My Work with Peter Schenzel: Reminiscence of a Friendship and a Cooperation 1979 - 2017

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Joint Work with Peter Schenzel

[1] M.Brodmann, P.Schenzel: ***Curves of degree r+2 in P(r): Cohomological, geometric and homological aspects,*** Journal of Algebra 242 (2001) 577-623

[2] M.Brodmann, P.Schenzel: ***On projective curves of maximal regularity,*** Mathematische Zeitschrift 244 (2003) 271-289

[3] M.Brodmann, P.Schenzel: ***On varieties of almost minimal degree in small codimension,*** Journal of Algebra 305 (2006) 789-801

[4] M.Brodmann, P.Schenzel: ***Arithmetic properties of projective varieties of almost minimal degree,*** Journal of Algebraic Geometry 16m (2007) 347-400

[5] M.Brodmann, P.Schenzel: ***Projective curves with maximal regularity and applications to syzygies and surfaces,*** Manuscripta Mathematica 135 (2011) 469-495

[6] M.Brodmann, E.Park, P.Schenzel: ***On varieties of almost minimal degree II: A rank-depth formula,*** Proceedings ot the AMS 139 (2011) 2025-2032

[7] M.Brodmann, P.Schenzel: ***Projective surfaces of degree r+1 in projective r-space and almost non-singular projections,*** Journal of Pure and Applied Algebra 216 (2012) 2241-2255

[8] M.Brodmann, W.Lee, E.Park, P.Schenzel: ***Projective varieties of maximal sectional regularity,*** Journal of Pure and Applied Algebra 221 (2017) 98-118

[9] M.Brodmann, W.Lee, E.Park, P.Schenzel: ***On surfaces of maximal sectional regularity,*** Taiwanese Journal Mathematics 21, No 3, (2017) 549-567

[10] M.Brodmann, W.Lee, E.Park, P.Schenzel: ***On surfaces of maximal sectional regularity in P(4),*** in Preparation

[11] M.Brodmann, P.Schenzel: ***Families of regular blowups of the real affine plane: Classification, isotopies and visualizations,*** in Preparation

1978 – 1979: The Beginnings

Brandeis University
Waltham, Mass. USA 1978:
(Meeting Wolfgang Vogel)

GDR (DDR) 1979:
Mathematatis does not care on
„Cold War“ and „Politics“

1979: First Visit in Halle – Followed by Visits in Halle, Berlin, Leipzig

Halle / Saale (Marktplatz, Roter Turm)

„an der Saale“ (March 2011)

Leipzig (Gewandhaus, MDR, Universität)

Berlin (Museuminsel mit Spree)

1980: Visit of Peter at the FIM ETHZ

FIM ETHZH

IMATH UNIBS

„Poly-Terrasse“ Zurich 1980

Former IMATH UZH

A Few Further Occasions to Meet ...

1) Halle: 1979,1990,1991,1996,…, 2015
2) Leipzig: 1979,1990,1997,2005,…,2015,2016,2017
3) Krupina-Bratislava: 1981,1984

Krupina-Bratislava 1981

… in Various Countries ...

4) Eisenach: 1991
5) Berlin (Humboldt-University): 1995, 2009
6) Constanza (Romania): 1996
7) IPM Tehran (Iran): 1997

Tehran,1997

… and Continents – in West, ...

8) Mamaya (Romania): 2002
9) Snowbird (Utah, USA): 2005

Snowbird, 2005

… East and Far East, ...

10) Seoul, Daejeon (South Korea): 2006,2008,2009,2012

View from Daedunsan

Silla-Graves Gyeongiu (7.& 8. Century)

Seoul

… in Grieve and ...

11) Lahore (Pakistan): 2009

Lahore, 2009: Terrorist Attack to Cricket Team from Sri Lanka

… Working and Living at Good Moods ...

12) Pohan (South Korea): 2012
13) Busan (South Korea): 2016

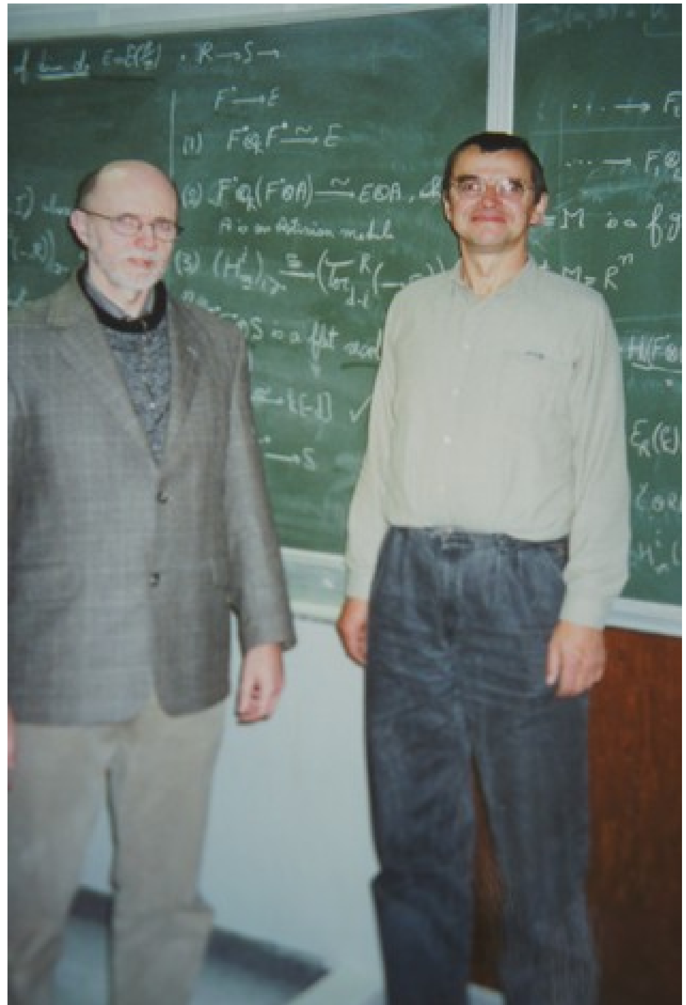
Korea University Seoul

Euisung Park (with Son Juitsu)

Pukyong National University Busan

Wanseok Lee

… Friendly Welcome at New Living Places ...



