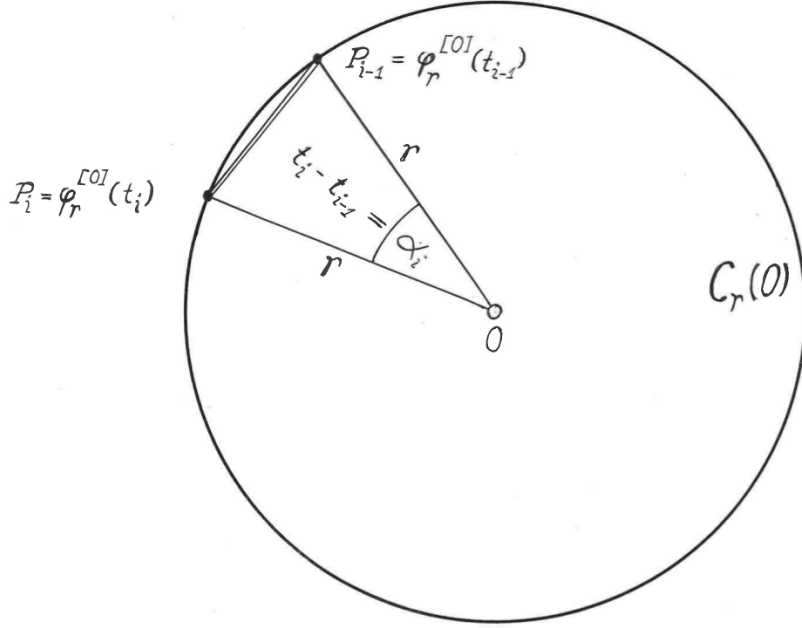


Figure 1.10: Edge, Central Angle and Circumference of a Polygon on  $\mathcal{C}$

(C) (*Regular Polygons*) We keep the previous notations and hypotheses. We say that the  $n$ -partition  $t_\bullet$  of  $[0, 2\pi]$  is *regular* or *equidistant* if  $t_i - t_{i-1}$  takes the same value for all  $i \in \{1, 2, \dots, n\}$ . So, in this case, we may write

$$t_i = \frac{2\pi}{n}i \text{ for all } i \in \{0, 1, \dots, n-1, n\}.$$

In this situation, we say that the induced  $n$ -gon  $P_\bullet = \varphi(t_\bullet) = \varphi_r^{[O]}(t_\bullet)$  is a *regular  $n$ -gon with vertices on  $\mathcal{C}$* . Prove that for a regular  $n$ -gon  $P_\bullet$  with vertices on  $\mathcal{C}$  the following statements hold:

- (a) For each  $i \in \{1, 2, \dots, n\}$  the  $i$ -th central angle is given by

$$\alpha_i = \frac{2\pi}{n}.$$

So all central angles of  $P_\bullet$  are the same.

- (b) For each  $i \in \{1, 2, \dots, n\}$  the length of the  $i$ -th edge is given by

$$f_{P_\bullet} = 2r \sin\left(\frac{\pi}{n}\right).$$

So, all edges of  $P_\bullet$  have the same length.



(c) The circumference of  $P_\bullet$  is given by

$$c_n := c^{[P_\bullet]} = 2nr \sin\left(\frac{\pi}{n}\right)$$

Finally use the previous results to show that

(d) The sequence  $(c_n)_{n \in \mathbb{N}}$  of the circumferences  $c_n$  of regular  $n$ -gons with vertices on  $\mathcal{C}$  is convergent and satisfies

$$\lim_{n \rightarrow \infty} c_n = 2r\pi = \pi d.$$

**Hints for Teachers.** Below, we shall include a few hints and suggestions, which could be helpful to transfer the material treated in this section to the level of High School Teaching. Obviously, we will not give some “preparations for lectures”, but rather some encouragement and initiation how certain of the treated subjects could help to get new ideas for your teaching. Most of this things will be not completely new for you, but we hope that they help you to get some new views to the subjects you regularly teach at High School.

**1.7 Didactic Comments.** (A) (*Transfer to the Classroom*) The material presented in this chapter is rather directly transferable to the classroom. It is most important to notice, that the metric definition of the circle given in Definition 1.1 (A) is already a nice opportunity to illustrate how Mathematics approaches the problem of defining its objects of investigation. The author remembers well, that the metric definition of the circle sounded to him like a piece of poetry when he heard it the first time at the age of at about 12, clearly in a slightly old-fashioned German.

Try to make this definition sound like a short poem to your class: “The circle is the geometric locus of all points in the plane which have the same distance from a given point”. But what is the relation between mathematics and poetry? It is the *word*, which expresses a *thought*. In case of our circle, this thought is at the same time very simple and very deep: simple, as it defines the circle by a very easy and comprehensible property – deep, as it reveals the essential property of a circle and needs no further explanation.

So, the metric definition of circle is like a prototype of how a mathematical definition should be shaped. Moreover, as we already realized earlier, the metric definition of the circle also gives an immediate hint how to draw circles in practice. May be, it was on the base of such thoughts, that in ancient Cultures the circle was considered to be the most fundamental and most beautiful geometric object. As a teacher, you should not be anxious, to let this kind of thoughts enter your teaching. Such ideas could rise the interest in Mathematics, Philosophy or Science the minds of your pupils. Do not forget, that a deep understanding of simple things is a sound basis for the understanding of complex phenomena.

Also, the notion of metric, as introduced and illustrated by examples in Definition, Remark and Exercise 1.3 is not beyond the level of High School Mathematics. In particular, the “strange metrics” and the resulting “strange circles” (see Definition and Exercise 1.4) are subjects which attract the attention of

High School students, as the author could experience at several instances. Below, we shall say a few more words on this.

The standard parametrization of the circle (see Exercise and Definition 1.5) is a theme often treated on advanced High School level. So, we have not to comment a lot on this. We hope, that our presentation encourages you to put impact on this important subject, which relates Geometry, Trigonometry and Calculus in an appealing way.

We hope, that our approach to polygons with vertices on a circle (see Definition and Exercise 1.6) gives you some ideas for your teaching. The suggested exercises to this subject can be solved on use of elementary Trigonometry and they could give rise to concrete examples, like calculating the circumference of a  $n$ -gon for some small values of  $n$ , like 1, 2, 3, 4, 5, 6, 7 . . .

(B) (*More about Metrics*) If guided properly, your students also can invent new metrics, which are not among the above examples. Another such metric could be the *Swiss railway tunnel metric*: You draw a line (the mountain ridge), then you delete one point in this line (the tunnel). If two points are on the same side of your line, you take as their distance the usual one. If the points are on different sides of the line, you relate them by the shortest path passing through the removed point (the tunnel) and take the length of the connecting path as their distance. So, in principle you measure the length of a journey between two points, under the condition that you have to use the tunnel to pass the mountain. Clearly, the game can be made more realistic by taking several tunnels and tunnels of positive length.

The circular disks in such a metric tell you, how far you can travel maximally from a given point if you have to observe the constraints of your traffic network. Now your students must realize, that the concept of metric and of circle or disk with respect to such a metric is more than an idle toy for mathematicians: it is a tool of great practical impact.

Let us hint another application of metrics: we look at all possible *words* of given length, say  $n$ , written down in our (24 -lettered) alphabet. Here by word we mean an arbitrary string of  $n$  letters, independent on the fact that this string forms a word in the common sense. Then let  $\mathbb{W}_n$  be the set of all these words. We write each word in the form

$$L_{\bullet} = L_1 L_2 \dots L_n \text{ where } L_i \text{ is the letter at the } i - \text{th position.}$$

If we have two words  $L_{\bullet}, M_{\bullet} \in \mathbb{W}_n$ , we define the *distance* between these two words as the numbers of positions, at which the letters of the two words are different, thus by

$$d_{[W]}(L_{\bullet}, M_{\bullet}) := \#\{i \in \{1, 2, \dots, n\} \mid L_i \neq M_i\}, \quad (\forall L_{\bullet}, M_{\bullet} \in \mathbb{W}_n.$$

This defines a metric on  $\mathbb{W}_n$ , the so-called *word metric*.

A problem in telecommunication networks is the fact, that letters in word may be transmitted wrongly. But there are ingenious mathematical procedures, which allow to encode words by an assignment

$$\gamma : \mathbb{W}_n \longrightarrow \mathbb{W}_m \text{ with } m \in \mathbb{N}_{\geq n}, \text{ with an associated number } \delta = \delta(\gamma)$$

such that the following holds:

- (a) For each word  $L'_\bullet \in \mathbb{W}_m$  there is at most one word  $L_\bullet$  such that

$$d_{[W]}(L'_\bullet, \gamma(L_\bullet)) \leq \delta$$

This means, that  $L'_\bullet \in \mathbb{W}_m$  is an erroneous transmission of  $L_\bullet \in \mathbb{W}_n$  with at most  $\delta$  errors, then the correct transmission  $\gamma(L_\bullet)$  of  $L_\bullet$  can be recovered. So, the code in question may correct transmission errors, if not more than  $\delta$  of them do occur. Such *error correcting codes* are used today at many instances. One technical device, which makes use of them, are CD-players. Here, these codes are used to “repair” a CD which has “not to many” recorded disturbances. This is one more of many other practical applications in which the concept of metric comes up. Why not to tell such a thing in a “relaxing moment” in your class, just do let the students know, that Mathematics goes much beyond what one usually sees directly in every-days life, and that it is a kind of “omnipresent but hidden key technology”.

**1.8 Conclusive Remarks.** In Exercise and Definition 1.5 and Definition and Exercise 1.6 we have already used the number  $\pi$  in several ways. In the Chapters to come, we shall indeed revisit this number and try to regain and explain it in strict mathematical terms. This will give us a chance, to discuss a number of fundamental notions and ideas from Geometry, Analysis and Algebra. In Chapter 4 we then will come back to the preparing considerations made above.

## 2. Approaching $\pi$

**First Approaches.** Let  $\mathcal{C} = \mathcal{C}_Z(r)$  be a circle of radius  $r$  and diameter  $d = 2r$ . Then it is taught to us already in early Secondary School, that the *circumference* or *perimeter* of  $\mathcal{C}$  is given by the formula

$$c = \pi d, \text{ where } \pi = 3.1415\dots$$

We all are using and applying this formula many times, very often without thinking much about its mathematical and philosophical background. In particular, the occurring number  $\pi$  was an object of fascination since the ancient age. How can this number be determined, computed or at least approximated? Classically there were two ways to approach these questions:

- (a) *By physical experiment*, for example by wrapping a rope or string around a circular disc or a cylinder of known diameter  $d$  and by measuring the length  $\tilde{c}$  of the string and forming the ratio  $\tilde{c}/d$ .
- (b) *By mathematical approximation*, for example by computing for large  $n$ , the circumference  $c_n$  of a regular  $n$ -gon with vertices on the circle  $\mathcal{C}$  of diameter  $d$  – and then finally forming the ratio  $c_n/d$ .

The “experimental” approach clearly is more ancient and sufficient for most practical applications. The same applies for other experimental methods, like comparing the distance between two points and the number of turns of a wheel of known diameter needed to travel this distance... We shall come back to this experimental approach in the exercises at the end of this section.

**Using Inscribed Regular  $2^n$ -Gons.** We now aim to consider the approach by mathematical approximation on use of regular  $n$ -gons inscribed to the circle  $\mathcal{C}$  – thus with vertices on  $\mathcal{C}$  – as mentioned above. To express the circumference  $c_n$  of an arbitrary regular  $n$ -gon with vertices on the given circle  $\mathcal{C}$  in terms of the diameter  $d$  of this circle, needs some trigonometry. In order to avoid this, we prefer to approximate the number  $\pi$  only by means of the circumferences  $c_{2^n}$  of regular  $2^n$ -gons for “large”  $n$ . More precisely, we aim to approximate the number  $\pi$  by the ratio

$$\pi_n := \frac{c_{2^n}}{d} \text{ for } n \text{ large } n.$$

A heuristic justification for this approach is the premathematical observation, that the regular  $2^n$ -gons with vertices on  $\mathcal{C}$  “better and better approximate the circle  $\mathcal{C}$  if  $n$  becomes larger and larger”. From a strict mathematical point of view, this justification is not satisfactory, as it does not explain in precise terms what it means, to “approximate the circle  $\mathcal{C}$ ”. Moreover, this justification does not tell us, how the length of a curve (like the circle) should be understood in stringent mathematical terms. It is only in Chapter 3 and 4, that we will take up these two questions.

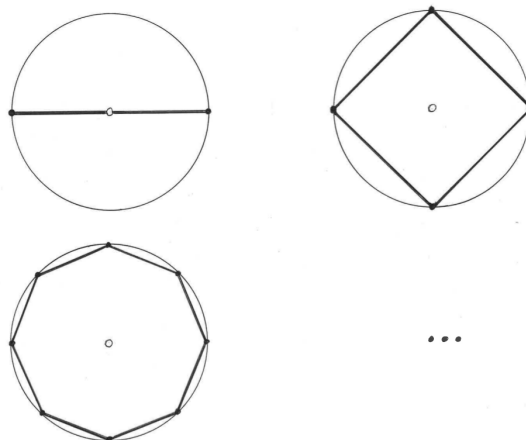


Figure 2.1: Regular  $2^n$ -Gons Inscribed to  $\mathcal{C}$ , ( $n = 1, 2, 3, 4, \dots$ )

As we shall see below, the sequence of the ratios

$$\left(\pi_n\right)_{n \in \mathbb{N}} = \left(\frac{c_{2^n}}{d}\right)_{n \in \mathbb{N}}, \quad (\text{with } c_{2^n} = \pi_n d \text{ for all } n \in \mathbb{N})$$

satisfies a recursive formula, which can be deduced by elementary ideas, not relying on trigonometry. More precisely, we can say:

**2.1 Proposition.** *For each  $n \in \mathbb{N}$  let  $c_{2^n}$  denote the circumference of a regular  $2^n$ -gon with vertices on the circle  $\mathcal{C}$  of diameter  $d$  and set  $\pi_n := \frac{c_{2^n}}{d}$ . Then we have*

- (a)  $\pi_1 = 2$ .
- (b) For all  $n \in \mathbb{N}$  it holds:

$$\pi_{n+1} = 2^n \sqrt{2} \sqrt{1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}.$$

*Proof.* (a): If  $n = 1$ , we are in the “degenerate” case of a  $2^1 = 2$ -gon, which consists just of two points  $P, Q \in \mathcal{C}$  lying opposite to each other on  $\mathcal{C}$ , this means on a line through the center  $Z$  of  $\mathcal{C}$ .

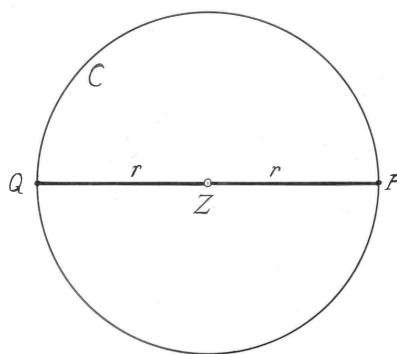


Figure 2.2: A regular 2-Gon on the Circle  $\mathcal{C}$

Clearly, in this situation, the circumference  $c_2 = \pi_1 d$  of the 2-gon in question is twice the distance between  $P$  and  $Q$ , thus  $\pi_1 d = c_2 = 2d$ , hence  $\pi_1 = 2$ . This proves statement (a).

(b): For each  $n \in \mathbb{N}$  we denote the length of one edge of a regular  $2^n$ -gon with vertices on  $\mathcal{C}$  by  $f_n$ , so that

$$f_n = \frac{c_{2^n}}{2^n} \quad \text{and} \quad f_{n+1} = \frac{c_{2^{n+1}}}{2^{n+1}}.$$

We now pass from the  $2^n$ -gon to the  $2^{n+1}$ -gon by introducing a new vertex  $Q$  in the middle of each arc on  $\mathcal{C}$  which relates two consecutive vertices  $P, P'$  of the given  $2^n$ -gon. This leaves us with the situation sketched in the following figure, in which  $Q'$  denotes the point in which the line segment between  $Z$  and  $Q$  and the edge between  $P$  and  $P'$  intersect each other:

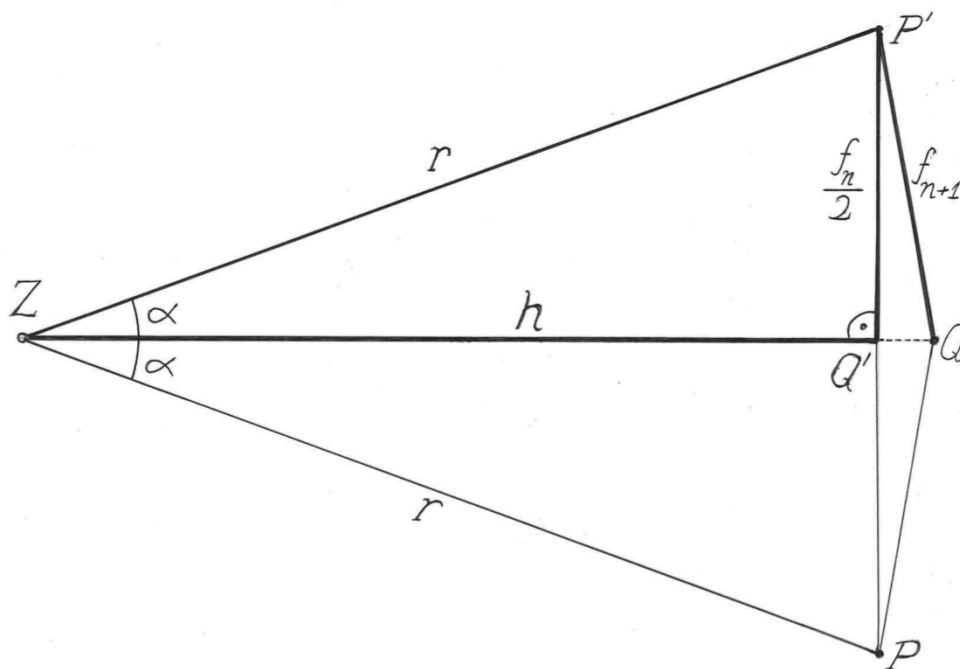


Figure 2.3: Relating  $f_{n+1}$  and  $f_n$

Observe, that the triangle  $ZQ'P'$  has a right angle in  $Q'$  and that the hypotenuse  $ZP'$  has length  $r$ , whereas the cathete  $Q'P'$  has length  $\frac{f_n}{2}$ . So, by the Theorem of Pythagoras the remaining cathete  $ZQ'$  has length

$$h := \sqrt{r^2 - \left(\frac{f_n}{2}\right)^2}$$

Observe, that the triangle  $Q'P'Q$  has a right angle in  $Q'$  and that its cathetes  $Q'P'$  and  $Q'Q$  are respectively of length  $\frac{f_n}{2}$  and  $r - h$ . So, again by Pythagoras,

the hypotenuse  $QP'$  of this triangle has length

$$\begin{aligned}
f_{n+1} &= \sqrt{\left(\frac{f_n}{2}\right)^2 + (r-h)^2} = \\
&= \sqrt{\left(\frac{f_n}{2}\right)^2 + r^2 - 2rh + h^2} = \\
&= \sqrt{\left(\frac{f_n}{2}\right)^2 + r^2 - 2r\sqrt{r^2 - \left(\frac{f_n}{2}\right)^2} + r^2 - \left(\frac{f_n}{2}\right)^2} = \\
&= \sqrt{2r^2 - 2r\sqrt{r^2 - \left(\frac{f_n}{2}\right)^2}} = \\
&= \sqrt{2r^2 - 2r^2\sqrt{1 - \left(\frac{f_n}{2r}\right)^2}} = \\
&= \sqrt{2r}\sqrt{1 - \sqrt{1 - \left(\frac{f_n}{2r}\right)^2}} = \\
&= \frac{1}{\sqrt{2}}d\sqrt{1 - \sqrt{1 - \left(\frac{f_n}{d}\right)^2}}.
\end{aligned}$$

As

$$\pi_{n+1} = \frac{c_{2^{n+1}}}{d} = \frac{2^{n+1}f_{n+1}}{d} \quad \text{and} \quad f_n = \frac{c_{2^n}}{2^n} = \frac{\pi_n d}{2^n}$$

it follows

$$\begin{aligned}
\pi_{n+1} &= \frac{2^{n+1}f_{n+1}}{d} = \frac{2^{n+1}}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \left(\frac{f_n}{d}\right)^2}} = \\
&= \frac{2^{n+1}}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \left(\frac{\pi_n d}{2^n d}\right)^2}} = \\
&= 2^n\sqrt{2}\sqrt{1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}.
\end{aligned}$$

This is the requested recursive relation.  $\square$

**2.2 Application and Exercise.** (A) (*Numerical Computation of the Numbers  $\pi_n$* ) On use the formulas given in Proposition 2.1 we may successively compute the values of  $\pi_n$  by means of a calculator for  $n = 1, 2, \dots, 8$  and present them in the table below. Verify these values by means of your calculator and compute  $\pi_{10}$ .

$n$	$\pi_n$ (approximate value)	number of vertices	name of the polygon
1	2.000000...	2	double edge
2	2.828427...	4	square
3	3.061467...	8	octagon
4	3.121445...	16	hexadecagon
5	3.136554...	32	
6	3.140337...	64	
7	3.141346...	128	
8	3.141583...	256	

(B) Write a computer program to compute the consecutive values of  $\pi_n$ .

