

# IDLE THOUGHTS ON LOCAL COHOMOLOGY AND RELATED TOPICS

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ABSTRACT. We collect a few thoughts related to local cohomology modules over Noetherian rings. These came up when we tried in vain to prove some conjectures. We hope however, that some of them may be useful – or at least worth not to be just thrown in the waste bucket.

## 1. INTRODUCTION

**Notation 1.1.** (A) By  $\mathbb{Z}$  we denote the set of integers. If  $c \in \mathbb{Z}$  we set  $\mathbb{Z}_{\leq c} := \{n \in \mathbb{Z} \mid n \leq c\}$  and  $\mathbb{Z}_{\geq c} := \{n \in \mathbb{Z} \mid n \geq c\}$ . We write  $\mathbb{N} := \mathbb{Z}_{\geq 1}$  and  $\mathbb{N}_0 := \mathbb{Z}_{\geq 0}$ . Moreover we use the standard convention that  $\sup \emptyset = -\infty$  and  $\inf \emptyset = \infty$ . Concerning local cohomology, we use without further mention the notations of [BS,1998] and [BS,2013].

(B) By  $R$  we always denote a commutative unitary Noetherian ring. By  $\mathcal{C}(R)$  we denote the category of  $R$ -modules. If  $\mathfrak{a} \subseteq R$  is an ideal we consider the (covariant, left exact)  $\mathfrak{a}$ -torsion functor

$$\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{a}}(\bullet) := \lim_{\rightarrow n} \text{Hom}_R(R/\mathfrak{a}^n, \bullet) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R),$$

and – for each  $i \in \mathbb{N}_0$  – the  $i$ -th local cohomology functor with respect to  $\mathfrak{a}$ , which we introduce as the  $i$ -right derived functor of the  $\mathfrak{a}$ -torsion functor  $\Gamma_{\mathfrak{a}}$ , thus:

$$H_{\mathfrak{a}}^i = H_{\mathfrak{a}}^i(\bullet) := \mathcal{R}^i \Gamma_{\mathfrak{a}} = \mathcal{R}^i \Gamma_{\mathfrak{a}}(\bullet) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R).$$

As usually, we identify  $\Gamma_{\mathfrak{a}} = H_{\mathfrak{a}}^0$ . If  $M$  is an  $R$ -module, we call  $H_{\mathfrak{a}}^i(M)$  the  $i$ -local cohomology module of  $M$  with respect to  $\mathfrak{a}$ .

We also introduce the (covariant left exact)  $\mathfrak{a}$ -transform functor

$$D_{\mathfrak{a}} = D_{\mathfrak{a}}(\bullet) := \lim_{\rightarrow n} \text{Hom}_R(\mathfrak{a}^n, \bullet) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R),$$

which is related to local cohomology by the natural four term exact sequence

$$0 \longrightarrow H_{\mathfrak{a}}^0(\bullet) \longrightarrow \bullet \longrightarrow D_{\mathfrak{a}}(\bullet) \longrightarrow H_{\mathfrak{a}}^1(\bullet) \longrightarrow 0$$

and natural isomorphisms

$$\mathcal{R}^i D_{\mathfrak{a}} \cong H_{\mathfrak{a}}^{i+1} \text{ for all } i \in \mathbb{N}.$$

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**Definitions and Remarks 1.2.** (A) Let the notations be as above. Then, the *cohomological dimension* of the  $R$ -module  $M$  with respect to the ideal  $\mathfrak{a} \subseteq R$  is defined by

$$\mathrm{cd}_{\mathfrak{a}}(M) := \sup\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \neq 0\}.$$

Observe that

$$\mathrm{cd}_{\mathfrak{a}}(M) \leq \mathrm{ara}(\mathfrak{a}) := \min\{r \in \mathbb{N}_0 \mid \exists x_1, x_2, \dots, x_r \in R \text{ such that } \sqrt{\sum_{i=1}^r Rx_i} = \sqrt{\mathfrak{a}}\}.$$

(B) Let the notations be as above, and assume in addition, that the  $R$ -module  $M$  is finitely generated. Then, the *cohomological finiteness dimension* of  $M$  with respect to  $\mathfrak{a}$  is given by

$$\mathrm{f}_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \mid \text{the } R\text{-module } H_{\mathfrak{a}}^i(M) \text{ is not finitely generated}\}.$$

**Notation 1.3.** We now consider a Noetherian homogeneous ring

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n = R_0[x_1, \dots, x_r] \quad (r \in \mathbb{N} \text{ and } x_1, \dots, x_r \in R_1)$$

with irrelevant ideal

$$R_+ := \bigoplus_{n \in \mathbb{N}} R_n = R_1 R = \sum_{k=1}^r Rx_k.$$

Keep in mind, that in this situation, the base ring  $R_0$  is Noetherian. Let  ${}^*\mathcal{C}(R)$  denote the category of graded  $R$ -modules, and for each  $i \in \mathbb{N}_0$  let

$$H_{R_+}^i(\bullet) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$$

denote the  $i$ -th local cohomology functor with respect to the irrelevant ideal  $R_+$ . Keep in mind that – by restriction – we get an induced functor of graded  $R$ -modules

$$H_{R_+}^i(\bullet) = \bigoplus_{n \in \mathbb{Z}} H_{R_+}^i(\bullet)_n : {}^*\mathcal{C}(R) \longrightarrow {}^*\mathcal{C}(R)$$

**Reminder 1.4.** (A) Let the notations and hypotheses as in Notation 1.3 and let  $M := \bigoplus_{n \in \mathbb{Z}} M_n$  be a finitely generated graded  $R$ -module. Keep in mind that for each  $i \in \mathbb{N}_0$  the  $n$ -th graded component  $H_{R_+}^i(M)_n$  of the graded local cohomology module  $H_{R_+}^i(M) = \bigoplus_{n \in \mathbb{Z}} H_{R_+}^i(M)_n$  is a finitely generated  $R_0$ -module, which vanishes in addition for all  $n \gg 0$ .

**Definitions and Remarks 1.5.** (A) Let the notations be as above. Then, the *cohomological dimension* of the graded  $R$ -module  $M$  is defined by

$$\mathrm{cd}(M) := \mathrm{cd}_{R_+}(M)$$

Observe that in view of Definition 1.2 (A) we have

$$\mathrm{cd}(M) \leq \mathrm{ara}(R_+) \leq \inf\{n \in \mathbb{N}_0 \mid \exists x_1, x_2, \dots, x_n \in R_1 \text{ such that } R_1 = \sum_{i=1}^n Rx_i\}.$$

If  $M$  is finitely generated, we have in addition

$$\begin{aligned} \text{cd}(M) &= \sup\{\dim_{R_{\mathfrak{p}_0}/\mathfrak{p}_0 R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \text{Spec}(R_0)\} = \\ &= \sup\{\dim_{R_{\mathfrak{m}_0}/\mathfrak{m}_0 R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}/\mathfrak{m}_0 M_{\mathfrak{m}_0}) \mid \mathfrak{m}_0 \in \text{Max}(R_0)\}. \end{aligned}$$

(B) Let the notations be as above, and assume again, that the graded  $R$ -module  $M$  is finitely generated. Then, the *cohomological finiteness dimension* of  $M$  is given by

$$\begin{aligned} \text{f}(M) &:= \text{f}_{R_+}(M) = \inf\{i \in \mathbb{N}_0 \mid \text{the } R\text{-module } H_{R_+}^i(M) \text{ is not finitely generated}\} = \\ &= \inf\{i \in \mathbb{N}_0 \mid H_{R_+}^i(M)_n \neq 0 \text{ for infinitely many } n \in \mathbb{Z}\}. \end{aligned}$$

Moreover, the *cohomological finite length dimension* of  $M$  is defined by

$$\text{g}(M) := \inf\{i \in \mathbb{N}_0 \mid \text{length}_{R_0}(H_{R_+}^i(M)_n) = \infty \text{ for infinitely many } n \in \mathbb{Z}\}.$$

Observe that

$$\text{f}(M) \leq \text{g}(M) \text{ and - in case } M \text{ does not vanish - } \text{g}(M) \leq \text{cd}(M).$$

(C) Finally, for each  $k \in \mathbb{N}_0$  the *Castelnuovo-Mumford regularity at and above level  $k$*  of the finitely generated graded  $R$ -module  $M$  is defined by

$$\text{reg}^k(M) := \sup\{\text{end}(H_{R_+}^i(M) + i \mid i \geq k)\} \quad (< \infty),$$

whereas the *Castelnuovo-Mumford regularity* of  $M$  is defined by

$$\text{reg}(M) := \text{reg}^0(M) = \sup\{\text{end}(H_{R_+}^i(M) + i \mid i \in \mathbb{N}_0)\}.$$

**Reminder 1.6.** (A) (We refer to [H,1977], Chapters II, III or [BS,2013], Chapter 20) Let

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n = R_0[R_1] \text{ be a Noetherian homogeneous ring ,}$$

and let

$$X := \text{Proj}(R)$$

denote the *projective scheme* defined by  $R$ . If

$$S := R_0[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_r]$$

is a standard graded polynomial ring, we set

$$\mathbb{P}_{R_0}^r := \text{Proj}(S).$$

and call  $\mathbb{P}_{R_0}^r$  the *projective  $r$ -space* over  $R_0$ . If there is a surjective homomorphism

$$R_0[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_r] \xrightarrow{\pi} R$$

the graded ideal

$$I_X := \text{Ker}(\pi)^{\text{sat}} := \bigcup_{n \in \mathbb{N}} (\text{Ker}(\pi) :_S (S_+)^n) \subseteq S$$

is called the *homogeneous vanishing ideal* of  $X$ . Observe, that in this situation we have an isomorphism of graded  $R$ -algebras

$$S/I_X \cong R/\Gamma_{R_+}(R) \text{ and } \text{Proj}(S/I_X) = \text{Proj}(R/\Gamma_{R_+}(R)) = X \subseteq \mathbb{P}_{R_0}^r \text{ is a closed subscheme.}$$

(B) Keep the notations and hypotheses of part (A). If  $M$  is a graded  $R$ -module, let  $\widetilde{M}$  denote the sheaf *induced* over  $X$  by  $M$ . Keep in mind, that the sheaf

$$\mathcal{O}_X := \widetilde{R}$$

induced by the  $R$ -module  $R$  carries a natural structure of sheaf of  $R_0$ -algebras, and is called the *structure sheaf* of the scheme  $X$ . Moreover, for each graded  $R$ -module  $M$ , the induced sheaf  $\widetilde{M}$  carries a natural structure of sheaf of  $\mathcal{O}_X$ -modules. The sheaves of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  which are induced by some graded  $R$ -module  $M$  are precisely the *quasi-coherent* sheaves of  $\mathcal{O}_X$ -modules. Those sheaves of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  which are induced by a finitely generated graded  $R$ -module  $M$ , are precisely the *coherent* sheaves of  $\mathcal{O}_X$ -modules.

Now, let  $\mathcal{F} = \widetilde{M}$  be the quasi-coherent sheaf induced by the graded  $R$ -module  $M$  and let  $n \in \mathbb{Z}$ . Then – up to isomorphism of sheaves of  $\mathcal{O}_X$ -modules – the sheaf  $\widetilde{M(n)}$  induced by the  $n$ -th *shift* of  $M$  does only depend on  $\mathcal{F}$  and on  $n$ . Therefore, we may define the  $n$ -th *twist* of  $\mathcal{F}$  by

$$\mathcal{F}(n) := \widetilde{M(n)}.$$

If  $X = \text{Proj}(R) \subseteq \mathbb{P}_{R_0}^r$  is as in part (A), the coherent sheaf of ideals

$$\mathcal{I}_X := \widetilde{I}_X \subseteq \mathcal{O}_{\mathbb{P}_{R_0}^r}$$

is called the *sheaf of vanishing ideals* of  $X$ .

(C) Keep the previous notations and hypotheses. Then, for each  $i \in \mathbb{N}_0$  and each quasi-coherent sheaf  $\mathcal{F}$  of  $\mathcal{O}_X$ -modules, we consider the

$i$  – th *cohomology group*  $H^i(X, \mathcal{F})$  of  $X$  with coefficients in the quasi-coherent sheaf  $\mathcal{F}$ .

Then  $H^i(X, \mathcal{F})$  carries a natural structure of  $R_0$ -module. Moreover if  $\mathcal{F} = \widetilde{M}$  for some graded  $R$ -module  $M$ , the *Serre-Grothendieck Correspondence* gives rise to isomorphisms of  $R_0$ -modules:

- (a)  $H^0(X, \mathcal{F}(n)) \cong D_{R_+}(M)_n$  for all  $n \in \mathbb{Z}$  and
- (b)  $H^i(X, \mathcal{F}(n)) \cong H_{R_+}^{i+1}(M)_n$  for all  $n \in \mathbb{Z}$  and all  $i \in \mathbb{N}$ .

If  $\pi : X \rightarrow X_0 := \text{Spec}(R_0)$  is the induced projective morphism, if  $\pi_*$  denotes the functor of taking direct images with respect to  $\pi$  and if  $(M_0)^\sim$  denotes quasi-coherent sheaf of  $\mathcal{O}_{X_0}$ -modules induced by an  $R_0$ -module  $M_0$ , we also may write the above relations in the form:

- (a\*)  $\pi_* \mathcal{F}(n) \cong (D_{R_+}(M)_n)^\sim$  for all  $n \in \mathbb{Z}$  and
- (b\*)  $(\mathcal{R}^i \pi_*)(\mathcal{F}(n)) \cong (H_{R_+}^{i+1}(M)_n)^\sim$  for all  $n \in \mathbb{Z}$  and all  $i \in \mathbb{N}$ .

If  $\mathcal{F}$  is coherent, we may chose the graded  $R$ -module  $M$  to be finitely generated. So, it follows from statements (a) and (b) by what we know about local cohomology:

- (c) For all  $i \in \mathbb{N}_0$  and all  $n \in \mathbb{Z}$ , the  $R_0$ -module  $H^i(X, \mathcal{F}(n))$  is finitely generated.
- (d) For all  $i \in \mathbb{N}$  and all  $n \gg 0$  it holds  $H^i(X, \mathcal{F}(n)) = 0$ .

(D) Keep the above notations and hypotheses. Let  $\mathcal{F} = \widetilde{M}$  be a coherent sheaf of  $\mathcal{O}_X$ -modules, induced by the finitely generated graded  $R$ -module  $M$ . Then the *Castelnuovo-Mumford regularity* of the sheaf  $\mathcal{F}$  is defined by

$$\text{reg}(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{F}(n-i)) = 0 \text{ for all } i \in \mathbb{N}\}.$$

The Serre Grothendieck-Correspondence now yields that

$$\text{reg}(\mathcal{F}) = \text{reg}^2(M).$$

If  $X = \text{Proj}(R) \subseteq \mathbb{P}_{R_0}^r$  is as in part (A), the *Castelnuovo-Mumford regularity* of  $X$  is defined by

$$\text{reg}(X) := \text{reg}(\mathcal{I}_X).$$

On use of the Serre-Grothendieck Correspondence it is easy to see, that

$$\text{reg}(X) = \text{reg}(I_X).$$

(E) Keep the above hypotheses, but assume in addition, that the base ring  $R_0$  is Artinian. Assume that  $\mathcal{F}$  is a coherent sheaf of  $\mathcal{O}_X$ -modules. Then, by statement (d) of part (D), we may define

$$h^i(X, \mathcal{F}(n)) := \text{length}_{R_0}(H^i(X, \mathcal{F}(n))) \text{ for all } i \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

Let us also mention, that in this case we have

$$\dim(X) = \dim(R) - 1.$$

## 2. FUNCTORS APPLIED TO TOP GRADED LOCAL COHOMOLOGY MODULES

Throughtout this section, let the notations and hypotheses be as in the introduction. Here, we are interested in the behavior of local cohomology with respect to the irrelevant ideal of finitely a generated graded modules over a homogeneous Noetherian ring.

**Notation 2.1.** We now always shall consider a Noetherian homogeneous ring  $R = \bigoplus_{n \in \mathbb{N}_0} R_n = R_0[x_1, \dots, x_r]$  ( $r \in \mathbb{N}$  and  $x_1, \dots, x_r \in R_1$ ) with irrelevant ideal  $R_+ := \bigoplus_{n \in \mathbb{N}} R_n = R_1 R = \sum_{k=1}^r R x_k$ .

**Remark and Definition 2.2.** (A) Let

$$F = F(\bullet) : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0)$$

be a covariant functor of  $R_0$ -modules, which commutes with direct sums. Observe that for each  $R$ -module  $M$ , the  $R_0$ -module  $F(M)$  carries a natural structure of  $R$ -module with scalar multiplication given by  $xm := F(x\text{id}_M)(m)$  for all  $x \in R$  and all  $m \in F(M)$ . Moreover, if  $f : M \longrightarrow N$  is a homomorphism of  $R$ -modules, the homomorphism of  $R$ -modules  $F(f) : F(M) \longrightarrow F(N)$  is a homomorphism of  $R$ -modules. So, there is an induced covariant functor of  $R$ -modules  $F : \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ . Finally, for each graded  $R$ -module  $M = \bigoplus_{n \in \mathbb{Z}} M_n$ , the  $R$ -module  $F(M) = \bigoplus_{n \in \mathbb{Z}} F(M_n)$  carries a natural grading with  $n$ -th graded Component  $F(M)_n := F(M_n)$  for all  $n \in \mathbb{Z}$ , and for each homomorphism of graded  $R$ -modules  $f : M \longrightarrow N$ , the homomorphism  $F(f) : F(M) \longrightarrow F(N)$  is a homomorphism of graded  $R$ -modules. So, the functor  $F$  induces covariant functor of graded  $R$ -modules

$$F = F(\bullet) : {}^*\mathcal{C}(R) \longrightarrow {}^*\mathcal{C}(R).$$

For later use, keep in mind that the graded  $R$ -module  $F(M)$  is  $R_+$ -torsion, whenever  $M = \bigoplus_{n \in \mathbb{Z}} M_n$  is a graded  $R$ -module such that  $M_n = 0$  for all  $n \gg 0$ .

(B) Let the functor  $F : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0)$  be as in part (A). We say that  $F$  is an  $\mathfrak{A}$ -functor, if turns finitely generated  $R_0$ -modules in Artinian  $R_0$ -modules and if the induced functor of graded  $R$ -modules  $F : {}^*\mathcal{C}(R) \longrightarrow {}^*\mathcal{C}(R)$  turns graded Artinian  $R$ -modules in graded Artinian  $R$ -modules. It is immediate, that then the functor  $F : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0)$  turns Artinian  $R_0$ -modules in Artinian  $R_0$ -modules. We say that  $F$  is an  $\mathfrak{F}$  functor, if it turns finitely generated  $R_0$ -modules in  $R_0$ -modules of finite length. If  $F$  is an  $\mathfrak{A}$  and a  $\mathfrak{F}$ -functor, we call it an  $\mathfrak{A}\mathfrak{F}$ -functor.

(C) Let  $M$  is a graded  $R$ -module whose components  $M_n$  are all finitely generated  $R_0$ -modules, and assume that  $F : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0)$  is an  $\mathfrak{A}$ -functor. Then, all the components  $F(M)_n$  of the graded  $R$ -module  $F(M)$  are Artinian  $R_0$ -modules. If  $F$  is an  $\mathfrak{A}\mathfrak{F}$ -functor, these components are even  $R_0$ -modules of finite length. Note, that this applies in particular if  $M$  is a finitely generated graded  $R$ -module. If  $M$  is finitely generated and concentrated in finitely many degrees, it follows that the graded  $R$ -module  $F(M)$  is Artinian, and even of fine length over  $R_0$  (and hence over  $R$ ) if  $F$  is an  $\mathfrak{A}\mathfrak{F}$ -functor.

