

**AROUND
CASTELNUOVO-MUMFORD
REGULARITY**

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PREFACE

Castelnuovo-Mumford regularity is one of the most fundamental invariants in *Commutative Algebra* and *Algebraic Geometry*. In fact, already in the late 19th century this invariant was tacitly present, a long time before it was properly defined.

One of its first hidden appearances may be found in Castelnuovo's work on *linear systems on smooth projective space curves* of 1893 [C]. Castelnuovo's result gives a sharp upper bound on the largest degree r such that the complete linear system of the r -fold plane sections on the given curve is not cut out by surfaces of degree r . Although this result is of fairly geometric appearance, Castelnuovo's method of proof has a rather algebraic flavour.

Another early invisible occurrence of Castelnuovo-Mumford regularity was initiated by Hilbert's *Syzygientheorie* of 1890 [Hi1], and comes up notably in the work of Hentzelt-Noether (1923) [Hen-Noe] and of Hermann (1926) [Herm]. In fact based on the ideas of Hentzelt-Noether, Grete Hermann did answer in an affirmative way the *problem of the finitely many steps* ("Problem der endlich vielen Schritte"), which at that time was a much controversial issue caused by Hilbert's syzygy theory. The results of Hermann show that the minimal free resolution of an ideal generated by finitely many homogeneous polynomials "can be computed" in a (finite) number of steps which depends only on the number of indeterminates of the ambient ring and the maximal degree of the given polynomials.

Hermann's work is not at all constructive, and so it does not give rise to an explicit algorithm. It was indeed only around 1980, when such algorithms became practicable, based on *Gröbner base* techniques, implemented in Computer Algebra Systems like *Macaulay*, *Cocoa*, *Singular* and powered by high performance computers. And indeed: Castelnuovo-Mumford regularity provides the ultimate bound of complexity for these algorithms (see for example [Bu] or [Ma-Me]).

It was only in 1966, when Mumford gave a first proper definition of Castelnuovo-Mumford regularity (see [Mu1]), which he called *Castelnuovo regularity*. In fact, Mumford did define the notion of *being m -regular in the sense of Castelnuovo* for a *coherent sheaf of ideals over a projective space* and a given integer m . More precisely, a sheaf of ideals over a projective space is called *m -regular* if for all positive values of i the i -th *Serre cohomology group* of the $(m - i)$ -fold twist of this sheaf vanishes. The minimal possible value of m is what today usually is called the *Castelnuovo-Mumford regularity* of the sheaf of ideals in question. Moreover Mumford (loc.cit) did prove a fundamental bounding result, namely:

The Castelnuovo-Mumford regularity of a coherent sheaf of ideals over a projective space is bounded by the Hilbert polynomial of this ideal.

In fact Mumford's arguments allow to make this bound explicit. Hence his ideas paved the way for a new kind of algorithmic considerations in Algebraic Geometry, notably in the theory of *Hilbert schemes*. Now, with the powerful machinery of *sheaf cohomology for algebraic varieties* (introduced 1955 by Serre [Se]) at hands on had a good chance to link algorithmic and geometric aspects of projective varieties in a new way. Although Castelnuovo-Mumford regularity was originally defined in terms of sheaf cohomology, it may be expressed in terms of degrees of syzygies and hence is of basic significance in "classical" *Projective Algebraic Geometry*. So, it is not surprising, that the seminal article "*What can be computed in Algebraic Geometry?*" of Bayer-Mumford [B-Mu] (published in 1993, but known in preliminary form a number of years earlier) emphasizes a lot upper bounds for the Castelnuovo-Mumford regularity.

Castelnuovo-Mumford regularity also found much interest in Commutative Algebra. In 1982 Ooishi [O] did define the Castelnuovo-Mumford regularity of a graded module in terms of certain *local cohomology modules*. His definition essentially corresponds to Mumford's via the *Serre-Grothendieck Correspondence* between local cohomology and sheaf cohomology. In 1984 Eisenbud-Goto [E-G] made explicit the link between this "algebraic" Castelnuovo-Mumford regularity of a graded module over a polynomial ring and its *minimal free resolution*.

In their same paper, Eisenbud and Goto made the conjecture (or rather did ask the corresponding question), that the Castelnuovo-Mumford regularity $\text{reg}(X)$ (of the homogeneous vanishing ideal) of a projective variety X cannot exceed the value $\text{deg}(X) - \text{codim}(X) + 1$. What Castelnuovo did show in his paper of 1893 is precisely, that for smooth curves in projective 3-space this conjecture holds. Forever he characterized in geometric terms the curves for which $\text{reg}(X) = \text{deg}(X) - \text{codim}(X) - 1$. In 1983 it was shown by Gruson-Lazarsfeld-Peskine [Gru-La-P] that an irreducible curve in a projective space of arbitrary dimension satisfies the requested inequality, and that also in this more general setting the curves for which equality holds can be characterized geometrically. For smooth projective surfaces in characteristic 0 the conjecture has proved to be true by Pinkham 1986 [Pi] and Lazarsfeld 1987 [La], the latter paper containing again an investigation on the surfaces for which "equality holds". Meanwhile the conjecture of Eisenbud-Goto has become one of the great challenges of classical Projective Algebraic Geometry, which yet waits to find its general answer. The particularity of this conjecture is, that arbitrary graded polynomial ideals may have very large Castelnuovo-Mumford regularity (compared with the degree of generators) (see [Ma-Me]), whereas for graded prime ideals this invariant is expected to be very small.

In this course, we shall attack the subject of Castelnuovo-Mumford regularity from the algebraic side, starting with Ooishi's definition. So we expect the reader to have a sound background in basic Commutative Algebra (as found in [Br-Bo-Ro], [Sh], [Kun1], [E1] or [Mat] for example). Clearly we also have

to expect a solid footing in Local Cohomology Theory consisting at least of the material presented in [Br-Fu-Ro] or (even better) in [Br-Sh1]. This includes also some basic notions of sheaf cohomology, at least over projective schemes as they are presented in [Br-Fu-Ro](Chapters 11,12) or [Br-Sh1](Shaper 20). Clearly we also shall discuss a number of results which refer to Algebraic Geometry. So, some basic (see [Br0] for example) or more advanced (see [H1]) knowledge of this field will make this course more profitable. But for the understanding of the course, these prerequisites are optional.

We allow ourselves to present a number of results, which we do not prove. When we give proves, they usually will only rely on the mentioned prerequisites. We first will present a number of basic results about local cohomology, which are not given in [Br-Fu-Ro] but only in [Br-Sh1]. We will proceed in a way that may be seen as a direct continuation of our fairly self-contained approach [Br-Fu-Ro], but we also shall rely on a number of "eside entries" from [Br-Sh1], which we do not prove. In the cases, where we reprove a result which is already proved in [Br-Sh1], we use a different approach in this course. So, in this respect we also continue the policy pursued in [Br-Fu-Ro].

According to the nature of the subject, many results of this course shall give upper bounds for the Castelnuovo-Mumford regularity, as such bounds are the driving force of the whole theory. In addition we shall restrict ourselves to consider only standard graded rings and modules. Indeed, during the last decade, multi-graded local cohomology has seen a fast development, mainly driven by the investigation of *Toric Varieties* and *Toric Schemes* (see [Ro] for example) and correspondingly there are versions of multi-graded Castelnuovo-Mumford regularity. But at the moment, these matters seem not yet to be at the state of maturity to teach them in a regular Master course.

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1. SOME PREREQUISITES FROM LOCAL COHOMOLOGY

In this section we recall a few facts about Local Cohomology Theory. Our basic references for this are [Br-Fu-Ro] and [Br-Sh1]. In our reminders we primarily shall quote the corresponding results of [Br-Fu-Ro]. Concerning basic notions of Commutative Algebra, we recommend to consult [Br-Bo-Ro], or alternatively [Sh], [Kun1], or also [E1] or [Mat]. As a basic reference in Homological Algebra we recommend [Rot].

1.1. Notation and Reminder. (*Local Cohomology and Torsion Functors*) A) Throughout this section let R be a commutative unitary Noetherian ring and let $\mathfrak{a} \subseteq R$ be an ideal of R . For each $n \in \mathbb{N}_0$ let $H_{\mathfrak{a}}^n = H_{\mathfrak{a}}^n(\bullet)$ denote the n -th local cohomology functor with respect to \mathfrak{a} (see [Br-Fu-Ro](2.14)).

B) Let $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{a}}(\bullet)$ denote the \mathfrak{a} -torsion-functor (see [Br-Fu-Ro](1.15)). Keep in mind that this functor is left exact and that for each R -module M we have (see [Br-Fu-Ro] (1.19), (1.2))

$$\text{Gamma}_{\mathfrak{a}}(M) = \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n).$$

Moreover for each $n \in \mathbb{N}$ the n -th local cohomology functor $H_{\mathfrak{a}}^n = H_{\mathfrak{a}}^n(\bullet)$ is nothing else than the n -th right derived functor $\mathcal{R}^n \Gamma_{\mathfrak{a}} = \mathcal{R}^n \Gamma_{\mathfrak{a}}(\bullet)$ of the torsion functor $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{a}}(\bullet)$ (see [Br-Fu-Ro] (2.14)). In particular all functors $H_{\mathfrak{a}}^n(\bullet)$ are linear functors of R -modules and we may identify $H_{\mathfrak{a}}^0(\bullet) = \Gamma_{\mathfrak{a}}(\bullet)$ (see [Br-Fu-Ro] (3.4),(2.13)C)).

1.2. Exercise and Definition. (*Quasi-Divisible Modules*) A) For an element $s \in R$ let

$$(\bullet)_s : \{s^n | n \in \mathbb{N}_0\}^{-1}(\bullet) : (M \xrightarrow{h} N) \mapsto (M_s \xrightarrow{h_s} N_s)$$

denote the (exact) functor of R -modules to R_s -modules given by taking up powers of s as denominators and consider the natural transformation

$$\eta_s : Id \rightarrow (\bullet)_s : M \mapsto (M \xrightarrow{\eta_{s,M}} M_s),$$

where $\eta_{s,M} : M \rightarrow M_s, m \mapsto 1/m$ is the canonical map. Fix the element $s \in R$ and show, that for an R -module M the following statements are equivalent:

- (i) The natural map $\eta_{s,M} : M \rightarrow M_s$ is surjective.
- (ii) The multiplication map $s : M/\Gamma_{\langle s \rangle}(M) \rightarrow M/\Gamma_{\langle s \rangle}(M)$ is an isomorphism.
- (iii) $H_{\langle s \rangle}^1(M) = 0$.

(Observe that $s \in \text{NZD}_R(M/\Gamma_{\langle s \rangle}(M))$ and that $H_{\langle s \rangle}^1(M) \cong H^1(M/\Gamma_{\langle s \rangle}(M))$.)

B) Keep the previous notations. An R -module M which satisfies the equivalent conditions (i),(ii) and (iii) of part A) is said to be *quasi-divisible with respect to s* . If $S \subseteq R$, the R -module M is said to be *quasi-divisible with respect to S* if it quasi-divisible with respect to all $s \in S$. The R -module M is said to

be *quasi-divisible* at all, if it is quasi-divisible with respect to R . Prove the following statements:

- a) The set of all elements $s \in R$ with respect to which M is quasi-divisible is closed under multiplication and contains all $s \in R$ for which $sM = M$ or $s^n M = 0$ for some $n \in \mathbb{N}$.
- b) Each injective R -module I is quasi-divisible.
- c) If M is quasi-divisible with respect to $S \subseteq R$ and $h : M \rightarrow P$ is an epimorphism of R -modules, then P is quasi-divisible with respect to S .

1.3. Reminder. (*Triads and their Derived Sequences*) (See [Br-Fu-Ro] (4.13)) Let R' be a second ring and let F, G, H be three additive functors from R -modules to R' -modules. Let $\mu : F \rightarrow G$ and $\nu : G \rightarrow H$ be two natural transformations. We then call

$$\Delta : F \xrightarrow{\mu} G \xrightarrow{\nu} H$$

a *triad of functors*, if for each injective R -module I the sequence

$$0 \rightarrow F(I) \xrightarrow{\mu_I} G(I) \xrightarrow{\nu_I} H(I) \rightarrow 0$$

is exact. In this case, for each R -module M there is a natural exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{R}^0 F(M) &\xrightarrow{\mathcal{R}^0 \mu_M} \mathcal{R}^0 G(M) \xrightarrow{\mathcal{R}^0 \nu_M} \mathcal{R}^0 H(M) \\ \xrightarrow{\delta_M^{0,\Delta}} \mathcal{R}^1 F(M) &\xrightarrow{\mathcal{R}^1 \mu_M} \mathcal{R}^1 G(M) \xrightarrow{\mathcal{R}^1 \nu_M} \mathcal{R}^1 H(M) \\ \xrightarrow{\delta_M^{1,\Delta}} \mathcal{R}^2 F(M) &\xrightarrow{\mathcal{R}^2 \mu_M} \mathcal{R}^2 G(M) \xrightarrow{\mathcal{R}^2 \nu_M} \dots \end{aligned}$$

where for each $n \in \mathbb{N}_0$ the n -th *right derived transformations* of μ and ν are denoted respectively by $\mathcal{R}^n \mu$ and $\mathcal{R}^n \nu$. We call this sequence the *right derived sequence of (the triad) Δ associated to M* (see [Br-Fu-Ro] (4.13)) and we sometimes denote it by $\mathcal{R}\Delta(M)$.

1.4. Construction and Exercise. (*The Comparison Sequences*) A) Fix an element $b \in R$. Observe that we have a natural transformation

$$\eta^{a,b} : \Gamma_a(\bullet) \rightarrow \Gamma_a(\bullet)_b,$$

given by

$$M \mapsto (\eta_M^{a,b} = \eta_b^{\Gamma_a(M)} : \Gamma_a(M) \rightarrow \Gamma_a(M)_b).$$

In addition consider the natural transformation

$$\iota = \iota^{a,b} : \Gamma_{a+(b)}(\bullet) \xrightarrow{\iota = \iota^{a,b}} \Gamma_a(\bullet),$$

given by

$$\iota = \iota^{a,b} : M \mapsto (\Gamma_{a+(b)}(M) \xrightarrow{\iota_M = \iota_M^{a,b}} \Gamma_a(M)),$$

where $\iota_M = \iota_M^{a,b}$ denotes the inclusion map. Observe that for each R -module M we have $\text{Ker}(\iota_M^{a,b}) = \Gamma_{a+(b)}(M)$ and hence an exact sequence

$$0 \rightarrow \Gamma_{a+(b)}(M) \xrightarrow{\iota_M^{a,b}} \Gamma_a(M) \xrightarrow{\eta_M^{a,b}} \Gamma_a(M)_b.$$

Now, use [Br-Fu-Ro] (3.14) and (1.2) B) to show that we have the triad of functors

$$\Delta = \Delta^{a,b} : \Gamma_{a+\langle b \rangle}(\bullet) \xrightarrow{\iota} \iota^{a,b} \Gamma_a(\bullet) \xrightarrow{\eta^{a,b}} \Gamma_a(\bullet)_b,$$

the so called *comparison triad of \mathbf{a} with respect to b* .

B) Keep the above notations and hypotheses, fix an R -module M and consider the right derived sequence $\mathcal{R}\Delta(M) = \mathcal{R}\Delta^{a,b}(M)$ of the triad $\Delta = \Delta^{a,b}$ associated to M , which clearly takes the shape

$$\begin{aligned} 0 \rightarrow H_{a+\langle b \rangle}^0(M) &\xrightarrow{\mathcal{R}^0 \iota_M} H_a^0(M) \xrightarrow{\mathcal{R}^0 \eta_M} \mathcal{R}^0(\Gamma_a(\bullet)_b)(M) \\ \xrightarrow{\delta_M^{\Delta,0}} H_{a+\langle b \rangle}^1(M) &\xrightarrow{\mathcal{R}^1 \iota_M} H_a^1(M) \xrightarrow{\mathcal{R}^1 \eta_M} \mathcal{R}^1(\Gamma_a(\bullet)_b)(M) \\ \xrightarrow{\delta_M^{\Delta,1}} H_{a+\langle b \rangle}^2(M) &\xrightarrow{\mathcal{R}^2 \iota_M} H_a^2(M) \xrightarrow{\mathcal{R}^2 \eta_M} \dots \end{aligned}$$

Now, observe that the functor $\Gamma_a(\bullet)_b$ is nothing else than the composition $(\bullet)_b \circ \Gamma_a$ of the exact functor $(\bullet)_b$ with the torsion functor Γ_a . So for each $n \in \mathbb{N}_0$ we have a natural equivalence of functors (see [Br-Fu-Ro] (5.3))

$$\gamma^n = \gamma^{n,(\bullet)_b, \Gamma_a} : H_a^n(\bullet)_b = (\bullet)_b \circ \mathcal{R}^n \Gamma_a \xrightarrow{\cong} \mathcal{R}^n((\bullet)_b \circ \Gamma_a) = \mathcal{R}^n(\Gamma_a(\bullet)_b) =: U^n(\bullet).$$

In particular, for each $n \in \mathbb{N}_0$ and each R -module M as above, we have a natural isomorphism

$$\gamma_M^n =: H_a^n(M)_b \xrightarrow{\cong} \mathcal{R}^n(\Gamma_a(\bullet)_b)(M) = U^n(M).$$

Conclude that multiplication with b yields an isomorphism

$$b : U^n(M) \xrightarrow{\cong} U^n(M),$$

that

$$\Gamma_{\langle b \rangle}(U^n(M)) = 0$$

and that $U^n(M)$ is quasi-divisible with respect to b . Use the fact that $H_{a+\langle b \rangle}^n(M)$ is $\langle b \rangle$ -torsion to show that in the right derived sequence $\mathcal{R}\Delta(M) = \mathcal{R}\Delta^{a,b}(M)$ of part B), we have

$$\text{Im}(\mathcal{R}^n \iota_M) = \text{Ker}(\mathcal{R}^n \eta_M) = \Gamma_{\langle b \rangle}(H_a^n(M))$$

for all $n \in \mathbb{N}_0$ all R -modules M .

C) Keep the previous notations and hypotheses, let $n \in \mathbb{N}_0$ and let M be an R -module. Observe that the derived sequence $\mathcal{R}\Delta(M)$ together with the equalities obtained at the end of part B) gives rise to an exact sequence

$$0 \rightarrow U^{n-1}(M)/\text{Im}(\mathcal{R}^{n-1} \eta_M) \xrightarrow{\varepsilon_M^n} H_{a+\langle b \rangle}^n(M) \xrightarrow{\pi_M^n = \pi_M^{n,a,b}} \Gamma_{\langle b \rangle}(H_a^n(M)) \rightarrow 0,$$

where the occurring maps ε_M^n and π_M^n are respectively induced by $\delta_M^{\Delta, n-1}$ and $\mathcal{R}^n \iota_M$. Now, consider the obvious short exact sequence

$$\mathbb{S}_M : 0 \rightarrow \text{Im}(\mathcal{R}^{n-1} \eta_M) \rightarrow U^{n-1}(M) \rightarrow U^{n-1}(M)/\text{Im}(\mathcal{R}^{n-1} \eta_M) \rightarrow 0,$$

apply cohomology with respect to $\langle b \rangle$ and occlude that we get the exact sequence

$$0 \rightarrow H_{\langle b \rangle}^0(U^{n-1}(M)/\text{Im}(\mathcal{R}^{n-1}\eta_M)) \xrightarrow{\delta_M^0} H_{\langle b \rangle}^1(\text{Im}(\mathcal{R}^{n-1}\eta_M)) \rightarrow H_{\langle b \rangle}^1(U^{n-1}(M)),$$

in which δ_M^0 is the 0-th connecting homomorphism with respect to the short exact sequence \mathbb{S}_M . Use the observations made in the last paragraph of part B) to show that we get an isomorphism

$$\delta_M^0 : U^{n-1}(M)/\text{Im}(\mathcal{R}^{n-1}\eta) \xrightarrow{\cong} H_{\langle b \rangle}^1(\text{Im}(\mathcal{R}^{n-1}\eta_M)).$$

Use the last statement of part B) to show that there is an isomorphism

$$\kappa_M : H_{\mathfrak{a}}^{n-1}(M)/\Gamma_{\langle b \rangle}(H_{\mathfrak{a}}^{n-1}(M)) \xrightarrow{\cong} \text{Im}(\mathcal{R}^{n-1}\eta_M),$$

induced by $\mathcal{R}^{n-1}\eta_M$. Let $p_M : H_{\mathfrak{a}}^{n-1}(M) \rightarrow H_{\mathfrak{a}}^{n-1}(M)/\Gamma_{\langle b \rangle}(H_{\mathfrak{a}}^{n-1}(M))$ be the canonical map and show that we have an isomorphism

$$\mu_M := (\delta_M^0)^{-1} \circ H_{\langle b \rangle}^1(\kappa_M) \circ H_{\langle b \rangle}^1(p_M) : H_{\langle b \rangle}^1(H_{\mathfrak{a}}^{n-1}(M)) \xrightarrow{\cong} U^{n-1}(M)/\text{Im}(\mathcal{R}^{n-1}\eta_M).$$

Set $\lambda_M^n = \lambda_M^{n,\mathfrak{a},b} := \varepsilon_M \circ \mu_M$, to end up with the short exact sequence

$$0 \rightarrow H_{\langle b \rangle}^1(H_{\mathfrak{a}}^{n-1}(M)) \xrightarrow{\lambda_M^{n,\mathfrak{a},b}} H_{\mathfrak{a}+\langle b \rangle}^n(M) \xrightarrow{\pi_M^{n,\mathfrak{a},b}} \Gamma_{\langle b \rangle}(H_{\mathfrak{a}}^n(M)) \rightarrow 0,$$

which we call the *n-th comparison sequence of \mathfrak{a} with respect to b and associated to M* .

D) Observe that the three homomorphisms $\mathcal{R}^{n-1}\eta_M$, $\mathcal{R}^n\iota_M$, p_M as well as the connecting homomorphisms δ_M^0 (see [Br-Fu-Ro] (3.9)C)) and $\delta_M^{n-1,\Delta}$ (see [Br-Fu-Ro] (4.13)D)) constitute natural transformations if M runs through all R -modules. Conclude that the homomorphisms $\lambda_M^n = \lambda_M^{n,\mathfrak{a},b}$ and $\pi_M^n = \pi_M^{n,\mathfrak{a},b}$ constitute natural homomorphisms, too. So the above comparison sequence is *natural* for all $n \in \mathbb{N}$.

E) Now, let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded Noetherian ring, let the ideal $\mathfrak{a} \subseteq R$ be graded and let the element $b \in R$ be homogeneous. For an arbitrary graded ideal $\mathfrak{b} \subseteq R$ consider the graded torsion functor ${}^*\Gamma_{\mathfrak{b}}(\bullet)$, as introduced in [Br-Fu-Ro] (8.8)B). Show that this time we get a triad of graded modules

$${}^*\Delta = {}^*\Delta^{\mathfrak{a},b} : {}^*\Gamma_{\mathfrak{a}+\langle b \rangle}(\bullet) \xrightarrow{\iota = \iota^{\mathfrak{a},\langle b \rangle}} {}^*\Gamma_{\mathfrak{a}}(\bullet) \xrightarrow{\eta^{\mathfrak{a},b}} {}^*\Gamma(\bullet)_b$$

in the sense of [Br-Fu-Ro] (11.9)B). Deduce from this, that by following the ideas of [Br-Fu-Ro](11.9)C) for each graded R -module M , we end up with a right derived triad sequence $\mathcal{R}{}^*\Delta(M) = \mathcal{R}{}^*\Delta^{\mathfrak{a},b}(M)$ which looks as the sequence in Part B), with all occurrences of H replaced by *H , where ${}^*H_{\mathfrak{b}}^n := \mathcal{R}{}^*\Gamma_{\mathfrak{b}}$ for each $n \in \mathbb{N}_0$ and each graded ideal $\mathfrak{b} \subseteq R$. Now make sure, that in this graded setting the arguments performed in part C) still work, so that in our comparison sequence we may replace H by *H and Γ by ${}^*\Gamma$ at all occurrences. Finally, observe that [Br-Fu-Ro] (8.24) shows that the n-th comparison sequence becomes a sequence of graded R -modules in our graded situation.

We now want to use the comparison sequences to establish a vanishing result for the local cohomology of modules, which satisfy a certain quasi-divisibility condition. We first give the following definition.

1.5. Definition. Let $S \subseteq R$. The ideal $\mathfrak{a} \subseteq R$ is called an S -ideal if it is generated by elements of S .

Now, we can prove the following result.

1.6. Proposition. Let $S \subseteq R$ and let M be an R -module such that $\Gamma_{\mathfrak{a}}(M)$ is quasi-divisible with respect to S for each S -ideal $\mathfrak{a} \subseteq R$. Then $H_{\mathfrak{a}}^n(M) = 0$ for all $n > 0$ and each S -ideal $\mathfrak{a} \subseteq R$.

Proof. Let $\mathfrak{a} = \langle a_1, a_2, \dots, a_r \rangle$, with $a_1, a_2, \dots, a_r \in S$. We show by induction on r , that $H_{\mathfrak{a}}^n(M) = 0$ for all $n > 0$. If $r = 0$ we have $\mathfrak{a} = 0$ and our claim is clear. So, let $r > 0$ and set $\mathfrak{b} = \langle a_1, a_2, \dots, a_{r-1} \rangle$. The comparison sequence of \mathfrak{b} with respect to a_r associated to M now gives an exact sequence

$$H_{\langle a_r \rangle}^1(H_{\mathfrak{b}}^{n-1}(M)) \rightarrow H_{\mathfrak{a}}^n(M) \rightarrow \Gamma_{\langle a_r \rangle}(H_{\mathfrak{b}}^n(M)).$$

If $n = 1$, the first module in this sequence vanishes as $H_{\mathfrak{b}}^0(M)$ is quasi-divisible with respect to a_r . Now we may conclude as by induction $H_{\mathfrak{b}}^m(M) = 0$ for all $m > 0$. \square

1.7. Exercise and Remark. A) Assume now, that $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is a Noetherian graded ring and let $R^h := \bigcup_{n \in \mathbb{Z}} R_n$ denote the set of homogeneous elements of R . Let I be a *injective (graded) R -module (see [Br-Fu-Ro] (8.12)). Show that I is quasi-divisible with respect to R^h . Use (1.6) and the fact that $\Gamma_{\mathfrak{a}}(I)$ is *injective for each graded ideal $\mathfrak{a} \subseteq R$ (see [Br-Fu-Ro] (8.22)B)) to show that $H_{\mathfrak{a}}^n(I) = 0$ for each graded ideal $\mathfrak{a} \subseteq R$.

B) What we have shown in part A) is nothing else than Theorem 8.23 of [Br-Fu-Ro]. This hints, that the concept of quasi-divisibility and the comparison sequences provide a powerful tool for proving (not necessarily new) vanishing results in Local Cohomology Theory. A collection of such proofs (as well as a detailed introduction to comparison sequences and quasi-divisibility) is given in [Tru].

C) To illustrate what we said in part B), we suggest to reprove [Br-Fu-Ro] (4.19) just on use of the comparison sequences and to pay attention to the fact, that now the induction argument can be started with $r = 0$, so that the auxiliary result [Br-Fu-Ro](4.18) is not needed.

We know that left-composition with exact functors commutes with right derivation. More precisely, if E is an (additive covariant) left exact functor from R -modules to R' -modules and F is an exact functor from R' -modules to R'' -modules, there are natural equivalences $\gamma^n : F \circ \mathcal{R}^n E \xrightarrow{\cong} \mathcal{R}^n(F \circ E)$ (see [Br-Fu-Ro] (5.3)). The construction below is devoted to the reverse situation, in which the "innere" functor E is exact, and hence it concerns right-composition with exact functors.

1.8. Construction and Exercise. (*Right-Composition with Exact Functors*)
 A) Let R , R' and R'' be rings (which need not be Noetherian this time), let E be a (covariant, additive) exact functor from R -modules to R' -modules and let F be a (covariant, additive) left exact functor from R' -modules to R'' -modules. Observe that the composite functor $F \circ E$ from R -modules to R'' -modules is left exact and that we may identify

$$\mathcal{R}^0(F \circ E) = F \circ E = (\mathcal{R}^0 F) \circ E.$$

Our first aim is to show that there is a unique family

$$(\nu^n = \nu^{n,F,E} : \mathcal{R}^n(F \circ E) \rightarrow (\mathcal{R}^n F) \circ E)_{n \in \mathbb{N}_0}$$

of natural transformations which satisfies the following requirements:

a) For each R -module M

$$\nu_M^{0,F,E} = id_{F \circ E(M)} : \mathcal{R}^0(F \circ E)(M) \rightarrow ((\mathcal{R}^0 F) \circ E)(M) = \mathcal{R}^0 F(E(M))$$

is the identity map.

b) For each $n \in \mathbb{N}$ and each exact sequence of R -modules

$$\mathbb{I} : 0 \rightarrow M \xrightarrow{h} I \xrightarrow{l} P \rightarrow 0$$

in which I is injective, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{R}^{n-1}(F \circ E)(P) & \xrightarrow{\delta_{\mathbb{I}}^{n-1, F \circ E}} & \mathcal{R}^n(F \circ E)(M) \\ \nu_P^{n-1, F, E} \downarrow & & \nu_M^{n, F, E} \downarrow \\ \mathcal{R}^{n-1}F(E(P)) & \xrightarrow{\delta_{E(\mathbb{I})}^{n-1, F}} & \mathcal{R}^n F(E(M)) \end{array}$$

where $\delta_{\mathbb{I}}^{n-1, F \circ E}$ is the $(n-1)$ -th connecting homomorphism with respect to $F \circ E$ associated to the exact sequence \mathbb{I} and $\delta_{E(\mathbb{I})}^{n-1, F}$ is the $(n-1)$ -th connecting homomorphism with respect to F associated to the exact sequence $E(\mathbb{I})$ (see [Br-Fu-Ro] (3.8)B).

B) The transformations ν^n can be constructed recursively on n . For each R -module M set $\nu^0 = id_M$. Then assume that $n > 0$ and that the transformations $\nu^0, \nu^1, \dots, \nu^{n-1}$ are already constructed such that the above requirements A) a) and b) are satisfied. Fix an R -module M and chose an exact sequence

$$\mathbb{I} : 0 \rightarrow M \rightarrow I \rightarrow P \rightarrow 0$$

in which the R -module I is injective. Show that there is a unique homomorphism of R'' -modules

$$\nu_{\mathbb{I}}^n : \mathcal{R}^n(F \circ E)(M) \rightarrow \mathcal{R}^n F(E(M))$$

such that the diagram in the requirement A)b) commutes if one replaces the right vertical map in this diagram by $\nu_{\mathbb{I}}^n$. (You may proceed like in the first part

of the proof of [Br-Fu-Ro] (8.21)). Now chose a homomorphism of R -modules $h : M \rightarrow N$ and a short exact sequence

$$\mathbb{J} : 0 \rightarrow N \rightarrow J \rightarrow Q \rightarrow 0$$

in which J is injective and show that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}^n(F \circ E)(M) & \xrightarrow{\nu_{\mathbb{J}}^n} & \mathcal{R}^n F(E(M)) \\ \mathcal{R}^n(F \circ E)(h) \downarrow & & \downarrow \mathcal{R}^n F(E(h)) \\ \mathcal{R}^n(F \circ E)(N) & \xrightarrow{\nu_{\mathbb{J}}^n} & \mathcal{R}^n F(E(N)) \end{array}$$

in which $\nu_{\mathbb{J}}^n$ is defined accordingly to $\nu_{\mathbb{I}}^n$. (For this you might get inspiration by looking at the second half of the proof of [Br-Fu-Ro] (8.21).) Conclude from this, that the homomorphism $\nu_{\mathbb{I}}^n$ depends only on M and not on the chosen exact sequence \mathbb{I} . Therefore, we may define:

$$\nu_M^n = \nu_M^{n,F,E} := \nu_{\mathbb{I}}^n.$$

Now, conclude that the assignment

$$\nu^{n,F,E} : M \mapsto (\nu_M^{n,F,E} : \mathcal{R}^n(F \circ E)(M) \rightarrow \mathcal{R}^n F(E(M)))$$

defines indeed a natural transformation

$$\nu^n = \nu^{n,F,E} : \mathcal{R}^n(F \circ E) = \mathcal{R}^n(F \circ E)(\bullet) \rightarrow \mathcal{R}^n F(E(\bullet)) = (\mathcal{R}^n F) \circ E.$$

C) Next, reconsult your previous arguments to prove the following:

a) If $\mathcal{R}^n F(E(I)) = 0$ for all injective R -modules I and all $n > 0$, then all the transformations $\nu^{n,F,E}$ are natural equivalences.

D) Finally, assume that the three rings $R = \bigoplus_{n \in \mathbb{Z}} R_n$, $R' = \bigoplus_{n \in \mathbb{Z}} R'_n$ and $R'' = \bigoplus_{n \in \mathbb{Z}} R''_n$ are graded and that the two functors E and F now correspondingly are functors of graded modules and hence convert homomorphisms of graded R (respectively R')-modules to homomorphisms of graded R' (respectively R'')-modules. Reconsider your previous arguments to make clear, that all what has been stated in parts A),B) and C) translates mutatis mutandis to the “*graded setting*”, clearly with “injective” replaced by “*injective”.

E) It is natural to ask, whether one has the commutative diagram occurring in requirement A)b) without the restriction that the middle module I in the exact sequence \mathbb{I} is injective. This is indeed true. One way to prove this, is to start with an arbitrary short exact sequence of R -modules

$$\mathbb{S} : 0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0,$$

and to use an injective resolution of \mathbb{S} (see [Br-Fu-Ro](3.6)) in order to end up with a commutative diagram \mathbb{D} consisting of three short exact rows and three short exact columns, having \mathbb{S} as the bottom row and having three injective modules in the middle row. Then apply the right derived sequences with respect to $F \circ E$ associated to all rows and columns of the diagram \mathbb{D} . Use

the fact (or prove it, if you are courageous), that you get a diagram in which all squares which consist of connecting homomorphisms are anti-commutative (see [Rot](11.24) for example). Then form the commutative diagram $E(\mathbb{D})$, apply the right derived sequences with respect to F and observe again the mentioned anti-commutativity. Then build up the appropriate cube diagram and do not forget that $(-1)(-1) = 1 \dots$

Our next aim is to consider (in the situation where R' is an R -algebra) exact linear functors from R -modules to R' modules which commute with taking torsion with respect to ideals in R and their extensions to R' .

1.9. Remark and Definition. (*Torsion-Faithful Functors*) A) Let $f : R \rightarrow R'$ be a homomorphism of Noetherian rings and let $\mathfrak{a} \subseteq R$ be an ideal of R . Let E be a covariant linear functor from R -modules to R' -modules. We say that the functor E is *torsion-faithful with respect to \mathfrak{a}* if the R' -module $E(\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}R'$ -torsion for all R -modules M . We say that E is *torsion-faithful at all*, if it is torsion-faithful with respect to all ideals $\mathfrak{a} \subseteq R$.

B) Keep the notations and hypotheses of part A) and let E' be a covariant linear functor from R' -modules to R -modules. Similarly as above we say that E' is *torsion-faithful with respect to \mathfrak{a}* if for each R' -module M' the R -module $E'(\Gamma_{\mathfrak{a}R'}(M'))$ is \mathfrak{a} -torsion. Again we say that E' is *torsion-faithful at all* if it is torsion faithful with respect to all ideals $\mathfrak{a} \subseteq R$.

C) Clearly, the notions defined in part A) and B) may be defined completely analogous in the graded setting: namely, assume in addition, that the two rings R and R' are graded and that f is a homomorphism of graded rings, so that $f(R_n) \subseteq R'_n$ for all $n \in \mathbb{Z}$. Let the functors E and E' be functors of graded R -modules which commute with shifting and scalar multiplication with homogeneous elements (see [Br-Fu-Ro] (8.6D)). Then we define the notion of **torsion-faithfulness* of E and E' with respect to graded ideals $\mathfrak{a} \subseteq R$ as in part A) and B), just with $*\Gamma_{\mathfrak{a}R'}$ instead of $\Gamma_{\mathfrak{a}R'}$ and $*\Gamma_{\mathfrak{a}}$ instead of $\Gamma_{\mathfrak{a}}$ respectively.

1.10. Examples and Exercise. A) Let $S \subseteq R$ be multiplicatively closed and let $\eta_S : R \rightarrow S^{-1}R$ be the canonical homomorphism. Show that the (exact) functor

$$S^{-1}\bullet : (M \xrightarrow{h} N) \rightsquigarrow (S^{-1}M \xrightarrow{S^{-1}h} S^{-1}N)$$

of taking up denominators at S (see (1.14) B)) is torsion-faithful (see also [Br-Fu-Ro] (5.6)).

B) Let $f : R \rightarrow R'$ be a homomorphism of Noetherian rings and let

$$\bullet \upharpoonright_R : (M' \xrightarrow{h'} N') \rightsquigarrow (M' \upharpoonright_R \xrightarrow{h' \upharpoonright_R} N' \upharpoonright_R)$$

be the (exact) functor of scalar restriction by means of f (see [Br-Fu-Ro] (1.14)C)). Show that this functor is torsion-faithful.

C) Let $f : R \rightarrow R'$ be a homomorphism of R -modules and let $\mathfrak{a} \subseteq R$ be an ideal. Prove the following statements

- a) Let E be a covariant linear exact functor from R -modules to R' modules. Assume that for each \mathfrak{a} -torsion module M and each element $x \in E(M)$ there is a finitely generated submodule $N \subseteq M$ such the $x \in \text{Im}(E(\iota))$, where $\iota : N \rightarrow M$ is the inclusion map. Then E is torsion-faithful with respect to \mathfrak{a} .
- b) Let E' be a covariant linear exact functor from R' -modules to R -modules. Assume that for each $\mathfrak{a}R'$ -torsion module M' and each element $x' \in E'(M')$ there is a finitely generated submodule $N' \subseteq M'$ such that $x' \in \text{Im}(E'(\iota'))$, where $\iota' : N' \rightarrow M'$ is the inclusion map. Then E' is torsion-faithful with respect to \mathfrak{a} .

D) There are obvious “graded versions” of the examples given in part A), B) and C). Formulate and justify these, keeping in mind (1.9)C).

1.11. **Exercise and Remark.** A) Let R and R' be rings (not necessarily Noetherian this time) and let E be a covariant additive exact functor from R -modules to R' -modules. Let M be an R -module and let $U, V \subseteq M$ be submodules. For each submodule $N \subseteq M$ let $\iota_N : N \rightarrow M$ denote the inclusion map. Use the short exact sequence

$$0 \rightarrow U \cap V \xrightarrow{\alpha} U \oplus V \xrightarrow{\beta} M, \alpha : x \mapsto (x, x), \beta : (u, v) \mapsto u - v$$

to show that

- a) $\text{Im}(E(\iota_{U \cap V})) = \text{Im}E((\iota_U)) \cap \text{Im}(E(\iota_V))$.
- b) The functor E commutes with finite intersections: if $N_1, \dots, N_r \subseteq M$ are finitely many submodules, then

$$\text{Im}(E(\iota_{\bigcap_{i=1, \dots, r} N_i})) = \bigcap_{i=1, \dots, r} \text{Im}(E(\iota_{N_i})).$$

B) Now, let $f : R \rightarrow R'$ be a homomorphism of Noetherian rings, let E be a covariant linear exact functor from R -modules to R' -modules and let E' be a covariant linear functor from R' -modules to R -modules. For each ideal $\mathfrak{a} \subseteq R$ each R -module M and each R' -module M' let $\iota_M^{\mathfrak{a}} : \Gamma_{\mathfrak{a}}(M) \rightarrow M$ and $\iota_{M'}^{\mathfrak{a}R'} : \Gamma_{\mathfrak{a}R'}(M') \rightarrow M'$ denote the inclusion maps. Prove the following claims:

- a) If E is torsion-faithful with respect to aR for some $a \in R$, then, for each R -module M we have

$$\text{Im}(E(\iota_M^{aR})) = \Gamma_{aR'}(E(M)).$$

- b) If $\mathfrak{a} = \langle a_1, \dots, a_r \rangle$ and E is torsion-faithful with respect to $\langle a_i \rangle$ for all $i \in \{1, \dots, r\}$, then for each R -module M we have

$$\text{Im}(E(\iota_M^{\mathfrak{a}})) = \Gamma_{\mathfrak{a}R'}(E(M)).$$

In particular E is torsion-faithful with respect to \mathfrak{a} .

- c) If E' is torsion-faithful with respect to aR' for some $a \in R$, then, for each R' -module M' we have

$$\text{Im}(E(\iota_{M'}^{aR'})) = \Gamma_{aR}(E'(M')).$$

- d) If $\mathfrak{a} = \langle a_1, \dots, a_r \rangle$ and E' is torsion-faithful with respect to $\langle a_i \rangle$ for all $i \in \{1, \dots, r\}$, then, for each R' -module M' we have

$$\text{Im}(\iota_{M'}^{\mathfrak{a}R'}) = \Gamma_{\mathfrak{a}}(E'(M'))$$

In particular E' is torsion faithful with respect to \mathfrak{a} .