**Approximating  $\pi$  – A Gem for the Classroom.** Approximating  $\pi$  as suggested above or in some other elementary way, can be considered as a real gem for the teaching of Mathematics at High School. Below, we want to justify this point of view, and we hope to encourage you to think about further details how you can bring this matter to become fertile for your teaching.

**2.3 Didactic Comments.** (A) (*Transfer to the Classroom*) A valuable approach could be to explain that we aim to approximate the number  $\pi$  by the the numbers  $\pi_n = \frac{c2^n}{d}$  for large  $n$  and to make some sketches for  $n = 1, 2, 3$  thus for the regular 2, 4, 8-gon. Then one could write down the formulas given in Proposition 2.1 without any prove. But it is most important to explain, what these formulas achieve, concerning the sequence  $(\pi_n)_{n \in \mathbb{N}}$ :

- They give us the initial value  $\pi_1$  (see statement (a) of Proposition 2.1).
- If we have already computed the  $n$ -th value  $\pi_n$  in our sequence, we can compute the next value  $\pi_{n+1}$  (see statement (b) of Proposition 2.1).

This means, that the recursive relations given in Proposition 2.1 allow to compute successively the numbers  $\pi_1, \pi_2, \pi_3, \dots$  as already performed in Application and Exercise 2.2.

Also, it is most important to illustrate the use of this idea by putting it to practice, and to use the recursive formulas of Proposition 2.1 to compute successively a few of the numbers  $\pi_n$  and comparing the outcome with the the above table for these values. Also, you may discuss the outcome of the said computation, by consulting a numerical approximation of  $\pi$  given in some table or looked up at internet.

Next, one also could hint the notion of convergent sequence and write down and speak out (for the class eventually for the first time) the conjectured (!! ) relation

$$\pi = \lim_{n \rightarrow \infty} \pi_n.$$

Finally, one could discuss the notion of recursively defined sequence – and point out to the class the great meaning of the *Principle of Recursion* and



the related *Principle of Complete Induction*, which are two of the most fundamental principles of Mathematics at all. As a number of students could have already some experience with computers, bring it to their mind, that in fact almost each computer program they have written or applied, made use (may be in disguised form) of the Principle of Recursion. It is really worth to discuss this basic theme of Mathematics – namely the Principle of Recursion – at “any occasion”, and here is such an occasion!

An ultimate step could be to prove Proposition 2.1. Clearly, this is a challenge which might be on the upper edge of comprehension of your class. But do not forget: It is very good to perform then and now something, which may stress the comprehension of the average student – as long as such “extras” are short and “rare events”. In doing so, you challenge the most gifted students and help them to increase their enthusiasm for Mathematics. In relation to this, it could be recommendable to really use the notions of Proposition and Proof and telling to your students, that proving conjectured or claimed results – hence Propositions or Theorems – is the basic issue of Mathematics.

So, by performing the mentioned proof, you can demonstrate them “Mathematics at work”. This also could be done individually for one single student or a smaller group of specially motivated students, by giving them the proof in written form, make them prepare a presentation of it and perform the proof later in front of the class.

Observe, that the proposition and the proof in question use only very basic notions of Geometry and a certain ability to handle basic algebraic terms. As geometric and algebraic considerations are closely linked in our arguments, the problem under discussion is of “interdisciplinary” nature, too. Besides the mentioned basic notions of Mathematics, the proof in question also requires a certain maturity in mathematical thinking: namely, to follow the track of an argument, that cannot be performed just by a straight forward application a well known formula.

At the end, let us summarize the didactic highlights of the theme:

- The idea of approximating a number by a (convergent) sequence of numbers.
- The notion of a recursively defined sequence of numbers.
- The experience, that geometric problems can be approached by Algebra.
- The experience to perform a proof of a stated claim.

(B) (*The Relation to Trigonometry*) The geometric arguments performed in the proof of Proposition 2.1 can also be performed in a more general context. Namely, let  $P, P'$  two points on the circle  $\mathcal{C}$  whose distance is  $f'(\leq d)$ . Let  $Q$  be the point on  $\mathcal{C}$  which is located precisely in the middle of the shorter arc between  $P$  and  $P'$  on  $\mathcal{C}$ . Denote the distance between  $P$  and  $Q$  by  $f$ . Then, the argument given in the proof of Proposition 2.1 in order to relate  $f_{n+1}$  and

$f_n$ , applies literally to show that

$$f = \frac{1}{\sqrt{2}}d\sqrt{1 - \sqrt{1 - \left(\frac{f'}{d}\right)^2}}.$$

Denote the angle  $\angle PZP' \leq 180^\circ$  of the triangle  $PZP'$  at  $Z$  by  $\alpha$  and observe that the angle  $\angle PZQ$  of the triangle  $PZQ$  at  $Z$  equals  $\frac{\alpha}{2}$ . In particular we have

$$f' = 2\sin\left(\frac{\alpha}{2}\right)r \text{ and } f = 2\sin\left(\frac{\alpha}{4}\right)r.$$

An easy computation shows, that this leads to the formula.

$$\begin{aligned} \sin\left(\frac{\alpha}{4}\right) &= \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \sin\left(\frac{\alpha}{2}\right)^2}}, \text{ or, with } \beta := \frac{\alpha}{2} \\ \sin\left(\frac{\beta}{2}\right) &= \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1 - \sin(\beta)^2}} = \frac{1}{\sqrt{2}}\sqrt{1 - \cos(\beta)}. \end{aligned}$$

This is nothing else than the well-known trigonometric ‘‘Half-Angle Formula’’ for the Sinus. So, our proof of Proposition 2.1 (b) could also be used to establish this latter formula with students who have already basic notions of Trigonometry.

Conversely, the above relation between  $f$  and  $f'$  can easily be deduced on use the trigonometric Half-Angle Formula. So, for a class, knowing already this latter formula, this gives another approach to the previously shown relation between  $f_{n+1}$  and  $f_n$ , and hence to the proof of Proposition 2.1.

(C) (*Further Polygonal Approximations*) Finally, we could use another variant of Proposition 2.2 to approximate  $\pi$ . Namely, assume that – for some  $m \in \mathbb{N}$  – we may express the circumference  $c_m$  of the regular  $m$ -gon on  $\mathcal{C}$  in terms of the diameter  $d$ , so that we can compute the number

$$\pi^{[m]} := \frac{c_m}{d}.$$

Then, we can indeed approximate  $\pi$  by the numbers

$$\pi_n^{[m]} := \frac{c_{m2^{n-1}}}{d} \text{ for large } n.$$

Namely, going once more through the arguments of the proof of Proposition 2.1 we namely immediately may derive the following recursive relations:

- (a)  $\pi_1^{[m]} = \pi^{[m]}$ .
- (b)  $\pi_{n+1}^{[m]} = 2^n\sqrt{2}\sqrt{1 - \sqrt{1 - \left(\frac{\pi_n^{[m]}}{2^n}\right)^2}}$  for all  $n \in \mathbb{N}$ .

Observe, that with  $m = 2$  we get back the recursive relations of Proposition 2.1, so that we have found a generalization of this proposition.

These new relations can be successfully applied for example in the case  $m = 6$ , thus in the case, where ‘‘the circle  $\mathcal{C}$  is approximated by the regular  $6 \cdot 2^n$ -gons  $n = 1, 2, \dots$ ’’.

**A Few Concrete Occurrences of  $\pi$ .** After this first strictly formal approach to the number  $\pi$ , we now also will discuss some practical occurrences of  $\pi$ .

**2.4 Remarks and Exercises.** (A) (*A View to the Thora*) In the First Book of Kings, Chapter 7, Verses 23-26 (thus in the Old Testament of the Bible, hence the Hebrew Thora) the “Sea” is described: a huge vessel made of bronze, set up in the Temple of Jerusalem. In verse 23 it is written: “Its measure was 10 ells (or ”cubits“, that is ”king size ells“, 52.5 cm each) from one edge to the other; it was completely round and its height was five ells. A string of thirty ells could surround it completely”. Does this mean, that the Bible (respectively the Thora) claims that  $\pi = 3$ ? Here come two ways, how the apparent contradiction can be understood:

- The “Sea” had not a cylindric shape!

Indeed in verse 26 we read: “Its (upper) border was shaped as the border of a calice, similar to a lily flower.” So, we can imagine that the diameter was measured by spanning a transverse string at the “upper edge of the calice”, but that the circumference was measured by wrapping a string around the “taille of the calice”. Sketch the “Sea” and compute the diameter  $d$  of its “taille” (in ells and in centimeters), using the approximate value of  $\pi$  computed in the last line of the table in 2.2 (A). If you feel challenged as an artist, try to complete the sketch of the “Sea” (may be, even with its “Support”) according to the details you find in the First Book of Kings.

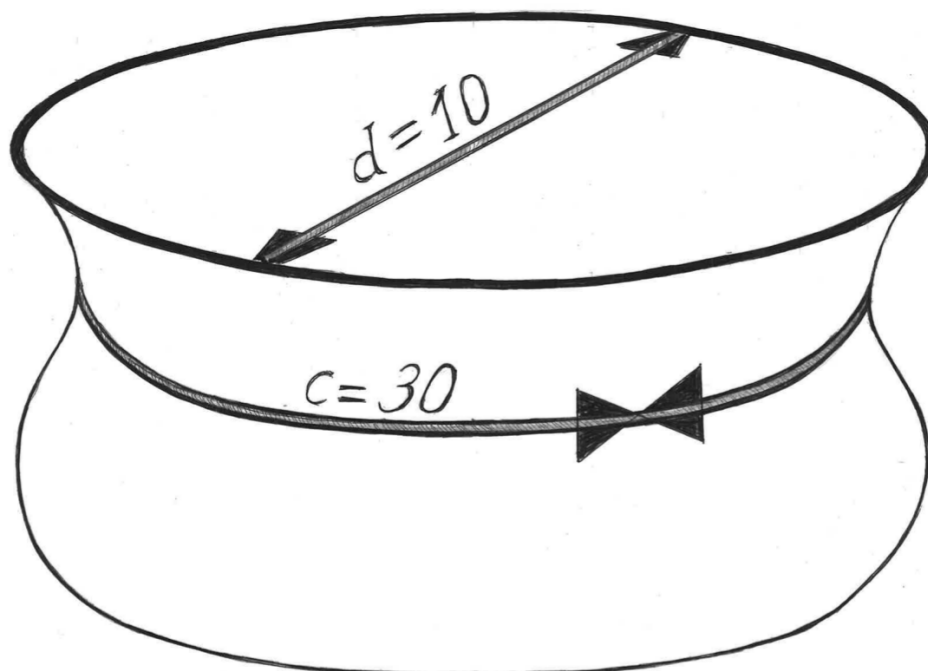


Figure 2.4: The Sea in the Temple of Jerusalem, “Shaped as Lily Flower”

Recently, at the occasion of the “International Conference on Commutative Algebra and its Interactions to Algebraic Geometry and Combinatorics”, which took place at Hanoi in December 2013, the author had breakfast with his colleague Gennady Lyubeznik from the University of Minnesota, an internationally reknown specialist in Commutative Algebra – and of Jewish faith. Gennady has an excellent knowledge of the Thora, which he reads in Hebrew Language. So, the author did ask him his opinion about the question, whether the Thora claims  $\pi = 3$ ? His answer was:

- The “Sea” was curved everywhere ( = “completely round”) but not circular, shaped like an ellipse.

So, assume that the (upper edge) of the “Sea” is an ellipse, which maximal diameter  $d = 10$  (ells) and minimal diameter  $d' < d$ . Determine  $d'$  (approximately) such that the perimeter  $c$  of this “elliptic upper edge” takes the value of 30 ells. Use various approximative formulas for the perimeter of the ellipse to do this, for example:

$$c = \pi\sqrt{dd'}, \quad c = \pi\frac{d+d'}{2} \text{ or } c = \pi\frac{\sqrt{d^2+d'^2}}{2}.$$

Comment your findings.

(B) (*An Inspiring Anecdote*) Rather an anecdote must be the story, that on the Campus of some University in Michigan it was carved in stone:

- *”The circumference of a circle is three times its diameter”.*

My colleague Gennady Lyubeznik and me also discussed this claim at the occasion of the mentioned breakfast. Our discussion lead us to the conclusion, that this obviously is true for a large circle on Earth, provided that the diameter is measured on the ground, this means on the surface of the Earth. We model this surface by a sphere  $\mathcal{S}$  of circumference  $C \approx 40'000$  Km. Actually to solve our problem, we need not specify a concrete value for  $C$ .

So, we assume that our circle  $\mathcal{C}$  of circumference  $c$  lies on a sphere  $\mathcal{S}$  of circumference  $C > c$ . Let  $\tilde{d}$  be the *spherical diameter* of  $\mathcal{C}$  on  $\mathcal{S}$ . So  $\tilde{d}$  is the length of the shortest arc on the sphere  $\mathcal{S}$  which connects to opposite points on our circle  $\mathcal{C}$ . The situation is sketched in Figure 2.5.

Now, our task is to find the right position of  $\mathcal{C}$  on  $\mathcal{S}$  – more precisely the right value of the proportion  $c : C$  – such that

$$\frac{c}{\tilde{d}} = 3.$$

We leave this as an exercise, but we will provide below some hints how to solve it.

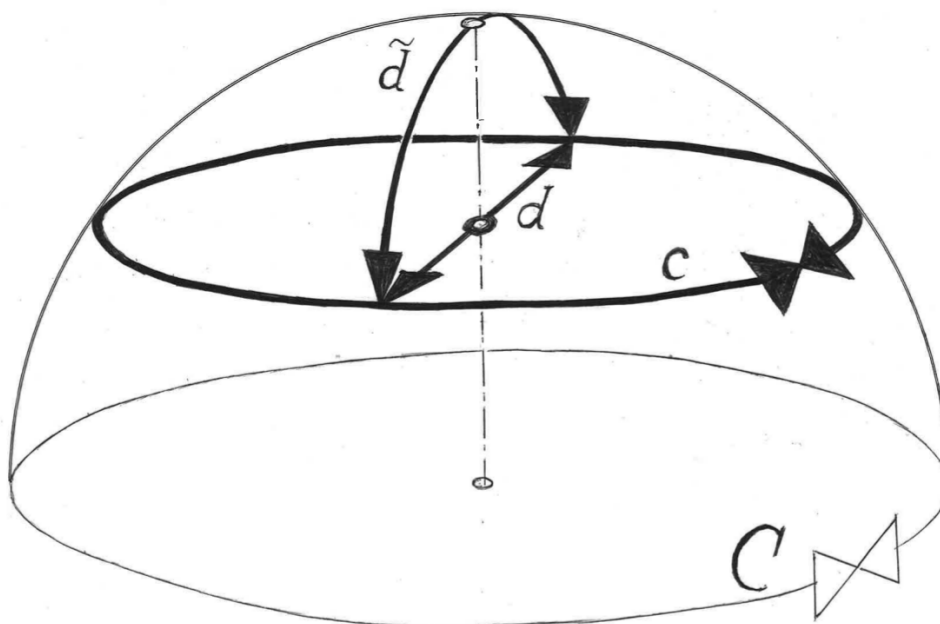


Figure 2.5: Circle on a Sphere

Express the ratio

$$\tilde{\pi} = \tilde{\pi}(\gamma) = \tilde{\pi}\left(\frac{c}{C}\right) := \frac{c}{\tilde{d}} = \frac{\pi d}{\tilde{d}} = \pi \frac{d}{\tilde{d}}$$

in terms of the ratio  $\gamma := \frac{c}{C}$ . Determine the limit

$$\pi_0 := \lim_{\gamma \rightarrow 0} \tilde{\pi}(\gamma).$$

Determine  $\gamma$  such that  $\tilde{\pi}(\gamma) = 3$ . To do so, set

$$\gamma := \cos(\beta), \text{ with } 0 \leq \beta \leq \frac{\pi}{2}$$

and show that with this choice we must have

$$\frac{\pi}{\tilde{\pi}(\gamma)} = \frac{d}{\tilde{d}} = \frac{\pi - 2\beta}{2\cos(\beta)}.$$

Finally assume that the circle  $C$  is located on the “Northern hemisphere” and has its center of the “axis of the Earth”. Compute its “Northern geographic altitude”.

(C) (*A Rolling Wheel*) A wheel of diameter  $d = 1$  meter, and carrying a marked point  $P$  on its periphery, is used to approximate experimentally the value  $\pi$  by rolling it along a straight line on a plane. When the wheel begins to roll, the point  $P$  coincides with some point  $S$  in the plane. The curve described by the point  $P$  is called a *cycloid*.

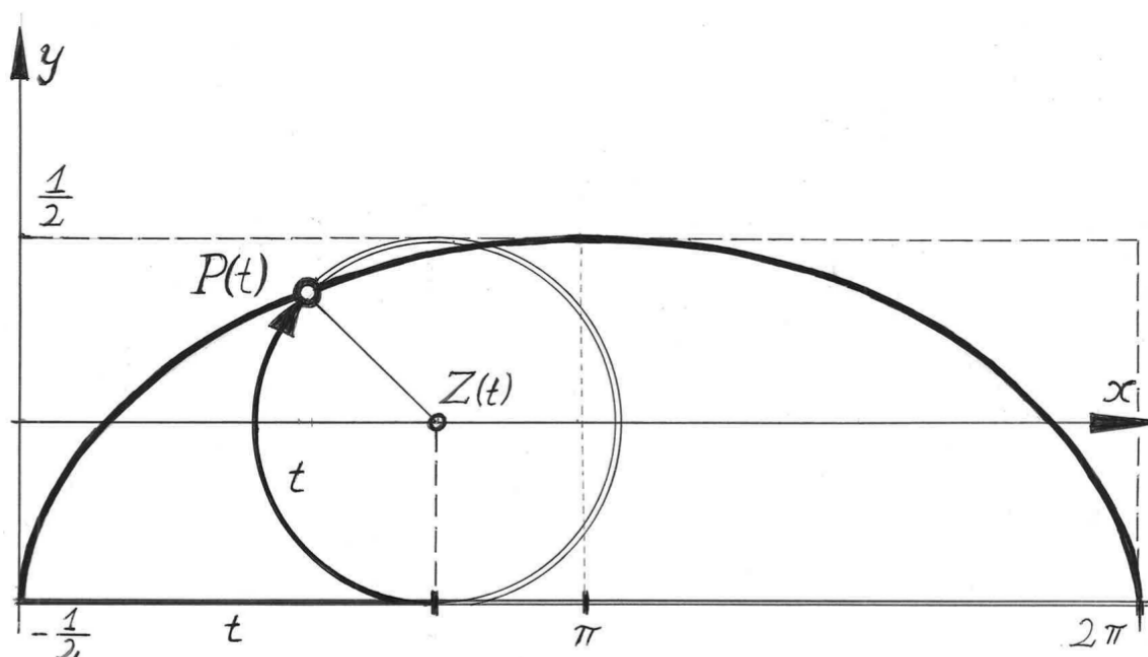


Figure 2.6: Cycloid

Suppose that the (center  $Z$  of) the wheel moves with constant speed of 1 – for example 1 meter/second – and that the center  $Z(0)$  of the wheel lies precisely over the point  $S$  if  $t = 0$ . Introduce a Cartesian system of coordinates with origin in  $Z(0)$ , horizontal  $x$ -axis and vertical  $y$ -axis. Let  $Z(t)$  denote the center of our circle at time  $t$ . Moreover let

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

be the position of the point  $P$  at the time  $t$ . Give a *parametrization* of the cycloid, by making explicit the two *coordinate functions*

$$t \mapsto x(t) \text{ and } t \mapsto y(t) \text{ for all } t \geq 0.$$

Determine the *vectorial*, and the *absolute velocity*, as well as the *vectorial* and the *absolute acceleration* of the point  $P(t)$ .

Assume that after  $\frac{3}{8}$  turns of the wheel from its starting position a drop of water which moved with the point  $P(t)$  flies away. At which distance from  $S$ , the water drop falls onto the plane? What is the maximal height over the plane of the drop during its flight?

Assume that the position at which the point  $P$  lies in the base plane can be determined only up to an error of 1cm. The wheel is stopped after 100 full turns - in the position where  $P$  can be assumed to touch again the base plane. Up to what error  $\pi$  can be determined by this experiment?

(C) (*Bi-Hexagonal Approximations of  $\pi$* ) Compute the values of  $\pi'_n := \pi_n^{[6]}$  for  $n = 1, 2, \dots, 7$  (see Didactic Comments 2.3 (C)) to approximate  $\pi$  and present the result in the following table.

$n$	$\pi'_n$ (approximate value)	number of vertices	name of the polygon
1			
2			
3			
4			
5			
6			
7			

(D) (*Approximating an Approximation*) Already in the Ancient Age, the number  $\pi$  was approximated by the fraction  $\frac{22}{7}$ . Determine  $n \leq 8$  and  $n' \leq 7$  such that  $\pi_n$  resp.  $\pi'_n$  is closest to the approximating fraction  $\frac{22}{7}$ .

(E) (*Completing Some Proofs*) Perform in detail the arguments hinted in the Didactic Comments 2.3 (C) and (B).

### 3. Defining $\pi$ as a Limit

**The Problem of Convergence.** In the Didactic Comment 2.3 (A) we have already hinted the conjectured relation  $\pi = \lim_{n \rightarrow \infty} \pi_n$ . From a premathematical point of view, this seems an obvious relation, as the regular  $2^n$ -gons on  $\mathcal{C}$  “better and better approximate the circle  $\mathcal{C}$ ”. But, from the strictly mathematical point of view, this heuristic observation is not yet satisfactory. Namely, viewed with the mathematicians eyes, we are left now with the following three questions:

- (Q1) Is the sequence  $(\pi_n)_{n \in \mathbb{N}}$  indeed convergent at all?
- (Q2) If the sequence  $(\pi_n)_{n \in \mathbb{N}}$  is convergent, can we estimate the difference between its limit and the approximating number  $\pi_n$  for all  $n \in \mathbb{N}$ ?
- (Q3) If the sequence  $(\pi_n)_{n \in \mathbb{N}}$  is convergent, is it justified to say that its limit is the quotient of the circumference of  $\mathcal{C}$  by the diameter of  $\mathcal{C}$ , hence what we defined heuristically as the number  $\pi$ ?