(D) Let  $G = G(\bullet) : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0)$  be a second covariant functor of  $R_0$ -modules, which commutes with direct sums. We say, that  $(F, G)$  is a *connected pair* of functors, if there is a natural assignement

$$\delta_{\bullet}^{(F,G)} : (\mathbb{S} : 0 \rightarrow N \xrightarrow{h} M \xrightarrow{l} P \rightarrow 0) \quad \mapsto \quad (F(P) \xrightarrow{\delta_{\mathbb{S}}^{(F,G)}} G(N)),$$

which to each short exact sequence of  $R_0$ -modules  $\mathbb{S}$  as above assigns a homomorphism of  $R_0$ -modules  $\delta_{\mathbb{S}}^{(F,G)} = \delta$  such that there is an exact sequence

$$F(N) \xrightarrow{F(h)} F(M) \xrightarrow{F(l)} F(P) \xrightarrow{\delta} G(N) \xrightarrow{G(h)} G(M) \xrightarrow{G(l)} G(P).$$

**Example 2.3.** (A) Let  $U$  be an  $R_0$ -module of finite length. Then, the functors

$$\mathrm{Tor}_i^{R_0}(U, \bullet) : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0), \quad (i \in \mathbb{N}_0)$$

are all  $\mathfrak{A}\mathfrak{F}$ -functors. Moreover, for each  $i \in \mathbb{N}$

$$(\mathrm{Tor}_i^{R_0}(U, \bullet), \mathrm{Tor}_{i-1}^{R_0}(U, \bullet)) \text{ is a connencted pair of functors.}$$

(B) Let  $U$  be an  $R_0$ -module of finite length. Then, the functors

$$\mathrm{Ext}_{R_0}^i(U, \bullet) : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0), \quad (i \in \mathbb{N}_0)$$

are all  $\mathfrak{A}\mathfrak{F}$ -functors. Moreover, for each  $i \in \mathbb{N}_0$

$$(\mathrm{Ext}_{R_0}^i(U, \bullet), \mathrm{Ext}_{R_0}^{i+1}(U, \bullet)) \text{ is a connencted pair of functors.}$$

(C) Assume that the base ring  $(R_0, \mathfrak{m}_0)$  is local. Then, the local cohomology functors

$$H_{\mathfrak{m}_0}^i(\bullet) : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0), \quad (i \in \mathbb{N}_0)$$

turn finitely generated  $R_0$ -modules to Artinian  $R_0$ -modules and  $\mathfrak{m}_0$ -torsion modules to 0 if  $i > 0$  and hence are  $\mathfrak{A}$ -functors. Moreover  $H_{\mathfrak{m}_0}^0(\bullet)$  is an  $\mathfrak{A}\mathfrak{F}$ -functor. In addition, for each  $i \in \mathbb{N}_0$

$$(H_{\mathfrak{m}_0}^i(\bullet), H_{\mathfrak{m}_0}^{i+1}(\bullet)) \text{ is a connencted pair of functors.}$$

**Proposition 2.4.** *Assume that the base ring  $(R_0, \mathfrak{m}_0)$  is local. Let  $F, G : \mathcal{C}(R_0) \longrightarrow \mathcal{C}(R_0) :$  be two  $\mathfrak{A}$ -functors. Let  $M \neq 0$  be a finitely generated graded  $R$ -module, set  $c := \mathrm{cd}_{R_+}(M)$  and  $g := \mathfrak{g}(M)$ . Then it holds*

- (a) *If  $F$  is a right exact, then  $F(H_{R_+}^c(M))$  is a graded Artinian  $R$ -module.*
- (b) *If  $g = c$  and  $(F, G)$  is a connected pair of functors, then the graded  $R$ -module  $F(H_{R_+}^c(M))$  is Artinian.*

*Proof.* As  $M \neq 0$ , we have  $c \geq 0$ . We prove both statements by induction on  $c$ . Assume first, that  $c = 0$ . As  $H_{R_+}^0(M)$  is finitely generated and concentrated in finitely many degrees, we may conclude by Remark and Definition 2.2 (C).

So, let  $c > 0$ . As  $H_{R_+}^c(M) \cong H_{R_+}^c(M/\Gamma_{R_+}(M))$ , we may assume that  $\Gamma_{R_+}(M) = 0$ , and hence that  $R_+$  is not contained in any of the associated primes of  $M$ . As  $\dim_{R/\mathfrak{m}_0 R}(M/\mathfrak{m}_0 M) = c > 0$ , none of the primes in the finite set  $\mathcal{S} := \{\mathfrak{p} \in \mathrm{Ass}_R(M/\mathfrak{m}_0 M) \mid \dim(R/\mathfrak{p}) = c\}$  contains  $R_+$ . So, by homogeneous prime avoidance there is some  $t \in \mathbb{N}$  and some  $x \in R_t$  which avoids all  $\mathfrak{p} \in \mathrm{Ass}_R(M) \cup \mathcal{S}$ . It follows that

$$\mathrm{cd}_{R_+}(M/xM) = \dim_{R/\mathfrak{m}_0 R}((M/xM)/\mathfrak{m}_0(M/xM)) = \dim_{R/\mathfrak{m}_0 R}(M/(\mathfrak{m}_0 M + xM)) = c - 1$$

and that there is an exact sequence of graded  $R$ -modules

$$0 \longrightarrow M(-t) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0.$$

Now, the induced exact sequences of graded  $R$ -modules

$$H_{R_+}^{i-1}(M) \longrightarrow H_{R_+}^{i-1}(M/xM) \longrightarrow H_{R_+}^i(M)(-t) \xrightarrow{x} H_{R_+}^i(M) \longrightarrow H_{R_+}^i(M/xM)$$

show that  $g_{R_+}(M/xM) \geq g - 1$  and imply the exact sequence

$$(\textcircled{a}) \quad H_{R_+}^{c-1}(M) \longrightarrow H_{R_+}^{c-1}(M/xM) \longrightarrow H_{R_+}^c(M)(-t) \xrightarrow{x} H_{R_+}^c(M) \longrightarrow 0.$$

To prove statement (a), we apply the right exact functor  $F$  to this sequence and get the exact sequence

$$F(H_{R_+}^{c-1}(M/xM)) \longrightarrow F(H_{R_+}^c(M)(-t)) \xrightarrow{x} F(H_{R_+}^c(M)).$$

By induction, the first module in this sequence is Artinian, so that the graded  $R$ -module

$$[0 :_{F(H_{R_+}^c(M))} x]$$

is Artinian. The graded components  $H_{R_+}^c(M)_n$  of  $H_{R_+}^c(M)$  vanish for all  $n \gg 0$ , and hence the  $R$ -module  $F(H_{R_+}^c(M))$  is  $x$ -torsion (see Remark and Definition 2.2(A)). It follows by Melkerssons Lemma that  $F(H_{R_+}^c(M))$  is indeed Artinian.

In order to prove statement (b), we may again proceed as above and restrict ourselves to show that the graded  $R$ -module

$$[0 :_{G(H_{R_+}^c(M))} x]$$

is Artinian. The sequence  $(\textcircled{a})$  implies a short exact sequence of graded  $R$ -modules

$$0 \longrightarrow H_{R_+}^{c-1}(M/xM)/W \longrightarrow H_{R_+}^c(M)(-t) \xrightarrow{x} H_{R_+}^c(M) \longrightarrow 0,$$

in which  $W$  is a graded homomorphic image of  $H_{R_+}^{c-1}(M)$ . We thus get an exact sequence of graded  $R$ -modules

$$F(H_{R_+}^{c-1}(M/xM)/W) \longrightarrow F(H_{R_+}^c(M)(-t)) \xrightarrow{x} F(H_{R_+}^c(M)).$$

It hence suffices to show that the  $R$ -module  $F(H_{R_+}^{c-1}(M/xM)/W)$  is Artinian. The exact sequence of graded  $R$ -modules  $0 \rightarrow W \rightarrow H_{R_+}^c(M/xM) \rightarrow H_{R_+}^{c-1}(M/xM)/W \rightarrow 0$  gives rise to an exact sequence of graded  $R$ -modules

$$F(H_{R_+}^{c-1}(M/xM)) \longrightarrow F(H_{R_+}^c(M/xM)/W) \longrightarrow G(W).$$

As  $M/xM \neq 0$ , we have  $c - 1 = g - 1 \leq g(M/xM) \leq c_{R_+}(M/xM) = c - 1$  and hence  $g(M/xM) \leq c_{R_+}(M/xM) = c - 1$ . So, by induction the  $R$ -module  $F(H_{R_+}^{c-1}(M/xM))$  is Artinian. It thus remains to show that  $G(W)$  is Artinian. As  $c - 1 < g = g(M)$ , there is some  $r \in \mathbb{Z}$  such that the graded  $R$ -module  $H_{R_+}^{c-1}(M)/H_{R_+}^{c-1}(M)_{\geq r}$  is Artinian (see [BRSa,2005], Proposition 3.4). So, the graded  $R$ -module  $W/W_{\geq r}$  is Artinian and  $W_{\geq r}$  is a finitely generated graded  $R$ -module concentrated in finitely many degrees. The short exact sequence of graded  $R$ -modules  $0 \rightarrow W_{\geq r} \rightarrow W \rightarrow W/W_{\geq r} \rightarrow 0$  implies an exact sequence

$$G(W_{\geq r}) \longrightarrow G(W) \rightarrow G(W/W_{\geq r}).$$

As  $G$  is an  $\mathfrak{A}$ -functor the graded  $R$ -modules  $G(W_{\geq r})$  and  $G(W/W_{\geq r})$  are both Artinian (see also Remark and Definition 2.2(C)). So, the graded  $R$ -module  $G(W)$  is indeed Artinian.  $\square$



**Corollary 2.5.** *Assume that the base ring  $(R_0, \mathfrak{m}_0)$  is local. Let  $M \neq 0$  be a finitely generated graded  $R$ -module, let  $U$  be an  $R_0$ -module of finite length, set  $c := \text{cd}_{R_+}(M)$  and  $g := \text{g}(M)$ . Then it holds*

- (a) *The graded  $R$ -module  $U \otimes_{R_0} H_{R_+}^c(M)$  is Artinian with graded components of finite length.*
- (b) *If  $g = c$  and  $i \in \mathbb{N}_0$ , then*
  - (1) *The graded  $R$ -module  $\text{Tor}_i^{R_0}(U, H_{R_+}^c(M))$  is Artinian with graded components of finite length.*
  - (2) *The graded  $R$ -module  $\text{Ext}_{R_0}^i(U, H_{R_+}^c(M))$  is Artinian with graded components of finite length.*
  - (3) *The graded  $R$ -module  $H_{\mathfrak{m}_0}^i(H_{R_+}^c(M))$  is Artinian.*

*Proof.* (a): This follows by Proposition 2.4(a) and Example 2.3(A).

(b): Claim (1) follows by statement (a), Proposition 2.4(b) and Example 2.3(A).  
 Claim (2) follows by Proposition 2.4(b) and Example 2.3(B).  
 Claim (3) follows by Proposition 2.4(b) and Example 2.3(C). □

**Remark 2.6.** (A) Choosing  $U = R_0/\mathfrak{m}_0$  in statement (a) of the previous result, we get back Theorem 2.1 of [RotSeg,2005], (see also [BRoS,2005] and [B,2005]).

(B) Claim (1) of statement (b) of the above corollary generalizes statement (a) if  $g = c$ . Note that as a consequence of claims (1) and (2) of statement (b) there are polynomials

$$P_{M,U}^i(X), Q_{M,U}^i(X) \in \mathbb{Q}[X] \text{ such that}$$

$$\text{length}_{R_0}(\text{Tor}_i^{R_0}(U, H_{R_+}^c(M)_n)) = P_{M,U}^i(n) \text{ and}$$

$$\text{length}_{R_0}(\text{Ext}_{R_0}^i(U, H_{R_+}^c(M)_n)) = Q_{M,U}^i(n) \text{ for all } n \ll 0.$$

(C) Observe that all three claims of statement (b) also can be obtained on use of the Grothendieck spectral sequences for composed functors. Observe also, that statement claim (3) is shown in [HasJZa,2009] – on use of the previously mentioned spectral sequence – under the stronger hypotheses that  $c = \text{f}(M)$ . As a consequence it is shown there, that  $\text{depth}_{R_0}(H_{R_+}^c(M)_n)$  is asymptotically stable for  $n \rightarrow -\infty$ , provided that  $c = \text{f}(M)$ . Observe that claims (2) and (3) of statement (b) of the above corollary yield the same conclusion under the weaker hypotheses that  $c = \text{g}(M)$ .

### 3. ASYMPTOTIC DEPTH OF COMPONENTS OF GRADED LOCAL COHOMOLOGY

**Convention and Notation 3.1.** (A) As in the previous Section, let  $R := \bigoplus_{n \in \mathbb{N}_0} R_n$  be a standard graded Noetherian ring, with local base ring  $(R_0, \mathfrak{m}_0)$ . So  $R_0$  is Noetherian,  $R_1$  is a

finitely generated  $R_0$ -module and we have  $R = R_0[R_1]$ . Again, let  $R_+ : \bigoplus_{n \in \mathbb{N}} R_n = R_1 R \subset R$  be the irrelevant ideal of  $R$  and let  $M = \bigoplus_{n \in \mathbb{Z}} M_n$  be a finitely generated graded  $R$ -module.

(B) As previously, for each  $i \in \mathbb{Z}$  let  $H_{R_+}^i(M) = \bigoplus_{n \in \mathbb{Z}} H_{R_+}^i(M)_n$  be the local cohomology module of  $M$  with respect to  $R_+$ , furnished with its natural grading. Keep in mind, that for each  $i \in \mathbb{N}_0$  the  $R_0$ -module  $H_{R_+}^i(M)_n$  is finitely generated and vanishes for all  $n \gg 0$ .

(C) Keep in mind that the *cohomological dimension*, the *cohomological finiteness dimension*, and the *cohomological finite length dimension* of  $M$  are given respectively by (see Definitions and Remarks 1.5)

$$\begin{aligned} \text{cd}(M) &:= \sup\{i \in \mathbb{Z} \mid H_{R_+}^i(M) \neq 0\}, \\ f(M) &:= \inf\{i \in \mathbb{Z} \mid H_{R_+}^i(M)_n \neq 0 \text{ for infinitely many } n \in \mathbb{Z}\} \text{ and} \\ g(M) &:= \inf\{i \in \mathbb{Z} \mid \text{length}(H_{R_+}^i(M)_n) = \infty \text{ for infinitely many } n \in \mathbb{Z}\}, \end{aligned}$$

with the usual convention, that  $\sup \emptyset = -\infty$  and  $\inf \emptyset = \infty$ .

**Remark 3.2.** (A) Keep the above notations and hypotheses. Then, clearly

$$0 < f(M) \leq g(M) \text{ and } f(M), g(M) \in \{1, 2, \dots, \text{cd}(M), \infty\}.$$

In addition, we can say:

$$\text{If } \text{cd}(M) > 0, \text{ then } f(M) \leq \text{cd}(M).$$

Moreover, with the convention that the 0-module has dimension  $-\infty$ , we have (see [BH,2005])

$$\text{cd}(M) = \dim(M/\mathfrak{m}_0 M).$$

This implies in particular

$$\text{If } \text{cd}(M) > 0, \text{ then } \text{cd}(M/\Gamma_{\mathfrak{m}_0 R}(M)) = \text{cd}(M).$$

(B) Keep the above hypotheses and notations. Following [BRSa,2005] we say, that a graded  $R$ -module  $U = \bigoplus_{n \in \mathbb{Z}} U_n$  is *K-Artinian*, if  $U$  is Artinian and  $\text{length}_{R_0}(U_n) < \infty$  for all  $n \in \mathbb{Z}$ . Using this terminology, we can say (see Proposition 4.2 of [BRSa,2005]):

$$\text{If } i \leq g(M), \text{ then the graded } R\text{-module } \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M)) \text{ is K-Artinian.}$$

Moreover, by Theorem 4.10 of [BRSa,2005] we it holds:

$$\text{If } i \leq g(M), \text{ the set } \text{Ass}(H_{R_+}^i(M)_n) \text{ is asymptotically stable for } n \rightarrow -\infty.$$

(C) Next, consider the canonical epimorphism of graded  $R$ -modules

$$\pi : M \twoheadrightarrow M/\Gamma_{\mathfrak{m}_0 R}(M).$$

Then, the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}(M) \longrightarrow M \xrightarrow{\pi} M/\Gamma_{\mathfrak{m}_0 R}(M) \longrightarrow 0$$

together with the fact, that the graded  $R$ -modules  $H_{R_+}^j(\Gamma_{\mathfrak{m}_0 R}(M))$  are K-Artinian for all  $j \in \mathbb{Z}$  yields:

The kernel and the cokernel of the induced homomorphism of graded  $R$ -modules

$$H_{R_+}^i(\pi) : H_{R_+}^i(M) \longrightarrow H_{R_+}^i(M/\Gamma_{\mathfrak{m}_0 R}(M))$$

are both K-Artinian for all  $i \in \mathbb{Z}$ .

As an immediate consequence of this, we obtain the relation

$$g(M/\Gamma_{\mathfrak{m}_0 R}(M)) = g(M).$$

(D) Let  $(R_0, \mathfrak{m}'_0)$  be a local Noetherian flat  $R_0$ -algebra, consider the standard graded Noetherian  $R'_0$ -algebra  $R' := R'_0 \otimes_{R_0} R = \bigoplus_{n \in \mathbb{N}_0} R'_0 \otimes_{R_0} R_n$  and the finitely generated graded  $R'$ -module  $M' := R'_0 \otimes_{R_0} M = \bigoplus_{n \in \mathbb{Z}} R'_0 \otimes_{R_0} M_n$ . Then, from the graded flat base change property of local cohomology it follows

$$\text{cd}(M') = \text{cd}(M) \text{ and } f(M') = f(M).$$

Moreover, we can say:

$$\text{If } \dim(R'_0/\mathfrak{m}'_0 R'_0) = 0, \text{ then } g(M') = g(M).$$

**Notation and Remark 3.3.** For each  $R_0$ -module  $V$  we use the notation

$$\bar{V} := V/\Gamma_{\mathfrak{m}_0}(V).$$

So, if  $U = \bigoplus_{n \in \mathbb{Z}} U_n$  is a graded  $R$ -module, we have

$$\bar{U} = U/\Gamma_{\mathfrak{m}_0 R}(U) = \bigoplus_{n \in \mathbb{Z}} \bar{U}_n = \bigoplus_{n \in \mathbb{Z}} \bar{U}_n.$$

**Lemma 3.4.** *Let  $g := g(M) < \infty$ . Then we have*

(a) *If  $j > 1$ , then*

$$H_{\mathfrak{m}_0}^j(H_{R_+}^g(M)_n) \cong H_{\mathfrak{m}_0}^j(H_{R_+}^i(\bar{M})_n) \text{ for all } n \ll 0.$$

(b) *Either*

$$H_{\mathfrak{m}_0}^1(H_{R_+}^g(M)_n) \neq 0 \text{ for all } n \ll 0 \text{ or else}$$

$$H_{\mathfrak{m}_0}^1(H_{R_+}^g(M)_n) \cong H_{\mathfrak{m}_0}^1(H_{R_+}^g(\bar{M})_n) \text{ for all } n \ll 0.$$

*Proof.* According to Remark 3.2 (C) we have an exact sequence of graded  $R$ -modules

$$0 \longrightarrow K \longrightarrow H_{R_+}^g(M) \longrightarrow H_{R_+}^g(\bar{M}) \longrightarrow C \longrightarrow 0$$

in which  $K = \bigoplus_{n \in \mathbb{Z}} K_n$  and  $C = \bigoplus_{n \in \mathbb{Z}} C_n$  are K-Artinian. Now, statement (a) follows immediately from Notation and Remark 3.3 (B).

Next, we prove statement (b). The above exact sequence also yields a short exact sequence of graded  $R$ -modules

$$0 \longrightarrow D \longrightarrow H_{\mathfrak{m}_0}^1(H_{R_+}^g(M)) \longrightarrow H_{\mathfrak{m}_0}^1(H_{R_+}^g(\overline{M})) \longrightarrow 0$$

in which  $D = \bigoplus_{m \in \mathbb{Z}} D_m$  is a homomorphic image of  $C$  and hence K-Artinian.