Leipzig, 2015

… Enjoying Dinners ...

Busan, 2016

… Exploring New Horizons – in and ...

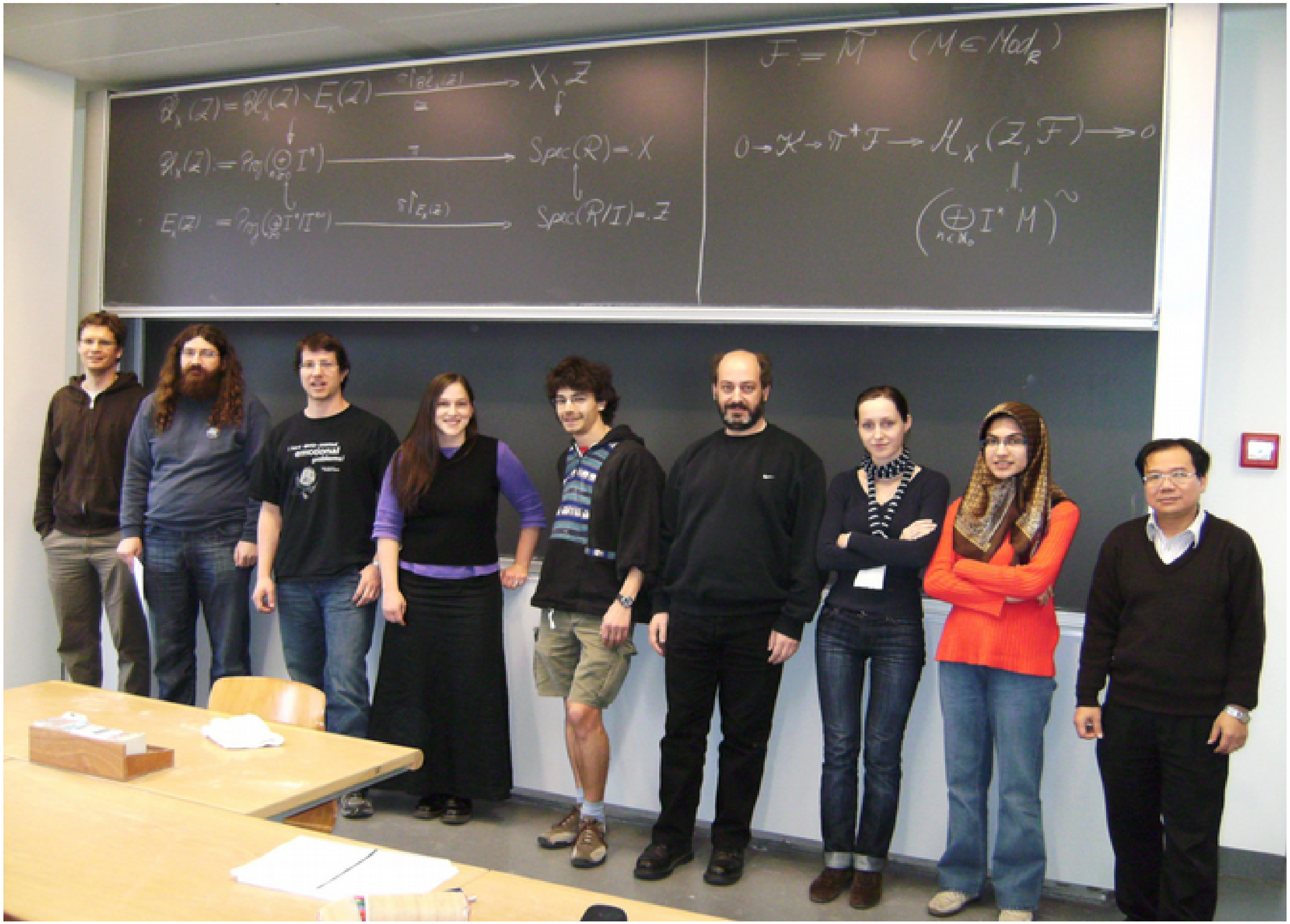
14) Irinjalakuda (Kerala/India): 2016
15) Hanoi, Ha Long (Vietnam): 2017

Ha Long, 2017

… after Work, in Vietnam ...

… and Many Times in Zurich

Seminar Höhere Algebra UZH, 2008 (with Blowup)



1. Introduction

(1.1) Notation: (A) $K = \overline{K}$ a field; $R = K[x_0, \ldots, x_r]$ a polynomial ring; $X \subseteq \mathbb{P}^r = \mathrm{Proj}(R)$ a non-degenerate, non conic irreducible projective variety of degree d and codimension $c \geq 2$.

(B) $I := \bigoplus_{n \geq 0} H^0(\mathbb{P}^r, \mathcal{J}_X(n)) \subseteq R$ the homogeneous vanishing ideal of X; $A := R/I$ the homogeneous coordinate ring of X. •

(1.2) Remark: (A) $d - c \geq 1$. If $d - c = 1$, X is of minimal degree.