**Using Circumscribed Regular  $2^n$ -Gons.** We now attack the first two of of these three questions, which both can indeed be approached by mathematical tools from basic High School (Pre-)Calculus, provided we make right use of these tools.

The third question is of essentially more mathematical depth and exceeds – at least in its full width – what we can expect to be understood by High School Students. Therefore, we later shall treat this question in an extra Chapter, namely Chapter 4.

For, each  $n \in \mathbb{Z}_{\geq 2}$  we consider the “smallest regular  $2^n$ -gon which contains the circle  $\mathcal{C}$ ”, hence a regular  $2^n$ -gon whose edges are all tangent to  $\mathcal{C}$  and touch  $\mathcal{C}$  in their midpoint. We denote the circumference of such a circumscribed regular  $2^n$ -gon by  $\bar{c}_{2^n}$  and set

$$\bar{\pi}_n := \frac{\bar{c}_{2^n}}{d} \text{ for all } n \in \mathbb{N}.$$

Our heuristic idea is to approximate the circumference of our circle simultaneously by the circumferences of its inscribed and circumscribed regular  $2^n$ -gons. This idea indeed carries further and will lead us to a strict proof of question (Q1) and a satisfactory affirmative answer to question (Q2).

**3.1 Remark.** Observe that we have:

$$\bar{\pi}_2 = 4.$$

Indeed, in the case  $n = 2$ , the regular circumscribed polygon in question is just a square whose 4 edges touch the circle  $\mathcal{C}$ , hence a square which is circumscribed to this circle.



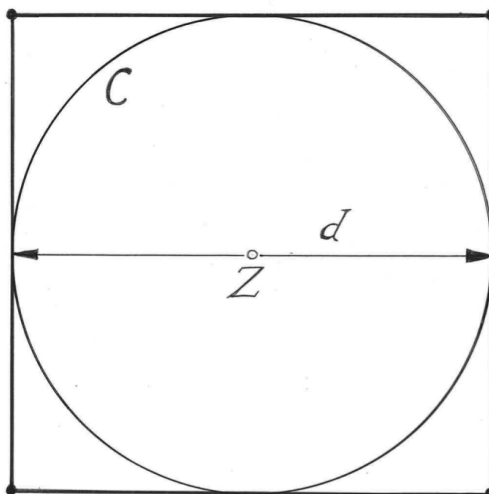


Figure 3.1: Square Circumscribed to the Circle

Clearly, this square has edges of length  $d$ , so that its circumference  $\bar{c}_{2^2} = \bar{c}_4$  equals  $4d$ . Consequently

$$\bar{\pi}_2 = \frac{\bar{c}_{2^2}}{d} = \frac{4d}{d} = 4.$$

**Comparing the “Inside” and the “Outside” Approach.** The essential step, needed to approach the previously mentioned two questions (Q1) and (Q2), is given by the following result, which compares the values of our two (hopefully) approximating sequences for  $\pi$  and estimates the difference between corresponding values in these sequences:

**3.2 Proposition.** *For each  $n \in \mathbb{N}$ , let  $c_{2^n}$  denote the circumference of a regular  $2^n$ -gon with vertices on the circle  $\mathcal{C}$  of diameter  $d$  and set  $\pi_n := \frac{c_{2^n}}{d}$ . For each  $n \in \mathbb{Z}_{\geq 2}$  let  $\bar{c}_{2^n}$  denote the circumference of a regular  $2^n$ -gon whose edges are all tangent to the circle  $\mathcal{C}$  and set  $\bar{\pi}_n := \frac{\bar{c}_{2^n}}{d}$ . Then we have*

(a) *For all  $n \geq 2$  it holds:*

$$\bar{\pi}_n = \pi_n \frac{1}{\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}.$$

(b) *For all  $n \geq 2$  it holds  $\pi_n < \bar{\pi}_n$ .*

(c) *For all  $n \geq 1$  it holds  $\pi_n < \pi_{n+1}$ .*

(d) *For all  $n \geq 2$  it holds  $\bar{\pi}_n > \bar{\pi}_{n+1}$ .*

(e) *For all  $n > 3$  it holds  $\bar{\pi}_n - \pi_n < \frac{1}{2^{2n-8}\sqrt{15}}$ .*

*Proof.* : (a): For each  $n \geq 1$  (respectively  $\geq 2$ ) let – once more –

$$f_n = \frac{\pi_n}{2^n}d \text{ respectively } \bar{f}_n = \frac{\bar{\pi}_n}{2^n}d$$

denote the length of a single edge of a regular  $2^n$ -gon with vertices on  $\mathcal{C}$  respectively with edges tangent to  $\mathcal{C}$ . Then, we have the situation sketched in Figure 3.3, in which  $Q$  and  $\bar{Q}$  respectively denote the mid-points of the the edges of the two  $2^n$ -gons under consideration.

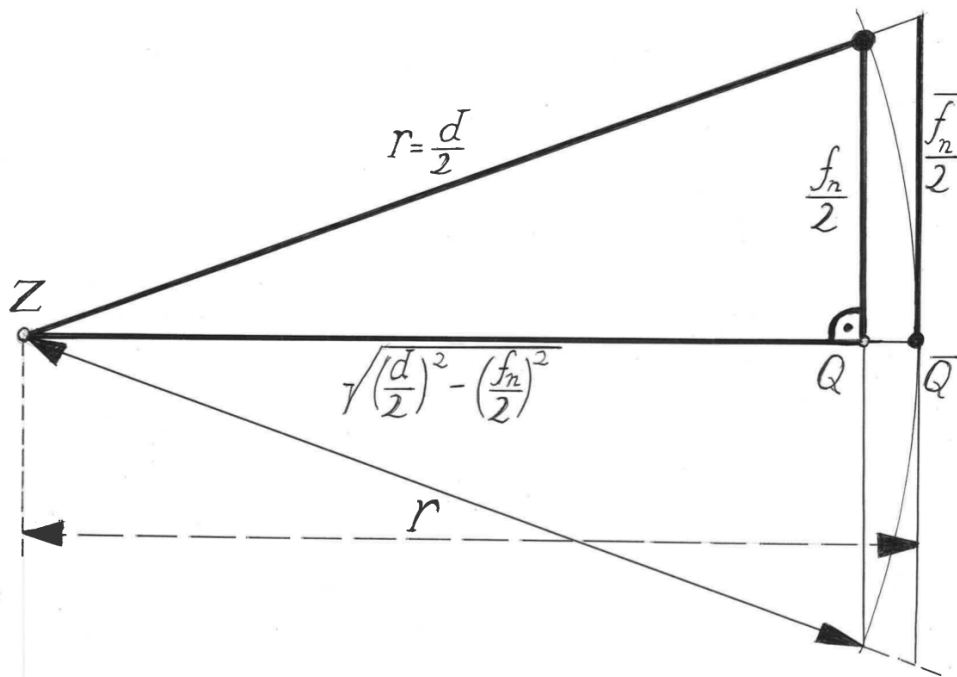


Figure 3.2: Relating  $\bar{f}_n$  and  $f_n$

Observe, that by Pythagoras the line segments from  $Z$  to  $Q$  and from  $Z$  to  $\bar{Q}$  have length

$$\sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{f_n}{2}\right)^2} = \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{\pi_n d}{2^n \cdot 2}\right)^2} = \frac{d}{2} \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} \text{ respectively } \frac{d}{2}.$$

It follows by similarity that

$$\bar{f}_n : f_n = \frac{d}{2} : \frac{d}{2} \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} = 1 : \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}, \text{ hence}$$

$$\bar{\pi}_n : \pi_n = \frac{\bar{\pi}_n d}{2^n} : \frac{\pi_n d}{2^n} = \bar{f}_n : f_n = 1 : \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2},$$

so that, indeed

$$\bar{\pi}_n = \pi_n \frac{1}{\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}.$$

(b): This is immediate by statement (a), as  $\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} < 1$ .

(c): We offer an algebraic and a geometric proof for this statement.

1. *Algebraic Proof:* Let  $n \in \mathbb{N}$ . Observe that

$$\begin{aligned}\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} &= \sqrt{1 - 2\frac{\pi_n^2}{2^{2n+1}}} < \sqrt{1 - 2\frac{\pi_n^2}{2^{2n+1}} + \frac{\pi_n^4}{(2^{2n+1})^2}} = \\ &= \sqrt{\left(1 - \frac{\pi_n^2}{2^{2n+1}}\right)^2} = 1 - \frac{\pi_n^2}{2^{2n+1}}.\end{aligned}$$

It follows that

$$1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} > 1 - \left(1 - \frac{\pi_n^2}{2^{2n+1}}\right) = \frac{\pi_n^2}{2^{2n+1}}.$$

So, by Proposition 2.1 (b) we obtain

$$\begin{aligned}\pi_{n+1}^2 &= \left[2^n \sqrt{2} \sqrt{1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}\right]^2 = 2^{2n+1} \left(1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}\right)^2 > \\ &> 2^{2n+1} \frac{\pi_n^2}{2^{2n+1}} = \pi_n^2,\end{aligned}$$

hence  $\pi_{n+1}^2 > \pi_n^2$ . As  $\pi_n$  and  $\pi_{n+1}$  are both positive, it follows that, indeed  $\pi_{n+1} > \pi_n$ .

2. *Geometric Proof:* Let  $n \in \mathbb{N}$ .

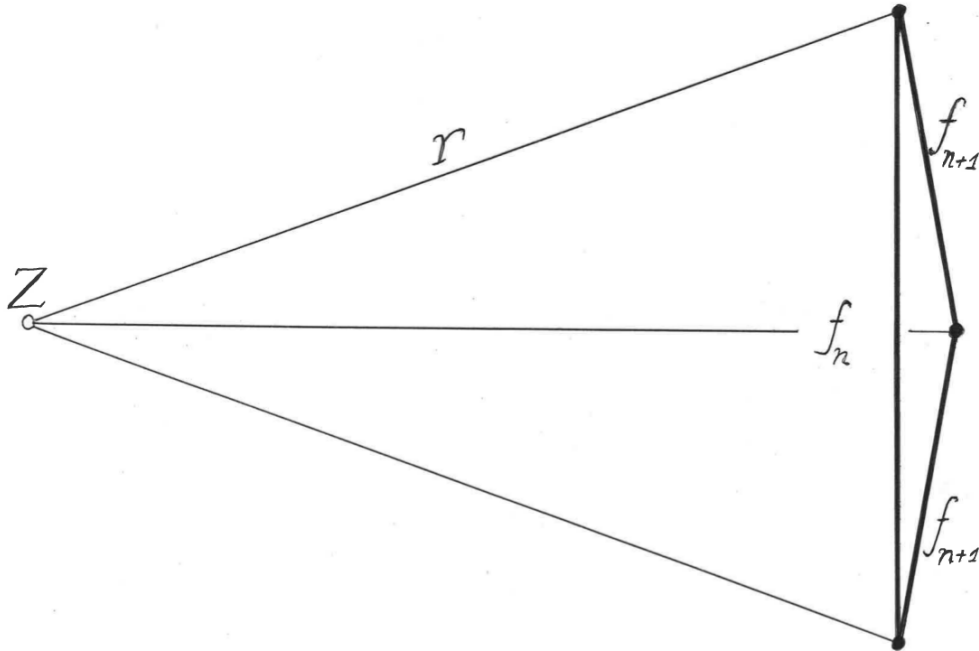


Figure 3.3: Relating  $f_n$  and  $2f_{n+1}$

According to Figure 3.4 we have an equilateral triangle with base  $f_n$  and with two edges of length  $f_{n+1}$ . Therefore we get

$$2f_{n+1} > f_n,$$

and hence

$$2\frac{\pi_{n+1}}{2^{n+1}}d > \frac{\pi_n}{2^n}d.$$

It follows that  $\pi_{n+1} > \pi_n$ .

(d): Again, we offer an algebraic and a geometric proof.

1. *Algebraic Proof:* Let  $n \in \mathbb{Z}_{\geq 2}$ . Observe that by statement (a) and Proposition 2.1 (b) we may write:

$$\begin{aligned} \bar{\pi}_{n+1}^2 &= \frac{\pi_{n+1}^2}{1 - \frac{\pi_{n+1}^2}{2^{2n+2}}} = \frac{2^{2n+1}(1 - \sqrt{1 - (\frac{\pi_n}{2^n})^2})}{1 - \frac{1}{2^{2n+2}} \cdot 2^{2n+1}(1 - \sqrt{1 - (\frac{\pi_n}{2^n})^2})} = \\ &= \frac{2^{2n+1}(1 - \sqrt{1 - (\frac{\pi_n}{2^n})^2})}{\frac{1}{2}(2 - (1 - \sqrt{1 - (\frac{\pi_n}{2^n})^2}))} = \frac{2^{2n+2}(1 - \sqrt{1 - (\frac{\pi_n}{2^n})^2})}{1 + \sqrt{1 - (\frac{\pi_n}{2^n})^2}} = \\ &= \frac{2^{2n+2}(1 - \sqrt{1 - (\frac{\pi_n}{2^n})^2})(1 + \sqrt{1 - (\frac{\pi_n}{2^n})^2})}{(1 + \sqrt{1 - (\frac{\pi_n}{2^n})^2})^2} = \\ &= \frac{2^{2n+2}(1 - (1 - (\frac{\pi_n}{2^n})^2))}{(1 + \sqrt{1 - (\frac{\pi_n}{2^n})^2})^2} = \frac{2^{2n+2} \frac{\pi_n^2}{2^{2n}}}{(1 + \sqrt{1 - (\frac{\pi_n}{2^n})^2})^2} = \frac{2^2 \pi_n^2}{(1 + \sqrt{1 - (\frac{\pi_n}{2^n})^2})^2}. \end{aligned}$$

Moreover, by statement (a), we get

$$\bar{\pi}_n^2 = \frac{\pi_n^2}{1 - (\frac{\pi_n}{2^n})^2}.$$

Setting

$$u := \sqrt{1 - (\frac{\pi_n}{2^n})^2},$$

we thus obtain

$$\bar{\pi}_{n+1}^2 = \frac{4}{(1+u)^2} \pi_n^2 \quad \text{and} \quad \bar{\pi}_n^2 = \frac{1}{u^2} \pi_n^2.$$

Observe, that  $0 < u < 1$ , so that  $3u^2 - 2u - 1 < 0$ , hence  $4u^2 < u^2 + 2u + 1 = (1+u)^2$ , thus

$$\frac{4}{(1+u)^2} < \frac{1}{u^2}.$$

Therefore we have that  $\bar{\pi}_{n+1}^2 < \bar{\pi}_n^2$ . As  $\bar{\pi}_{n+1}$  and  $\bar{\pi}_n$  are both positive, it follows indeed, that  $\bar{\pi}_{n+1} < \bar{\pi}_n$ .

2. *Geometric Proof:* Let  $n \in \mathbb{Z}_{\geq 2}$ . If we reconsider the situation sketched in Figure 3.2, we see immediately that (see Figure 3.5 below)

$$\bar{f}_{n+1} = 2 \frac{\bar{f}_{n+1}}{2} < \frac{\bar{f}_n}{2}.$$

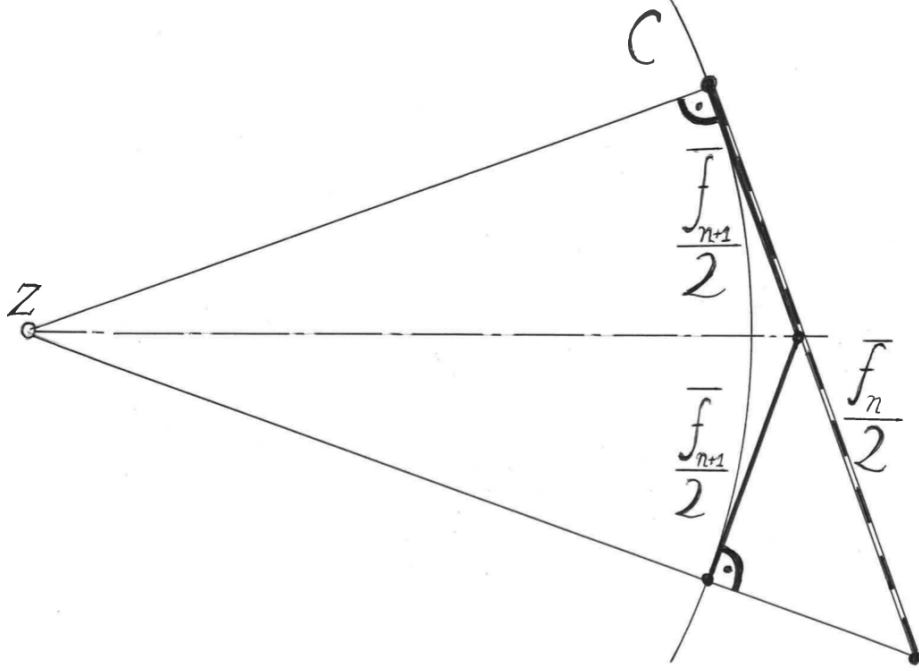


Figure 3.4: Comparing  $\bar{f}_{n+1}$  and  $\frac{\bar{f}_n}{2}$

Therefore, indeed:

$$\bar{\pi}_{n+1} = \frac{2^{n+1}\bar{f}_{n+1}}{d} < \frac{2^{n+1}\frac{\bar{f}_n}{2}}{d} = \frac{2^n\bar{f}_n}{d} = \bar{\pi}_n.$$

(e): Let  $n > 3$ . According to statement (a) we have

$$\begin{aligned} \bar{\pi}_n - \pi_n &= \pi_n \frac{1}{\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}} - \pi_n = \pi_n \left( \frac{1}{\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}} - 1 \right) = \\ &= \pi_n \frac{1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}{\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}. \end{aligned}$$

By statements (b) and (d) and by Remark 3.1 we have

$$\pi_n < \bar{\pi}_n \leq \bar{\pi}_2 = 4 = 2^2.$$

It follows that

$$\left(\frac{\pi_n}{2^n}\right)^2 < \left(\frac{2^2}{2^n}\right)^2 = \left(\frac{1}{2^{n-2}}\right)^2 = \frac{1}{2^{2n-4}}.$$

Consequently, as  $\frac{1}{2^{2n-4}} \leq \frac{1}{16}$ , we get

$$\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} > \sqrt{1 - \frac{1}{2^{2n-4}}} \geq \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}.$$

Moreover, as  $0 < 1 - \left(\frac{\pi_n}{2^n}\right)^2 < 1$ , we have

$$\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} > 1 - \left(\frac{\pi_n}{2^n}\right)^2 > 1 - \frac{1}{2^{2n-4}},$$

whence

$$1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} < 1 - \left(1 - \frac{1}{2^{2n-4}}\right) = \frac{1}{2^{2n-4}}.$$

Altogether, we thus obtain, indeed

$$\bar{\pi}_n - \pi_n = \pi_n \frac{1 - \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}}{\sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}} < 2^2 \frac{\frac{1}{2^{2n-4}}}{\frac{\sqrt{15}}{4}} = \frac{1}{2^{2n-8}\sqrt{15}},$$

as requested.  $\square$

**Answering Questions (Q1) and (Q2).** Now, we get the following consequence of Proposition 3.2, which will allow to answer affirmatively the two mentioned questions. This will be done by showing that our two approximating sequences converge to a common limit and by giving an estimate of the difference between the values of these sequences and their common limit.

**3.3 Corollary.** *Let the notations and hypotheses be as in Proposition 3.2. Then, the two sequences*

$$(\pi_n)_{n \geq 1} \text{ and } (\bar{\pi}_n)_{n \geq 2}$$

*are both convergent and have the same limit, say*

$$\pi := \lim_{n \rightarrow \infty} \pi_n = \lim_{n \rightarrow \infty} \bar{\pi}_n.$$

*Moreover:*

- (a) *For all  $n \geq 2$  it holds  $\pi_n < \pi < \bar{\pi}_n$ .*
- (b) *For all  $n > 3$  it holds*

$$0 < \pi - \pi_n, \quad \bar{\pi}_n - \pi < \frac{1}{2^{2n-8}\sqrt{15}}.$$

*Proof.* For each  $n \geq 2$  we consider the closed interval

$$\mathbb{I}_n := [\pi_n, \bar{\pi}_n] = \{x \in \mathbb{R} \mid \pi_n \leq x \leq \bar{\pi}_n\}.$$

By statements (b), (c) and (d) of Proposition 3.2 it follows immediately, that

$$\mathbb{I}_{n+1} \subseteq \mathbb{I}_n \text{ for all } n \geq 2.$$

This means, that

$$(\mathbb{I}_n)_{n \geq 2} \text{ is a sequence of nested closed intervals.}$$

So, by the completeness of  $\mathbb{R}$ , there is some

$$\pi \in \bigcap_{n \geq 2} \mathbb{I}_n, \text{ whence } \pi_n < \pi < \bar{\pi}_n \text{ for all } n \geq 2.$$

Moreover, statement (e) of Proposition 3.2 now yields that

$$0 < \pi - \pi_n, \quad \bar{\pi}_n - \pi < \frac{1}{2^{2n-8}\sqrt{15}} \text{ for all } n > 3.$$

It thus remains to show that the number  $\pi$  is the common limit of the two sequences  $(\pi_n)_{n \geq 1}$  and  $(\bar{\pi}_n)_{n \geq 2}$ , hence that

$$\pi = \lim_{n \rightarrow \infty} \pi_n = \lim_{n \rightarrow \infty} \bar{\pi}_n.$$

To this end, chose an arbitrary real number  $\varepsilon > 0$ . As

$$\lim_{n \rightarrow \infty} \frac{1}{2^{2n-8}\sqrt{15}} = 0,$$

there is some integer  $n(\varepsilon) > 3$  such that

$$\frac{1}{2^{2n-8}\sqrt{15}} < \varepsilon \text{ for all } n \geq n(\varepsilon).$$

So, as  $\pi - \pi_n$  and  $\bar{\pi}_n - \pi_n$  are both positive, we finally obtain

$$|\pi - \pi_n| = \pi - \pi_n < \frac{1}{2^{2n-8}\sqrt{15}} < \varepsilon \text{ for all } n \geq n(\varepsilon)$$

and

$$|\pi - \bar{\pi}_n| = \bar{\pi}_n - \pi < \frac{1}{2^{2n-8}\sqrt{15}} < \varepsilon \text{ for all } n \geq n(\varepsilon).$$

As  $\varepsilon > 0$  was chosen arbitrarily it follows indeed, that the two sequences  $(\pi_n)_{n \geq 1}$  and  $(\bar{\pi}_n)_{n \geq 2}$  both converge to  $\pi$ , and this concludes our proof.  $\square$

As a further immediate consequence of this result we see, that we can answer affirmatively the first two of our three original questions. We namely have.