Assume first, that  $D_m \neq 0$  for infinitely many  $m \leq 0$ . As  $D$  is K-Artinian, it follows that  $D_n \neq 0$  for all  $n \ll 0$  and hence that

$$H_{\mathfrak{m}_0}^1(H_{R_+}^g(M)_n) \neq 0 \text{ for all } n \ll 0.$$

In the remaining case, we have  $D_n = 0$  for all  $n \ll 0$ , so that

$$H_{\mathfrak{m}_0}^1(H_{R_+}^g(M)_n) \cong H_{\mathfrak{m}_0}^1(H_{R_+}^g(\overline{M})_n) \text{ and all } n \ll 0.$$

□

**Definition and Remark 3.5.** Keep the above notations. Let  $U = \bigoplus_{n \in \mathbb{Z}} U_n$  be a graded  $R$ -modules, whose graded components  $U_n$  are all finitely generated  $R_0$ -modules. We define the *left inferior depth* of  $U$  as the inferior limit of the depths of the  $R_0$ -module  $U_n$  if  $n$  tends to  $-\infty$ , thus

$$\begin{aligned} \underline{\text{depth}}(U) &:= \liminf_{n \rightarrow -\infty} \text{depth}(U_n) \\ &= \inf\{c \in \mathbb{Z} \cup \{\infty\} \mid \text{depth}(U_n) = c \text{ for infinitely many } n < 0\} \\ &= \inf\{j \in \mathbb{Z} \mid H_{\mathfrak{m}_0}^j(U_n) \neq 0 \text{ for infinitely many } n < 0\}. \end{aligned}$$

**Definition and Remark 3.6.** Following [BJ,2012] we say that a graded  $R$ -module  $U = \bigoplus_{n \in \mathbb{Z}} U_n$  is *almost Artinian* if there is a graded submodule  $T = \bigoplus_{n \in \mathbb{Z}} T_n$  such that  $T_n = 0$  for all  $n \ll 0$  and  $T/N$  is Artinian. The property of being almost Artinian clearly is inherited by graded subquotients. Moreover each almost Artinian module  $U$  is *tame*, so that

$$\text{either } U_n \neq 0 \text{ for all } n \ll 0, \text{ or else } U_n = 0 \text{ for all } n \ll 0.$$

**Proposition 3.7.** *Let  $g := g(M) < \infty$ . Set  $c := \underline{\text{depth}}(\overline{H_{R_+}^g(M)})$ . Assume that the graded  $R$ -module  $H_{\mathfrak{m}_0}^{i-g-1}(H_{R_+}^i(M))$  is almost Artinian for all  $i \in \mathbb{Z}$  with  $g < i < g + c$ . Then,  $\text{depth}(\overline{(H_{R_+}^g(M))_n})$  is asymptotically stable for  $n \rightarrow -\infty$ .*

*Proof.* Observe that  $c > 0$ . The case  $c = \infty$  follows immediately by the asymptotic stability of  $\text{Ass}(H_{R_+}^g(M)_n)$  for  $n \rightarrow -\infty$  (see Remark 3.2 (B)). So, let us assume that  $c < \infty$ . Clearly, our claim is clear if  $\dim(H_{R_+}^g(M)_n) \leq 1$  for all  $n \ll 0$ . Hence, by the previously mentioned asymptotic stability we may assume that  $\dim(H_{R_+}^g(M)_n) > 1$  for all  $n \ll 0$ .

We now proceed by induction on  $d := \dim(M)$ . If  $d = 0$ , our claim is obvious. So, let  $d > 0$ . Let  $(\widehat{R}_0, \widehat{\mathfrak{m}}_0)$  be the completion of  $(R_0, \mathfrak{m}_0)$  and set

$$\widehat{R} := \widehat{R}_0 \otimes_{R_0} R \text{ and } \widehat{M} := \widehat{R}_0 \otimes_{R_0} M.$$

Then we have  $\dim(\widehat{M}) = d$  and the Graded Flat Base Change and the Graded Base Ring independence Property of local cohomology show that there are isomorphisms of graded  $\widehat{R}$ -modules

$$H_{\widehat{R}_+}^i(\widehat{M})_n \cong \widehat{R}_0 \otimes_{R_0} H_{R_+}^i(M)_n \text{ for all } n, i \in \mathbb{Z}.$$

From this it follows easily that neither our hypotheses nor one of the occurring invariants is affected if we replace  $R$  and  $M$  respectively by  $\widehat{R}$  and  $\widehat{M}$ . This allows to hence assume that the base ring  $(R_0, \mathfrak{m}_0)$  is complete.

K-Artinian (see Remark 3.2 (C)) we may replace  $M$  by  $\overline{M}$  and hence assume that  $\Gamma_{\mathfrak{m}_0 R}(M) = 0$ . As  $R_0$  is complete, the Countable Prime Avoidance Principle of [SV, ] allows to chose an element  $x_0 \in \mathfrak{m}_0$  which avoids all members of the countable set of primes

$$\left( \bigcup_{n \in \mathbb{Z}} \text{Ass}_{R_0}(M_n) \cup \bigcup_{i, n \in \mathbb{Z}} \text{Ass}_{R_0}(H_{R_+}^i(M)_n) \right) \setminus \{\mathfrak{m}_0\}.$$

As  $\Gamma_{\mathfrak{m}_0 R}(M) = 0$  it follows that

$$x_0 \in \text{NZD}(M), \text{ and } \dim(M/x_0M) = d - 1.$$

Moreover,  $x_0$  is filter-regular with respect to  $H_{R_+}^i(M)_n$  for all  $i, n \in \mathbb{Z}$ , so that for all  $i \in \mathbb{Z}$  we may say:

- (1)  $(0 :_{H_{R_+}^i(M)} x_0) \subseteq \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$ ;
- (2)  $x_0 \in \text{NZD}(\overline{H_{R_+}^i(M)})$ .

For each  $i \in \mathbb{Z}$  we have exact sequences of graded  $R$ -modules

- (3)  $0 \longrightarrow H_{R_+}^i(M)/x_0 H_{R_+}^i(M) \longrightarrow H_{R_+}^i(M/x_0M) \longrightarrow (0 :_{H_{R_+}^{i+1}(M)} x_0) \longrightarrow 0$ ;
- (4)  $0 \longrightarrow (0 :_{H_{R_+}^i(M)} x_0) \longrightarrow H_{R_+}^i(M) \longrightarrow x_0 H_{R_+}^i(M) \longrightarrow 0$ .

As  $\dim(H_{R_+}^g(M)_n) > 1$  for all  $n \ll 0$  the sequences (3) first imply (see also Remark 3.2 (B))

$$g(M/x_0M) = g \text{ and } \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^g(M/x_0M)) \text{ is K-Artinian.}$$

Moreover by the inclusion (1) and the sequence (3) we get exact sequences of graded  $R$ -modules

- (5)  $0 \longrightarrow K \longrightarrow \overline{H_{R_+}^g(M)/x_0 H_{R_+}^g(M)} \longrightarrow \overline{H_{R_+}^g(M/x_0M)} \longrightarrow C \longrightarrow 0$ ,
- (6)  $0 \longrightarrow \overline{H_{R_+}^g(M)/x_0 H_{R_+}^g(M)} \longrightarrow \overline{H_{R_+}^g(M/x_0M)} \longrightarrow D \longrightarrow 0$ ,

in which

- (5')  $K$  is a graded homomorphic image of  $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^g(M/x_0M))$  – hence K-Artinian, and
- (6')  $C$  and  $D$  are both graded subquotients of  $H_{\mathfrak{m}_0 R}^0(H_{R_+}^g(M))$ .

Applying (2) with  $i = g$ , we get exact sequences

$$0 \longrightarrow \overline{H_{R_+}^g(M)}_n \xrightarrow{x_0} \overline{H_{R_+}^g(M)}_n \longrightarrow \overline{H_{R_+}^g(M)}_n / x_0 \overline{H_{R_+}^g(M)}_n \longrightarrow 0$$

for all  $n \in \mathbb{Z}$ . Applying cohomology with respect to  $\mathfrak{m}_0$  we thus obtain for all  $n, j \in \mathbb{Z}$ :

(7) If  $H_{\mathfrak{m}_0}^{j-1}(\overline{H_{R_+}^g(M)_n}) = H_{\mathfrak{m}_0}^j(\overline{H_{R_+}^g(M)_n}) = 0$ , then we have

$$H_{\mathfrak{m}_0}^{j-1}(\overline{H_{R_+}^g(M)_n/x_0\overline{H_{R_+}^g(M)_n}}) = 0;$$

(8) If  $H_{\mathfrak{m}_0}^{j-1}(\overline{H_{R_+}^g(M)_n/x_0\overline{H_{R_+}^g(M)_n}}) = 0$ , then we have

$$H_{\mathfrak{m}_0}^j(\overline{H_{R_+}^g(M)_n}) = 0.$$

We now proceed by induction on  $c$ . Assume first, that  $c = 1$ . Then  $H_{\mathfrak{m}_0}^1(\overline{H_{R_+}^g(M)_n}) \neq 0$  for infinitely many  $n \leq 0$ . So, by (7), we have  $H_{\mathfrak{m}_0}^0(\overline{H_{R_+}^g(M)_n/x_0\overline{H_{R_+}^g(M)_n}}) \neq 0$  for infinitely many  $n \leq 0$ . Observe that the sequence (5) implies an isomorphism of graded  $R$ -modules

$$K \cong H_{\mathfrak{m}_0}^0(\overline{H_{R_+}^g(M)/x_0\overline{H_{R_+}^g(M)}}).$$

As  $K$  is K-Artinian it follows that  $H_{\mathfrak{m}_0}^0(\overline{H_{R_+}^g(M)_n/x_0\overline{H_{R_+}^g(M)_n}}) \neq 0$  for all  $n \ll 0$ . As  $H_{\mathfrak{m}_0}^0(\overline{H_{R_+}^g(M)}) = 0$  it follows by (7) that  $H_{\mathfrak{m}_0}^1(\overline{H_{R_+}^g(M)_n}) \neq 0$  for all  $n \ll 0$ , and this proves our claim if  $c = 1$ .

Now, let  $c > 1$ . Then, by our hypotheses and by (6') the graded  $R$ -module  $D$  is almost Artinian. Now, the sequence (6) implies a short exact sequence of graded  $R$ -modules

$$(9) \quad 0 \longrightarrow D \longrightarrow H_{\mathfrak{m}_0R}^1(\overline{H_{R_+}^g(M)/x_0\overline{H_{R_+}^g(M)}}) \longrightarrow H_{\mathfrak{m}_0R}^1(\overline{H_{R_+}^g(M/x_0M)}) \longrightarrow 0$$

and isomorphisms of graded  $R$ -modules

$$(10) \quad H_{\mathfrak{m}_0R}^{j-1}(\overline{H_{R_+}^g(M)/x_0\overline{H_{R_+}^g(M)}}) \cong H_{\mathfrak{m}_0R}^{j-1}(\overline{H_{R_+}^g(M/x_0M)}) \text{ for all } j > 2.$$

Assume first, that  $D_n \neq 0$  for all  $n \ll 0$ . Then by (9) we have

$$H_{\mathfrak{m}_0}^1(\overline{H_{R_+}^g(M)_n/x_0\overline{H_{R_+}^g(M)_n}}) \neq 0 \text{ for all } n \ll 0.$$

As  $c > 1$  it follows by (7) that  $H_{\mathfrak{m}_0}^2(\overline{H_{R_+}^g(M)_n}) \neq 0$  for all  $n \ll 0$ . This means that  $c = 2$  and proves the required asymptotic stability.

So, we may assume that  $D_n = 0$  for all  $n \ll 0$ . But now, it follows from (9) and (10), that we have isomorphisms

$$(11) \quad H_{\mathfrak{m}_0R}^{j-1}(\overline{H_{R_+}^g(M)_n/x_0\overline{H_{R_+}^g(M)_n}}) \cong H_{\mathfrak{m}_0R}^{j-1}(\overline{H_{R_+}^g(M/x_0M)_n}), \quad \forall j > 1 \text{ and } \forall n \ll 0.$$

By (7) and (8) it now follows that  $H_{\mathfrak{m}_0R}^{c-1}(\overline{H_{R_+}^g(M/x_0M)_n}) \neq 0$  for infinitely many  $n \leq 0$  and  $H_{\mathfrak{m}_0R}^{j-1}(\overline{H_{R_+}^g(M/x_0M)_n}) = 0$  for all  $j \in \mathbb{Z}$  with  $1 < j < c - 1$ . As  $\overline{H_{R_+}^g(M/x_0M)}$  has no  $\mathfrak{m}_0$ -torsion it follows that

$$\text{depth}(\overline{H_{R_+}^g(M/x_0M)}) = c - 1.$$

Now, for each  $i > g$  the inclusions (1) and the sequences (3) and (4) respectively imply epimorphisms of graded  $R$ -modules

$$H_{\mathfrak{m}_0R}^{i-g-1}(H_{R_+}^i(M)/x_0\overline{H_{R_+}^i(M)}) \twoheadrightarrow H_{\mathfrak{m}_0R}^{i-g-1}(H_{R_+}^i(M/x_0M));$$

$$H_{\mathfrak{m}_0 R}^{i-g-1}(H_{R_+}^i(M)) \twoheadrightarrow H_{\mathfrak{m}_0 R}^{i-g-1}(H_{R_+}^i(M)/x_0 H_{R_+}^i(M)).$$

These show, that

$$H_{\mathfrak{m}_0 R}^{i-g-1}(H_{R_+}^i(M/x_0 M)) \text{ is almost Artinian for all } i \in \mathbb{Z} \text{ with } g < i < g + (c-1) - 1.$$

Hence, the module  $M/xM$  satisfies again our hypotheses. So, we may apply induction on  $c$  in order to see that indeed

$$H_{\mathfrak{m}_0 R}^{c-1}(\overline{H_{R_+}^g(M/x_0 M)}_n) \neq 0 \text{ for all } n \ll 0.$$

But now, another use of the isomorphisms (11) and the implications (7) shows that

$$H_{\mathfrak{m}_0}^c(\overline{H_{R_+}^g(M)}_n) \neq 0 \text{ for all } n \ll 0,$$

and this proves our claim.  $\square$

**Corollary 3.8.** *Let the notations and hypotheses be as in Proposition 3.7. Then  $\text{depth}(H_{R_+}^g(M)_n)$  is asymptotically stable for  $n \rightarrow -\infty$ .*

*Proof.* This is clear by Proposition 3.7 and Remark 3.2 (B).  $\square$

**Corollary 3.9.** *Let the notations be as in Proposition 3.7. Then  $\text{depth}(H_{R_+}^g(M)_n)$  is asymptotically stable for  $n \rightarrow -\infty$ , provided that one of the following conditions holds:*

- (i)  $\text{depth}(H_{R_+}^i(M)) \geq i - g$  for all  $i \in \mathbb{Z}$  with  $g < i < g + c$ ;
- (ii)  $\dim(H_{R_+}^i(M)_n) < i - g$  for all  $i \in \mathbb{Z}$  with  $g < i < g + c$  and infinitely many  $n \leq 0$ ;
- (iii)  $g = \text{cd}(M)$ ;
- (iv)  $g = \text{cd}(M) - 1$  and  $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^{\text{cd}(M)}(M))$  is almost Artinian.

*Proof.* All these statements follow immediately from Corollary 3.8 and by standard vanishing theorems of local cohomology.  $\square$

#### 4. ASYMPTOTIC PRIME DIVISORS OF COMPONENTS OF GRADED LOCAL COHOMOLOGY

**Notation 4.1.** We keep all our previous notations and conventions, but we admit now, that the Noetherian  $R_0$  base ring of our standard Noetherian ring  $R = \bigoplus_{n \in \mathbb{N}_0} R_n$  is not necessarily local. Keep in mind, that

$$\dim(H_{R_+}^i(M)_n) \leq 0 \text{ for all } i < g \text{ and all } n \leq n_0,$$

for an appropriate  $n_0 \in \mathbb{Z}$ . We once for all fix such an integer  $n_0 = n_0(M)$ .

**Lemma 4.2.** *Let  $\mathfrak{a}_0 \subset R_0$  be an ideal and let  $x_0 \in R_0$ . Then for each  $i \in \mathbb{N}_0$  and each  $n \in \mathbb{Z}$  there is an exact sequence of  $R_0$ -modules*

$$0 \longrightarrow H_{x_0 R_0}^1(H_{\mathfrak{a}_0 + R_+}^{i-1}(M)_n) \longrightarrow H_{x_0 R_0 + \mathfrak{a}_0 + R_+}^i(M)_n \longrightarrow H_{x_0 R_0}^0(H_{\mathfrak{a}_0 + R_+}^i(M)_n) \longrightarrow 0.$$

*Proof.* This is an immediate consequence of the graded version of the comparison sequence [BS,2013] Proposition 8.1.  $\square$

**Lemma 4.3.** *Assume that  $(R_0, \mathfrak{m}_0)$  is local. Let  $\mathfrak{a}_0 \subseteq \mathfrak{m}_0$  be an ideal. Then*

- (a) *For all  $i < g$  and for all  $n \leq n_0$  we have  $\dim(H_{\mathfrak{a}_0+R_+}^i(M)_n) \leq 0$ .*
- (b) *For all  $n \leq n_0$  there is an isomorphism of  $R_0$ -modules*

$$H_{\mathfrak{a}_0+R_+}^g(M)_n \cong H_{\mathfrak{a}_0}^0(H_{R_+}^g(M)_n).$$

- (c) *For all  $n \leq n_0$  the  $R_0$  module  $H_{\mathfrak{a}_0+R_+}^g(M)_n$  is finitely generated.*

*Proof.* (a): We proceed by induction on the number  $\mu$  of generators of  $\mathfrak{a}_0$ . If  $\mu = 0$  we have  $\mathfrak{a}_0 = 0$  and our claim is clear by our choice of  $n_0$ . So, let  $\mu > 0$  and write  $\mathfrak{a}_0 = x_0R_0 + \mathfrak{b}_0$  where  $x_0 \in \mathfrak{m}_0$  and  $\mathfrak{b}_0 \subseteq \mathfrak{m}_0$  is an ideal generated by  $< \mu$  elements. The comparison sequences of Lemma 4.2 applied to the ideal  $\mathfrak{b}_0 \subseteq R_0$  now read as

$$0 \longrightarrow H_{x_0R_0}^1(H_{\mathfrak{b}_0+R_+}^{i-1}(M)_n) \longrightarrow H_{\mathfrak{a}_0+R_+}^i(M)_n \longrightarrow H_{x_0R_0}^0(H_{\mathfrak{b}_0+R_+}^i(M)_n) \longrightarrow 0.$$

By induction we have

$$\dim(H_{\mathfrak{b}_0+R_+}^{i-1}(M)_n) \text{ and } \dim(H_{\mathfrak{b}_0+R_+}^i(M)_n) \leq 0 \text{ for all } i < g \text{ and all } n \leq n_0.$$

Therefore  $H_{x_0R_0}^1(H_{\mathfrak{b}_0+R_+}^{i-1}(M)_n) = 0$  and  $\dim(H_{x_0R_0}^0(H_{\mathfrak{b}_0+R_+}^i(M)_n)) \leq 0$ , and this proves our claim.

(b): We proceed again by induction on the number  $\mu$  of generators of  $\mathfrak{a}_0$ . If  $\mu = 0$ , we have  $\mathfrak{a}_0 = 0$  and our claim is obvious. If  $\mu > 0$ , we write again  $\mathfrak{a}_0 = x_0R_0 + \mathfrak{b}_0$ , where  $x_0 \in \mathfrak{m}_0$  and  $\mathfrak{b}_0 \subseteq \mathfrak{m}_0$  is an ideal generated by  $< \mu$  elements. So, by induction we have a isomorphisms

$$H_{\mathfrak{b}_0+R_+}^g(M)_n \cong H_{\mathfrak{b}_0R_0}^0(H_{R_+}^g(M)_n) \text{ for all } n \leq n_0.$$

Now, we apply the comparison sequence of the proof of statement (a) with  $i = g$  and obtain exact sequences

$$0 \longrightarrow H_{x_0R_0}^1(H_{\mathfrak{b}_0+R_+}^{g-1}(M)_n) \longrightarrow H_{\mathfrak{a}_0+R_+}^g(M)_n \longrightarrow H_{x_0R_0}^0(H_{\mathfrak{b}_0+R_+}^g(M)_n) \longrightarrow 0$$

for all  $n \leq n_0$ . By statement (a) we have  $H_{x_0R_0}^1(H_{\mathfrak{b}_0+R_+}^{g-1}(M)_n) = 0$  for all  $n \leq n_0$ . We thus get isomorphisms

$$H_{\mathfrak{a}_0+R_+}^g(M)_n \cong H_{x_0R_0}^0(H_{\mathfrak{b}_0+R_+}^g(M)_n) \cong H_{x_0R_0}^0(H_{\mathfrak{b}_0}^0(H_{R_+}^g(M)_n)) = H_{\mathfrak{a}_0}^0(H_{R_+}^g(M)_n)$$

for all  $n \leq n_0$ , and our claim follows.