- C) Keep the notations and hypotheses of part B). Let $a \in R$. Show

- a) If E is torsion-faithful with respect to $\langle a \rangle$, and M is an R -module which is quasi-divisible with respect to a , then the R' -module $E(M)$ is quasi-divisible with respect to $f(a)$.
- b) If E' is torsion-faithful with respect to $\langle a \rangle$ and M' is an R' -module which is quasi-divisible with respect to $f(a)$, then the R -module $E'(M)$ is quasi-divisible with respect to a .

- D) Translate and verify all statements made in parts A), B) and C) to the graded setting, keeping in mind (1.9)C).

Now, we finally can give the results which are the main objective of this section.

1.12. Theorem. *Let $f : R \rightarrow R'$ be a homomorphism of Noetherian rings, let $a_1, a_2, \dots, a_r \in R$ and set $\langle a_1, a_2, \dots, a_r \rangle =: \mathfrak{a} \subseteq R$.*

- a) *If E is a covariant, linear exact functor from R -modules to R' -modules which is torsion-faithful with respect to $\langle a_i \rangle$ for all $i \in \{1, \dots, r\}$, then for each $n \in \mathbb{N}_0$ there is a natural equivalence*

$$\tau^{n, \mathfrak{a}, E} : E(H_{\mathfrak{a}}^n(\bullet)) \xrightarrow{\cong} H_{\mathfrak{a}R'}^n(E(\bullet)).$$

- b) *If E' is a covariant linear exact functor from R' -modules to R -modules which is torsion-faithful with respect to $\langle a_i \rangle$ for all $i \in \{1, \dots, r\}$, then for each $n \in \mathbb{N}_0$ there is a natural equivalence*

$$\sigma^{n, \mathfrak{a}, E'} : E'(H_{\mathfrak{a}R'}^n(\bullet)) \xrightarrow{\cong} H_{\mathfrak{a}}^n(E'(\bullet)).$$

Proof. eea): We set $S := \{a_1, \dots, a_r\}$. For each R -module M and each S -ideal $\mathfrak{b} \subseteq R$ (see (1.5)) let $\iota_M^{\mathfrak{b}} : \Gamma_{\mathfrak{b}}(M) \hookrightarrow M$ denote the inclusion map. According to (1.11)B)b) the functor E is torsion-faithful with respect to each S -ideal \mathfrak{b} , and for each S -ideal \mathfrak{b} and each R -module M , we get an isomorphism

$$\varepsilon_M^{\mathfrak{b}} : E(\Gamma_{\mathfrak{b}}(M)) \xrightarrow{\cong} \Gamma_{\mathfrak{b}R'}(E(M))$$

induced by $E(\iota_M^{\mathfrak{b}})$. In particular we have a natural equivalence

$$\varepsilon^{\mathfrak{a}} = \varepsilon : E(\Gamma_{\mathfrak{a}}(\bullet)) \xrightarrow{\cong} \Gamma_{\mathfrak{a}R'}(E(\bullet)).$$

So, by right-derivation we get for each $n \in \mathbb{N}_0$ a natural equivalence

$$\mathcal{R}^n \varepsilon : \mathcal{R}^n(E \circ \Gamma_{\mathfrak{a}})(\bullet) \xrightarrow{\cong} \mathcal{R}^n(\Gamma_{\mathfrak{a}R'} \circ E)(\bullet).$$

As E is exact, we also have a natural equivalence (see [Br-Fu-Ro] (5.3)B))

$$\gamma^n : (E \circ H_{\mathfrak{a}}^n)(\bullet) = (E \circ \mathcal{R}^n \Gamma_{\mathfrak{a}})(\bullet) \xrightarrow{\cong} \mathcal{R}^n(E \circ \Gamma_{\mathfrak{a}})(\bullet).$$

Now, let I be an injective R -module and let $\mathfrak{b} \subseteq R$ be an S -ideal. Then $\Gamma_{\mathfrak{b}}(I)$ is injective (see [Br-Fu-Ro] (3.14)) and hence quasi-divisible (see (1.2)B)). By (1.11)C)a) it follows that the R' -module $E(\Gamma_{\mathfrak{b}}(I))$ is quasi-divisible with respect to $f(S)$. In view of the above isomorphism $\varepsilon_I^{\mathfrak{b}}$ it thus follows, that $\Gamma_{\mathfrak{b}R'}(E(I))$ is quasi-divisible with respect to $f(S)$. This means that the R' -module $\Gamma_{\mathfrak{b}'}(E(I))$ is quasi-divisible with respect to $f(S)$ for each $f(S)$ -ideal $\mathfrak{b}' \subseteq R'$. As $\mathfrak{a}R'$ belongs to these ideals, it follows by (1.6), that

$$\mathcal{R}^n \Gamma_{\mathfrak{a}R'}(E(I)) = H_{\mathfrak{a}R'}^n(E(I)) = 0$$

for all $n > 0$. But in view of (1.8), this means that for each $n \in \mathbb{N}_0$ we have a natural equivalence

$$\nu^n : \mathcal{R}^n(\Gamma_{\mathfrak{a}R'} \circ E)(\bullet) \xrightarrow{\cong} (\mathcal{R}^n \Gamma_{\mathfrak{a}R'} \circ E)(\bullet) = H_{\mathfrak{a}R'}^n(E(\bullet)).$$

So, for each $n \in \mathbb{N}_0$ we end indeed up with a natural equivalence

$$\tau^{n,\mathfrak{a},E} = \nu^n \circ \mathcal{R}^n \varepsilon \circ \gamma^n : E(H_{\mathfrak{a}}^n(\bullet)) \xrightarrow{\cong} H_{\mathfrak{a}R'}^n(E(\bullet)).$$

ebe)”: The proof of this statement is similar to the proof of statement a), and we leave it as an exercise. \square

In the next result we use the notation $*H^n := \mathcal{R}^n * \Gamma$, as done already earlier.

1.13. Theorem. *Let $f : R = \bigoplus_{n \in \mathbb{Z}} R_n \rightarrow R' = \bigoplus_{n \in \mathbb{Z}} R'_n$ be a homomorphism of graded Noetherian rings. Let $a_1, a_2, \dots, a_r \in R^h = \bigcup_{n \in \mathbb{Z}} R_n$ be homogeneous elements and consider the graded ideal $\langle a_1, a_2, \dots, a_r \rangle =: \mathfrak{a} \subseteq R$.*

a) *If E is a covariant linear exact functor from graded R -modules to graded R' -modules which is torsion-faithful with respect to $\langle a_i \rangle$ for all $i \in \{1, \dots, r\}$, then, for each $n \in \mathbb{N}_0$ there is a natural equivalence*

$$*\tau^{n,\mathfrak{a},E} : E(*H_{\mathfrak{a}}^n(\bullet)) \xrightarrow{\cong} *H_{\mathfrak{a}R'}^n(E(\bullet)).$$

b) *If E' is a covariant linear exact functor from graded R' -modules to graded R -modules which is torsion-faithful with respect to $\langle a_i \rangle$ for all $i \in \{1, \dots, r\}$, then for each $n \in \mathbb{N}_0$ there is a natural equivalence*

$$*\sigma^{n,\mathfrak{a},E'} : E'(*H_{\mathfrak{a}R'}^n(\bullet)) \xrightarrow{\cong} *H_{\mathfrak{a}}^n(E'(\bullet)).$$

Proof. This proof is similar as the one of (1.12) and is obtained just by “translation to the graded setting”. We suggest it as an exercise. \square

1.14. **Remark and Exercise.** A) The natural equivalences established in (1.12) and (1.13) are indeed even natural with respect to taking right derived sequences. This means, that they also commute with connecting homomorphisms in cohomology sequences associated to short exact sequences. To see this, one has to prove the corresponding naturality of the natural equivalences γ^n and ν^n , which came up in the proof of (1.12). As for the equivalences ν^n , we gave some hint to this in (1.8)E).

B) If we apply (1.12)b) in the situation, where E' is the functor $\bullet \downarrow_R$ of scalar restriction (see (1.10)B)), and observe what is said in part A), we obtain the *Base Ring Independence of Local Cohomology*, where as applying (1.13)b) to the graded scalar restriction functor, we get the *Graded Base Ring Independence of Local Cohomology*.

C) Now, let $f : R \rightarrow R'$ be a flat homomorphism of Noetherian rings, so that the (covariant, linear, right exact) *tensor product functor* with R' from R -modules to R' -modules

$$R' \otimes_R \bullet : (M \xrightarrow{h} N) \mapsto (R' \otimes_R M \xrightarrow{R' \otimes_R h} R' \otimes_R N),$$

(where $R' \otimes_R h$ is given by $x' \otimes m \mapsto x' \otimes h(m)$) is exact. It is easy to verify, that this functor satisfies the requirement (1.10)C)a) for each ideal $\mathfrak{a} \subseteq R$. Therefore we may say on use of (1.10)C)a) and (1.12)a):

- a) The exact functor $R' \otimes_R \bullet$ is torsion-faithful.
- b) For each ideal $\mathfrak{a} \subseteq R$ and all $n \in \mathbb{N}_0$ there is a natural equivalence

$$\tau^{n, \mathfrak{a}, R' \otimes_R \bullet} : R' \otimes_R H_{\mathfrak{a}}^n(\bullet) \xrightarrow{\cong} H_{\mathfrak{a}R'}^n(R' \otimes_R \bullet).$$

This is nothing else than the *Flat Base Change Property of Local Cohomology*.

In these lectures, we shall mainly have to use a special graded version of the Flat Base Change Property of Local Cohomology. We pave the way for this in our next remark.

1.15. **Remark and Exercise.** A) Now, assume that the Noetherian ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is positively graded and let $f_0 : R_0 \rightarrow R'_0$ be a flat homomorphisms of Noetherian rings. We consider the R'_0 -algebra $R' := R'_0 \otimes_{R_0} R$ which carries a canonical grading and thus may be written in the form

$$R' := R'_0 \otimes_{R_0} R = \bigoplus_{n \in \mathbb{N}_0} R'_0 \otimes R_n.$$

We canonically identify $R'_0 \otimes_{R_0} R_0 = R'_0$. As R is Noetherian and positively graded, there are finitely many homogeneous elements $x_1, \dots, x_r \in R$ such that

$$R = R_0[x_1, \dots, x_r].$$

From this it follows, that

$$R' = R'_0[1 \otimes x_1, \dots, 1 \otimes x_r].$$

As R' is Noetherian, we see in particular, that R' is Noetherian, too. Observe in addition, that we have a homomorphism of graded rings

$$f : R \rightarrow R' = R'_0 \otimes_{R_0} R, x \mapsto 1 \otimes x.$$

B) Keep the previous hypotheses and notations. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R -module. Then $M' = R'_0 \otimes_{R_0} M = R' \otimes_{R_0} M \downarrow_{R_0}$ is an R' -module (we omit to write the functor \downarrow_{R_0} henceforth) carries a natural grading and thus may be written in the form

$$M' = R'_0 \otimes_{R_0} M = \bigoplus_{n \in \mathbb{Z}} R'_0 \otimes_{R_0} M_n.$$

Observe, that the R' -module M' is finitely generated, if the R -module M is. As R'_0 is a flat R_0 -algebra, we now get a (covariant, linear) exact functor from graded R -modules to graded R' -modules

$$R'_0 \otimes_{R_0} \bullet : (M \xrightarrow{h} N) \mapsto (R'_0 \otimes_{R_0} M \xrightarrow{R'_0 \otimes_{R_0} h} R'_0 \otimes_{R_0} N).$$

It is again easy to verify, that this functor satisfies the graded version of the requirement in (1.10)C)a) for each graded ideal $\mathfrak{a} \subseteq R$. So, by the graded version of (1.10)C)a) and by (1.13)a) we may conclude

- a) The exact functor $R'_0 \otimes_{R_0} \bullet$ is $*$ torsion-faithful.
- b) For each graded ideal $\mathfrak{a} \subseteq R$ and all $n \in \mathbb{N}_0$ there is a natural equivalence

$$*\tau^n = *\tau^{n, \mathfrak{a}, R'_0 \otimes_{R_0} \bullet} : R'_0 \otimes_{R_0} *H_{\mathfrak{a}}^n(\bullet) \xrightarrow{\cong} *H_{\mathfrak{a}R'}^n(R'_0 \otimes_{R_0} \bullet).$$

- c) For each graded ideal $\mathfrak{a} \subseteq R$, each choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ and each graded R -module M , there is a natural isomorphism of R'_0 -modules

$$(*\tau_M^n)_t : R'_0 \otimes_{R_0} *H_{\mathfrak{a}}^n(M)_t \xrightarrow{\cong} *H_{\mathfrak{a}R'}^n(R'_0 \otimes_{R_0} M)_t.$$

C) Keep the above notations and hypotheses. Let $\mathfrak{a} \subseteq R$ be a graded ideal and let M be a graded R -module. As usually, we may use the $*$ equivalence of [Br-Fu-Ro] (8.24) to identify $H_{\mathfrak{a}}^n(M) = *H_{\mathfrak{a}}^n(M)$ and correspondingly $H_{\mathfrak{a}R'}^n(M) = *H_{\mathfrak{a}R'}^n(M)$. In doing so, we may reformulate B)c) as follows

- a) For each graded ideal $\mathfrak{a} \subseteq R$, each choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ and each graded R -module M , there is a natural isomorphism of R'_0 -modules

$$(\tau_M^n)_t : R'_0 \otimes_{R_0} H_{\mathfrak{a}}^n(M)_t \xrightarrow{\cong} H_{\mathfrak{a}R'}^n(R'_0 \otimes_{R_0} M)_t.$$

D) Finally, we consider the specific case, which concerns taking up denominators in the base ring R_0 , and actually is incorporated in what we said in parts A), B) and C). Namely, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be as in part A), let $S_0 \subseteq R_0$ be a multiplicatively closed subset and consider the canonical homomorphism $\eta_0 := \eta_{S_0} : R_0 \rightarrow S_0^{-1}R_0$. Observe that we have a natural equivalence of functors

$$S_0^{-1} \bullet : \xrightarrow{\cong} S_0^{-1}R_0 \otimes_{R_0} \bullet.$$

As the functor $S_0^{-1}\bullet$ is exact, the homomorphism η_0 is flat, and we may apply what has been said in parts A), B) and C) with $f_0 = \eta_0$. In particular we can say:

- a) For each graded ideal $\mathfrak{a} \subseteq R$, each choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ and each graded R -module M there is a natural isomorphism of S_0^{-1} -modules

$$S_0^{-1}(H_{\mathfrak{a}}^n(M)_t) \xrightarrow{\cong} H_{\mathfrak{a}_{S_0^{-1}R}}^n(S_0^{-1}M)_t.$$

1.16. Remark. A) (*Extensions to the Multi-Graded Case*) It would be interesting and very useful in many respects (for example in the understanding of toric schemes (see [Ro]) to extend the concepts and results which we presented in this section just for \mathbb{Z} -graded rings and modules to rings and modules graded by arbitrary (finitely generated) Abelian groups G .

B) (*Extensions to Non-Noetherian Rings*) Also, inspired by [Ro] one could try to extend the concepts of quasi-divisibility and its application to situations in which the (G -graded) ring R is not necessarily Noetherian.

2. SUPPORTING DEGREES OF COHOMOLOGY

In this section, we shall do our first step toward the algebraic definition of Castelnuovo-Mumford regularity. This naturally leads us to look at the *supporting degrees* of the local cohomology modules $H_{R_+}^i(M)$ of a finitely generated graded R -module M with respect to the irrelevant ideal R_+ of a Noetherian homogeneous ring R , hence the integers $n \in \mathbb{Z}$ for which the n -th graded component $H_{R_+}^i(M)_n$ of $H_{R_+}^i(M)$ does not vanish. This shall lead us to generalize some results in sections 8 and 9 of [Br-Fu-Ro] which were proved there only for Noetherian homogeneous algebras over infinite fields. We start with a few preparations.

2.1. Notation and Reminder. A) Throughout this section, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a *positively graded Noetherian ring*. Keep in mind that the base-ring R_0 is Noetherian and

$$R = R_0[x_1, x_2, \dots, x_r]$$

for finitely many homogeneous elements $x_1, x_2, \dots, x_r \in R^h := \bigcup_{n \in \mathbb{N}_0} R_n$. Let

$$R_+ = \bigoplus_{n \in \mathbb{N}} R_n$$

denote the *irrelevant ideal* of R .

B) Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R -module. Let us recall the notion of *generating degree* of M (see [Br-Fu-Ro] (9.6D)) defined as

$$\text{gendeg}(M) := \inf\{t \in \mathbb{Z} \mid M = \sum_{n \leq t} RM_n\}.$$

In addition, let us introduce the *beginning* and the *end* of M , which are defined respectively by

$$\begin{aligned} \text{beg}(M) &:= \inf\{n \in \mathbb{Z} \mid M_n \neq 0\}, \\ \text{end}(M) &:= \sup\{n \in \mathbb{Z} \mid M_n \neq 0\}. \end{aligned}$$

Observe the following facts:

- a) If $M \neq 0$ is a graded R -module, then $\text{beg}(M) \leq \text{gendeg}(M) \leq \text{end}(M)$.
- b) If $M \neq 0$ is a finitely generated graded R -module, then

$$-\infty < \text{beg}(M) \leq \text{gendeg}(M) < \infty.$$

C) Finally observe that we have the *Graded Nakayama Lemma*, which can be stated as follows (and proved as an exercise):

- a) If M is a graded R -module with $-\infty < \text{beg}(M)$, $N \subseteq M$ is a graded submodule and $\mathfrak{a} \subseteq R_+$ is a graded ideal such that $M = N + \mathfrak{a}M$, then $N = M$.

D) We also shall repeatedly use the *Homogeneous Prime Avoidance Principle* (see [Br-Fu-Ro] (10.13)):

a) Let $\mathfrak{a} \subseteq R_+$ be a graded ideal and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r \in \text{Spec}(R)$ such that $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for $i \in \{1, \dots, r\}$. Then, there exists some $t \in \mathbb{N}$ and some $x \in \mathfrak{a}_t \setminus \bigcup_{1 \leq i \leq r} \mathfrak{p}_i$.

E) If $\mathfrak{a} \subseteq R$ is a graded ideal, then as done already previously, for each graded R -module M and all $n \in \mathbb{N}_0$ we always shall identify

$$H_{\mathfrak{a}}^n(M) = {}^*H_{\mathfrak{a}}^n(M) = \mathcal{R}^n {}^*\Gamma_{\mathfrak{a}}(M)$$

and hence consider $H_{\mathfrak{a}}^n(M)$ as a graded R -module by means of the $*$ equivalence shown in [Br-Fu-Ro] (8.24).

We now prove a first result which concerns the supporting degrees of local cohomology modules over Noetherian positively graded rings with respect to the irrelevant ideal.

2.2. Proposition. *Let M be a finitely generated graded R -module and let $i \in \mathbb{N}_0$. Then*

- a) *For all $n \in \mathbb{Z}$ the R_0 -module $H_{R_+}^i(M)_n$ is finitely generated.*
- b) *For all $n \gg 0$ we have $H_{R_+}^i(M)_n = 0$, so that $\text{end}(H_{R_+}^i(M)) < \infty$.*

Proof. We proceed by induction on i . First, let $i = 0$. Clearly $H_{R_+}^0(M) = \Gamma_{R_+}(M)$ is a graded submodule of M , and hence finitely generated, as R is Noetherian. Therefore all the graded components $H_{R_+}^0(M)_n$ of $H_{R_+}^0(M)$ are finitely generated R_0 -modules (see [Br-Fu-Ro](9.6)C)). As $H_{R_+}^0(M)$ is generated by finitely many homogeneous elements each of which is annihilated by some power of R_+ , we also have $H_{R_+}^0(M)_n = 0$ for all $n \gg 0$. This proves statements a) and b) for $i = 0$. Now, let $i > 0$. In view of the natural isomorphism of graded R -modules

$$H_{R_+}^i(M) \xrightarrow{\cong} H_{R_+}^i(M/\Gamma_{R_+}(M))$$

we may assume, as usually, that $\Gamma_{R_+}(M) = 0$. This means that $R_+ \not\subseteq \mathfrak{p}$ for each of the finitely many members \mathfrak{p} of $\text{Ass}_R(M)$ (see [Br-Fu-Ro] (1.9)). So by the Homogeneous Prime Avoidance Principle (2.1)D) we find some $t \in \mathbb{N}$ and some $x \in R_t$ such that x does not belong to $\bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} \mathfrak{p} = \text{ZD}_R(M)$. Therefore we have an exact sequence of graded R -modules

$$0 \rightarrow M(-t) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0.$$

Applying cohomology, we get for each $n \in \mathbb{Z}$ an exact sequences of R_0 -modules

$$H_{R_+}^{i-1}(M/xM)_n \rightarrow H_{R_+}^i(M)_{n-t} \xrightarrow{x} H_{R_+}^i(M)_n.$$

By induction, we find some $n_0 \in \mathbb{Z}$ such that $H_{R_+}^{i-1}(M/xM)_n = 0$ for all $n \geq n_0$. So for all these n the multiplication map $x : H_{R_+}^i(M)_{n-t} \rightarrow H_{R_+}^i(M)_n$ is injective. As the R -module $H_{R_+}^i(M)$ is R_+ -torsion, it follows that $H_{R_+}^i(M)_{n-t} = 0$ for all $n \geq n_0$. This proves statement b). Now one proves statement a) by

descending induction starting at n_0 by means of the above sequences and observing that by induction the R_0 -module $H_{R_+}^{i-1}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$. \square

Castelnuovo-Mumford regularity is a notion which basically applies over homogeneous Noetherian rings. We therefore shall give now a number of preparations which will allow us to perform certain repeatedly used replacement arguments relying on flat base ring changes.

2.3. Exercise and Remark. (*Flat and Faithfully Flat Base Ring Changes*)

A) Let $f_0 : R_0 \rightarrow R'_0$ be a flat homomorphism of Noetherian rings and consider the Noetherian positively graded ring (see (1.14))

$$R' := R'_0 \otimes_{R_0} R = \bigoplus_{n \in \mathbb{N}_0} R'_0 \otimes_{R_0} R_n.$$

By what is said in (1.15)A) it is easy to verify:

- a) $R'_+ = R_+ R'$.
- b) If R is a Noetherian homogeneous R_0 -algebra, then $R' = R'_0 \otimes_{R_0} R$ is a Noetherian homogeneous R' -algebra.

B) Let the notations and hypotheses be as in part A). In view of (1.15)C) and (2.2) we can say

- a) For each choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ and each graded R -module M there is a natural isomorphism of R'_0 -modules

$$(\tau_M^n)_t : R'_0 \otimes_{R_0} H_{R_+}^n(M)_t \xrightarrow{\cong} H_{R'_+}^n(R'_0 \otimes_{R_0} M)_t.$$

- b) If the graded R -module M is finitely generated, then the R'_0 -modules occurring in statement a) are finitely generated and vanish for all sufficiently large values of t .

C) Very often, we shall apply what is said in (1.15)C) in the special case of *localization at a prime in the base ring*. More precisely, we choose $\mathfrak{p}_0 \in \text{Spec}(R_0)$ and perform what is said in (1.15)C) with $S_0 := R_0 \setminus \mathfrak{p}_0$. In this situation we obviously use the traditional but slightly abusive notation $(S_0)^{-1} \bullet =: \bullet_{\mathfrak{p}_0}$. By (1.15)C) we can say:

- a) Let $\mathfrak{p}_0 \in \text{Spec}(R_0)$. Then, for each choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ and each graded R -module M , there is a natural isomorphism of $(R_0)_{\mathfrak{p}_0}$ -modules

$$(H_{R_+}^n(M)_t)_{\mathfrak{p}_0} \xrightarrow{\cong} H_{(R_{\mathfrak{p}_0})_+}^n(M_{\mathfrak{p}_0})_t.$$

We express this by saying that *localization in the base ring commutes component-wise with taking local cohomology*.

D) Keep in mind that the homomorphism $f_0 : R_0 \rightarrow R'_0$ is *faithfully flat* if it is flat and if for each R_0 -module $M \neq 0$ the R'_0 -module $R'_0 \otimes_{R_0} M$ does not vanish, too. Now, in view of B)a) we can say

- a) Let the homomorphism $f_0 : R_0 \rightarrow R'_0$ be faithfully flat and let $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$. Then the R'_0 -module $H_{R'_+}^n(R' \otimes_{R_0} M)_t$ vanishes if and only if the R_0 -module $H_{R_+}^n(M)_t$ does.

Later, we shall mainly use two types of faithfully flat base changes. The first of these concerns the case in which the base ring R_0 is a field and the ring R'_0 is an extension field. We now pave the way for this.

2.4. Exercise and Remark. (*Base Field Changes*) A) Let K be a field and let $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a positively graded Noetherian K -Algebra. Let K' be an extension field of K and keep in mind that the inclusion homomorphism $K \hookrightarrow K'$ is faithfully flat. We consider the Noetherian positively graded ring

$$R' : K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \oplus \dots$$

If $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is graded R -module, we write $M' = \bigoplus_{n \in \mathbb{Z}} M'_n$ for the graded R' -module $K' \otimes_K M = \bigoplus_{n \in \mathbb{Z}} K' \otimes_K M_n$. Now, as the K' -vector spaces M'_t and $K' \otimes_K M_t$ coincide and in view of (2.4)B)a) we can say:

- a) For all $t \in \mathbb{Z}$ we have $\dim_{K'}(M'_t) = \dim_K(M_t)$.
 b) For all $n \in \mathbb{N}_0$ and all $t \in \mathbb{Z}$ we have $\dim_{K'}(H_{R'_+}^n(M')_t) = \dim_K(H_{R_+}^n(M)_t)$.

B) Keep all hypotheses and notations of part A). Assume that the extension field K' of K is infinite. Observe that R' is homogeneous if R is (see (2.3)A)), and that the graded R' -module M' is finitely generated if the graded R -module M is. Use (2.2) to show that for each $i \in \mathbb{N}_0$ and any (that is, not necessarily infinite) field K the notion of *i -th cohomological Hilbert function*

$$h_M^i : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto h_M^i(n) := \dim_K(H_{R_+}^i(M)_n)$$

as introduced in [Br-Fu-Ro] (9.13) makes sense for any finitely generated graded module over a positively graded Noetherian K -algebra R . Then, prove that in this general setting one always may define the *characteristic function*

$$\chi_M : \mathbb{Z} \rightarrow \mathbb{Z}, \quad n \mapsto \chi_M(n) := \dim_K(M_n) - \sum_{i \in \mathbb{N}_0} (-1)^i h_M^i(n)$$

of M as defined in [Br-Fu-Ro] (9.14). Prove on use of A)b) that this function is *additive* in the sense of [Br-Fu-Ro] (9.15) for any finitely generated graded module M over a homogeneous Noetherian K -algebra R and is *presented by a polynomial* in the sense of [Br-Fu-Ro] (9.17) for an arbitrary field K . So, also in this more general setting one has for the *Hilbert polynomial* $P_M \in \mathbb{Q}[X]$ with $P_M(n) = \chi_M(n)$ for all $n \in \mathbb{Z}$ and $P_M(n) = \dim_K(M_n)$ for all $n \gg 0$.

C) Let the notations be as in parts A) and B) and assume that the Noetherian positively graded K -algebra R is homogeneous. Keep in mind that then for any

finitely generated graded R -module $M \neq 0$ one has the equality $\dim_R(M) = \deg(P_M) + 1$ for the (*Krull*) *dimension* of M , under the convention the 0-polynomial has degree -1 . Show that for any extension field K' of K and any finitely generated graded R -module M one has in the above notations:

- a) $P_{M'} = P_M$
- b) $\dim_R(M) = \dim_{R'}(M')$.

We also shall make use of the Graded Base Ring Independence of Local Cohomology (see (1.14)B)) in two special instances, which we mention below.

2.5. Remark and Exercise. A) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian positively graded ring and let M be a graded R -module. Let $\mathfrak{a} = \bigoplus_{n \in \mathbb{Z}_0} \mathfrak{a}_n \subseteq R$ be graded ideal such that $\mathfrak{a}M = 0$. Then, we may consider M canonically as a graded R/\mathfrak{a} -module. If we do so, the graded R -module M is obtained from the graded R/\mathfrak{a} -module M by means of scalar restriction by the canonical homomorphism of graded rings $f : R \rightarrow R/\mathfrak{a}$. As $(R/\mathfrak{a})_+ = (R_+ + \mathfrak{a})/\mathfrak{a} = (R_+)/\mathfrak{a}$, and $(R/\mathfrak{a})_0 = R_0/\mathfrak{a}_0$, the Graded Base Ring Independence of Local Cohomology (see (1.14)B) and (1.13)b) and the identification made in (2.1) E), allow to say:

- a) For any choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$, any graded ideal $\mathfrak{a} \subseteq R$ and any graded R -module M with $\mathfrak{a}M = 0$ there is an isomorphism of R_0/\mathfrak{a}_0 -modules (and hence of R_0 -modules)

$$H_{(R/\mathfrak{a})_+}^n(M)_t \xrightarrow{\cong} H_{R_+}^n(M)_t.$$

In particular:

- b) If K is a field, R is a Noetherian positively graded K -algebra, M is a finitely generated graded R -module and $\mathfrak{a} \subseteq R$ is a graded ideal such that $\mathfrak{a}M = 0$, the cohomological Hilbert functions $h_M^i : \mathbb{Z} \rightarrow \mathbb{Z}$ ($i \in \mathbb{N}_0$) are the same, if M is considered as a graded R/\mathfrak{a} -module or as a graded R -module.

B) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be as in part A), let $R' = \bigoplus_{n \in \mathbb{N}_0} R'_n$ be a positively graded ring which is a finite integral extension of R and let the inclusion map $j : R \hookrightarrow R'$ be a homomorphism of graded rings, so that $R_n \subseteq R'_n$ for all $n \in \mathbb{Z}$. Observe that R' is Noetherian and make clear that

$$R_+R' \subseteq \sqrt{R_+R'} = \sqrt{R'_+}$$

Conclude that $H_{R'_+}^n(M') = H_{R_+R'}^n(M')$ for all $n \in \mathbb{N}_0$ and all R' -modules M' . Now, if we apply (1.14)B), (1.13)b) and (2.1)E) to the functor $\bullet \upharpoonright_R$ of scalar restriction by means of j , we can say

- a) For each choice of integers $n \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ and for each graded R' -module M' there is an isomorphism of R_0 -modules

$$H_{R'_+}^n(M')_t \upharpoonright_{R_0} \xrightarrow{\cong} H_{R_+}^n(M' \upharpoonright_R)_t.$$

In particular

- b) If K is a field and $R \subseteq R'$ are both Noetherian positively graded K -algebras such that $R_n \subseteq R'_n$ for all $n \in \mathbb{N}_0$ and R' is integral over R , then for each finitely generated graded R' -module and all $i \in \mathbb{N}_0$ we have

$$h_{M'}^i = h_{M' \upharpoonright_R}^i.$$

2.6. Reminder. We also shall use the *Homogeneous Normalization Lemma* which we state in the following form

- a) Let K be an infinite field and let R be a Noetherian homogeneous K -algebra. Then there exist elements $x_1, x_2, \dots, x_d \in R_1$ algebraically independent over K such that R is a finite integral extension of its graded subring $K[x_1, x_2, \dots, x_d]$
- b) In the situation of a) we have $d = \dim(R)$ and $R_+ = \sqrt{\langle x_1, x_2, \dots, x_d \rangle} R$.

2.7. Reminder and Remark. (*Cohomological Dimension*) A) If $\mathfrak{a} \subseteq R$ is an ideal of the Noetherian ring R and M is an R -module, the *cohomological dimension* of M with respect to \mathfrak{a} is defined by (see [Br-Fu-Ro](Section 4))

$$\text{cd}_{\mathfrak{a}}(M) := \sup\{n \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^n(M) \neq 0\}.$$

Keep in mind that by Hartshorne's Vanishing Theorem (see [Br-Fu-Ro](4.21)) we have $\text{cd}_{\mathfrak{a}}(M) < \infty$.

B) Looking at supporting degrees of the local cohomology modules $H_{R_+}^i(M)$ of a finitely generated graded module M over a Noetherian homogeneous ring R naturally leads to study the cohomological dimension $\text{cd}_{R_+}(M)$ of M with respect to R_+ . Indeed $\text{cd}_{R_+}(M)$ is the largest value of i for which $H_{R_+}^i(M)$ has supporting degrees at all.

Our aim is to express $\text{cd}_{R_+}(M)$ in terms of “non-cohomological invariants” which is in fact possible in this situation. We first treat the following special case:

2.8. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Then*

$$\text{cd}_{R_+}(M) = \dim_R(M).$$

Proof. By Grothendieck's Vanishing Theorem [Br-Fu-Ro](4.11) we already have $\text{cd}_{R_+}(M) \leq \dim_R(M)$. It thus remains to show that $\text{cd}_{R_+}(M) \geq \dim_R(M)$. If $M = 0$, we are done, as $\text{cd}_{R_+}(0) = \dim_R(0) = -\infty$. So let $d := \dim_R(M) \geq 0$. Assume first that $d = 0$. Then $M_n = 0$ for all $n \gg 0$ and hence M is R_+ -torsion. Therefore $H_{R_+}^0(M) \cong M \neq 0$ and $H_{R_+}^i(M) = 0$ for all $i > 0$. This gives our claim if $d = 0$.

So, let $d > 0$. We proceed by induction on d . Let K' be an infinite extension field of K . Consider the Noetherian homogeneous K' -algebra $R' := K' \otimes_K R$

and the finitely generated graded R' -module $M' := K' \otimes_K M$. Now, in view of (2.4)A)b) and (2.4)C)b) we have $\text{cd}_{R'_+}(M') = \text{cd}_{R_+}(M)$ and $\dim_{R'}(M') = d$. This allows to replace R and M by R' and M' respectively and hence to assume that K is infinite. Now, let $\mathfrak{a} := (0 :_R M)$ be the (graded) annihilator ideal of M . Then clearly $\dim(R/\mathfrak{a}) = \dim_{R/\mathfrak{a}}(M) = d$ and $\text{cd}_{(R/\mathfrak{a})_+}(M) = \text{cd}_{R_+}(M)$ (see (2.5)A)a)). This allows to replace R by R/\mathfrak{a} and hence to assume that $\dim(R) = d$. According to the Homogeneous Normalization Lemma (see (2.6)) we find elements $x_1, x_2, \dots, x_d \in R_1$ algebraically independent over K such that R is a finite integral extension of its graded subring $S := K[x_1, x_2, \dots, x_d]$. Now, clearly $M \upharpoonright_S$ is a finitely generated graded S -module of dimension d and by (2.5)B)b) it holds $\text{cd}_{S_+}(M \upharpoonright_S) = \text{cd}_{R_+}(M)$. So, we may replace R and M respectively by S and $M \upharpoonright_S$ and hence finally assume that $R = K[x_1, x_2, \dots, x_d]$ is a polynomial ring over the field K .

Suppose now, that our claim is not true. Then there is a maximal graded submodule $U \subseteq M$ such that our claim fails for the graded R -module M/U . By induction we then must have $\dim_R(M/U) = d$. This allows to replace M by M/U and hence to assume that $\text{cd}_{R_+}(M/N) = \dim_R(M/N)$ for each non-zero graded submodule $N \subseteq M$.

Our next aim is to show that $\text{Ass}_R(M) = \{0\}$. Assume that this is not the case. As R is a domain of dimension $d = \dim(M)$ we clearly have $0 \in \text{Ass}_R(M)$. As $\text{Ass}_R(M)$ consists of finitely many graded primes (see [Br-Fu-Ro] (10.3)C)), we have the proper non-zero graded ideal

$$\mathfrak{q} := \bigcap_{\mathfrak{p} \in \text{Ass}_R(M) \setminus \{0\}} \mathfrak{p} \subseteq R.$$

Now $N := \Gamma_{\mathfrak{q}}(M) \subseteq M$ is a graded submodule with $\text{Ass}_R(N) \subseteq \text{Var}(\mathfrak{q})$ and $\text{Ass}_R(M/N) = \text{Ass}_R(M) \setminus \text{Var}(\mathfrak{q}) = \{0\}$ (see [Br-Fu-Ro](1.9)). In particular we have $\dim(M/N) = d$, and $N \subseteq M$ is a non-zero graded submodule with $\dim_R(N) < d$. By our assumption on M , the graded module M/N satisfies our claim, so that $\text{cd}_{R_+}(M/N) = d$. In particular we have $H_{R_+}^d(M/N) \neq 0$. As $\dim_R(N) < d$, Grothendieck's Vanishing Theorem [Br-Fu-Ro](4.11) implies that $H_{R_+}^i(N) = 0$ for all $i \geq d$. If we apply cohomology to the short exact sequence of graded R -modules

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0,$$

we thus get an isomorphism

$$H_{R_+}^d(M) \cong H_{R_+}^d(M/N),$$

which shows that $H_{R_+}^d(M) \neq 0$, and thus contradicts our assumption that $\text{cd}_{R_+}(M) < d$. So, we have indeed $\text{Ass}_R(M) = \{0\}$ and hence M is torsion-free over the domain R .

Now, as M is non-zero, torsion-free and finitely generated, we find an integer $r \in \mathbb{N}$ and a monomorphism of R -modules

$$\iota : M \hookrightarrow R^{\oplus r} =: F$$

such that $P := \text{Coker}(\iota)$ satisfies $\dim_R(P) < d$, and hence also $H_{R_+}^d(P) = 0$ by Grothendieck's Vanishing Theorem. As the irrelevant ideal R_+ of our polynomial ring $R = K[x_1, x_2, \dots, x_r]$ is generated by the R -sequence x_1, x_2, \dots, x_r we have $H_{R_+}^d(R) \neq 0$ (see [Br-Fu-Ro](4.6)) and hence $H_{R_+}^d(F) \cong H_{R_+}^d(R)^{\oplus r} \neq 0$. If we apply cohomology with respect to R_+ to the short exact sequence

$$0 \rightarrow M \xrightarrow{\iota} F \rightarrow P \rightarrow 0,$$

we thus get $H_{R_+}^d(M) \neq 0$, which finally contradicts our assumption that $\text{cd}_{R_+}(M) < d$. \square

Now, we aim to generalize the previous result to the case where the base field K is replaced by a local Noetherian ring R_0 .

2.9. Proposition. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let M be a finitely generated graded R -module. Then*

$$\text{cd}_{R_+}(M) = \dim_R(M/\mathfrak{m}_0M).$$

Proof. If $M = 0$, our claim is again obvious. So, let $M \neq 0$. Then $M_n \neq 0$ for some $n \in \mathbb{Z}$. As the R_0 -module M_n is finitely generated, it follows by Nakayama that $(M/\mathfrak{m}_0M)_n = M_n/\mathfrak{m}_0M_n \neq 0$, so that $M/\mathfrak{m}_0M \neq 0$. Therefore we may proceed by induction on $d := \dim_R(M/\mathfrak{m}_0M)$ starting with $d = 0$. If $d = 0$, we have $M_n/\mathfrak{m}_0M_n = (M/\mathfrak{m}_0M)_n = 0$ for all $n \gg 0$ and we see again by Nakayama that $M_n = 0$ for all $n \gg 0$. It follows that M is a non-zero R_+ -torsion module, and hence once more that $\text{cd}_{R_+}(M) = 0$.

So, let $d > 0$. We first show by induction on d that $\text{cd}_{R_+}(M) \leq d$. As $d > 0$, clearly M cannot be R_+ -torsion, so that $U := M/\Gamma_{R_+}(M)$ is a finitely generated non-zero R_+ -torsion-free graded R -module. Hence by the Graded Nakayama Lemma it follows that $R_+U \neq U$ (see (2.1C)). Therefore we have $H_{R_+}^i(U) \neq 0$ for some $i \in \mathbb{N}_0$ (see [Br-Fu-Ro](4.7)). As U has no R_+ -torsion, we must have $i > 0$. In view of the natural isomorphisms $H_{R_+}^j(M) \cong H_{R_+}^j(U)$ for all $j > 0$, we therefore have $\text{cd}_{R_+}(U) = \text{cd}_{R_+}(M)$. Moreover, the kernel of the canonical epimorphism $M/\mathfrak{m}_0M \rightarrow U/\mathfrak{m}_0U$ is R_+ -torsion and hence of dimension at most 0. This shows that $\dim(U/\mathfrak{m}_0U) = d$. So, we may replace M by U and hence assume that $\Gamma_{R_+}(M) = 0$.