(B) (Bertini et al, 19th Century) X of minimal degree $\Longrightarrow$ X is either:

(1) the Veronese surface $[\mathbb{P}^2]^{(2)} \subseteq \mathbb{P}^5$ or else

(2) a smooth rational normal scroll $\mathbb{P}(\bigoplus_{i=1}^{r-c} \mathcal{O}_{\mathbb{P}^1}(a_i)) = S(a_1, \ldots, a_{r-c}) \subseteq \mathbb{P}^r$;

$(0 < a_1 \leq a_2 \leq \cdots \leq a_{r-c}\ ;\ \sum_{i=1}^{r-c} a_i = d = c+1)$. •

(1.3) Basic Program: Study X if $d - c > 1$ if either

(1) $d - c$ is "small", or

(2) $X \cap \mathbb{E}$ is a "well understood curve" for general $\mathbb{E} \in \mathbb{G}(c+1, \mathbb{P}^r)$.

NB: Focus: Geometry, Homology (= Syzygies), Cohomology, Singularities. •

2. Curves with $d-c=3$ (hence $d=r+2$) (see [1], 2001)

(2.1) **Remark:** Curves with $d-c=2$ are "classically" well understood. ●

(2.2) **Notation:** $nct := (h^1(\mathbb{P}^r, \mathcal{J}_X(1)), h^1(\mathbb{P}^r, \mathcal{J}_X(2)))$ = "numerical cohomology type" of X. ●

(2.3) **Classification and Structure of Curves with $d-c=3$** (for simplicity: $r \geq 4$):

Case I nct: $(0,0)$	$A = CM$ $\text{type}(A)=2$	$X_0 \xrightarrow[\cong]{\lvert D\rvert} X$ X_0 smooth, $g(X_0)=2$ $D \in Div_{r+2}(X_0)$	i	1	$1 \leq i \leq r-3$	$r-2$	$r-1$	r
			β_{1i}	$\binom{r}{2}-2$	$\binom{r}{i+1}-2\binom{r-1}{i-1}$	$r-1$	2	0
			β_{2i}	0	0	$r-2$	0	0
Case II nct: $(1,0)$	$A = BB$ $i_{BB}(A)=1$	$\tilde{X} \xrightarrow[\cong]{\bullet\mathbb{P}^0} X$ $\tilde{X} \subseteq \mathbb{P}^{r+1}$ $\tilde{X}$ = A Gor	i	1	$1 \leq i \leq r-3$	$r-2$	$r-1$	r
			β_{1i}	$\binom{r}{2}-3$	$\leq i\binom{r}{i+1}-2\binom{r-1}{i-1} =: a_i$	0 or $r-2$	0	0
			β_{2i}	$*$	$\leq \binom{r}{i}$	$\binom{r+1}{2}-1$	$r+2$	1
Case III nct: $(2,0)$	$A = BB$ $i_{BB}(A)=2$	$\tilde{X} \xrightarrow[\cong]{\bullet\mathbb{P}^1} X$ $\tilde{X} \subseteq \mathbb{P}^{r+2}$, RNC. $\nexists\, \mathbb{P}^3: \mathbb{P}^1 \subseteq \mathbb{P}^3 \wedge$ $\text{length}(\tilde{X} \cap \mathbb{P}^3) \geq 4$	i	1	$1 \leq i \leq r-3$	$r-2$	$r-1$	r
			β_{1i}	$\binom{r}{2}-4$	$\leq a_i$	0 or $r-2$	0	0
			β_{2i}	$*$	$\leq 2\binom{r}{i}$	r^2-1	$2r+2$	2
Case IV nct: $(2,1)$	$A = 2$-BB H-R module has simple socle	$\tilde{X} \xrightarrow[\cong]{\bullet\mathbb{P}^1} X$ $\tilde{X} \subseteq \mathbb{P}^{r+2}$, RNC. $\exists\, \mathbb{P}^3: \mathbb{P}^1 \subseteq \mathbb{P}^3 \wedge$ $\text{length}(\tilde{X} \cap \mathbb{P}^3) = 4$	i	1	$1 \leq i \leq r-3$	$r-2$	$r-1$	r
			β_{1i}	$\binom{r}{2}-3$	$\leq a_i$	0 or $r-2$	0	0
			β_{2i}	$\leq r-1$	$\leq \binom{r-1}{i}$	$2r-2$	3	0
			β_{3i}	1	$\binom{r-1}{i-1}$	$\binom{r-1}{2}$	$r-1$	1

3. Curves of Maximal Regularity and Applications (see [2], 2003; [5], 2011; ... [9], 2017; [10] 2017; ...) ③

(3.1) Remark: (Gruson, Lazarsfeld, Peskine, 1983) (A) X = curve $\Rightarrow \operatorname{reg}(X) \leq d-c+1$.
(B) $\operatorname{reg}(X) = d-c+1$ ("X of maximal regularity") and $d-c \geq 3$, $r \geq 3 \Longrightarrow$
X is smooth and rational and has a $(d-c+1)$-secant line $\mathbb{L} = \mathbb{P}^1$ (= "extremal secant line"). •

(3.2) Extremal Secant Lines: (X of maximal regularity with $r \geq 4$ and $d-c \geq 3$)

(a) $\exists_1$ extremal secant line $\mathbb{L}$ of X and $Y := X \cup \mathbb{L}$ is linearly normal; $\operatorname{reg}(Y) \leq d-c$.
(b) Y ACM $\Leftrightarrow$ H-R module of X has simple socle $\Leftrightarrow \operatorname{reg}(Y) = 3$.
(c) $d < 2r-1 \Rightarrow Y$ is ACM; (NB: $\not\Leftarrow$).
(d) B = homogeneous coordinate ring of $Y \underset{(1\leq i\leq r)}{\Rightarrow} \operatorname{Tor}_i^R(K,A) \cong \operatorname{Tor}_i^R(K,B) \oplus K^{\binom{r-1}{i-1}}(-i-d+r-1)$. ▪

(3.3) Remark: If Y is ACM and $\mathbb{H} = \mathbb{P}^{r-1} \subseteq \mathbb{P}^r$ general, then $Y \cap \mathbb{H} \subseteq \mathbb{H} = \mathbb{P}^{r-1}$ is a scheme of d points in general position with the same Betti numbers as Y. •

(3.4) Example: $X = \{(s^{11} : s^{10}t : \cdots : s^5t^6 : (st^{10} + s^2t^9) : t^{11}) \mid (s,t) \in K^2 \setminus \{(0,0)\}\}$: $d = 11$, $r = 8$.