**3.4 Corollary.** *Let the notations and hypotheses be as in Corollary 3.3.*

- (a) *The sequence  $(\pi_n)_{n \geq 1}$  converges.*
- (b) *With  $\pi := \lim_{n \rightarrow \infty} \pi_n$  we have the estimate*

$$0 < \pi - \pi_n < \frac{1}{2^{2n-8}\sqrt{15}} \text{ for all } n > 3.$$

**3.5 Exercises and Remarks.** (A) (*A Recursive Formula for the Numbers  $\bar{\pi}_{n+1}$ : An Algebraic Proof*) Going back to the algebraic proof of statement (d) of Proposition 3.2 you will be able to immediately establish the equality

$$\bar{\pi}_{n+1} = \frac{2\pi_n}{1 + \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2}} \text{ for all } n \geq 2.$$

On use of statement (a) of Proposition 3.2 you also will derive immediately the equality

$$\pi_n = \frac{\bar{\pi}_n}{\sqrt{1 + \left(\frac{\bar{\pi}_n}{2^n}\right)^2}} \text{ for all } n \geq 2.$$

Use this, to derive the recursive relation

$$\bar{\pi}_{n+1} = \frac{\bar{\pi}_n}{1 + \sqrt{1 + \left(\frac{\bar{\pi}_n}{2^n}\right)^2}} \text{ for all } n \geq 2.$$

(B) (*A Recursive Formula for the Numbers  $\bar{\pi}_{n+1}$ : A Geometric Proof*) Let the notations be as in part (A) and as in the proof of statement (d) of Proposition 3.2. verify first, that

$$\frac{\bar{\pi}_n}{1 + \sqrt{1 + \left(\frac{\bar{\pi}_n}{2^n}\right)^2}} = \frac{2^{2n+1}}{\bar{\pi}_n} \left( \sqrt{1 + \left(\frac{\bar{\pi}_n}{2^n}\right)^2} - 1 \right) \text{ for all } n \geq 2.$$

Then, use the figure below (in which  $\alpha = \frac{2\pi}{2^n}$ ) to prove the recursion formula of part (A).

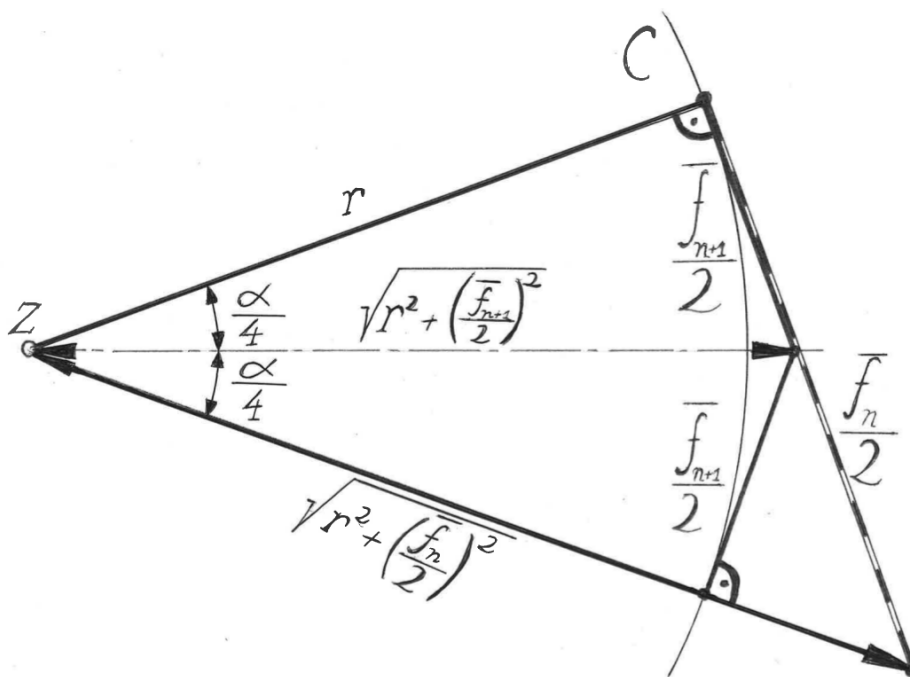


Figure 3.5: Relating  $\bar{f}_{n+1}$  and  $\bar{f}_n$

(C) (*Approximating  $\pi$  by the numbers  $\bar{\pi}_n$* ). Compute the numbers  $\bar{\pi}_n$ , the differences  $\bar{\pi}_n - \pi_n$  and the upper bound  $1/(2^{2n-8}\sqrt{15})$  for these differences for  $n = 2, 3, \dots, 8$  – either by using the values computed in Application and Exercise 2.2 (A) and the formula of Proposition 3.2 (a) – or else on use of the recursion formula of part (A). Write down the result in the following table.



$n$	$\bar{\pi}_n$	$\bar{\pi}_n - \pi_n$	$1/(2^{2n-8}\sqrt{15})$
2			
3			
4			
5			
6			
7			
8			

(D) (*The Area of the Circular Disk*) Let  $A_n$  and  $\bar{A}_n$  denote the area of a regular  $n$ -gon inscribed respectively circumscribed to the circle  $\mathcal{C}$  of diameter  $d$  and radius  $r$ . Prove that for all  $n \geq 2$  we have

$$\bar{A}_{2n} = \bar{\pi}_n r^2 \quad \text{and} \quad A_{2n} = \pi_n \sqrt{1 - \left(\frac{\pi_n}{2^n}\right)^2} r^2.$$

Use this to explain heuristically why it is justified to say that the area of the circular disk of radius  $r$  is given by

$$A = \pi r^2.$$

(E) (*Approximating  $\pi$  up to 6 Digits*) Use the inequality of Corollary 3.3 (b) in order determine some integer  $n_0 > 3$  such that

$$\bar{\pi}_n - \pi_n < 10^{-6} \quad \text{and hence} \quad \bar{\pi}_n - \pi, \pi - \pi_n < 10^{-6} \quad \text{for all } n \geq n_0.$$

**Not Loosing Courage.** At first glance, you might feel that this chapter is not very helpful for your teaching at High School. Nevertheless, below, we try to encourage you to dare make enter some of the treated matters to your teaching.

**3.6 Didactic Comments.** (A) (*Transfer to the Classroom*) At a first glance, Proposition 3.2 may seemingly be beyond of the reach of High School Mathematics, at least concerning its proof. But again, you could proceed in two steps, and first just give the statements, in particular the formula given in statement (a). Then you could already attack Exercise 3.5 (C). In a second turn you could prove (some of) the statements of Proposition 3.2 – obviously with the presented *geometric arguments*. These arguments surely are accessible to what you can do in the classroom.

Corollary 3.3 first of all is addressed to you, this means the teachers. So, consider its proof and reflect it in the sense of a reminder what you have learned in your Undergraduate Calculus course. This brings you a few steps ahead of your class and makes you argue in a way which slightly exceeds what you likely can do in the classroom. But as an extra challenge you could give the consideration of this proof as an object of study for a (group of) highly motivated students. Or you could take the challenge on your own, and reduce the arguments of the proof to a level which is accessible to the majority of your class.

(B) (*A General Consideration*) Let the students share the insight, how much Mathematics is hidden behind a simple object as the circle, and do not be afraid to tell them, that there is indeed a substantial portion of much deeper and much more advanced Mathematics related to the circle and the number  $\pi$ . This could help to promote the scientific curiosity of your students, at least of those, who have already a particular interest in Mathematics.

## 4. What is the Circumference?

**Approximation by Regular Polygons.** In this chapter we shall consider the third initial question posed at the beginning of Chapter 3: How to justify the idea that the circumference of the circle can be considered as the limit

$$\pi := \lim_{n \rightarrow \infty} \pi_n = \lim_{n \rightarrow \infty} \bar{\pi}_n.$$

This *justification problem* is related to a much more general question, namely the problem of defining properly the *length of a curve* or – in other words the problem of *rectification of curves*. An exhaustive treatment of this problem would by far exceed the framework of this course. Nevertheless, we will hint some details, which could help to brush up a few things you likely have learned once when attending University. We believe, that such a brushing up opportunity does not harm to a Mathematics teacher, and can even reveal – once more, hopefully – the intellectual beauty of the field he is teaching.

**4.1 Remarks.** (A) (*How to Define the Circumference?*) In Chapter 3 we fixed a circle  $\mathcal{C}$  of diameter  $d$  and defined  $\pi_n := \frac{c_{2^n}}{d}$ , where  $c_{2^n}$  was the circumference of a regular  $2^n$ -gon with vertices on  $\mathcal{C}$ . (See also Definition and Exercise 1.6.) Then we used the heuristic idea that the circumferences  $c_{2^n}$  of these polygons should converge to the circumference  $c$  of our circle  $\mathcal{C}$ . Yet heuristically – this gave us the idea that the sequence  $\pi_n = \frac{c_{2^n}}{d}$  must converge to the ratio  $\frac{c}{d}$  between the circumference  $c$  of the circle  $\mathcal{C}$  and the diameter  $d$  of this circle. Then, we provided a proper proof of the fact that the sequence  $(\pi_n)_{n \in \mathbb{N}}$  converges at all. We denoted the limit of this sequence by  $\pi$ . In view of our heuristic considerations we were ready to believe that this limit is indeed the requested ratio  $\frac{c}{d}$ , or – equivalently – that  $c = \pi d$ . But actually, in this way we have computed the circumference  $c$  (or the mentioned ratio) without having defined the notion of circumference at all. More precisely, we have left open the question:

(Q4) How can we define the notion of circumference of a circle?

Clearly, a possible way to define this circumference would just be to say:

(A4) The circumference of the circle  $\mathcal{C}$  with diameter  $d$  is the number  $c := \pi d$ , where  $\pi := \lim_{n \rightarrow \infty} \pi_n$ , is the limit of our convergent sequence

$$\left(\pi_n = \frac{c_{2^n}}{d}\right)_{n \in \mathbb{N}}.$$

(B) (*Sequences of Regular Polygons*) If we agree to accept the previous definition of circumference, we are faced with a new problem. Namely, let

$$\pi'_n := \frac{c_{3^n}}{d}, \quad (n \in \mathbb{N})$$

be the ratio between the circumference  $c_{3^n}$  of a regular  $3^n$ -gon with vertices on  $\mathcal{C}$  and the  $d$  diameter of this circle. Why should it be forbidden, to show that the sequence  $(\pi'_n)_{n \in \mathbb{N}}$  converges to some limit  $\pi'$ , and then to define the circumference of  $\mathcal{C}$  as  $c' := \pi' d$ . But this would leave us with the task to show

that  $\pi' = \pi$ .

Otherwise there would be no sensible way to define the circumference of  $\mathcal{C}$ . Indeed,  $2^n$ -gonists would have to use a different circumference from that of  $3^n$ -gonists, not to speak about  $5^n$ -gonists or  $7^n$ -gonists... But also simple  $n$ -gonists, or  $3n$ -gonists or even  $5n - 4$ -gonists might come along and claim to compete for the definition of the circumference. This means, that finally one would have to admit to the pace for the computation of  $\pi$  all sequences of the form

$$\left(\frac{c_{n_i}}{d}\right)_{i \in \mathbb{N}}, \text{ for which } n_i \xrightarrow{i \rightarrow \infty} \infty,$$

where  $c_{n_i}$  denotes the circumference of a regular  $n_i$ -gon with vertices on  $\mathcal{C}$ . In other words, all “regular polygonists” could claim to have the right value of circumference of our circle  $\mathcal{C}$ .

But indeed, at least those among you who have solved exercise (d) of Definition and Exercise 1.6 (C), have already achieved a reconciliation among all possible types of regular polygonists. Namely, by what you (should) have shown there, it follows (as you hopefully will prove now):

(R1) For all sequences  $(n_i)_{i \in \mathbb{N}}$  of integers  $n_i \in \mathbb{Z}_{\geq 2}$  with  $\lim_{i \rightarrow \infty} n_i = \infty$  the sequence  $\left(\frac{c_{n_i}}{d}\right)_{i \in \mathbb{N}}$  converges to the limit  $\pi$  of (A4).

That this common limit exists, shows that all regular polygonists have found a convincing way to define the circumference of our circle  $\mathcal{C}$ .

**Daring More: Approximations by Irregular Polygons.** If you content yourself to be a member of the regular polygonists union, you may skip the rest of the chapter, on taking the risk, that you could miss some interesting features. But if you agree that one should also admit arbitrarily shaped or *irregular* polygons with vertices on  $\mathcal{C}$  to compete for the computation of the circumference, you are invited to follow our further developments.

**4.2 Remarks.** (A) (*Admitting Sequences of Irregular Polygons*) To introduce the new situation, we go back once more to the concepts we have developed already in Exercise and Definition 1.5 and Definition and Exercise 1.6. We first introduce the radius  $r = \frac{d}{2}$  of  $\mathcal{C}$ . For the sake of simplicity we also assume that the circle  $\mathcal{C}$  is centered at  $O = (0, 0)$ . Then we consider the standard parametrization

$$\begin{aligned} \varphi = \varphi_r^{[O]} : [0, 2\pi] &\longrightarrow \mathbb{R}^2 = \mathbb{E}, \quad t \mapsto \varphi(t) = \varphi_r^{[O]}(t) := (x(t), y(t)) \text{ with} \\ x(t) &:= r \cos(t) \quad \text{and} \quad y(t) := r \sin(t) \quad \text{for all } t \in [0, 2\pi], \end{aligned}$$

where  $\cos(t)$  and  $\sin(t)$  are again understood in the “rad”-mode, as usually in Calculus. Then, we consider an arbitrary  $n$ -partition of the closed interval  $[0, 2\pi]$ :

$$t_\bullet := (t_i)_{i=0}^n \in [0, 2\pi]^{n+1} \text{ with } 0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n.$$

In addition, we consider the induced  $n$ -gon with vertices on  $\mathcal{C}$ :

$$P_\bullet = (P_i)_{i=0}^n \text{ with } P_i := \varphi(t_i) = \varphi_r^{[O]}(t_i) \in \mathcal{C}, \quad (i = 0, 1, 2, \dots, n-1, n),$$

which we shall write again in the form

$$\varphi(t_\bullet) = \varphi_r^{[O]}(t_\bullet) = (\varphi(t_i))_{i=0}^n = (\varphi_r^{[O]}(t_i))_{i=0}^n.$$

Finally, we consider the circumference of the polygon  $P_\bullet$  (see Definition and Exercise 1.6 (B)(b)):

$$\begin{aligned} c^{[P_\bullet]} = c^{[\varphi(t_\bullet)]} &:= \sum_{i=1}^n \text{dist}(P_i, P_{i-1}) = \\ &= \sum_{i=1}^n \text{dist}(\varphi(t_i), \varphi(t_{i-1})) = 2r \sum_{i=1}^n \sin\left(\frac{t_i - t_{i-1}}{2}\right). \end{aligned}$$

Now, our aim must be to show that the circumferences of all sequences of polygons with vertices on  $\mathcal{C}$  which are defined “by finer and finer partitions” converge to the same value.

(B) (*Reformulating our Task*) We now aim to reformulate the reconciliation task mentioned at the end of part (A) in precise terms. First of all, if

$$t_\bullet := (t_i)_{i=0}^n \in [0, 2\pi]^{n+1} \text{ with } 0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n.$$

is an arbitrary partition of the interval  $[0, 2\pi]$ , we introduce the *fineness* of  $t_\bullet$  by

$$f(t_\bullet) := \max\{t_i - t_{i-1} \mid i = 1, 2, \dots, n\}.$$

Now, in order to solve the task we gave ourselves in part (A), we aim to show the following:

(R2) There is a number  $c \in \mathbb{R}$  such that for all sequences of partitions

$$(t_\bullet^{[k]})_{k \in \mathbb{N}} \text{ where } t_\bullet^{[k]} = (t_i^{[k]})_{i=0}^{n_k}, \text{ with } 0 = t_0^{[k]} < t_1^{[k]} < \dots < t_{n_k-1}^{[k]} < t_{n_k}^{[k]} = 2\pi$$

such that  $\lim_{k \rightarrow \infty} f(t_\bullet^{[k]}) = 0$  it holds

$$\lim_{k \rightarrow \infty} 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) = c.$$

Indeed, we have the following result.

**4.3 Proposition.** *The reconciliation result (R2) mentioned in Remark 4.2 (B) holds.*

*Proof.* First of all, let us recall the well known fact – usually established in basic calculus – that

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

So, for each  $\varepsilon \in \mathbb{R}_{>0}$  there is some  $\delta(\varepsilon) \in \mathbb{R}_{>0}$  such that

$$\left| \frac{\sin(x)}{x} - 1 \right| < \varepsilon \text{ for all } x \in \mathbb{R} \setminus \{0\} \text{ with } |x| < \delta(\varepsilon).$$

It follows that

$$(\textcircled{a}) \quad |\sin(x) - x| < |x|\varepsilon \text{ for all } x \in \mathbb{R} \text{ with } |x| < \delta(\varepsilon).$$

Now, let the sequence of partitions  $(t_{\bullet}^{[k]})_{k \in \mathbb{N}}$  be such that  $\lim_{k \rightarrow \infty} f(t_{\bullet}^{[k]}) = 0$ . Then, it follows that for each  $\varepsilon \in \mathbb{R}_{>0}$  there is some  $k_0(\varepsilon) \in \mathbb{N}$  such that

$$f(t_{\bullet}^{[k]}) < \delta(\varepsilon) \text{ for all } k > k_0(\varepsilon).$$

As  $\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2} \leq \frac{f(t_{\bullet}^{[k]})}{2}$  for all  $i \in \{1, 2, \dots, n_k\}$  it follows that

$$(\textcircled{\text{@@}}) \quad \frac{t_i^{[k]} - t_{i-1}^{[k]}}{2} < \frac{\delta(\varepsilon)}{2} \text{ for all } k > k_0(\varepsilon) \text{ and all } i \in \{1, 2, \dots, n_k\}.$$

If we apply statements  $(\textcircled{\text{@}})$  and  $(\textcircled{\text{@@}})$  with  $x := \frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}$  and with  $\frac{2\varepsilon}{r}$  instead of  $\varepsilon$ , we thus obtain

$$\begin{aligned} \left| \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - \frac{t_i^{[k]} - t_{i-1}^{[k]}}{2} \right| &< \left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) \frac{2\varepsilon}{r} = \\ &= (t_i^{[k]} - t_{i-1}^{[k]}) \frac{\varepsilon}{r} \text{ for all } k > k_0\left(\frac{2\varepsilon}{r}\right) \text{ and all } i \in \{1, 2, \dots, n_k\}. \end{aligned}$$

Consequently, for all  $\varepsilon \in \mathbb{R}_{>0}$  and all  $k > k_0\left(\frac{2\varepsilon}{r}\right)$  we have

$$\begin{aligned} \left| 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - 2r\pi \right| &= \left| 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - r \sum_{i=1}^{n_k} (t_i^{[k]} - t_{i-1}^{[k]}) \right| = \\ &= \left| 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - 2r \sum_{i=1}^{n_k} \frac{t_i^{[k]} - t_{i-1}^{[k]}}{2} \right| = \\ &= 2r \left| \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - \frac{t_i^{[k]} - t_{i-1}^{[k]}}{2} \right| \leq \\ &\leq 2r \sum_{i=1}^{n_k} \left| \sin\left(\frac{(t^{[k]})_i - t_{i-1}^{[k]}}{2}\right) - \frac{t_i^{[k]} - t_{i-1}^{[k]}}{2} \right| < \\ &< 2r \sum_{i=1}^{n_k} (t_i^{[k]} - t_{i-1}^{[k]}) \frac{\varepsilon}{r} = 2r \cdot 2\pi \frac{\varepsilon}{r} = 4\pi\varepsilon. \end{aligned}$$

If we apply this with  $\frac{\varepsilon}{4\pi}$  instead of  $\varepsilon$  we finally obtain

$$\left| 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - 2r\pi \right| < \varepsilon \text{ for all } \varepsilon > 0 \text{ and all } k > k_0\left(\frac{\varepsilon}{2\pi r}\right).$$

Setting  $k_1(\varepsilon) := k_0\left(\frac{\varepsilon}{2\pi r}\right)$ , we thus have

$$\left| 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) - 2r\pi \right| < \varepsilon \text{ for all } \varepsilon > 0 \text{ and all } k > k_1(\varepsilon).$$

This shows that

$$\lim_{k \rightarrow \infty} 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) = 2r\pi.$$

□

As an immediate application of the previous result we get:

**4.4 Corollary.** *Let  $\mathcal{C} = \mathcal{C}_r(O)$  be the circle of radius  $r$ , diameter  $d$  and center  $O$ . Then, for all sequences of partitions*

$$(t_{\bullet}^{[k]})_{k \in \mathbb{N}}, \quad t_{\bullet}^{[k]} = (t_i^{[k]})_{i=0}^{n_k}, \quad (0 = t_0^{[k]} < t_1^{[k]} < \dots < t_{n_k-1}^{[k]} < t_{n_k}^{[k]} = 2\pi)$$

of  $[0, 2\pi]$  such that

$$\lim_{k \rightarrow \infty} f(t_{\bullet}^{[k]}) = 0$$

it holds

$$\lim_{k \rightarrow \infty} c^{[\varphi(t_{\bullet}^{[k]})]} = \lim_{k \rightarrow \infty} 2r \sum_{i=1}^{n_k} \sin\left(\frac{t_i^{[k]} - t_{i-1}^{[k]}}{2}\right) = 2r\pi,$$

where  $c^{[\varphi(t_{\bullet}^{[k]})]}$  denotes the circumference of the polygon  $P_{\bullet}^k := \varphi(t_{\bullet}^{[k]})$

*Proof.* This is an immediate consequence of Proposition 4.3 and its proof.  $\square$

**Taking a Higher Challenge: Rectifying Smooth Curves.** Now, we have properly solved the *rectification problem for the circle*, this means the problem to define the circumference of a circle. It is natural to ask, whether the previous ideas generalize to more general curves. First one could consider closed curves, as the circle. But it would be more appealing, to solve the *rectification problem* for non-closed curves too, by replacing the notion of circumference by the notion of length. So, more precisely, we may ask: is there a larger class of plane curves  $\mathcal{S}$ , for which we can define the notion of length of  $\mathcal{S}$  in an appropriate way. In the case of the circle, the clue to the solution of this problem was the use of the standard parametrization, a parametrization which we shall recognize below as being *smooth*. Therefore, we shall follow this track and introduce the notion of *length of smoothly parametrized curves*. You will recognize, that you most likely had studied this subject in your former Calculus Course at University, or that you have actually treated the subject in a more heuristic way with your own classes, at least to a certain extend.