Now consider the finite sets of graded primes

$$\mathcal{A} := \text{Ass}_R(M), \mathcal{B} := \min(0 :_R M/\mathfrak{m}_0M) \subseteq {}^*\text{Spec}(R).$$

As $\Gamma_{R_+}(M) = 0$, we have $R_+ \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{A}$ (see [Br-Fu-Ro](1.9)). Now, let $\mathfrak{p} \in \mathcal{B}$. Then $\mathfrak{m}_0 \subseteq \mathfrak{p} \subsetneq \mathfrak{m}_0 + R_+$, as equality in the second inclusion would imply that the set \mathcal{B} consists only of the unique graded maximal ideal

$\mathfrak{m}_0 + R_+$ of R , and hence lead to the contradiction that $d = 0$. This implies that $R_+ \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{B}$. So for each $\mathfrak{p} \in \mathcal{A} \cup \mathcal{B}$ we have $R_+ \not\subseteq \mathfrak{p}$, and hence by the Homogeneous Prime Avoidance Principle (2.1)D) we find some $t \in \mathbb{N}$ and some

$$x \in R_t \setminus \bigcup_{\mathfrak{p} \in \mathcal{A} \cup \mathcal{B}} \mathfrak{p}.$$

As x avoids all members of \mathcal{B} we have $\dim_R((M/\mathfrak{m}_0M)/x(M/\mathfrak{m}_0M)) \leq d - 1$ and in view of the canonical isomorphisms

$$(M/xM)/\mathfrak{m}_0(M/xM) \cong M/(\mathfrak{m}_0M + xM) \cong (M/\mathfrak{m}_0M)/x(M/\mathfrak{m}_0M)$$

we obtain

$$\dim_R((M/xM)/\mathfrak{m}_0(M/xM)) \leq d - 1.$$

So, by induction we have $\text{cd}_{R_+}(M/xM) \leq d - 1$ and hence $H_{R_+}^{i-1}(M/xM) = 0$ for all $i > d$. As x avoids all members of \mathcal{A} , we have $x \in \text{NZD}_R(M)$ and hence a short exact sequence

$$0 \rightarrow M(-t) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0.$$

Applying cohomology we see that the multiplication map

$$H_{R_+}^i(M)(-t) \xrightarrow{x} H_{R_+}^i(M)$$

is injective for all $i > d$. This implies as usually that $H_{R_+}^i(M) = 0$ for all $i > d$, and hence that indeed $\text{cd}_{R_+}(M) \leq d$.

It thus remains to show that $\text{cd}_{R_+}(M) \geq d$, hence that $H_{R_+}^d(M) \neq 0$. To this end, consider the short exact sequence

$$0 \rightarrow \mathfrak{m}_0M \rightarrow M \rightarrow M/\mathfrak{m}_0M \rightarrow 0$$

and the exact sequence of R -modules

$$H_{R_+}^d(M) \xrightarrow{\pi} H_{R_+}^d(M/\mathfrak{m}_0M) \rightarrow H_{R_+}^{d+1}(\mathfrak{m}_0M)$$

induced in cohomology. As $(0 :_R M/\mathfrak{m}_0M)\mathfrak{m}_0M \subseteq \mathfrak{m}_0^2M$, we have

$$(0 :_R M/\mathfrak{m}_0M) \subseteq (0 :_R \mathfrak{m}_0M/\mathfrak{m}_0^2M)$$

and hence $\dim_R(\mathfrak{m}_0M/\mathfrak{m}_0^2M) \leq d$. By what we have already shown it follows that $\text{cd}_{R_+}(\mathfrak{m}_0M) \leq d$, so that $H_{R_+}^{d+1}(\mathfrak{m}_0M) = 0$ and hence π is an epimorphism. Observe that $\dim_{R/\mathfrak{m}_0R}(M/\mathfrak{m}_0M) = d$ and that R/\mathfrak{m}_0R is a Noetherian homogeneous algebra over the field R_0/\mathfrak{m}_0 . Therefore by (2.8) we have $H_{(R/\mathfrak{m}_0R)_+}^d(M/\mathfrak{m}_0M) \neq 0$. In view of the base ring independence stated in (2.5)A)a) (applied with $\mathfrak{a} = \mathfrak{m}_0R$) it follows that $H_{R_+}^d(M/\mathfrak{m}_0M) \neq 0$ and so the epimorphism π yields that $H_{R_+}^d(M) \neq 0$. \square

Now, we can give the announced characterization of the cohomological dimension of a finitely generated graded module over a Noetherian homogeneous ring in its full generality.

2.10. Theorem. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring and let M be a finitely generated graded R -module. Then*

$$\begin{aligned} \text{cd}_{R_+}(M) &= \max\{\dim_R(M/\mathfrak{m}_0 M) \mid \mathfrak{m}_0 \in \text{Max}(R_0)\} \\ &= \max\{\dim_{R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}/\mathfrak{m}_0 M_{\mathfrak{m}_0}) \mid \mathfrak{m}_0 \in \text{Max}(R_0)\} \\ &= \max\{\dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \text{Spec}(R_0)\}. \end{aligned}$$

Proof. We set

$$\begin{aligned} c &:= \text{cd}_{R_+}(M), a' := \max\{\dim_R(M/\mathfrak{m}_0 M) \mid \mathfrak{m}_0 \in \text{Max}(R_0)\}, \\ a &:= \max\{\dim_{R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}/\mathfrak{m}_0 M_{\mathfrak{m}_0}) \mid \mathfrak{m}_0 \in \text{Max}(R_0)\}, \end{aligned}$$

and

$$b := \max\{\dim_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}/\mathfrak{p}_0 M_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \text{Spec}(R_0)\}.$$

Now, let $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then, in view of (2.3)C)a) and the Local-Global Principle for the Vanishing of R_0 -modules the following statements are equivalent:

- (i) $H_{R_+}^i(M)_n = 0$;
- (ii) $H_{(R_{\mathfrak{m}_0})_+}^i(M_{\mathfrak{m}_0})_n \cong (H_{R_+}^i(M)_n)_{\mathfrak{m}_0} = 0$ for all $\mathfrak{m}_0 \in \text{Max}(R_0)$;
- (iii) $H_{(R_{\mathfrak{p}_0})_+}^i(M_{\mathfrak{p}_0})_n \cong (H_{R_+}^i(M)_n)_{\mathfrak{p}_0} = 0$ for all $\mathfrak{p}_0 \in \text{Spec}(R_0)$.

On use of (2.9) it follows immediately that $c = a = b$. To prove $a = a'$ observe that for each $\mathfrak{m}_0 \in \text{Max}(R_0)$ and each R -module M we have a canonical isomorphism of R -modules $M/\mathfrak{m}_0 M \xrightarrow{\cong} M_{\mathfrak{m}_0}/\mathfrak{m}_0 M_{\mathfrak{m}_0}$. \square

Now, we can prove a basic result on the supporting degrees of local cohomology modules over Noetherian homogeneous rings, which later will justify our definition of Castelnuovo-Mumford regularity. We begin with a few preparations, which we will be of use in its proof.

2.11. Exercise and Remark. (*Faithfully Flat Local Homomorphisms*) A) A homomorphism of local rings $f : R \rightarrow R'$ is a homomorphism f of rings such that (R, \mathfrak{m}) and (R', \mathfrak{m}') are both local and $f(\mathfrak{m}) \subseteq \mathfrak{m}'$. Show that a flat homomorphism of local rings is faithfully flat.

B) Let (R, \mathfrak{m}) be a local ring, let X be an indeterminate and show the following.

- a) The ring $R' := R[X]_{\mathfrak{m}R[X]}$ is local with maximal ideal $\mathfrak{m}' := \mathfrak{m}R'$ and $\#R'/\mathfrak{m}' = \infty$.
- b) The canonical map $f : R \rightarrow R'$, given by $a \mapsto a/1$ is a faithfully flat homomorphism of local rings.

2.12. Exercise. (*Strict Homogeneous Prime Avoidance*) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous graded ring, whose base ring (R_0, \mathfrak{m}_0) is local and has infinite residue field R_0/\mathfrak{m}_0 . Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r \in {}^* \text{Spec}(R)$ be finitely

many graded prime ideals such that $R_+ \not\subseteq \mathfrak{p}_i$ for all $i \in \{1, 2, \dots, r\}$. Show (if you like on use of [Br-Fu-Ro](9.8a)), that there is some element

$$x \in R_1 \setminus \bigcup_{1 \leq i \leq r} \mathfrak{p}_i.$$

2.13. Theorem. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring and let M be a finitely generated graded R -module. Set $\bar{M} := M/\Gamma_{R_+}(M)$.*

- a) *If $l \in \mathbb{N}$ and $r \in \mathbb{Z}$ are such that $H_{R_+}^i(M)_{r+1-i} = 0$ for all $i \geq l$, then $\text{end}(H_{R_+}^i(M)) \leq r - i$ for all $i \geq l$.*
- b) *If $l \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $H_{R_+}^i(M)_{r-1-i} = 0$ for all $i \in \{1, 2, \dots, l\}$ and $\bar{M}_{r-1} = 0$, then $\text{beg}(\bar{M}) \geq r$ and $\text{beg}(H_{R_+}^i(M)) \geq r - i$ for all $i \in \{1, 2, \dots, l\}$.*
- c) *If $c := \text{cd}_{R_+}(M) > 0$, then $a := \text{end}(H_{R_+}^c(M)) \in \mathbb{Z}$ and $H_{R_+}^c(M)_n \neq 0$ for all $n \leq a$.*
- d) *If $\bar{M} \neq 0$, then $H_{R_+}^1(M)_n \neq 0$ for all n with $\text{beg}(H_{R_+}^1(M)) \leq n < \text{beg}(\bar{M})$ and $\bar{M}_n \neq 0$ for all $n \geq \text{beg}(\bar{M})$.*

Proof. “a)”: Let $l \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $H_{R_+}^i(M)_{r+1-i} = 0$ for all $i \geq l$. We have to show that $H_{R_+}^i(M)_{s+1-i} = 0$ for all $i \geq l$ and for all $s \geq r$. By the Local-Global Principle for the vanishing of R_0 -modules it suffices to show that (see also (2.3)C)a)

$$H_{(R_{\mathfrak{p}_0})_+}^i(M_{\mathfrak{p}_0})_{s+1-i} \cong (H_{R_+}^i(M)_{s+1-i})_{\mathfrak{p}_0} = 0$$

for all $i \geq l$, all $s \geq r$ and all $\mathfrak{p}_0 \in \text{Spec}(R_0)$ under the hypothesis that we have this vanishing statement in the case $s = r$. So, we may fix $\mathfrak{p}_0 \in \text{Spec}(R_0)$ and replace R and M respectively by $R_{\mathfrak{p}_0}$ and $M_{\mathfrak{p}_0}$. This allows to assume that the base ring (R_0, \mathfrak{m}_0) is local and to restrict ourselves to show that $H_{R_+}^i(M)_{s+1-i}$ vanishes for all $i \geq l$ and all $s \geq r$, provided it does so in the case $s = r$.

By (2.11)B) there is a faithfully flat Noetherian local R_0 -algebra (R'_0, \mathfrak{m}'_0) with infinite residue field R'_0/\mathfrak{m}'_0 . Now, consider the Noetherian homogeneous R'_0 -algebra $R' := R'_0 \otimes R$ and the finitely generated graded R' -module $M' := R'_0 \otimes M$. Then, for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ the R_0 -module $H_{R_+}^i(M)_n$ vanishes if and only if the R'_0 -module $H_{R'_+}^i(M')_n$ does (see (2.3)D)a)). This allows to replace R and M respectively by R' and M' and hence to assume that the local base ring (R_0, \mathfrak{m}_0) has infinite residue field R_0/\mathfrak{m}_0 .

As R is a homomorphic image of a polynomial ring over the Noetherian local (and hence finite-dimensional) ring R_0 , we have $\dim(R) < \infty$ and hence $d := \dim_R(M) < \infty$. So, we may prove our claim by induction on d . The case $d \leq 0$ is clear as then $H_{R_+}^i(M) = 0$ for all $i > 0$ by Grothendieck's Vanishing Theorem.

So, let $d > 0$. As our claim concerns only local cohomology modules $H_{R_+}^i(M)$ with $i > 0$, and as $\dim_R(M/\Gamma_{R_+}(M)) \leq \dim_R(M) = d$ we can as usually replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. Therefore we have $R_+ \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R(M)$ (see [Br-Fu-Ro](1.9)). Hence by the Strict Homogeneous Prime Avoidance Principle (2.12) we find some

$$x \in R_1 \setminus \bigcup_{\mathfrak{p} \in \text{Ass}_R(M)} \mathfrak{p} \subseteq \text{NZD}_R(M).$$

So, we have

$$\dim_R(M/xM) \leq d - 1$$

(see [Br-Fu-Ro](4.10)B)c)) and a short exact sequence of graded R -modules

$$0 \rightarrow M(-1) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0.$$

If we apply cohomology to this sequence, we get short exact sequences of R_0 -modules

$$H_{R_+}^i(M)_{(s-1)+1-i} \rightarrow H_{R_+}^i(M)_{s+1-i} \rightarrow H_{R_+}^i(M/xM)_{s+1-i} \rightarrow H_{R_+}^{i+1}(M)_{s+1-(i+1)}$$

for all $i \geq l$ and all $s \geq r$. If we choose $s = r$ and consider the last three terms in the resulting sequences, we see that $H_{R_+}^i(M/xM)_{r+1-i} = 0$ for all $i \geq l$. Therefore, by induction we have $H_{R_+}^i(M/xM)_{s+1-i} = 0$ for all $i \geq l$ and all $s \geq r$. So, for all $i \geq l$ and all $s > r$ we have an epimorphism

$$H_{R_+}^i(M)_{(s-1)+1-i} \rightarrow H_{R_+}^i(M)_{s+1-i} \rightarrow 0.$$

As $H_{R_+}^i(M)_{r+1-i} = 0$ for all $i \geq l$, we now get our claim.

“b)”: We have to show that

$$H_{R_+}^i(M)_{s-1-i} = 0, \forall i \in \{1, 2, \dots, l\}$$

and

$$\bar{M}_{s-1} = 0$$

for all $s \leq r$, provided we have these vanishing statements for $s = r$. As in the proof of statement a), we can assume that the base ring (R_0, \mathfrak{m}_0) is local with infinite residue field R_0/\mathfrak{m}_0 and proceed by induction on $d := \dim_R(M)$. Again, the case $d \leq 0$ is clear, so that we can assume that $d > 0$. As in the proof of statement a) we can again assume that $\Gamma_{R_+}(M) = 0$ and hence $M = \bar{M}$, and also as in the proof of statement a) we thus find some element $x \in R_1 \cap \text{NZD}_R(M)$. Consequently we have $\dim_R(M/xM) \leq d - 1$ and short exact sequences of R_0 -modules

$$H_{R_+}^{i-1}(M)_{s-i} \rightarrow H_{R_+}^{i-1}(M/xM)_{s-i} \rightarrow H_{R_+}^i(M)_{s-1-i} \rightarrow H_{R_+}^i(R)_{(s+1)-1-i}$$

for all $i \in \{1, 2, \dots, l\}$ and all $s \leq r$.

Now first of all, as $\Gamma_{R_+}(M) = 0$ and $M_{r-1} = 0$, we have $\bar{M}_{s-1} = M_{s-1} = 0$ for all $s \leq r$. It remains to show that $H_{R_+}^i(M)_{s-1-i} = 0$ for all i with $1 \leq i \leq l$

and all $s \leq r$. By what we just have shown, it follows that $(M/xM)_{s-1} = 0$ and hence

$$[(M/xM)/\Gamma_{R_+}(M/xM)]_{s-1} = 0, \Gamma_{R_+}(M/xM)_{s-1} = H_{R_+}^0(M/xM)_{s-1} = 0$$

for all $s \leq r$. Moreover, if we apply the above exact sequence with $s = r$ we see that $H_{R_+}^{i-1}(M/xM)_{r-1-(i-1)} = 0$ for all $i \in \{2, 3, \dots, l\}$. So, by induction, we have $H_{R_+}^{i-1}(M/xM)_{s-1-(i-1)} = 0$ for all these i . So, for all i with $1 \leq i \leq l$ and all $s < r$ we have a monomorphism

$$0 \rightarrow H_{R_+}^i(M)_{s-1-i} \rightarrow H_{R_+}^i(M)_{(s+1)-1-i}.$$

As $H_{R_+}^i(M)_{r-1-i} = 0$ for all i with $1 \leq i \leq l$, we get our claim.

“c)”: This follows immediately from statement a).

“d)”: This follows easily from statement b). \square

We now aim to extend of our last result to sheaf cohomology. We start with a few preparations.

2.14. Reminder and Exercise. A) (*The Serre-Grothendieck Correspondence*) (See [Br-Fu-Ro] (Section 12)) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring and set $X := \text{Proj}(R)$. Let \mathcal{F} be a *coherent sheaf* of \mathcal{O}_X -modules so that $\mathcal{F} = \widetilde{M}$ for some finitely generated graded R -module M . For each $n \in \mathbb{Z}$ let $\mathcal{F}(n)$ denote the *n-th twist* of \mathcal{F} . Keep in mind that by [Br-Fu-Ro](9.5C) we can write $\mathcal{F}(n) = \widetilde{M(n)}$. We now make the identification of the composed functor $H^i(X, \tilde{\bullet})$ with the functor $H^i(X, \tilde{\bullet}) := \mathcal{R}^i(\tilde{\circ}(X))(\bullet)$ as suggested in [Br-Fu-Ro](12.9C). Then the *Serre-Grothendieck Correspondence* [Br-Fu-Ro](11.14) yields:

a) There is an exact sequence of R_0 -modules

$$0 \rightarrow H_{R_+}^0(M)_n \rightarrow M_n \rightarrow H^0(X, \mathcal{F}(n)) \rightarrow H_{R_+}^1(M)_n \rightarrow 0.$$

b) For each $i \in \mathbb{N}$ there is an isomorphism of R_0 -modules

$$H^i(X, \mathcal{F}(n)) \cong H_{R_+}^{i+1}(M)_n.$$

B) (*Zero Sheaves*) Keep the notations and hypotheses of part A). Let \mathbb{U}_X denote the set of all open subsets $U \subseteq X$. The *zero sheaf* is the sheaf of \mathcal{O}_X -modules defined by the assignment $U \mapsto 0$ for all $U \in \mathbb{U}_X$. The zero sheaf is denoted by 0 . Show that the property of being the zero sheaf is indicated by the vanishing of stalks:

a) For a sheaf \mathcal{F} of \mathcal{O}_X -modules one has $\mathcal{F} = 0$ if and only if $\mathcal{F}_x = 0$ for all $x \in X$.

Show in addition

b) If M is a graded R -module which is R_+ -torsion, then $\widetilde{M} = 0$.

It is more challenging to show that indeed also the converse of statement b) holds, namely:

c) If M is a graded R -module with $\widetilde{M} = 0$, then M is R_+ -torsion.

C) (*Homomorphisms of Sheaves of Modules*) Let \mathcal{F} and \mathcal{G} be sheaves of \mathcal{O}_X -modules. A *homomorphism of sheaves* of \mathcal{O}_X -modules from \mathcal{F} to \mathcal{G} is a homomorphism $h : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of Abelian groups such that for each $U \in \mathbb{U}_X$ the homomorphism of Abelian groups $h(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules. Make clear that the composition of homomorphisms of \mathcal{O}_X -modules is again a homomorphism of sheaves of \mathcal{O}_X -modules. Make clear, that for each sheaf \mathcal{F} of \mathcal{O}_X -modules the identity homomorphism $\text{id}_{\mathcal{F}}$ is a homomorphism of sheaves of \mathcal{O}_X -modules. Show that for two homomorphisms $g, h : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of \mathcal{O} -modules one has

a) $g = h$ if and only if $h_x = g_x$ for all $x \in X$.

Clearly as usually a homomorphism $h : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of \mathcal{O}_X -modules is called an *isomorphism* if there is a homomorphism $g : \mathcal{G} \rightarrow \mathcal{F}$ of sheaves of \mathcal{O}_X -modules such that $g \circ h = \text{id}_{\mathcal{F}}$ and $h \circ g = \text{id}_{\mathcal{G}}$. In this situation, g is uniquely determined by h , also an isomorphism of sheaves of \mathcal{O}_X -modules, denoted by h^{-1} and called the *inverse* of h . Make clear that the composition $h \circ g$ of two isomorphisms of sheaves of \mathcal{O}_X -modules is again an isomorphism of sheaves of \mathcal{O}_X -modules and that $(h \circ g)^{-1} = g^{-1} \circ h^{-1}$. Show that for a homomorphism $h : \mathcal{F} \rightarrow \mathcal{G}$ of \mathcal{O}_X -modules the following statements are equivalent

- (i) h is an isomorphism of sheaves of \mathcal{O}_X -modules.
- (ii) $h_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is an isomorphism of $\mathcal{O}_{X,x}$ -modules for all $x \in X$.
- (iii) $h(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is an isomorphism of $\mathcal{O}_X(U)$ -modules for all $U \in \mathbb{U}_X$.

As usually, we say that two sheaves \mathcal{F} and \mathcal{G} of \mathcal{O}_X -modules are *isomorphic* and write $\mathcal{F} \cong \mathcal{G}$ if there is an isomorphism of sheaves of \mathcal{O}_X -modules $h : \mathcal{F} \xrightarrow{\cong} \mathcal{G}$.

D) (*Induced Homomorphisms of Sheaves of Modules*) Make clear, that the functor

$$\widetilde{\bullet} : (M \xrightarrow{h} N) \mapsto (\widetilde{M} \xrightarrow{\widetilde{h}} \widetilde{N})$$

of taking induced sheaves (see [Br-Fu-Ro](12.9)A)) is indeed an exact functor from graded R -modules to sheaves of \mathcal{O}_X -modules. Show the following facts:

- a) If $0 \rightarrow U \rightarrow M \xrightarrow{h} N \rightarrow V \rightarrow 0$ is an exact sequence of graded R -modules such that U and V are R_+ -torsion modules, then $\widetilde{h} : \widetilde{M} \rightarrow \widetilde{N}$ is an isomorphism of sheaves of \mathcal{O}_X -modules.
- b) If M is a graded R -module, then $\widetilde{M} \cong \widetilde{M/\Gamma_{R_+}(M)}$.

2.15. **Definition.** (*Cohomological Patterns*) Keep the above notations and hypotheses of (2.14). Then we define the *cohomological pattern* of \mathcal{F} by:

$$\mathcal{P}(X, \mathcal{F}) = \mathcal{P}(\mathcal{F}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\}.$$

We define the *cohomological dimension* of \mathcal{F} as

$$\text{cd}_X(\mathcal{F}) := \sup\{i \in \mathbb{N}_0 \mid (\{i\} \times \mathbb{Z}) \cap \mathcal{P}(\mathcal{F}) \neq \emptyset\}.$$

By statement (2.14)A)b) we see that $\mathcal{F} \neq 0$ implies $\text{cd}_X(\mathcal{F}) = \text{cd}_{R_+}(M) - 1$.

Now we are ready to prove the announced application of (2.13) to sheaf cohomology. We do this in the form of a structure result on cohomological patterns.

2.16. **Theorem.** Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be Noetherian homogeneous ring, let $X := \text{Proj}(R)$, let $\mathcal{F} \neq 0$ be coherent sheaf of \mathcal{O}_X -modules and let $\mathcal{P} = \mathcal{P}(X, \mathcal{F})$ denote the cohomological pattern of \mathcal{F} . Then

- a) There is some $n \in \mathbb{Z}$ with $(0, n) \in \mathcal{P}$
- b) For all $i \in \mathbb{N}$ and all $n \gg 0$ it holds $(i, n) \notin \mathcal{P}$.
- c) If $(i, n) \in \mathcal{P}$, then there is some $k \geq i$ such that $(k, n - k + i - 1) \in \mathcal{P}$.
- d) If $(i, n) \in \mathcal{P}$, then there is some $l \leq i$ such that $(l, n - l + i + 1) \in \mathcal{P}$.

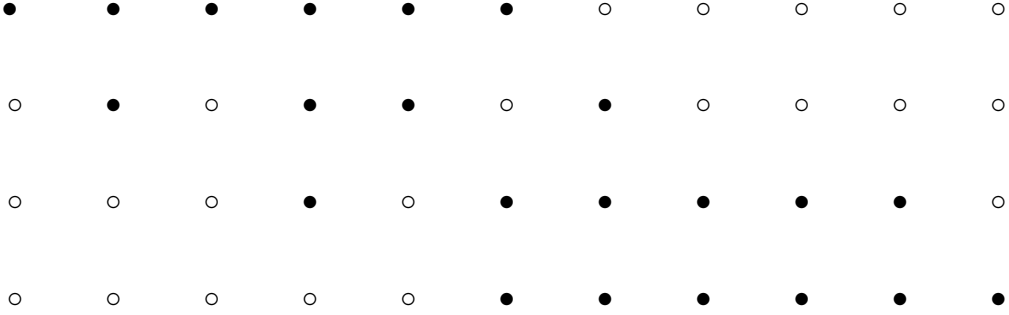
Proof. All three statements follow readily from (2.13) by means of the Serre-Grothendieck Correspondence (2.14)A)a),b) and the obvious replacement of M by $M/\Gamma_{R_+}(M)$ allowed by (2.14)D)b). We suggest to perform this as an exercise. \square

2.17. **Remark.** (*Around Cohomological Patterns*) A) Let the notations be as in (2.16). One might present the cohomological pattern \mathcal{P} of the sheaf of \mathcal{O}_X -modules $\mathcal{F} \neq 0$ in a diagram with horizontal n -axis and vertical i -axis, marking the place $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ by \bullet if $(i, n) \in \mathcal{P}$ and by \circ otherwise. Then, the four statements of (2.16) respectively say:

- a) One finds a \bullet on the bottom row.
- b) Except on the bottom row one finds only \circ 's far out to the right.
- c) If there is a diagonal consisting entirely of \circ 's above a certain level i , there are no \bullet 's right of this diagonal above level i .
- d) If there is a diagonal consisting entirely of \circ 's below a certain level i , there are no \bullet 's left of this diagonal below level i .

Observe in particular, that as a consequence of these properties of \mathcal{P} we get:

- e) If there is a \bullet on the bottom level, then right of it on the bottom level there are only \bullet 's.
- f) If there is a \bullet on the top level $c := \text{cd}_X(\mathcal{F})$, then left of it on the top level c there are only \bullet 's.



B) (*Tameness*) Keep the above notations and hypotheses. Let $i \in \mathbb{N}_0$. Then, the cohomological pattern \mathcal{P} is said to be *tame at level i* , if one of the following requirements is satisfied:

- (i) $(i, n) \in \mathcal{P}$ for all $n \ll 0$;
- (ii) $(i, n) \notin \mathcal{P}$ for all $n \ll 0$.

We express this also by saying, that $\mathcal{F} = \widetilde{M}$ is *cohomologically tame at level i* . According to the Serre-Grothendieck Correspondence (see (2.14)) this is equivalent to the fact that the R_0 -module $H_{R_+}^{i+1}(M)_n$ either vanishes for all $n \ll 0$ or else does not vanish for all $n \ll 0$. We express this by saying that the finitely generated graded R -module M is *cohomologically tame at level $i + 1$* . We say that the pattern \mathcal{P} is *tame at all*, if it is tame at all levels i . Correspondingly we say that the coherent sheaf \mathcal{F} of \mathcal{O}_X -modules respectively the finitely generated graded R -module M is (*cohomologically*) *tame at all*, if it is so at all levels i . Now, let $r \in \mathbb{N}$. Then we have the following *Realization Result for Tame Patterns* (see [Br-He]):

- a) Let $\mathcal{P} \subseteq \{0, 1, \dots, r\} \times \mathbb{Z}$ be an arbitrary set which satisfies the pattern requirements a), b), c), d) of (2.16) and the above tameness condition at all levels $i \in \{0, 1, \dots, r\}$. Let K be a field and let $\mathbb{P}_K^r = \text{Proj}(K[X_0, X_1, \dots, X_r])$ be the projective r -space over K . Then, there is a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules \mathcal{F} such that

$$\mathcal{P}(\mathbb{P}_K^r, \mathcal{F}) = \mathcal{P}.$$

C) (*The Tameness Problem*) It is quite natural to ask, whether at least over a polynomial ring $R = K[X_0, X_1, \dots, X_r]$ over a field K , one can characterize in combinatorial terms all sets $\mathcal{P} \subseteq \{0, 1, \dots, r\} \times \mathbb{Z}$ which occur as the cohomological pattern of a coherent sheaf \mathcal{F} of $\mathcal{O}_{\mathbb{P}_K^r} = \mathcal{O}_{\text{Proj}(R)}$ -modules. If we knew, that all such coherent sheaves (or equivalently: all finitely generated graded R -modules) were cohomologically tame, then statement B)a) would answer affirmatively this question. For a while it was indeed an open problem, whether all cohomological patterns are a fortiori tame (see [Br4], [Br6]) (at all levels). There are indeed many results, proving tameness of a finitely

generated graded module M over a Noetherian homogeneous ring R at particular levels or under certain assumptions on R - or else on M (see [Br6], [Br7], [Br-Fu-Lim], or also [Br-He], [Lim3], [Rott-Seg] for example). Nevertheless in [Ch-Cu-Her-Sr] a striking counter-example is constructed. Namely, it is shown there:

- a) There exists a Noetherian homogeneous domain $R = \bigoplus_{n \in \mathbb{N}_0} R_n$, of finite type over the complex field \mathbb{C} with $\dim(R) = 4$ and $\dim(R_0) = 3$ such that $M = R$ is not cohomologically tame at level 2 (or equivalently $\mathcal{O}_{\text{Proj}(R)}$ is not cohomologically tame at level 1).

This immediately shows, that even over polynomial rings over \mathbb{C} the mentioned Tameness Problem finds a negative answer.

D) (*The Realization Problem for Smooth Complex Projective Varieties*) Let $X = \text{Proj}(R)$ be a smooth connected complex projective variety of dimension at least 2, so that R is a Noetherian homogeneous integral \mathbb{C} -algebra such that the local ring $\mathcal{O}_{X,x} = R_{(\mathfrak{p})}$ is regular for all $x = \mathfrak{p} \in X = \text{Proj}(R)$. Then, by the Vanishing Theorem of Kodaira [Ko] one has $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i < \dim(X) = \dim(R) - 1$ and all $n < 0$. By another result of Mumford and Ramanujam [Mu2] one has the same vanishing statement for $i = 1$ under the weaker assumption that X is normal. So, one is naturally lead to ask the following realization question:

- a) Let $d \geq 2$ be an integer and let $\mathcal{P} \subseteq \{0, 1, \dots, d\} \times \mathbb{Z}$ be a set which satisfies the pattern requirements (2.16)a),b),c),d) and the additional *positivity* condition that $(i, n) \notin \mathcal{P}$ if $i < d$ and $n < 0$. Does there exist a smooth (or only normal) complex projective variety X (of dimension d) such that $\mathcal{P}_X(X, \mathcal{O}_X) = \mathcal{P}$?

We do not know the answer to this question, even in the surface case, that is in the case $d = 2$. In [M] a method is given, which allows to realize by smooth surfaces a great variety of positive patterns as discussed above. We also should mention that by the *Non-Rigidity Theorem* of Evans-Griffiths [Ev-Gri] (see also [Mi-N-P]) there are realization results of the above type in which indeed more than the cohomological pattern is described. Nevertheless, these results allow a realization only up to an eventual shift of the pattern and do not allow to control the last supporting degree the top cohomology groups. Therefore they do not answer our question. Another, local realization result, similar to those just quoted, is given in [Br-Sh2].

E) (*Extensions to the Multi-Graded Case*) The study of supporting degrees over rings who carry more general gradings is a surprisingly complex subject, which found much attention in the past two decades, partly motivated by toric geometry (see [Ro]). We just want to mention here [Br-Sh3] which concerns the case of \mathbb{Z}^n -gradings.

3. CASTELNUOVO-MUMFORD REGULARITY

Now, we are ready to define the notion of Castelnuovo-Mumford regularity and to derive some of its general properties. We first give a purely algebraic definition of Castelnuovo-Mumford regularity in terms of ends of local cohomology modules as given by Ooishi [O]. We observe a few basic properties of this new invariant. Then, we prove that Castelnuovo-Mumford regularity provides an upper bound for the generating degree and treat the basic example of a polynomial ring. After this purely algebraic exposition, we turn to the original sheaf-theoretic definition of Castelnuovo-Mumford regularity as given by Mumford [Mu1]. Then we attack the main result of this section, which says that Castelnuovo-Mumford regularity provides an upper bound for the least order needed to twist a coherent sheaf to become generated by global sections.

3.1. Notation. Throughout this section let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring.

3.2. Definition. (*Castelnuovo-Mumford Regularity*) Let M be a finitely generated graded R -module and let $l \in \mathbb{N}_0$. We define the *Castelnuovo-Mumford regularity of the finitely generated graded R -module M at and above level l* by

$$\text{reg}^l(M) := \sup\{\text{end}(H_{R_+}^i(M)) + i \mid i \geq l\}.$$

Observe that by (2.2) and as $\text{cd}_{R_+}(M) < \infty$, we have

$$\text{reg}^l(M) \in \mathbb{Z} \cup \{-\infty\}.$$

The *Castelnuovo-Mumford regularity of M* is defined by

$$\text{reg}(M) := \text{reg}^0(M).$$

From now on, we prefer just to speak of *regularity* instead of *Castelnuovo-Mumford regularity*.

In the following exercise we collect a few simple facts which we shall repeatedly use later.

3.3. Exercise. A) (*Properties of Generating Degrees*) Let M be a finitely generated graded R -module. Prove the following statements.

- a) For all $n \in \mathbb{Z}$ one has $\text{gendeg}(M(n)) = \text{gendeg}(M) - n$.
- b) If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of finitely generated graded R -modules, then

$$\text{gendeg}(N) \leq \text{gendeg}(M) \leq \max\{\text{gendeg}(L), \text{gendeg}(N)\}.$$

- c) $\text{gendeg}(M) = \max\{\text{gendeg}(M_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \text{Spec}(R_0)\}$.
- d) If R'_0 is a Noetherian faithfully flat R_0 -algebra, the finitely generated graded module

$$M' := R'_0 \otimes_{R_0} M = \bigoplus_{n \in \mathbb{Z}} R'_0 \otimes_{R_0} M_n$$

over the Noetherian homogeneous ring

$$R' := R'_0 \otimes_{R_0} R = \bigoplus_{n \in \mathbb{N}_0} R'_0 \otimes_{R_0} R_n$$

satisfies

$$\text{gendeg}(M') = \text{gendeg}(M).$$

- e) If $\mathfrak{b} \in R$ is a graded ideal such that $\mathfrak{b}M = 0$, the generating degree of M as an R/\mathfrak{b} -module is the same as the generating degree of M as an R -module.

B) (*Properties of Regularity*) Now, let M be a finitely generated graded R -module and let $l, k \in \mathbb{N}_0$. Then, concerning regularities, one has the following statements:

- a) If $k \geq l$ then $\text{reg}^k(M) \leq \text{reg}^l(M)$.
- b) For all $n \in \mathbb{Z}$ one has $\text{reg}^l(M(n)) = \text{reg}^l(M) - n$.
- c) $\text{reg}(M) = \max\{\text{end}(\Gamma_{R_+}(M)), \text{reg}^1(M)\}$.
- d) $\text{reg}(M/\Gamma_{R_+}(M)) = \text{reg}^1(M/\Gamma_{R_+}(M)) = \text{reg}^1(M) \leq \text{reg}(M)$.
- e) $M = \Gamma_{R_+}(M)$ if and only if $\text{reg}^1(M) = -\infty$.
- f) $M = 0$ if and only if $\text{reg}(M) = -\infty$.
- g) $\text{reg}^l(M) = \max\{\text{reg}^l(M_{\mathfrak{p}_0}) \mid \mathfrak{p}_0 \in \text{Spec}(R_0)\}$.
- h) If R'_0, R' and M' are as in statement A)d), then $\text{reg}^l(M') = \text{reg}^l(M)$.
- i) If $\mathfrak{b} \subseteq R$ is as in A)e), $\text{reg}^l(M)$ does not change if we consider M as an R/\mathfrak{b} -module.

C) (*Behaviour of Regularity in Short Exact Sequences*) Now let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be an exact sequence of finitely generated graded R -modules and let $l \in \mathbb{N}_0$. Then

- a) $\text{reg}(L) \leq \max\{\text{reg}(M), \text{reg}(N) + 1\}$.
- b) $\text{reg}^{l+1}(L) \leq \max\{\text{reg}^{l+1}(M), \text{reg}^l(N) + 1\}$.
- c) $\text{reg}^l(M) \leq \max\{\text{reg}^l(L), \text{reg}^l(N)\}$.
- d) $\text{reg}^l(N) \leq \max\{\text{reg}^{l+1}(L) - 1, \text{reg}^l(M)\}$.

D) (*Alternative Characterization of Regularity*) Finally, use (2.13) to show that for all $l \in \mathbb{N}_0$ and each finitely generated graded R -module M one has:

$$\text{reg}^l(M) = \inf\{r \in \mathbb{Z} \mid H_{R_+}^i(M)_{r+1-i} = 0, \forall i \geq l\}.$$

Our first aim is to compare generating degrees with regularity.

3.4. Proposition. *Let M be a finitely generated graded R -module. Then*

$$\text{gendeg}(M) \leq \text{reg}(M).$$

Proof. By (3.3)A)c) and (3.3)B)g) we may immediately assume that the Noetherian base ring (R_0, \mathfrak{m}_0) is local. Now, by (2.11) there is a Noetherian local faithfully flat R_0 -Algebra (R'_0, \mathfrak{m}'_0) with infinite residue field R'_0/\mathfrak{m}'_0 . So, in view of (3.3)A)d) and (3.3)B)h) we may assume that R_0/\mathfrak{m}_0 is infinite. If $M = 0$, our claim is obvious. So, let $M \neq 0$.

We proceed by induction on $d := \dim_R(M) (\in \mathbb{N}_0)$. If $d = 0$, we have $H_{R_+}^0(M)_n = M \neq 0$ and $H_{R_+}^i(M) = 0$ for all $i > 0$, and hence $\text{gendeg}(M) \leq \text{end}(M) = \text{reg}(M)$. So, let $d > 0$ and consider the short exact sequence

$$0 \rightarrow \Gamma_{R_+}(M) \rightarrow M \rightarrow M/\Gamma_{R_+}(M) \rightarrow 0.$$

As $\text{gendeg}(\Gamma_{R_+}(M)) \leq \text{end}(H_{R_+}^0(M)) \leq \text{reg}(M)$ and in view of (3.3)A)b) it suffices to show that $\text{gendeg}(M/\Gamma_{R_+}(M)) \leq \text{reg}(M)$. In view of (3.3)A)d) it is indeed even enough to show that $\text{gendeg}(M/\Gamma_{R_+}(M)) \leq \text{reg}(M/\Gamma_{R_+}(M))$. As $\dim(M/\Gamma_{R_+}(M)) \leq d$ and in view of the isomorphisms of graded R -modules $H_{R_+}^i(M) \xrightarrow{\cong} H_{R_+}^i(M/\Gamma_{R_+}(M))$ for all $i > 0$ we thus may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. So, as usually by the strict Graded Prime Avoidance Principle (2.12) we find some element $x \in R_1 \cap \text{NZD}_R(M)$. We consider the short exact sequence of graded R -modules

$$0 \rightarrow M(-1) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0.$$

As $\dim(M/xM) < d$ we get by induction that

$$g' := \text{gendeg}(M/xM) \leq \text{reg}(M/xM).$$

By (3.3)B)a),b) we have $\text{reg}(M(-1)) \leq \text{reg}(M) + 1$ and so (3.3)C)d) implies that

$$\text{reg}(M/xM) \leq \max\{\text{reg}^1(M(-1)) - 1, \text{reg}(M)\} \leq \max\{\text{reg}(M), \text{reg}(M)\},$$

so that $g' \leq \text{reg}(M)$. With $N := \sum_{n \leq g'} RM_n$ we thus have $(N + xM)/xM = M/xM$, hence $M = N + xM$. So by (2.1)C)a) we end up with $N = M$ and hence with $\text{gendeg}(M) \leq g' \leq \text{reg}(M)$. \square

3.5. Example. Let $r \in \mathbb{N}_0$ and consider the polynomial ring

$$R := R_0[X_1, X_2, \dots, X_r].$$

We aim to show that

$$\text{reg}(R) = \text{reg}(R_0[X_1, X_2, \dots, X_r]) = 0.$$

By (3.4) we already know that $\text{reg}(R) \geq \text{gendeg}(R) = 0$. It thus remains to show that $\text{reg}(R) \leq 0$. Observe that the ideal $R_+ \subseteq R$ is generated by the R -sequence X_1, X_2, \dots, X_r so that $H_{R_+}^i(R) = 0$ for all $i \neq r$ (see [Br-Fu-Ro](4.3) and (4.19)). It thus remains to show that $\text{end}(H_{R_+}^r(R)) \leq -r$. We do this by induction on r .

If $r = 0$ our claim is clear as $H_0^0(R) = R = R_0$ in this case. So, let $r > 0$ and consider the canonical homomorphism of graded R_0 -algebras

$$\pi : R = R_0[X_1, X_2, \dots, X_r] \rightarrow R' := R_0[X_1, X_2, \dots, X_{r-1}]$$

given by $X_i \mapsto X_i$ for all $i \in \{1, 2, \dots, r-1\}$ and $X_r \mapsto 0$. By induction, we have $\text{end}(H_{R'_+}^{r-1}(R')) \leq -r+1$. If we consider R' as a graded R -module by means of π and keep in mind that $R_+R' = R'_+$, the Graded Base Ring Independence of Local Cohomology (see (1.14)B)) teaches us, that we have $\text{end}(H_{R_+}^{r-1}(R')) \leq -r+1$. But now the short exact sequence of graded R -modules

$$0 \rightarrow R(-1) \xrightarrow{X_r} R \rightarrow R' \rightarrow 0$$

induces short exact sequences of R_0 -modules

$$H_{R_+}^{r-1}(R')_n \rightarrow H_{R_+}^r(R)_{n-1} \xrightarrow{X_r} H_{R_+}^r(R)_n$$

which show that multiplication by X_r yields a monomorphism $H_{R_+}^r(R)_{n-1} \hookrightarrow H_{R_+}^r(R)_n$ for all $n \geq -r+2$. But this shows a usually that $H_{R_+}^r(R) = 0$ for all $n \geq -r+1$.