(c): This is an immediate consequence of statement (b) and the fact that the  $R_0$ -modules  $H_{R_+}^g(M)_n$  are finitely generated.  $\square$



**Notation and Remark 4.4.** According to Remark 3.2 (B) we know that there is some integer  $n_1 \leq n_0$  such that

$$\text{Ass}(H_{R_+}^g(M)_n) = \text{Ass}(H^g(M)_{n_1}) \text{ for all } n \leq n_1$$

. We fix such an integer  $n_1 = n_1(M)$ . In addition we write:

$$\mathfrak{S} = \mathfrak{S}(M) := \text{Ass}(H^g(M)_{n_1}).$$

**Proposition 4.5.** *Let  $(R_0, \mathfrak{m}_0)$  be local and let  $\mathfrak{a}_0 \subseteq \mathfrak{m}_0$  be an ideal. Then*

$$\text{Ass}_{R_0}(H_{\mathfrak{a}_0+R_+}^g(M)_n) = \mathfrak{S}(M) \cap \text{Var}(\mathfrak{a}_0) \text{ for all } n \leq n_1.$$

*Proof.* This is immediate by Lemma 4.3 (b). □

**Lemma 4.6.** *Let  $V$  be an  $R_0$ -module and let  $x_0 \in R_0$ . Then*

$$\text{Ass}(H_{x_0R_0}^1(V)) = \text{Ass}(V/(\Gamma_{x_0R_0}(V) + x_0V)).$$

*Proof.* Let  $\mathfrak{p}_0 \in \text{Spec}(R_0)$ . We have to show, that

$$\mathfrak{p}_0 \in \text{Ass}(H_{x_0R_0}^1(V)) \text{ if and only if } \mathfrak{p}_0 \in \text{Ass}(V/(\Gamma_{x_0R_0}(V) + x_0V)).$$

By the Flat Base Change Property of local cohomology we may localize at  $\mathfrak{p}_0$  and hence assume that  $(R_0, \mathfrak{m}_0)$  is local and  $\mathfrak{p}_0 = \mathfrak{m}_0$ . If  $x_0 \notin \mathfrak{m}_0$  our claim is obvious as both of the occurring modules vanish. So, let  $x_0 \in \mathfrak{m}_0$ . We write  $W := V/\Gamma_{x_0R_0}(V)$ , so that  $H_{x_0R_0}^1(V) \cong H_{x_0R_0}^1(W)$ . This allows to replace  $V$  by  $W$  and hence to assume that  $\Gamma_{x_0R_0}(V) = 0$  and hence  $x_0 \in \text{NZD}(V)$ . In particular, we have a canonical isomorphism  $H_{x_0R_0}^1(V) \cong V_{x_0}/V$ .

If  $x_0V = V$  our claim is obvious. So, we may assume that  $x_0V \subsetneq V$ . Hence, for each  $w \in V_{x_0} \setminus V$  there is a unique  $n(w) \in \mathbb{N}$  such that  $x_0^{n(w)} \in V$  and  $x_0^{n(w)-1} \notin V$ .

First, let  $\mathfrak{m}_0 \in \text{Ass}(H_{x_0R_0}^1(V)) = \text{Ass}(V_{x_0}/V)$ . Then  $\mathfrak{m}_0w \subseteq V$  for some  $w \in V_{x_0} \setminus V$ . It follows that  $\mathfrak{m}_0x_0^{n(w)}w \subseteq x_0^{n(w)}V$  and  $x_0^{n(w)}w \in V \setminus x_0^{n(w)}V$ . But this implies that  $\mathfrak{m}_0 \in \text{Ass}(V/x_0^{n(w)}V) = \text{Ass}(V/x_0V)$ . Conversely, let  $\mathfrak{m}_0 \in \text{Ass}(V/x_0V)$ . Then  $\mathfrak{m}_0v \subseteq x_0^0V$  for some  $v \in V \setminus x_0V$ . It follows, that  $\mathfrak{m}_0\frac{v}{x_0} \in V$  and  $\frac{v}{x_0} \in V_{x_0} \setminus V$ , hence  $\mathfrak{m}_0 \in \text{Ass}(V_{x_0}/V) = \text{Ass}(H_{x_0R_0}^1(V))$ . □

## 5. SUPPORTS OF DEFICIENCY MODULES, PSEUDO-SUPPORTS AND CANONICAL MODULES

We keep the previous notations. First we recall a few facts on deficiency modules.

**Reminder 5.1.** (As basic references we recommend [Sc,1982], [Sc,1996] and also [BrHe,1998].)

(A) Assume that our Noetherian ring  $R$  is local of dimension  $d$  and with maximal ideal  $\mathfrak{m}$ . Let  $(R', \mathfrak{m}')$  be a local Gorenstein Ring of dimension  $d' \geq d$  and suppose that there is a

surjective ring homomorphism  $f : R' \twoheadrightarrow R$ . For each  $i \in \mathbb{N}_0$  consider the contravariant functor

$$\mathrm{Ext}_{R'}^i(\bullet, R') : \mathcal{C}(R') \longrightarrow \mathcal{C}(R').$$

Observe, that by means of scalar restriction with respect to the homomorphism  $f : R' \twoheadrightarrow R$ , we get an induced contravariant functor

$$K_R^i(\bullet) = \mathrm{Ext}_{R'}^{d-i}(\bullet, R') : \mathcal{C}(R) \longrightarrow \mathcal{C}(R) \text{ for all } i \leq d.$$

Up to natural equivalence, each of these functors is independent on the choice of the surjective homomorphism from a local Gorenstein ring to  $R$ , and it is called the  $i$ -th *deficiency functor*. For each  $R$ -module  $M$  and each non-negative integer  $i \leq d$ , the  $R$ -module  $K_R^i(M)$  is called the  $i$ -th *deficiency module* of  $M$ .

(B) Keep the previous notations and hypotheses. Keep in mind, that the deficiency functors constitute a strongly negative connected sequence of functors. More precisely, every short exact sequence of  $R$ -modules

$$\mathbb{S} : 0 \longrightarrow N \xrightarrow{h} M \xrightarrow{l} P \longrightarrow 0$$

induces naturally an exact sequence

$$\begin{aligned} 0 &\longrightarrow K_R^d(P) \xrightarrow{K_R^d(l)} K_R^d(M) \xrightarrow{K_R^d(h)} K_R^d(N) \\ &\xrightarrow{\delta_{\mathbb{S}}^d} K_R^{d-1}(P) \xrightarrow{K_R^{d-1}(l)} K_R^{d-1}(M) \xrightarrow{K_R^{d-1}(h)} K_R^{d-1}(N) \\ &\xrightarrow{\delta_{\mathbb{S}}^{d-1}} K_R^{d-2}(P) \xrightarrow{K_R^{d-2}(l)} K_R^{d-2}(M) \longrightarrow \dots \\ &\xrightarrow{\delta_{\mathbb{S}}^2} K_R^1(P) \xrightarrow{K_R^1(l)} K_R^1(M) \xrightarrow{K_R^1(h)} K_R^1(N) \\ &\xrightarrow{\delta_{\mathbb{S}}^1} K_R^0(P) \xrightarrow{K_R^0(l)} K_R^0(M) \xrightarrow{K_R^0(h)} K_R^0(N) \longrightarrow 0, \end{aligned}$$

the *deficiency sequence* associated to the short exact sequence  $\mathbb{S}$ .

(C) Keep the previous notations and hypotheses, and let  $M$  be a finitely generated  $R$ -module. Let  $i \leq d$  be a positive integer and let  $\mathfrak{p} \in \mathrm{Spec}(R)$ . Then, the following statements hold

- (a)  $K_R^i(M)$  is finitely generated and vanishes if  $i > \dim(M)$ .
- (b)  $\dim(K_R^i(M)) \begin{cases} \leq i, & \text{if } i < \dim(M); \\ = i, & \text{if } i = \dim(M). \end{cases}$
- (c) If  $E_R(N)$  denotes the injective envelope of the  $R$ -module  $N$  – by *local duality* – there is an isomorphism of  $R$ -modules

$$H_{\mathfrak{m}}^i(M) \cong \mathrm{Hom}_R(K_R^i(M), E_R(R/\mathfrak{m})).$$

(d) If  $i \leq \dim(R/\mathfrak{p})$ , there is a natural isomorphism of  $R_{\mathfrak{p}}$ -modules

$$K_R^i(M)_{\mathfrak{p}} \cong K_{R_{\mathfrak{p}}}^{i-\dim(R/\mathfrak{p})}(M_{\mathfrak{p}}).$$

(e) If  $i = \dim(R/\mathfrak{p})$ , then, the following statements are equivalent:

- (i)  $\mathfrak{p} \in \text{Ass}_R(M)$ ;
- (ii)  $\mathfrak{p} \in \text{Ass}_R(K_R^i(M))$ ;
- (iii)  $\mathfrak{p} \in \text{Supp}_R(K_R^i(M))$ .

**Proposition 5.2.** *Assume that  $R$  is a homomorphic image of a local Gorenstein ring. Let  $\mathfrak{p} \in \text{Spec } R$  and let  $i \leq d$  be a non-negative integer with  $\dim(R/\mathfrak{p}) \leq i$ . Let  $M$  and  $N$  be two finitely generated  $R$ -modules such that  $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ . Then it holds*

$$\mathfrak{p} \in \bigcup_{\dim(R/\mathfrak{p}) \leq j \leq i} \text{Supp}_R(K_R^j(M)) \text{ if and only if } \mathfrak{p} \in \bigcup_{\dim(R/\mathfrak{p}) \leq j \leq i} \text{Supp}_R(K_R^j(N)).$$

*Proof.* By Reminder 5.1(C)(d) we may assume that  $\mathfrak{p} = \mathfrak{m}$ . We show by induction on  $i$  the implication

$$\exists k \leq i \text{ such that } K_R^k(M) \neq 0 \Rightarrow \exists j \leq i \text{ such that } K_R^j(N) \neq 0.$$

We proceed by induction on  $i$ . Assume first, that  $i = 0$ . By Reminder 5.1(C)(e) it follows that  $\mathfrak{m} \in \text{Ass}_R(M)$ . By our hypothesis it follows that  $\mathfrak{m} \in \text{Ass}_R(N)$ , whence another use of Reminder 5.1(C)(e) yields that  $\mathfrak{m} \in \text{Supp}_R(K_R^0(N))$ , thus  $K_R^0(N) \neq 0$ .

Next, let  $i > 0$ . By induction we may assume that  $K_R^i(M) \neq 0$ . Assume first, that  $\text{depth}_R(M) = 0$ . Then, by our hypothesis we have  $\text{depth}_R(N) = 0$ , and hence Reminder 5.1(C)(b) and (c) show that  $K_R^0(N)$  is an  $R$ -module of dimension 0, whence  $K_R^0(N) \neq 0$ , as requested. Assume now, that  $\text{depth}_R(M) > 0$ . Then  $\text{depth}_R(N) > 0$  and hence there is some  $x \in \mathfrak{m} \cap \text{NZD}(M) \cap \text{NZD}(N)$ . So by Reminder 5.1(B) the two short exact sequences

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

imply exact sequences

$$(@_{M,j}) \quad K_R^{j+1}(M) \longrightarrow K_R^j(M/xM) \longrightarrow K_R^j(M) \xrightarrow{x} K_R^j(M) \longrightarrow K_R^{j-1}(M/xM)$$

$$(@_{N,j}) \quad K_R^{j+1}(N) \longrightarrow K_R^j(N/xN) \longrightarrow K_R^j(N) \xrightarrow{x} K_R^j(N) \longrightarrow K_R^{j-1}(N/xN)$$

for all  $j \geq 0$  – with the convention that the right-hand side modules vanish if  $j = 0$ . Applying Nakayama in the sequence  $(@_{M,i})$  yields that  $K_R^{j-1}(M/xM) \neq 0$ . As  $\text{depth}_R(M/xM) = \text{depth}_R(N/xN)$  it follows by induction that  $K_R^k(N/xN) \neq 0$  for some  $k \leq i - 1$ . By the sequence  $(@_{N,k})$  it follows that either  $K_R^{k+1}(M) \neq 0$  or  $K_R^k(M) \neq 0$ .  $\square$

As an immediate consequence of Proposition 5.2 we now get.

**Corollary 5.3.** *Assume that  $R$  is a homomorphic image of a local Gorenstein domain. Let  $i \in \mathbb{N}_0$ , and let  $M$  and  $N$  be two finitely generated  $R$ -modules such that  $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$  for all  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim(R/\mathfrak{p}) \leq i$ . Then*

$$\bigcup_{0 \leq j \leq i} \text{Supp}_R(K_R^j(M)) = \bigcup_{0 \leq j \leq i} \text{Supp}_R(K_R^j(N)).$$

As we shall see in a moment, the previous proposition and corollary are totally non-surprising, and both follow directly from a more general result, which is almost immediate. In order to formulate this latter result, we introduce a few more notions.

**Reminder 5.4.** (As a basic reference we recommend [BS,2002].) (A) Assume that our Noetherian ring  $R$  is local with maximal ideal  $\mathfrak{m}$ . Let  $M$  be a finitely generated  $R$ -module and let  $i \in \mathbb{N}_0$ . Then, the  $i$ -th *pseudo-support* of  $M$  is defined by

$$\text{Psupp}^i(M) := \{\mathfrak{p} \in \text{Spec}(R) \mid H_{\mathfrak{p}R_{\mathfrak{p}}}^{i-\dim(R/\mathfrak{p})}(M_{\mathfrak{p}}) \neq 0\},$$

whereas the  $i$ -th *pseudo-dimension* of  $M$  is defined by

$$\text{psd}^i(M) := \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Psupp}^i(M)\}.$$

(B) Keep the notations and hypotheses of part (A). Observe the following general facts.

- (a)  $\text{Psupp}^i(M) \subset \text{Supp}(M)$ .
- (b)  $\{\mathfrak{p} \in \text{Supp}(M) \mid \dim(R/\mathfrak{p}) = \dim(M)\} \subseteq \text{Psupp}^{\dim(M)}(M)$ .
- (c) If  $i > \dim(M)$ , then  $\text{Psupp}^i(M) = \emptyset$ .
- (d)  $\text{psd}^i(M) \leq i$ .

An easy, but important property of pseudo-supports is:

- (e) For all  $\mathfrak{p} \in \text{Spec } R$  with  $\dim(R/\mathfrak{p}) \leq i$  it holds

$$\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq i - \dim(R/\mathfrak{p}) \text{ and only if } \mathfrak{p} \in \bigcup_{\dim(R/\mathfrak{p}) \leq j \leq i} \text{Psupp}^j(M).$$

Bearing in mind the previous statement (d), we get as an immediate consequence for all  $i \in \mathbb{N}_0$  the equality

$$(\text{@@}) \quad \{\mathfrak{p} \in \text{Spec } R \mid \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq i - \dim(R/\mathfrak{p})\} = \bigcup_{0 \leq j \leq i} \text{Psupp}^j(M).$$

(C) Some important properties of pseudo-supports depend on the nature of the local ring  $R$ . So, for every  $i \in \mathbb{N}_0$  we can say:

- (a) If  $R$  is catenary, then  $\text{Supp}^i(M)$  is closed under specialization. But there is an example of a (non-catenary) Noetherian 3-dimensional local domain  $R$  for which  $\text{Psupp}^3(R)$  is not closed under specialisation and  $\text{Psupp}^2(R)$  is not closed.
- (b) If  $R$  is universally catenary, and all its formal fibers are Cohen-Macaulay, then

(1) The  $i$ -th pseudo-support of  $M$  is closed, more precisely

$$\text{Supp}^i(M) = \text{Cosupport}(H_{\mathfrak{m}}^i(M)) \quad \left( := \bigcup_{\mathfrak{p} \in \text{Att}_R(H_{\mathfrak{m}}^i(M))} \text{Var}(\mathfrak{p}) \right);$$

(2)  $\text{psd}^i(M) = \dim(H_{\mathfrak{m}}^i(M))$ .

However, there is an example of a universally catenary Noetherian 3-dimensional local domain  $R$  (whose formal fibers are not all Cohen-Macaulay) such that  $\text{Psupp}^2(R)$  is not closed.

(c) If  $R$  is a homomorphic image of a local Gorenstein ring, then

$$\text{Psupp}^i(M) = \text{Supp}(K_R^i(M)).$$

**Remark 5.5.** (A) Assume that  $(R, \mathfrak{m})$  is local and let  $M$  and  $N$  be two finitely generated  $R$ -modules. Then Remark 5.4(B)(e) implies the following statement.

(a) For all  $\mathfrak{p} \in \text{Spec } R$  with  $\dim(R/\mathfrak{p}) \leq i$  and  $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$  it holds

$$\mathfrak{p} \in \bigcup_{\dim(R/\mathfrak{p}) \leq j \leq i} \text{Psupp}^j(M) \text{ if and only if } \mathfrak{p} \in \bigcup_{\dim(R/\mathfrak{p}) \leq j \leq i} \text{Psupp}^j(N).$$

Moreover, by the relation (@@) of Remark 5.4(B) we get the following statement.

(b) Let  $i \in \mathbb{N}_0$ , and assume that  $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$  for all  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim(R/\mathfrak{p}) \leq i$ . Then

$$\bigcup_{0 \leq j \leq i} \text{Psupp}^j(M) = \bigcup_{0 \leq j \leq i} \text{Psupp}^j(N).$$

But now, in view of Remark 5.4 (C)(c) Proposition 5.2 is an immediate consequence of statement (a), whereas Corollary 5.3 is an immediate consequence of statement (b).

(B) The example of universally catenary 3-dimensional Noetherian local domain  $R$  with non-closed second pseudo-support  $\text{Psupp}^2(R)$  mentioned in Remark 5.4(C)(b) has the property that  $\text{Psupp}^i(R) = \emptyset$  for  $i = 0, 1$ , that  $\text{psd}^2(R) = 1$  and that there are infinitely many minimal primes  $\mathfrak{p} \in \text{Supp}^2(R)$  with  $\dim(R/\mathfrak{p}) = 1$  (see [BS,2002]). So, in this case the union

$$\bigcup_{j=0}^2 \text{Supp}^j(R) = \text{Supp}^2(R) \text{ is not closed.}$$

(C) Now, fix  $i \in \mathbb{N}_0$  and  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim(R/\mathfrak{p}) \leq i$  and  $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq i - \dim(R/\mathfrak{p})$ . Moreover, let  $\mathfrak{q} \in \text{Var}(\mathfrak{p})$ . Then we have the well known relations:

$$\text{height}(\mathfrak{q}/\mathfrak{p}) \leq \dim(R/\mathfrak{p}) - \dim(R/\mathfrak{q}) \text{ and } \text{depth}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \text{height}(\mathfrak{q}/\mathfrak{p}).$$

These imply that  $\text{depth}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq i - \dim(R/\mathfrak{p})$ . In view of statement (@@) of Reminder 5.4(B) this implies:

$$\bigcup_{0 \leq j \leq i} \text{Psupp}^j(M) \text{ is closed under specialization.}$$

We now return to the deficiency module, more precisely there special case of canonical module

**Reminder 5.6.** (As basic references we recommend [Sc,1982], [Sc,1996], [BrHe,1998] and [BS,2013].) (A) Assume, as in Reminder 5.1, that our Noetherian ring  $R$  is local of dimension  $d$  and with maximal ideal  $\mathfrak{m}$ . Let  $(R', \mathfrak{m}')$  be a local Gorenstein Ring of dimension  $d' \geq d$  and suppose that there is a surjective ring homomorphism  $f : R' \twoheadrightarrow R$ . Let  $M \neq 0$  be a finitely generated  $R$ -module. The *canonical module* of  $M$  is defined by

$$K(M) := K_R^{\dim(M)}(M).$$

Keep in mind that

- (a)  $\text{Ass}_R(K(M)) = \{\mathfrak{p} \in \text{Ass}_R(M) \mid \dim(R/\mathfrak{p}) = \dim(M)\}$ ;
- (b)  $K(M) \cong K(M/U_M(0))$ , where  $U_M(0) = \Gamma_{\mathfrak{p} \in \text{Ass}_R(M) \mid \dim(R/\mathfrak{p}) < \dim(M)} \mathfrak{p}(M)$  is (unique) maximal submodule of  $M$  whose dimension is strictly less than the dimension of  $M$ .

Observe, that in the above notations we have

- (c)  $\text{Ass}_R(M/U_M(0)) = \{\mathfrak{p} \in \text{Ass}_R(M) \mid \dim(R/\mathfrak{p}) = \dim(M)\}$ ;
- (d)  $U_M(0) = 0$  if and only if  $\dim(R/\mathfrak{p}) = \dim(M)$  for all  $\mathfrak{p} \in \text{Ass}_R(M)$ .