**4.5 Remark, Reminder, Definition and Exercise.** (A) (*Polygons on Smoothly Parametrized Curves*) Let  $\mathcal{S} \subseteq \mathbb{E} = \mathbb{R}^2$ . Assume that there is a closed interval

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} \in \mathbb{R} \text{ with } a < b$$

and a map

$$\varphi : [a, b] \longrightarrow \mathbb{E} = \mathbb{R}^2, \quad t \mapsto \varphi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

such that

- (a)  $\mathcal{S} = \varphi([a, b])$ .
- (b) The two coordinate functions  $x, y : [a, b] \longrightarrow \mathbb{R}$  of  $\varphi$  are both continuously differentiable.

Condition (a) means, that the set  $\mathcal{S}$  is *parametrized* by the map  $\varphi$ , so that we can regard it as a curve.

Condition (b) means that the parametrization map  $\varphi$  is *smooth*. So, we assume that the derivatives

$$\dot{x}(t) = \frac{d}{dt}x(t) \text{ and } \dot{y}(t) = \frac{d}{dt}y(t) \text{ exist for all } t \in [a, b]$$

and the two derivation functions

$$\dot{x}, \dot{y} : [a, b] \rightarrow \mathbb{R}, \text{ given by } t \mapsto \dot{x}(t) \text{ respectively } t \mapsto \dot{y}(t) \text{ are continuous.}$$

In this situation, we call  $\mathcal{S}$ , or more precisely

$$\varphi : [a, b] \rightarrow \mathcal{S},$$

a *smoothly parametrized curve*. Observe, that

- (c) The standard parametrization  $\varphi = \varphi_r^{[Z]} : [0, 2\pi] \rightarrow \mathcal{C} = \mathcal{C}_r(Z)$  of the circle  $\mathcal{C}$  is smooth and hence turns  $\mathcal{C}$  into a smoothly parametrized curve.

Now, we aim to formulate the *rectification problem for smoothly parametrized curves* and we shall do this by generalizing what we did in the case of the circle. Again, we start with the notion of partition. More precisely, by a *partition* of the interval  $[a, b]$  we mean a finite sequence

$$t_\bullet = (t_0, t_1, \dots, t_n) \text{ with } a = t_0 < t_1 < \dots < t_{n-1} < t_n = b.$$

We now consider the sequence

$$P_\bullet = (P_i)_{i=0}^n \in \mathcal{S}^n \text{ with } P_i := \varphi(t_i) \in \mathcal{S} \text{ for all } i = 0, 1, \dots, n,$$

which we call *the polygon on  $\mathcal{S}$  induced by the partition  $t_\bullet$* . The points  $P_i \in \mathcal{S}$  are again called the *vertices* of our polygon. Again, we also shall write

$$P_\bullet = \varphi(t_\bullet).$$

Now, the *polygonal path induced by the partition  $t_\bullet$*  is the sequence

$$(|P_{i-1}P_i|)_{i=1}^n \text{ of line segments } |P_{i-1}P_i| \quad (i = 1, 2, \dots, n)$$

between all pairs of consecutive vertices of our polygon on  $\mathcal{S}$ . Such paths are just referred to as *polygonal paths on  $\mathcal{S}$* . Instead of the circumferences of polygons with vertices on the circle, we now shall use the total lengths of polygonal paths on the curve  $\mathcal{S}$  to approximate the length of this curve. Below, we shall make this more precise.



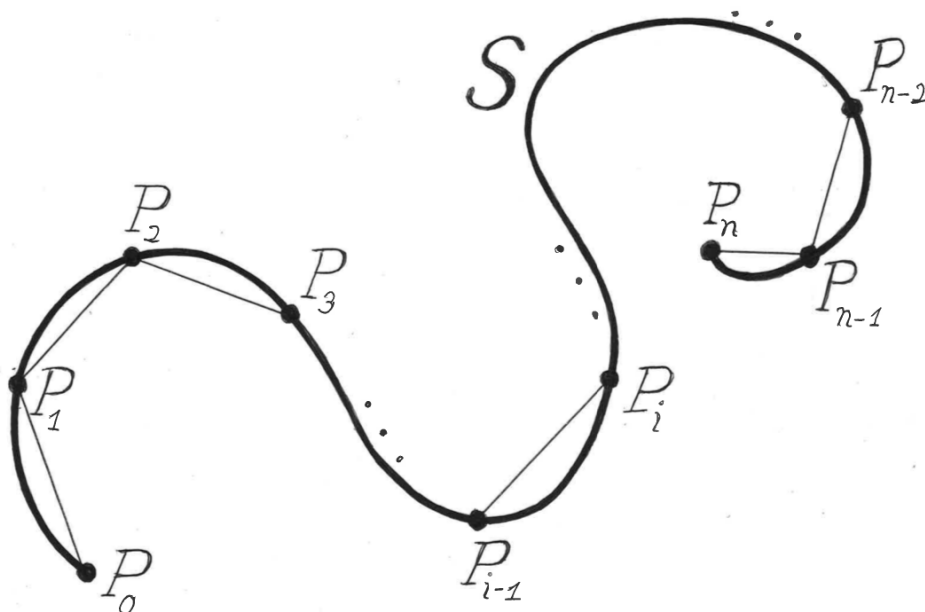


Figure 4.1: Polygonal Path  $P_0, P_1, \dots, P_{i-1}, P_i, \dots, P_{n-1}, P_n$  on  $\mathcal{S}$

The *length* of the polygonal path induced by the partition  $t_\bullet$  is obviously given by:

$$\begin{aligned} \lambda(P_\bullet) &= \lambda(\varphi(t_\bullet)) := \sum_{i=1}^n \text{dist}(P_i, P_{i-1}) = \\ &= \sum_{i=1}^n \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}. \end{aligned}$$

Now, again our aim must be to show, that the lengths of all polygonal paths with vertices on  $\mathcal{S}$  which are defined “by finer and finer partitions” converge to the same value.

(B) (*Reformulating the Rectification Problem for Curves*) We follow the trace hinted by what we did in Remark 4.2 (B) for the circle. So, again, the *fineness* of the partition  $t_\bullet = (t_0, \dots, t_n)$  of  $[a, b]$  is defined by

$$f(t_\bullet) := \max\{t_i - t_{i-1} \mid i = 1, \dots, n\}.$$

Now, to solve the task we gave ourselves in part (A), we aim to show the following:

(RC) There is a number  $\lambda(\varphi) \in \mathbb{R}$  such that for all sequences of partitions

$$(t_\bullet^{[k]})_{k \in \mathbb{N}}, \text{ where } t_\bullet^{[k]} = (t_i^{[k]})_{i=0}^{n_k}, \text{ with } a = t_0^{[k]} < t_1^{[k]} < \dots < t_{n_k-1}^{[k]} < t_{n_k}^{[k]} = b$$

such that  $\lim_{k \rightarrow \infty} f(t_\bullet^{[k]}) = 0$  it holds

$$\begin{aligned}
\lim_{k \rightarrow \infty} \lambda(\varphi(t_{\bullet}^{[k]})) &= \lim_{k \rightarrow \infty} \sum_{i=1}^n \text{dist}(\varphi(t_i^{[k]}), \varphi(t_{i-1}^{[k]})) = \\
&= \lim_{k \rightarrow \infty} \sum_{i=1}^n \sqrt{(x(t_i^{[k]}) - x(t_{i-1}^{[k]}))^2 + (y(t_i^{[k]}) - y(t_{i-1}^{[k]}))^2} = \\
&= \lambda(\varphi).
\end{aligned}$$

The solution of this problem can be given by showing that the requested number  $\lambda(\varphi)$  can be represented as an integral. Below, we shall formulate this solution, without proving it. In case our problem finds an affirmative answer, the number  $\lambda(\varphi)$  we are looking for is called the *length of the curve  $\mathcal{S}$  with respect to the parametrization  $\varphi$* .

Observe that the number  $\lambda(\varphi)$  may heuristically be understood as the *total length of the trajectory travelled by the point  $\varphi(t)$  if  $t$  runs from  $a$  to  $b$* . This is particularly illuminating heuristic point of view, if  $t$  is considered as time (see Exercise (d) in Part (C) below).

(C) (*Computing the Curve Length by Integration*) A basic theorem – usually proved in Advanced Calculus courses, and made evident in Undergraduate Calculus courses – claims:

(RC1) For each smooth parametrization

$$\varphi : [a, b] \rightarrow \mathcal{S}$$

of the curve  $\mathcal{S}$ , the requested curve length of  $\mathcal{S}$  with respect to  $\varphi$  exists and is given by

$$\lambda(\varphi) = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt = \int_a^b \|\dot{\varphi}\| dt,$$

where  $\|\dot{\varphi}\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$  denotes the *norm* of the (*vectorial*) *derivative*

$$\dot{\varphi}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

of  $\varphi$  at  $t$ .

We shall give a proper proof of this statement later. For the moment, we content ourselves to illustrate the meaning of this claim by some exercises.

- (a) Use the integral formula of (RC1) to compute the circumference of a circle.
- (b) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuously differentiable function. Prove by the integral formula given in (RC1), that the length of the graph

$$\text{Graph}(f) := \{(x, f(x)) \in \mathbb{R}^2 \mid x \in [a, b]\}$$

is given by

$$\int_a^b \sqrt{1 + (f'(x))^2} dx.$$

- (c) Give a heuristic argument to justify the integral formula for  $\lambda(\varphi)$  given in (RC1). To do so, make evident and then use that

$$\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2} \approx \sqrt{\dot{x}(t_i)^2 + \dot{y}(t_i)^2}(t_i - t_{i-1})$$

if  $t_i \approx t_{i-1}$ ,

where “ $\approx$ ” is used to denote the relation “approximately equal”.

- (d) Assume that the variable  $t$  in the parametrization  $\varphi : [a, b] \rightarrow \mathcal{S}$  is time, so that  $\varphi(t)$  is the position of a point on the curve  $\mathcal{S}$  at time  $t$ . Give a physically evident explanation for the integral formula for  $\lambda(\varphi)$  given in (RC1).

(D) (*Revisiting the Cycloid*) We consider the cycloid  $\mathcal{S} \subseteq \mathbb{E} = \mathbb{R}^2$  introduced in Part (B) of Remark and Exercise 2.4 in a slightly more general setting, namely parametrized by

$$\varphi : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \mapsto \varphi(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ with}$$

$$x(t) := rt - r\sin(t) \quad \text{and} \quad y(t) := -r\cos(t), \quad (\forall t \in [0, 2\pi]),$$

for a fixed value  $r \in \mathbb{R}_{>0}$ . Make clear, that the map

$$\varphi : [0, 2\pi] \rightarrow \mathcal{S} := \varphi([0, 2\pi])$$

is a smooth parametrization of  $\mathcal{S}$ . Prove that

- (a) For all  $t \in [0, 2\pi]$ , the vectorial derivative

$$\dot{\varphi}(t) := \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \text{ satisfies the equality}$$

$$\|\dot{\varphi}(t)\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} = r\sqrt{2}\sqrt{1 - \cos(t)}.$$

- (b) Use the relation  $-\cos(t) = \cos(\pi - t)$  to show that

$$\sqrt{1 - \cos(t)} = \sqrt{2}\cos\left(\frac{\pi - t}{2}\right) \text{ for all } t \in [0, 2\pi].$$

- (c) Use statements (a) and (b) to compute the length

$$\lambda(\varphi) = \int_0^{2\pi} \|\dot{\varphi}(t)\| dt$$

of the parametrized cycloid  $\mathcal{S}$ .

(E) (*The Absolute Curve Length*) If one likes to have a notion of curve length, which is independent on the choice of a smooth parametrization, one may define:

$$\lambda(\mathcal{S}) := \inf(\{\lambda(\varphi) \mid \varphi : [a, b] \rightarrow \mathcal{S} \text{ is a smooth parametrization of } \mathcal{S}\}).$$

Here  $\inf(\mathbb{S}) \in \mathbb{R} \cup \infty$  is used to denote the *infimum* of a set  $\mathbb{S} \subseteq \mathbb{R}$ . This new invariant  $\lambda(\mathcal{S})$  – which equals  $\infty$  if  $\mathcal{S}$  does not admit a smooth parametrization

– is called the *absolute length of  $\mathcal{S}$* .

We say that a smooth parametrization  $\varphi : [a, b] \rightarrow \mathcal{S}$  is *simple*, if

$$\|\dot{\varphi}(t)\| \text{ for all } t \in ]a, b[.$$

It can be shown that the following holds:

- (a) If  $\varphi : [a, b] \rightarrow \mathcal{S}$  is a smooth and simple parametrization, then the absolute length of  $\mathcal{S}$  is given by

$$\lambda(\mathcal{S}) = \lambda(\varphi) = \int_a^b \|\dot{\varphi}(t)\| dt.$$

Clearly, the standard parametrization  $\varphi_r^{[Z]} : [0, 2\pi] \rightarrow \mathcal{C} = \mathcal{C}_r(Z)$  of the circle is smooth and simple. So, in view of Corollary 4.4 we may say:

- (b) The circumference of the circle  $\mathcal{C}_r(Z)$  is its absolute length, thus

$$\lambda(\mathcal{C}_r(Z)) = 2r\pi.$$

Now, prove the following statements:

- (c) The smooth parametrization  $\varphi : [0, 2\pi] \rightarrow \mathcal{S}$  of the cycloid  $\mathcal{S}$  given in part (D) is simple.  
 (d) The absolute length  $\lambda(\mathcal{S})$  of the above cycloid  $\mathcal{S}$  coincides with the value  $\lambda(\varphi)$  computed in (D)(c).  
 (e) A motorcyclist is running with constant speed of 45km/h. The wheels of his bike have a diameter of 48cm. How long does he have to run until a marked point on the running face of tire has travelled 1 km further than himself?

Now, we shall establish the claim (RC1). We add this proof as a kind of refresher for those of you who like to through a glance “of mathematical nostalgia” back to their time at the University, where strict arguing and stringent reasoning were basic issues.

**4.6 Proposition** *The claim (RC1) of Remark, Reminder, Definition and Exercise 4.5 (C) holds.*

*Proof.* Let

$$\varphi : [a, b] \rightarrow \mathcal{S} \subseteq \mathbb{R}^2, \quad t \mapsto \varphi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

be a smooth parametrization. Then the derivation functions

$$\dot{x}, \dot{y} : [a, b] \rightarrow \mathbb{R}, \quad t \mapsto \dot{x}(t), \quad t \mapsto \dot{y}(t) \quad (\forall t \in [a, b])$$

exist and are both continuous. So the modulus or norm function

$$\|\dot{\varphi}\| : [a, b] \rightarrow \mathbb{R}, \quad t \mapsto \|\dot{\varphi}(t)\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \quad (\forall t \in [a, b])$$

is continuous to. Thus, we know from Calculus, that the integral

$$\lambda(\varphi) := \int_a^b \|\dot{\varphi}(t)\| dt = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt$$

exists. Now, let

$$(t_{\bullet}^{[k]})_{k \in \mathbb{N}}, \text{ where } t_{\bullet}^{[k]} = (t_i^{[k]})_{i=0}^{n_k}, \text{ with } a = t_0^{[k]} < t_1^{[k]} < \dots < t_{n_k-1}^{[k]} < t_{n_k}^{[k]} = b$$

be a sequence of partitions of  $[a, b]$  such that

$$(\textcircled{A}) \quad \lim_{k \rightarrow \infty} f(t_{\bullet}^{[k]}) = 0.$$

Our aim is to show that (see Remark, Reminder, Definition and Exercise 4.5 (B)):

$$(\textcircled{!}) \quad \lim_{k \rightarrow \infty} \sum_{i=1}^n \sqrt{(x(t_i^{[k]}) - x(t_{i-1}^{[k]}))^2 + (y(t_i^{[k]}) - y(t_{i-1}^{[k]}))^2} = \lambda(\varphi).$$

By the *Mean Value Theorem of Differential Calculus*, for all  $i \in \{1, 2, \dots, n_k\}$ , we find

$$\alpha_i^{[k]}, \beta_i^{[k]} \in [t_{i-1}^{[k]}, t_i^{[k]}] \text{ such that}$$

$$\frac{x(t_i^{[k]}) - x(t_{i-1}^{[k]})}{t_i^{[k]} - t_{i-1}^{[k]}} = \dot{x}(\alpha_i^{[k]}) \text{ and } \frac{y(t_i^{[k]}) - y(t_{i-1}^{[k]})}{t_i^{[k]} - t_{i-1}^{[k]}} = \dot{y}(\beta_i^{[k]}).$$

So, for all  $k \in \mathbb{N}$  and all  $i \in \{1, 2, \dots, n_k\}$  we have:

$$x(t_i^{[k]}) - x(t_{i-1}^{[k]}) = \dot{x}(\alpha_i^{[k]})(t_i^{[k]} - t_{i-1}^{[k]}) \text{ and } y(t_i^{[k]}) - y(t_{i-1}^{[k]}) = \dot{y}(\beta_i^{[k]})(t_i^{[k]} - t_{i-1}^{[k]}).$$

Therefore we can say:

(\textcircled{A}1) For all  $k \in \mathbb{N}$  and all  $i \in \{1, 2, \dots, n\}$  it holds

$$\sqrt{(x(t_i^{[k]}) - x(t_{i-1}^{[k]}))^2 + (y(t_i^{[k]}) - y(t_{i-1}^{[k]}))^2} = (t_i^{[k]} - t_{i-1}^{[k]}) \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2}.$$

By what we have learned in Calculus about the Riemann integral we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2} &= \\ &= \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt = \lambda(\varphi). \end{aligned}$$

It thus remains to show that (see (\textcircled{!}) above):

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2} &= \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^n \sqrt{(x(t_i^{[k]}) - x(t_{i-1}^{[k]}))^2 + (y(t_i^{[k]}) - y(t_{i-1}^{[k]}))^2}. \end{aligned}$$

Expressing the summands in the second limit by means of statement (\textcircled{A}1) above, we thus may restrict ourselves to show that

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2} &= \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2}. \end{aligned}$$

Passing to differences between the two involved sums in the limits, we thus are left with the task to show that

$$(!!!) \quad \lim_{k \rightarrow \infty} \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) (\sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2} - \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2}) = 0.$$

To do so, we first use the well known inequality

$$|\sqrt{u} - \sqrt{v}| \leq \sqrt{|u - v|} \quad (\forall u, v \in \mathbb{R}_{\geq 0})$$

in order to establish the following statement.