Now, we shall define Castelnuovo-Mumford regularity in sheaf theoretic terms, as this was originally done by Mumford.

3.6. Definition. (*Castelnuovo-Mumford Regularity of Sheaves*) A) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be our Noetherian homogeneous ring, set $X := \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. We define the *Castelnuovo-Mumford regularity* of \mathcal{F} by:

$$\text{reg}(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{F}(r-i)) = 0, \forall i > 0\}.$$

As in the case of modules, we speak just of *regularity* from now on.

3.7. Exercise and Remark. A) (*Comparison with Regularity of Modules*) Let the notations and hypotheses be as in (3.6). Keep in mind, that there is a finitely generated graded R -module M such that $\mathcal{F} = \widetilde{M}$ (see [Br-Fu-Ro] (12.2)D)). Show on use of the Serre-Grothendieck Correspondence that

$$\text{reg}(\mathcal{F}) = \text{reg}^2(M).$$

and conclude that

$$\text{reg}(\mathcal{F}) \in \mathbb{Z} \cup \{-\infty\}.$$

B) (*Regularity and Patterns*) Let the notations and hypotheses be as above. Let $\mathcal{P} = \mathcal{P}(X, \mathcal{F})$ denote the cohomological pattern of the coherent sheaf of \mathcal{O}_X -modules \mathcal{F} . We describe the pattern \mathcal{P} by \bullet 's and \circ 's as suggested in (2.17). Make clear that $\text{reg}(\mathcal{F})$ is the first place in the bottom row of \mathcal{P} lying on a diagonal which contains only \circ 's above the bottom level. Observe that

all “later” diagonals also consist completely of \circ 's above the bottom level.

$$\begin{array}{cccccccccc}
 \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 \circ & \circ & \bullet & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\
 \bullet & \bullet & \bullet & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\
 \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \text{reg}(\mathcal{F}) & \bullet & \bullet
 \end{array}$$

Our next goal is to deduce the announced relation between the regularity of a coherent sheaf and the generation by global sections of its twists. To do so, we first need to develop further our sheaf-theoretic machinery.

3.8. Construction and Exercise. (*Total Modules of Sections*) A) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be our Noetherian homogeneous ring, let $X := \text{Proj}(R)$ and let \mathcal{F} be a sheaf of \mathcal{O}_X -modules. Let $U \in \mathbb{U}_X$, where \mathbb{U}_X denotes the set of open subsets of X . We consider the R_0 -module (see [Br-Fu-Ro](12.2),(12.8)A))

$$\Gamma_*(U, \mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(U, \mathcal{F}(n)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}(n)(U),$$

which we call the *total module of sections* in \mathcal{F} over U . Now, let $m, n \in \mathbb{Z}$ and let

$$a = (a_l)_{l \in R_1} \in \mathcal{O}_X(m)(U); f = (f_l)_{l \in R_1} \in \mathcal{F}(n)(U)$$

be families on m -sections of \mathcal{O}_X respectively of n -sections of \mathcal{F} over U (see [Br-Fu-Ro](12.5)A)). Make clear that

$$af := (a_l f_l)_{l \in R_1} \in \mathcal{F}(m+n)(U)$$

is a family of $(m+n)$ -sections of \mathcal{F} over U .

B) Keep the above notations and hypotheses. Let $m, m', n, n' \in \mathbb{Z}$ and show that in $\Gamma_*(U, \mathcal{F})$ we have

a) For all $a \in \mathcal{O}_X(m)(U)$, all $a' \in \mathcal{O}_X(m')(U)$ and all $f \in \mathcal{F}(n)(U)$:

$$(a + a')f = af + a'f.$$

b) For all $a \in \mathcal{O}_X(m)(U)$, all $f \in \mathcal{F}(n)(U)$ and all $f' \in \mathcal{F}(n')(U)$:

$$a(f + f') = af + af'.$$

Conclude that the R_0 -module $\Gamma_*(U, \mathcal{O}_X)$ is in fact a (unitary, commutative) R_0 -algebra, by means of the multiplication defined by:

c) For all $(a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}} \in \Gamma_*(U, \mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(U, \mathcal{O}_X(n))$:

$$(a_n)_{n \in \mathbb{Z}}(b_n)_{n \in \mathbb{Z}} := \left(\sum_{i+j=n} a_i b_j \right)_{n \in \mathbb{Z}}.$$

Clearly, the ring $\Gamma_*(U, \mathcal{O}_X)$ carries a natural \mathbb{Z} -grading, given by

$$d) \Gamma_*(U, \mathcal{O}_X)_n = \Gamma(U, \mathcal{O}_X(n)) \text{ for all } n \in \mathbb{Z}.$$

Show that the R_0 -module $\Gamma_*(U, \mathcal{F})$ is turned into a $\Gamma_*(U, \mathcal{O}_X)$ -module, by means of the scalar multiplication defined by:

e) For all

$$(a_n)_{n \in \mathbb{Z}} \in \Gamma_*(U, \mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(U, \mathcal{O}_X(n))$$

and all

$$(f_n)_{n \in \mathbb{Z}} \in \Gamma_*(U, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(U, \mathcal{F}(n)) :$$

$$(a_n)_{n \in \mathbb{Z}} (f_n)_{n \in \mathbb{Z}} := \left(\sum_{i+j=n} a_i f_j \right)_{n \in \mathbb{Z}}.$$

Finally, the $\Gamma_*(U, \mathcal{O}_X)$ -module $\Gamma_*(U, \mathcal{F})$ carries a natural \mathbb{Z} -grading, given by

$$f) \Gamma_*(U, \mathcal{F})_n = \Gamma(U, \mathcal{F}(n)) \text{ for all } n \in \mathbb{Z}.$$

C) Now, for each $n \in \mathbb{Z}$ we may consider the following natural homomorphism of R_0 -modules (see [Br-Fu-Ro] (11.5)C), (12.5)C)f)

$$\varepsilon_{R,n}^U : R_n = R(n)_0 \xrightarrow{\varphi^U \circ \eta_{R(n)}^U} \mathcal{O}_X(n)(U),$$

given by $c \mapsto (c_h)_{h \in R_1} \in \prod_{h \in R_1} \mathcal{O}_X(U_h)$, where for the germs we have

$$(c_h)_x = \frac{c}{h^n} \in \mathcal{O}_{X,x} = R_{(\mathfrak{p})}, \forall h \in R_1, \forall x = \mathfrak{p} \in U_h.$$

Make clear, that the map

$$\varepsilon_{R,*}^U : R \rightarrow \Gamma_*(U, \mathcal{O}_X), (c_n)_{n \in \mathbb{Z}} \mapsto (\varepsilon_{R,n}^U(c_n))_{n \in \mathbb{Z}}.$$

is a homomorphism of graded rings, which allows to view $\Gamma_*(U, \mathcal{F})$ as a graded R -module.

D) Consider a homomorphism of sheaves of \mathcal{O}_X -modules, $f : \mathcal{F} \rightarrow \mathcal{G}$. Show that for each $n \in \mathbb{Z}$, there is a homomorphism of \mathcal{O}_X -modules

$$h(n)(U) : \Gamma_*(U, \mathcal{F})_n = \mathcal{F}(n)(U) \rightarrow \mathcal{G}(n)(U) = \Gamma_*(U, \mathcal{G})_n,$$

given by $(f_i)_{i \in R_1} \mapsto (h(U_i)(f_i))_{i \in \mathbb{Z}}$ for each family of n -sections

$$(f_i)_{i \in \mathbb{Z}} \in \prod_{i \in R_1} \mathcal{F}(U_i)$$

of \mathcal{F} over U . Show, that there is a homomorphism of graded $\Gamma_*(U, \mathcal{O}_X)$ -modules

$$\Gamma_*(U, h) : \Gamma_*(U, \mathcal{F}) \rightarrow \Gamma_*(U, \mathcal{G}); (\gamma_n)_{n \in \mathbb{Z}} \mapsto (h(U)(\gamma_n))_{n \in \mathbb{Z}}.$$

Prove, that we now have defined a (linear, covariant) left exact functor

$$\Gamma_*(U, \bullet) : (\mathcal{F} \xrightarrow{h} \mathcal{G}) \mapsto (\Gamma_*(U, \mathcal{F}) \xrightarrow{\Gamma_*(U, h)} \Gamma_*(U, \mathcal{G}))$$

from sheaves of \mathcal{O}_X -modules to graded $\Gamma_*(U, \mathcal{O}_X)$ -modules and hence to graded R -modules (see part C)).

E) Let M be a graded R -module with $\widetilde{M} = \mathcal{F}$. Then again, for each $n \in \mathbb{Z}$ we may consider the natural map (see [Br-Fu-Ro] (11.5)C), (12.5)Cf))

$$\varepsilon_{M,n}^U : M_n = M(n)_0 \xrightarrow{\varphi^U \circ \eta_{M(n)}^U} \mathcal{F}(n)(U),$$

given by $m \mapsto (m_h)_{h \in R_1} \in \prod_{h \in R_1} \mathcal{F}(U_h)$, where for the germs we have

$$(m_h)_x = \frac{m}{h^n} \in \mathcal{F}_x = M_{(\mathfrak{p})}, \forall h \in R_1, \forall x = \mathfrak{p} \in U_h.$$

Make clear, that we have a homomorphisms of graded R -modules

$$\varepsilon_{M,*}^U : M \rightarrow \Gamma_*(U, \mathcal{F}), (m_n)_{n \in \mathbb{Z}} \mapsto (\varepsilon_{M,n}^U(m_n))_{n \in \mathbb{Z}}.$$

Now prove that we have a natural transformation of functors of graded R -modules

$$\varepsilon_*^U : \bullet \rightarrow \Gamma_*(U, \bullet), M \mapsto (\varepsilon_{M,*}^U : M \rightarrow \Gamma_*(U, \widetilde{M})).$$

F) Finally, let $\mathfrak{a} \subseteq R_+$ be a graded ideal such that $U = \mathcal{U}(\mathfrak{a}) \subseteq \text{Proj}(R) = X$ is the open set defined by \mathfrak{a} (see [Br-Fu-Ro] (11.4)C)a)). Let M be a graded R -module and set $\mathcal{F} = \widetilde{M}$. Conclude by [Br-Fu-Ro](11.13) that for each $n \in \mathbb{Z}$ there is an exact sequence of R_0 -modules

$$0 \rightarrow \Gamma_{\mathfrak{a}}(M)_n \rightarrow M_n \xrightarrow{\varepsilon_{M,n}^U} \Gamma_*(U, \mathcal{F})_n \rightarrow H_{\mathfrak{a}}^1(M)_n \rightarrow 0.$$

Draw the following conclusions:

- a) $\text{Ker}(\varepsilon_{M,*}^U) = \Gamma_{\mathfrak{a}}(M)$.
- b) If M is \mathfrak{a} -torsion, then $\Gamma_*(U, \mathcal{F}) = 0$.

3.9. Lemma. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let $\mathfrak{a} \subseteq R_+$ be a graded ideal, let $U = \mathcal{U}(\mathfrak{a}) \in \mathbb{U}_X$ be the open set defined in $X = \text{Proj}(R)$ by \mathfrak{a} and let \mathcal{F} be a sheaf of \mathcal{O}_X -modules. Then $H_{\mathfrak{a}}^0(\Gamma_*(U, \mathcal{F})) = 0$.*

Proof. Let $n \in \mathbb{Z}$ and let $f \in H_{\mathfrak{a}}^0(\Gamma_*(U, \mathcal{F}))_n$. Then $f \in \Gamma_*(U, \mathcal{F})_n = \mathcal{F}(n)(U)$ and $\mathfrak{a}^t f = 0$ for some $t \in \mathbb{N}$. We write $f = (f_l)_{l \in R_1} \in \prod_{l \in R_1} \mathcal{F}(U_l)$ as a family of n -sections in \mathcal{F} over U . Let $\mathfrak{a}^t = \langle a_1, a_2, \dots, a_r \rangle$ with $a_i \in R_{d_i}$ and $d_i \in \mathbb{N}$ for all $i \in \{1, 2, \dots, r\}$. Then $0 = a_i f = \varepsilon_{R,*}^U(a_i) f = \varepsilon_{R,d_i}^U(a_i) f$. So for all $i \in \{1, 2, \dots, r\}$ and all $l \in R_1$ we have (see (3.8)E))

$$\frac{a_i}{l^{d_i}} f_{l,x} = 0(\in \mathcal{F}_x), \forall x \in U_l.$$

Now let $x \in U$. Then $x \in U_l$ for some $l \in R_1$. As $U = \mathcal{U}(\mathfrak{a}) = \mathcal{U}(\mathfrak{a}^t)$, there is some $i \in \{1, 2, \dots, r\}$ such that $x \notin \text{Var}(Ra_i)$. We thus have $a_i, l \notin \mathfrak{m}_{X,x}$ and hence $\frac{l^{d_i}}{a_i} \in \mathcal{O}_{X,x}$. So we obtain

$$f_{l,x} = \frac{l^{d_i}}{a_i} \frac{a_i}{l^{d_i}} f_{l,x} = 0(\in \mathcal{F}_x).$$

This proves that $f_l \in \mathcal{F}(U_l)$ is vanishing for all $l \in R_1$. Therefore $f = 0$. \square

3.10. Proposition. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let $X = \text{Proj}(R)$, let M be a finitely generated graded R -module and set $\Gamma := \Gamma_*(X, \widetilde{M})$. Then*

a) *The natural homomorphism of graded R -modules $\varepsilon_M = \varepsilon_{M,*}^X : M \rightarrow \Gamma$ induces an isomorphism of sheaves*

$$\widetilde{\varepsilon}_M : \widetilde{M} \xrightarrow{\cong} \widetilde{\Gamma}.$$

b) *The natural homomorphism of graded R -modules*

$$\varepsilon_\Gamma = \varepsilon_{\Gamma,*}^X : \Gamma \rightarrow \Gamma_*(X, \widetilde{\Gamma})$$

is an isomorphism.

c) $H_{R_+}^0(\Gamma) = H_{R_+}^1(\Gamma) = 0$.

Proof. “a)”: For each $n \in \mathbb{Z}$ we have an exact sequence

$$0 \rightarrow H_{R_+}^0(M)_n \rightarrow M_n \xrightarrow{\varepsilon_{M,n}} \Gamma_n \rightarrow H_{R_+}^1(M)_n \rightarrow 0.$$

(see (3.8)F)). As $H_{R_+}^i(M)_n = 0$ for all $n \gg 0$ and all $i \in \mathbb{N}_0$ (see (2.2)) we thus have an exact sequence of graded R -modules

$$0 \rightarrow U \rightarrow M \xrightarrow{\varepsilon_M} \Gamma \rightarrow V \rightarrow 0$$

in which U and V are both R_+ -torsion. Bearing in mind (2.14)D)a) we get indeed the requested isomorphism $\widetilde{\varepsilon}_M : \widetilde{M} \xrightarrow{\cong} \widetilde{\Gamma}$.

“b)”: Keep in mind that $\Gamma_{R_+}(\Gamma) = 0$ (see (3.9)). In view of (3.8)F)a) it therefore follows that the homomorphism ε_Γ is injective. It remains to show that ε_Γ is surjective. By the naturality of the homomorphisms $\varepsilon_{\bullet,*}^X$ (see (3.8)E)) we get the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\varepsilon_M} & \Gamma \\ \downarrow \varepsilon_M & & \downarrow \varepsilon_\Gamma \\ \Gamma & \xrightarrow{f := \Gamma_*(X, \widetilde{\varepsilon}_M)} & \Gamma_*(X, \widetilde{\Gamma}) \end{array}$$

in which f is an isomorphism by statement a). Now, let $\delta \in \Gamma_*(X, \widetilde{\Gamma})$. We find some $\gamma \in \Gamma$ such that $f(\gamma) = \delta$. In the proof of part a) we have already seen, that the cokernel of ε_M is R_+ -torsion. We therefore find some $n \in \mathbb{N}$ such that $(R_+)^n \gamma \subseteq \text{Im}(\varepsilon_M)$. Now let $u_1, u_2, \dots, u_t \in R_+$ such that $(R_+)^n = \langle u_1, u_2, \dots, u_t \rangle$. Then for each index $i \in \{1, 2, \dots, t\}$ we find some element $m_i \in M$ such that $u_i \gamma = \varepsilon_M(m_i)$. Consequently in view of the above diagram we get for each $i \in \{1, 2, \dots, t\}$ the relations

$$u_i f(\gamma) = f(u_i \gamma) = f(\varepsilon_M(m_i)) = \varepsilon_\Gamma(\varepsilon_M(m_i)) = \varepsilon_\Gamma(u_i \gamma) = u_i \varepsilon_\Gamma(\gamma).$$

So, for each $i \in \{1, 2, \dots\}$ we finally obtain

$$u_i(\delta - \varepsilon_\Gamma(\gamma)) = u_i f(\gamma) - u_i \varepsilon_\Gamma(\gamma) = 0.$$

But this implies that $(R_+)^n(\delta - \varepsilon_\Gamma(\gamma)) = 0$. As $\Gamma_{R_+}(\Gamma_*(X, \widetilde{\Gamma})) = 0$ (see (3.9)) it follows that $\delta - \varepsilon_\Gamma(\gamma) = 0$ and hence $\delta = \varepsilon_\Gamma(\gamma)$. This proves that ε_Γ is surjective.

“c)”: This follows immediately from statement b) on use of the exact sequences of (3.8)F). \square

3.11. Lemma. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let $X = \text{Proj}(R)$, let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules and let $t \in \mathbb{Z}$. Consider the graded R -module $\Gamma := \Gamma_*(X, \mathcal{F})$ and its graded submodule*

$$\Gamma_{\geq t} := \bigoplus_{n \geq t} \Gamma_n \subseteq \Gamma.$$

Then:

- a) *The graded R -module $\Gamma_{\geq t}$ is finitely generated.*
- b) *$H_{R_+}^0(\Gamma_{\geq t}) = 0$ and $\text{end}(H_{R_+}^1(\Gamma_{\geq t})) < t$.*
- c) *$\widetilde{\Gamma_{\geq t}} \cong \widetilde{\Gamma} \cong \mathcal{F}$.*
- d) *$\text{gendeg}(\Gamma_{\geq t}) \leq \max\{t, \text{reg}(\mathcal{F})\}$.*
- e) *$\text{gendeg}(\Gamma) \leq \text{reg}(\mathcal{F})$.*

Proof. “a)”: We find a finitely generated graded R -module M with $\widetilde{M} = \mathcal{F}$. Again by the exact sequences of (3.8)F) and by the fact that $H_{R_+}^i(M)_n = 0$ for all $n \gg 0$ we get isomorphisms of R_0 -modules

$$\varepsilon_{M,n} : M_n \xrightarrow{\cong} \Gamma_n, \forall n \gg 0,$$

so that the natural homomorphism of graded R -modules $\varepsilon_{M,n} : M \rightarrow \Gamma$ is an isomorphism in large degrees. As M is finitely generated, it follows immediately, that $\Gamma_{\geq t}$ is finitely generated.

“b)”: By (3.10)c) we have $H_{R_+}^0(\Gamma) = 0$ and as $\Gamma_{\geq t}$ is a submodule of Γ , we get $H_{R_+}^0(\Gamma_{\geq t}) = 0$. Observe, that we also have an exact sequence of graded R -modules

$$0 \rightarrow \Gamma_{\geq t} \rightarrow \Gamma \rightarrow P \rightarrow 0$$

in with $\text{end}(P) < t$. If we apply cohomology and observe that $H_{R_+}^i(\Gamma) = 0$ for $i = 1, 2$ (see (3.10)c)) we get an isomorphism of graded R -modules $P \cong H_{R_+}^1(\Gamma_{\geq t})$ so that indeed $\text{end}(H_{R_+}^1(\Gamma_{\geq t})) < t$.

“c)”: The second isomorphism is clear by (3.10)a). The first isomorphism follows by (2.14)D)a) applied the exact sequence used in the proof of statement b).

“d)”: By statement c) and (3.7) we have $\text{reg}(\mathcal{F}) = \text{reg}^2(\Gamma_{\geq t})$. By statement b) it follows

$$\text{reg}(\Gamma_{\geq t}) \leq \max\{t, \text{reg}^2(\Gamma_{\geq t})\} = \max\{t, \text{reg}(\mathcal{F})\}.$$

Now we may conclude by (3.4).

“e)”: This follows easily from the fact that statement d) holds for any choice of t . \square

3.12. Exercise and Definition. (*Generation of Sheaves by Global Sections*)

A) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian positively graded ring, let $X = \text{Proj}(R)$ and let \mathcal{F} be a sheaf of \mathcal{O}_X -modules. Let $\mathcal{S} \subseteq \Gamma(X, \mathcal{F})$. We say that the sheaf \mathcal{F} is generated by \mathcal{S} if

$$\mathcal{F}_x = \sum_{f \in \mathcal{S}} \mathcal{O}_{X,x} f_x, \forall x \in X.$$

B) Keep the notations and hypotheses of part A) and prove the following statements

- a) If $\mathcal{S} \subseteq \mathcal{T} \subseteq \Gamma(X, \mathcal{F})$, and \mathcal{F} is generated by \mathcal{S} , then it is also generated by \mathcal{T} .
- b) \mathcal{F} is generated by \mathcal{S} if and only if it is generated by the R_0 -module $\langle \mathcal{S} \rangle = \sum_{f \in \mathcal{S}} R_0 f \subseteq \Gamma(X, \mathcal{F})$.

C) Keep the previous notations and hypotheses. We say that \mathcal{F} is *generated by global sections* if there is some set $\mathcal{S} \subseteq \Gamma(X, \mathcal{F})$ such that \mathcal{F} is generated by \mathcal{S} . It is obviously equivalent to say that \mathcal{F} is generated by $\Gamma(X, \mathcal{F})$. If \mathcal{F} is generated by a finite set $\mathcal{S} \subseteq \Gamma(X, \mathcal{F})$, we say that \mathcal{F} is *generated by finitely many global sections*.

Now we are ready to formulate and to prove the result on the global generation of twists of coherent sheaves over projective schemes we are heading for.

3.13. Theorem. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let $X = \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Then for all $n \geq \text{reg}(\mathcal{F})$ the sheaf $\mathcal{F}(n)$ is generated by (finitely many) global sections.*

Proof. Let $n \geq \text{reg}(\mathcal{F})$. We set $\Gamma := \Gamma_*(X, \mathcal{F})$. According to (3.11)a), the graded R -module $\Gamma_{\geq n}$ is finitely generated. So, the R_0 module $\Gamma(X, \mathcal{F}(n)) = \Gamma_n$ is finitely generated. Let $f_1, f_2, \dots, f_r \in \Gamma_n$ be such that $\Gamma_n = \sum_{1 \leq i \leq r} R_0 f_i$. According to (3.11)d) we have $\text{gendeg}(\Gamma_{\geq n}) \leq n$. As the ring R is homogeneous it follows by [Br-Fu-Ro](9.6)Ea) that

$$\text{a) } \Gamma_t = R_{t-n} \Gamma_n = \sum_{1 \leq i \leq r} R_{t-n} f_i \text{ for all } t \geq n.$$

Now, let $x = \mathfrak{p} \in X$ and let $\gamma \in \mathcal{F}_x$. By (3.11)c) we may write $\mathcal{F} = \widetilde{\Gamma}$ and hence $\mathcal{F}(n) = \widetilde{\Gamma}(n) = \widetilde{\Gamma(n)}$ (see [Br-Fu-Ro](12.5)C)). Therefore the stalk $\mathcal{F}(n)_x$ of $\mathcal{F}(n)$ at x coincides with the homogeneous localization $\Gamma(n)_{(\mathfrak{p})}$ of the graded R -module $\Gamma(n)$ at the prime $\mathfrak{p} \in \text{Proj}(R)$ (see [Br-Fu-Ro](12.4)B)). So, we find some $m \in \mathbb{N}_0$, some $s \in R_m \setminus \mathfrak{p}$ and some $f \in \Gamma(n)_m = \Gamma_{m+n}$ such that in $\mathcal{F}(n)_x = \Gamma(n)_{(\mathfrak{p})}$ we have $\gamma = \frac{f}{s}$. Applying the above observation a)

with $t = m + n$ we thus find some elements $a_1, a_2, \dots, a_r \in R_m$ such that $f = \sum_{1 \leq i \leq r} a_i f_i$. It follows that

$$\gamma = \frac{f}{s} = \sum_{1 \leq i \leq r} \frac{a_i f_i}{s \cdot 1},$$

with $\frac{a_i}{s} \in R_{(p)} = \mathcal{O}_{X,x}$ and $\frac{f_i}{1} \in \Gamma(n)_{(p)} = \mathcal{F}(n)_x$ for all $i \in \{1, 2, \dots, r\}$. According to [Br-Fu-Ro](12.4)B) we may write $\frac{f_i}{1} = (f_i)_x \in \Gamma(n)_{(p)} = \mathcal{F}(n)_x$. Therefore

$$\gamma = \sum_{1 \leq i \leq r} \frac{a_i f_i}{s \cdot 1} = \sum_{1 \leq i \leq r} \frac{a_i}{s} (f_i)_x \in \sum_{1 \leq i \leq r} \mathcal{O}_{X,x} (f_i)_x.$$

So, the sheaf $\mathcal{F}(n)$ is generated by the finite set $\{f_1, f_2, \dots, f_r\} \subseteq \Gamma(X, \mathcal{F}(n))$. \square

3.14. Corollary. *Let $X = \text{Proj}(R)$, where $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is a Noetherian homogeneous ring, and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Then the sheaf $\mathcal{F}(n)$ is generated by (finitely many) global sections for all $n \gg 0$.*

Proof. This is immediate by (3.13). \square

In order to illustrate the results of this section we now give a number of examples, presented in the form of exercises.

3.15. Examples and Exercises. A) (*Algebras of Regularity Zero*) In 3.5 we have seen that polynomial rings are of regularity 0. We now want to establish a partial converse of this. So let K be an infinite field and let R be a homogeneous K -algebra of dimension d such that $\text{reg}(R) = 0$. Show by induction on d , that there are d elements $x_1, x_2, \dots, x_d \in R_1$ such that $R = K[x_1, x_2, \dots, x_d]$ and conclude that R can be viewed as a polynomial ring over K .

B) (*Preservation of Global Generation Under Positive Twists*) Once more, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let $X = \text{Proj}(R)$, let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Write $\mathcal{F} = \widetilde{M}$ with some finitely generated graded R -module M . Prove the following:

a) For all $n \in \mathbb{Z}$, all $\mathfrak{p} \in \text{Proj}(R)$ and all $f \in R_1 \setminus \mathfrak{p}$, the multiplication map

$$f : M(n)_{(p)} \rightarrow M(n+1)_{(p)}$$

is an isomorphism of $R_{(p)}$ -modules.

b) If $m \in \mathbb{Z}$ and $\mathcal{F}(m)$ is generated by $\mathcal{S} \subseteq \Gamma(X, \mathcal{F}(m))$, then $\mathcal{F}(n)$ is generated by $R_{n-m} \mathcal{S}$ for all $n \geq m$.

c) If $\mathcal{F}(m)$ is generated by global sections for some $m \in \mathbb{Z}$, then it is so for all $n \geq m$.

C) (*Global Generation of Twisted Structure Sheaves*) Let the notations be as in part B) but assume in addition that $\dim(R) > 1$ and $H_{R_+}^i(R) = 0$ for $i = 0, 1$. Show that $\mathcal{O}_X(n)$ is generated by global sections if and only if $n \geq 0$. Use

(3.5) to see that the bound given in (3.13) is sharp if $R = R_0[X_1, X_2, \dots, X_d]$ is a polynomial ring with $d > 1$ and $\mathcal{F} = \mathcal{O}_X(m)$ for an arbitrary integer m . Now, let K be field, let $d, r \in \mathbb{N}$ with $d > 2$, let $f \in K[X_1, X_2, \dots, X_d]_r \setminus 0$. Show that

$$\text{reg}(R) := K[X_1, X_2, \dots, X_d]/\langle f \rangle = r$$

Conclude from this, that the bound given in (3.13) is not sharp in this case for $\mathcal{F} = \mathcal{O}_X$.

D) (*Alternative Characterization of Generation by Global Sections*) Let the notations be as in part B). Let $\mathcal{G}_1, \dots, \mathcal{G}_r$ be sheaves of \mathcal{O}_X -modules. Make clear that the assignment $U \mapsto \bigoplus_{1 \leq i \leq r} \mathcal{G}_i(U)$ for all $U \in \mathbb{U}_X$ defines a sheaf $\bigoplus_{1 \leq i \leq r} \mathcal{G}_i$ of \mathcal{O}_X -modules, the *direct sum* of the sheaves $\mathcal{G}_1, \dots, \mathcal{G}_r$. Show that:

- a) For all $x \in X$ we have $(\bigoplus_{1 \leq i \leq r} \mathcal{G}_i)_x \cong \bigoplus_{1 \leq i \leq r} (\mathcal{G}_i)_x$.
- b) If $\mathcal{G}_1, \dots, \mathcal{G}_r$ are coherent, then so is $\bigoplus_{1 \leq i \leq r} \mathcal{G}_i$.
- c) If M_1, \dots, M_r are graded R -modules, then

$$\widetilde{\bigoplus_{1 \leq i \leq r} M_i} \cong \bigoplus_{1 \leq i \leq r} \widetilde{M_i}.$$

If \mathcal{F} is a sheaf of \mathcal{O}_X -modules and $r \in \mathbb{N}$ we write $\mathcal{F}^{\oplus r} := \bigoplus_{1 \leq i \leq r} \mathcal{F}$. Using this notation prove the following:

- d) The coherent sheaf of \mathcal{O}_X -modules \mathcal{F} is generated by r global sections if and only if there is a surjective homomorphism of sheaves of \mathcal{O}_X -modules $\mathcal{O}^{\oplus r} \rightarrow \mathcal{F}$.

E) (*Total Modules of Sections*) Let the notations and hypotheses be as in part B) and set $\Gamma = \Gamma_*(X, \mathcal{F})$, where $\mathcal{F} = \widetilde{M}$ for some finitely generated graded R -module M . Show that the following statements are equivalent:

- (i) The R -module Γ is finitely generated.
- (ii) $H_{R_+}^1(M)$ is a finitely generated R -module.
- (iii) $\text{beg}(H_{R_+}^1(M)) > -\infty$.

Assume that the equivalent conditions (i),(ii),(iii) are satisfied and show that

- a) $\text{beg}(\Gamma) = \min\{\text{beg}(M/\Gamma_{R_+}(M)), \text{beg}(H_{R_+}^1(M))\}$.
- b) $\text{gendeg}(\Gamma) \leq \max\{\text{gendeg}(M), \text{end}(H_{R_+}^1(M))\}$.

4. HILBERT-SERRE COEFFICIENTS

In (3.4) we have seen that the generating degree of a finitely generated graded module over a Noetherian homogeneous ring is bounded from above by the regularity of this module. In many cases finitely generated graded modules are given by a presentation, so that their generating degree is a fortiori known. Clearly the bounding result (3.4) is not of any interest in these cases. It would be much more interesting in this situation, to find an upper bound for the regularity in terms of the generating degree and eventually some additional numerical invariants of the module under consideration. There are indeed examples given in [Ma-Me], which show that the generating degree alone cannot be used to bound the regularity. In this section, we shall prove a bounding result of this type, which applies over Noetherian homogeneous algebras over a field. More precisely, we shall prove that for a finitely generated graded module M over such a ring the regularity at and above level 1 is bounded in terms of the generating degree and the Hilbert polynomial of M . As an application we shall get back Mumford's regularity bound for coherent sheaves of ideals \mathcal{I} over a projective space [Mu1] in terms of the Hilbert polynomial of \mathcal{I} .

4.1. Notation. Throughout this section let $R = K \oplus R_1 \oplus R_2 \oplus R_3 \oplus \dots$ be a Noetherian homogeneous ring with base field K . For a finitely generated R -module M let $P_M \in \mathbb{Q}[X]$ denote the Hilbert polynomial of M . For each $i \in \mathbb{N}_0$ let $h_M^i : \mathbb{Z} \rightarrow \mathbb{N}_0$ denote the i -th cohomological Hilbert function of M and let $\chi_M : \mathbb{Z} \rightarrow \mathbb{Z}$ denote the characteristic function of M (see (2.4)B)).

4.2. Reminder and Exercise. (*Numerical Polynomials*) A) A polynomial $P \in \mathbb{R}[X]$ is called a *numerical polynomial* if $P(\mathbb{Z}) \subseteq \mathbb{Z}$. Show that for a polynomial $P \in \mathbb{R}[X]$ of degree $s \geq 0$ the following statements are equivalent:

- (i) P is a numerical polynomial.
- (ii) There is an integer $n \in \mathbb{Z}$ such that $P(n+i) \in \mathbb{Z}$ for all $i \in \{0, 1, \dots, s\}$
- (iii) There are integers $e_0^P, e_1^P, \dots, e_s^P \in \mathbb{Z}$ such that $e_0^P \neq 0$ and

$$P(X) = \sum_{0 \leq i \leq s} (-1)^i e_i^P \binom{X+s-i}{s-i}.$$

In this situation, the integers e_i^P are uniquely determined by P and are called the *binomial coefficients* of P .

B) If $P \in \mathbb{R}[X]$ is a numerical polynomial of degree $s \geq 0$ we can say:

- a) $P \in \mathbb{Q}[X]$.
- b) $e_0^P = \lim_{n \rightarrow \infty} \frac{s!P(n)}{n^s}$.
- c) $e_0^P > 0$ if and only if $P(n) > 0$ for all $n \gg 0$.
- d) $e_0^P < 0$ if and only if $P(n) < 0$ for all $n \gg 0$.

C) Let $P \in \mathbb{Q}[X]$ be a numerical polynomial of degree $s > 0$. Show that the *first difference polynomial*

$$\Delta P := P(X) - P(X - 1) \in \mathbb{Q}[X]$$

is a numerical polynomial of degree $s - 1$ such that

a) $e_i^{\Delta P} = e_i^P$ for all $i \in \{0, 1, \dots, s - 1\}$.

4.3. Reminder and Exercise. (*Hilbert-Serre Coefficients*) A) Let M be a finitely generated graded R -module. We set

$$e_i(M) := \begin{cases} e_i^{P_M}, & \forall i \in \{0, 1, \dots, \deg(P_M)\} \\ 0, & \forall i \in \mathbb{Z}_{>\deg(P_M)} \end{cases}$$

and call this number the i -th *Hilbert-Serre coefficient* of M .

If $\deg(P_M) \geq 0$ or-equivalently-if $\dim_R(M) > 0$ the number $e_0(M)$ is called the *Hilbert-Serre multiplicity* $\text{mult}(M)$ of M . If $\dim_R(M) \leq 0$ or-equivalently-if M is R_+ -torsion, the Hilbert-Serre multiplicity of M is defined as the (finite) K -vector space dimension $\dim_K(M)$ of M . Thus

$$\text{mult}(M) := \begin{cases} e_0(M), & \dim_R(M) > 0 \\ \dim_K(M), & \dim_R(M) \leq 0 \end{cases}$$

B) Prove that

- a) $e_0(M) \in \mathbb{N}$ if and only if $\deg(P_M) \geq 0$.
- b) $P_M(n) = \chi_M(n) = \sum_{i \in \mathbb{N}_0} (-1)^i e_i(M) \binom{n + \deg(P_M) - i}{\deg(P_M) - i}$ for all $n \in \mathbb{Z}$.
- c) $e_i(M) = e_i(M/\Gamma_{R_+}(M))$ for all $i \in \mathbb{N}_0$.
- d) If $x \in \text{NZD}_R(M) \cap R_1$, then $P_{M/xM} = \Delta P_M$ and $e_i(M/xM) = e_i(M)$ for all $i < \deg(P_M)$.
- e) $\text{mult}(M) \in \mathbb{N}$ if and only if $M \neq 0$.

C) Later, we often shall have to perform a base field change with our Noetherian homogeneous K -algebra R . We now wish to develop in the form of exercises a number of facts which shall be useful in this respect. So let K' be an extension field of K , consider the Noetherian homogeneous K' -algebra $R' := K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \dots$ and the finitely generated graded R' -module $K' \otimes_K M = \bigoplus_{n \in \mathbb{Z}} K' \otimes_K M_n$. Use the observations made in (2.4) to prove the following facts

- a) $e_i(M') = e_i(M)$ for all $i \in \mathbb{N}_0$.
- b) $\text{mult}(M') = \text{mult}(M)$.

We now prove the announced bounding result in the special case of a graded module of dimension 1, where it takes a particularly simple form.

4.4. Lemma. *Let M be a finitely generated graded R -module of dimension ≤ 1 . Then*

$$\text{reg}^1(M) \leq \text{gendeg}(M) + e_0(M).$$

Proof. If $\dim_R(M) \leq 0$ the left hand side of the stated inequality takes the value $-\infty$ and we are done. So, let $\dim_R(M) = 1$. By the reduction arguments of (2.4)C)a),b) and (3.3)A)d),B)h) we may replace K by one of its infinite extension fields and hence assume at once, that K is infinite. As $\deg(P_M) = \dim(M) - 1 = 0$ (see (2.4)C)b)) we may write $P_M = e_0(M)$. Now, in view of (4.3)c), (3.3)B)d) and (3.3)A)b) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. So, we find some $x \in \text{NZD}_R(M) \cap R_1$ (see [Br-Fu-Ro](9.9)). As $P_{M/xM} = \Delta P_M = 0$ (see (4.3)d)) we have $(M/xM)_n = 0$ for all large n and hence

$$M_{n+1} = xM_n, \forall n \gg 0.$$

Now, let $m \geq \text{gendeg}(M)$ such that $M_{m+1} = xM_m$ and let $f_1, f_2, \dots, f_r \in R_1$ be such that $R_1 = \sum_{1 \leq i \leq r} Kf_i$. As $m \geq \text{gendeg}(M)$ and R is homogeneous we obtain (see [Br-Fu-Ro](9.6)E)a))

$$\begin{aligned} M_{m+2} &= R_1 M_{m+1} = \sum_{1 \leq i \leq r} f_i M_{m+1} = \sum_{1 \leq i \leq r} f_i x M_m \\ &= x \sum_{1 \leq i \leq r} f_i M_m = x R_1 M_m = x M_{m+1}. \end{aligned}$$

By induction on m it now follows:

- a) If $m_0 \geq \text{gendeg}(M)$ such that the multiplication map $x : M_{m_0} \rightarrow M_{m_0+1}$ is an isomorphism, then the multiplication map $x : M_m \rightarrow M_{m+1}$ is an isomorphism for all $m \geq m_0$.

This shows that the function $m \mapsto \dim_K(M_m)$ is strictly increasing in the range $m \geq \text{gendeg}(M)$ until it reaches its constant value $e_0(M)$. Consequently we obtain

$$\dim_K(M_n) = e_0(M) = P_M = \chi_M, \forall n \geq \text{gendeg}(M) + e_0(M).$$

As $H_{R_+}^i(M) = 0$ for all $i \neq 1$ we have $e_0(M) = \chi_M = \dim_K(M_n) + h_M^1(n)$ for all $n \in \mathbb{Z}$. It thus follows:

$$h_M^1(n) = e_0(M) - \dim_K(M_n) = 0, \forall n \geq \text{gendeg}(M) + e_0(M).$$

Therefore $\text{end}(H_{R_+}^1(M)) < \text{gendeg}(M) + e_0(M)$. As $H_{R_+}^i(M) = 0$ for all $i > 1$ it follows that $\text{reg}^1(M) \leq \text{gendeg}(M) + e_0(M)$. \square

Now we want to approach our announced bounding result in the general situation. We begin with a few technical prerequisites.

4.5. Construction and Exercise. A) (*A Family of Bounding Polynomials*) Let $(U_i)_{i \in \mathbb{N}_0}$ be a family of independent indeterminates. We recursively define a family $(Q_t)_{t \in \mathbb{N}}$ of polynomials $Q_t \in \mathbb{Q}[U_0, U_1, \dots, U_{t-1}]$ as follows

- a) $Q_1 = Q_1(U_0) := U_0 \in \mathbb{Z}[U_0]$;
 b) $Q_t = Q_t(U_0, U_1, \dots, U_{t-1}) := Q_{t-1} + \sum_{0 \leq i \leq t-1} (-1)^i U_i \binom{Q_{t-1} + t - 2 - i}{t-1-i}, \forall t > 1$.

Check that

- c) $\deg(Q_1) = 1$;
 d) $\deg(Q_t) = 1 + \deg(Q_{t-1})(t-1), \forall t > 1$.
 e) $(t-1)! \leq \deg(Q_t) < t!, \forall t > 1$.

These polynomials will be used in the main result we are heading for.