(B) Keep the notations and hypotheses of part (A). We consider the natural map

$$\varepsilon : \text{Hom}_R(M, M) \longrightarrow \text{Hom}_R(K(M), K(M)), \quad h \mapsto K^{\dim(M)}(h) \text{ for all } h \in \text{Hom}_R(M, M).$$

Assume that  $\dim(R/\mathfrak{p}) = \dim(M)$  for all  $\mathfrak{p} \in \text{Ass}_R(M)$ , so that  $U_0(M) = 0$ . Now, chose  $h \in \text{Hom}_R(M, M) \setminus \{0\}$ . Then  $h(M) \neq 0$  and hence  $\dim(h(M)) = \dim(M)$ , thus  $K_R^{\dim(M)}(h(M)) = K(h(M)) \neq 0$ . Now, the shoet exact sequence

$$0 \longrightarrow h(M) \xrightarrow{j} M \longrightarrow M/h(M) \longrightarrow M/h(M) \longrightarrow 0,$$

in which  $j$  denotes the inclusion map, induces the exact sequence

$$K(M) \xrightarrow{K_R^{\dim(M)}(j)} K(h(M)) \xrightarrow{\delta} K_R^{\dim(M)-1}(M/h(M)).$$

By Reminder 5.1 (C)(b), the last module in this sequence has dimension  $< \dim(M) = \dim(K(h(M)))$ . Therefore, the map  $\delta$  is  $\neq 0$ . The short exact sequence

$$0 \longrightarrow \text{Ker}(h) \longrightarrow M \xrightarrow{\bar{h}} h(M) \longrightarrow 0,$$

with  $h = j \circ \bar{h}$  gives rise to a monomorphism

$$0 \longrightarrow K(h(M)) \xrightarrow{K_R^{\dim(M)}(\bar{h})} K(M).$$

It follows that

$$\varepsilon(h) = K_R^{\dim(M)}(h) = K_R^{\dim(M)}(j \circ \bar{h}) = K_R^{\dim(M)}(\bar{h}) \circ K_R^{\dim(M)}(j) \neq 0.$$

Now, the two exact sequences

$$0 \longrightarrow \text{Ann}_R(M) \longrightarrow R \xrightarrow{\gamma} \text{Hom}_R(M, M), \quad (\gamma(x) := x\text{Id}_M, \forall x \in R) \quad \text{and}$$

$$0 \longrightarrow \text{Ann}_R(K(M)) \longrightarrow R \xrightarrow{\bar{\gamma}} \text{Hom}_R(K(M), K(M)), \quad (\bar{\gamma}(x) := x\text{Id}_{K(M)}, \forall x \in R),$$

together with the fact that  $\varepsilon \circ \gamma = \bar{\gamma}$ , show that  $\text{Ann}_R(K(M)) = \text{Ann}_R(M)$ . Therefore, by statements (b) and (c) of part (A) we can say:

$$(**) \quad \text{Ann}_R(K(M)) = \text{Ann}_R(M/U_M(0)).$$

## 6. COMPONENTS OF GRADED MODULES

**Notation 6.1.** (A) We keep the previous notations. Throughout this section, let

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n \text{ be a Noetherian homogenous ring}$$

and let

$$M = \bigoplus_{n \in \mathbb{Z}} M_n \text{ be a finitely generated graded } R\text{-module.}$$

(B) We introduce the *generating degree* of  $M$ , which is defined by

$$\text{gendeg}(M) := \inf\{n \in \mathbb{Z} \mid M = R \sum_{m \leq n} M_m\}.$$

Keep in mind that

$$\text{gendeg}(M) \leq \text{reg}(M).$$

**Lemma 6.2.** *Let  $n_0 \geq \max\{\text{end}(\Gamma_{R_+}(M) + 1), \text{gendeg}(M)\}$ . Then*

$$\text{Ann}_{R_0}(M_n) = \text{Ann}_{R_0}(M_{n_0}) \text{ for all } n \geq n_0.$$

*Proof.* Let  $\mathfrak{p} \in \text{Spec}(R_0)$ . Then, for each  $n \in \mathbb{Z}$  we have  $\text{Ann}_{R_{0\mathfrak{p}}}(M_{n\mathfrak{p}}) = \text{Ann}_{R_0}(M_n)R_{\mathfrak{p}}$ . Moreover  $\text{end}(\Gamma_{R_{\mathfrak{p}+}}(M_{\mathfrak{p}})) \leq \text{end}(\Gamma_{R_+}(M))$  and  $\text{gendeg}(M_{\mathfrak{p}}) \leq \text{gendeg}(M)$ . This allows to assume that  $R_0$  is local, with maximal ideal say  $\mathfrak{m}_0$ .

There is a Noetherian local faithfully flat extension ring  $(R'_0, \mathfrak{m}')$  of  $(R, \mathfrak{m})$  such that  $R'/\mathfrak{m}'$  is infinite. Now, by faithful flatness we may replace  $R$  and  $M$  respectively by  $R'_0 \otimes_{R_0} R$  and  $R'_0 \otimes_{R_0} M$  and hence assume in addition that the residue field  $R_0/\mathfrak{m}$  is infinite.

Now, there is some  $x \in R_1$  such that the multiplication map  $x : M_m \rightarrow M_{m+1}$  is injective for all  $m \geq n_0 > \text{end}(\Gamma_{R_+}(M))$ . This shows that

$$\text{Ann}_{R_0}(M_{m+1}) \subseteq \text{Ann}_{R_0}(M_m) \text{ for all } m \geq n_0.$$

On the other hand we have  $M_{m+1} = R_1 M_m$  for all  $m \geq n_0 \geq \text{gendeg}(M)$ , and hence

$$\text{Ann}_{R_0}(M_m) \subseteq \text{Ann}_{R_0}(M_{m+1}) \text{ for all } m \geq n_0.$$

□

In view of Notation 6.1 (B) we now immediately get.

**Proposition 6.3.** *Let  $n_0 > \text{reg}(M)$ . Then*

$$\text{Ann}_{R_0}(M_n) = \text{Ann}_{R_0}(M_{n_0}) \text{ for all } n \geq n_0.$$

As a further application we now get

**Corollary 6.4.** *Assume that  $R$  is a homomorphic image of a local Gorenstein ring. Then, there is a integer  $n_0$  such that*

$$\text{Ann}_{R_0}(K(M_n)) = \text{Ann}_{R_0}(K(M_{n_0})) \text{ for all } n \geq n_0.$$

*Proof.* For each  $n \in \mathbb{Z}$  we set

$$\mathcal{S}_n := \{\mathfrak{p}_0 \in \text{Ass}_{R_0} \mid \dim(R_0/\mathfrak{p}_0) = \dim(M_n)\}.$$

As  $\text{Ass}_{R_0}(M_n)$  is asymptotically stable for  $n \rightarrow \infty$ , there is some  $m_0 \in \mathbb{Z}$  such that  $\mathcal{S}_n = \mathcal{S}_{m_0}$  for all  $n \geq m_0$ . We consider the finitely generated graded  $R$ -module

$$\overline{M} := M/\Gamma_{(\cap_{\mathfrak{p} \in \mathcal{S}_{m_0}} \mathfrak{p})}(M).$$

Then, we have  $\overline{M}_n = M_n/U_M(0)$  for all  $n \geq m_0$ . If we set  $n_0 := \max\{m_0, \text{reg}(\overline{M})\}$  we get our claim by Proposition 6.3 and statement (\*\*) of Reminder 5.6 (B). □

For our convenience we now reprove in a concise manner a result which is well known and found in various versions at different places (see for example [B,1990,1]).

**Proposition 6.5.** *Let  $\mathfrak{b} \subseteq R_0$  be an ideal. Then, there is some  $n_0 \in \mathbb{Z}$  such that  $\text{grade}_{M_n}(\mathfrak{b}) = \text{grade}_{M_{n_0}}(\mathfrak{b})$  for all  $n \geq n_0$ .*

*Proof.* Let  $\delta := \liminf_{n \rightarrow \infty} \text{grade}_{M_n}(\mathfrak{b}) \in \mathbb{N}_0 \cup \{\infty\}$ . It remains to show that  $\text{grade}_{M_n}(\mathfrak{b}) = \delta$  for all  $n \gg 0$ .

Assume first, that  $\delta = \infty$ . Then  $\text{grade}_{M_n}(\mathfrak{b}) = \infty$ , hence  $\mathfrak{b}M_n = M_n$  and finally  $M_n/\mathfrak{b}M_n = 0$  for infinitely many  $n > 0$ . As the graded  $R$ -module  $M\mathfrak{b}M = \bigoplus_{n \in \mathbb{Z}} M_n/\mathfrak{b}M_n$  is finitely generated, it follows that  $M_n/\mathfrak{b}M_n = 0$ , hence  $\mathfrak{b}M_n = M_n$  and finally  $\text{grade}_{M_n}(\mathfrak{b}) = \infty$ , for



all  $n \gg 0$ .

So, let  $\delta \in \mathbb{N}_0$ . We proceed by induction on  $\delta$ . If  $\delta = 0$ , We have  $\text{grade}_{M_n}(\mathfrak{b}) = 0$ , hence

$$\mathfrak{b} \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}_{R_0}(M_n)} \mathfrak{p} \text{ for infinitely many } n > 0$$

As  $\text{Ass}_{R_0}(M_n)$  is asymptotically stable for  $n \rightarrow \infty$  the above inclusion holds for all  $n \gg 0$ , so that  $\text{grade}_{M_n}(\mathfrak{b}) = 0$  for all  $n \gg 0$ .

Finally, let  $\delta \in \mathbb{N}$ . Then  $\text{grade}_{M_n}(\mathfrak{b}) = \delta > 0$ , hence

$$\mathfrak{b} \setminus \left[ \bigcup_{\mathfrak{p} \in \text{Ass}_{R_0}(M_n)} \mathfrak{p} \right] \neq \emptyset \text{ for infinitely many } n > 0.$$

As  $\text{Ass}_{R_0}(M_n)$  is asymptotically stable for  $n \rightarrow \infty$ , it follows that there is some  $x \in \mathfrak{b}$ , such that  $x \in \text{NZD}_{R_0}(M_n)$  for all  $n \gg 0$ . So, we obtain

$$\text{grade}_{(M/xM)_n}(\mathfrak{b}) = \text{grade}_{M_n/xM_n}(\mathfrak{b}) = \text{grade}_{M_n}(\mathfrak{b}) - 1 = \delta - 1 \text{ for infinitely many } n > 0.$$

In particular we have  $\liminf_{n \rightarrow \infty} \text{grade}_{M_n}(\mathfrak{b}) = \delta - 1$ . Thus, by induction the previous equalities hold for all  $n \gg 0$ , so that  $\text{grade}_{M_n}(\mathfrak{b}) = \delta$  for all  $n \gg 0$ .  $\square$

We aim to extend this result to a situation, in which  $R$  is not homogeneous. To this end we first recall a basic notion on graded rings.

**Notation and Reminder 6.6.** (A) (See [BSh,2013] 13.5.9) Let  $R = \bigoplus_{n \in \mathbb{Z}} R_n$  be a  $\mathbb{Z}$ -graded Ring, let  $r \in \mathbb{N}_n$  and consider  $r$ -th *Veronese subring*

$$R^{(r)} := \bigoplus_{n \in \mathbb{Z}} \mathcal{R}_{rn}, \text{ furnished with its natural grading.}$$

Observe, that for each homogeneous element  $x \in R$  we have  $x^r \in R^{(r)}$ , so that  $R$  is an integral extension of  $R^{(r)}$ . Moreover, if  $R$  is of finite type over  $R_0$ , the  $R$  is a finite integral extension of  $R^{(r)}$ .

(B) Keep the above notations and hypotheses. Let  $M = \bigoplus_{n \in \mathbb{Z}} M_n$  be a graded  $R$ -module. For each  $s \in \mathbb{Z}$  consider the graded  $R^{(r)}$ -module

$$M^{(r,s)} := \bigoplus_{n \in \mathbb{Z}} M_{rn+s}.$$

If  $\bullet \downarrow_{R^{(r)}}: {}^* \mathcal{C}(R) \rightarrow {}^* \mathcal{C}(R^{(r)})$  denotes the (exact) functor of taking scalar restriction with respect to the inclusion homomorphism  $R^{(r)} \rightarrow R$ , we have a canonical isomorphism of graded  $R^{(r)}$ -modules

$$\bigoplus_{i=0}^{r-1} M^{(r,i)} \xrightarrow{\cong} M \downarrow_{R^{(r)}}.$$

Moreover, if  $R$  is of finite type over  $R_0$ , The graded  $R^{(r)}$ -module  $M \downarrow_{R^{(r)}}$  is finitely generated, and so are the modules  $M^{(r,s)}$ , which are all direct summands of  $M \downarrow_{R^{(r)}}$ . (C) assume now,

that  $R = \bigoplus_{n \in \mathbb{N}_0} R_n$  is Noetherian and positively graded. Then  $R_0$  is Noetherian and of finite type over  $R_0$ . We thus may write

$$R = R_0[x_i \mid i = 1, \dots, t], \text{ with } t \in \mathbb{N} \text{ and } x_i \in R_{n_i}, \text{ with } n_i \in \mathbb{N} \text{ for } i = 1, \dots, t.$$

Then we have (see for example [Ma,2009] 3.25)

$$\text{For } r := t \cdot \text{lcm}(n_1, \dots, n_t) \text{ it holds } R^{(r)} = R_0[R_r], \text{ hence } R^{(r)} \text{ is homogeneous.}$$

**Corollary 6.7.** *Let  $R = \bigoplus_{n \in \mathbb{N}_0} R_n$  be a Noetherian and positively graded ring. Let  $\mathfrak{b} \subseteq R_0$  be an ideal and let  $M = \bigoplus_{n \in \mathbb{Z}} M_n$  be a finitely generated graded  $R$ -module. Then, there is some  $r \in \mathbb{N}$  and some  $n_0 \in \mathbb{Z}$  such that for all  $s = 0, 1, \dots, r-1$  and all  $n \geq n_0$  it holds  $\text{grade}_{M_{rn+s}}(\mathfrak{b}) = \text{grade}_{M_{rn_0+s}}(\mathfrak{b})$*

*Proof.* According to Notation and Reminder 6.6 (C) we find some  $r \in \mathbb{N}$  such that  $R^{(r)}$  is homogeneous. By Notation and Reminder 6.6 (B) the graded  $R^{(r)}$ -modules  $M^{(r,s)} := \bigoplus_{n \in \mathbb{Z}} M_{rn+s}$  ( $s = 0, 1, \dots, r-1$ ) are finitely generated. Now, our claim follows easily on application of Proposition 6.5.  $\square$

## 7. IDEAL TRANSFORMS OF REES RINGS AND REES MODULES

**Notation and Remark 7.1.** (A) Let  $R$  be a Noetherian ring, let  $\mathfrak{a} \subseteq R$  be an ideal, let  $t$  be an indeterminate and let  $M$  be an  $R$ -module. The *Rees ring* of  $R$  with respect to  $\mathfrak{a}$  is defined as the  $\mathbb{Z}$ -graded subring of  $R[t, t^{-1}]$  given by

$$\mathcal{R}'(\mathfrak{a}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{R}'(\mathfrak{a})_n := R[\mathfrak{a}t, t^{-1}], \text{ with } \mathcal{R}'(\mathfrak{a})_n = \mathfrak{a}^{\max\{0, n\}} t^n \text{ for all } n \in \mathbb{Z}.$$

The *truncated Rees ring* of  $R$  with respect to  $\mathfrak{a}$  is defined as the  $\mathbb{N}_0$ -graded subring of  $R[t]$  given by

$$\mathcal{R}(\mathfrak{a}) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{R}(\mathfrak{a})_n := R[\mathfrak{a}t], \text{ with } \mathcal{R}(\mathfrak{a})_n = \mathfrak{a}^n t^n \text{ for all } n \in \mathbb{N}_0.$$

Observe that  $\mathcal{R}(\mathfrak{a})$  is a positively graded homogeneous Noetherian ring, that  $\mathcal{R}(\mathfrak{a})$  is a graded subring of  $\mathcal{R}'(\mathfrak{a})$  and that – considering  $\mathcal{R}'(\mathfrak{a})$  as a graded  $\mathcal{R}(\mathfrak{a})$ -module – we have  $\mathcal{R}(\mathfrak{a}) = \mathcal{R}'(\mathfrak{a})_{\geq 0}$ .

(B) We consider the Noetherian  $\mathbb{Z}$ -graded ring  $R[t, t^{-1}]$  and the  $\mathbb{Z}$ -graded  $R[t, t^{-1}]$ -module  $M[t, t^{-1}] := M \otimes_R R[t, t^{-1}]$ . Moreover, we identify  $M = M \otimes_R R = M \otimes_R R[t, t^{-1}]_0 = (M[t, t^{-1}t])_0$ , so that

$$M[t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} M[t, t^{-1}]_n, \text{ with } M[t, t^{-1}]_n = Mt^n \text{ for all } n \in \mathbb{Z}.$$

We now consider  $M[t, t^{-1}]$  as graded  $\mathcal{R}'(\mathfrak{a})$ -module and define the *Rees module* of  $M$  with respect to  $\mathfrak{a}$  as the graded  $\mathcal{R}'(\mathfrak{a})$ -module of  $M[t, t^{-1}]$  given by

$$\mathcal{R}'_M(\mathfrak{a}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{R}'_M(\mathfrak{a})_n := M\mathcal{R}'(\mathfrak{a}), \text{ with } \mathcal{R}'_M(\mathfrak{a})_n = \mathfrak{a}^{\max\{0, n\}}Mt^n \text{ for all } n \in \mathbb{Z}.$$

Accordingly, we define the *truncated Rees module* of  $M$  with respect to  $\mathfrak{a}$  as the graded  $\mathcal{R}(\mathfrak{a})$ -sub-module of  $M[t, t^{-1}]$  given by

$$\mathcal{R}_M(\mathfrak{a}) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{R}_M(\mathfrak{a})_n, \text{ with } \mathcal{R}_M(\mathfrak{a})_n = \mathfrak{a}^n Mt^n \text{ for all } n \in \mathbb{N}_0.$$

Observe that  $\mathcal{R}'_M(\mathfrak{a})$  and  $\mathcal{R}_M(\mathfrak{a})$  are generated over  $\mathcal{R}'(\mathfrak{a})$  respectively over  $\mathcal{R}(\mathfrak{a})$  by finitely many elements of degree 0 if  $M$  is finitely generated over  $R$ . Moreover  $\mathcal{R}_M(\mathfrak{a})$  is a graded  $\mathcal{R}(\mathfrak{a})$ -submodule of  $\mathcal{R}'_M(\mathfrak{a})$  and under this aspect we have  $\mathcal{R}_M(\mathfrak{a}) = \mathcal{R}'_M(\mathfrak{a})_{\geq 0}$ .

(C) Let  $N \subset M$  be a submodule and observe that  $N[t, t^{-1}]$  is a graded  $R[t, t^{-1}]$ -submodule of  $M[t, t^{-1}]$  and hence carries natural structures of graded  $\mathcal{R}(\mathfrak{a})$ - and  $\mathcal{R}'(\mathfrak{a})$ -submodules. So we may define the *strict transform* of  $N$  in  $M$  with respect to  $\mathfrak{a}$  as the graded  $\mathcal{R}'(\mathfrak{a})$ -submodule of  $\mathcal{R}'_M(\mathfrak{a})$  given by

$$\mathcal{S}'_M^N(\mathfrak{a}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{S}'_M^N(\mathfrak{a})_n := \mathcal{R}'_M(\mathfrak{a}) \cap N[t, t^{-1}], \text{ with } \mathcal{S}'_M^N(\mathfrak{a})_n = (N \cap \mathfrak{a}^{\max\{0, n\}}M)t^n, (\forall n \in \mathbb{Z}).$$

Correspondingly, the *truncated strict transform* of  $N$  in  $M$  with respect to  $\mathfrak{a}$  is defined of the graded  $\mathcal{R}(\mathfrak{a})$ -submodule of  $\mathcal{R}_M(\mathfrak{a})$  given by

$$\mathcal{S}_M^N(\mathfrak{a}) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{S}_M^N(\mathfrak{a})_n := \mathcal{R}_M(\mathfrak{a}) \cap N[t], \text{ with } \mathcal{S}_M^N(\mathfrak{a})_n = (N \cap \mathfrak{a}^n M)t^n, (\forall n \in \mathbb{N}_0).$$

Clearly, we may consider  $\mathcal{S}_M^N(\mathfrak{a})$  as a graded  $\mathcal{R}(\mathfrak{a})$ -submodule of  $\mathcal{S}'_M^N(\mathfrak{a})$  and write  $\mathcal{S}_M^N(\mathfrak{a}) = \mathcal{S}'_M^N(\mathfrak{a})_{n \geq 0}$ . Observe that we have the canonical isomorphisms of graded  $\mathcal{R}'(\mathfrak{a})$  respectively  $\mathcal{R}(\mathfrak{a})$ -modules

$$\mathcal{R}'_{M/N}(\mathfrak{a}) \cong \mathcal{R}'_M(\mathfrak{a})/\mathcal{S}'_M^N(\mathfrak{a}) \quad \text{and} \quad \mathcal{R}_M(\mathfrak{a}) \cong \mathcal{R}(\mathfrak{a})/\mathcal{S}_M^N(\mathfrak{a}).$$

(D) Observe that we have the canonical isomorphisms of graded  $\mathcal{R}(\mathfrak{a})$ -modules

$$\mathcal{R}'(\mathfrak{a})/t^{-1}\mathcal{R}'(\mathfrak{a}) \cong \mathcal{R}(\mathfrak{a})/\mathfrak{a}\mathcal{R}(\mathfrak{a}) \cong G_M(\mathfrak{a}),$$

where

$$G_M(\mathfrak{a}) = \bigoplus_{n \in \mathbb{N}_0} G_M(\mathfrak{a})_n, \text{ with } G_M(\mathfrak{a})_n = \mathfrak{a}^n/\mathfrak{a}^{n+1} \text{ for all } n \in \mathbb{N}_0$$

is the *associated graded module* of  $M$  with respect to  $\mathfrak{a}$ .