The Betti diagram of X is given by: ($\operatorname{reg}(X) = 5$, Y = ACM) (Computed by SINGULAR)

24	84	126	84	20	0	0	0
0	0	0	20	36	21	4	0
0	0	0	0	0	0	0	0
1	7	21	35	35	21	7	1

NB: As $\operatorname{reg}(Y) < \operatorname{reg}(X)$, the last "binomial line" survives in all Betti diagrams. •

(3.5) Remark: Application: Approximation of Betti numbers of surfaces whose general hyperplane section is a curve of maximal regularity. •

4. Varieties of Almost (and Almost-Almost) Minimal Degree (see [3], 2006; [4], 2007; [6], 2011; ([7], 2012))

(4.1) Remark: X is of "almost minimal degree" if $d - c = 2$. Such varieties were studied first by Fujita (1982) and Hoa-Stückrad-Vogel (1991).

(4.2) The Two Types (X a variety of almost minimal degree, $t := \operatorname{depth}(A)$) Either

(a) Case I: X is normal and A Gor (hence a normal Del Pezzo variety); or else

(b) Case II: $\exists \tilde{X} \subseteq \mathbb{P}^{r+1}$ of minimal degree, $p \in \mathbb{P}^{r+1} \setminus \tilde{X}$ with $\operatorname{Sec}_p(\tilde{X}) = \mathbb{P}^{t-1}$ and $\nu: \tilde{X} \xrightarrow{\pi_p} X$ is the normalization of X.

(4.3) Study of Case II (Case I studied by Fujita)

(a) $\operatorname{Sing}(\nu) = \mathbb{P}^{t-2} \subseteq X$; $\operatorname{Sing}(\nu) = X \setminus S_2(X) = X \setminus CM(X)$, provided $t \leq r - c$.

(b) $\nu^{-1}(\operatorname{Sing}(\nu))$ $(= \operatorname{Sec}_p(\tilde{X}) \cap \tilde{X})$ is a quadric hypersurface in $\mathbb{P}^{t-1} = \operatorname{Sec}_p(\tilde{X})$.

(c) The generic point $x \in X$ of $\operatorname{Sing}(\nu)$ is of Goto type: $H^i_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) = \begin{cases} 0; & i \neq r-c-t+2 \\ k(x), & i = r-c-t+2 \end{cases}$.

(d) $\exists Y \subseteq \mathbb{P}^r$, cone over a rational normal scroll, with $X \subset Y$ and $\operatorname{codim}_Y(X) = 1$ (an embedding scroll of X).
Moreover with $h := \dim \operatorname{vert}(\tilde{X})$, $l := \dim \operatorname{vert}(Y)$ it holds $h \leq l \leq h+3$ and $t \leq h+5$.

(e) A satisfactory approximation of the Betti numbers of X is possible on use of $\nu: \tilde{X} \twoheadrightarrow X$ and $Y \supset X$.

(4.4) Remark: Surfaces with $d - c = 3$ are studied in [7]. We distinguish 11 cases there!

2008 & 2011: Joint „Research in Pairs“ at the MFO (Oberwolfach)

Research on:

1) Projections of Rational Normal Surface Scrolls, their Syzigies and their Cohomology.

2) Surfaces of Maximal Sectional Regularity – later merging in a long term joint project with E. Park and W. Lee.

5. Varieties of Maximal Sectional Regularity: VMSR (see [8], [9], [10], 2017)

(5.1) **Definition and Remark:** (A) $\mathcal{U} \subseteq \mathbb{G}(c+1, \mathbb{P}^r)$ largest open subset such that $(\forall \Lambda \in \mathcal{U})\ \mathcal{C}_\Lambda := X \cap \Lambda \subseteq \Lambda = \mathbb{P}^{c+1}$ is a curve of maximal regularity.

$\boxed{X = \text{VMSR} :\Longleftrightarrow \mathcal{U} \neq \emptyset}$ Study of VMSR motivated by $\begin{cases} \text{Gruson-Lazarsfeld-Peskine (1983)} \\ \text{Bertin (2002)} \\ \text{(1.3)(2) and (3.1-5)} \end{cases}$

(B) $c \geq 3$ and $d-c \geq 3 \overset{(3.2)(a)}{\Longrightarrow} \forall \Lambda \in \mathcal{U}\ \exists_1\ \mathbb{L}_\Lambda$ = extremal secant line to $\mathcal{C}_\Lambda$.

$^*\Sigma^\circ := \{\mathbb{L}_\Lambda \mid \Lambda \in \mathcal{U}\} \subseteq \mathbb{G}(1, \mathbb{P}^r)$; $\mathbb{F} := \overline{\bigcup_{\Lambda \in \mathcal{U}} \mathbb{L}_\Lambda}$ = "extremal secant variety of X".

(5.2) **Classification of VMSR with $c \geq 3$, $d-c \geq 3$** $(n := r-c = \dim(X))$

(a) If $n = 2$ or $\text{char}(K) = 0$, there is a smooth rational normal scroll $\tilde{X} = S(a_1, \cdots, a_n) \subseteq \mathbb{P}^{d+n-1}$ and a space $\Gamma \in \mathbb{G}(d+n-r-2, \mathbb{P}^{d+n-1})$ with $X \cap \Gamma = \emptyset$ such that the normalization of X is given by projecting $\tilde{X}$ from Γ: $\tilde{X} \xrightarrow{\pi_\Gamma = \nu} X$.