(@2) For all  $k \in \mathbb{N}$  and all  $i \in \{1, 2, \dots, n_k\}$  we have

$$\begin{aligned} & \left| \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2} - \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2} \right| \leq \\ & \leq \sqrt{|\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2 - \dot{x}(\alpha_i^{[k]})^2 - \dot{y}(\alpha_i^{[k]})^2|} = \\ & = \sqrt{|\dot{y}(\beta_i^{[k]})^2 - \dot{y}(\alpha_i^{[k]})^2|}. \end{aligned}$$

As the function  $\dot{y}^2 : [a, b] \rightarrow \mathbb{R}$  is continuous and defined on a closed interval, it is *uniformly continuous*. Therefore we can say:

(@3) For all  $\varepsilon \in \mathbb{R}_{>0}$  there is some  $\delta(\varepsilon) \in \mathbb{R}_{>0}$  such that  $|\dot{y}(t)^2 - \dot{y}(s)^2| < \varepsilon$  for all  $s, t \in [a, b]$  with  $|t - s| < \delta(\varepsilon)$ .

By our hypotheses (@1) we now can say

(@4) For all  $\varepsilon \in \mathbb{R}_{>0}$  there is some  $k_0(\varepsilon) \in \mathbb{N}$  such that  $f(t_{\bullet}^{[k]}) < \delta(\varepsilon)$  for all  $k \geq k_0(\varepsilon)$ .

As  $\alpha_i^{[k]}, \beta_i^{[k]} \in [t_{i-1}^{[k]}, t_i^{[k]}]$  for all  $k \in \mathbb{N}$  and all  $i \in \{1, 2, \dots, n_k\}$ , it follows that

$$|\beta_i^{[k]} - \alpha_i^{[k]}| < \delta(\varepsilon) \text{ for all } k \geq k_0(\varepsilon) \text{ and all } i \in \{1, 2, \dots, n_k\}.$$

Applying (@3) with  $s = \alpha_i^{[k]}$  and  $t = \beta_i^{[k]}$  we see that

$$|\dot{y}(\beta_i^{[k]})^2 - \dot{y}(\alpha_i^{[k]})^2| < \varepsilon \text{ for all } \varepsilon \in \mathbb{R}_{>0}, \text{ all } k \geq k_0(\varepsilon) \text{ and all } i \in \{1, 2, \dots, n_k\}.$$

On application of the above statement (@2) we thus get – for all  $\varepsilon \in \mathbb{R}_{>0}$ , all  $k \geq k_0(\varepsilon)$  and all  $i \in \{1, 2, \dots, n_k\}$  – the inequality

$$\begin{aligned} & \left| \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2} - \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2} \right| \leq \\ & \leq \sqrt{|\dot{y}(\beta_i^{[k]})^2 - \dot{y}(\alpha_i^{[k]})^2|} < \sqrt{\varepsilon}. \end{aligned}$$

But this implies that for all  $\varepsilon \in \mathbb{R}_{>0}$  and all  $k \geq k_0(\varepsilon)$  we have

$$\begin{aligned} & \left| \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) (\sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2} - \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2}) \right| \leq \\ & \leq \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) \left| \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\beta_i^{[k]})^2} - \sqrt{\dot{x}(\alpha_i^{[k]})^2 + \dot{y}(\alpha_i^{[k]})^2} \right| \leq \\ & < \sum_{i=1}^n (t_i^{[k]} - t_{i-1}^{[k]}) \sqrt{\varepsilon} = (b-a)\sqrt{\varepsilon}. \end{aligned}$$

This implies our claim (!!!). □

### Being Even More Courageous: Attacking Non-Parametrized Curves.

Obviously, one might also ask, whether the notion of (absolute) curve length can be introduced without using a parametrization. This is indeed possible. Below, we offer a glance to this idea.

**4.7 Remark, Definition and Exercise.** (A) (*The Path Length of a Point Set*) Let  $\mathcal{S}$  be any non-empty set in the plane  $\mathbb{E}$ . Then it is possible to assign a “number”

$$\bar{\lambda}(\mathcal{S}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

to the set  $\mathcal{S}$ , which can be considered in a natural way as the *length* or the *path length* of  $\mathcal{S}$ . The process of determining this length  $\bar{\lambda}(\mathcal{S})$  is usually again called the *rectification* of  $\mathcal{S}$ . We briefly recall this concept.

To do so, fix some  $n \in \mathbb{Z}_{\geq 2}$  and let us consider the set

$$\mathcal{S}^{n+1} := \{(P_0, P_1, \dots, P_n) \mid P_0, P_1 \dots P_n \in \mathcal{S}\}$$

of all sequences of length  $n+1$  consisting of points  $P_0, P_1, \dots, P_n$  of  $\mathcal{S}$ . (Observe, that we do not insist that  $P_i \neq P_j$  if  $i \neq j$ .) Now, fix such a sequence  $(P_0, P_1, \dots, P_n) \in \mathcal{S}^{n+1}$ . We use the notation

$$\bar{\lambda}(P_0, P_1, \dots, P_n)$$

to denote the length of the “*shortest connected path*” in the plane which passes through all the points  $P_0, P_1, \dots, P_n$ . Clearly, this path must consist of line segments between some points  $P_i$  and  $P_j$  and it must run through all the points  $P_i$  in an appropriate order, such that its length becomes minimal.

To make this more precise, let

$$\mathfrak{S}_{n+1} := \{\sigma : \{0, 1, \dots, n\} \rightarrow \{0, 1, \dots, n\} \mid \sigma \text{ is a bijective map}\}$$

be the set of all *permutations* of the numbers  $0, 1, \dots, n$ . (Observe, that we just have defined the notion of permutation in stringent terms as a bijective map.) Then we can chose a permutation  $\sigma \in \mathfrak{S}_{n+1}$  and run through our points in the order

$$P_{\sigma(0)}, P_{\sigma(1)}, P_{\sigma(2)}, \dots, P_{\sigma(n-1)}, P_{\sigma(n)}$$

with a path consisting of the  $n$  consecutive line segments

$$|P_{\sigma(i-1)}P_{\sigma(i)}| \text{ with } i = 1, 2, \dots, n.$$

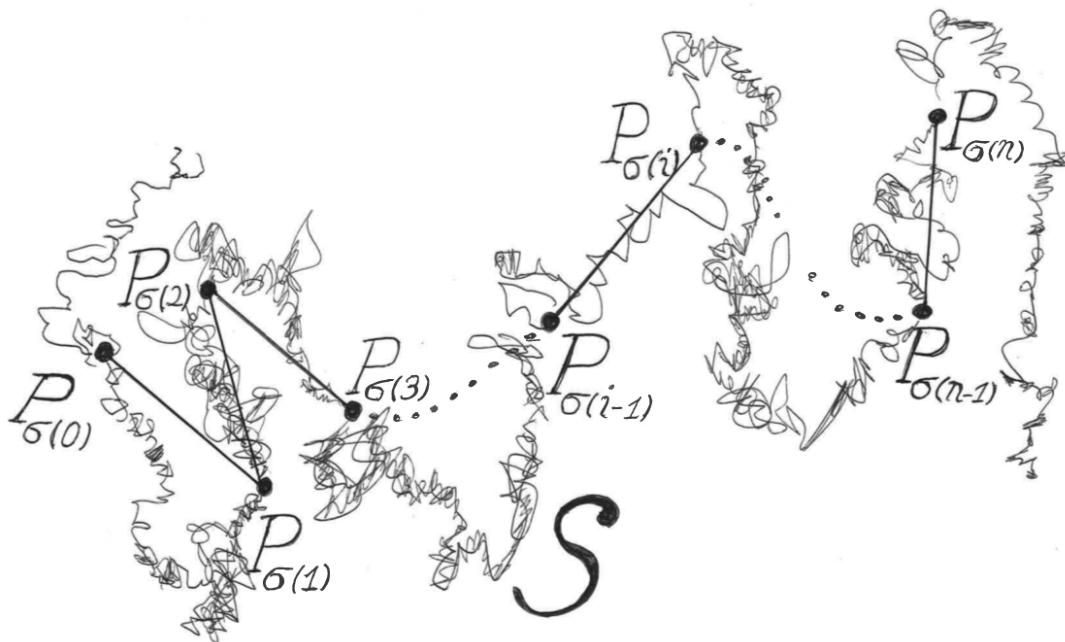


Figure 4.2: Path Through  $P_{\sigma(0)}, P_{\sigma(1)}, P_{\sigma(2)}, \dots, P_{\sigma(n-1)}, P_{\sigma(n)}$

Now, obviously the length of this path is given by the sum

$$\sum_{i=1}^n \text{dist}(P_{\sigma(i-1)}, P_{\sigma(i)}).$$

To determine  $\lambda(P_0, P_1, \dots, P_n)$ , we must try all possible permutations  $\sigma \in \mathfrak{S}_{n+1}$  and chose one for which the above sum becomes minimal. In formal terms, we thus have:

$$\bar{\lambda}(P_0, P_1, \dots, P_n) := \min \left\{ \sum_{i=1}^n \text{dist}(P_{\sigma(i-1)}, P_{\sigma(i)}) \mid \sigma \in \mathfrak{S}_{n+1} \right\}.$$

Now, let

$$\bar{\Lambda}(\mathcal{S}) := \{ \bar{\lambda}(P_0, P_1, \dots, P_n) \mid n \in \mathbb{N}_{\geq 2} \text{ and } P_0, P_1, \dots, P_n \in \mathcal{S} \} \subseteq \mathbb{R}_{\geq 0}$$

be the set of all path-lengths  $\bar{\lambda}(P_0, P_1, \dots, P_n)$  defined by finite sequences in  $\mathcal{S}$ . Assume first, that the set  $\bar{\Lambda}(\mathcal{S})$  is bounded. Then, there exists some upper bound  $b$  of  $\bar{\Lambda}(\mathcal{S})$  – thus a number  $b \in \mathbb{R}$  such that  $\bar{\lambda} \leq b$  for all  $\bar{\lambda} \in \bar{\Lambda}(\mathcal{S})$ . In this case – by the completeness of  $\mathbb{R}$  – the set  $\bar{\Lambda}(\mathcal{S})$  has a *least upper bound*



or *supremum*, which we denote by  $\sup(\bar{\Lambda}(\mathcal{S}))$ . So, we define the previously mentioned *path length* of the connected non-empty set  $\mathcal{S}$  by

$$\bar{\lambda}(\mathcal{S}) := \begin{cases} \sup(\bar{\Lambda}(\mathcal{S})), & \text{if } \bar{\Lambda}(\mathcal{S}) \text{ is bounded} \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly one now would have to show, that this new and more general notion of curve length coincides with the notion of absolute length for smoothly parametrized curves, in order to reconcile the now notion of length with the former one.

(B) (*Length of Smoothly Parametrized Simple Curves*) Assume now, that there is a smooth simple parametrization (see Remark, Reminder, Definition and Exercise 4.5 (A),(E))

$$\varphi : [a, b] \rightarrow \mathcal{S} \subseteq \mathbb{E} = \mathbb{R}^2, \quad t \mapsto \varphi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then, one can again prove (we omit this here) that:

$$\bar{\lambda}(\mathcal{S}) = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt = \int_a^b \|\dot{\varphi}\| dt,$$

where  $\|\dot{\varphi}\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$  denotes the modulus of the (vectorial) derivative

$$\dot{\varphi}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

of  $\varphi$  at  $t$ .

So, by what we said in Remark, Reminder, Definition and Exercise 4.5 (E) we can say:

(a) If  $\mathcal{S}$  admits a smooth simple parametrization  $\varphi : [a, b] \rightarrow \mathcal{C}$ , then

$$\bar{\lambda}(\mathcal{S}) = \lambda(\varphi) = \lambda(\mathcal{S}).$$

This is the requested reconciliation, which (as already said above) we do not attempt to prove here.

(C) (*Examples by Exercise*) Determine  $\bar{\lambda}(\mathcal{S})$  for the following sets  $\mathcal{S} \subseteq \mathbb{E} = \mathbb{R}^2$ .

- (a)  $\mathcal{S} := \mathcal{C}_r(Z)$  ( $r \in \mathbb{R}_{>0}$ ,  $Z \in \mathbb{E}$ ).
- (b) The cycloid  $\mathcal{S}$  introduced in Remark, Reminder, Definition and Exercise 4.5 (D).
- (c)  $\mathcal{S} := \{P, Q\}$  with  $P, Q \in \mathbb{E}$ .
- (d)  $\mathcal{S} := |PQ|$  with  $P, Q \in \mathbb{E}$ .
- (e)  $\mathcal{S} := \{P, Q, R\}$  with  $P, Q, R \in \mathbb{E}$ .
- (f)  $\mathcal{S} := \left\{ \left( r \cos\left(\frac{2k\pi}{n}\right), r \sin\left(\frac{2k\pi}{n}\right) \right) \mid k = 0, 1, \dots, n \right\}$  for  $r \in \mathbb{R}_{>0}$  and  $n \in \mathbb{N}$ .
- (g)  $\mathcal{S} := |PQ| \cup |QS|$  with  $P, Q, S \in \mathbb{E}$  not collinear.
- (h)  $\mathcal{S} := |PQ| \cup |RS|$  with  $P, Q, R, S \in \mathbb{E}$  and  $|PQ| \cap |RS| = \{T\}$  with  $T \notin \{P, Q, R, S\}$ .
- (i)  $\mathcal{S} := \{(\cos(t), \sin(t)) \mid 0 \leq t \leq \pi\} \cup \{(0, 0), (0, 2), (2, 0)\}$
- (k)  $\mathcal{S} := \mathcal{D}_r(Z)$  with  $r \in \mathbb{R}_{>0}$  and  $Z \in \mathbb{E}$ .

**Resuming what is of use for High School Teaching.** Clearly, a great deal of our serious arguments in this chapter are “hard core Analysis”. These arguments are presented in order to remind the readers of a few things they once learned at University. These things clearly are not intended to enter your everyday High School teaching directly. But then and now, teachers should get a chance to revisit strict formal arguments which are behind what is taught by heuristic considerations at High School. We hope, that we have done this with the proofs of Proposition 4.3 and 4.6. Most mathematicians believe, that what you keep in the back of your mind also influences the way you teach, may be even inconsciously. The author shares this view, and this was his motivation to add the mentioned propositions and there proofs.

But beside of this rather abstract matters, who likely will have an indirect impact to your teaching, there are a number of concrete things treated in this chapter, which you could use as an immediate source of inspiration for your teaching. We give some hints to this in the conclusive didactic comments of this chapter.

**4.8 Didactic Comments (A)** (*The Pace of Polygonists*) The comments made under Remarks 4.1 (A),(B),(C) might throughout find their way to your classroom. It is a valuable approach, to introduce deep mathematical problems by academic anecdotes. Even, if pupils miss to see the depth of the whole question (which is likely the case for their majority) they hear what “kind of strange questions mathematicians are caring about”. This need not have a negative effect to their judgement of Mathematics at all. And for the teacher, such an approach could be an opportunity to develop his individual views on presenting mathematical features.

(B) (*Admitting Irregular Polygons*) We also believe, that our considerations on irregular polygons, suggested in Remarks 4.2 (A) and (B) are not completely beyond the comprehension of Secondary School Students at A-Level. Clearly, in our presentation, we are a bit formal, as we address ourselves to you, that is to the teachers. But could it not be worth to investigate say 30 minutes to in a try to understand the meaning of the reconciliation statement (R2) of part (B) of the mentioned remarks? For those of your students, who aim to do University studies in Mathematics, this would be very nice preview to the kind of thinking, they will be faced with when doing basic Analysis. Do not forget, that it may be worth to do this for the real profit of one or of two students. You might also suggest to prepare a short presentation of the material as a cooperate home work to a selected group of students. Then, you also should say, that the reconciliation property (R2) holds, and that this can be proved by means of tools coming from calculus. What you actually need from calculus, is very little, namely:

- (1) the fact that  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$ , and

(2) the precise definition of limit for a sequence of real numbers.

But the main reason, why one has to give a strict formal proof for a statement like (R2), which – at a first glance – seems quite easy from the heuristic point of view, can be explained as follows: If all edges of a polygon with vertices on the circle  $\mathcal{C}$  become shorter, clearly, the differences between the length of each edge and the length of the corresponding arc on  $\mathcal{C}$  over this edge, becomes smaller. So, for each single edge there is a smaller and smaller error, if we replace the arc length over this edge (thus the quantity we are interested in) by the length of the edge. But, on the other hand, if all edges become shorter and shorter, there must be more and more of them. So, we have a larger and larger number of smaller and smaller errors, if we replace the arc lengths by the edge length. But, finally, who wins? The “smallness“ of the of errors or the ”higher and higher number“ of the small errors. This kind of competing problem between ”smaller and smaller but more and more errors“ is in fact a basic issue of Analysis. In a certain sense, this kind of question became even one of the driving forces for the development of Analysis at all.

(B) (*Rectification of Curves*) Our comments on the rectification of curves, as we made them under Remark, Reminder, Definition and Exercise 4.5 (A) – (E) are mostly subjects you could treat in your class, provided students dispose on the basic notions of Integral Calculus. Indeed, the integral formula of Exercise (b) of part (C) occurs in most standard Calculus textbooks for High Schools. The heuristic justification of this formula is usually given as suggested in Exercise (c) of part (C). The very important (!) Physical interpretation of the integral formula for  $\lambda(\varphi)$ , as suggested in Exercise (d) of part (C) is also presented in many of Textbooks for High Schools.

(C) (*Path Length of Sets*) Part (C) of Remark, Definition and Exercise 4.7 could also find their way to the class room. Obviously, if you want to set this to practice, you must introduce the path length  $\bar{\lambda}(\mathcal{S})$  of a set  $\mathcal{S} \subseteq \mathbb{E} = \mathbb{R}^2$  in completely elementary way – just by means of examples. So, you could give yourself finitely many points  $P_0, P_1, \dots, P_n$  in the plane and ask for the shortest path passing through all of them. Clearly if doing so, you should begin with the case  $n = 1$ , and then try to do the case  $n = 2$  and – may be – even the case  $n = 3$  in detail. In this case, you will have

$$\bar{\lambda}(\mathcal{S}) = \bar{\lambda}(\{P_0, P_1, \dots, P_n\}) = \bar{\lambda}(P_0, P_1, \dots, P_n).$$

Clearly, you need not write this down in your class. Discuss just the examples. Let your students find by experiment some (shortest) path lengths in the suggested cases with say 4 or even 5 points...

This could help to explain our notion of path length  $\bar{\lambda}(\mathcal{S})$  for a finite set  $\mathcal{S} \subseteq \mathbb{E}$ .

In a next step, you could be more daring and admit sets  $\mathcal{S}$  which contain finitely many (single points and) line sections. You could do this by discussing exercises (g) and (g) in part (D) of Remark, Definition and Exercise 4.7. Finally, you could even give a short hint to the case, in which  $\mathcal{S}$  contains even

finitely many curve segments (see Remark, Definition and Exercise 4.7 (C)(i)). The whole subject should approach to your class the idea, that we are – once more – asking for a number of great practical significance:

*The length of a shortest journey which comes across a sample of given places!*

## 5. Squaring the Circle

**Compasses and Straightedge versus  $\pi$ .** A classical problem of Geometry – already discussed in ancient times – is the following question:

- (Q1) Given a circle of radius  $r$  and center  $Z$ . Is there a construction by compasses and straightedge of a square which has the same area as the circular disk  $\mathcal{D} := \mathcal{D}_r(Z)$ ?

As the area of the disk  $\mathcal{D}$  is given by  $A = r^2\pi$  (see also Exercises and Remark 3.5 (D)), question (Q1) can be reformulated as follows:

- (Q2) Given a line segment of length  $r$ . Is there a construction by compasses and straightedge of a line segment of length  $r\sqrt{\pi}$ ?

It is easy to see, that it suffices to solve his problem in the case  $r = 1$ . So question (Q2) is actually equivalent to the following question, in which we use the notion of *unit line segment* for a line segment of length 1:

- (Q3) Is it possible, to construct by compasses and straightedge a line segment of length  $\sqrt{\pi}$  out of a unit line segment?

Some of you may remember, that for any line segment of length  $a$  one can construct by compasses and straightedge a line segment of length  $\sqrt{a}$  (provided a unit line segment is given). Those among you, who cannot remember this fact at the moment, get another chance to brush up their memory later on. If we accept the stated fact for the moment, we finally can reduce our question (Q3) to the following problem

- ( $C\pi$ ) (*Constructibility of  $\pi$* ) Is it possible to construct by compasses and straightedge a line segment of length  $\pi$  out of a unit line segment?

**Between Arithmetics and Geometry.** The present chapter is devoted to this constructibility question. It turns out, that the *general constructibility question*, whether a line segment of given length  $a$  can be constructed by compasses and straightedge from a unit line segment is not a geometric question. This constructibility question is in fact an algebraic question on the nature of the number  $a$ . The first to discover the algebraic nature of this general constructibility question and to realize the great importance of this fact, was the great German Mathematician C.F.Gauss (1777 - 1855). He made this discovery and did put his findings to practice – at the age of 18 – when he gave a construction for the regular 17-gon in the year 1795.

We shall start our investigation by a number of basic facts on constructions with compasses and straightedge, facts you are familiar with from what you teach at school. Nevertheless we prefer to recall some of these basic features in order to introduce the subject in a self-contained way. Also, we hope that our discussions shed some light to things you are already familiar with.

**5.1 Remark, Reminders, Exercises and Definition (A)** (*Basic Constructions by Compasses and Straightedge*) We first recall some basic constructions which can be performed on use of compasses and straightedge. We list these constructions below in the form of exercise, in order to encourage you to think

a moment on them. Then we shall proceed to slightly more advanced constructions. At first place, we suggest that you perform – only on use of compasses and straightedge – the following basic constructions.