B) (*Shifted Numerical Polynomials*) Now, let $P \in \mathbb{Q}[X]$ be a numerical polynomial of degree ≥ 0 . Let Y be a second indeterminate and consider the polynomial $P(X+Y) \in \mathbb{Q}[X, Y]$. Observe that $\deg(P(X+Y)) = s$ and that the family of polynomials

$$(Y^j \binom{X+s-i}{s-i})_{(i,j) \in \mathbb{N}_0^2: i+j \leq s}$$

forms a \mathbb{Q} -basis of the space

$$\mathbb{Q}[X, Y]_{\leq s} := \{g \in \mathbb{Q}[X, Y] \mid \deg(g) \leq s\}.$$

From this we may conclude:

- a) There is a uniquely determined family $(E_i^P)_{0 \leq i \leq s}$ of polynomials $E_i^P \in \mathbb{Q}[Y]$ such

$$\begin{aligned} \deg(E_i^P) &= s - i, \forall i \in \{0, 1, \dots, s\}; \\ P(X+Y) &= \sum_{0 \leq i \leq s} (-1)^{-i} E_i^P(Y) \binom{X+s-i}{s-i}. \end{aligned}$$

Observe in particular that

- b) $E_i^P(0) = e_i^P, \forall i \in \{0, 1, \dots, s\}$

More generally, let $c \in \mathbb{Z}$. Then $P(X+c) \in \mathbb{Q}[X]$ is again a numerical polynomial of degree s and we get:

- c) $E_i^P(c) = e_i^{P(X+c)}, \forall i \in \{0, 1, \dots, s\}$.

C) (*Shifted Binomial Coefficients*) Now, let $k \in \mathbb{N}_0$. We consider the numerical polynomial of degree k given by $\binom{X+k}{k}$ and set:

- a) $E_{k,j}(Y) := E_j^{\binom{X+k}{k}}(Y), \forall j \in \{0, 1, \dots, k\}$.

So, we have $\deg(E_{k,j}(Y)) = k - j$ for all $j \in \{0, 1, \dots, k\}$ and may write

- b) $\binom{X+Y+k}{k} = \sum_{0 \leq j \leq k} (-1)^j E_{k,j}(Y) \binom{X+k-j}{k-j}$.

Now, it follows by (4.2)A) that

$$c) P(X + Y) = \sum_{0 \leq i \leq s} (-1)^i (\sum_{0 \leq j \leq i} e_j^P E_{s-j, i-j}(Y)) \binom{X+s-i}{s-i}.$$

In particular by statement B)a) we get:

$$d) E_i^P(Y) = \sum_{0 \leq j \leq i} e_j^P E_{s-j, i-j}(Y), \forall i \in \{0, 1, \dots, s\}.$$

As a consequence of statement B)c) we finally obtain:

$$e) e_i^{P(X+c)} = \sum_{0 \leq j \leq i} e_j^P E_{s-j, i-j}(c), \forall c \in \mathbb{Z}, \forall i \in \{0, 1, \dots, s\}.$$

D) (*Shifted Hilbert Coefficients*) We use the notation of part A) and define for each $t \in \mathbb{N}$ the following polynomial in $\mathbb{Q}[U_0, U_1, \dots, U_{t-1}, Y]$:

$$a) F_t = F_t(U_0, U_1, \dots, U_{t-1}, Y) := Q_t(V_0, V_1, \dots, V_{t-1}) + Y, \text{ where}$$

$$V_i := \sum_{0 \leq j \leq i} U_j E_{t-1-j, i-j}(Y), \forall i \in \{0, 1, \dots, t-1\}.$$

Observe that by A)e) we have

$$b) (t-1)! \leq \deg(Q_t) \leq \deg(F_t) \leq t \deg(Q_t) < t!t.$$

Now, let M be a finitely generated graded R -module and let $c \in \mathbb{Z}$. Assume that $\dim_R(M) = t \in \mathbb{N}$. Then on use of [Br-Fu-Ro](9.14)D) we have $P_{M(c)}(X) = P_M(X+c)$. So, as $\deg(P_M) = t-1$ we get

$$c) e_i(M(c)) = \sum_{0 \leq j \leq i} e_j(M) E_{t-1-j, i-j}(c), \forall i \in \{0, 1, \dots, t-1\}.$$

In view of our definition of the polynomials F_t in statement a) we thus end up with:

$$d) F_t(e_0(M), \dots, e_{t-1}(M), c) = Q_t(e_0(M(c)), \dots, e_{t-1}(M(c))) + c.$$

We now treat a few further simple preliminaries, which shall be useful in the proof of our announced main result.

4.6. Exercise. A) (*Finite Direct Sums*) Let $(M^{(i)})_{1 \leq i \leq r}$ be a finite family of graded R -modules and keep in mind that the R -module $\bigoplus_{0 \leq i \leq r} M^{(i)}$ carries a natural grading, given by $(\bigoplus_{1 \leq i \leq r} M^{(i)})_n = \bigoplus_{1 \leq i \leq r} (M^{(i)})_n$, for all $n \in \mathbb{Z}$. Assume now, that the graded R -modules $M^{(i)}$ are all finitely generated. Show that

$$a) \text{gendeg}(\bigoplus_{1 \leq i \leq r} M^{(i)}) = \max\{\text{gendeg}(M^{(i)}) \mid 1 \leq i \leq r\}.$$

$$b) P_{\bigoplus_{1 \leq i \leq r} M^{(i)}} = \sum_{1 \leq i \leq r} P_{M^{(i)}}.$$

$$c) \dim_R(\bigoplus_{1 \leq i \leq r} M^{(i)}) = \max\{\dim_R(M^{(i)}) \mid 1 \leq i \leq r\}.$$

$$d) \text{reg}^l(\bigoplus_{1 \leq i \leq r} M^{(i)}) = \max\{\text{reg}^l(M^{(i)}) \mid 1 \leq i \leq r\}, \text{ for all } l \in \mathbb{N}_0.$$

e) If all the R -modules $M^{(i)}$ have the same dimension, then

$$e_j\left(\bigoplus_{1 \leq i \leq r} M^{(i)}\right) = \sum_{1 \leq i \leq r} e_j(M^{(i)}), \quad \forall j \in \mathbb{Z}.$$

f) If all the R -modules $M^{(i)}$ have the same dimension, then

$$\text{mult}\left(\bigoplus_{1 \leq i \leq r} M^{(i)}\right) = \sum_{1 \leq i \leq r} \text{mult}(M^{(i)}).$$

B) (*Regularity of Kernels*) Let M be a finitely generated graded R -module and let $m_1, m_2, \dots, m_s \in M$ be a homogeneous system of generators of M such that $m_i \in M_{a_i}$ for an appropriate integer a_i for all $i \in \{1, 2, \dots, s\}$. Observe that there is an epimorphism of graded R -modules

$$\pi : \bigoplus_{1 \leq i \leq s} R(-a_i) \rightarrow M, (v_1, \dots, v_s) \mapsto \sum_{1 \leq i \leq s} v_i m_i.$$

Show that

a) $\text{gendeg}(\text{Ker}(\pi)) \leq \max(\{\text{reg}(R) + a_i \mid 1 \leq i \leq s\} \cup \{\text{reg}(M) + 1\})$.

b) If $R = R_0[X_0, \dots, X_r]$ is a polynomial ring, then

$$\text{gendeg}(\text{Ker}(\pi)) \leq \max(\{a_i \mid 1 \leq i \leq s\} \cup \{\text{reg}(M) + 1\}).$$

Now we are ready to formulate and to prove our main result. The occurring bounding polynomials are as defined in (4.5)D).

4.7. Theorem. *Let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous algebra over a field K and let M be a finitely generated graded R -module of dimension $t > 0$ and let $g \in \mathbb{Z}$ such that $\text{gendeg}(M) \leq g$. Then*

$$\text{reg}^1(M) \leq F_t(e_0(M), e_1(M), \dots, e_{t-1}(M), g).$$

Proof. Let K' be an algebraically closed extension fields of K and consider the Noetherian homogeneous K' -algebra $R' = K' \otimes_K R$ and the finitely generated graded R' -module $M' = R' \otimes_K M$. According to (2.4)C) we have $\dim_{R'}(M') = t$ and $P'_M = P_M$, so that $e_j(M') = e_j(M)$ for all $j \in \mathbb{N}_0$. Moreover by (3.3)A)d) we have $\text{gendeg}(M') = \text{gendeg}(M)$, whereas by (3.3)B)h) we have $\text{reg}^1(M') = \text{reg}^1(M)$. So, we may replace R and M by R' and M' respectively and hence assume that the base field K is algebraically closed.

Clearly, we find a polynomial ring $S := K[X_0, \dots, X_r]$ and a graded ideal $\mathfrak{a} \subseteq S$ such that we can write $R = S/\mathfrak{a}$. So, we may consider M as an S -module such that $\mathfrak{a}M = 0$. Clearly we have by the Base Ring Independence of Dimension that $\dim_S(M) = t$. Moreover, a family of homogeneous elements generates M over R if and only if it does over S . So the generating degrees of M as an S -module and as an R -module are the same. Clearly the Hilbert function $n \mapsto \dim_K(M_n)$ is independent on whether we consider M as a module over S or over R . Finally, by (2.5)A)b) the cohomological Hilbert functions of M and hence also $\text{reg}^1(M)$ are not affected if we consider M as a module

over S instead over R . So, we may replace R by S and hence assume that $R = K[X_0, X_1, \dots, X_r]$ is a polynomial ring over the algebraically closed field K .

As $\text{reg}^1(M(g)) = \text{reg}^1(M) - g$ (see (3.3)B)b)) and in view of the equality (4.5)D)d) it suffices to show that

$$\text{reg}^1(M(g)) \leq Q_t(e_0(M(g)), \dots, e_{t-1}(M(g))).$$

As $\text{gendeg}(M(g)) = \text{gendeg}(M) - g \leq 0$ (see (3.3)A)a)) and $\dim(M(g)) = t$ we thus may assume that $\text{gendeg}(M) \leq 0$ and content ourselves to prove

$$\text{a) } \text{reg}^1(M) \leq Q_t(e_0(M), \dots, e_{t-1}(M)).$$

Observe that we have an exact sequence of finitely generated graded R -modules

$$0 \rightarrow M_{\geq 0} \rightarrow M \rightarrow V \rightarrow 0$$

in which V is R_+ -torsion and hence satisfies $\text{reg}^1(V) = -\infty$. So, by (3.3)C)c) we have $\text{reg}^1(M) \leq \text{reg}^1(M_{\geq 0})$. Therefore, it suffices to show that

$$\text{reg}^1(M_{\geq 0}) \leq Q_t(e_0(M), \dots, e_{t-1}(M)).$$

As $V_n = 0$ for all $n \gg 0$ the above sequence yields that $P_{M_{\geq 0}} = P_M$ so that $\dim_R(M_{\geq 0}) = \dim_R(M) = t$ and $e_i(M_{\geq 0}) = e_i(M)$ for all $i \in \mathbb{N}_0$. Clearly, we also have $\text{gendeg}(M_{\geq 0}) = 0$. So, we may replace M by $M_{\geq 0}$ and hence assume that

$$\text{b) } \text{beg}(M) = \text{gendeg}(M) = 0.$$

We now prove statement a) under the additional assumption b) by induction on t . First, let $t = 1$. Then (4.4) implies that

$$\text{reg}^1(M) \leq e_0(M) = Q_1(e_0(M))$$

and we are done. So, let $t > 1$. We set

$$\mathcal{P} := \text{Ass}_R(M) \cap (\text{mProj}(R) \cup \{R_+\}), \quad \mathfrak{a} := \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}.$$

By [Br-Fu-Ro](10.3)C) we know that $\text{Ass}_R(M)$ consists of graded primes, and hence $\mathfrak{a} \subseteq R$ is a graded ideal. Therefore the module $\bar{M} := M/\Gamma_{\mathfrak{a}}(M)$ is graded. By [Br-Fu-Ro](1.9)b) and a) we respectively have $\text{Ass}_R(\bar{M}) = \text{Ass}_R(M) \setminus \mathcal{P}$ and $\text{Ass}_R(\Gamma_{\mathfrak{a}}(M)) = \mathcal{P}$. As $\dim(R/\mathfrak{p}) \leq 1$ for all $\mathfrak{p} \in \mathcal{P}$ and as $t = \dim_R(M) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \text{Ass}_R(M)\}$ it follows

$$\text{c) } \dim_R(\bar{M}) = t \text{ and } \dim_R(\Gamma_{\mathfrak{a}}(M)) \leq 1.$$

As there is an epimorphism of graded R -modules $M \rightarrow \bar{M} \neq 0$ we get

$$\text{d) } \text{gendeg}(\bar{M}) = \text{beg}(\bar{M}) = 0.$$

Moreover, by our choice of \mathcal{P} we have

$$\text{e) } \text{Ass}_R(\bar{M}) \cap \text{mProj}(R) = \emptyset \text{ and } \Gamma_{R_+}(\bar{M}) = 0.$$

From this, it follows by [Br-Fu-Ro](10.5) that there are two elements $x, y \in R_1$ such that

$$f) \quad \alpha x + \beta y \in \text{NZD}_R(\bar{M}) \text{ for all } (\alpha, \beta) \in K^2 \setminus \{(0, 0)\}.$$

In view of the exact sequence of graded R -modules

$$0 \rightarrow \Gamma_{\mathfrak{a}}(M) \rightarrow M \rightarrow \bar{M} \rightarrow 0$$

we get by the additivity of characteristic functions (see [Br-Fu-Ro](9.15)) that $\chi_{\bar{M}} = \chi_M - \chi_{\Gamma_{\mathfrak{a}}(M)}$ and hence (see also c))

$$g) \quad P_{\bar{M}} = P_M - P_{\Gamma_{\mathfrak{a}}(M)} \text{ and } \deg(P_{\Gamma_{\mathfrak{a}}(M)}) \leq \dim_R(\Gamma_{\mathfrak{a}}(M)) - 1 \leq 0.$$

So, it follows by the definition (4.2)A) that

$$h) \quad e_i(\bar{M}) = e_i(M) \text{ for all } i \in \{0, \dots, t-2\}.$$

Now, we choose a pair $(\alpha, \beta) \in K^2 \setminus \{(0, 0)\}$ and set $z := \alpha x + \beta y$. As $z \in \text{NZD}_R(\bar{M})$ we have $\dim_R(\bar{M}/z\bar{M}) = \dim_R(\bar{M}) - 1 = t - 1$ (see c)). By (4.3)B)d) and by the previous statement h) we also get $e_i(\bar{M}/z\bar{M}) = e_i(M)$ for all $i \in \{0, \dots, t-2\}$. Moreover $\text{gendeg}(\bar{M}/z\bar{M}) \leq \text{gendeg}(\bar{M}) \leq 0$ (see d) and (3.3)A)b)) and $\text{beg}(\bar{M}/z\bar{M}) \geq \text{beg}(\bar{M}) \geq 0$ (see d)), so that $\bar{M}/z\bar{M}$ satisfies our hypothesis b). Therefore, by induction we obtain

$$i) \quad \text{reg}^1(\bar{M}/z\bar{M}) \leq Q_{t-1}(e_0(M), e_1(M), \dots, e_{t-2}(M)) =: p.$$

Now, the short exact sequence of graded R -modules

$$0 \rightarrow \bar{M}(-1) \xrightarrow{z} \bar{M} \rightarrow \bar{M}/z\bar{M} \rightarrow 0$$

shows that $\text{reg}^2(\bar{M}) + 1 = \text{reg}^2(\bar{M}(-1)) \leq \max\{\text{reg}^2(\bar{M}), \text{reg}^1(\bar{M}/z\bar{M}) + 1\}$ (see (3.3)B)b), C)b)). Therefore $\text{reg}^2(\bar{M}) \leq \text{reg}^1(\bar{M}/z\bar{M})$. Hence by statement g) we get

$$j) \quad \text{reg}^2(\bar{M}) \leq p.$$

By i) we also have $H_{R_+}^1(\bar{M}/z\bar{M})_n = 0$ for all $n \geq p$. So, if we apply cohomology to the above exact sequence, we get an epimorphism of K -vector spaces

$$H_{R_+}^1(\bar{M})_{n-1} \xrightarrow{z=\alpha x+\beta y} H_{R_+}^1(\bar{M})_n, \quad \forall n \geq p, \forall (\alpha, \beta) \in K^2 \setminus \{(0, 0)\}.$$

By [Br-Fu-Ro](10.7) we conclude from this, that

$$h_{\bar{M}}^1(n) \leq \max\{0, h_{\bar{M}}^1(n-1) - 1\}, \forall n \geq p.$$

Therefore we finally obtain

$$\text{end}(H_{R_+}^1(\bar{M})) \leq p + h_{\bar{M}}^1(p-1) - 1.$$

By statement j) we have $h_{\bar{M}}^i(p-1) = 0$ for all $i \geq 2$. So, as $H_{R_+}^0(\bar{M}) = 0$ (see statement e)) and as $P_{\Gamma_{\mathfrak{a}}(M)} = \chi_{\Gamma_{\mathfrak{a}}(M)}$ is constant (see statement g)) we obtain

$$\dim_K(\bar{M}_{p-1}) + h_{\bar{M}}^1(p-1) = \chi_{\bar{M}}(p-1) = \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)}$$

and hence

$$0 \leq h_M^1(p-1) \leq \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)}.$$

Therefore

$$\text{end}(H_{R_+}^1(\bar{M})) \leq p + \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)} - 1$$

and

$$\chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)} \geq 0.$$

So, as $H_{R_+}^0(\bar{M}) = 0$ and in view statement i) we get

$$\text{k) } \text{reg}(\bar{M}) = \text{reg}^1(\bar{M}) \leq p + \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)}.$$

Our next aim is to show the inequality

$$\text{l) } \text{reg}(M) \leq p + \chi_M(p-1) + 1.$$

We first show that

$$\text{m) } p + \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)} + 1 > 0.$$

To do so, observe that $U := (\bar{M}/z\bar{M})/\Gamma_{R_+}(\bar{M}/z\bar{M})$ is an R_+ -torsion-free finitely generated graded R -module $\neq 0$, so that by statement i) we have $p \geq \text{reg}^1(\bar{M}/z\bar{M}) = \text{reg}(U) \geq \text{gendeg}(U) \geq \text{beg}(U) \geq \text{beg}(\bar{M}) \geq \text{beg}(M) = 0$ (see also (3.3)B)b) and (3.4)). As we already know that $\chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)} \geq 0$ this proves statement m).

Now, let us prove statement l). In view of our assumption b) we find an integer $s \in \mathbb{N}$ and an epimorphism of graded R -modules $R^{\oplus s} \xrightarrow{\pi} M$ which is incorporated in the following commutative diagram of graded R -modules, in which both rows are exact, u is the inclusion map and w is the canonical epimorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker}(\pi) & \longrightarrow & R^{\oplus s} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow \text{Id} & & \downarrow w & & \\ 0 & \longrightarrow & \text{Ker}(\bar{\pi}) & \longrightarrow & R^{\oplus s} & \xrightarrow{\bar{\pi}} & \bar{M} & \longrightarrow & 0 \end{array}$$

If we apply (4.6)B)b) to the second row of this diagram and keep in mind statements k) and m) we get

$$\text{gendeg}(\text{Ker}(\bar{\pi})) \leq \max\{0, p + \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)} + 1\} = p + \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)} + 1.$$

By the Snake Lemma we have $\text{Coker}(u) = \text{Ker}(w) = \Gamma_{\mathfrak{a}}(M)$ and therefore we obtain

$$\text{gendeg}(\Gamma_{\mathfrak{a}}(M)) \leq p + \chi_M(p-1) - \chi_{\Gamma_{\mathfrak{a}}(M)}.$$

As $\Gamma_{\mathfrak{a}}(M) \subseteq M$ is a submodule of dimension ≤ 1 we have $\chi_{\Gamma_{\mathfrak{a}}(M)} = e_0(\Gamma_{\mathfrak{a}}(M))$. Consequently we obtain

$$\text{gendeg}(\Gamma_{\mathfrak{a}}(M)) \leq p + \chi_M(p-1) - e_0(\Gamma_{\mathfrak{a}}(M)) + 1.$$

But now, by (4.4) we get

$$\operatorname{reg}^1(\Gamma_{\mathfrak{a}}(M)) \leq p + \chi_M(p-1) + 1.$$

If apply the sequence mentioned just before statement g) and bear in mind that $\chi_{\Gamma_{\mathfrak{a}}(M)} = e_0(\Gamma_{\mathfrak{a}}(M)) > 0$, we get by statement k) that $\operatorname{reg}^1(M) \leq p + \chi_M(p-1) + 1$, that is statement l).

Finally, by our previous definition of p (see statement i)) and by the definition (4.5)A)b) of the bounding polynomial Q_t it follows at once that

$$p + \chi_M(p-1) + 1 = Q_t(e_0(M), e_1(M), \dots, e_{t-1}(M)).$$

So, by statement l) we get the requested inequality a). \square

4.8. Corollary. *Let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous algebra over a field K , let $t \in \mathbb{N}$, $g \in \mathbb{Z}$ and let*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be an exact sequence of finitely generated graded R -modules such that $\dim_R(N) = t$ and $\operatorname{gendeg}(N) \leq g$. Then

$$\operatorname{reg}^2(L) \leq \max\{\operatorname{reg}^2(M), F_t(e_0(N), \dots, e_{t-1}(N), g) + 1\}.$$

Proof. This follows immediately by (4.7) and (3.3)C)b). \square

We now aim to apply the previous result to ideals in polynomial rings. We start with a few preparations.

4.9. Exercise and Definition. (*Saturation of Graded Submodules*) A) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian positively graded ring, let M be a graded R -module and let $N \subseteq M$ be a graded submodule. Make clear that

$$N^{\operatorname{sat}_M} = N^{\operatorname{sat}} := \bigcup_{n \in \mathbb{N}} (N :_M (R_+)^n) \subseteq M$$

is a graded submodule of M , the so called *saturation* of N in M . Make clear that the following hold

- a) For each graded submodule $N \subseteq M$ we have $N \subseteq N^{\operatorname{sat}}$.
- b) $M^{\operatorname{sat}} = M$ and $0^{\operatorname{sat}} = \Gamma_{R_+}(M)$.
- c) If $L \subseteq N \subseteq M$ are graded submodules, then $L^{\operatorname{sat}} \subseteq N^{\operatorname{sat}}$.
- d) For each graded submodule $N \subseteq M$ we have $(N^{\operatorname{sat}})^{\operatorname{sat}} = N^{\operatorname{sat}}$.

We say that the graded R -submodule $N \subseteq M$ is *saturated*, if $N^{\operatorname{sat}} = N$. According to statement d), the graded saturated submodules of M are precisely those, which are the saturation of some graded submodule of M .

B) Let R , M and $N \subseteq M$ be as in part A) and show:

- a) $N^{\operatorname{sat}}/N = \Gamma_{R_+}(M/N)$.

b) If $H_{R_+}^0(M) = H_{R_+}^1(M) = 0$, we have an exact sequence of graded R -modules

$$0 \rightarrow N \xrightarrow{\subseteq} N^{\text{sat}} \rightarrow H_{R_+}^1(N) \rightarrow 0.$$

c) If M is finitely generated, then $P_N = P_{N^{\text{sat}}}$, $P_{M/N} = P_{M/N^{\text{sat}}}$ and

$$e_i(N) = e_i(N^{\text{sat}}), \quad e_i(M/N) = e_i(M/N^{\text{sat}}), \quad \forall i \in \mathbb{N}_0.$$

d) If $N \neq 0$, then $\dim_R(N) = \dim_R(N^{\text{sat}})$.

e) If M/N is not R_+ -torsion, then $\dim_R(M/N^{\text{sat}}) = \dim_R(M/N)$.

Now, we are ready to formulate and to prove the announced application of (4.8) to polynomial ideals.

4.10. Corollary. *Let $r \in \mathbb{N}$, let $R = K[X_0, X_1, \dots, X_r]$ be a polynomial ring over the field K and let $\mathfrak{a} \subseteq R$ be a graded saturated ideal such that $\dim(R/\mathfrak{a}) =: t > 0$. Then*

a) $\text{reg}(R/\mathfrak{a}) \leq Q_t(e_0(R/\mathfrak{a}), e_1(R/\mathfrak{a}), \dots, e_{t-1}(R/\mathfrak{a}))$.

b) $\text{reg}(\mathfrak{a}) \leq Q_t(e_0(R/\mathfrak{a}), e_1(R/\mathfrak{a}), \dots, e_{t-1}(R/\mathfrak{a})) + 1$.

Proof. According to (4.9)B)a) we have $H_{R_+}^0(R/\mathfrak{a}) = 0$, so that $\text{reg}(R/\mathfrak{a}) = \text{reg}^1(R/\mathfrak{a})$. Observe that $\text{gendeg}(R/\mathfrak{a}) = 0$ and that by (4.5)D)d)

$$F_t(e_0(R/\mathfrak{a}), e_1(R/\mathfrak{a}), \dots, e_{t-1}(R/\mathfrak{a}), 0) = Q_t(e_0(R/\mathfrak{a}), e_1(R/\mathfrak{a}), \dots, e_{t-1}(R/\mathfrak{a})).$$

Now statement a) follows from (4.7), applied with $g = 0$. Clearly we have $H_{R_+}^0(\mathfrak{a}) = 0$. By (4.9)B)b) we also have $H_{R_+}^1(\mathfrak{a}) = 0$. Therefore $\text{reg}(\mathfrak{a}) = \text{reg}^2(\mathfrak{a})$ and so statement b) follows by (4.8), applied with $g = 0$ and bearing in mind the fact that $\text{reg}(R) = 0$ (see (3.5)). \square

Now, we aim to translate our previous results to the language of sheaves. We begin with a some preparations.

4.11. Exercise and Definition. A) (*Cohomological Hilbert Functions of Coherent Sheaves*) A) Let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous algebra over the field K , let $X = \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules, so that $\mathcal{F} = \widetilde{M}$ for some finitely generated graded R -module M . Now, we may extend what was done in [Br-Fu-Ro](12.12) only in the case of an infinite base field K to arbitrary base fields. Namely, the fact that M_n and $H_{R_+}^i(M)_n$ are K -vector spaces of finite dimension for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (see (2.4) and [Br-Fu-Ro](9.6)C)) and the Serre-Grothendieck Correspondence (2.14)A) tell us that

$$h^i(X, \mathcal{F}(n)) := \dim_K(H^i(X, \mathcal{F}(n))) \in \mathbb{N}_0, \quad \forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z}.$$

So, for each $i \in \mathbb{N}_0$ we may again define the i -th cohomological Hilbert function

$$h_{\mathcal{F}}^i : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto h^i(X, \mathcal{F}(n))$$

of $(X$ with respect to the coherent sheaf of coefficients) \mathcal{F} . Now, by (2.16)b) we can say, that these functions $h_{\mathcal{F}}^i : \mathbb{Z} \rightarrow \mathbb{N}_0$ are again right-vanishing.

B) (*Characteristic Functions, Serre Polynomials and Serre Coefficients*). Keep all the notations and hypotheses of part A). As $\text{cd}_X(\mathcal{F}) = \text{cd}_{R_+}(M) < \infty$ (see (2.15)) we may again define the *characteristic function* of \mathcal{F} :

$$\chi_{\mathcal{F}} : \mathbb{Z} \rightarrow \mathbb{Z}, \quad n \mapsto \sum_{i \in \mathbb{N}_0} (-1)^i h_{\mathcal{F}}^i(n) = \sum_{i \in \mathbb{N}_0} h^i(X, \mathcal{F}(n)).$$

Check on use of the Serre-Grothendieck Correspondence that in the notation used in part A) we have

$$\chi_{\mathcal{F}} = \chi_M$$

so that there is a numerical polynomial

a) $P_{\mathcal{F}} = P_M \in \mathbb{Q}[X]$ such that $P_{\mathcal{F}}(n) = \chi_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}$.

which also is characterized by the property

b) $P_{\mathcal{F}}(n) = h_{\mathcal{F}}^0(n) = h^0(X, \mathcal{F}(n))$ for all $n \gg 0$.

This numerical polynomial $P_{\mathcal{F}}$ is called the *Serre polynomial* of \mathcal{F} . Now, using the notation introduced in (4.2)A), for each $i \in \mathbb{N}_0$ we may define the *i-th Serre coefficient* of \mathcal{F} by

$$e_i(\mathcal{F}) := \begin{cases} e_i^{P_{\mathcal{F}}}, & \forall i \in \{0, 1, \dots, \deg(P_{\mathcal{F}})\} \\ 0, & \forall i \in \mathbb{Z}_{>\deg(P_{\mathcal{F}})} \end{cases}$$

Verify that in the notation of part A) we have

c) $e_i(\mathcal{F}) = e_i(M)$ for all $i \in \mathbb{N}_0$.

C) (*Support and Dimension of Sheaves*) For a moment let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be an arbitrary Noetherian positively graded ring, let $X = \text{Proj}(R)$ and let $\mathcal{F} = \widetilde{M}$ be a coherent sheaf of \mathcal{O}_X -modules, induced by the finitely generated graded R -module M . Then, we define the *support* of \mathcal{F} by:

$$\text{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

As $\text{Supp}(\mathcal{F}) \subseteq \text{Spec}(R)$, it has a *Krull dimension* and so we may define the *dimension* of \mathcal{F} as follows:

$$\dim(\mathcal{F}) := \dim(\text{Supp}(\mathcal{F})).$$

Now, prove the following statements:

a) $\text{Supp}(\mathcal{F}) = \text{Proj}(R) \cap \text{Var}(0 :_R M)$.

b) If $\mathfrak{p}, \mathfrak{q} \in {}^*\text{Spec}(R)$ with $\mathfrak{p} \subseteq \mathfrak{q}$ and $\text{ht}(\mathfrak{q}/\mathfrak{p}) > 1$, then, there is some $\mathfrak{s} \in {}^*\text{Spec}(R)$ with $\mathfrak{p} \subsetneq \mathfrak{s} \subsetneq \mathfrak{q}$.

Finally, let the notations and hypotheses again be as in part A) and show that:

- c) If $\mathcal{F} \neq 0$, then $\dim_R(M) > 0$ and $\dim(\mathcal{F}) = \dim_R(M) - 1$.
d) If $\mathcal{F} \neq 0$, then $\dim_R(\mathcal{F}) = \deg(P_{\mathcal{F}})$.

Now, we prove the announced sheaf-theoretic version of (4.7). In our statement the total module of sections (see (3.8)) occurs and we use the fact that this module has generating degree $< \infty$ (see (3.11)e)).

4.12. Corollary. *Let $X = \text{Proj}(R)$, where $R = K \oplus R_1 \oplus R_2 \cdots$ is a Noetherian homogeneous algebra over the field K , let $d \in \mathbb{N}_0$, $\gamma \in \mathbb{Z}$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules such that $\dim(\mathcal{F}) = d$ and $\text{gendeg}(\Gamma_*(X, \mathcal{F})) \leq \gamma$. Then*

$$\text{reg}(\mathcal{F}) \leq F_{d+1}(e_0(\mathcal{F}), e_1(\mathcal{F}), \dots, e_d(\mathcal{F}), \gamma).$$

Proof. Let $\Gamma = \Gamma_*(X, \mathcal{F})$.

Clearly, as R is homogeneous, we have $\text{gendeg}(\Gamma_{\geq \gamma}) = \gamma$. Moreover $\mathcal{F} = \widetilde{\Gamma_{\geq \gamma}}$ (see (3.11)c)) and $\text{reg}(\mathcal{F}) = \text{reg}^2(\Gamma_{\geq \gamma}) \leq \text{reg}^1(\Gamma_{\geq \gamma})$ (see (3.7) and (3.3)B)a)).

As $\dim(\Gamma_{\geq \gamma}) = d + 1$ (see (4.11)C)c)) and in view of the coincidence of Hilbert-Serre coefficients (see (4.11)B)c)) our claim follows if we apply (4.7) to the finitely generated graded R -module $\Gamma_{\geq \gamma}$. \square

Another application to sheaves is given by the following result.

4.13. Corollary. *Let $X = \text{Proj}(R)$, where $R = K \oplus R_1 \oplus R_2 \dots$ is a Noetherian homogeneous algebra over the field K , let $d \in \mathbb{N}_0$, $\rho \in \mathbb{Z}$ and let*

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{H} \rightarrow 0$$

be an exact sequence of sheaves of coherent \mathcal{O}_X -modules such that $\dim(\mathcal{H}) = d$ and $\text{reg}(\mathcal{F}) \leq \rho$. Then

- a) $\text{reg}(\mathcal{H}) \leq F_{d+1}(e_0(\mathcal{H}), e_1(\mathcal{H}), \dots, e_d(\mathcal{H}), \rho)$.
b) $\text{reg}(\mathcal{G}) \leq \max\{\rho, F_{d+1}(e_0(\mathcal{H}), e_1(\mathcal{H}), \dots, e_d(\mathcal{H}), \rho) + 1\}$.

Proof. For each $n \in \mathbb{Z}$ we get an induced exact sequence of K -vector spaces

$$0 \rightarrow H^0(X, \mathcal{G}(n)) \rightarrow H^0(X, \mathcal{F}(n)) \rightarrow H^0(X, \mathcal{H}(n)) \rightarrow H^1(X, \mathcal{G}(n))$$

in which the last term vanishes for all $n \gg 0$. Passing over to total modules of sections we thus get an exact sequence

$$0 \rightarrow \Gamma_*(X, \mathcal{G}) \rightarrow \Gamma_*(X, \mathcal{F}) \rightarrow N \rightarrow 0,$$

where $N \subseteq \Gamma_*(X, \mathcal{H})$ is a graded submodule such that $N_n = \Gamma_*(X, \mathcal{H})_n$ for all $n \gg 0$. In particular we have $\mathcal{H} \cong \widetilde{N}$ (see (2.14)D)a)) and $\text{gendeg}(N) \leq \text{gendeg}(\Gamma_*(X, \mathcal{F})) \leq \rho$ (see (3.11)e)), so that (4.7) and (3.7)A) (and also (3.3)B)a)) yield statement a). Statement b) follows likewise from (4.8). \square

4.14. Remark and Exercise. A) (*Regularity of Quotient Modules*) The principal significance of the results of these section is the fact that they provide *uniform bounds on the regularity of quotient modules*. To make this explicit, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous algebra over the field K and let M be a finitely generated graded R -module. A graded R -module N is called a *quotient of M* if there is an epimorphism of graded R -modules $M \rightarrow N \rightarrow 0$. Clearly, in this situation the module N is also finitely generated and we have $\text{gendeg}(N) \leq \text{gendeg}(M)$. Now, as an immediate consequence of (4.7) we can say

- a) Let $P \in \mathbb{Q}[X]$ be a numerical polynomial. Then, there exists an integer β such that $\text{reg}^1(N) \leq \beta$ for all quotients N of M with $P_N = P$.

In particular we obtain the following application

- b) Let $P \in \mathbb{Q}[X]$ be a numerical polynomial. Then, there is an integer β such that $\text{reg}^1(R/\mathfrak{a}) \leq \beta$ for all graded ideals $\mathfrak{a} \subseteq R$ with $P_{R/\mathfrak{a}} = P$.

This is a quantitative and algebraic extension of Mumford's original bounding result for the regularity of coherent sheaves of ideals over a projective space [Mu1]. We shall turn back to this later.

We suggest the following example to make clear, that in statement b) one cannot replace $\text{reg}^1(R/\mathfrak{a})$ by $\text{reg}(R/\mathfrak{a})$. Namely, let $R = K[X, Y]$ be a polynomial ring. For each $r \in \mathbb{N}$ consider the graded ideal

$$\mathfrak{a}^{(r)} := \langle X \rangle \cap \langle X, Y \rangle^r \subseteq R.$$

Calculate $P_{R/\mathfrak{a}^{(r)}}$ and $\text{end}(R/\mathfrak{a}^{(r)})$ and conclude, that statement b) fails if reg^1 is replaced by reg .

B) (*Regularity of Saturated Submodules*) Keep the notations and hypotheses of part A). Prove the following statements:

- a) Let $L \subseteq M$ be a graded saturated submodule with $\dim(M/L) = t > 0$ and assume that $\text{gendeg}(M) \leq g \in \mathbb{Z}$. Then

$$\text{reg}(M/L) \leq F_t(e_0(M/L), e_1(M/L), \dots, e_{t-1}(M/L), g).$$

- b) If $L \subsetneq M$ is a graded saturated submodule then $P_L \neq P_M$ and if $\text{gendeg}(M) \leq g \in \mathbb{Z}$ and setting $s = \deg(P_M - P_L)$ we have

$$\text{reg}(M/L) \leq F_{s+1}(e_0^{P_M - P_L}, e_1^{P_M - P_L}, \dots, e_s^{P_M - P_L}, g).$$

- c) If M, L, s are as in statement b) and if in addition $H_{R_+}^1(M) = 0$ and $\text{reg}(M) \leq \rho \in \mathbb{Z}$ then

$$\text{reg}(L) \leq \max\{\rho, F_{s+1}(e_0^{P_M - P_L}, e_1^{P_M - P_L}, \dots, e_s^{P_M - P_L}, \rho) + 1\}.$$

- d) Let $P \in \mathbb{Q}[X]$ be a numerical polynomial. Then there is an integer β such that $\text{reg}(M/L) \leq \beta$ for all saturated graded submodules $L \subseteq M$ with $P_{M/L} = P$.

- e) Assume that $H_{R_+}^1(M) = 0$ and let $Q \in \mathbb{Q}[X]$ be a numerical polynomial. Then there is an integer γ such that $\text{reg}(L) \leq \gamma$ for all saturated graded submodules $L \subseteq M$ with $P_L = Q$.

C) (*Sheaves of Submodules*) Let R and X be as in part A) and let \mathcal{F} be a sheaf of \mathcal{O}_X -modules. We say that \mathcal{G} is a *sheaf of submodules of \mathcal{F}* if

- a) $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ is a submodule of the $\mathcal{O}_X(U)$ -module $\mathcal{F}(U)$ for all $U \in \mathbb{U}_X$.
b) For all $U, V \in \mathbb{U}_X$ with $V \subseteq U$ we have the commutative diagram

$$\begin{array}{ccc} \mathcal{G}(U) & \xrightarrow{\subseteq} & \mathcal{F}(U) \\ \downarrow \rho_{U,V}^{\mathcal{G}} & & \downarrow \rho_{U,V}^{\mathcal{F}} \\ \mathcal{G}(V) & \xrightarrow{\subseteq} & \mathcal{F}(V) \end{array}$$

Prove that in this situation, there is an injective homomorphism of sheaves of \mathcal{O}_X -modules

$$\text{incl}_{\mathcal{G},\mathcal{F}} : \mathcal{G} \rightarrow \mathcal{F}, \quad U \mapsto (\mathcal{G}(U) \xrightarrow{\subseteq} \mathcal{F}(U)), \forall U \in \mathbb{U}_X,$$

the so called *inclusion homomorphism*. Show the following statements

- c) If M is a graded R -module and $N \subseteq M$ is a graded submodule, then \widetilde{N} is sheaf of submodules of \widetilde{M} .
d) If M and N are as in statement c), then $\widetilde{N^{\text{sat}}} = \widetilde{N}$.
e) If $L \subseteq N$ are graded submodules of M such that $\widetilde{L} = \widetilde{N}$, then $N \subseteq L^{\text{sat}}$.
f) If $L, N \subseteq M$ are graded submodules with $\widetilde{L} = \widetilde{N}$, then $L^{\text{sat}} = N^{\text{sat}}$.

Now, assume that $\mathcal{F} = \widetilde{M}$ is coherent, with M finitely generated and let $\mathcal{G} = \widetilde{N}$ be a quasi-coherent sheaf of submodules of \mathcal{F} , with N a graded R -module. Use the diagram (see (3.8)D),E))

$$\begin{array}{ccc} N & & M \\ \downarrow \varepsilon_{N,*}^X & & \downarrow \varepsilon_{M,*}^X \\ \Gamma_*(X, \mathcal{G}) & \xrightarrow{\Gamma_*(X, \text{incl}_{\mathcal{G},\mathcal{F}})} & \Gamma_*(X, \mathcal{F}) \end{array}$$

to show that there is a graded submodule $L \subseteq M$ such that $\mathcal{G} = \widetilde{L}$. Use this to prove

- g) The quasicohereant sheaves of submodules of a coherent sheaf \mathcal{F} of \mathcal{O}_X -modules are all coherent.
h) If $\mathcal{F} = \widetilde{M}$ with a finitely generated graded R -module M , the assignment $L \mapsto \widetilde{L}$ gives a bijection between the graded saturated submodules of M and the coherent sheaves of submodules of \mathcal{F} .