(b) If $\text{char}(K) = 0$, then either:

Case I: $c = 3$; $X \subseteq \mathbb{F} = S(0, \cdots, 0, 1, 1, 1)$ = a (singular) scroll of dimension $n+1$ and degree 3. Then X is linearly equivalent to $H + (d-3)F \in \text{Div}(\mathbb{F})$, where $H \subseteq \mathbb{F}$ is the hyperplane divisor and $F \subseteq \mathbb{F}$ a linear n-space.

Case II: $\mathbb{F}$ is a linear n-space and $X \cap \mathbb{F} \subseteq \mathbb{F}$ is a hypersurface of degree $d-c+1$.

(c) $\dim(^*\Sigma^\circ) = 2n-2 = \dim(\overline{\Sigma^\circ})$; $(\Sigma^\circ = \{\mathbb{L} \in \mathbb{G}(1, \mathbb{P}^r) \mid \text{length}(X \cap \mathbb{L}) = d-c+1\})$.

6. Surfaces of Maximal Sectional Regularity: SMSR (see [9], [10], 2017)

(6.1) Classification and Properties of SMSR with $r \geq 5$, $d \geq r+1$

Case I: $r=5$, $X \subseteq S(1,1,1) = \mathbb{F}$ = a smooth rational 3-scroll, X is smooth with

Betti table

β_{1i}	3	2			
$\vdots$	0	0	0	0	0
$\beta_{(d-2)i}$	$\binom{d-1}{2}$	$2(d-1)(d-3)$	$3(d^2-5d+5)$	$2(d-2)(d-4)$	$\binom{d-3}{2}$

Case II: $\mathbb{F} = \mathbb{P}^2$, $X \cap \mathbb{F} \subseteq \mathbb{F}$ is a curve of degree $d-r+3 = \mathrm{reg}(X)$, X is linearly normal;

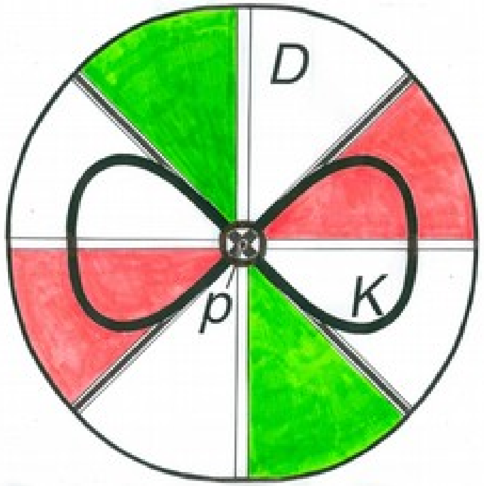
$\binom{d-r+2}{2} \overset{(*)}{\leq} e(X) := \sum_{\substack{x \in X \\ \text{closed}}} h^1_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) = h^2(\mathbb{P}^r, \mathcal{J}_X(j)), (\forall j \leq 0)$, and equality holds in $(*)$ if and only if $\bigoplus_{j \in \mathbb{Z}} H^2(\mathbb{P}^r, \mathcal{J}_X(j))$ has simple socle.

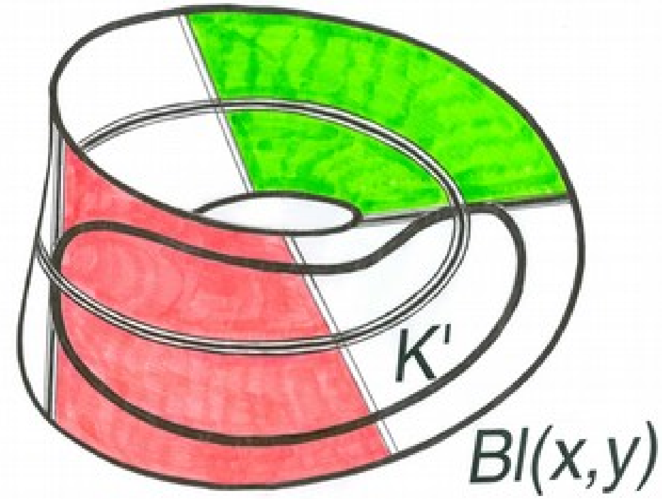
Moreover X and $Y := X \cup \mathbb{F}$ are in close homological and cohomological relation... (in analogy to curves of maximal regularity)

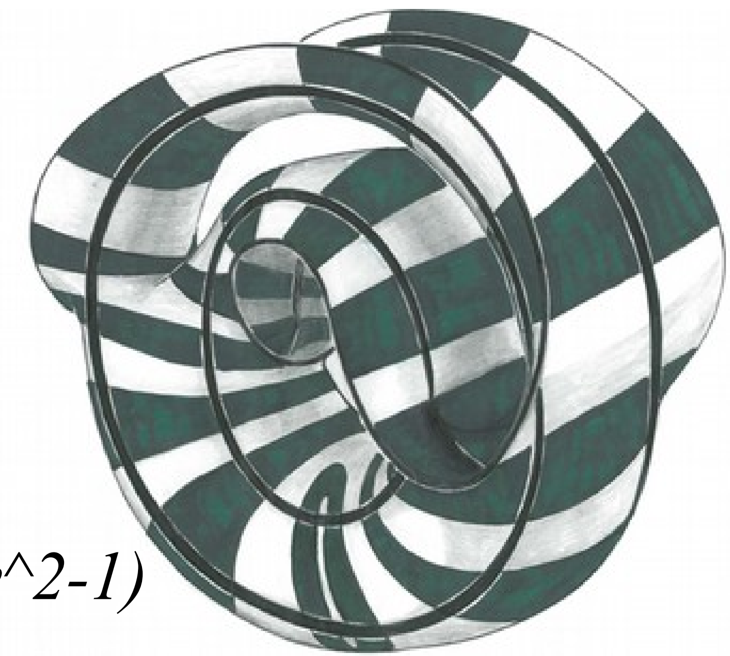
(6.2) Classification of SMSR with $r=4$ and $d \geq 5$ (Ongoing research)