- (a) (*Transport of a Line Segment*) Let  $g \subseteq \mathbb{E}$  be a line, let  $P \in g$  be a point, let  $A, B \in \mathbb{E}$  two distinct points. Construct the two points  $Q, Q' \in g$  with  $\text{dist}(P, Q) = \text{dist}(P, Q') = \text{dist}(A, B)$ . This allows you to transport any line segment to a line segment contained in  $g$  and having  $P$  as an extremal point.
- (b) (*Bisection of Line Segments*) Let  $P, Q \in \mathbb{E}$  be two distinct points. Construct the *midpoint* of the line segment  $|PQ|$ , hence the unique point  $M \in |PQ|$  with  $\text{dist}(P, M) = \text{dist}(M, Q)$ . This allows you to bisect a line segment in two equally long pieces.
- (c) (*Drawing Parallel Lines*) Let  $g \subseteq \mathbb{E}$  be a line and let  $P \in \mathbb{E} \setminus g$  be a point. Construct a line  $h \subseteq \mathbb{E}$  which is *parallel* to  $g$  and contains the point  $P$ , hence the unique line  $h \subseteq \mathbb{E}$  with

$$h \parallel g \quad \text{and} \quad P \in h.$$

Remember, that by definition, the parallelity relation  $h \parallel g$  of two lines  $g, h \subseteq \mathbb{E}$  is defined by the condition that either  $h = g$  or else  $h \cap g = \emptyset$ .

- (d) (*Constructing Right Angles*) Let  $g \subseteq \mathbb{E}$  be a line and let  $P \in \mathbb{E}$  be a point. Construct the unique line  $h \subseteq \mathbb{E}$  which is *perpendicular* on the line  $g$  and contains  $P$ . This allows you to construct a line  $h$ , which intersects a given line  $g$  under a *right angle* in a given point  $P$  of  $g$ .

(B) (*Arithmetics by Compasses and Straightedge*) We now always assume that a unit line segments and two line segments of length  $a, b \in \mathbb{R}_{>0}$  are given. For the sake of simplicity, we always assume that  $a \geq b$ . Below, we suggest constructions which can be performed by compasses and straightedge and which respectively furnish line segments of the following lengths:

$$a + b, \quad a - b, \quad ab, \quad \frac{a}{b} \quad \text{and} \quad \sqrt{a}.$$

This means, that the four basic arithmetic operations of addition, subtraction, multiplication and division, but also the operation of taking square roots, may be performed on use of compasses and straightedge. Below, we hint in sketches how these geometric realizations of arithmetic operations are put to practice, and we suggest it as exercises for you to prove our claims. You also must make sure, that the suggested constructions can be performed only on use of compasses and straightedge. For this you will need what you already did in part (A).

Here you also find reconciliation for the fact already used in the introductory comment to this chapter: If we can construct  $\pi$  by means of compasses and straightedge, so we can construct  $\sqrt{\pi}$  in this way. This means, our reduction of the problem of squaring the circle to the constructability question for  $\pi$  will now have found full justification.

You also might have met some of the geometric constructions below performed

in a different way. If this is the case you should also make clear, that "your" construction gives indeed the result of the underlying elementary arithmetic operation.

(a) (*Taking Sums: The Addition*)

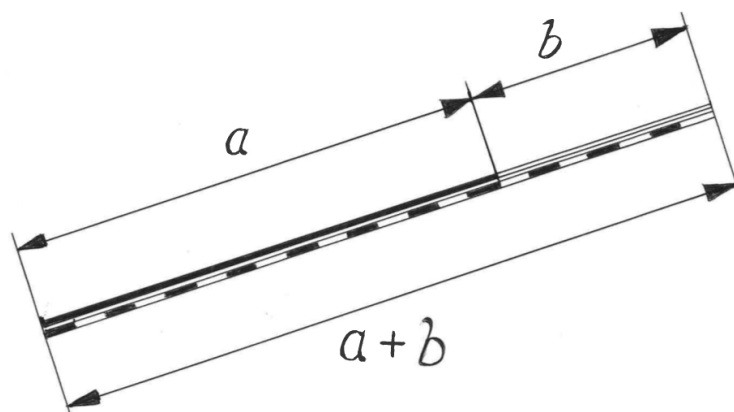


Figure 5.1: Construction of  $a + b$

This construction clearly is self-evident. A particular case of it is the construction of  $2a$ . More generally one can get the construction of  $na$  for any  $n \in \mathbb{N}$  on repeated application of the suggested construction.

(b) (*Taking Differences: The Subtraction*)

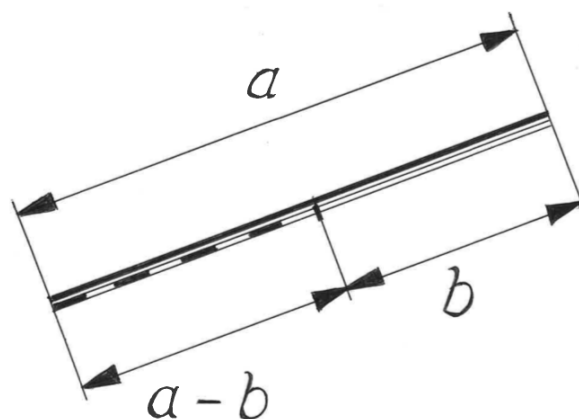


Figure 5.2: Construction of  $a - b$

Also here, we meet a self-evident construction. make clear here also how you proceed in the case where  $b \geq a$ . Also, discuss the occurring practical difficulty if one aims to construct  $n(a - b)$  in the case where

$$a \gg 0, a \approx b \text{ and } n \in \mathbb{N} \text{ with } n \gg 0.$$

Do not forget, that even a carefully working designer barely can perform constructions by compasses and straightedge with a precision below 0.5mm. Perform the construction of  $20(a - b)$  with  $a = 10\text{cm}$  and  $b = 9.7\text{cm}$ .

(c) (*Taking Products: The Multiplication*)

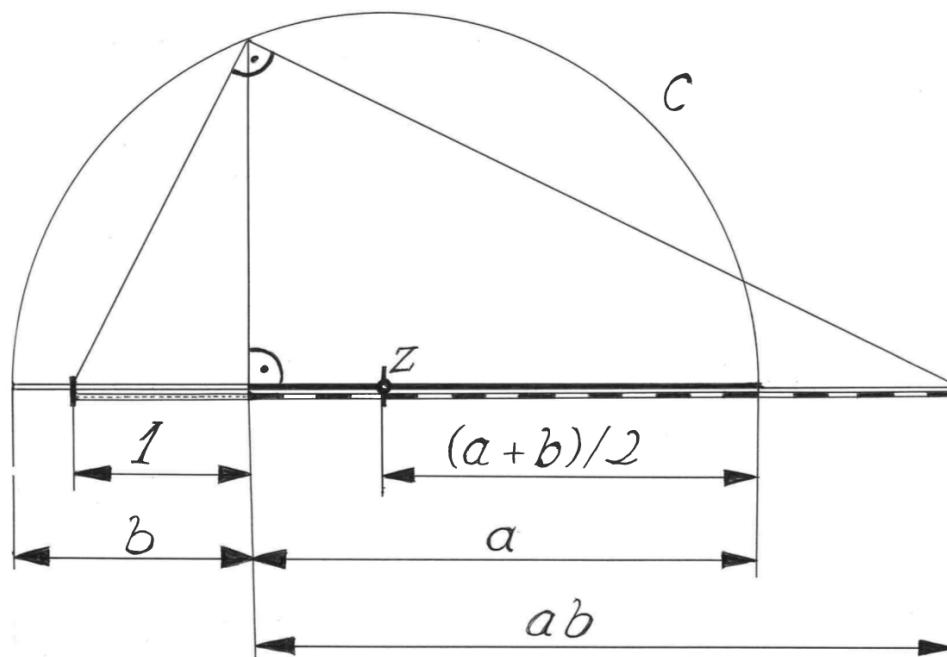


Figure 5.3: Construction of  $ab$

This construction is not at all self-evident, and its explanation needs basic knowledge on right-angled triangles. Observe, that the explanation of the construction first must use another right-angled triangle, which is not drawn in our figure. But the upper semicircle shown in the figure should hint, where the "invisible right-angled triangle" can be found. The semicircle in question just must be understood as half of the *Thales circle* over its diameter. What you need here to complete the explanation of the construction is the *Height Theorem for right-angled triangles* (" $h^2 = pq$ "), which your students clearly know. So the construction in question and its explanation perfectly fit to what you do in class. If you usually treat this construction in your classes, we encourage you to go on with doing so. If you do not treat and discuss the construction in your classes, we encourage you to do so in future.

Similarly as in the construction of the difference, there is also a precision problem in the construction of the product: If  $a + b$  is large (compared to 1), the construction cannot be performed precisely – as you may explain easily by looking at the geometric situation. Here again, in the class-room you could construct  $ab$  for different values of  $a$  and  $b$  – in the range of 1, 2, ..., 5[cm] and make students compare the constructed value of  $ab$  with its actual value.

A way to gain some precision would be – for example – to construct  $\frac{1}{n}ab$  for  $n \in \mathbb{Z}_{>1}$  (by replacing the unite line segment in the construction by a line



segment of length  $n \in \{2, 3, \dots\}$ ) and then to add up  $n$ -times the result by the construction given in (a).

Finally keep in mind, that the two right angles occurring in the suggested construction, can be obtained by a straightedge and compasses construction. So, what we suggest is indeed a construction of the requested type. However, in practice one would prefer to use a right angle ruler instead of our previously suggested construction by straightedge and compasses.

(d) (*Taking Quotients: The Division*)

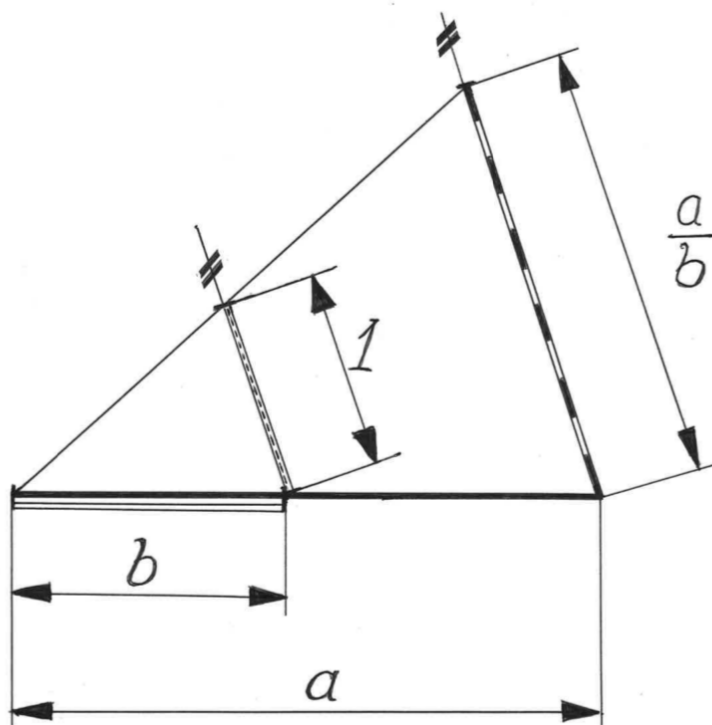


Figure 5.4: Construction of  $\frac{a}{b}$

Clearly, this construction relies on *Geometric Similarity*. Generally, if  $b$  is

small (with respect to  $1$ ) and  $a$  is large (with respect to  $1$ ), the suggested construction will tend to be in-precise. Also here, we recommend you to think on this problem of precision and to discuss it. Again, in the class rooms a performance of a practical test with simple values of  $a$  and  $b$  can be quite illustrative.

Observe also that for  $a < b$ , the figure looks slightly different. Finally, there is a nice direct and simple construction for the value of  $\frac{a}{n}$ , where  $n \in \mathbb{N}$ , which you likely will know.

Observe that in our construction, one has to draw a line passing through a given point and being parallel to a given line. As we did recall earlier, this is a task which can be performed merely on use of straightedge and compasses.

However, also here, in practice, one likewise will prefer the well known method of *parallel transport* of a given line.

(e) (*Taking Square Roots*)

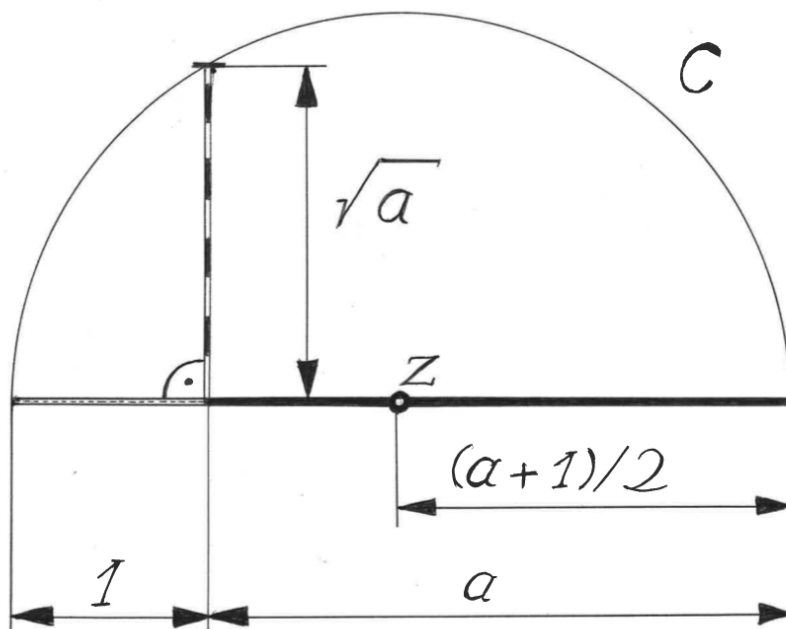


Figure 5.5: Construction of  $\sqrt{a}$

This construction relies again directly on the *Height Theorem for right-angled triangles* already used in (c). Observe, that if you replace the unit line segment by a line segment of length  $b$ , the result of the construction is  $\sqrt{ab}$ . So, our construction is a special case of the well known classical *construction of (the edge of) a square whose area equals the area of a given rectangle*.

(C) (*Basic Arithmetic Constructions by Compasses and Straightedge*) We call the five constructions presented in (a) – (e) the *basic arithmetic constructions*.

**Constructible Numbers.** As already said above, the originally geometric constructibility question is rather an arithmetic question concerning a certain type of numbers. So, we now will introduce and study these numbers, and hence move from Geometry to Arithmetics and in fact even get a flavour of Abstract Algebra.

**5.2 Remark, Definition and Exercises (A)** (*Constructibly Closed Fields of Numbers*) Inspired by what we learned in Remark, Reminder and Exercises 5.1 we now aim to look at subsets  $\mathbb{S} \subseteq \mathbb{R}$  which satisfy the following properties:

- (a) (*Unitality*)  $1 \in \mathbb{S}$ .
- (b) (*Closedness under Addition*) If  $a, b \in \mathbb{S}$ , then  $a + b \in \mathbb{S}$ .
- (c) (*Closedness under Subtraction*) If  $a, b \in \mathbb{S}$ , then  $a - b \in \mathbb{S}$ .
- (d) (*Closedness under Multiplication*) If  $a, b \in \mathbb{S}$ , then  $ab \in \mathbb{S}$ .

- (e) (*Closedness under Division*) If  $a, b \in \mathbb{S}$  with  $b \neq 0$ , then  $\frac{a}{b} \in \mathbb{S}$   
 (f) (*Closedness under Formation of Square Roots*) If  $a \in \mathbb{S}$  with  $a \geq 0$ , then  $\sqrt{a} \in \mathbb{S}$ .

First observe, that the properties (a)–(e) imposed on the set  $\mathbb{S} \subseteq \mathbb{R}$  imply that  $\mathbb{S}$  is a *subfield* of (the field)  $\mathbb{R}$  (of real numbers). The notions of field and subfield you know from *Linear Algebra* or from *Abstract Algebra*. Property (f) is sometimes spoken by saying that the field  $\mathbb{S}$  is *quadratically closed*.

Prove the following fact, which justifies to call a set  $\mathbb{S}$  which satisfies the above relations (a)–(f) *constructibly closed*.

- (g) (*Closedness under Construction by Compass and Straightedge*) Let  $\mathbb{S} \subseteq \mathbb{R}$  be a constructibly closed subfield and let

$$a_1, a_2, \dots, a_n \in \mathbb{S}$$

be lengths of line sections. Assume that  $s$  is the length of a line section which is obtained from the given  $n$  line sections by repeated application of the five basic arithmetic constructions presented in Remark Reminder and Exercises 5.1 (B). Then

$$s \in \mathbb{S}.$$

So, we can say: If we perform in  $\mathbb{S}$  those arithmetic operations, which can be realized by arithmetic constructions, we cannot get out of  $\mathbb{S}$ . Finally, show the following:

- (h)  $\mathbb{R}$  is a constructibly closed subfield of  $\mathbb{R}$ .  
 (i)  $\mathbb{Q}$  is not a constructibly closed subfield of  $\mathbb{R}$ .  
 (k) If  $\mathbb{S}$  is a constructibly closed subfield of  $\mathbb{R}$ , then  $\mathbb{Q} \subseteq \mathbb{S}$ .  
 (l) If  $\mathbb{S}$  is a constructibly closed subfield of  $\mathbb{R}$  and  $q \in \mathbb{S}$  with  $q > 0$ , then

$$q^{\frac{1}{2^n}} \in \mathbb{S} \text{ for all } n \in \mathbb{N}.$$

(B) (*The Field of Constructible Numbers*) We start introducing the announced notion by some exercises, which invite you to prove the following claims:

- (a) Let  $\mathfrak{S}$  be a set of all constructibly closed subfields  $\mathbb{S} \subseteq \mathbb{R}$ . Show that the intersection of all members of  $\mathfrak{S}$ , hence the set

$$\bigcap_{\mathbb{S} \in \mathfrak{S}} \mathbb{S} \text{ is a constructibly closed subfield of } \mathbb{R}.$$

- (b) The intersection

$$\mathbb{K} := \bigcap_{\{\mathbb{S} \subseteq \mathbb{R} \mid \mathbb{S} \text{ is a constructibly closed subfield}\}} \mathbb{S}$$

of all constructibly closed sub-fields of  $\mathbb{R}$  is a constructibly closed sub-field of  $\mathbb{R}$ .

- (c) If  $\mathbb{S} \subseteq \mathbb{R}$  is a constructibly closed subfield and if  $\mathbb{K}$  is defined as in statement (b), then  $\mathbb{K} \subseteq \mathbb{S}$ . So,  $\mathbb{K}$  is the *least constructibly closed subfield of  $\mathbb{R}$* .

- (d) The constructibly closed sub-field  $\mathbb{K} \subseteq \mathbb{R}$  defined in statement (b) consists of all numbers which are obtained from 1 by performing finitely many of the 5 arithmetic operations mentioned in part (A)(b)–(f).
- (e) The constructibly closed subfield  $\mathbb{K} \subseteq \mathbb{R}$  defined in statement (b) consists of all numbers which are obtained from rational numbers by performing finitely many of the five operations mentioned in part (A)(a)–(f).
- (f) If  $a \in \mathbb{K}$  is the length of a line section, then this line section can be constructed from the unit line section on use of compasses and straightedge.

The field  $\mathbb{K}$  defined in statement (b) and characterized in statements (c), (d) and (e) is called the *subfield of constructible numbers*. The numbers  $a \in \mathbb{K}$  are called *constructible numbers*. The justification for these names is given by statement (f). Observe that constructible numbers may consist of fairly complicated terms, build up from rational numbers, square roots and combining these by the four basic operations. An example is the number

$$c = \sqrt{\frac{1 + \sqrt{2} + \frac{\sqrt{1+\sqrt{7}}}{5+\sqrt{11}}}{\sqrt{3} - 1}}$$

Observe, that a number  $a \in \mathbb{R}$  is constructible if and only if a line segment of length  $|a|$  can be constructed by compasses and straightedge out of a unit line segment. If we perform the construction of such a line segment, we just say, that we *construct the number*  $a$ .

(C) (*Examples by Exercise*) Construct the following numbers  $a$ :

- (a)  $a = 3$   
 (b)  $a = \frac{2}{3}$   
 (c)  $a = \sqrt{5}$   
 (d)  $a = \sqrt{1 + \sqrt{3}}$   
 (e)  $a = \frac{\sqrt{2}}{\sqrt{1+\sqrt{5}}}$  and ...  
 (f)  $a = c$ , where  $c$  is the example of constructible number given previously in part (B).

Sometimes, the construction of a number from two given numbers may be performed in fewer steps, if you try a direct geometric (or algebraic approach) instead of using consecutive basic arithmetic constructions. Moreover transformation of terms may help to simplify matters, too. Put this to practice in the following examples, in which  $0 < a < b$ :

- (g)  $\sqrt{a^2 + b^2}$   
 (h)  $\sqrt{b^2 - a^2}$   
 (i)  $\sqrt{ab}$   
 (k)  $a + \frac{a}{2} + \frac{a}{4} + \frac{a}{8} + \frac{a}{16} + \dots$   
 (l)  $a + \frac{a^2}{2} + \frac{a^3}{4} + \frac{a^4}{8} + \frac{a^5}{16} + \dots$  ( $0 < a < 2$ ).

(m)  $\text{asin}(\frac{\pi}{8})$ .

(D) (*Rephrasing the Constructability Problem for  $\pi$* ) In view of the previous development we now reformulate the mentioned constructability problem (C $\pi$ ) in the foillowing form:

( $\mathbb{K}\pi$ ) Is it true that  $\pi \in \mathbb{K}$ ?

We now shall prepare the answer to this question, although the proof of this answer exceeds by far what we can achieve in this series of lectures.