**Remark 7.2.** (A) Keep the previous notations and hypotheses. We always consider  $\mathcal{R}'_M(\mathfrak{a})$  as a graded  $\mathcal{R}(\mathfrak{a})$ -module. Observe that

$$\Gamma_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a})) = \mathcal{S}_M^{\Gamma_{\mathfrak{a}}(M)}(\mathfrak{a}) \quad \text{and} \quad \Gamma_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a})) = \mathcal{S}_M^{\Gamma_{\mathfrak{a}}(M)}(\mathfrak{a}).$$

In particular we have

$$\Gamma_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a}))_n = (\Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0,n\}}M)t^n \text{ for all } n \in \mathbb{Z}$$

and

$$\Gamma_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n = (\Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^n M)t^n \text{ for all } n \in \mathbb{N}_0.$$

Moreover, the canonical isomorphisms of 7.1 (C) give rise to canonical homomorphisms of graded  $\mathcal{R}(\mathfrak{a})$ -modules

$$\mathcal{R}'_{M/\Gamma_{\mathfrak{a}}(M)}(\mathfrak{a}) \cong \mathcal{R}'_M(\mathfrak{a})/\Gamma_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a})) \quad \text{and} \quad \mathcal{R}_{M/\Gamma_{\mathfrak{a}}(M)}(\mathfrak{a}) \cong \mathcal{R}_M(\mathfrak{a})/\Gamma_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a})).$$

As a consequence we get canonical isomorphisms of graded  $\mathcal{R}(\mathfrak{a})$ -modules

$$D_{\mathcal{R}(\mathfrak{a})}(\mathcal{R}'_{M/\Gamma_{\mathfrak{a}}(M)}(\mathfrak{a})) \cong D_{\mathcal{R}(\mathfrak{a})}(\mathcal{R}'_M(\mathfrak{a})) \quad \text{and} \quad D_{\mathcal{R}(\mathfrak{a})}(\mathcal{R}_{M/\Gamma_{\mathfrak{a}}(M)}(\mathfrak{a})) \cong D_{\mathcal{R}(\mathfrak{a})}(\mathcal{R}_M(\mathfrak{a})).$$

(B) Observe, that in view of the observations made in part (A), the canonical exact sequence of graded  $\mathcal{R}(\mathfrak{a})$ -modules

$$0 \longrightarrow \mathcal{R}_M(\mathfrak{a}) \longrightarrow \mathcal{R}'_M(\mathfrak{a}) \longrightarrow \mathcal{R}'_M(\mathfrak{a})/\mathcal{R}_M(\mathfrak{a}) \longrightarrow 0$$

gives rise to exact sequences of  $R$ -modules

$$0 \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n \longrightarrow H^1_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a}))_n \longrightarrow 0 \text{ for all } n < 0,$$

to isomorphisms of  $R$ -modules

$$H^1_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n \cong H^1_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a}))_n \text{ for all } n \geq 0$$

and to isomorphisms of graded  $\mathcal{R}(\mathfrak{a})$ -modules

$$D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a})) \cong D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a})) \text{ and} \\ H^i_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a})) \cong H^i_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a})) \text{ for all } i > 1.$$

**Notation and Remark 7.3.** (A) Keep the above notations and hypotheses. For all  $n \in \mathbb{Z} \cup \{\infty\}$  and all  $m \in \mathbb{N}_0$  Consider the  $R$ -module

$$\mathcal{U}_M(\mathfrak{a})_n^m := \text{Hom}_R(\mathfrak{a}^m, \mathfrak{a}^{\max\{0,m+n\}}M).$$

Fix  $n \in \mathbb{Z} \cup \{-\infty\}$ , let  $m, m' \in \mathbb{N}_0$  with  $m \leq m'$  and let  $h \in \mathcal{U}_M(\mathfrak{a})_n^m$ . Then,  $h(\mathfrak{a}^m) \subseteq \mathfrak{a}^{\max\{0,m+n\}}M$  together with the inequality  $m' - m + \max\{0, m+n\} \geq \max\{0, m'+n\}$  implies that  $h(\mathfrak{a}^{m'}) \subseteq \mathfrak{a}^{\max\{0,m'+n\}}M$ . Hence, by restriction and coadstriction we get an induced homomorphism of  $R$ -modules

$$\iota_n^{m,m'} : \mathcal{U}_M(\mathfrak{a})_n^m \longrightarrow \mathcal{U}_M(\mathfrak{a})_n^{m'}.$$

The family of homomorphisms of  $R$ -modules

$$(\iota_n^{m,m'} : \mathcal{U}_M(\mathfrak{a})_n^m \longrightarrow \mathcal{U}_M(\mathfrak{a})_n^{m'})_{m,m' \in \mathbb{N}_0, m \leq m'}$$

is a direct system, and so we may form the direct limit

$$\mathcal{U}_M(\mathfrak{a})_n := \lim_{m \in \mathbb{N}_0} \mathcal{U}_M(\mathfrak{a})_n^m,$$

together with the induced homomorphisms

$$\iota_n^m : \mathcal{U}_M(\mathfrak{a})_n^m \longrightarrow \mathcal{U}_M(\mathfrak{a})_n \text{ for all } m \in \mathbb{N}_0.$$

Observe that

$$\mathcal{U}_M(\mathfrak{a})_{-\infty} = D_{\mathfrak{a}}(M).$$

Now, fix  $m \in \mathbb{N}_0$  and let  $n, n' \in \mathbb{Z} \cup \{-\infty\}$  with  $n \leq n'$ . Then, the inclusion  $\mathfrak{a}^{\max\{0, m+n'\}} M \subseteq \mathfrak{a}^{\max\{0, m+n\}} M$  gives rise to a monomorphism of  $R$ -modules

$$\varepsilon_{n, n'}^m : \mathcal{U}_M(\mathfrak{a})_{n'}^m \longrightarrow \mathcal{U}_M(\mathfrak{a})_n^m.$$

As the family  $(\varepsilon_{n, n'}^m : \mathcal{U}_M(\mathfrak{a})_{n'}^m \longrightarrow \mathcal{U}_M(\mathfrak{a})_n^m)_{m \in \mathbb{N}_0}$  is a homomorphism of direct systems, we get an induced monomorphism

$$\varepsilon_{n, n'} : \mathcal{U}_M(\mathfrak{a})_{n'} \longrightarrow \mathcal{U}_M(\mathfrak{a})_n.$$

In particular, for each  $n \in \mathbb{Z}$  we have the monomorphism

$$\varepsilon_n := \varepsilon_{-\infty, n} : \mathcal{U}_M(\mathfrak{a})_n \longrightarrow \mathcal{U}_M(\mathfrak{a})_{-\infty} = D_{\mathfrak{a}}(M).$$

If we view  $\mathcal{U}_M(\mathfrak{a})_n$  as a submodule of  $D_{\mathfrak{a}}(M)$  by means of  $\varepsilon_n$ , we may write

$$\mathcal{U}_M(\mathfrak{a})_{n'} \subseteq \mathcal{U}_M(\mathfrak{a})_n \subseteq D_{\mathfrak{a}}(M) \text{ for all } n, n' \in \mathbb{Z} \cup \{-\infty\} \text{ with } n \leq n'.$$

(B) Keep the above notations and hypotheses. Let  $\emptyset \neq S \subseteq \text{NZD}_R(M)$  be multiplicatively closed such that  $S \cap \mathfrak{a} \neq \emptyset$ . Let  $n \in \mathbb{Z} \cup \{\infty\}$ , let  $m \in \mathbb{N}_0$  and let  $h \in \mathcal{U}_M(\mathfrak{a})_n^m := \text{Hom}_R(\mathfrak{a}^m, \mathfrak{a}^{\max\{0, m+n\}} M)$ . We consider  $\mathfrak{a}^{\max\{0, m+n\}} M$  as a submodule of  $S^{-1}M$ . Then, for each  $s \in S \cap \mathfrak{a}$  the element  $\alpha_n^m(h) := \frac{h(s^m)}{s^m} \in S^{-1}M$  is independent of our choice of  $s \in S \cap \mathfrak{a}$  and for each  $x \in \mathfrak{a}^m$  we have

$$x\alpha_n^m(h) = \frac{xh(s^m)}{s^m} = h(x) \in \mathfrak{a}^{\max\{0, m+n\}} M, \text{ so that } \alpha_n^m(h) \in (\mathfrak{a}^{\max\{0, m+n\}} M :_{S^{-1}M} \mathfrak{a}^m)$$

and the assignment  $h \mapsto \alpha_n^m(h)$  defines a homomorphism of  $R$ -modules  $\alpha_n^m : \mathcal{U}_M(\mathfrak{a})_n^m \longrightarrow (\mathfrak{a}^{\max\{0, m+n\}} M :_{S^{-1}M} \mathfrak{a}^m)$

Conversely, for each  $y \in (\mathfrak{a}^{\max\{0, m+n\}} M :_{S^{-1}M} \mathfrak{a}^m)$ , and each  $x \in \mathfrak{a}$  we have  $xy \in \mathfrak{a}^{\max\{0, m+n\}} M$ . So the assignment  $x \mapsto xy$  defines a homomorphism of  $R$ -modules  $\beta_n^m(y) \in \mathcal{U}_M(\mathfrak{a})_n^m$  and finally the assignment  $y \mapsto \beta_n^m(y)$  defines a homomorphism  $(\mathfrak{a}^{\max\{0, m+n\}} M :_{S^{-1}M} \mathfrak{a}^m) \longrightarrow \mathcal{U}_M(\mathfrak{a})_n^m$ . So, we have the isomorphisms

$$\alpha_n^m : \mathcal{U}_M(\mathfrak{a})_n^m \longrightarrow (\mathfrak{a}^{\max\{0, m+n\}} M :_{S^{-1}M} \mathfrak{a}^m) \text{ for all } n \in \mathbb{Z} \cap \{-\infty\} \text{ and all } m \in \mathbb{N}_0,$$

which allow to identify

$$\mathcal{U}_M(\mathfrak{a})_n^m = (\mathfrak{a}^{\max\{0, m+n\}} M :_{S^{-1}M} \mathfrak{a}^m) \text{ for all } n \in \mathbb{Z} \cap \{-\infty\} \text{ and all } m \in \mathbb{N}_0.$$

If we do so, the homomorphisms  $\iota_n^{m, m'}$  of part (A) become the inclusion maps and so we may write now

$$\mathcal{U}_M(\mathfrak{a})_n^m \subseteq \mathcal{U}_M(\mathfrak{a})_n^{m'} \text{ for all } n \in \mathbb{Z} \cup \{-\infty\} \text{ and all } m, m' \in \mathbb{N}_0 \text{ with } m \leq m'.$$

As the family of isomorphisms  $[\alpha_n^m : \mathcal{U}_M(\mathfrak{a})_n^m \longrightarrow (\mathfrak{a}^{\max\{0, m+n\}}M :_{S^{-1}M} \mathfrak{a}^m)]_{m \in \mathbb{N}_0}$  is an isomorphism of direct systems, we get an induced isomorphism of  $R$ -modules

$$\alpha_n : \mathcal{U}_M(\mathfrak{a})_n \xrightarrow{\cong} \bigcup_{m \in \mathbb{N}_0} (\mathfrak{a}^{\max\{0, m+n\}}M :_{S^{-1}M} \mathfrak{a}^m) \text{ for all } n \in \mathbb{Z} \cup \{-\infty\},$$

which allows to identify

$$\mathcal{U}_M(\mathfrak{a})_n = \bigcup_{m \in \mathbb{N}_0} (\mathfrak{a}^{\max\{0, m+n\}}M :_{S^{-1}M} \mathfrak{a}^m) \text{ for all } n \in \mathbb{Z} \cup \{-\infty\}.$$

(C) Keep the above notations and hypotheses. We consider  $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$  as a submodule of  $D_{\mathfrak{a}}(M) = D_{\mathfrak{a}}(\overline{M})$ . Then, for all  $n \in \mathbb{Z} \cup \{-\infty\}$  and all  $m \in \mathbb{N}_0$  we have  $\mathfrak{a}^{\max\{0, m+n\}}\overline{M} \subseteq D_{\mathfrak{a}}(M)$ . Observe, that for all  $n$  and  $n$  as aboe, we have a canonical short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0, m+n\}}M \longrightarrow \mathfrak{a}^{\max\{0, m+n\}}M \longrightarrow \mathfrak{a}^{\max\{0, m+n\}}\overline{M} \longrightarrow 0,$$

and hence an induced exact sequence

$$(\textcircled{a}) \quad \begin{cases} 0 & \longrightarrow \text{Hom}_R(\mathfrak{a}^m, \mathfrak{a}^{\max\{0, m+n\}}\overline{M}) \xrightarrow{\pi_n^m} \text{Hom}_R(\mathfrak{a}^m, \mathfrak{a}^{\max\{0, m+n\}}M) \longrightarrow \\ & \longrightarrow \text{Hom}_R(\mathfrak{a}^m, \Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0, m+n\}}M). \end{cases}$$

Now, let  $h \in \text{Hom}_R(\mathfrak{a}^m, \Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0, m+n\}}M)$ . Then  $h(\mathfrak{a}^m)$  is a finitely generated  $R$ -submodule of  $\Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0, m+n\}}M$ . So, there is some  $m' \in \mathbb{N}_0$  such that  $m \leq m'$  and  $\mathfrak{a}^{m'-m}h(\mathfrak{a}^m) = 0$ , so that the restriction map  $h \upharpoonright \in \text{Hom}_R(\mathfrak{a}^{m'}, \Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0, m'+n\}}M)$  vanishes. Therefore  $\lim_{\overrightarrow{m}} \text{Hom}_R(\mathfrak{a}^m, \Gamma_{\mathfrak{a}}(M) \cap \mathfrak{a}^{\max\{0, m+n\}}M) = 0$  and hence the above short exact sequences  $(\textcircled{a})$  induce an isomorphism of  $R$ -modules

$$\pi_n : \mathcal{U}_{\overline{M}}(\mathfrak{a})_n \xrightarrow{\cong} \mathcal{U}_M(\mathfrak{a})_n \text{ for all } n \in \mathbb{Z} \cup \{-\infty\},$$

which allows us to identify

$$(\textcircled{a}\textcircled{a}) \quad \mathcal{U}_{\overline{M}}(\mathfrak{a})_n = \mathcal{U}_M(\mathfrak{a})_n \text{ for all } n \in \mathbb{Z} \cup \{-\infty\}.$$

(D) Keep the previous notations and hypotheses. Assume that  $M$  is finitely generated with  $\Gamma_{\mathfrak{a}}(M) = 0$ . Then  $\text{NZD}_R(M) \cap \mathfrak{a} \neq \emptyset$  and  $M \subseteq D_{\mathfrak{a}}(M) \subseteq \text{NZD}_R(M)^{-1}M$ . If we view  $\mathcal{U}_M(\mathfrak{a})_n$  as a submodule of  $D_{\mathfrak{a}}(M)$  by means of the monomorphism  $\varepsilon_n : \mathcal{U}_M(\mathfrak{a})_n \longrightarrow D_{\mathfrak{a}}(M)$  of part (A) and apply the isomorphism  $\alpha_n$  of part (B) with  $S = \text{NZD}_R(M)$  we thus can say:

- (\textcircled{a}) If  $M$  is finitely generated with  $\Gamma_{\mathfrak{a}}(M) = 0$  and  $n \in \mathbb{Z} \cup \{-\infty\}$ , then it holds:
- (a)  $\mathcal{U}_M(\mathfrak{a})_n^m = (\mathfrak{a}^{\max\{0, m+n\}}M :_{D_{\mathfrak{a}}(M)} \mathfrak{a}^m)$  for all  $m \in \mathbb{N}_0$ ;
  - (b)  $\mathcal{U}_M(\mathfrak{a})_n^m \subseteq \mathcal{U}_M(\mathfrak{a})_n^{m'}$  for all  $m, m' \in \mathbb{N}_0$  with  $m \leq m'$ ;
  - (c)  $\mathcal{U}_M(\mathfrak{a})_n = \bigcup_{m \in \mathbb{N}_0} \mathcal{U}_M(\mathfrak{a})_n^m$ ;
  - (d)  $\bigcup_{n \in \mathbb{Z}} \mathcal{U}_M(\mathfrak{a})_n = D_{\mathfrak{a}}(M)$ .

Passing to direct limits one can avoid in the previous statements the hypothesis that  $M$  is finitely generated.

**Remark 7.4.** (A) Let  $\mathcal{R} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{R}_n$  be a homogeneous Noetherian ring with base ring  $\mathcal{R}_0 = R$ . Let  $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$  be a graded  $\mathcal{R}$ -module and let  $\mathcal{S} \subseteq \text{NZD}_{\mathcal{R}}(\mathcal{M}) \cap (\bigcup_{n \in \mathbb{N}_0} \mathcal{R}_n)$  be multiplicatively closed and such that  $\mathcal{S} \cap \mathcal{R}_+ \neq \emptyset$ . Then (see )

$$D_{\mathcal{R}_+}(\mathcal{M}) = \bigcup_{m \in \mathbb{N}_0} (\mathcal{M} :_{\mathcal{S}^{-1}\mathcal{M}} (\mathcal{R}_+)^m) = \bigcup_{m \in \mathbb{N}_0} (\mathcal{M} :_{\mathcal{S}^{-1}\mathcal{M}} \mathcal{R}_m)$$

is a graded  $\mathcal{R}$ -submodule of  $\mathcal{S}^{-1}\mathcal{M}$  such that

$$D_{\mathcal{R}_+}(\mathcal{M})_n = \bigcup_{m \in \mathbb{N}_0} (\mathcal{M}_{m+n} :_{(\mathcal{S}^{-1}\mathcal{M})_n} \mathcal{R}_m) \text{ for all } n \in \mathbb{Z}.$$

Keep in mind the following observation:

If  $\mathcal{M}$  is finitely generated and  $n \in \mathbb{Z}$ , the  $R$ -module  $D_{\mathcal{R}_+}(\mathcal{M})_n$  is finitely generated.

(B) Now, let  $R$  be a Noetherian ring, let  $\mathfrak{a} \subseteq R$  be an ideal, let  $M$  be an  $R$ -module and let  $s \in \text{NZD}_R(M) \cap \mathfrak{a}$ . Set

$$\mathcal{R} := \mathcal{R}(\mathfrak{a}), \quad \mathcal{M} := \mathcal{R}'_M(\mathfrak{a}), \quad S := \{s^p \mid p \in \mathbb{N}_0\} \text{ and } \mathcal{S} := \{s^p t^n \mid p, n \in \mathbb{N}_0\} = \bigcup_{n \in \mathbb{N}_0} St^n.$$

Then, clearly  $\mathcal{S}$  is multiplicatively closed and  $\mathcal{S} \subseteq \text{NZD}_{\mathcal{R}}(\mathcal{M}) \cap (\bigcup_{n \in \mathbb{N}_0} \mathcal{R}_n)$  with  $\mathcal{S} \cap \mathcal{R}_+ \neq \emptyset$ . Now, obviously by Notation and Remark 7.3 (B) we have

$$(\mathcal{M}_{m+n} :_{(\mathcal{S}^{-1}\mathcal{M})_n} \mathcal{R}_m) = (\mathfrak{a}_{\max\{0, m+n\}} M t^{n+m} :_{\mathcal{S}^{-1} M t^n} \mathfrak{a}^m t^m) = \mathcal{U}(\mathfrak{a})_n^m t^n$$

and hence by part (A) and Remark 7.2 (B) it follows that

$$(\textcircled{a}) \quad D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n = D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a}))_n = \mathcal{U}(\mathfrak{a})_n t^n.$$

Let the notations and hypotheses be as above, assume in addition that  $M$  is finitely generated and set  $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$ . Then, there is some  $s \in \text{NZD}_R(\overline{M}) \cap \mathfrak{a}$ . So, if we apply what is said above to the graded  $R$ -module  $\overline{M}$  and observe Remark 7.2 (A) and Notation and Remark 7.3 (C), we see that the previous equalities hold, too.

Keep in mind, that from the above equalities and the last observation made in part (A) it follows

If  $M$  is finitely generated, then  $\mathcal{U}_M(\mathfrak{a})_n$  is finitely generated for all  $n \in \mathbb{Z}$ .