- i) If \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules with $\text{reg}(\mathcal{F}) \leq \rho \in \mathbb{Z}$ and $\mathcal{G} \neq \mathcal{F}$ is a coherent sheaf of submodules, then $P_{\mathcal{G}} \neq P_{\mathcal{F}}$ and with $s := \deg(P_{\mathcal{F}} - P_{\mathcal{G}})$ we have

$$\text{reg}(\mathcal{G}) \leq \max\{\rho, F_{s+1}(e_0^{P_{\mathcal{F}}-P_{\mathcal{G}}}, e_1^{P_{\mathcal{F}}-P_{\mathcal{G}}}, \dots, e_s^{P_{\mathcal{F}}-P_{\mathcal{G}}}, \rho) + 1\}.$$

D) (*Sheaves of Ideals*) Let the notations and hypotheses be as above. A *sheaf of ideals* over X is a sheaf \mathcal{I} of submodules of the structure sheaf \mathcal{O}_X of X . If $\mathcal{I} \neq \mathcal{O}_X$, we call \mathcal{I} a *proper sheaf of ideals* over X . Now, let $r \in \mathbb{N}$ let K be a field, consider the polynomial ring $R = K[X_0, X_1, \dots, X_r]$ and the corresponding *projective r -space over K* , hence the scheme

$$\mathbb{P}_K^r := \text{Proj}(K[X_0, X_1, \dots, X_r]).$$

Prove the following results

- a) If \mathcal{I} is a proper coherent sheaf of ideals over \mathbb{P}_K^r then $T_{\mathcal{I}} := P_{\mathcal{I}} - \binom{X+r}{r} \neq 0$ and with $s := \deg(T_{\mathcal{I}})$ we have:

$$\text{reg}(\mathcal{I}) \leq Q_{s+1}(e_0^{T_{\mathcal{I}}}, e_1^{T_{\mathcal{I}}}, \dots, e_s^{T_{\mathcal{I}}}) + 1.$$

- b) For all numerical polynomials $P \in \mathbb{Q}[X]$, there is an integer γ such that $\text{reg}(\mathcal{I}) \leq \gamma$ for all coherent sheaves \mathcal{I} of ideals over \mathbb{P}_K^r .

The first of these statements is a quantitative version of Mumford's bounding result for coherent sheaves of ideals (see [Mu1] and part A)). The second statement is the original form of Mumford's result.

4.15. Remark. (*Regularity of Annihilators*) Let us mention one more result, which is of the same spirit as the results presented in this section, namely (see the Diploma thesis [Sei])

- a) The regularity of the annihilator of a finitely generated graded module M over a polynomial ring $R = K[X_1, X_2, \dots, X_r]$ is bounded in terms of the Hilbert polynomial (and hence of the Hilbert coefficients) of M and the postulation number

$$P(M) := \sup\{n \in \mathbb{Z} \mid \dim_K(M_n) \neq P_M(n)\}$$

of M .

This result was actually motivated by a question from the theory of D -modules and Weyl algebras (brought to us by M. Bächtold, a former a PhD student in our Department's research group in Mathematical Physics, see [Bäc]) : Is there an upper bound on the degree of the equations defining (set theoretically) the characteristic variety of a D -module, only in terms of the Hilbert function associated to the chosen filtration on D ? This nicely fits to a statement made by a leading Mathematical Physicist at a workshop on Commutative Algebra and Algebraic at the Max-Planck-Institute in Leipzig: "What Physicists like very much in Algebraic Geometry is the fact that it produces a huge number of invariants".

5. FILTER-REGULAR SEQUENCES

In this section we aim to introduce a basic tool for the study of Castelnuovo-Mumford regularity: the so called *filter-regular sequences*. In view of our subject, we do not introduce these sequences in the most general setting, that is relative to arbitrary ideals in Noetherian rings. Instead we consider only the case where the filter-ideal is the irrelevant ideal of a homogeneous Noetherian ring. Moreover we consider only filter-regular sequences consisting of homogeneous elements in this ideal. Our main result shall be (a generalized version of) the *Regularity Criterion of Bayer-Stillman* (see [B-St]). In order to avoid too much technicalities, we do not prove the most general form of this criterion which is given in [Br5].

In a shorter second part of this section we also shall prove that filter-regular sequences are *systems of multiplicity parameters* and provide an example which shows that the converse implication is not true. This shows, that filter-regular sequences are not only a powerful tool to study Castelnuovo-Mumford regularity, but also can be applied in Multiplicity Theory.

5.1. Exercise and Definition. A) (*Filter-Regular Elements*) Fix a Noetherian homogeneous ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ and let $R_+^h := \bigcup_{n \in \mathbb{N}} R_n$ denote the set of homogeneous elements of positive degree in R . Moreover let M be a finitely generated graded R -module. Show that for a given element $f \in R_t \subseteq R_+^h$ with $t \in \mathbb{N}$ the following statements are equivalent:

- (i) $f \in \text{NZD}_R(M/\Gamma_{R_+}(M))$.
- (ii) $f \notin \bigcup_{\mathfrak{p} \in \text{Ass}_R(M) \cap \text{Proj}(R)} \mathfrak{p}$.
- (iii) $\frac{f}{1} \in \text{NZD}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \text{Proj}(R)$.
- (iv) $(0 :_M f) \subseteq \Gamma_{R_+}(M)$.
- (v) $\text{end}(0 :_M f) < \infty$.
- (vi) The multiplication map $f : M_n \rightarrow M_{n+t}$ is injective for all $n \gg 0$.

If the homogeneous element $f \in R_t$ satisfies these equivalent conditions, it is called a *filter-regular element* with respect to M .

B) (*Properties of Filter-Regular Elements*) Let the notations and hypotheses be as in part A). Prove the following facts:

- a) If $f \in \text{NZD}_R(M)$, then f is filter-regular with respect to M .
- b) If $N \subseteq \Gamma_{R_+}(M)$ is a graded submodule, then f is filter-regular with respect to M/N if and only if it is with respect to M .
- c) If f is filter-regular with respect to M , then f^n is filter-regular with respect to M for all $n \in \mathbb{N}$.
- d) If $f \in R_t$ is filter regular with respect to M , then $(0 :_M f)_n = \Gamma_{R_+}(M)_n$ for all $n \geq \text{end}(\Gamma_{R_+}(M)) - t + 1$.

- e) (*Flat Base Change Property of Filter-Regular Elements*) If $f \in R_t$ is filter-regular with respect to M and R'_0 is a flat Noetherian R_0 -algebra, then the element $1_{R'_0} \otimes f \in (R'_0 \otimes_{R_0} R)_t = R'_0 \otimes R_t$ is filter-regular with respect to the finitely generated graded module $R'_0 \otimes_{R_0} M$ over the Noetherian homogeneous ring $R'_0 \otimes_{R_0} R$.
- f) (*Base Ring Independence of Filter-Regular Elements*) If $\mathfrak{b} \subseteq R$ is a graded ideal such that $\mathfrak{b}M = 0$, then f is filter-regular with respect to the R -module M if and only if $f + \mathfrak{b} \in R/\mathfrak{b}$ is filter regular with respect to the R/\mathfrak{b} -module M .

5.2. Exercise and Definition. A) (*Filter-Regular Sequences*) Let R and M be as above. Let $f_1, f_2, \dots, f_r \in R_+^h$. Show that the following statements are equivalent:

- (i) f_i is filter-regular with respect to $M/\sum_{j<i} f_j M$ for all $i \in \{1, 2, \dots, r\}$.
(ii) $\frac{f_1}{1}, \frac{f_2}{1}, \dots, \frac{f_r}{1} \in R_{\mathfrak{p}}$ form an $M_{\mathfrak{p}}$ -sequence for all $\mathfrak{p} \in \text{Proj}(R)$.

If these equivalent conditions hold, we say that f_1, f_2, \dots, f_r form a *filter-regular sequence with respect to M* .

B) (*Properties of Filter-Regular Sequences*) Keep the notations and hypotheses of part A) and let $f_1, f_2, \dots, f_r \in R_+^h$. Prove the following statements:

- a) If f_1, f_2, \dots, f_r form an M -sequence, then they form a filter-regular sequence with respect to M .
b) If $N \subseteq \Gamma_{R_+}(M)$ is a graded submodule, then f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M/N if and only if they do with respect to M .
c) If f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M , then $f_1^{n_1}, f_2^{n_2}, \dots, f_r^{n_r}$ form a filter-regular sequence with respect to M for any choice $n_1, n_2, \dots, n_r \in \mathbb{N}$.
d) (*Flat base Change Property of Filter-Regular Sequences*) If f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M and if R'_0 is a flat R_0 -algebra, then $1_{R'_0} \otimes f_1, 1_{R'_0} \otimes f_2, \dots, 1_{R'_0} \otimes f_r \in (R'_0 \otimes_{R_0} R)_+^h$ form a filter-regular sequence with respect to $R'_0 \otimes_{R_0} M$.
e) (*Base Ring Independence of Filter-Regular Sequences*) If $\mathfrak{b} \subseteq R$ is a graded ideal such that $\mathfrak{b}M = 0$, then f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M if and only if $f_1 + \mathfrak{b}, f_2 + \mathfrak{b}, \dots, f_r + \mathfrak{b} \in R/\mathfrak{b}$ do.

5.3. Lemma. Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let M be a finitely generated graded R -module, let $r \in \mathbb{N}$, let $t_1, t_2, \dots, t_r \in \mathbb{N}$ and let $f_i \in R_{t_i}$ for all $i \in \{1, 2, \dots, r\}$ such that f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M . Then, for all $k \in \mathbb{N}_0$ and all $i \in \{0, 1, \dots, r\}$ we have

a) $\text{end}(H_{R_+}^k(M/\sum_{j=1}^i f_j M)) \leq \max_{j=0}^i \{\text{end}(H_{R_+}^{k+j}(M)) + j\} - i + \sum_{j=1}^i t_j$.

- b) $\text{end}(H_{R_+}^{k+i}(M)) + \sum_{j=1}^1 t_j \leq \text{end}(H_{R_+}^k(M/\sum_{j=1}^i f_j M))$.
c) $\text{reg}^k(M/\sum_{j=1}^i f_j M) \leq \text{reg}^k(M) - i + \sum_{j=1}^i t_j$.
d) $\text{reg}^{k+i}(M) \leq \text{reg}^k(M/\sum_{j=1}^i f_j M) + i - \sum_{j=1}^i t_j$.

Proof. It suffices to show statements a) and b). For $i = 0$, both statements are clear. So, let $i > 0$ and let $l \in \mathbb{N}_0$. As f_1 is filter-regular with respect to M we have $(0 :_M f_1) \subseteq \Gamma_{R_+}(M)$ and hence get an epimorphism of graded R -modules

$$H_{R_+}^l(M) \rightarrow H_{R_+}^l(M/(0 :_M f_1)) \rightarrow 0$$

and an isomorphism of graded R -modules

$$H_{R_+}^{l+1}(M) \cong H_{R_+}^{l+1}(M/(0 :_M f_1)).$$

If we apply cohomology to the short exact sequence of graded R -modules

$$0 \rightarrow (M/(0 :_M f_1))(-t_1) \rightarrow M \rightarrow M/f_1 M \rightarrow 0$$

we therefore obtain

$$\text{end}(H_{R_+}^l(M/f_1 M)) \leq \max\{\text{end}(H_{R_+}^l(M)), \text{end}(H_{R_+}^{l+1}(M)) + t_1\}$$

hence

$$(i) \quad \text{end}(H_{R_+}^l(M/f_1 M)) \leq \max\{\text{end}(H_{R_+}^l(M)), \text{end}(H_{R_+}^{l+1}(M)) + 1\} + t_1 - 1,$$

and moreover

$$(ii) \quad \text{end}(H_{R_+}^{l+1}(M)) + t_1 \leq \text{end}(H_{R_+}^l(M/f_1 M)).$$

Applying the estimate (i) with $M/\sum_{j=1}^{i-1} f_j M$ instead of M and with f_i instead of f_1 , we thus get

$$\begin{aligned} & \text{end}(H_{R_+}^l(M/\sum_{j=1}^i f_j M)) \leq \\ & \leq \max\{\text{end}(H_{R_+}^l(M/\sum_{j=1}^{i-1} f_j M)), \text{end}(H_{R_+}^{l+1}(M/\sum_{j=1}^{i-1} f_j M)) + 1\} + t_i - 1. \end{aligned}$$

By induction on i applied to statement a), we get

$$\text{end}(H_{R_+}^k(M/\sum_{j=1}^{i-1} f_j M)) \leq \max_{j=0}^{i-1}\{\text{end}(H_{R_+}^{k+j}(M)) + j\} - i + 1 + \sum_{j=1}^{i-1} t_j,$$

with $k = l, l + 1$. Combining this with the previous estimate, we get

$$\text{end}(H_{R_+}^l(M/\sum_{j=1}^i f_j M)) \leq \max_{j=0}^{i-1}\{\text{end}(H_{R_+}^{l+j}(M)) + j\} - i + \sum_{j=1}^i t_j.$$

This proves statement a). By induction on i applied to statement b) with M/f_1M instead of M , we have

$$\text{end}(H_{R_+}^{k+i-1}(M/f_1M)) + \sum_{j=2}^i t_j \leq \text{end}(H_{R_+}^k(M/\sum_{j=1}^i f_jM)).$$

If we apply the estimate (ii) with $l = k + i - 1$, we now get statement b). \square

5.4. Exercise and Definition. A) (*Saturated Filter-Regular Sequences*) Let the notations and hypotheses be as in (5.2). Let f_1, f_2, \dots, f_r be a filter-regular sequence with respect to M . Show that the following statements are equivalent:

- (i) $M/\sum_{i \leq r} f_i M$ is R_+ -torsion.
- (ii) $R_+ \subseteq \sqrt{(0 :_R M) + \langle f_1, f_2, \dots, f_r \rangle}$.

If these equivalent conditions hold, f_1, f_2, \dots, f_r is called a *saturated filter-regular sequence with respect to M* .

B) (*Properties of Saturated Filter-Regular Sequences*) Keep all notations and hypotheses of part A). Show the following statements:

- a) Let $N \subseteq \Gamma_{R_+}(M)$ be a graded submodule. Then f_1, f_2, \dots, f_r form a saturated filter-regular sequence with respect to M/N if and only if they do with respect to M .
- b) (*Base Ring Independence of Saturated Filter-Regular Sequences*) Let $\mathfrak{b} \subseteq R$ be a graded ideal such that $\mathfrak{b}M = 0$. Then f_1, f_2, \dots, f_r form a saturated filter regular sequence with respect to M if and only if $f_1 + \mathfrak{b}, f_2 + \mathfrak{b}, \dots, f_r + \mathfrak{b} \in R/\mathfrak{b}$ do.
- c) If f_1, f_2, \dots, f_r form a saturated filter-regular sequence with respect to M , then $H_{R_+}^i(M) = H_{\langle f_1, f_2, \dots, f_r \rangle}^i(M)$ for all $i \in \mathbb{N}_0$ and $r \geq \text{cd}_{R_+}(M)$.

5.5. Lemma. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Let f_1, f_2, \dots, f_r be a filter-regular sequence with respect to M . Then*

- a) *If $M \neq 0$, then $\dim_R(M/\sum_{i \leq s} f_i M) = \max\{0, \dim_R(M) - s\}, \forall s \leq r$.*
- b) *The filter-regular sequence f_1, f_2, \dots, f_r is saturated if and only if $r \geq \dim_R(M)$.*

Proof. "a)": We first treat the cases with $s \leq d := \dim_R(M)$. We do this by induction on s . The case $s = 0$ is obvious. So let $s > 0$. Set $\bar{M} := M/\sum_{i \leq s-1} f_i M$. By induction we have $\dim_R(\bar{M}) = d - s + 1 > 0$. In particular all minimal members of (the non-empty set) $\text{Ass}_R(\bar{M})$ belong to $\text{Proj}(R)$. As f_s is filter-regular with respect to \bar{M} it avoids all these minimal members so that $\dim_R(\bar{M}/f_s \bar{M}) = \dim_R(\bar{M}) - 1 = d - s$. As $M/\sum_{i \leq s} f_i M \cong \bar{M}/f_s \bar{M}$ we get our claim. Now, let $s > d$. Then clearly, by what we have already shown

$\dim_R(M/\sum_{i \leq s} f_i M) \leq \dim_R(M/\sum_{i \leq d} f_i M) \leq 0$. As $M \neq 0$ and $f_i \in R_+$ for all $i \in \{1, 2, \dots, r\}$ it follows from the Graded Nakayama Lemma (2.1C)a) that $M/\sum_{i \leq s} f_i M_i \neq 0$ so that $\dim_R(M/\sum_{i \leq s} f_i M) = 0$.

"b)": This follows immediately by statement a), as the module $M/\sum_{i \leq r} f_i M$ is R_+ -torsion if and only if it has dimension ≤ 0 . \square

We now prove a basic existence result for filter-regular sequences.

5.6. Proposition. *Let $r \in \mathbb{N}_0$, let K be an infinite field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K algebra, let M be a finitely generated graded R -module and let $\mathfrak{a} \subseteq R_+$ be a graded ideal with*

$$R_+ \subseteq \sqrt{\mathfrak{a} + (0 :_R M)}.$$

Let $t_1, t_2, \dots, t_r \in \mathbb{Z}_{\geq \text{gendeg}(\mathfrak{a})}$ and let $\mathcal{P} \subseteq \text{Spec}(R) \setminus \text{Var}(\mathfrak{a})$ be a finite set. Then, there are elements

$$f_i \in \mathfrak{a}_{t_i} \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p} \quad (i = 1, 2, \dots, r)$$

such that f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M . Moreover this filter-regular sequence is saturated if and only if $r \geq \dim_R(M)$.

Proof. By (5.5)b) it suffices to prove the existence of the requested filter-regular sequence. We do this by induction on r . For $r = 0$, there is nothing to show. So, let $r > 0$. Let

$$\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_s\} := (\text{Ass}_R(M) \cap \text{Proj}(R)) \cup \mathcal{P}, \quad (s \in \mathbb{N}_0).$$

We first aim to show that

$$\mathfrak{a} \not\subseteq \mathfrak{p}_j, \quad \forall j \in \{1, 2, \dots, s\}.$$

So, let $j \in \{1, 2, \dots, s\}$. If $\mathfrak{p}_j \in \text{Ass}_R(M) \cap \text{Proj}(R)$ we have $(0 :_R M) \subseteq \mathfrak{p}_j$ and $\mathfrak{p}_j \not\subseteq R_+ \subseteq \sqrt{\mathfrak{a} + (0 :_R M)}$ which implies that $\mathfrak{a} \not\subseteq \mathfrak{p}_j$. If $\mathfrak{p}_j \in \mathcal{P}$ this latter inclusion is clear by our hypothesis.

But now we get $\mathfrak{a}_{\geq t_1} \not\subseteq \mathfrak{p}_j$ for all $j \in \{1, 2, \dots, s\}$. As $t_1 \geq \text{gendeg}(\mathfrak{a})$ we also have $\mathfrak{a}_{\geq t_1} = \langle \mathfrak{a}_{t_1} \rangle$ and hence $\mathfrak{a}_{t_1} \not\subseteq \mathfrak{p}_j$ for all $j \in \{1, 2, \dots, s\}$. Therefore $\mathfrak{a}_{t_1} \cap \mathfrak{p}_j \subsetneq \mathfrak{a}_{t_1}$ for all $j \in \{1, 2, \dots, s\}$. As K is infinite, we thus find some element

$$f_1 \in \mathfrak{a}_{t_1} \setminus \bigcup_{j \in \{1, 2, \dots, s\}} \mathfrak{a}_{t_1} \cap \mathfrak{p}_j = \mathfrak{a}_{t_1} \setminus \bigcup_{\mathfrak{p} \in (\text{Ass}_R(M) \cap \text{Proj}(R)) \cup \mathcal{P}} \mathfrak{p}.$$

So,

$$f_1 \in \mathfrak{a}_{t_1} \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$$

is filter-regular with respect to M . Observe that $(0 :_R M) \subseteq (0 :_R M/f_1 M)$, so that

$$R_+ \subseteq \sqrt{\mathfrak{a} + (0 :_R M/f_1 M)}.$$

Therefore, by induction we find elements

$$f_i \in \mathfrak{a}_{t_i} \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p} \quad (i = 2, 3, \dots, r)$$

such that f_2, f_3, \dots, f_r form a filter-regular sequence with respect to M/f_1M . Therefore f_1, f_2, \dots, f_r form a filter-regular sequence with respect to M . \square

We now focus on (saturated) filter-regular sequences which consist of homogeneous elements of degree 1.

5.7. Proposition. *Let $r \in \mathbb{N}$, let $m \in \mathbb{Z}$, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let $f_1, f_2, \dots, f_r \in R_1$ and let M be a finitely generated graded R -module. Then the following statements are equivalent:*

- (i) $\text{reg}(M) < m$ and f_1, f_2, \dots, f_r is a saturated filter-regular sequence with respect to M .
- (ii) $\text{end}(0 :_{(M/\sum_{j < k} f_j M)} f_k) < m, \forall k \leq r$ and $\text{end}(M/\sum_{j \leq r} f_j M) < m$.

Proof. Assume first, that condition (i) holds. And let $i \in \{1, 2, \dots, r\}$. Then, by (5.3)c) we have

$$\text{end}(H_{R_+}^0(M/\sum_{j \leq i} f_j M)) \leq \text{reg}(M/\sum_{j \leq i} f_j M) \leq \text{reg}(M) < m.$$

Now, let $k \in \{1, 2, \dots, r\}$. As f_k is filter-regular with respect to $M/\sum_{j < k} f_j M$ and on application of the previous estimate with $i = k - 1$ we thus get

$$\text{end}(0 :_{(M/\sum_{j < k} f_j M)} f_j) \leq \text{end}(H_{R_+}^0(M/\sum_{j < i} M f_j)) < m.$$

As the filter-regular sequence f_1, f_2, \dots, f_r is saturated and on use of the above inequality with $k = r$ we also get

$$\text{end}(M/\sum_{j \leq r} f_j M) = \text{end}(H_{R_+}^0(M/\sum_{j \leq r} f_j M)) < m.$$

So, condition (ii) holds.

Assume now, that condition (ii) holds. Then

$$\text{end}(0 :_{(M/\sum_{j < k} f_j M)} f_k) < m < \infty, \quad \forall k \in \{1, 2, \dots, r\}$$

shows that f_1, f_2, \dots, f_r is a filter-regular sequence with respect to M . As $\text{end}(M/\sum_{j \leq r} f_j M) < m < \infty$, the module $M/\sum_{j \leq r} f_j M$ is R_+ -torsion, so that the filter-regular sequence f_1, f_2, \dots, f_r is saturated. In particular we have $H_{R_+}^i(M) = 0$ for all $i > 0$ (see (5.4)B)c). By (5.3)b) (applied with $k = 0$) we have

$$\text{end}(H_{R_+}^i(M)) + i \leq \text{end}(H_{R_+}^0(M/\sum_{j \leq i} f_j M)), \quad \forall i \in \{0, 1, \dots, r\}.$$

If we apply this with $i = r$ and bear in mind that $M/\sum_{j \leq i} f_j M$ is R_+ -torsion and has an end $< m$, we get $\text{end}(H_{R_+}^r(M)) + r < m$. Finally, let $k \in \{1, 2, \dots, r\}$. As $f_k \in R_1$ is filter-regular with respect to $M/\sum_{j < k} f_j M$ we have (see (5.1)Bd))

$$\text{end}(H_{R_+}^0(M/\sum_{j < k} f_j M)) = \text{end}(0 :_{(M/\sum_{j < k} f_j M)} f_k) < m.$$

If we apply the above estimate with $i = k - 1$ we thus get $\text{end}(H_{R_+}^{k-1}(M)) + (k - 1) < m$. Altogether we thus have $\text{reg}(M) < m$. Therefore condition (i) holds. \square

Later, we shall have to consider the situation where M is a graded submodule of a given graded module V . The following consequence of (5.7) will help to pave the way to this.

5.8. Corollary. *Let $r \in \mathbb{N}$, let $m \in \mathbb{Z}$, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let V be a finitely generated graded R -module with $\text{reg}(V) < m$ and let $f_1, f_2, \dots, f_r \in R_1$. Then the following statements are equivalent:*

- (i) $\text{reg}(M) \leq m$ and f_1, f_2, \dots, f_r is a saturated filter-regular sequence with respect to V/M .
- (ii) $((M + \sum_{j < i} f_j V) :_V f_i)_{\geq m} = (M + \sum_{j < i} f_j V)_{\geq m}$, $\forall i \in \{1, 2, \dots, r\}$ and $(M + \sum_{j \leq r} f_j V)_{\geq m} = V_{\geq m}$.

Proof. Observe that by (3.3)C)a),d) we have

$\text{reg}(M) \leq \max\{\text{reg}(V), \text{reg}(V/M) + 1\}$, $\text{reg}(V/M) \leq \max\{\text{reg}(M) - 1, \text{reg}(V)\}$, so that $\text{reg}(V/M) < m$ if and only if $\text{reg}(M) \leq m$. Therefore, condition (i) is equivalent to the fact that V/M satisfies condition (i) of (5.7). Moreover condition (i) is obviously equivalent to the fact that V/M satisfies condition (ii) of (5.7). Now, we may conclude by (5.7). \square

Upper bounds on the generating degree of an intersection of two graded submodules of a given graded module are a basic issue in computational algebraic geometry. Here comes a corresponding bounding result, which will be used as an important tool for our investigation of filter-regular sequences.

5.9. Proposition. *Let $m \in \mathbb{Z}$, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let V be a finitely generated graded R -module and let $M, N \subseteq V$ be two graded submodules such that*

$$\text{gendeg}(M), \text{gendeg}(N) \leq m, \quad \text{reg}(M + N) < m.$$

Then $\text{gendeg}(M \cap N) \leq m$.

Proof. We find a polynomial ring $S := R_0[X_1, X_2, \dots, X_r]$ and a graded ideal $\mathfrak{a} \subseteq S$ such that $R = S/\mathfrak{a}$. If we consider $V, M, N, M + N$ and $M \cap N$ as graded S -modules their regularities and generating degrees remain the same

(see (3.3)A)e) and (3.3)B)i)). This allows to replace R by S and hence to assume that $R = R_0[X_1, X_2, \dots, X_r]$ is a polynomial ring. As M and N are generated in degrees $\leq m$, there are epimorphisms of graded R -modules

$$F \xrightarrow{\pi} M \rightarrow 0, \quad G \xrightarrow{\rho} N \rightarrow 0$$

in which

$$F = \bigoplus_{i=1}^r R(-a_i), \quad a_1 \leq a_2 \leq \dots \leq a_r = \text{gendeg}(M),$$

$$G = \bigoplus_{i=1}^s R(-a_{r+i}), \quad a_{r+1} \leq a_{r+2} \leq \dots \leq a_{r+s} = \text{gendeg}(N)$$

are graded free R -modules of finite rank with

$$\text{gendeg}(F), \text{gendeg}(G) \leq m.$$

So

$$F \oplus G = \bigoplus_{j=1}^{r+s} R(-a_j), \quad a_j \leq m, \forall j \in \{1, 2, \dots, r+s\}$$

is a graded free R -module of finite rank with $\text{gendeg}(M \oplus N) \leq m$. As $\text{reg}(R) = 0$ (see (3.5)), we thus have $\text{reg}(F \oplus G) \leq m$ (see (4.6)A)d)). So, the exact sequence of graded R -modules

$$0 \rightarrow \text{Ker}(\pi + \rho) \rightarrow F \oplus G \xrightarrow{\pi + \rho} (M + N) \rightarrow 0$$

yields that $\text{reg}(\text{Ker}(\pi + \rho)) \leq m$ (see (3.3)C)a)). Therefore $\text{gendeg}(\text{Ker}(\pi + \rho)) \leq m$, (see (3.4)). Now, the commutative diagram

$$\begin{array}{ccc} F \oplus G & \xrightarrow{\text{Id}_{F \oplus G}} & F \oplus G \\ \pi \oplus \rho \downarrow & & \downarrow \pi + \rho \\ M \oplus N & \xrightarrow{\sigma := \text{Id}_M + \text{Id}_N} & M + N \end{array}$$

shows that

$$(\pi \oplus \rho)(\text{Ker}(\pi + \rho)) = \text{Ker}(\sigma).$$

Therefore

$$\text{gendeg}(\text{Ker}(\sigma)) = \text{gendeg}((\pi \oplus \rho)(\text{Ker}(\pi + \rho))) \leq \text{gendeg}(\text{Ker}(\pi + \rho)) \leq m.$$

In view of the isomorphism of graded R -modules

$$\text{Ker}(\sigma) \xrightarrow{\cong} M \cap N, \quad (m, n) \mapsto m$$

we finally get that $\text{gendeg}(M \cap N) \leq m$, as requested. \square

We now are ready to prove the crucial result of this section, which shall immediately lead to the announced generalized version of the Regularity Criterion of Bayer-Stillman. As a last preparative step, we prove the following auxiliary result.

5.10. **Lemma.** *Let $m \in \mathbb{Z}$, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let V be a finitely generated graded R -module, let $M \subseteq V$ be a graded submodule and let $f \in R_1$ be a filter-regular element with respect to V . Assume that*

$$\text{gendeg}(M), \text{reg}(V), \text{reg}(M + fV) \leq m.$$

Then $\text{gendeg}(M :_V f) \leq m$.

Proof. As (see (3.3) A)a),b), (3.4))

$$\text{gendeg}(fV) \leq \text{gendeg}(V) + 1 \leq \text{reg}(V) + 1 \leq m + 1$$

our previous proposition (5.9) implies that $\text{gendeg}(M \cap fV) \leq m + 1$ and hence $\text{gendeg}(M(1)) \leq m$ (see (3.3)A)a)). As

$$M \cap fV = f(M :_V f)$$

we have an exact sequence of graded R -modules

$$0 \rightarrow (0 :_V f) \rightarrow (M :_V f) \rightarrow (M \cap fV)(1) \rightarrow 0.$$

As $f \in R_1$ is filter-regular with respect to V we also have (see (5.1)A)):

$$\text{gendeg}(0 :_V f) \leq \text{end}(0 :_V f) \leq \text{end}(H_{R_+}^0(V)) \leq \text{reg}(V) \leq m.$$

So, in view of the above exact sequence we obtain (see (3.3)A)b)

$$\text{gendeg}(M :_V f) \leq m.$$

□

Now, we are ready to formulate and to prove the announced main results.

5.11. **Theorem.** *Let $r \in \mathbb{N}$, let $m \in \mathbb{Z}$, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let V be a finitely generated graded R -module, let $M \subseteq V$ be a graded submodule and let $f_1, f_2, \dots, f_r \in R_1$ be filter-regular elements with respect to V . Assume that $\text{reg}(V) < m$ and $\text{gendeg}(M) \leq m$. Then, the following statements are equivalent:*

- (i) $\text{reg}(M) \leq m$ and f_1, f_2, \dots, f_r is a saturated filter-regular sequence with respect to V/M .
- (ii) $((M + \sum_{j < i} f_j V) :_V f_i)_m = (M + \sum_{j < i} f_j V)_m$, $\forall i \in \{1, 2, \dots, r\}$ and $(M + \sum_{j \leq r} f_j V)_m = V_m$.

Proof. By (5.8) statement (i) implies statement (ii). We prove the reverse implication by induction on r . First, let $r = 1$ and assume that statement (ii) holds. Then $(M + f_1 V)_m = V_m$. As $\text{gendeg}(V) \leq \text{reg}(V) \leq m$ (see (3.4)) and R is homogeneous, it follows $(M + f_1 V)_{\geq m} = V_{\geq m}$ and hence $\text{end}(V/(M + f_1 V)) < m < \infty$. In particular the module $V/(M + f_1 V)$ is R_+ -torsion, so that $\text{reg}(V/(M + f_1 V)) = \text{end}(V/(M + f_1 V)) < m$. Hence, in view of (3.3)C)a) the short exact sequence of graded R -modules

$$0 \rightarrow (M + f_1 V) \rightarrow V \rightarrow V/(M + f_1 V) \rightarrow 0$$

implies that $\text{reg}(M + f_1V) \leq m$. Now, by (5.10) it follows that

$$\text{gendeg}(M :_V f_1) \leq m.$$

By our assumption we have $(M :_V f_1)_m = M_m$. As R is homogeneous it thus follows $(M :_V f_1)_{\geq m} = M_{\geq m}$. But this means that in our situation statement (ii) of (5.8) holds for $r = 1$. So, by (5.8) we see that $\text{reg}(M) \leq m$ and that f_1 constitutes a saturated filter-regular sequence with respect to V/M . This proves the requested implication if $r = 1$.

So, let $r > 1$ and assume that statement (ii) holds. As $\text{gendeg}(f_1V) \leq \text{gendeg}(V) + 1 \leq \text{reg}(V) + 1 \leq m$, we have $\text{gendeg}(M + f_1V) \leq m$. Applying induction to the graded submodule $M + f_1V \subseteq V$ and the sequence $f_2, f_3, \dots, f_r \in R_1$ of elements which all are filter-regular with respect to V , we obtain that $\text{reg}(M + f_1V) \leq m$ and that f_2, f_3, \dots, f_r is a saturated filter-regular sequence with respect to $V/(M + f_1V)$. Hence, by (5.8) we have

$$((M + \sum_{j<i} f_jV) :_V f_i)_{\geq m} = (M + \sum_{j<i} f_jV)_{\geq m}, \quad \forall i \in \{2, 3, \dots, r\}$$

and in addition

$$(M + \sum_{j \leq r} f_jV)_{\geq m} = V_{\geq m}.$$

By (5.10) we also have $\text{gendeg}(M :_V f_1) \leq m$. As R is homogeneous, as $(M :_V f_1)_m = M_m$ and as $\text{gendeg}(M) \leq m$, this implies that

$$(M :_V f_1)_{\geq m} = M_{\geq m}.$$

Now, by another use of (5.8) we get statement (ii). \square

Finally we can prove the announced extension of the Regularity Criterion of Bayer-Stillman.

5.12. Theorem. *Let $m \in \mathbb{Z}$, let K be an infinite field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let V be a finitely generated graded R -module and let $M \subseteq V$ be a graded submodule. Assume that $\text{reg}(V) < m$ and $\text{gendeg}(M) \leq m$. Then, the following statements are equivalent:*

- (i) $\text{reg}(M) \leq m$.
- (ii) *There is an integer $r \in \mathbb{N}_0$ and there are elements $f_1, f_2, \dots, f_r \in R_1$ which are filter-regular with respect to V and such that*

$$((M + \sum_{j<i} f_jV) :_V f_i)_m = (M + \sum_{j<i} f_jV)_m, \quad \forall i \in \{1, 2, \dots, r\}$$

and

$$(M + \sum_{j \leq r} f_jV)_m = V_m.$$

Moreover, a sequence $f_1, f_2, \dots, f_r \in R_1$ which satisfies the conditions mentioned in statement (ii) is a saturated filter-regular sequence with respect to V/M .

Proof. Assume first, that statement (i) holds. If $\dim_R(V/M) \leq 0$ the module V/M is R_+ -torsion and so, by (3.3)C)d)

$$\text{end}(V/M) = \text{reg}(V/M) \leq \max\{\text{reg}(M) - 1, \text{reg}(V)\} < m.$$

So we get statement (ii) with $r = 0$.

Now, let $r := \dim_R(V/M) > 0$ and set $\mathcal{P} := \text{Ass}_R(V) \cap \text{Proj}(R)$. Then, by (5.6) (applied with $\mathfrak{a} = R_+$ and $t_1 = t_2 = \dots = t_r = 1$ to the module V/M) we find elements

$$f_1, f_2, \dots, f_r \in R_1 \setminus \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$$

which constitute a saturated filter-regular sequence with respect to V/M . By (5.11) these elements satisfy the requirements of statement (ii).

Assume now, that statement (ii) holds. If $r = 0$ we see that V/M is R_+ -torsion with $\text{reg}(V/M) = \text{end}(V/M) < m$, so that by (3.3)C)a) we get

$$\text{reg}(M) \leq \max\{\text{reg}(V), \text{reg}(V/M) + 1\} \leq m$$

and hence statement (ii). If $r > 0$ statement (i) follows immediately by (5.11).

The remaining statement of our theorem is also immediate by (5.11) if $r > 0$ and clear by what we were saying above in the case $r = 0$. \square

5.13. Remark. (*Around the Regularity Criterion of Bayer-Stillman*) A) Let $m \in \mathbb{N}$, let $R = K[X_1, X_2, \dots, X_s]$ be a polynomial ring over the infinite field K and let $\mathfrak{a} \subsetneq R$ be a graded ideal with $\text{gendeg}(\mathfrak{a}) \leq m$. As $\text{reg}(R) = 0$ (see (3.5)) we may apply (5.12) with $V = R$ and $M = \mathfrak{a}$ and thus get, that $\text{reg}(\mathfrak{a}) \leq m$ if and only if there are linear forms $f_1, f_2, \dots, f_r \in R \setminus \{0\}$ such that

$$((\mathfrak{a} + \langle f_1, f_2, \dots, f_{i-1} \rangle) :_R f_i)_m = (\mathfrak{a} + \langle f_1, f_2, \dots, f_{i-1} \rangle)_m, \forall i \in \{1, 2, \dots, r\}$$

and

$$(\mathfrak{a} + \langle f_1, f_2, \dots, f_r \rangle)_m = R_m.$$

Moreover we know that the linear forms f_1, f_2, \dots, f_r satisfy the above requirements if and only if they constitute a filter-regular sequence with respect to R/\mathfrak{a} . In addition by (5.6) we always can choose $r = \dim(R/\mathfrak{a})$. This is essentially what has by been shown by Bayer and Stillmann [B-St]. In fact their statement says that for a "generic family" of $r = \dim(R/\mathfrak{a})$ elements the above requirements hold. The genericity condition means that there is a non-empty (Zariski-) open subset $U \subseteq (R_1)^r = K^{rs}$ such that for all $(f_i)_{i \in \{1, 2, \dots, r\}} \in U$ the above requirements hold. Indeed, a generic family of r linear forms is always a filter-regular sequence with respect to R/\mathfrak{a} . We leave the proof of this to those readers who have already some background in Algebraic Geometry. Our result gives more information, as it precisely characterizes the sequences which satisfy the above requirements.

B) In [Br-L2] we did prove that the Regularity Criterion of Bayer-Stillman holds for graded submodules of free modules over a polynomial ring over a

field, as we needed the criterion in this more general form. In [Br5] we did establish the criterion in question in greater generality as given in (5.11). We namely did show that this criterion holds over any Noetherian homogeneous ring $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ whose base ring R_0 has infinite residue fields, which means that the ring R_0/\mathfrak{p}_0 is infinite for all $\mathfrak{p}_0 \in \text{Spec}(R_0)$.

C) Finally, let us point out the significance of the Regularity Criterion of Bayer-Stillman. Observe that this criterion tells that the requirements given in statement (5.12)(ii) are satisfied for all degrees $m \geq m_0$ if they hold for $m = m_0$. So, this criterion includes a *persistence result* for the requirements in question. There is in fact a classical idea hidden behind this criterion: Already in [Herm] one finds similar criteria to bound the *saturation degree* (see (4.9))

$$\text{satdeg}(\mathfrak{a}) := \sup\{n \in \mathbb{Z} \mid (\mathfrak{a}^{\text{sat}})_n = \mathfrak{a}_n\} = \text{end}(H_{R_+}^0(R/\mathfrak{a}))$$

of a graded ideal $\mathfrak{a} \subseteq R = K[X_1, X_2, \dots, X_r]$ in a polynomial ring over a field K .

We now aim to investigate the relation between multiplicities and filter-regular sequences. We start with a number of preliminaries, which will lead us immediately to the corresponding main result, which claims that non-saturated filter-regular sequences with respect to graded modules over Noetherian homogeneous K -algebras are systems of multiplicity parameters.

5.14. Exercise and Definition. A) (*Homogeneous Parameters*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be finitely generated graded R -module. We set

$$\text{Ass}_R^{(0)}(M) := \{\mathfrak{p} \in \text{Ass}_R(M) \mid \dim(R/\mathfrak{p}) = \dim_R(M)\}.$$

Now, let $t \in \mathbb{N}$, assume that $\dim(M) > 0$ and show that for all elements $f \in R_t$ the following statements are equivalent:

- (i) $\dim(M/fM) < \dim(M)$.
- (ii) $\dim(M/fM) = \dim(M) - 1$.
- (iii) $f \notin \bigcup_{\mathfrak{p} \in \text{Ass}_R^{(0)}(M)} \mathfrak{p}$.

If these equivalent conditions hold, the element $f \in R_t$ is called a *homogeneous parameter (of degree t) with respect to M* .

B) (*Multiplicity Parameters*) Keep the notations and hypotheses of part A), assume that $\dim_R(M) > 0$ and let $f_t \in R_t$ be a homogeneous parameter of degree $t \in \mathbb{N}$ with respect to M . Prove the following claims

- a) $\deg(P_{M/fM}) = \deg(P_M) - 1$.
- b) $P_M(n) - P_M(n-t) \leq P_{M/fM}(n)$ for all $n \gg 0$.
- c) If $\dim(M) > 1$, then $e_0(M/fM) \geq t e_0(M)$.

If $\dim(M) > 1$ and $e_0(M/fM) = te_0(M)$, we call f a *multiplicity parameter (of degree t) with respect to M* .

C) (*Filter-Regular Elements as Multiplicity Parameters*) Keep the notations of part A) and assume that $\dim_R(M) > 0$. Prove the following statements:

- a) If $f \in R_t$ is filter-regular with respect to M , then it is a homogeneous parameter with respect to M and

$$P_{M/fM}(X) = P_M(X) - P_M(X - t).$$

- b) If $\dim_R(M) > 1$ and $f \in R_t$ is filter regular with respect to M , then f is a multiplicity parameter with respect to M .