Case I: $h^0(\mathbb{P}^4, \mathcal{J}_X(2)) = 1$ and $X \subset \mathbb{F} = S(0,1,1)$ = a singular quadric.	**Case II:** $h^0(\mathbb{P}^4, \mathcal{J}_X(2)) = 0$ and $\mathbb{F} = \mathbb{P}^2$, $X \cap \mathbb{F} \subseteq \mathbb{F}$ a curve of degree $d-1$.
Case III: $h^0(\mathbb{P}^4, \mathcal{J}_X(2)) = 0$, $d=5$ and $\mathbb{F}$ = finite union of 3-spaces. $\exists ?$	**Case IV:** $h^0(\mathbb{P}^4, \mathcal{J}_X(2)) = 0$ and $X \subset \mathbb{F}$ = irreducible hypersurface of degree > 2. $\exists ?$

The Blowup Story: Muenster 1979 ...

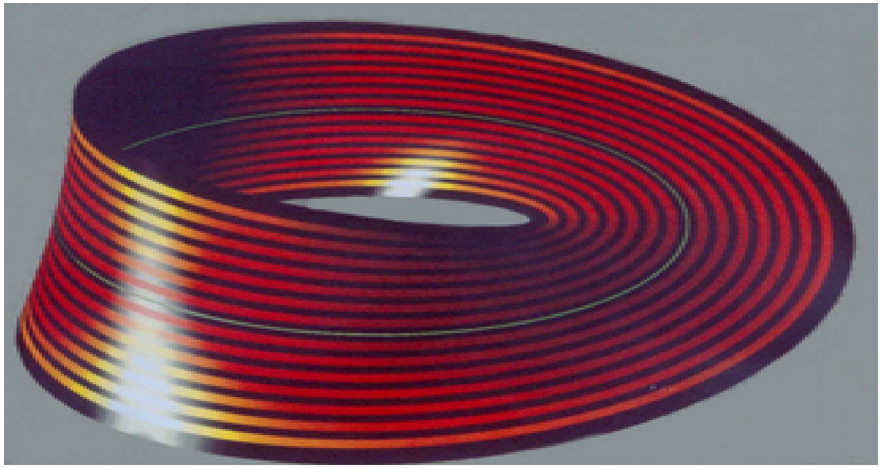


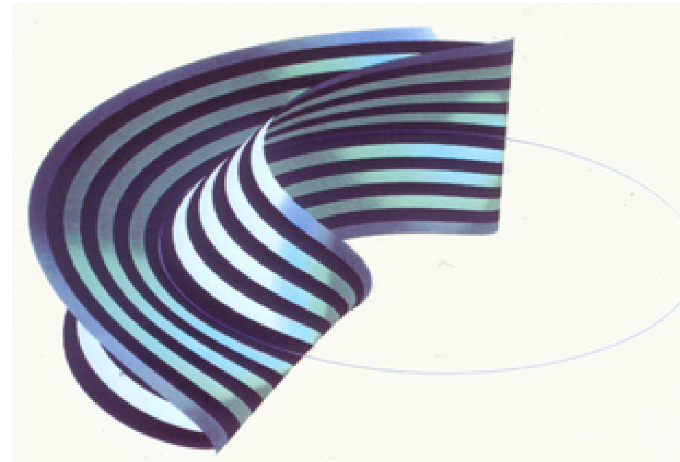


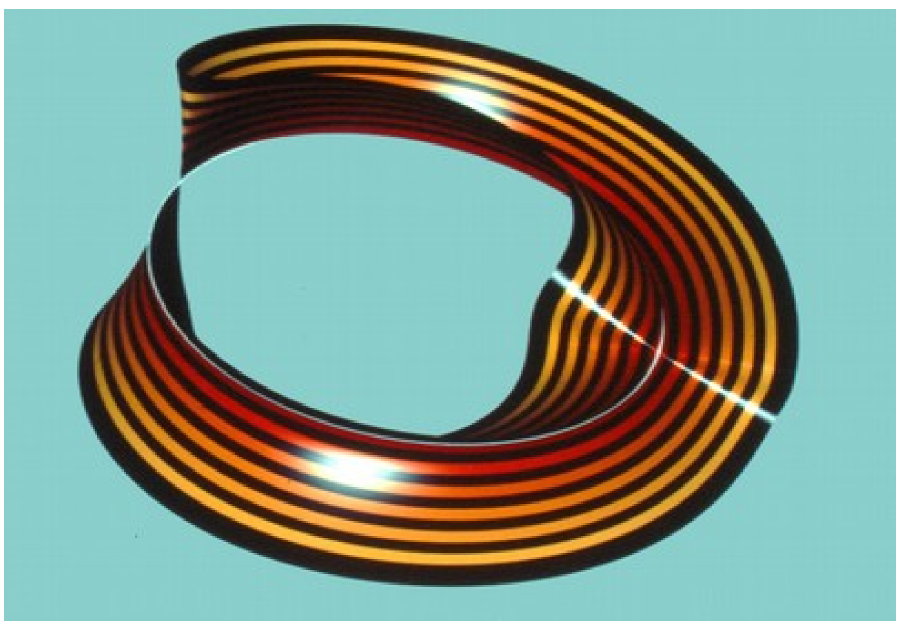


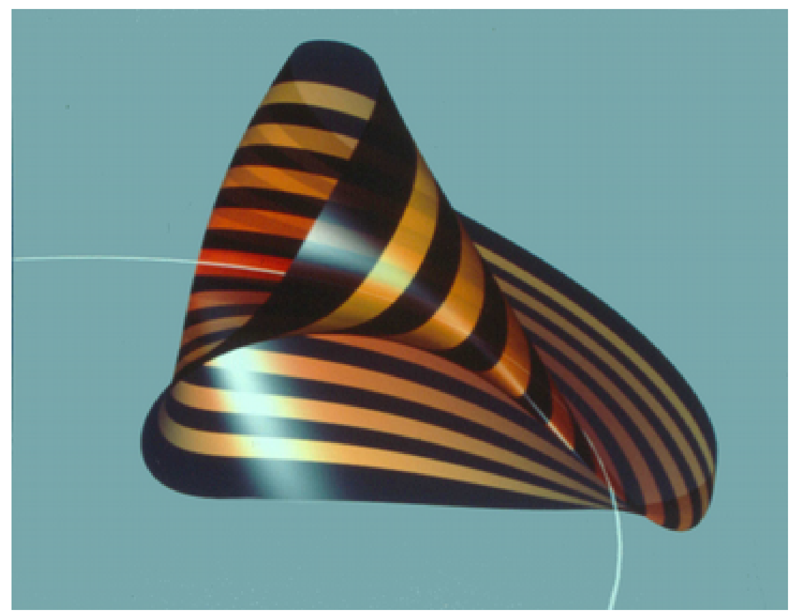
Bl(x^2-1,y^2-1)

Zurich 1991 (Heureka – 700 Years of 🇨🇭) ...





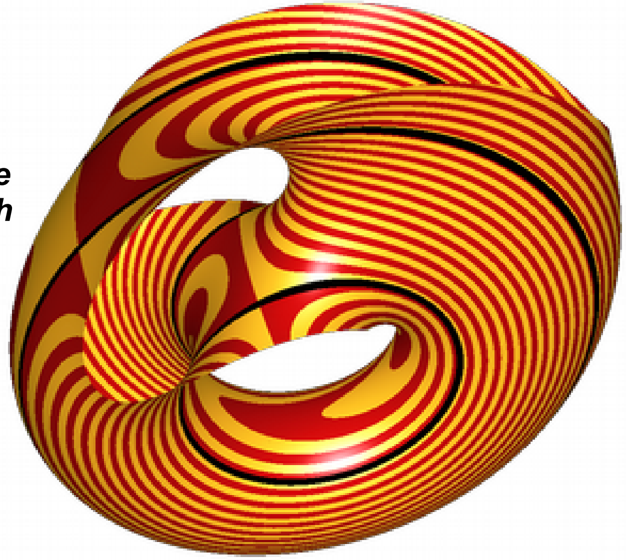


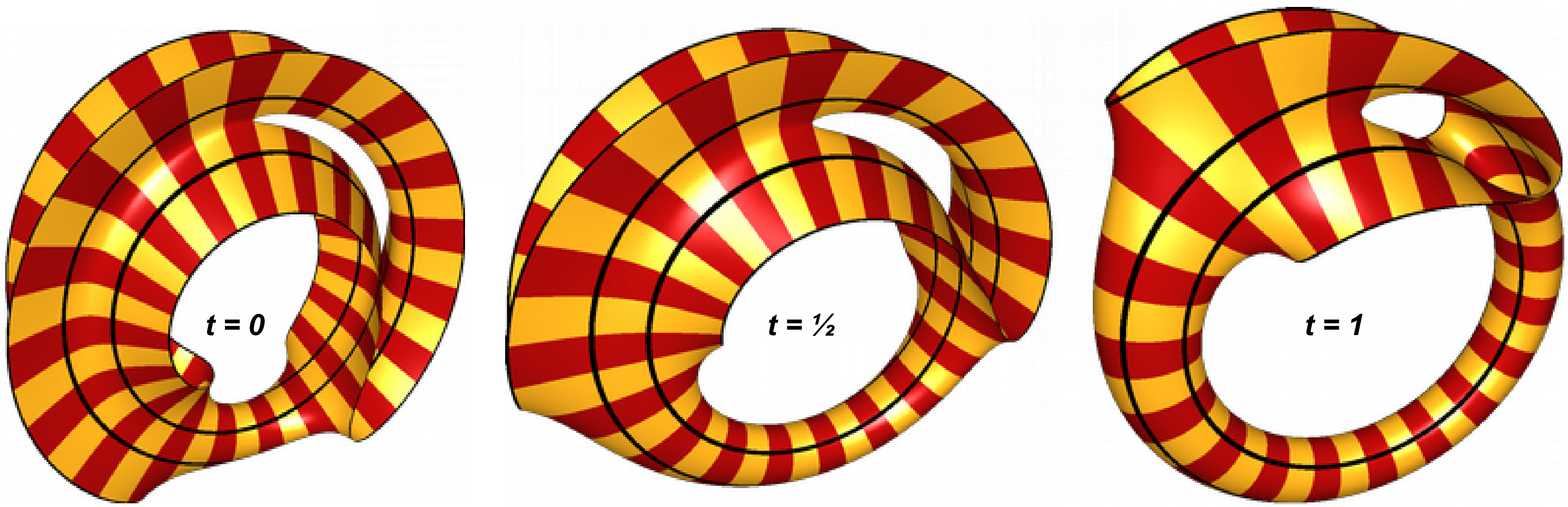


… Halle / Leipzig 2008 - 2017

P.Schenzel, Ch.Stussak (2013):

High Resolution Static and Real-Time Dynamic Visualizations of Embedded Blowups of the Real Affine Plane with Respect to a Pair of Polynomials !