(C) Keep the notations and hypotheses of part (B) and consider the blowup morphism

$$\pi : X := \text{Proj}(\mathcal{R}(\mathfrak{a})) \longrightarrow \text{Spec}(R) := X_0$$

and the coherent sheaf of  $\mathcal{O}_X$ -modules

$$\mathcal{F} := \widetilde{\mathcal{R}_M(\mathfrak{a})}.$$

Then by statement (B)(\textcircled{a}) and by Reminder 1.6 (C) we can say:

$$\Gamma(X, \mathcal{F}(n)) = H^0(X, \mathcal{F}(n)) \cong \mathcal{U}_M(\mathfrak{a})_n \quad \text{and} \quad \pi_* \mathcal{F}(n) \cong (\mathcal{U}_M(\mathfrak{a})_n)^\sim.$$

**Lemma 7.5.** *Let  $R$  be a Noetherian ring, let  $\mathfrak{a} \subseteq R$  be an ideal and let  $M$  be a finitely generated  $R$ -module and let  $n \in \mathbb{Z}$ . Then, it holds*

$$\mathcal{U}_M(\mathfrak{a})_{n-1} = (\mathcal{U}_M(\mathfrak{a})_n :_{D_{\mathfrak{a}}(M)} \mathfrak{a}) \cong \text{Hom}_R(\mathfrak{a}, \mathcal{U}_M(\mathfrak{a})_n).$$

*Proof.* According to Notation and Remark 7.3 (C) we may assume that  $\Gamma_{\mathfrak{a}}(M) = 0$  so that  $M \subseteq D := D_{\mathfrak{a}}(M)$  and (see Notation and Remark 7.3 (D)(@))  $\mathcal{U}_M(\mathfrak{a})_k = \bigcup_{m \in \mathbb{N}_0} \mathcal{U}_M(\mathfrak{a})_k^m$  with  $\mathcal{U}_M(\mathfrak{a})_k^m \subseteq \mathcal{U}_M(\mathfrak{a})_k^{m'}$  and  $\mathcal{U}_M(\mathfrak{a})_k^m = (\mathfrak{a}^{\max\{m+k, 0\}} M :_D \mathfrak{a}^m)$  for all  $k \in \mathbb{Z}$  and all  $m, m' \in \mathbb{N}_0$  with  $m \leq m'$ . As the  $R$ -module  $\mathcal{U}_M(\mathfrak{a})_k$  is finitely generated for all  $k \in \mathbb{Z}$  (see Remark 7.4 (B)) we may conclude from Notation and Remark 7.3 (D)(@)(c) that there is some  $m \in \mathbb{N}_0$  with:

$$m + n - 1 \geq 0, \quad \mathcal{U}_M(\mathfrak{a})_{n-1} = \mathcal{U}_M(\mathfrak{a})_{n-1}^m \text{ and } \mathcal{U}_M(\mathfrak{a})_n = \mathcal{U}_M(\mathfrak{a})_n^{m-1}.$$

It follows:

$$\begin{aligned} \mathcal{U}_M(\mathfrak{a})_{n-1} &= \mathcal{U}_M(\mathfrak{a})_{n-1}^m = (\mathfrak{a}^{m+(n-1)} M :_D \mathfrak{a}^m) = \\ &= (\mathfrak{a}^{(m-1)+n} M :_D \mathfrak{a}^{(m-1)}) = ((\mathfrak{a}^{(m-1)+n} M :_D \mathfrak{a}^{(m-1)}) :_D \mathfrak{a}) = \\ &= (\mathcal{U}_M(\mathfrak{a})_n :_D \mathfrak{a}) = (\mathcal{U}_M(\mathfrak{a})_n :_{D_{\mathfrak{a}}(M)} \mathfrak{a}). \end{aligned}$$

Finally observe that with  $S := \text{NZD}_R(M)$  we have  $S \cap \mathfrak{a} \neq 0$ , and  $\mathcal{U}_M(\mathfrak{a})_n \subseteq D_{\mathfrak{a}}(M) = (D_{\mathfrak{a}}(M) :_{S^{-1}M} \mathfrak{a})$ , hence  $(\mathcal{U}_M(\mathfrak{a})_n :_{D_{\mathfrak{a}}(M)} \mathfrak{a}) = (\mathcal{U}_M(\mathfrak{a})_n :_{S^{-1}\mathcal{U}_M(\mathfrak{a})_n} \mathfrak{a})$ . Now, on application of the isomorphism  $\alpha_{n-1}^1$  of Remark 7.3 (C) to the  $R$ -module  $\mathcal{U}_M(\mathfrak{a})_n$  get  $\mathcal{U}_M(\mathfrak{a})_{n-1} \cong \text{Hom}_R(\mathfrak{a}, \mathcal{U}_M(\mathfrak{a})_n)$ .  $\square$

**Proposition 7.6.** *Let  $R$  be a Noetherian ring, let  $\mathfrak{a} \subseteq R$  be an ideal, let  $M$  be a finitely generated  $R$ -module and let  $n \in \mathbb{Z}$ . Set  $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$  and  $D := D_{\mathfrak{a}}(M)$ . Then, it holds:*

- (a) *The following statements are equivalent:*
  - (i)  $\mathcal{U}_M(\mathfrak{a})_{n-1} = \mathcal{U}_M(\mathfrak{a})_n$
  - (ii)  $\mathcal{U}_M(\mathfrak{a})_n = D$ .
  - (iii)  $\mathcal{U}_M(\mathfrak{a})_k = D$  for all  $k \in \mathbb{Z} \cup \{-\infty\}$  with  $k \leq n$ .
  - (iv)  $\mathcal{U}_M(\mathfrak{a})_k = \mathcal{U}_M(\mathfrak{a})_n$  for all  $k \in \mathbb{Z} \cup \{-\infty\}$  with  $k \leq n$ .
- (b) *If  $n, r \in \mathbb{Z}$  with  $n \leq r$ , then  $\mathcal{U}_M(\mathfrak{a})_n = (\mathcal{U}_M(\mathfrak{a})_{r+1} :_D \mathfrak{a}^{r-n+1})$ .*
- (c) *If  $n \in \mathbb{Z}$ , then  $\mathfrak{a}^{\max\{0, n\}} \overline{M} \subseteq \mathcal{U}_M(\mathfrak{a})_n$  with equality if and only if  $H_{\mathcal{R}(\mathfrak{a})_+}^1(\mathcal{R}'_M(\mathfrak{a}))_n = 0$ .*
- (d) *If  $r \geq \text{end}[H_{\mathcal{R}(\mathfrak{a})_+}^1(\mathcal{R}'_M(\mathfrak{a}))]$  and  $n \in \mathbb{Z}$  we have*
  - (1)  $\mathcal{U}_M(\mathfrak{a})_n = \begin{cases} \mathfrak{a}^{\max\{0, n\}} \overline{M} & \text{if } n > r \\ (\mathfrak{a}^{\max\{0, r+1\}} \overline{M} :_D \mathfrak{a}^{r-n+1}) = \mathcal{U}_M(\mathfrak{a})_n^{r-n+1} & \text{if } n \leq r \end{cases}$ .
  - (2) *If  $0 \leq n \leq r$ , then  $\mathfrak{a}^{r-n+1} \mathcal{U}_M(\mathfrak{a})_n = \mathfrak{a}^{r+1} \overline{M}$ .*
- (e) *If  $n, r \in \mathbb{Z}$  with  $n \leq r$  and  $H_{\mathcal{R}(\mathfrak{a})_+}^1(\mathcal{R}'_M(\mathfrak{a}))_{r+1} = 0$ , then:*
  - (1)  $H_{\mathcal{R}(\mathfrak{a})_+}^1(\mathcal{R}'_M(\mathfrak{a}))_n = 0$  if and only if  $\mathfrak{a}^{\max\{0, n\}} \overline{M} = (\mathfrak{a}^{\max\{r+1, 0\}} \overline{M} :_D \mathfrak{a}^{r-n+1})$ .
  - (2)  $\mathfrak{a}^{r-n+1} D \subseteq \mathfrak{a}^{\max\{r+1, 0\}} \overline{M}$  if and only if  $\mathcal{U}_M(\mathfrak{a})_n = D$ .
- (f)  $\mathcal{U}_M(\mathfrak{a})_n = D$  for all  $n \ll 0$  if and only if  $H_{\mathfrak{a}}^1(M)$  is finitely generated.



*Proof.* (a): "(i)  $\Rightarrow$  (ii)": Assume that (i) holds. Then Lemma 7.5 yields that  $\mathcal{U}_M(\mathfrak{a})_n = (\mathcal{U}_M(\mathfrak{a})_n :_D \mathfrak{a})$ . So, by induction we obtain that  $\mathcal{U}_M(\mathfrak{a})_n = (\mathcal{U}_M(\mathfrak{a})_n :_D \mathfrak{a}^k)$  for all  $k \in \mathbb{N}_0$ , whence

$$\mathcal{U}_M(\mathfrak{a})_n = \bigcup_{k \in \mathbb{N}_0} (\mathcal{U}_M(\mathfrak{a})_n :_D \mathfrak{a}^k) = D_{\mathfrak{a}}(\mathcal{U}_M(\mathfrak{a})_n) \subseteq D.$$

As  $\mathfrak{a}^{\max\{n,0\}}\overline{M} \subseteq \mathcal{U}_M(\mathfrak{a})$  we now obtain

$$D = D_{\mathfrak{a}}(\overline{M}) = D_{\mathfrak{a}}(\mathfrak{a}^{\max\{n,0\}}\overline{M}) \subseteq D_{\mathfrak{a}}(\mathcal{U}_M(\mathfrak{a})_n) = \mathcal{U}_M(\mathfrak{a})_n.$$

and hence  $\mathcal{U}_M(\mathfrak{a})_n = D$ .

"(ii)  $\Rightarrow$  (iii)": This is clear by Notation and Remark 7.3 (A), (D)(@)(d).

"(iii)  $\Rightarrow$  (iv)": This is clear by Notation and Remark 7.3 (A).

"(iv)  $\Rightarrow$  (i)": This is obvious.

(b): This follows immediately from Lemma 7.5 by induction on  $r - n$ .

(c): This is immediate by the isomorphisms of  $R$ -modules:

$$\mathcal{U}_M(\mathfrak{a})_n / \mathfrak{a}^{\max\{n,0\}}\overline{M} \cong D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}'_M(\mathfrak{a}))_n / \mathcal{R}'_M(\mathfrak{a})_n \cong H_{\mathcal{R}(\mathfrak{a})_+}^1(\mathcal{R}'_M(\mathfrak{a}))_n.$$

(d): Claim (1) follows immediately from statements (b) and (c) and keeping in mind the isomorphism  $\alpha_n^{r-m_1}$  given in Notation and Remark 7.3 (B). Let  $0 \leq n \leq r$ . Then, the inclusion  $\mathfrak{a}^{r-n+1}\mathcal{U}_M(\mathfrak{a})_n \subseteq \mathfrak{a}^{r+1}\overline{M}$  is immediate by claim (1). By the last observation of Notation and Remark 7.3 (B) – applied to the module  $\overline{M}$  – we have  $\mathfrak{a}^n\overline{M} \subseteq \mathcal{U}_M(\mathfrak{a})_n$ , and it follows  $\mathfrak{a}^{r+1}\overline{M} = \mathfrak{a}^{r-n+1}\mathfrak{a}^n\overline{M} \subseteq \mathfrak{a}^{r-n+1}\mathcal{U}_M(\mathfrak{a})_n$ .

(e): This is clear by statements (b) and (c).

(f): This is clear by statements (a) and (e)(2) as  $H_{\mathfrak{a}}^1(M) \cong D/\overline{M}$  is finitely generated if and only if it is annihilated by some power of  $\mathfrak{a}$ .  $\square$

**Remark 7.7.** (A) (See [Ma,2009]) Let  $R$  be a Noetherian ring, let  $\mathfrak{a}, \mathfrak{b} \subseteq R$  be ideals and let  $M$  be a finitely generated  $R$ -module. One of the questions which remained open in the cited master thesis is, whether  $\text{grade}_{D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n}(\mathfrak{b})$  is *asymptotically stable* for  $n \rightarrow -\infty$ .

More precisely

(Q1) Is there some  $n_0 \in \mathbb{Z}$  such that for all  $n \leq n_0$  it holds

$$\text{grade}_{D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n}(\mathfrak{b}) = \text{grade}_{D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_{n_0}}(\mathfrak{b})?$$

Indeed, it suffices to treat the special case in which  $(R, \mathfrak{m})$  is local and  $\mathfrak{b} = \mathfrak{m}$  (see [BaB,2009]). So, as  $\text{grade}_{\bullet}(\mathfrak{m}) =: \text{depth}(\bullet)$ , we may ask

(Q2) Assume that  $(R, \mathfrak{m})$  is local. Is there some  $n_0 \in \mathbb{Z}$  such that for all  $n \leq n_0$  it holds

$$\text{depth}[D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_n] = \text{depth}[D_{\mathcal{R}(\mathfrak{a})_+}(\mathcal{R}_M(\mathfrak{a}))_{n_0}]?$$

In view of Remark 7.4 (B)(@) we may reformulate the last question as follows:

(Q3) Assume that  $(R, \mathfrak{m})$  is local. Is there some  $n_0 \in \mathbb{Z}$  such that for all  $n \leq n_0$  it holds

$$\text{depth}(\mathcal{U}_M(\mathfrak{a})_n) = \text{depth}(\mathcal{U}_M(\mathfrak{a})_{n_0})?$$

(B) Keep the previous notations and hypotheses. Then, in view of Proposition 7.6 (f) we can say:

(PA) Assume that  $(R, \mathfrak{m})$  is local and that the  $R$ -module  $H_{\mathfrak{a}}^1(M)$  is finitely generated. Then, there is some  $n_0 \in \mathbb{Z}$  such that for all  $n \leq n_0$  it holds

$$\mathcal{U}_M(\mathfrak{a})_n = \mathcal{U}_M(\mathfrak{a})_{n_0} \text{ and hence } \text{depth}(\mathcal{U}_M(\mathfrak{a})_n) = \text{depth}(\mathcal{U}_M(\mathfrak{a})_{n_0}).$$

In the special case, in which  $H_{\mathfrak{a}}^1(M)$  is finitely generated (or – equivalently –  $D := D_{\mathfrak{a}}(M)$  is finitely generated), Question (Q3) of part (A) has a positive answer by (PA). But what about the case in which  $H_{\mathfrak{a}}^1(M)$  is not finitely generated? Here is a question which seems particularly challenging in this respect:

(Q3') Assume that  $(R, \mathfrak{m})$  is local and that the functor  $D_{\mathfrak{a}}(\bullet)$  is exact (or – equivalently – the quasi-affine scheme  $\text{Spec}(R) \setminus \text{Var}(\mathfrak{a})$  is affine). Is there some  $n_0 \in \mathbb{Z}$  such that for all  $n \leq n_0$  it holds

$$\text{depth}(\mathcal{U}_M(\mathfrak{a})_n) = \text{depth}(\mathcal{U}_M(\mathfrak{a})_{n_0})?$$

In the special case, in which  $\text{Spec}(R) \setminus \text{Var}(\mathfrak{a})$  is an elementary open affine subscheme of  $\text{Spec}(R)$  (hence  $\mathfrak{a}$  is radically principal) (Q3') and hence (Q3) clearly has an affirmative answer.

Finally note, that (Q3) finds an affirmative answer if either  $\dim(R) \leq 1$  or else if  $\dim(R) = 2$  and  $R$  is a finite integral extension of a domain or essentially of finite type over a field (see [BFT,2003], [BFLi,2004] and [BaB,2009]).

(C) In view of Proposition 7.6 (d)(1) and setting  $\overline{M} := M/\Gamma_{\mathfrak{a}}(M)$  and  $D := D_{\mathfrak{a}}(M)$  we may reformulate question (Q3) as follows:

(Q4) Assume that  $(R, \mathfrak{m})$  is local. Is there some  $r_0 \in \mathbb{N}_0$  such that for all  $r \geq r_0$  there is some  $n_r \in \mathbb{N}_0$  with the property that for all  $n \geq n_r$  it holds

$$\text{depth}(\mathfrak{a}^r \overline{M} :_D \mathfrak{a}^n) = \text{depth}(\mathfrak{a}^r \overline{M} :_D \mathfrak{a}^{n_r})?$$

Instead of this last question, one also could look at the following question:

(Q5) Assume that  $(R, \mathfrak{m})$  is local. Is there some  $n_0 \in \mathbb{N}_0$  such that for all  $n \geq n_0$  it holds

$$\text{depth}(\overline{M} :_D \mathfrak{a}^n) = \text{depth}(\overline{M} :_D \mathfrak{a}^{n_0})?$$

Clearly, if (Q5) has an affirmative answer for each finitely generated  $R$ -module  $M$ , then so has (Q4).

## 8. A FINITENESS CRITERION FOR IDEAL TRANSFORMS OF REES-ALGEBRAS

**Notation and Reminder 8.1.** (See Notation and Reminder 6.6 and [BSh,2013] 13.5.9) Let  $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} \mathcal{R}_n$  be a  $\mathbb{Z}$ -graded Noetherian Ring, let  $r \in \mathbb{N}_n$ , and consider  $r$ -th Veronese subring  $\mathcal{R}_{(r)} := \bigoplus_{n \in \mathbb{Z}} \mathcal{R}_{rn}$ . Let  $\mathfrak{a} \subseteq \mathcal{R}$  be a graded ideal and consider the graded ideal

$$\mathfrak{a}^{(r)} := \mathfrak{a} \cap \mathcal{R}^{(r)} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}_{rn} = \mathfrak{a}^{(r,0)} \subseteq \mathcal{R}, \text{ with } \sqrt{\mathfrak{a}^{(r)}\mathcal{R}} = \sqrt{\mathfrak{a}}.$$

In this situation we have natural isomorphism of graded  $\mathcal{R}^{(r)}$ -modules

$$H_{\mathfrak{a}^{(r)}}^i(\mathcal{M}^{(r,s)}) \cong H_{\mathfrak{a}}^i(\mathcal{M})^{(r,s)} \text{ and } D_{\mathfrak{a}^{(r)}}(\mathcal{M}^{(r,s)}) \cong D_{\mathfrak{a}}(\mathcal{M})^{(r,s)} \quad (\forall i \in \mathbb{N}_0, \forall s \in \mathbb{Z}).$$

So, we get an isomorphisms of graded  $\mathcal{R}^{(r)}$ -modules

$$H_{\mathfrak{a}^{(r)}}^i(\mathcal{M} \upharpoonright_{\mathcal{R}^{(r)}}) \cong H_{\mathfrak{a}}^i(\mathcal{M}) \upharpoonright_{\mathcal{R}^{(r)}} \text{ and } D_{\mathfrak{a}^{(r)}}(\mathcal{M} \upharpoonright_{\mathcal{R}^{(r)}}) \cong D_{\mathfrak{a}}(\mathcal{M}) \upharpoonright_{\mathcal{R}^{(r)}} \quad (\forall i \in \mathbb{N}_0).$$

**Lemma 8.2.** Let  $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} \mathcal{R}_n$  be a  $\mathbb{Z}$ -graded Noetherian Ring and let  $\mathfrak{a} \subseteq \mathcal{R}$  be a graded ideal. then, the following conditions are equivalent:

- (i)  $\mathfrak{a}D_{\mathfrak{a}}(\mathcal{R}) = D_{\mathfrak{a}}(\mathcal{R})$ .
- (ii) The  $\mathfrak{a}$ -transform functor  $D_{\mathfrak{a}} : \mathcal{C} \rightarrow \mathcal{C}$  is exact.
- (iii)  $H_{\mathfrak{a}}^i(\mathcal{R}) = 0$  for all  $i \geq 2$ .
- (iv)  $H_{\mathfrak{a}}^2(\mathcal{M}) = 0$  for each finitely generated graded  $\mathcal{R}$ -module  $\mathcal{M}$ .
- (v)  $H_{\mathfrak{a}}^2(\mathcal{M}) = 0$  for each  $\mathcal{R}$ -module  $\mathcal{M}$ .
- (vi)  $H_{\mathfrak{a}}^i(\mathcal{M}) = 0$  for each  $\mathcal{R}$ -module  $\mathcal{M}$  and all  $i \geq 2$ .
- (vii) The scheme  $U := \text{Spec}(\mathcal{R}) \setminus \text{Var}(\mathfrak{a})$  is affine.

Moreover, if these equivalent conditions hold, then  $D_{\mathfrak{a}}(\mathcal{R})$  is a finitely generated  $\mathcal{R}$ -algebra and  $D_{\mathfrak{a}}(\mathcal{M})$  is a finitely generated (graded)  $D_{\mathfrak{a}}(\mathcal{R})$ -module for each finitely generated (graded)  $\mathcal{R}$ -module  $\mathcal{M}$ .

*Proof.* The equivalence “(i)  $\Leftrightarrow$  (ii)” is given in [BSh,2013] 6.3.5. The equivalences “(ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v)  $\Leftrightarrow$  (ii)” are clear by [BSh,2013] 6.3.1.