5.15. **Exercise and Definition.** A) (*Homogeneous Systems of Parameters*)

As in (5.14), let K be a field and let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module, let $r \in \mathbb{N}$ with $r \leq \dim_R(M)$, let $t_1, t_2, \dots, t_r \in \mathbb{N}$ and let $f_i \in R_{t_i}$ for all $i \in \{1, 2, \dots, r\}$. Prove that the following statements are equivalent;

- (i) f_i is a homogeneous parameter with respect to $M/\sum_{j<i} f_j M$ for all $i \in \{1, 2, \dots, r\}$.
- (ii) $\dim_R(M/\sum_{i \leq s} f_i M) = \dim_R(M) - s$ for all $s \in \{0, 1, \dots, r\}$.
- (iii) $\dim_R(M/\sum_{i \leq r} f_i M) = \dim_R(M) - r$.
- (iv) $\dim_R(M/\sum_{i \leq r} f_i M) \geq \dim_R(M) - r$.

If the equivalent conditions (i)-(iv) are satisfied, the sequence f_1, f_2, \dots, f_r is called a *homogeneous system of parameters with respect to M* .

B) (*Properties of Homogeneous Systems of Parameters*) Let the notations and hypotheses be as in part A). Prove the following statements:

- a) If f_1, f_2, \dots, f_r is a homogeneous system of parameters with respect to M , then so is f_1, f_2, \dots, f_s for all $s \in \{1, 2, \dots, r\}$.
- b) If f_1, f_2, \dots, f_r is a homogeneous system of parameters with respect to M , then so is $f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(r)}$ for each permutation σ of $\{1, 2, \dots, r\}$.
- c) If $r < \dim_R(M)$, then $e_0(M/\sum_{i \leq r} f_i M) \geq t_1 t_2 \dots t_r e_0(M)$.

C) (*Systems of Multiplicity Parameters*) Keep the above notations and hypotheses. Assume in addition that $r < \dim_R(M)$ and that f_1, f_2, \dots, f_r is a homogeneous system of parameters with respect to M . Show that the following statements are equivalent:

- (i) $e_0(M/\sum_{i \leq r} f_i M) \leq t_1 t_2 \dots t_r e_0(M)$.
- (ii) $e_0(M/\sum_{i \leq s} f_i M) = t_1 t_2 \dots t_s e_0(M)$ for all $s \in \{1, 2, \dots, r\}$.
- (iii) f_i is a multiplicity parameter with respect to $M/\sum_{j<i} f_j M$ for all $i \in \{1, 2, \dots, r\}$.

It these three equivalent conditions are satisfied, the sequence f_1, f_2, \dots, f_r is called a *system of multiplicity parameters with respect to M* .

D) (*Properties of Systems of Multiplicity Parameters*) Keep all notations and hypotheses of part C) and prove the following:

- a) If f_1, f_2, \dots, f_r is a system of multiplicity parameters with respect to M , then so is f_1, f_2, \dots, f_s for all $s \in \{1, 2, \dots, r\}$.
- b) If f_1, f_2, \dots, f_r is a system of multiplicity parameters with respect to M , then so is $f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(r)}$ for each permutation σ of the set $\{1, 2, \dots, r\}$.

5.16. Theorem. *Let $r \in \mathbb{N}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module such that $\dim_R(M) > r$ and let f_1, f_2, \dots, f_r be a filter-regular sequence with respect to M . Then f_1, f_2, \dots, f_r is a system of multiplicity parameters with respect to M .*

Proof. This follows easily from (5.14)C)a) on use of the equivalences shown in (5.15)A),C). \square

5.17. Example and Exercise. Let K be a field, let X, Y, Z, W be indeterminates and consider the Noetherian homogeneous K -algebra

$$R := K[X, Y, Z, W]/\langle X \rangle \cap \langle X^2, Y, Z \rangle = K[X, Y, Z, W]/X\langle X, Y, Z \rangle,$$

consider the graded primes

$$\mathfrak{p} := XR, \quad \mathfrak{q} := \langle X, Y, Z \rangle R, \quad \mathfrak{s} := \langle X, W \rangle R \in \text{Proj}(R)$$

and the two elements

$$f_1 := W1_R, \quad f_2 := Z1_R \in R_1.$$

Show that

- a) $\text{Ass}_R(R) = \{\mathfrak{p}, \mathfrak{q}\}$ and $\dim(R) = 3$.
- b) $\text{Ass}_R(R/f_1R) = \{\mathfrak{s}, R_+\}$.
- c) f_1, f_2 is a filter-regular sequence with respect to R .
- d) f_2, f_1 is not a filter-regular sequence with respect to R .
- e) f_1, f_2 and f_2, f_1 are systems of multiplicity parameters with respect to R .
- f) $e_0(R) = e_0(R/\langle f_1, f_2 \rangle) = 1$.

This example teaches us, that filter-regular sequences are not permutable (see statements c) and d)). It thus shows the most important difference between filter regular sequences and systems of multiplicity parameters: the latter are always permutable, whereas the former need not be. In particular, we see by this example, that there are systems of multiplicity parameters which are not filter-regular sequences.

We have already observed in (5.2)B)a) that M -sequences with respect to a finitely generated graded module over a Noetherian homogeneous ring are filter-regular sequences. We now want to prove a partial converse of this. We begin with the following preparation.

5.18. Reminder. (*Grade with Respect to the Irrelevant Ideal*) (See [Br-Fu-Ro] (4.5),(4.6)) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring and let M be a finitely generated graded R -module. Then the *grade of R_+ with respect to M* is defined as the supremum of lengths of all M -sequences which consist of elements of R_+ and is denoted by $\text{grade}_M(R_+)$. Keep in mind that

$$\text{grade}_M(R_+) = \inf\{i \in \mathbb{N}_0 \mid H_{R_+}^i(M) \neq 0\}.$$

Using this notation we now can say:

5.19. Proposition. *Let $r \in \mathbb{N}_0$, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let M be a finitely generated graded R -module and let $f_1, f_2, \dots, f_r \in R_+^h$ form a filter regular sequence with respect to M . Then, the following statements are equivalent:*

- (i) $r \leq \text{grade}_M(R_+)$.
- (ii) f_1, f_2, \dots, f_r is an M -sequence.

Proof. Assume that condition (i) is satisfied. Suppose first, that $\text{grade}_M(R_+) = \infty$. Then $R_+M = M$ (see [Br-Fu-Ro] (4.7)). By the Graded Nakayama Lemma (2.1)C)a) it thus follows $M = 0$, and so f_1, f_2, \dots, f_r is an M -sequence.

Assume now that $g := \text{grade}_M(R_+) < \infty$. We prove by induction on r that f_1, f_2, \dots, f_r is an M -sequence. If $r = 0$, there is nothing to show. So, let $r > 0$. Then we have $g > 0$ and hence $\text{NZD}_R(M) \cap R_+ \neq \emptyset$. Therefore $\text{Ass}_R(M) \cap \text{Var}(R_+) = \emptyset$ and hence $\text{Ass}_R(M) \subseteq \text{Proj}(R)$. As f_1 is filter-regular with respect to M it thus avoids all members of $\text{Ass}_R(M)$ and therefore we get $f_1 \in \text{NZD}_R(M)$. Now, we have $r - 1 \leq g - 1 = \text{grade}_{M/f_1M}(R_+)$ and f_2, f_3, \dots, f_r is a filter-regular sequence with respect to M/f_1M . So, by induction f_2, f_3, \dots, f_r is an M/f_1M -sequence. It follows that f_1, f_2, \dots, f_r is an M -sequence.

The reverse implication is immediate by the definition of grade. □

We now come to the last result of this section, which may be seen as a complement of (5.16).

5.20. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module with $r := \dim_R(M) = \text{grade}_M(R_+) > 0$, let $t_1, t_2, \dots, t_r \in \mathbb{N}$ and let*

$$f_i \in R_{t_i}, \quad (i = 1, 2, \dots, r)$$

be such that f_1, f_2, \dots, f_r is an M -sequence. Then

$$\dim_K(M/\langle f_1, f_2, \dots, f_r \rangle M) = \prod_{i=1,2,\dots,r} t_i e_0(M).$$

Proof. We first treat the case $r = 1$. Observe that for all $n \in \mathbb{Z}$ there is an exact sequence of K -vector spaces

$$0 \rightarrow M_{n-t_1} \xrightarrow{f_1} M_n \rightarrow (M/f_1M)_n \rightarrow 0$$

with $M_n = 0$ for all $n \ll 0$ and all $\dim_K(M_n) = e_0(M)$ for all $n \gg 0$. So, for all $m \in \mathbb{Z}$ we have

$$\sum_{k \in \mathbb{Z}} \dim_K((M/f_1M)_{m+kt_1}) = \sum_{k \in \mathbb{Z}} \dim_K(M_{m+kt_1}) - \dim_K(M_{m+kt_1-t_1}) = e_0(M).$$

and hence

$$\begin{aligned} \dim_K(M/f_1M) &= \sum_{n \in \mathbb{Z}} \dim_K((M/f_1M)_n) = \\ &= \sum_{m=1}^{t_1} \sum_{k \in \mathbb{Z}} \dim_K((M/f_1M)_{m+kt_1}) = t_1 e_0(M). \end{aligned}$$

This proves the case $r = 1$.

Now, let $r > 1$ and set $M' = M/\langle f_1, f_2, \dots, f_{r-1} \rangle M$. By (5.2)B)a) and (5.16) we know that f_1, f_2, \dots, f_{r-1} form a system of multiplicity parameters with respect to M , so that

$$\dim_R(M') = 1, \quad e_0(M') = \prod_{i < r} t_i e_0(M).$$

Moreover, $f_r \in \text{NZD}_R(M')$ and

$$M/\langle f_1, f_2, \dots, f_r \rangle M \cong M'/f_r M'.$$

If we apply what we have shown in the case $r = 1$ to the module M' instead of M and to the element f_r instead of f_1 , our claim follows. \square

Now, using the notion of Hilbert-Serre multiplicity as introduced in (4.3)A), we can conclude.

5.21. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K algebra, let $r \leq \text{grade}_M(R_+)$, let $t_1, t_2, \dots, t_r \in \mathbb{N}$ and let*

$$f_i \in R_{t_i}, \quad (i = 1, 2, \dots, r)$$

be such that f_1, f_2, \dots, f_r is a filter-regular sequence with respect to M . Then

$$\text{mult}(M/\langle f_1, f_2, \dots, f_r \rangle M) = \prod_{i=1}^r t_i \text{mult}(M)$$

Proof. If $r < \dim_R(M)$, we may conclude by (5.16). If $r = \dim_R(M)$ we may conclude by (5.18), (5.19) and (5.20). \square

6. REGULARITY OF SUBMODULES AND GENERATING DEGREES

This section is motivated by a classical question, namely: to give a "good" bound on the regularity of a graded ideal \mathfrak{a} in a polynomial ring

$$R = K[X_1, X_2, \dots, X_r]$$

over a field K in terms of the generating degree $\text{gendeg}(\mathfrak{a})$ of this ideal. We shall actually study a more general situation and bound the regularity of a graded submodule M of a finitely generated graded module V over a Noetherian homogeneous *Cohen-Macaulay* K -algebra R in terms of the generating degrees of M and of the ideal $(M :_R V)$ and some further numerical invariants of the ring R and the ambient R -module V . Specializing our main result to the case where $M := \mathfrak{a} \subseteq K[X_1, X_2, \dots, X_r] =: R =: V$ we shall get that

$$\text{reg}(\mathfrak{a}) \leq (2\text{gendeg}(\mathfrak{a}))^{2^{r-2}},$$

a bound which was established by Galligo and Giusti for fields K of characteristic 0 (see [G], [Gi]) in 1979 and 1984 and by Caviglia-Sbarra [Cav-Sb] in 2005 for fields K of arbitrary characteristic.

We also shall derive a number of further bounding results, which apply in the case case in which the base ring R is not necessarily a Cohen-Macaulay ring or in which the Annihilator of V/M is not known. We also derive a bounding result for the regularity of M in terms of the discrete data of a *presentation* of M . This will give us the opportunity to comment once more on the *Problem of the Finitely Many Steps* and hence to turn our view for a very short moment back to the beginning of Algorithmic Algebraic Geometry.

We begin with an investigation on the ends and the lengths of R_+ -torsion modules.

6.1. Proposition. *Let $t \in \mathbb{N}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let $M \neq 0$ be a finitely generated graded R_+ -torsion R -module. Let $\mathfrak{a} \subseteq R$ be a graded ideal such that $\mathfrak{a}M = 0$, $\sqrt{\mathfrak{a}} = R_+$ and $\text{gendeg}(\mathfrak{a}) \leq t$. Then*

$$\text{end}(M) \leq \text{reg}(R) + \text{gendeg}(M) + (t - 1)\dim(R).$$

Proof. We set

$$r := \dim(R), \quad g := \text{gendeg}(M).$$

As $M \neq 0$ we have $g \in \mathbb{Z}$.

Let K' be an infinite extension field of K , consider the Noetherian homogeneous K' -algebra $R' := K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \dots$, the finitely generated graded R' -module $M' := K' \otimes_K M = \bigoplus_{n \in \mathbb{Z}} K' \otimes_K M_n$ and the graded ideal $\mathfrak{a}' := K' \otimes \mathfrak{a} = \bigoplus_{n \in \mathbb{N}} K' \otimes_K \mathfrak{a}_n \subseteq R'$. Then clearly $\text{reg}(R') = \text{reg}(R)$ (see (3.3)Bh), $\dim(R') = \dim(R) = r$ (see (2.4)Cb)) and $\text{end}(M') = \text{end}(M) < \infty$ (see (2.4)A)). In particular M' is R'_+ -torsion. Finally it is easy to see that $\mathfrak{a}'M' = 0$ and $\sqrt{\mathfrak{a}'} = R'_+$. By (3.3)Ad) we have

$\text{gendeg}(M') = \text{gendeg}(M) = g$ and $\text{gendeg}(\mathfrak{a}') = \text{gendeg}(\mathfrak{a}) \leq t$. So, we may replace R , M and \mathfrak{a} respectively by R' , M' and \mathfrak{a}' and hence assume that K is infinite.

We find a short exact sequence of graded R -modules

$$0 \rightarrow N \xrightarrow{\subseteq} F \rightarrow M \rightarrow 0$$

in which

$$F = \bigoplus_{i=1}^k R(-a_i), \quad a_1 \leq a_2 \leq \dots \leq a_k = g$$

is a graded free R -module of finite rank k with $\text{gendeg}(F) = g$. Now, clearly $\dim_R(F) = r$ (see (4.6)A)c) and $\text{reg}(F) = \text{reg}(R) + g$ (see (3.3)B)b) and (4.6)A)d)). As $\mathfrak{a}M = 0$ we also have $\mathfrak{a}F \subseteq N$.

If we apply (5.6) to the graded R -module F , we find elements $f_1, f_2, \dots, f_r \in \mathfrak{a}_t$ which form a saturated filter-regular sequence with respect to F . In particular

$$G := F / \langle f_1, f_2, \dots, f_r \rangle F$$

is R_+ -torsion, and (5.3)c) applied to the R -module F with $k = 0$ and $i = r$ yields that

$$\text{end}(G) = \text{reg}(F / \sum_{j \leq r} f_j F) \leq \text{reg}(F) - r + rt \leq \text{reg}(R) + g + (t-1)r.$$

As $\langle f_1, f_2, \dots, f_r \rangle F \subseteq \mathfrak{a}F \subseteq N$, we have an epimorphism of graded R -modules $G \rightarrow M \rightarrow 0$, and therefore $\text{end}(M) \leq \text{end}(G) \leq \text{reg}(R) + g + (t-1)r$. \square

Before we prove our next result, we introduce a few notions.

6.2. Exercise and Definition. (*Minimal Numbers of Homogeneous Generators*) A) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. We then consider the number

$$\mu_R(M) = \mu(M) := \dim_K(M/R_+M) \quad (\in \mathbb{N}_0).$$

It follows easily from the Graded Nakayama Lemma, that $\mu(M)$ is the number of elements in all minimal homogeneous systems of generators of M . Therefore we call this number the *minimal number of homogeneous generators of M* .

B) Keep the notations and hypotheses of part A). Show the following statements:

- a) (*Base Field Change Property*) Let K' be an extension field of K , and consider the Noetherian homogeneous K -algebra $R' := K' \otimes_K R$ and the finitely generated graded R' -module $M' := K' \otimes_K M$. Then $\mu_{R'}(M') = \mu_R(M)$.
- b) (*Base Ring Independence*) Let $\mathfrak{b} \subseteq R$ be a graded ideal such that $\mathfrak{b}M = 0$. Then $\mu_{R/\mathfrak{b}}(M) = \mu_R(M)$.
- c) For all $n \in \mathbb{Z}$ we have $\mu(M(n)) = \mu(M)$.

- d) If $(M^{(i)})_{i \in \{1,2,\dots,s\}}$ is a finite family of finitely generated graded R -modules, then $\mu(\bigoplus_{i \in \{1,2,\dots,s\}} M^{(i)}) = \sum_{i \in \{1,2,\dots,s\}} \mu(M^{(i)})$.

6.3. Exercise and Definition. A) (*Graded Cohen-Macaulay Modules*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, and let $M \neq 0$ be a finitely generated graded R -module. Then we have

$$\text{grade}_M(R_+) \leq \dim_R(M),$$

(as one can easily see on use of (5.18) and Grothendieck's Vanishing Theorem [Br-Fu-Ro](4.11), for example). We say that M is a (*graded*) *Cohen-Macaulay module* or a *CM module* for short, if equality holds in the above inequality.

B) (*Properties of Graded CM-Modules*). Let the hypotheses and notations be as in part A). Prove the following claims:

- a) If $\dim_R(M) = 0$, then M is CM.
- b) If $\dim_R(M) = 1$, then M is CM if and only if $\Gamma_{R_+}(M) = 0$.
- c) (*Base Field Change Property*) Let $d \in \mathbb{N}_0$, let K' be a field extension of K and consider the Noetherian homogeneous K' -algebra $R' := K' \otimes_K R$ and the finitely generated graded R' -module $M' = R' \otimes_K M$. Then M' is CM of dimension d if and only if M is CM of dimension d .
- d) (*Base Ring Independence*) If $\mathfrak{b} \subseteq R$ is a graded ideal with $\mathfrak{b}M = 0$, then M is CM as an R/\mathfrak{b} -module if and only if it is as an R -module.
- e) Let $n \in \mathbb{Z}$. Then $M(n)$ is CM if and only if M is.
- f) Let $(M^{(i)})_{i \in \{1,2,\dots,s\}}$ be a finite family of finitely generated graded R -modules $M^{(i)}$ which have all the same dimension. Then $\bigoplus_{i \in \{1,2,\dots,s\}} M^{(i)}$ is CM if and only if all the $M^{(i)}$ are.
- g) Let $d = \dim_R(M)$. Then the following statements are equivalent:
 - (i) There is an M -sequence f_1, f_2, \dots, f_d consisting of elements $f_i \in R_+$
 - (ii) Each homogeneous system of parameters with respect to M is an M -sequence.
 - (iii) Each Filter-regular sequence f_1, f_2, \dots, f_s with $s \in \{1, 2, \dots, d\}$ is an M -sequence

C) (*Homogeneous Cohen-Macaulay-Algebras*) Let R be as in part A), we say that R is a (*homogeneous*) *CM-algebra (over K)* if R is CM as an R -module.

Prove the following facts.

- a) If $\dim(R) = 0$, R is CM.
- b) If $\dim(R) = 1$, then R is CM if and only if $\Gamma_{R_+}(R) = 0$.
- c) (*Base Field Change Property*) Let K' be an extension field of K and let $d \in \mathbb{N}_0$. Then, the following statements are equivalent
 - (i) R is CM and of dimension d .

- (ii) The Noetherian homogeneous K' -algebra $R' := K' \otimes_K R$ is CM and of dimension d .
- c) If $f_1, f_2, \dots, f_s \in R_+$ is a homogeneous R -sequence then R is CM if and only if $R/\langle f_1, f_2, \dots, f_s \rangle$ is.
- d) If R is CM of dimension d , then each graded free R -module $F = \bigoplus_{i=1}^k R(-a_i)$ of rank $k > 0$ is CM of dimension d .
- e) If $R = K[X_1, X_2, \dots, X_r]$ is a polynomial ring, it is CM.

We now give an estimate on the length, or –equivalently– the vector space dimension, or – also equivalently– the Hilbert-Serre multiplicity (see (4.3)A)) of torsion modules over Noetherian homogeneous K -algebras.

6.4. Proposition. *Let $t \in \mathbb{N}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra which is CM and let $M \neq 0$ be a finitely generated graded R_+ -torsion R -module. Let $\mathfrak{a} \subseteq R$ be a graded ideal such that $\mathfrak{a}M = 0$, $\sqrt{\mathfrak{a}} = R_+$ and $\text{gendeg}(\mathfrak{a}) \leq t$. Then*

$$\text{mult}(M) = \dim_K(M) \leq \text{mult}(R)\mu_R(M)t^{\dim(R)}.$$

Proof. We set

$$e := \text{mult}(R), \quad d := \dim(R), \quad k := \mu_R(M).$$

Clearly $e > 0$. As $M \neq 0$, we have $k > 0$.

Now, let K' be an infinite extension field of K and consider the Noetherian homogeneous K' -algebra $R' = K' \otimes_K R$, the graded ideal $\mathfrak{a}' = K' \otimes_K \mathfrak{a} \subseteq R'$ and the finitely generated graded R' -module $M' = K' \otimes_K M$. Observe that R' is again CM of dimension d (see (6.3)C)c)) and $\text{mult}(R') = \text{mult}(R) = e$ (see (4.3)C)b)). By (6.2)B)a) we have $\mu_{R'}(M') = \mu_R(M) = k$. Moreover, by (4.3)C)b) it also follows that $\dim_{K'}(M') = \dim_K(M)$. So, as in the proof of (6.1) we may replace R , \mathfrak{a} and M respectively by R' , \mathfrak{a}' and M' and hence assume that K is infinite.

As M is generated by k homogeneous elements, there is again an exact sequence of graded R -modules

$$0 \rightarrow N \xrightarrow{\subseteq} F \rightarrow M \rightarrow 0, \quad F = \bigoplus_{i=1}^k R(-a_i)$$

in which F is a graded free R -module of rank k and $\mathfrak{a}F \subseteq N$. In particular we have $\text{mult}(F) = \text{mult}(R)k = ek$ (see (4.6)A)e)). Moreover by (6.3)C)e) the module F is CM of dimension d .

By (5.6), applied to the R -module F , we find elements $f_1, f_2, \dots, f_d \in \mathfrak{a}_t$ which constitute a filter-regular sequence with respect to F . But now, by (6.3)B)g) f_1, f_2, \dots, f_d is an F -sequence. We set $G := F/\langle f_1, f_2, \dots, f_d \rangle F$. Then (5.21) implies that $\dim_K(G) = \text{mult}(F)t^d = ekt^d$. As $\langle f_1, f_2, \dots, f_d \rangle F \subseteq \mathfrak{a}F \subseteq N$

we have an epimorphism of graded R -modules $G \rightarrow M \rightarrow 0$ and thus get $\dim_K(M) \leq \dim_K(G) \leq ekt^d$. \square

Now, we want to study the length of some particular torsion modules, the so called *filter kernels*, which we will define now.

6.5. Definition. (*Filter Kernels*) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, let M be a finitely generated graded R -module and let $t \in \mathbb{N}$ and let $f \in R_t$ be a filter-regular element with respect to M . The graded R_+ -torsion submodule $(0 :_M f)$ is called the *filter kernel* of M with respect to f .

We also introduce another numerical invariant of graded modules.

6.6. Exercise and Definition. (*Span of Graded Torsion-Modules*) A) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring and let M be finitely generated graded R -module. We define the *span* of M by

$$\text{span}(M) := \begin{cases} 0, & M = 0 \\ \text{end}(M) - \text{beg}(M) + 1, & M \neq 0 \end{cases}$$

B) Let the notations and hypotheses be as in part A), let $t \in \mathbb{N}$, let $f \in R_t$ and let M in addition be R_+ -torsion. Show that

$$f^n M = 0, \forall n \geq \frac{\text{span}(M)}{t}.$$

6.7. Notation.

$$[a]^+ := \min\{n \in \mathbb{Z} \mid n \geq a\}, \quad (a \in \mathbb{R}).$$

6.8. Lemma. Let $t \in \mathbb{N}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian graded K -algebra, let M be a finitely generated R -module and let $f \in R_t$ be filter-regular with respect to M . Then

- $\dim_K(0 :_M f) \leq \dim_K(H_{R_+}^0(M/fM))$.
- $\dim_K(H_{R_+}^0(M)) \leq \left[\frac{\text{span}(H_{R_+}^0(M))}{t} \right]^+ \dim_K(H_{R_+}^0(M/fM))$.

Proof. "a)": As f is filter-regular with respect to M we have

$$(H_{R_+}^0(M) :_M f) = H_{R_+}^0(M)$$

and hence $fM \cap H_{R_+}^0(M) = f(H_{R_+}^0(M) :_M f) = fH_{R_+}^0(M)$, so that

$$(H_{R_+}^0(M) + fM)/fM \cong H_{R_+}^0(M)/fH_{R_+}^0(M).$$

As $(H_{R_+}^0(M) + fM)/fM \subseteq H_{R_+}^0(M/fM)$ we thus get a monomorphism of graded R -modules

$$H_{R_+}^0(M)/fH_{R_+}^0(M) \hookrightarrow H^0(M/fM)$$

and therefore

$$\dim_K(H_{R_+}^0(M)/fH_{R_+}^0(M)) \leq \dim_K(H_{R_+}^0(M/fM)).$$

Moreover, the exact sequence

$$0 \rightarrow (0 :_M f) \rightarrow H_{R_+}^0(M) \xrightarrow{f} (fH_{R_+}^0(M))(t) \rightarrow 0$$

implies

$$\begin{aligned} \dim_K(0 :_M f) &= \dim_K(H_{R_+}^0(M)) - \dim_K(fH_{R_+}^0(M)) \\ &= \dim_K(H_{R_+}^0(M)/fH_{R_+}^0(M)). \end{aligned}$$

Together with the above inequality, this proves our claim.

"b)": We set:

$$m := \left[\frac{\text{span}(H_{R_+}^0(M))}{t} \right]_+.$$

By (6.6)B) we then have $f^m H_{R_+}^0(M) = 0$. Therefore

$$\dim_K(H_{R_+}^0(M)) = \sum_{n=0}^{m-1} \dim_K(f^n H_{R_+}^0(M)/f^{n+1} H_{R_+}^0(M)).$$

In view of the epimorphism of graded R -modules

$$H_{R_+}^0(M)/fH_{R_+}^0(M) \xrightarrow{f^n} (f^n H_{R_+}^0(M)/f^{n+1} H_{R_+}^0(M))(nt) \rightarrow 0$$

we have an epimorphism of K -vector spaces

$$H_{R_+}^0(M)/fH_{R_+}^0(M) \rightarrow (f^n H_{R_+}^0(M)/f^{n+1} H_{R_+}^0(M)) \rightarrow 0,$$

so that

$$\dim_K(f^n H_{R_+}^0(M)/f^{n+1} H_{R_+}^0(M)) \leq \dim_K(H_{R_+}^0(M)/fH_{R_+}^0(M)), \quad \forall n \in \mathbb{N}_0$$

and hence

$$\dim_K(H_{R_+}^0(M)) \leq m \dim_K(H_{R_+}^0(M)/fH_{R_+}^0(M)).$$

In view of the inequality used in the proof of statement a), we now get our claim. \square

6.9. Lemma. *Let $r \in \mathbb{N}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let $M \neq 0$ be a finitely generated graded R -module with $\dim_R(M) = r > 0$ and let $f_1, f_2, \dots, f_r \in R_1$ be a filter-regular sequence with respect to M . Set $k := \dim_K(M/\sum_{j=1}^r f_j M)$.*

$$\text{a) } \dim_K(H_{R_+}^0(M)) \leq k \prod_{i=0}^{r-1} \text{span}(H_{R_+}^0(M/\sum_{j=1}^i f_j M)).$$

$$\text{b) } \dim_K(0 :_M f_1) \leq k \prod_{i=1}^{r-1} \text{span}(H_{R_+}^0(M/\sum_{j=1}^i f_j M)).$$

Proof. First of all observe that by (5.5) the R -module $M/\sum_{j=1}^r f_j M$ has dimension 0, so that $k \in \mathbb{N}$.

"a)": We proceed by induction on r . So, let $r = 1$. Then, $M/f_1 M = H_{R_+}^0(M/f_1 M)$ and by (6.8)b) it follows

$$\dim_K(H_{R_+}^0(M)) \leq \left[\frac{\text{span}(H_{R_+}^0(M))}{1} \right]^+ \dim_K(M/f_1 M) = k \text{span}(H_{R_+}^0(M)).$$

This is precisely our claim for $r = 1$.

Now, let $r > 1$. we set $\bar{M} := M/f_1 M$. Then $\dim_R(\bar{M}) = r - 1$ and $f_2, f_3, \dots, f_r \in R_1$ form a filter-regular sequence with respect to \bar{M} . Moreover, for each $i \in \{1, 2, \dots, r\}$ there is an isomorphism of graded R -modules $\bar{M}/\sum_{j=1}^i f_j \bar{M} \cong M/\sum_{j=1}^i f_j M$. So, if we apply induction to the R -module \bar{M} and the sequence $f_2, f_3, \dots, f_r \in R_1$ we get

$$\dim_K(H_{R_+}^0(\bar{M})) \leq k \prod_{i=1}^{r-1} \text{span}(H_{R_+}^0(M/\sum_{j=1}^i f_j M)).$$

By (6.8)b) we also have

$$\dim_K(H_{R_+}^0(M)) \leq \text{span}(H_{R_+}^0(M)) \dim_K(H_{R_+}^0(\bar{M})).$$

Both inequalities together give our claim.

"b)": First let $r = 1$. Then $M/f_1 M$ is R_+ -torsion, and so (6.8)a) yields that $\dim_K(0 :_M f_1) \leq \dim_K(M/f_1 M) = k$ and this is our claim. So, let $r > 1$. We set again $\bar{M} := M/f_1 M$. Then (6.8)a) gives us

$$\dim_K(0 :_M f_1) \leq \dim_K(H_{R_+}^0(\bar{M})).$$

If we apply our statement a) to the $(r - 1)$ -dimensional finitely generated graded R -module \bar{M} and the sequence $f_2, f_3, \dots, f_r \in R_1$ and bear in mind the isomorphisms of graded R -modules $\bar{M}/\sum_{j=2}^i f_j \bar{M} \cong M/\sum_{j=1}^i f_j M$ for all $i \in \{1, 2, \dots, r\}$ we obtain

$$\dim_K(H_{R_+}^0(\bar{M})) \leq k \prod_{i=1}^{r-1} \text{span}(H_{R_+}^0(M/\sum_{j=1}^i f_j M)).$$

Our claim follows. \square

Now, we are ready to prove our main results on the lengths of filter kernels.

6.10. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K algebra which is CM and of dimension d . Let M be a finitely generated graded R -module of dimension $r > 0$, let $f_1, f_2, \dots, f_r \in R_1$ be a filter-regular sequence with respect to R and to M , let $\mathfrak{b} \subseteq (0 :_R M)$ be graded ideal such that $\sqrt{\mathfrak{b}} = \sqrt{(0 :_R M)}$ and let $t \in \mathbb{N}$ with $\text{gendeg}(\mathfrak{b}) \leq t$. Then*

$$\dim_K(0 :_M f_1) \leq \text{mult}(R) \mu_R(M) t^{d-r} \prod_{i=1}^{r-1} \text{span}(H_{R_+}^0(M/\sum_{j=1}^i f_j M)).$$

Proof. Observe first, that as in the proof of (6.4) we have $e := \text{mult}(R), m := \mu_R(M) > 0$. According to (5.19) the sequence $f_1, f_2, \dots, f_r \in R_1$ is an R -sequence. So the Noetherian homogeneous K -algebra $\bar{R} := R/\langle f_1, f_2, \dots, f_r \rangle$ is CM of dimension $d - r$ (see (6.3)C)c). By (5.21) we also have $\text{mult}(\bar{R}) = \text{mult}(R) = e$. As the sequence f_1, f_2, \dots, f_r is a saturated filter-regular sequence with respect to M , the module

$$\bar{M} := M / \sum_{j=1}^r f_j M = M / \langle f_1, f_2, \dots, f_r \rangle M$$

is R_+ -torsion (see (5.5)). As $(0 :_R M), \langle f_1, f_2, \dots, f_r \rangle \subseteq R_+$ and $\sqrt{\mathfrak{b}} = \sqrt{(0 :_R M)}$ we have (see (5.4)A))

$$R_+ = \sqrt{(0 :_R M) + \langle f_1, f_2, \dots, f_r \rangle} = \sqrt{\mathfrak{b} + \langle f_1, f_2, \dots, f_r \rangle}.$$

If we set $\bar{\mathfrak{b}} := \mathfrak{b}\bar{R}$ we thus get

$$\bar{\mathfrak{b}}\bar{M} = 0, \quad \sqrt{\bar{\mathfrak{b}}} = \bar{R}_+, \quad \text{gendeg}(\bar{\mathfrak{b}}) \leq t.$$

If we apply (6.4) to the \bar{R} -module \bar{M} and the ideal \mathfrak{b} we thus get

$$\dim_K(M / \sum_{j=1}^r f_j M) = \dim_K(\bar{M}) \leq emt^{d-r}.$$

Now, we may conclude by (6.9)b). □

6.11. Corollary. *Let the notations and hypotheses be as in (6.10). Then*

$$\dim_K(0 :_M f_1) \leq \text{mult}(R)\mu_R(M)t^{d-r} \prod_{i=1}^{r-1} (\text{reg}(M / \sum_{j=1}^i f_j M) - \text{beg}(M) + 1).$$

Proof. This follows immediately from (6.10) on use of the definition of span and regularity. □

Now we are heading for the main result of this section. We start with the following auxiliary results.

6.12. Lemma. *Let K be an infinite field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let V be a finitely generated graded R -module, let $M \subseteq V$ be a graded submodule and let $f \in R_1$ be filter-regular with respect to V/M and to V . Let $m \in \mathbb{Z}$ such that*

$$\max\{\text{gendeg}(M), \text{reg}(V) + 1, \text{reg}(M + fV)\} \leq m, \quad (M :_V f)_m = M_m.$$

Then $\text{reg}(M) \leq m$.

Proof. By our hypothesis we have (see (3.4))

$$\text{reg}(V) < m, \quad \text{gendeg}(M + fV) \leq \text{reg}(M + fV) \leq m.$$

If we apply the Bayer-Stillman Criterion (5.12) to the modules $M + fV \subseteq V$ we thus find an integer $r \in \mathbb{N}_0$ and elements $f_2, f_3, \dots, f_r \in R_1$ which are filter-regular with respect to V and such that with $f_1 := f$ we have

$$\left((M + f_1V + \sum_{j=2}^{i-1} f_jV) :_V f_i \right)_m = \left(M + f_1V + \sum_{j=2}^{i-1} f_jV \right)_m$$

for all $i \in \{2, 3, \dots, r\}$ and

$$\left(M + f_1V + \sum_{j=2}^r f_jV \right)_m = V_m.$$

As $(M :_V f_1)_m = M_m$. We thus get

$$\left((M + \sum_{j=1}^{i-1} f_jV) :_V f_i \right)_m = \left(M + \sum_{j=1}^{i-1} f_jV \right)_m, \quad \forall i \in \{1, 2, \dots, r\}.$$

We also may write

$$\left(M + \sum_{j=1}^r f_jV \right)_m = V_m.$$

As $f_1 \in R_1$ is filter-regular with respect to V we now may apply the criterion (5.12) in the opposite direction to the modules $M \subseteq V$ and obtain $\text{reg}(M) \leq m$. \square

6.13. Lemma. *Let K, R, V, M and $f \in R_1$ be as in (6.12). Then*

$$\text{reg}(M) \leq \max\{\text{gendeg}(M), \text{reg}(V) + 1, \text{reg}(M + fV)\} + \dim_K((M :_V f)/M).$$

Proof. Let

$$d := \max\{\text{gendeg}(M), \text{reg}(V) + 1, \text{reg}(M + fV)\}$$

and observe that $((M :_V f)/M) = (0 :_{V/M} f)$ is R_+ -torsion so that indeed $\dim_K((M :_V f)/M) \in \mathbb{N}_0$. Consequently, there is an integer

$$m \in \{d, d + 1, \dots, d + \dim_K((M :_V f)/M)\}$$

such that

$$(M :_V f)_m / M_m = ((M :_V f)/M)_m = 0.$$

Now, we conclude by (6.12). \square

Now we are ready to state and to prove the announced main result.

6.14. Theorem. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous CM-algebra over K with $\dim(R) = r > 0$. Let V be a finitely generated graded R -module, let $M \subsetneq V$ be a graded submodule and let $\mathfrak{b} \subseteq (M :_R V)$ be a graded ideal such that $\sqrt{\mathfrak{b}} = \sqrt{(M :_R V)}$. Let $t \in \mathbb{N}$ and $g, \rho \in \mathbb{Z}$ with*

$$\text{gendeg}(\mathfrak{b}) \leq t, \quad \text{gendeg}(M) \leq g, \quad \max\{\text{reg}(V), \text{gendeg}(V) + \text{reg}(R)\} \leq \rho.$$

Moreover set

$$e := \text{mult}(R), \quad b := \text{beg}(V), \quad \mu := \mu_R(V), \quad s := \dim_R(V/M).$$

Then

$$\text{a) } s = 0: \quad \text{reg}(M) \leq \rho + (t-1)r + 1.$$

$$\text{b) } s > 0: \quad \text{reg}(M) \leq [\max\{g, \rho + (t-1)(r-s) + 1\} + \mu e t^{r-s} - b]^{2^{s-1}} + b.$$

Proof. Once more, let K' be an infinite extension field of K , consider the Noetherian homogeneous K' -algebra $R' := K' \otimes_K R$, the graded ideal $\mathfrak{b}' := K' \otimes_K \mathfrak{b} \subseteq R'$ and the finitely generated graded R' -module $M' := K' \otimes_K M$. By the same arguments as performed already at several instances we may replace R , \mathfrak{b} and M respectively by R , \mathfrak{b}' and M' and hence assume that K is infinite.

"a)": If $s = 0$, the module V/M is R_+ -torsion and so (6.1) yields

$$\begin{aligned} \text{reg}(V/M) &= \text{end}(V/M) \leq \text{reg}(R) + \text{gendeg}(V/M) + (t-1)r \\ &\leq \text{reg}(R) + \text{gendeg}(V) + (t-1)r \leq \rho + (t-1)r. \end{aligned}$$

Now, in view of the short exact sequence of graded R -modules

$$0 \rightarrow M \rightarrow V \rightarrow V/M \rightarrow 0$$

we get (see (3.3)C)a))

$$\text{reg}(M) \leq \max\{\text{reg}(V), \text{reg}(V/M)+1\} \leq \max\{\rho, \rho+(t-1)r+1\} = \rho+(t-1)r+1.$$

"b)": So, let $s > 0$. As

$$\begin{aligned} (M(b) :_R V(b)) &= (M :_R V), & \text{gendeg}(M(b)) &= \text{gendeg}(M) - b \leq g - b, \\ \text{reg}(V(b)) &= \text{reg}(V) - b, & \text{gendeg}(V(b)) &= \text{gendeg}(V) - b, & \text{beg}(V(b)) &= 0, \\ \mu_R(V) &= \mu, & \dim_R(V(b)/M(b)) &= s \end{aligned}$$

we may replace M and V respectively by $M(b)$ and $V(b)$ and hence assume that $\text{beg}(V) = 0$. Now, with

$$A := [\max\{g, \rho + (t-1)(r-s) + 1\} + \mu e t^{r-s}]$$

we have to show that

$$\text{reg}(M) \leq A^{2^{s-1}}.$$

We proceed by induction on s . Assume first that $s = 1$. According to (5.6) (applied with $\mathfrak{a} = R_+$) we find some $f \in R_1$ which is filter-regular with respect to $R \oplus V \oplus V/M$. So, f is filter-regular with respect to R , V and V/M . Now $V/(M + fV) \cong (V/M)/f(V/M)$ is of dimension $s - 1 = 0$ (see (5.5)a)), and hence R_+ -torsion. Setting $\mathfrak{a} := \mathfrak{b} + \langle f \rangle$ we clearly have $\mathfrak{a}(V/(M + fV)) = 0$. Moreover $\text{gendeg}(\mathfrak{a}) \leq t$ and as f forms a saturated filter-regular sequence with respect to V/M (see (5.5)b) we also have (see (5.4)A))

$$\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b} + \langle f \rangle} = \sqrt{(M :_R V) + \langle f \rangle} = \sqrt{(0 :_R V/M) + \langle f \rangle} = R_+.$$

Now, by (6.1), and bearing in mind that $\text{gendeg}(V/(M + fV)) \leq \text{gendeg}(V)$ we get

$$\begin{aligned} \text{reg}(V/(M + fV)) &= \text{end}(V/(M + fV)) \\ &\leq \text{reg}(R) + \text{gendeg}(V) + (t - 1)r \leq \rho + (t - 1)r. \end{aligned}$$

So, as $\text{reg}(V) \leq \rho$, the exact sequence of graded R -modules

$$0 \rightarrow (M + fV) \rightarrow V \rightarrow V/(M + fV) \rightarrow 0$$

yields that

$$\text{reg}(M + fV) \leq \rho + (t - 1)r + 1.$$

Hence, by (6.13) and bearing in mind that $\text{reg}(V) \leq \rho$ we get

$$\text{reg}(M) \leq \max\{g, \rho + (t - 1)r + 1\} + \dim_K((M :_V f)/M).$$

As $(M :_V f)/M = (0 :_{V/M} f)$ it remains to show that

$$\dim_K(0 :_{V/M} f) \leq \mu e t^{r-1}.$$

To this end, we just apply 6.10 to the one-dimensional R -module V/M , the graded ideal $\mathfrak{b} \subseteq R$ and the sequence of length one which consists of the single element f . So, we are done in the case $s = 1$.