[11] M.Brodmann, P.Schenzel: ***Families of regular blowups of the real affine plane: Classification, isotopies and visualizations,*** in Preparation

7. Families of Embedded Regular Blowups of the Real Affine Plane (s. [11, 2017])

(7.1) Notation: $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}^2$, $\mathcal{U}$ open, star-shaped, bounded; $\emptyset \neq Z \subseteq \mathcal{U}$, Z finite.

$P := \{\underline{f} = (f_0, f_1) \in \mathbb{R}[\underline{x},\underline{y}]^2 \mid Z(\underline{f}) := \{p \in \overline{\mathcal{U}} \mid f_0(p) = f_1(p) = 0\} = Z\}$.

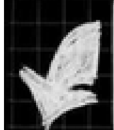
(7.2) Definition: $\forall \underline{f} \in P$: $\mathring{Bl}(\underline{f}) := \{(p, (f_0(p) : f_1(p))) \mid p \in \mathcal{U} \setminus Z\} \subset \mathcal{U} \times \mathbb{P}^1_{\mathbb{R}}$ and

$Bl(\underline{f}) = \mathring{Bl}(\underline{f}) \mathring{\cup} (Z \times \mathbb{P}^1_{\mathbb{R}})$ $\left(= \overline{\mathring{Bl}(f)}^{Zariski} \subseteq \mathcal{U} \times \mathbb{P}^1_{\mathbb{R}}\right)$ = "embedded blowup of $\mathcal{U}$ with respect to the pair $\underline{f} = (f_0, f_1) \in P$."

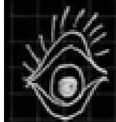
(7.3) Visualization of Embedded Blowups: $(0 < \rho < r)$

$\mathcal{U} \subseteq D := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < \rho^2\}$ = open disk in the plane;

$T := \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + (r - \sqrt{v^2+w^2})^2 < \rho^2\}$ = open solid torus in 3-space.



$$D \times \mathbb{P}^1_{\mathbb{R}} \xrightarrow[\cong]{\iota = \text{diffeo}} T; \quad ((x,y),(x_0:x_1)) \stackrel{\iota}{\mapsto} \left(x, (r-y)\frac{x_0^2 - x_1^2}{x_0^2+x_1^2}, (r-y)\frac{2x_0x_1}{x_0^2+x_1^2}\right); \text{ (NB: } T \subseteq \mathbb{R}^3\text{)}.$$



For each $\underline{f} = (f_0,f_1) \in P$, the blowup $Bl(\underline{f})$ is visualized by its diffeomorphic image $\iota(Bl(\underline{f})) \subseteq \iota(\mathcal{U}\times\mathbb{P}^1_{\mathbb{R}}) \subseteq T$, hence appears as a surface in T.

"Coloring" (of $\iota(\mathring{Bl}(\underline{f}))$): $\forall p \in \mathcal{U}\setminus Z$: $\gamma[\iota(p,(f_0(p):f_1(p)))] := \gamma(p)$ ($\gamma \,\hat{=}\,$ color)!

"Movies": Vary coefficients of $f_0, f_1 \in \mathbb{R}[x,y]$ in time!

(7.4) **Definition:** $\underline{f} = (f_0, f_1) \in P$ is a "regular pair" (on U with respect to Z), if

$$\partial \underline{f}(p) = \begin{bmatrix} \frac{\partial f_0}{\partial x}(p) & \frac{\partial f_1}{\partial x}(p) \\ \frac{\partial f_0}{\partial y}(p) & \frac{\partial f_1}{\partial y}(p) \end{bmatrix}$$ is of rank 2 for all $p \in Z$.

If $\underline{f} \in P$ is regular, $\mathcal{B}l(\underline{f})$ is a "regular embedded blowup" of U. •

(7.5) **Aim:** Study $\mathbb{B} := \{\mathcal{B}l(\underline{f}) \mid \underline{f} \in P \text{ regular}\}$. •

(7.6) **Definition:** (A) $(\forall B, C \in \mathbb{B})$ $B \cong C :\Longleftrightarrow \exists \varphi : U \times \mathbb{P}^1 \circlearrowleft$ (≅, orient. pres. rational U-autom.) $: C = \varphi(B)$

(B) $(\forall B \in \mathbb{B},\ B = \mathcal{B}l(\underline{f}))$ $sgn_B : Z \to \{\pm 1\};\ (p \mapsto sgn[\det(\partial \underline{f}(p))])$ "sign distribution of B". •

(7.7) **Classification Theorem:** $\forall B, C \in \mathbb{B}:\ B \cong C \Longleftrightarrow sgn_B = sgn_C$. ■

(7.8) **Deformation Theorem:** $B, C \in \mathbb{B}$ with $B \cong C \Longrightarrow$

$\exists (B^{(t)})_{t \in [0,1]} \subset \mathbb{B}$ with $B^{(0)} = B$, $B^{(1)} = C$ coming from a U-isotopy:

($\exists\ \Phi : U \times \mathbb{P}^1 \times [0,1] \xrightarrow{\text{rat.}} U \times \mathbb{P}^1$ such that for all $t \in [0,1]$:

$\varphi^{(t)}(\bullet) := \Phi(\bullet, t) : U \times \mathbb{P}^1 \circlearrowleft$ (≅, orient. pres. rational U-autom) and $B^{(t)} = \varphi^{(t)}(B)$.) ■

(7.9) **Comment:** If $B, C \in \mathbb{B}$ it is easy to check whether $B \cong C$ (by (7.7)). If $B \cong C$, (by (7.8)), "there is a movie" relating B and C within their common isomorphism class. •

Big Anchor – Big Hope ...

Busan, 2016

Dear Peter! We All Wish You …