**Algebraic Numbers.** We now shall introduce a new class of numbers, namely the so called algebraic numbers. We shall see, that these form a constructibly closed subfield of  $\mathbb{R}$ , so that all constructible numbers are algebraic. We begin with the following elementary result from (Abstract) Algebra.

**5.3 Proposition** *Let  $n \in \mathbb{N}$  and let  $a \in \mathbb{R}$ . Then the following statements are equivalent:*

(i) *There is an equation*

$$z_n a^n + z_{n-1} a^{n-1} + \cdots + z_1 a + z_0 = 0 \text{ with } z_0, z_1, \dots, z_{n-1} \in \mathbb{Z} \text{ and } z_n \in \mathbb{N}.$$

(ii) *There is an equation*

$$z_n a^n + z_{n-1} a^{n-1} + \cdots + z_1 a + z_0 = 0 \text{ with } z_0, z_1, \dots, z_n \in \mathbb{Z} \text{ and } z_n \neq 0.$$

(iii) *There is an equation*

$$q_n a^n + q_{n-1} a^{n-1} + \cdots + q_1 a + q_0 = 0 \text{ with } q_0, q_1, \dots, q_n \in \mathbb{Q} \text{ and } q_n \neq 0.$$

(iv) *There is an equation*

$$a^n + q_{n-1} a^{n-1} + \cdots + q_1 a + q_0 = 0 \text{ with } q_0, q_1, \dots, q_{n-1} \in \mathbb{Q}.$$

(v) *The elements  $1, a, \dots, a^{n-1}$  generate the  $\mathbb{Q}$  vector-space*

$$\mathbb{Q}[a] := \sum_{i \in \mathbb{N}_0} \mathbb{Q} a^i = \left\{ \sum_{j=0}^m q_j a^j \mid m \in \mathbb{N}_0 \text{ and } q_0, q_1, \dots, q_m \in \mathbb{Q} \right\} \subseteq \mathbb{R}$$

(vi) *The dimension of the  $\mathbb{Q}$ -vector space  $\mathbb{Q}[a]$  of statement (v) satisfies the inequality*

$$\dim_{\mathbb{Q}}(\mathbb{Q}[a]) \leq n.$$

*Proof.* : The equivalences

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$$

are easy to verify, and we leave it as an exercise to prove them.

(iv)  $\Rightarrow$  (v): Assume that there is an equation

$$a^n + q_{n-1} a^{n-1} + \cdots + q_1 a + q_0 = 0 \text{ with } q_0, q_1, \dots, q_{n-1} \in \mathbb{Q}.$$

Let

$$w := \sum_{j=0}^m s_j a^j \in \mathbb{Q}[a], \quad (m \in \mathbb{N}_0, \text{ and } s_0, s_1, \dots, s_m \in \mathbb{Q}).$$

We have to show that

$$(\textcircled{a}) \text{ There are } b_0, b_1, \dots, b_{n-1} \in \mathbb{Q} \text{ such that } w = \sum_{i=0}^{n-1} b_i a^i.$$

To do so, we consider the two polynomials

$$f(X) := X^n + q_{n-1}X^{n-1} + \dots + q_1X + q_0, \quad g(X) := \sum_{j=0}^m s_j X^j \in \mathbb{Q}[X].$$

By *Euclid's Remainder Theorem for Polynomials*, we find polynomials  $h(X), r(X) \in \mathbb{Q}[X]$  such that

$$g(X) = f(X)h(X) + r(X) \text{ and the degree of } r \text{ satisfies } \deg(r(X)) \leq n - 1.$$

We write

$$r(X) = \sum_{i=0}^{n-1} b_i X^i, \text{ with } b_0, b_1, \dots, b_{n-1} \in \mathbb{Q}.$$

As  $f(a) = a^n + q_{n-1}a^{n-1} + \dots + q_1a + q_0 = 0$  it follows that

$$w = \sum_{j=0}^m s_j a^j = g(a) = f(a)h(a) + r(a) = r(a) = \sum_{i=0}^{n-1} b_i a^i.$$

But this proves our claim  $(\textcircled{a})$ .

(v)  $\Rightarrow$  (vi): This is clear by the definition of dimension.

(vi)  $\Rightarrow$  (iii): Assume that  $\dim_{\mathbb{Q}}(\mathbb{Q}[a]) \leq n$ . As  $1, a, a^2, \dots \in \mathbb{Q}[a]$  it follows that the  $n + 1$  elements  $1, a, a^2, \dots, a^n$  are linearly dependent over  $\mathbb{Q}$ . So, we find some  $m \in \{1, 2, \dots, n - 1\}$  and elements  $t_0, t_1, \dots, t_m \in \mathbb{Q}$  such that

$$t_m a^m + t_{m-1} a^{m-1} + \dots, t_1 a + t_0 = 0 \text{ and } t_m \neq 0.$$

We set

$$q_i := \begin{cases} 0, & \text{for } i = 0, 1, \dots, n - m - 1; \\ t_{i-n+m}, & \text{for } i = n - m, n - m + 1, \dots, n. \end{cases}$$

Then, it holds  $q_n = t_m \neq 0$  and

$$\begin{aligned} q_n a^n + q_{n-1} a^{n-1} + \dots + q_1 a + q_0 &= \\ &= q_n a^n + q_{n-1} a^{n-1} + \dots + q_{n-m} a^{n-m} + q_{n-m-1} a^{n-m-1} + q_1 a + q_0 = \\ &= q_n a^n + q_{n-1} a^{n-1} + \dots + q_{n-m} a^{n-m} = \\ &= t_m a^n + t_{m-1} a^{n-1} + \dots + t_1 a^{n-n+1} + t_0 a^{n-m} = \\ &= (t_m a^m + t_{m-1} a^{m-1} + \dots, t_1 a + t_0) a^{n-m} = 0 a^{n-m} = 0. \end{aligned}$$

This shows, that condition (iii) holds.  $\square$

**5.4 Definition, Examples and Exercises.** (A) (*Algebraic Numbers and their Degree*) A number  $a \in \mathbb{R}$  is called an *algebraic number* if there is some  $n \in \mathbb{N}$  such that the equivalent conditions (i)–(vi) of Proposition 5.3 hold. The least  $n \in \mathbb{N}$  for which this happens, is called the *degree* of  $a$  and denoted by  $\deg(a)$ . We denote the set of algebraic numbers by  $\mathbb{A}$ , thus:

$$\mathbb{A} := \{a \in \mathbb{R} \mid a \text{ is an algebraic number}\}.$$

(B) (*Examples by Exercises*) Show the following statements.

- (a)  $\mathbb{Q} = \{a \in \mathbb{A} \mid \deg(a) = 1\}$ .  
 (b) For all  $q \in \mathbb{Q}_{\geq 0}$  and all  $r \in \mathbb{N}$  it holds

$$q^{\frac{1}{r}} \in \mathbb{A} \quad \text{and} \quad \deg(q^{\frac{1}{r}}) \leq r.$$

- (c) If  $a \in \mathbb{A}_{\geq 0}$  and  $r \in \mathbb{N}$ , then  $a^{\frac{1}{r}} \in \mathbb{A}$ .  
 (d) If  $a \in \mathbb{A} \setminus \{0\}$  with  $\deg(a) = n$ , then  $a$  satisfies an algebraic equation of the form

$$q_n a^n + q_{n-1} a^{n-1} + \dots + q_1 a + 1 = 0 \quad \text{with } q_1, q_2, \dots, q_n \in \mathbb{Q}.$$

- (e) If  $a \in \mathbb{A} \setminus \{0\}$ , then  $\frac{1}{a} \in \mathbb{A}$ .  
 (f) If  $z \in \mathbb{N}$  is not a square number, then  $\deg(\sqrt{z}) = 2$ .  
 (h)  $2^{\frac{1}{3}} \in \mathbb{A}$  with  $\deg(2^{\frac{1}{3}}) = 3$ .

**Constructible Numbers are Algebraic.** We now aim to give the second but last ingredient which is needed to show that our constructibility question for  $\pi$  cannot have an affirmative answer. We begin with the following result, whose proof makes again allusion to things you like have heard in your University Lectures in Abstract Algebra.

**5.5 Proposition.** *The set  $\mathbb{A}$  of algebraic numbers is a constructively closed subfield of  $\mathbb{R}$ .*

*Proof.* We first show that  $\mathbb{A}$  is closed under addition, subtraction and multiplication. So, let

$$a, b \in \mathbb{A} \quad \text{with } \deg(a) = m \text{ and } \deg(b) = n.$$

We consider the  $\mathbb{Q}$ -vector space

$$\begin{aligned} \mathbb{Q}[a, b] &:= \sum_{i, j \in \mathbb{N}_0} \mathbb{Q} a^i b^j = \\ &= \left\{ \sum_{i, j \in \mathbb{N}_0: i+j \leq p} q_{i,j} a^i b^j \mid p \in \mathbb{N}_0 \text{ and } q_{i,j} \in \mathbb{Q} \text{ for all } i, j \in \mathbb{N}_0 \text{ with } i+j \leq p \right\}. \end{aligned}$$

Our first aim is to show that

$$(\textcircled{a}) \quad \dim_{\mathbb{Q}}(\mathbb{Q}[a, b]) \leq mn.$$

To do so, we show that

( $\textcircled{a}\textcircled{a}$ ) The  $\mathbb{Q}$ -vector space  $\mathbb{Q}[a, b]$  is generated by the elements

$$a^i b^j \quad \text{with } 0 \leq i < m \text{ and } 0 \leq j < n.$$

As the  $\mathbb{Q}$ -vector space  $\mathbb{Q}[a, b]$  is generated by the elements  $a^k b^l$  with  $k, l \in \mathbb{N}_0$ , it suffices to show that each of these elements is a  $\mathbb{Q}$ -linear combination of the elements mentioned under (@@).

So, let  $k, l \in \mathbb{N}_0$ . Then, by Proposition 5.3 we have

$$a^k \in \mathbb{Q}[a] = \sum_{i=0}^{m-1} \mathbb{Q}a^i \text{ and } b^l \in \mathbb{Q}[b] = \sum_{j=0}^{n-1} \mathbb{Q}b^j.$$

Therefore, we find elements  $u_0, u_1, \dots, u_{m-1}, v_0, v_1, \dots, v_{n-1} \in \mathbb{Q}$  such that

$$a^k = \sum_{i=0}^{m-1} u_i a^i \text{ and } b^l = \sum_{j=0}^{n-1} v_j b^j.$$

As  $q_{i,j} := u_i v_j \in \mathbb{Q}$  for all  $i = 0, 1, \dots, m-1$  and all  $j = 0, 1, \dots, n-1$ , we get

$$\begin{aligned} a^k b^l &= \left( \sum_{i=0}^{m-1} u_i a^i \right) \left( \sum_{j=0}^{n-1} v_j b^j \right) = \\ &= \sum_{i \in \{0,1,\dots,m-1\}; j \in \{0,1,\dots,n-1\}} u_i v_j a^i b^j = \\ &= \sum_{i \in \{0,1,\dots,m-1\}; j \in \{0,1,\dots,n-1\}} q_{i,j} a^i b^j. \end{aligned}$$

This proves our claim (@@) and hence also claim (@).

Now, observe that for all  $r \in \mathbb{N}_0$  we have

$$\begin{aligned} (ab)^r &= a^r b^r \in \mathbb{Q}[a, b], \\ (a+b)^r &= \sum_{i=0}^r \binom{r}{i} a^i b^{r-i} \in \mathbb{Q}[a, b] \text{ and} \\ (a-b)^r &= \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} a^i b^{r-i} \in \mathbb{Q}[a, b]. \end{aligned}$$

It follows that

$$\mathbb{Q}[ab], \quad \mathbb{Q}[a+b], \quad \mathbb{Q}[a-b] \subseteq \mathbb{Q}[a, b].$$

So, by statement (@), we obtain that:

$$\dim_{\mathbb{Q}}(\mathbb{Q}[ab]), \quad \dim_{\mathbb{Q}}(\mathbb{Q}[a+b]), \quad \dim_{\mathbb{Q}}(\mathbb{Q}[a-b]) \leq mn.$$

Therefore, by Proposition 5.3 we obtain indeed

$$a+b, \quad a-b, \quad ab \in \mathbb{A},$$

and hence  $\mathbb{A}$  is closed under addition, subtraction and multiplication.

Now, it follows immediately by Definition, Examples and Exercises 5.4 (B)(e) that

$$\frac{a}{b} \in \mathbb{A} \text{ for all } a \in \mathbb{A} \text{ and all } b \in \mathbb{A} \setminus \{0\}.$$



So,  $\mathbb{A}$  is closed under division, too.

Finally, by Definition, Examples and Exercises 5.4 (B)(c) we have

$$\sqrt{a} \in \mathbb{A} \text{ for all } a \in \mathbb{A}_{\geq 0},$$

so that  $\mathbb{A}$  is closed under the formation of square roots, too.  $\square$

**5.6 Corollary.** *Constructible numbers are algebraic, thus*

$$\mathbb{K} \subseteq \mathbb{A}.$$

*Proof.* This follows immediately from Proposition 5.5 by Remark, Definition and Exercises 5.2 (B)(c).  $\square$

**5.7 Exercises.** (A) (*Behavior of Degrees*) Prove the following claims:

(a) If  $a, b \in \mathbb{A}$ , then  $a + b, a - b, ab \in \mathbb{A}$  and

$$\deg(a + b), \deg(a - b), \deg(ab) \leq \deg(a)\deg(b).$$

(Hint: Follow the track of the proof of Proposition 5.5.)

(b) If  $a, d \in \mathbb{A}$  with  $b \neq 0$ , then  $\frac{a}{b} \in \mathbb{A}$  and

$$\deg\left(\frac{a}{b}\right) \leq \deg(a)\deg(b).$$

(c) If  $a \in \mathbb{A}$  with  $a > 0$ , then  $\sqrt{a} \in \mathbb{A}$  and

$$\deg(\sqrt{a}) \leq 2\deg(a).$$

(B) (*Countability of Algebraic Numbers*) Let  $n \in \mathbb{N}$ . We say that  $a \in \mathbb{R}$  is an  $n$ -bounded (algebraic) number, if there are integers

$m \in \mathbb{N}$  and  $z_0, z_1, \dots, z_m \in \mathbb{Z}$  such that  $m \leq n, |z_i| \leq n$  for all  $i = 0, 1, \dots, m$ ,

$$z_m \neq 0 \text{ and, finally } z_m a^m + z_{m-1} a^{m-1} + \dots + z_1 a + z_0 = 0.$$

We write

$$\mathbb{A}_n := \{a \in \mathbb{R} \mid a \text{ is } n\text{-bounded}\}.$$

Prove the following statements:

(a) If  $m \in \mathbb{N}$  and  $z_0, z_1, \dots, z_m \in \mathbb{Z}$  with  $z_m \neq 0$ , then

$$\#\{a \in \mathbb{R} \mid z_m a^m + z_{m-1} a_{m-1} + \dots + z_1 a + z_0 = 0\} \leq m$$

(Hint: Use the Theorem of Vieta).

(b)  $\#\mathbb{A}_n \leq \sum_{m=1}^n m 2^m n^{m+1}$  (Hint: Use statement (a)).

(c)  $\mathbb{A} = \bigcup_{n \in \mathbb{N}} \mathbb{A}_n$ .

(d) The set  $\mathbb{A}$  of algebraic numbers is countable.

**Transcendence of  $\pi$ .** Non-algebraic numbers, hence numbers which belong to the set  $\mathbb{R} \setminus \mathbb{A}$  are called *transcendent numbers*. The main result of this Chapter says that  $\pi$  belongs to these transcendent numbers. Unfortunately, any try to prove this result would by far exceed what we could achieve in these lectures. So, we content ourselves with stating the result.

**5.8 Theorem.** (*F. Lindemann, 1882*) *The number  $\pi$  is transcendental, thus*

$$\pi \in \mathbb{R} \setminus \mathbb{A}.$$

**5.9 Corollary.** *The number  $\pi$  is not constructible, hence  $\pi \in \mathbb{R} \setminus \mathbb{K}$ . So, the constructability problem  $\mathbb{K}\pi$  of Remark, Definition and Exercises 5.2 (D) has a negative answer.*

*Proof.* This is clear by Theorem 5.8 and Corollary 5.6. □

**5.10 Exercises and Remark.** (A) (*In-Countability of Transcendent Numbers*) We begin with the following exercises:

- (a) Use *Cantor's In-Countability Theorem* for the set  $\mathbb{R}$  of real numbers and Exercise 5.7 (B)(d) to show that the set  $\mathbb{R} \setminus \mathbb{A}$  of transcendental numbers is in-countable.
- (b) Look at the internet for examples of other specified transcendental numbers and note when and by whom the transcendence of these numbers has been proved.

(B) (*A Paradox Situation*) If you consider what you have learned so far, you likely will come to draw the following philosophical conclusions:

- (a) “Almost all real numbers are transcendental” (see statement (a) of part (A)). So, you should find them on “random choice”.
- (b) To prove that a specified real number is transcendental is usually “very demanding” (see what we said on the Proof of Theorem 5.8 and bear in mind your findings made under point (b) of part (A)). So, transcendental numbers are “hard to find”

We leave it to the readers, to make themselves their own thoughts on these apparently controversial points of view.

**What Survives for Teaching at High School?** Clearly, our algebraic arguments to approach the set  $\mathbb{K}$  of constructible numbers exceeds what you may teach in class. Observe nevertheless for your own purpose, that the arguments in our proofs were all quite elementary, and did only rely on Euclid's Remainder Theorem and on basic facts of Linear Algebra. Only the proof of Lindemann's result needs a good portion of Advanced Analysis, so that we had to skip it. But in spite of these abstract features, this chapter also offers a number of aspects, which could enrich your teaching when speaking on the number  $\pi$  or on constructions with compasses and straightedge.

**5.10. Didactic Comments.** (A) (*Transfer to the Class Room*) The geometric constructions mentioned in Remark, Reminders and Exercises 5.1 (A) are part of High School Geometry. The basic arithmetic constructions presented in part (B) of this same exercise are a very nice subject for being taught at High School. These two themes can be introduced and treated without referring to the number  $\pi$  or even without referring to any constructability question. The subject seems didactically of great value, as it ties a link between Geometry and Algebra. To practice on concrete objects, you could treat the exercises

suggested in Remark, Definition and Exercises 5.2 (C),

If you have treated the subject so far, you eventually might ask the question, whether “all (real) numbers can be constructed on repeated (finitely many) applications of basic arithmetic constructions”. You could just give the answer by saying that for example  $\pi$  is a non-constructible number, and you could then mention the fact that this implies that the problem of squaring the circle has a negative answer. You also could mention the *Euler number*  $e$  as another example of non-constructible number, (a number which is indeed also transcendent, like  $\pi$ ).

Another non-constructible number is  $2^{\frac{1}{3}}$  - the third root of 2. This number is algebraic, but non constructible (things, you likewise will not speak about in your class). The fact, that the third root of 2 is not constructible, gives a negative answer to another classical constructability problem, namely the question:

- Is it possible to construct by compasses and straightedge (the edge of) a cube which has precisely half the volume of a given cube?

If you mention the third root of 2, you should make your class think on the relation between this number and the problem of constructing a cube of half volume.

(B) (*Thinking on the Nature of Numbers*) What we suggested under point (A) could furnish a good opportunity to speak to your class on the “building-up of the system of numbers”: How to get the set  $\mathbb{R}$  of all real numbers via a step-wise extension of the set  $\mathbb{N}$  of natural numbers. You likely would just repeat, what you already have treated. But it turns out in practice, that students tend to forget very quickly these matters, which results in wrong ideas about the nature of numbers. Then you could speak on rational and irrational numbers, furnishing examples for both. Finally you could come back to the set  $\mathbb{K}$  of constructible numbers and say, that there are further classes of real numbers, which usually are not treated in school (like the algebraic numbers, for example).

Whenever you treat the subject, do not miss to speak out the general idea, that the subject of constructible numbers ties a very important link between Geometry and Algebra, a fact, which we have already emphasized in part (A).

## Biographical Hints

*Euclid of Alexandria: 300 B.C.:* Greek Mathematician, living in the Province of Alexandria, recent Northern Egypt. Wrote the earliest textbook on Geometry, called by him the “Elements”. The Elements attempt to approach (Plane) Geometry by deducing all results by logical deduction from a small set of axioms. Euclid wrote further work on Geometry, Arithmetics, Logics and Philosophy.

*René Descartes (Cartesius): 1596-1650.* French Philosopher and Mathematician; mostly lived in the Netherlands. Considered as “Father of Cartesian Geometry” – thus Geometry based on the use of normal coordinates.

*Carl Friedrich Gauss: 1777-1855.* German Mathematician, Professor at the University of Göttingen, Germany. Considered at his time as the Greatest Mathematician since Antiquity. Contributed fundamentally to all fields of Mathematics of his time, notably: Number Theory, Geometry, Analysis, Differential Geometry, Theory of Magnetism, Statistics and Astronomy.

*David Hilbert: 1863-1943.* German Mathematician, Professor in Göttingen, Germany. Contributed fundamental and seminal work to all fields of Mathematics of his time, notably: Logic and Foundations of Mathematics, Geometry, Analysis, Number Theory, Algebra. Presented the first completely axiomatic build-up of Geometry.

**Suggestion.** Consult the Vitae of the mentioned (and further) Mathematicians at Wikipedia. Record some details out of it, which impress you most. Make use of your records to report in the class room. Mathematics does not exist on its own. It is developed and performed by human beings usually called Mathematicians. Knowing concrete instances of this, may turn Mathematics more attractive for your students.