For each  $\mathcal{R}$ -module  $\mathcal{M}$  and all  $i \geq 2$  we have  $H^{i-1}(U, \widetilde{\mathcal{M}}) \cong \mathcal{R}^i D_{\mathfrak{a}}(\mathcal{M}) \cong H_{\mathfrak{a}}^i(\mathcal{M})$  (see [BSh,2013] 2.2.6 and 20.3.11). So, the equivalence “(vi)  $\Leftrightarrow$  (vii)” is a consequence of the *Affineness Criterion of Serre* (see [Ha, 1977], III, Theorem 3.7).

Also the implication “(ii)  $\Rightarrow$  (vi)” is again a consequence of the natural equivalences  $\mathcal{R}^i D_{\mathfrak{a}}(\bullet) \cong H_{\mathfrak{a}}^{i+1}(\bullet) (\forall i \in \mathbb{N})$  (see [BSh,2013] 2.2.6). This proves that the conditions (i) - (vii) are equivalent.

Assume that these equivalent conditions hold. Then, indeed  $D_{\mathfrak{a}}(\mathcal{R})$  is a finitely generated graded  $\mathcal{R}$ -algebra by [BSh,2013] 6.3.4. Moreover the natural homomorphism of rings  $\eta_{\mathfrak{a}} :$

$\mathcal{R} \rightarrow D_a(\mathcal{R})$  respects gradings. Now, let  $\mathcal{M}$  be a finitely generated (graded)  $\mathcal{R}$ -module. Then, there is an epimorphism of (graded)  $\mathcal{R}$ -modules  $\mathcal{H} \twoheadrightarrow \mathcal{M}$ , where  $\mathcal{H}$  is (graded) free  $\mathcal{R}$ -module. As the functor  $D_a : \mathcal{C} \rightarrow \mathcal{C}$  is exact (and respects gradings) we thus get an epimorphism of (graded)  $\mathcal{R}$ -modules  $D_a(\mathcal{H}) \twoheadrightarrow D_a(\mathcal{M})$ . By the additivity of the functor  $D_a : \mathcal{C} \rightarrow \mathcal{C}$ , the module  $D_a(\mathcal{H})$  is a (graded) direct sum of finitely many copies of  $D_a(\mathcal{R})$ . Therefore  $D_a(\mathcal{M})$  is a finitely generated (graded)  $D_a(\mathcal{R})$ -module.  $\square$

## 9. BOUNDING ARITHMETIC GENERA OF CURVES

**Notation and Reminder 9.1.** (A) We keep the notations of the previous sections. Throughout this section, let  $R_0 := K$  be an algebraically closed field,

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n = K[R_1] \text{ be a 2-dimensional Noetherian homogeneous integral } K\text{-algebra.}$$

Moreover let

$$X := \text{Proj}(R) \subseteq \text{Proj}(\text{Sym}_K(R_1)) = \mathbb{P}_K^r$$

be the *projective curve* induced by  $R$ , where

$$r := \dim_K(R_1) - 1 \text{ is the } \textit{embedding dimension} \text{ of } X.$$

Moreover, let us consider

$$\text{the } \textit{degree } d := \deg_{\mathbb{P}_K^r}(X) \text{ and the } \textit{arithmetic genus } p := h^1(X, \mathcal{O}_X) \text{ of } X.$$

Finally consider the *first cohomological end* of  $X$ , which is defined by

$$e := \max(\{0\} \cup \{n \in \mathbb{Z} \mid H^1(X, \mathcal{O}_X(n)) \neq 0\}).$$

(B) Keep the notations and hypotheses of part (A). Keep in mind, that the *Hilbert polynomial* or *characteristic function* of  $X$  is given by

$$(a) \chi_X(t) = h^0(X, \mathcal{O}_X(t)) - h^1(X, \mathcal{O}_X(t)) = dt - p + 1 \text{ for all } t \in \mathbb{Z}.$$

In particular, we have

$$(b) h^0(X, \mathcal{O}_X(t)) = dt - p + 1 \text{ for all } t > e.$$

Moreover, for all  $l \in \mathbb{R}_1 \setminus \{0\}$  and all  $t \in \mathbb{Z}$  we have

$$\text{an injection } 0 \rightarrow H^0(X, \mathcal{O}_X(t)) \xrightarrow{l} H^0(X, \mathcal{O}_X(t+1))$$

$$\text{and a surjection } H^1(X, \mathcal{O}_X(t)) \xrightarrow{l} H^1(X, \mathcal{O}_X(t+1)) \rightarrow 0.$$

Therefore, by [B,1987,2] we have

$$(c) h^0(X, \mathcal{O}_X(t+1)) \geq h^0(X, \mathcal{O}_X(t)) + r \text{ for all } t \in \mathbb{N}_0, \text{ and}$$

$$(d) h^1(X, \mathcal{O}_X(t+1)) \leq \max\{0, h^1(X, \mathcal{O}_X(t)) - r\} \text{ for all } t \in \mathbb{Z}.$$

**Proposition 9.2.** *Let the notations and hypotheses be as in Notation and Reminder 9.1. Then it holds*

$$\min\{1, e\} + er \leq p \leq (e + 1)(d - r).$$

*Proof.* By Notation and Reminder 9.1 (B)(b),(c) and as  $h^0(X, \mathcal{O}_X(0)) = 1$ , we get

$$1 + (e + 1)r \leq h^0(X, \mathcal{O}_X(e + 1)) = d(e + 1) - p + 1.$$

If  $e > 0$  it follows from Notation and Reminder 9.1 (B)(b), that

$$1 + er \leq h^1(X, \mathcal{O}_X) = p.$$

Therefore we get indeed indeed

$$\min\{1, e\} + er \leq p \leq d(e + 1) - (e + 1)r = (e + 1)(d - r).$$

□

**Corollary 9.3.** *Let the notations and hypotheses be as in Notation and Reminder 9.1. Then it holds*

$$p \leq (d - r)^2.$$

*Proof.* According to the *regularity bound* of Gruson-Lazarsfeld-Peskine (see [GruLaP,1983], Theorem 1.1), the Castelnuovo-Mumford regularity of  $X$  satisfies the inequality

$$\text{reg}(R) + 1 = \text{reg}(\text{Ker}(\text{Sym}_K(R_1) \xrightarrow{\text{can}} R)) =: \text{reg}(X) \leq d - r + 2.$$

(See Definition 1.5 (C) for the notion of Castelnuovo-Mumford regularity of a finitely generated graded  $R$ -module.) As  $H_{R_+}^1(R)_n = H^1(X, \mathcal{O}_X(n))$  for all  $n \in \mathbb{Z}$ , it follows in particular, that

$$e = \max(\{0\} \cup \max\{n \in \mathbb{N} \mid H_{R_+}^2(R)_n \neq 0\}) \leq \text{reg}(R) - 2 \leq d - r - 1.$$

Now, we conclude by Proposition 9.2. □

**Remark 9.4.** Applying the previous estimate to a curve  $X \subset \mathbb{P}_K^3$  in three-space, we get  $p \leq (d - 3)^2$ . In the case, where  $X$  is smooth, Castelnuovo's bound for the genus of a smooth space curve indeed yields the sharper bound (see [C,1893] and [H,1977],IV,Theorem 6.4)

$$p \leq \frac{1}{4}(d^2 - \frac{1}{2}(1 - (-1)^d)) - d + 1.$$

10. SECTIONALLY RATIONAL SURFACES,  $S_2$ -COVERS AND LOCAL COHOMOLOGY

In this section, we aim to preserve an argument given [BLePS, 2013], not published in the final version [BLePS, 2017]. The argument hints a use of local cohomology in order to prove a structural result on certain projective surfaces. Throughout this section, let  $r \in \mathbb{N}_{\geq 4}$ , let  $K$  be an algebraically closed field, let  $S := K[x_0, x_1, \dots, x_r]$  be a polynomial ring endowed with its standard grading, let  $\mathbb{P}^r := \text{Proj}(S)$  be the projective  $r$ -space over  $K$ , and let  $X \subset \mathbb{P}^r$  be a non-degenerate irreducible projective surface of degree  $d \geq r$  with homogeneous vanishing ideal  $I \subset S$  and let  $A := S/I$  denote the homogeneous coordinate ring of  $X$ .

**Convention and Notation 10.1.** (A) Let  $r' \geq r$  and let  $X' \subset \mathbb{P}^{r'}$  be an irreducible, non-degenerate surface. If  $\Lambda = \mathbb{P}^{r'-r-1} \subset \mathbb{P}^{r'}$  is a linear subspace, such that  $X' \cap \Lambda = \emptyset$  and  $\pi'_\Lambda : \mathbb{P}^{r'} \setminus \Lambda \rightarrow \mathbb{P}^r$  is a *linear projection with center  $\Lambda$* , we write  $X'_\Lambda := \pi'_\Lambda(X')$  for the *projected image*  $\pi'_\Lambda(X') \subset \mathbb{P}^r$  of  $X'$  and  $\pi_\Lambda : X' \rightarrow X'_\Lambda$  for the finite morphism induced by the projection  $\pi'_\Lambda$ . If  $X = X'_\Lambda$ , we say that  $X$  is a *projection of  $X'$  (from the center  $\Lambda$ )* and call  $X' \subset \mathbb{P}^{r'}$  a *projecting surface of  $X$* .

(B) The *singular locus*  $\text{Sing}(\pi)$  of  $\pi$  is defined as the least closed subset  $Z \subseteq X$  such that the induced morphism  $\pi \upharpoonright : X' \setminus \pi^{-1}(Z) \rightarrow X \setminus Z$  is an isomorphism. We say that  $\pi$  is *almost non-singular* if  $\text{Sing}(\pi)$  is finite. If this is the case, we say that  $X$  is an *almost non-singular projection of  $X'$  (from the center  $\Lambda$ )*.

Next, we define the basic geometric concept of this section.

**Definition 10.2.** The surface  $X \subset \mathbb{P}^r$  is said to be *sectionally rational* if  $X \cap \mathbb{P}^{r-1} \subset \mathbb{P}^{r-1}$  is a (possibly singular) rational curve for a general hyperplane  $\mathbb{P}^{r-1} \in (\mathbb{P}^r)^* := \mathbb{G}(r-1, \mathbb{P}^r)$ .

**Notation and Reminder 10.3.** (A) (See [BS, 2012]) Let  $\mathfrak{a} \subseteq A_+ = S_+A$  be the graded radical ideal which defines the non-Cohen-Macaulay locus  $X \setminus \text{CM}(X)$  of  $X$ . Observe that  $\text{height } \mathfrak{a} \geq 2$ , so that the ideal transform

$$B(A) := D_{\mathfrak{a}}(A) = \varinjlim \text{Hom}_A(\mathfrak{a}^n, A) = \bigcup_{n \in \mathbb{N}} (A :_{\text{Quot}(A)} \mathfrak{a}^n) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\text{CM}(X), \mathcal{O}_X(n))$$

of  $A$  with respect to  $\mathfrak{a}$  is a positively graded finite birational integral extension domain of  $A$ . In particular  $B(A)_0 = K$ . Moreover  $B(A)$  has the second Serre-property  $S_2$ . As  $\tilde{X} := \text{Proj}(B(A))$  is of dimension 2, it thus is a locally Cohen-Macaulay scheme.

If  $E$  is a finite graded integral extension domain of  $A$  which satisfies the property  $S_2$ , we have  $A \subset B(A) \subset E$ . So  $B(A)$  is the least finite graded integral extension domain which has the property  $S_2$ . Therefore, we call  $B(A)$  the  *$S_2$ -cover of  $A$* . We also can describe  $B(A)$  as the endomorphism ring  $\text{End}(K(A), K(A))$  of the canonical module  $K(A) = K^3(A) = \text{Ext}_S^{r-2}(A, S(-r-1))$  of  $A$ .

(B) Let the notations be as in part (A). Then, the inclusion map  $A \rightarrow B(A)$  gives rise to a finite morphism

$$\pi : \tilde{X} := \text{Proj}(B(A)) \rightarrow X, \text{ with } \text{Sing}(\pi) = X \setminus \text{CM}(X).$$

In particular  $\pi$  is almost non-singular and hence birational. Moreover, for any finite morphism  $\rho : Y \twoheadrightarrow X$  such that  $Y$  is locally Cohen-Macaulay, there is a unique morphism  $\sigma : Y \rightarrow \tilde{X}$  such that  $\rho = \pi \circ \sigma$ . In addition  $\sigma$  is an isomorphism if and only if  $\text{Sing}(\rho) = X \setminus \text{CM}(X)$ . Therefore, the morphism  $\pi : \tilde{X} \twoheadrightarrow X$  is addressed as the *finite Macaulayfication* of  $X$ . Keep in mind, that – unlike to what happens with normalization – there may be proper birational morphisms  $\tau : Z \twoheadrightarrow X$  with  $Z$  locally Cohen-Macaulay, which do not factor through  $\pi$  (see [B,1986,2]).

(C) Let  $X \subset \mathbb{P}^r$  be as in part (A). We introduce the invariants

$$e_x(X) := \text{length}(H_{\mathfrak{m}_{X,x}}^1(\mathcal{O}_{X,x})), (x \in X \text{ closed}) \quad \text{and} \quad e(X) := \sum_{x \in X, \text{closed}} e_x(X).$$

Note that the latter counts the *number of non-Cohen-Macaulay points* of  $X$  in a weighted way. Keep in mind that

$$e(X) = h^1(X, \mathcal{O}_X(n)) \text{ for all } n \ll 0.$$

(D) Let  $X \subset \mathbb{P}^r$  and  $A = S/I$  be as above. We denote the *arithmetic depth* of  $X$  by  $\text{depth}(X)$ , hence  $\text{depth}(X) := \text{depth}(A)$ .

Now, we are ready to formulate and to prove the result we announced at the beginning of this section. It makes part of the more extended result Theorem 3.8 of [BLEPS, 2013] and essentially is statement (c) of that theorem. We first remind an important notion.

**Reminder 10.4.** Let  $X \subset \mathbb{P}^r$  be a non-degenerate irreducible projective surface. It is well-known that  $d := \deg(X) \geq \text{codim}(X) + 1 = r - 1$ . In case equality holds,  $X$  is called a *surface of minimal degree*. Surfaces of minimal degree were completely classified more than hundred years ago by P. del Pezzo and E. Bertini. The variety  $X \subset \mathbb{P}_K^r$  is of minimal degree if and only if it is either  $\mathbb{P}^2$  or a quadric hypersurface in  $\mathbb{P}^3$  a Veronese surface in  $\mathbb{P}^5$  or a rational (possibly singular) normal surface scroll. In case  $X$  is a singular rational normal scroll, it is a cone with a singleton vertex over a rational normal curve, hence over a curve of degree  $d$  in  $\mathbb{P}^d$ .

**Proposition 10.5.** *Assume that  $X \subset \mathbb{P}^r$  is sectionally rational and has finite non-normal locus  $X \setminus \text{Nor}(X)$  and that  $d \geq 5$ . Then*

*The  $S_2$ -cover  $B = B(A)$  of  $A = S/I$  is the homogeneous coordinate ring a surface of minimal degree  $\tilde{X} \subset \mathbb{P}^{d+1}$  – which is a scroll if  $d \geq 5$  – and there is a subspace  $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$  such that  $\tilde{X} \cap \Lambda = \emptyset$  and*

- (1)  $X = \tilde{X}_\Lambda$ ,
- (2)  $\text{Sing}(\pi_\Lambda : \tilde{X} \twoheadrightarrow X) = X \setminus \text{CM}(X)$ .

*Proof.* Our first aim is to show that  $\dim_K(B_1) = d + 2$ . As  $h \in S_1 \setminus \{0\}$  is general, we have  $\sqrt{(\mathfrak{a}, h)} = A_+$ . So, comparing local cohomology furnishes a short exact sequence of graded

local cohomology modules (see Proposition 8.1.2 (ii) of [BSh,2013]).

$$0 \rightarrow H_{(h)}^1(H_{\mathfrak{a}}^1(A)) \rightarrow H_{A_+}^2(A) \rightarrow H_{(h)}^0(H_{\mathfrak{a}}^2(A)) \rightarrow 0.$$

Observe that with  $D := D_{A_+}(A) = \varinjlim \mathrm{Hom}_A((A_+)^n, A) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ , the kernel of the natural map  $B/A \rightarrow B/D$  is  $S_+$ -torsion. As  $\sqrt{(\mathfrak{a}, h)} = A_+$ , it follows

$$H_{(h)}^1(H_{\mathfrak{a}}^1(A)) = H_{A_+}^1(H_{\mathfrak{a}}^1(A)) = H_{A_+}^1(B/A) = H_{A_+}^1(B/D).$$

By Lemma 2.4 of [BS, 2012] we have  $\dim_K((B/D)_n) = e(X)$  for all  $n \gg 0$ . Consequently  $\dim_K(D_{A_+}(B/D)_n) = e(X)$  for all  $n \in \mathbb{Z}$ . As  $(B/D)_0 = 0$  it follows that

$$\dim_K(H_{(h)}^1(H_{\mathfrak{a}}^1(A))_0) + \dim_K(H_{A_+}^1(B/D)_0) = e(X).$$

By statement (b) we also have  $\dim_K(H_{A_+}^2(A)_0) = H^1(X, \mathcal{O}_X) = e(X)$ . So, the above sequence shows that  $H_{(h)}^0(H_{\mathfrak{a}}^2(A))_0 = 0$ . Therefore the multiplication map  $h : H_{\mathfrak{a}}^2(A)_0 \rightarrow H_{\mathfrak{a}}^2(A)_1$  is injective. Now, applying the functor  $D_{\mathfrak{a}}(\bullet)$  to the exact sequence  $0 \rightarrow A(-1) \xrightarrow{h} A \rightarrow A/hA \rightarrow 0$  and observing once more that  $\sqrt{(\mathfrak{a}, h)} = A_+$ , we get the exact sequence of  $K$ -vector spaces

$$0 \rightarrow D_{\mathfrak{a}}(A)_0 \rightarrow D_{\mathfrak{a}}(A)_1 \rightarrow D_{A_+}(A/hA)_1 \rightarrow H_{\mathfrak{a}}^2(A)_0 \xrightarrow{h} H_{\mathfrak{a}}^2(A)_1.$$

As  $D_{\mathfrak{a}}(A)_0 = B_0 = K$  and as, in addition, the last map in this sequence is injective, we end up with  $\dim_K(B_1) = \dim_K(D_{\mathfrak{a}}(A)_1) = \dim_K(D_{A_+}(A/hA)_1) + 1$ . As  $C_h := \mathrm{Proj}(A/hA) \subset \mathrm{Proj}(S/hS) = \mathbb{P}^{r-1}$  is a non-degenerate smooth rational curve of degree  $d$ , the  $K$ -vector space  $D_{A_+}(A/hA)_1 \cong H^0(C_h, \mathcal{O}_{C_h}(1))$  has dimension  $d + 1$ , so that indeed  $\dim_K(B_1) = d + 2$ .

Now, consider the non-degenerate closed subscheme  $\tilde{X} := \mathrm{Proj}(K[B_1]) \subset \mathbb{P}^{d+1}$ . As  $K[B_1]$  is a finite birational integral extension domain of  $A$ , the scheme  $\tilde{X} \subset \mathbb{P}^{d+1}$  is a non-degenerate irreducible and reduced surface of degree  $d$ . It follows in particular that  $\tilde{X} \subset \mathbb{P}^{d+1}$  is a surface of minimal degree – and hence a (possibly singular) surface scroll if  $d \geq 5$ . In particular  $K[B_1]$  is a Cohen-Macaulay ring which contains  $A$  and is contained in the  $S_2$ -cover  $B(A)$  of  $A$ . Thus  $K[B_1] = B(A)$  (see Notation and Reminder 10.3 (A)) and hence  $B = B(A)$  is the homogeneous coordinate ring of the surface  $\tilde{X} \subset \mathbb{P}^{d+1}$ .

Moreover, the inclusion map  $A \rightarrow B(A)$  gives rise to a finite morphism  $\pi_{\Lambda} : \tilde{X} \rightarrow X$ , induced by a linear projection  $\pi'_{\Lambda} : \mathbb{P}^{d+1} \setminus \Lambda \rightarrow \mathbb{P}^r$  from a subspace  $\Lambda = \mathbb{P}^{d-r} \subset \mathbb{P}^{d+1}$  disjoint to  $\tilde{X} \subset \mathbb{P}^{d+1}$ , so that indeed  $X = \tilde{X}_{\Lambda}$ . Finally, by Notation and Reminder 10.3 (B), we have  $\mathrm{Sing}(\pi_{\Lambda} : \tilde{X} \rightarrow X) = X \setminus \mathrm{CM}(X)$ . So, statement (c) is shown.  $\square$

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