Now, let $s > 1$. By (5.6) (applied with $\mathfrak{a} = R_+$) we find a sequence of linear forms $f_1, f_2, \dots, f_s \in R_1$ which is filter-regular with respect to $R \oplus V \oplus V/M$. Clearly, this sequence is filter-regular with respect to R , V and V/M . Now, for each $i \in \{1, 2, \dots, s\}$, we consider the finitely generated graded R -modules

$$V^{(i)} := V / \sum_{j=1}^i f_j V, \quad M^{(i)} := (M + \sum_{j=1}^i f_j V) / \sum_{j=1}^i f_j V \subseteq V^{(i)}.$$

As $V^{(i)}/M^{(i)} \cong (V/M) / \sum_{j=1}^i f_j(V/M)$ and as the sequence f_1, f_2, \dots, f_s is filter-regular with respect to V/M we have (see (5.5))

$$\dim_R(V^{(i)}/M^{(i)}) = s - i, \quad \forall i \in \{1, 2, \dots, s\}.$$

Clearly we also have

$$\text{gendeg}(M^{(i)}) \leq \text{gendeg}(M), \quad \text{gendeg}(V^{(i)}) \leq \text{gendeg}(V), \quad \forall i \in \{1, 2, \dots, s\}.$$

By (5.3)c) (applied to the module V with $k = 0$) we also have

$$\text{reg}(V^{(i)}) \leq \text{reg}(V), \quad \forall i \in \{1, 2, \dots, s\}.$$

Finally, for each $i \in \{1, 2, \dots, s\}$ we consider the graded ideal

$$\mathfrak{b}^{(i)} := \mathfrak{b} + \langle f_1, f_2, \dots, f_i \rangle.$$

Then clearly

$$\mathfrak{b}^{(i)} \subseteq (M^{(i)} :_R V^{(i)}), \quad \text{gendeg}(\mathfrak{b}^{(i)}) \leq t, \quad \forall i \in \{1, 2, \dots, s\}.$$

Moreover, as $V^{(i)}/M^{(i)} \cong (V/M) / \sum_{j=1}^i f_j(V/M)$ we have

$$\sqrt{(M^{(i)} :_R V^{(i)})} = \sqrt{(0 :_R V^{(i)}/M^{(i)})}$$

$$\begin{aligned}
&= \sqrt{(0 :_R (V/M) / \sum_{j=1}^i f_j(V/M))} \\
&= \sqrt{(0 :_R V/M) + \langle f_1, f_2, \dots, f_i \rangle} = \sqrt{(M :_R V) + \langle f_1, f_2, \dots, f_i \rangle} \\
&= \sqrt{\mathfrak{b} + \langle f_1, f_2, \dots, f_i \rangle} = \sqrt{\mathfrak{b}^{(i)}}, \quad \forall i \in \{1, 2, \dots, s\}.
\end{aligned}$$

So, for each $i \in \{1, 2, \dots, s-1\}$ we may apply induction to the modules $M^{(i)} \subsetneq V^{(i)}$ and the ideal $\mathfrak{b}^{(i)} \subseteq R$ and get

$$\operatorname{reg}(M^{(i)}) \leq A^{2^{s-i-1}}, \quad \forall i \in \{1, 2, \dots, s-1\}.$$

Now, the short exact sequences of graded R -modules

$$0 \rightarrow M^{(i)} \rightarrow V^{(i)} \rightarrow V/(M + \sum_{j=1}^i f_j V) \rightarrow 0$$

and the fact that $\operatorname{reg}(V^{(i)}) \leq \operatorname{reg}(V) \leq \rho < A \leq A^{2^{s-i-1}}$ yield the inequalities (see (3.3)C)d)

$$\operatorname{reg}(V/(M + \sum_{j=1}^i f_j V)) \leq A^{2^{s-i-1}} - 1, \quad \forall i \in \{1, 2, \dots, s-1\}.$$

If we apply this for $i = 1$, we get

$$\operatorname{reg}(V/(M + f_1 V)) \leq A^{2^{s-2}} - 1.$$

So by means of the the short exact sequence of graded R -modules

$$0 \rightarrow (M + f_1 V) \rightarrow V \rightarrow V/(M + f_1 V) \rightarrow 0$$

and remembering that $\operatorname{reg}(V) < A^{2^{s-2}}$ we get (see (3.3)C)a)

$$\operatorname{reg}(M + f_1 V) \leq A^{2^{s-2}}.$$

Observe that for each $i \in \{1, 2, \dots, s-1\}$ there is an isomorphism of graded R -modules

$$(V/M) / \sum_{j=1}^i f_j(V/M) \cong V/(M + \sum_{j=1}^i f_j V),$$

so that we have

$$\operatorname{reg}((V/M) / \sum_{j=1}^i f_j(V/M)) \leq A^{2^{s-i-1}} - 1, \quad \forall i \in \{1, 2, \dots, s-1\}.$$

If we apply (6.10) to the s -dimensional R -module V/M , the ideal $\mathfrak{b} \subseteq R$ and the sequence $f_1, f_2, \dots, f_s \in R_1$ which is filter-regular with respect to R and to V/M and bear in mind that $(0 :_{V/M} f_1) = (M :_V f_1)/M$ we get

$$\dim_K((M :_V f_1)/M) \leq e\mu t^{r-s} \prod_{i=1}^{s-1} A^{2^{s-i-1}} = e\mu t^{r-s} A^{2^{s-1}-1}.$$

We now apply (6.13) with $f = f_1$ and obtain

$$\operatorname{reg}(M) \leq \max\{g, \operatorname{reg}(V) + 1, \operatorname{reg}(M + f_1V)\} + e\mu t^{r-s} A^{2^{s-1}-1}.$$

As

$$g, \operatorname{reg}(V) + 1, \quad 1 + e\mu t^{r-s} \leq A \leq A^{2^{s-2}}, \quad \operatorname{reg}(M + f_1V) \leq A^{2^{s-2}},$$

we thus get

$$\begin{aligned} \operatorname{reg}(M) &\leq A^{2^{s-2}} + e\mu t^{r-s} A^{2^{s-1}-1} = A^{2^{s-2}}(1 + e\mu t^{r-s} A^{2^{s-2}-1}) \\ &\leq A^{2^{s-2}}(1 + e\mu t^{r-s}) A^{2^{s-2}-1} \leq A^{2^{s-2}} A A^{2^{s-2}-1} = A^{2^{s-1}} \end{aligned}$$

and this is precisely our claim. \square

We now draw a number of conclusions from the above bounding result.

6.15. Corollary. *Let the notations and hypotheses be as in (6.14) and assume that $s < r \neq 1$. Then*

$$\operatorname{reg}(M) \leq [\max\{g, \rho + t\} + \mu et - b]^{2^{r-2}} + b.$$

Proof. As in the proof of (6.14) we may shift V appropriately in order to assume that $b = \operatorname{beg}(V) = 0$. Then clearly $\rho, g \geq 0$. We now set

$$B := \max\{g, \rho + t\} + \mu et$$

and must show that

$$\operatorname{reg}(M) \leq B^{2^{r-2}}.$$

Assume first that $s = 0$. Then by (6.14) we have

$$\operatorname{reg}(M) \leq \rho + (t - 1)r + 1.$$

If $r = 2$ we thus obtain

$$\operatorname{reg}(M) < \rho + 2t \leq \rho + t + \mu et \leq B$$

and our claim is shown if $s = 0$ and $r = 2$.

Now, still assume that $s = 0$, but let $r > 2$. Then we have $r - 1 \leq 2^{r-2}$ and thus may write

$$\begin{aligned} \operatorname{reg}(M) &< (\rho + t) + (r - 1)t \leq (\rho + t)^{2^{r-2}} + 2^{r-2}(\rho + t)^{2^{r-2}-1}t \\ &< ((\rho + t) + t)^{2^{r-2}} \leq ((\rho + t) + \mu et)^{2^{r-2}} \leq B^{2^{r-2}}. \end{aligned}$$

So, we have our claim if $s = 0$.

Now, let $s > 0$. Then $1 \leq c := r - s \leq r - 1$. We now set

$$A(c) := \max\{g, \rho + (t - 1)c + 1\} + \mu et^c.$$

In view of (6.14)b) it suffices to show that

$$A(c)^{2^{r-c-1}} \leq B^{2^{r-2}}.$$

We do this by induction on c . As $A(1) = B$, we are done in the case $c = 1$. So, let $2 \leq c \leq r - 1$. Then, we have

$$A(c) \leq t \max\{g, \rho + (t - 1)(c - 1) + 1\} + \mu e t^c \leq A(c - 1)^2$$

and hence

$$A(c)^{2^{r-c-1}} \leq A(c - 1)^{2^{r-(c-1)-1}}.$$

Now, we may conclude by induction. \square

Our next application concerns the case where the R -module V is free

6.16. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra of dimension $r > 0$ which is CM. Let F be a graded free R -module and let $M \subsetneq F$ be a graded submodule. Let $t \in \mathbb{N}$ and $g, \rho \in \mathbb{Z}$ with*

$$\text{gendeg}(M :_R F) \leq t, \quad \text{gendeg}(M) \leq g, \quad \text{gendeg}(F) + \text{reg}(R) \leq \rho.$$

Moreover set

$$e := \text{mult}(R), \quad b := \text{beg}(F), \quad \mu := \text{rank}(F), \quad s := \dim(F/M).$$

Then

- a) $s = 0$: $\text{reg}(M) \leq \rho + (t - 1)r + 1$.
- b) $s > 0$: $\text{reg}(M) \leq [\max\{g, \rho + (t - 1)(r - s) + 1\} + \mu e t^{r-s} - b]^{2^{s-1}} + b$.
- c) $s < r \neq 1$: $\text{reg}(M) \leq [\max\{g, \rho + t\} + \mu e t - b]^{2^{r-2}} + b$.

Proof. As

$$F = \bigoplus_{i=1}^{\mu} R(-a_i), \quad a_1 \leq a_2 \leq \dots \leq a_{\mu} = \text{gendeg}(F)$$

is a graded free R -module of finite rank, we have $\mu_R(F) = \text{rank}(F)$ (see (6.2)B)) and $\text{reg}(F) = \text{gendeg}(F) + \text{reg}(R)$ (see (4.6)a)). Now we may apply (6.14) and (6.15) with $\mathfrak{b} = (M :_R V)$. \square

As a special case of this latter result, we get back the bound of Galligo, Giusti and Caviglia-Sbarra mentioned at the beginning of this section.

6.17. Corollary. *Let $r > 1$ and let $0 \neq \mathfrak{a} \subsetneq K[X_1, X_2, \dots, X_r]$ be a non-zero graded ideal of a polynomial ring in r indeterminates over a field K . Then*

$$\text{reg}(\mathfrak{a}) \leq (2 \text{gendeg}(\mathfrak{a}))^{2^{r-2}}.$$

Proof. Apply (6.16)c) with $F = R = K[X_1, X_2, \dots, X_r]$ and $M = \mathfrak{a}$ bearing in mind that $\dim(R/\mathfrak{a}) < r$, $\text{reg}(R) = 0$ (see (3.5)) and $\text{mult}(R) = 1$. \square

6.18. **Remark.** A) (*Regularity and Generating Degrees of Ideals*) Bounds for the regularity of graded ideals in polynomial rings are a classical subject of Algebraic Geometry, which goes back much further than a well defined concept of regularity. The motivation to this was the controversy around the *Problem of the Finitely Many Steps* which was initiated by Hilbert's *Theory of Syzygies* [Hi1], [Hi2]. This controversy found its end through G. Hermann [Herm], who proved (in "syzygetic terms") that in the situation of (6.17) we have the following bound

$$\operatorname{reg}(\mathfrak{a}) \leq (2\operatorname{gendeg}(\mathfrak{a}))^{2^{(r-1)r}}.$$

In fact, the bound given in [Herm] is not calculated properly, and would give a smaller value on the right hand side. Putting Hermann's arguments right, one obtains the above estimate. Quite early the question came up, whether the "quadratically exponential bound" of Hermann could be replaced by a "linearly exponential bound", that is a bound of the form

$$\operatorname{reg}(\mathfrak{a}) \leq (2\operatorname{gendeg}(\mathfrak{a}))^{2^{Cr}}.$$

with some universal constant $C > 0$. Observe, that the bound of (6.17) has this property. As already pointed out, over base-fields of characteristic 0 such linearly exponential bounds were given already by Galligo [G] and Giusti [Gi], whereas over fields of arbitrary characteristic such bounds were given much later by Caviglia-Sbarra [Cav-Sb].

In between, there was a characteristic-free bound by Bayer-Mumford [B-Mu], which is considerably smaller than the bound of Hermann (as $(r-1)! \ll 2^{(r-1)r}$ for all $r \gg 0$), but still far away from being linearly exponential (as for all $C > 0$ one has $2^{Cr} \ll (r-1)!, \forall r \gg 0$), namely:

$$\operatorname{reg}(\mathfrak{a}) \leq (2\operatorname{gendeg}(\mathfrak{a}))^{(r-1)!}.$$

On the other hand it is known, that linearly exponential regularity bounds are best possible in general. Indeed according to Mayr-Meyer [Ma-Me] for each $r > 1$ there is a graded ideal $\mathfrak{a}^{(r)} \subseteq \mathbb{C}[X_1, X_2, \dots, X_r]$ such that

$$\operatorname{gendeg}(\mathfrak{a}^{(r)}) = 4, \quad \operatorname{reg}(\mathfrak{a}^{(r)}) > 8^{2^{\frac{r-2}{10}}}.$$

B) (*Generalizations to Modules*) In [Br-L2] we had to use an extension of the Bayer-Mumford bound to submodules of free modules. It says that for a non-zero graded submodule $M \subsetneq K[1, X_2, \dots, X_r]^{\oplus s}$ one has

$$\operatorname{reg}(M) \leq s^{r!} (2\operatorname{gendeg}(M))^{(r-1)!}.$$

In [Br5] we gave a generalization of this to arbitrary graded submodules of finitely generated graded submodules over Noetherian homogeneous rings R with Artinian local base ring R_0 . We do not spell out the further details of this result here. More general versions of (6.14), (16.17) and (16.18) are given in [Br-Gö]. In fact, also these bounds hold for Noetherian homogeneous rings R with Artinian local base ring R_0 . Finally, similar bounds (slightly sharper in

some cases) have been given by use of different methods by Chardin-Fall-Nagel [Ch-F-N].

In fact, our previous bounding results have further consequences. We begin with a bound in which the condition that R is CM is dropped.

6.19. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra with $r := \dim_K(R_1) > 1$, let V be a finitely generated graded R -module and let $M \subsetneq V$ be a graded submodule. Let $t \in \mathbb{N}$ and $g, \rho \in \mathbb{Z}$ with*

$$\max\{\operatorname{reg}(R) + 1, \operatorname{gendeg}(M :_R V)\} \leq t, \quad \operatorname{gendeg}(M) \leq g, \quad \operatorname{reg}(V) \leq \rho.$$

Moreover set

$$b := \operatorname{beg}(V), \quad \mu := \mu_R(V), \quad s := \dim_R(V/M).$$

Then

- a) $s = 0$: $\operatorname{reg}(M) \leq \rho + (t - 1)r + 1$
- b) $s > 0$: $\operatorname{reg}(M) \leq [\max\{g, \rho + (t - 1)(r - s) + 1\} + \mu t^{r-s} - b]^{2^{s-1}} + b.$
- c) $s < r$: $\operatorname{reg}(M) \leq [\max\{g, \rho + t\} + \mu t - b]^{2^{r-2}} + b.$

Proof. By our hypotheses there is a polynomial ring $S = K[X_1, X_2, \dots, X_r]$ and a graded ideal $\mathfrak{c} \subseteq S$ such that $R = S/\mathfrak{c}$. We also find a graded ideal $\mathfrak{b} \subseteq S$ with $\mathfrak{c} \subseteq \mathfrak{b}$ and $\mathfrak{b}/\mathfrak{c} = (M :_R V)$. Now, the short exact sequence of graded R -modules

$$0 \rightarrow \mathfrak{c} \rightarrow S \rightarrow R \rightarrow 0$$

together with the facts that $\operatorname{reg}(R)$ is also the regularity of the S -module R and that $\operatorname{reg}(S) = 0$ yields that $\operatorname{reg}(\mathfrak{c}) \leq \operatorname{reg}(R) + 1$ (see (3.3)C)a)). So, by (3.4) we obtain $\operatorname{gendeg}(\mathfrak{c}) \leq \operatorname{reg}(R) + 1$. Now, the short exact sequence of graded S -modules

$$0 \rightarrow \mathfrak{c} \rightarrow \mathfrak{b} \rightarrow (M :_R V) \rightarrow 0$$

shows that (see (3.3)A)b))

$$\operatorname{gendeg}(\mathfrak{b}) \leq \max\{\operatorname{reg}(R) + 1, \operatorname{gendeg}(M :_R V)\} \leq t.$$

Observe also that

$$\mathfrak{b} = (M :_S V).$$

and that the invariants

$$\operatorname{gendeg}(V) \leq \operatorname{reg}(V), \quad \operatorname{beg}(V), \quad \mu(V), \quad \operatorname{gendeg}(M), \quad \operatorname{reg}(M), \quad \dim(V/M)$$

remain the same if we consider V and M as S -modules. So, if we apply (6.14) and (6.15) to these S -modules and the above ideal $\mathfrak{b} \subseteq S$ and keep in mind that $\dim(S) = r$, $\operatorname{reg}(S) = 0$, $\operatorname{mult}(S) = 1$, we get our claim. \square

Our next application concerns the case where the base ring R is CM, but the annihilator of V/M is unknown.

6.20. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous CM-algebra over K with $\dim(R) = r > 0$, let V be a finitely generated graded R -module, let $M \subsetneq V$ be a graded submodule and set*

$$\alpha := \min\{\text{beg}(V), \text{reg}(V) - \text{reg}(R)\}, \quad \rho := \max\{\text{gendeg}(M), \text{reg}(V) + 1\},$$

$$e := \text{mult}(R), \quad \mu := \mu_R(V).$$

Then

$$\text{reg}(M) \leq [\rho + (\mu + 1)e - \alpha]^{2^{r-1}} + \alpha.$$

Proof. Consider M as a graded submodule of $W := V \oplus R(-\alpha)$ and observe that

$$\mu_R(W) = \mu + 1, \quad \text{beg}(W) = \alpha, \quad \dim_R(W/M) = r, \quad (M :_R W) = 0,$$

$$\text{reg}(W) = \max\{\text{reg}(V), \text{reg}(R(-\alpha))\} = \max\{\text{reg}(V), \text{reg}(R) + \alpha\} = \text{reg}(V).$$

Then apply (6.14)b) with $\mathfrak{b} = 0$ and $t = 1$. \square

The next application deals with a similar situation as (6.20), but without requiring that the K -algebra R is CM.

6.21. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra with $\dim_K(R_1) = r > 1$, let V be a finitely generated graded R -module, let $M \subsetneq V$ be a graded submodule and set*

$$\mu := \mu_R(V), \quad \alpha := \min\{\text{beg}(V), \text{reg}(V) - \text{reg}(R)\},$$

$$\sigma := \max\{\text{gendeg}(M), \text{reg}(R) + \text{reg}(V) + 1\}.$$

Then

$$\text{reg}(M) \leq [\sigma + \mu + 1 - \alpha]^{2^{r-1}} + \alpha.$$

Proof. This is shown similar as (6.20): namely, consider M as a graded submodule of $W = V \oplus R(-\alpha)$, observe that

$$\mu_R(W) = \mu + 1, \quad \text{beg}(W) = \alpha, \quad \dim_R(W/M) = r, \quad (M :_R W) = 0$$

and apply (6.19)b) with $\mathfrak{b} = 0$. \square

Now, we turn to regularity bounds in terms of discrete data of a presentation of a module.

6.22. Theorem. *Let K be a field and let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra. Let*

$$\bigoplus_{j=1}^{\nu} R(-b_j) \xrightarrow{h} \bigoplus_{i=1}^{\mu} R(-a_i) \rightarrow M \rightarrow 0$$

be an exact sequence of graded R -modules, such that $h \neq 0$ and with integers

$$b_1 \leq b_2 \leq \dots \leq b_{\nu}, \quad a_1 \leq a_2 \leq \dots \leq a_{\mu}$$

and set

$$\mu^* := \max\{i \in \{1, 2, \dots, \mu\} \mid a_i \leq b_{\nu}\}.$$

Then

a) If R is CM, with $r := \dim(R) > 0$, $e := \text{mult}(R)$ and $\rho := \text{reg}(R)$ we have

$$\text{reg}(M) \leq \max\{a_\mu + \rho, [b_\nu + \rho + 1 + (\mu^* + 1)e - a_1]^{2^{r-1}} + a_1 - 1\}.$$

b) If $r := \dim_K(R_1) > 1$ (and R is not necessarily CM), then with $\rho = \text{reg}(R)$ we have

$$\text{reg}(M) \leq \max\{a_\mu + \rho, [b_\nu + 2\rho + \mu^* + 2 - a_1]^{2^{r-1}} + a_1 - 1\}.$$

Proof. “a)”: As $h \neq 0$ we have $a_1 \leq b_\nu$ so that $\mu^* \in \mathbb{N}$. We set

$$W := \sum_{j=1}^{\nu} R(-b_j), \quad V := \sum_{i=1}^{\mu^*} R(-a_i), \quad F := \sum_{i=1}^{\mu} R(-a_i).$$

Clearly the map h factors through the submodule V of F , so that $\text{Im}(h) \subseteq V$. Observe that

$$\text{gendeg}(\text{Im}(h)) \leq \text{gendeg}(W) = b_\nu, \quad \text{reg}(V) = a_{\mu^*} + \text{reg}(R) \leq b_\nu + \text{reg}(R),$$

$$\mu_R(V) = \mu^*$$

and

$$\alpha := \min\{\text{beg}(V), \text{reg}(V) - \text{reg}(R)\} = \min\{a_1, a_{\mu^*} + \text{reg}(R) - \text{reg}(R)\} = a_1.$$

If $\text{Im}(h) \subsetneq V$, we apply (6.20) to the pair of graded modules $\text{Im}(h) \subsetneq V$ and obtain

$$\text{reg}(\text{Im}(h)) \leq [b_\nu + \text{reg}(R) + 1 + (\mu^* + 1)e - a_1]^{2^{r-1}} + a_1.$$

If $\text{Im}(h) = V$, this inequality is obvious. As $\text{reg}(F) = a_\mu + \text{reg}(R)$ we now get our claim by (3.3)C)d) and the exact sequence of graded R -modules

$$0 \rightarrow \text{Im}(h) \rightarrow F \rightarrow M \rightarrow 0.$$

“b)”: We may argue similar as in the proof of statement a), namely: again we may assume that $\text{Im}(h) \subsetneq V$ and use (6.21) together with the previously observed facts that $\text{gendeg}(\text{Im}(h)) \leq b_\nu$, $\text{reg}(V) \leq b_\nu + \text{reg}(R)$, $\alpha = a_1$, and $\mu_R(V) = \mu^*$ to see that

$$\text{reg}(\text{Im}(h)) \leq [b_\nu + 2\text{reg}(R) + \mu^* + 2 - a_1]^{2^{r-1}} + a_1,$$

which is obviously true if $\text{Im}(h) = V$. Then we conclude once more with the exact sequence of graded R -modules

$$0 \rightarrow \text{Im}(h) \rightarrow F \rightarrow M \rightarrow 0.$$

□

Our final result in this section is an application to the classical case in which R is a polynomial ring over a field. It says, that the discrete data of a presentation of a graded module over a polynomial ring bound the regularity of this module in a “linearly exponential way”. We shall give a more detailed explanation of this statement in final remark of the present section.

6.23. Corollary. *Let $r \in \mathbb{N}$, let K be a field, let $R = K[X_1, X_2, \dots, X_r]$ be a polynomial ring and let*

$$\bigoplus_{j=1}^{\nu} R(-b_j) \xrightarrow{h} \bigoplus_{i=1}^{\mu} R(-a_i) \rightarrow M \rightarrow 0$$

be an exact sequence of graded R -modules such that $h \neq 0$ and with integers

$$b_1 \leq b_2 \leq \dots \leq b_{\nu}, \quad a_1 \leq a_2 \leq \dots \leq a_{\mu}$$

and set

$$\mu^* := \max\{i \in \{1, 2, \dots, \mu\} \mid a_i \leq b_{\nu}\}.$$

Then

$$\operatorname{reg}(M) \leq \max\{a_{\mu}, [b_{\nu} + \mu^* + 2 - a_1]^{2^{r-1}} + a_1 - 1\}.$$

Proof. Apply (6.22)a) and keep in mind that $\operatorname{reg}(R) = 0$ and $\operatorname{mult}(R) = 1$. \square

6.24. Remark and Exercise. (*Presentations of Graded Modules*). A) Let $r \in \mathbb{N}$, let $R = K[X_1, X_2, \dots, X_r]$ be a polynomial ring over the field K and let $M \neq 0$ be a finitely generated graded R -module. By a *presentation* of M we mean an exact sequence of graded R -modules

$$\bigoplus_{j=1}^{\nu} R(-b_j) \xrightarrow{h} \bigoplus_{i=1}^{\mu} R(-a_i) \rightarrow M \rightarrow 0$$

with integers

$$b_1 \leq b_2 \leq \dots \leq b_{\nu}, \quad a_1 \leq a_2 \leq \dots \leq a_{\mu}.$$

We call the two sequences

$$(b_j)_{j=1}^{\nu}, \quad (a_i)_{i=1}^{\mu}$$

the *degree sequences* of the given presentation of M . These two sequences are considered as the *discrete data* of our presentation. Observe that $M = \operatorname{Coker}(h)$, so that M is indeed determined by the homomorphism of graded free R -modules

$$G := \bigoplus_{j=1}^{\nu} R(-b_j) \xrightarrow{h} \bigoplus_{i=1}^{\mu} R(-a_i) =: F$$

B) Keep the above hypotheses and notations and let

$$e_j^G := (\delta_{lj}(-b_j))_{l=1}^{\nu} \in G, \quad j \in \{1, 2, \dots, \nu\}$$

$$e_i^F := (\delta_{ki}(-a_i))_{k=1}^{\mu} \in F, \quad i \in \{1, 2, \dots, \mu\}$$

(with δ_{lk} the Kronecker symbol) be the canonical basis elements of the graded free R -modules G and F respectively. Then the map $h : G \rightarrow F$ is uniquely determined by a matrix

$$A = A[h] = [f_{ij} \mid 1 \leq i \leq \mu, 1 \leq j \leq \nu] \in R^{\mu \times \nu}$$

with

- a) $f_{ij} \in R_{b_j - a_i}, \quad \forall (i, j) \in \{1, 2, \dots, \mu\} \times \{1, 2, \dots, \nu\}$.
 b) $h(\sum_{j=1}^{\nu} u_j e_j^G) = \sum_{i=1}^{\mu} (\sum_{j=1}^{\nu} f_{ij} u_j) e_i^F, \quad \forall (u_1, u_2, \dots, u_{\nu}) \in R^{\nu}$.

The matrix $A[h] = A$ is called a *presentation matrix with respect to the degree sequences* $(b_j)_{j=1}^{\nu}$ and $(a_i)_{i=1}^{\mu}$ or more precisely *the presentation matrix* of h . If we fix our degree sequences, the assignment $h \mapsto A[h]$ yields a bijection between the set of graded homomorphisms h from G to F and the set of matrices $R^{\mu \times \nu}$ with homogeneous entries which satisfy the above requirement a). The presentation matrix $A[h]$ expresses the *continuous data* of our presentation.

C) Keep the above notations and hypothesis. Our bounding result (6.23) says that there is an upper bound on the regularity of M , linearly exponential in the number of indeterminates and only depending on the two degree-sequences of our presentation. In fact, in [Br-Gö] there is shown a more general result than (6.22), and a more general version of (6.23) is drawn there as a consequence. Observe also, that our bounding results (6.22) and (6.23) use only little information on the degree sequences of the given presentation. A finer bound, but using more information on these sequences has been shown in [Ch-F-N].

D) Finally let us remark, that with our result (6.23) we are back to the core of the classical controversy around the problem of the finitely many steps. Namely, still in our above notations and also the notation of (6.23) we can use the short exact sequence

$$0 \rightarrow \text{Ker}(h) \xrightarrow{\subseteq} G \rightarrow \text{Im}(h) \rightarrow 0$$

and the bound on $\text{reg}(\text{Im}(h))$ given in the proof of (6.22) to see that

$$\text{reg}(\text{Ker}(h)) \leq [b_{\nu} + \mu^* + 2 - a_1]^{2^{r-1}} + a_1.$$

As a consequence of this we get

$$\text{gendeg}(\text{Ker}(h)) \leq [b_{\nu} + \mu^* + 2 - a_1]^{2^{r-1}} + a_1.$$

This type of bound is indeed the crucial point in the positive answer of the problem of the finitely many steps. We namely can say in particular, that the generating degree of $\text{Ker}(h)$ is bounded in terms of the beginning of F , the rank of F and the generating degree of G .

E) We now come to the exercise part. Keep the above notations and hypotheses. From the last bound in part D) one may conclude the following:

a) If $g : R^\nu \rightarrow R^\mu$ is the R -linear map, given by a matrix

$$B = [b_{kl} \mid 1 \leq k \leq \mu, 1 \leq l \leq \nu] \in R^{\mu \times \nu},$$

whose entries $b_{kl} \in R$ are homogeneous polynomials, the generating degree

$$\min\{\max_{1 \leq i \leq t, 1 \leq j \leq \nu}(\deg(f_{ij})) \mid f_{ij} \in R : \text{Ker}(g) = \sum_{i=1}^t R(f_{i1}, f_{i2}, \dots, f_{i\nu})\}$$

of $\text{Ker}(g)$ is bounded only by the size (μ, ν) of B and the (maximal) degree

$$\deg(B) := \max_{1 \leq k \leq \mu, 1 \leq l \leq \nu}(\deg(b_{kl}))$$

of all entries of B .

b) In the situation of statement a), one even may drop the condition that the entries of B are homogeneous.

Indeed G. Hermann [Herm] has established a corresponding bound (not computed correctly, as mentioned already earlier). This type of bounds is also of great interest in the more general situation where the base field K is replaced by an appropriate ring. These more general bounds also apply in *Algebraic Number Theory* or *Arithmetic Geometry*, (see [Mas-W] for example).

7. PRESENTATIONS AND RESOLUTIONS

Towards the end of the last section we were lead to consider presentations of graded modules and hence dropped on a fundamental concept of Commutative Algebra and Algebraic Geometry. In particular the computational aspect of these theories is closely related to the notion of presentation and its natural extension, namely the notion of resolution.

Starting from the concept of *minimal resolution* we shall define the notion of *Betti vector* and of *homological dimension* of a finitely generated graded module M over a Noetherian homogeneous ring R with local base ring (R_0, \mathfrak{m}_0) and relate the ends of the Betti vectors of M to the Castelnuovo-Mumford regularity $\text{reg}(M)$ of M . This relation becomes particularly simple for finitely generated graded modules of finite homological dimension over a standard graded polynomial ring.

Here we also naturally shall be lead to prove Hilbert's "*Syzygiensatz*" for finitely generated graded modules $M \neq 0$ over a standard graded polynomial ring $R = K[X_1, X_2, \dots, X_r]$ over a field K - whose essential statement is that in this situation M has finite homological dimension. The main ingredient of our proof is the fact, that the (*graded*) *maximal Cohen-Macaulay modules* over the polynomial ring R are precisely the graded free R -modules of finite positive rank - and hence may be characterized in terms of the vanishing of the local cohomology modules $H_{R_+}^i(M)$ for $i = 1, 2, \dots, r - 1$.

Once having established this cohomological criterion the for freeness of graded modules over the polynomial ring $R = K[X_1, X_2, \dots, X_r]$ we can dare a detour in order to through a glance to *algebraic vector bundles* over projective spaces and to prove the *Splitting Criterion of Horrocks* and the *Splitting Theorem of Grothendieck* for such bundles. On our way to this, we also extend the *Vanishing Theorem of Severi-Enriques-Zariski-Serre* to projective schemes over arbitrary fields and rephrase it as a *Criterion for a Coherent Sheaf to be an Algebraic Vector Bundle* over a regular irreducible projective scheme.

Finally, we return to the computational significance of regularity in a number of conclusive remarks, by retrospecting once more the *Problem of the Finitely Many Steps*.

7.1. Exercise and Definition. A) (*Homogeneous Rings with Local Base Rings*) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) . Keep in mind that

$$\mathfrak{m} := \mathfrak{m}_0 \oplus R_+ = \mathfrak{m}_0 \oplus R_1 \oplus R_2 \dots$$

is the *unique graded maximal ideal of R* and that there is a canonical isomorphism of fields

$$R_0/\mathfrak{m}_0 \cong R/\mathfrak{m}.$$

B) (*Minimal Systems of Homogeneous Generators*) Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R -module, let $r \in \mathbb{N}$, let $d_1 \leq d_2 \leq \dots \leq d_r$ be integers and let $(m_i)_{i=1}^r$ be a family of homogeneous elements such that $m_i \in M_{d_i}$ for all $i \in \{1, 2, \dots, r\}$. We use the notational convention that $R_m = 0$ for all $m < 0$. Show that the following statements are equivalent

- (i) $M = \sum_{i=1}^r Rm_i$.
- (ii) $M_n = \sum_{i=1}^r R_{n-d_i}m_i$ for all $n \in \mathbb{Z}$.
- (iii) $M_n = \mathfrak{m}_0M_n + \sum_{i=1}^r R_{n-d_i}m_i$ for all $n \in \mathbb{Z}$.

Conclude that

- a) The elements $m_1, m_2, \dots, m_r \in M$ form a minimal system of homogeneous generators of the graded R -module M if and only if the classes $m_1 + \mathfrak{m}_0M, m_2 + \mathfrak{m}_0M, \dots, m_r + \mathfrak{m}_0M \in M/\mathfrak{m}_0M$ form a minimal homogeneous system of generators of the graded R/\mathfrak{m}_0R -module M/\mathfrak{m}_0M .
- b) The elements $m_1, m_2, \dots, m_r \in M$ form a minimal system of homogeneous generators of the graded R -module M if and only if the classes $m_1 + \mathfrak{m}M, m_2 + \mathfrak{m}M, \dots, m_r + \mathfrak{m}M \in M/\mathfrak{m}M$ form a basis of the R/\mathfrak{m} (and hence R_0/\mathfrak{m}_0)-vector space $M/\mathfrak{m}M$.

C) (*Vectors of Generating Degrees*) Keep the previous notations and hypotheses. For each integer n we consider the non-negative integer

$$\mu_{R,n}(M) = \mu_n(M) := \dim_{R_0/\mathfrak{m}_0}((M/\mathfrak{m}M)_n),$$

which we call the *minimal number of homogeneous generators of M in degree n* . Moreover we consider the family on non-negative integers

$$\mu_{R,*}(M) = \mu_*(M) := (\mu_{R,n}(M))_{n \in \mathbb{Z}},$$

which we call the *vector of generating degrees of M* . Prove the following statements:

- a) If m_1, m_2, \dots, m_r is a minimal homogeneous system of generators of the finitely generated graded R -module M , then

$$\mu_{R,n}(M) = \#\{i \in \mathbb{Z} \mid m_i \in M_n\}, \quad \forall n \in \mathbb{Z}.$$

- b) $\inf\{n \in \mathbb{Z} \mid \mu_{R,n}(M) \neq 0\} = \text{beg}(M)$.
- c) $\sup\{n \in \mathbb{Z} \mid \mu_{R,n}(M) \neq 0\} = \text{gendeg}(M)$.
- d) $\sum_{n \in \mathbb{Z}} \mu_{R,n}(M) = \mu_R(M)$.

- e) For a free graded R -module F of finite rank we have

$$F = \bigoplus_{n \in \mathbb{Z}} R(-n)^{\oplus \mu_{R,n}(F)}, \quad \text{rank}(F) = \sum_{n \in \mathbb{Z}} \mu_{R,n}(F) = \mu_R(F).$$

7.2. Exercise and Definition. A) (*Minimal Homomorphisms of Graded Modules*) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be graded R -modules

such that M is finitely generated. Show that for a homomorphism of graded R -modules $h : M \rightarrow N$ the following statements are equivalent:

- (i) For each minimal system $(m_i)_{i=1}^r$ of homogeneous generators of M the sequence $(h(m_i))_{i=1}^r$ is a minimal system of homogeneous generators of $\text{Im}(h)$.
- (ii) There is a system $(m_i)_{i=1}^r$ of homogeneous generators of M such that $(h(m_i))_{i=1}^r$ is a minimal system of homogeneous generators of $\text{Im}(h)$.
- (iii) $\mu_{R,*}(M) = \mu_{R,*}(\text{Im}(h))$.
- (iv) $\mu_R(M) = \mu_R(\text{Im}(h))$.
- (v) The induced R_0/\mathfrak{m}_0 -linear map

$$\bar{h} : M/\mathfrak{m}M \rightarrow \text{Im}(h)/\mathfrak{m}\text{Im}(h), \quad m + \mathfrak{m}M \mapsto h(m) + \mathfrak{m}\text{Im}(h)$$

is an isomorphism of vector spaces.

- (vi) $\text{Ker}(h) \subseteq \mathfrak{m}M$.

If the homomorphism of graded R -modules $h : M \rightarrow N$ satisfies these equivalent conditions (i)-(vi), it is called *minimal*.

B) (*First Properties of Minimal Homomorphisms*) Keep the notations and hypotheses of part A). Prove the following facts:

- a) If the zero map $M \xrightarrow{0} N$ is minimal, then $M = 0$.
- b) If $h : M \rightarrow N$ is minimal, then $\text{Ker}(h) \cap M_{\text{beg}(M)} \subseteq \mathfrak{m}_0 M$.
- c) If $h : M \rightarrow N$ is injective, then it is minimal.
- d) If $h : M \rightarrow N$ is surjective and $\mu_R(M) \leq \mu_R(N)$, then h is minimal.
- e) There is a graded free R -module F and a minimal epimorphism of graded R -modules $g : F \rightarrow M \rightarrow 0$.
- f) If $g : F \rightarrow M \rightarrow 0$ is as in statement e), then there is an isomorphism of graded R -modules

$$F \cong \bigoplus_{n \in \mathbb{Z}} R(-n)^{\mu_{R,n}(M)}$$

and moreover each homogeneous basis of F is mapped under g to a minimal homogeneous system of generators of M .

C) (*Minimal Epimorphisms from Graded Free Modules*) Let the notations and hypotheses as above. Prove the followings statements:

- a) If $f : M \xrightarrow{\cong} M'$ is an isomorphism of graded R -modules and if

$$0 \rightarrow N \xrightarrow{\iota} F \xrightarrow{\pi} M \rightarrow 0, \quad 0 \rightarrow N' \xrightarrow{\iota'} F' \xrightarrow{\pi'} M' \rightarrow 0$$

are exact sequences of graded R -modules in which F and F' are free of finite rank and π and π' are minimal, then there are isomorphisms of graded R -modules

$$g : F \xrightarrow{\cong} F', \quad h : N \xrightarrow{\cong} N',$$

which occur in the following commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & N & \xrightarrow{\iota} & F & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow h & & \downarrow g & & \downarrow f & & \\
 0 & \longrightarrow & N' & \xrightarrow{\iota'} & F' & \xrightarrow{\pi'} & M' & \longrightarrow & 0
 \end{array}$$

b) In the notations and under the hypotheses of statement a) we have

$$\mu_{R,*}(F) = \mu_{R,*}(F'), \quad \mu_{R,*}(N) = \mu_{R,*}(N').$$

c) If $0 \rightarrow N \xrightarrow{\iota} F \xrightarrow{\pi} M \rightarrow 0$ is as in statement a), then

$$\begin{aligned}
 \operatorname{reg}(N) &\leq \operatorname{reg}(M) + \max\{1, \operatorname{reg}(R)\}, \\
 \operatorname{reg}(M) &\leq \max\{\operatorname{reg}(N) - 1, \operatorname{gendeg}(M) + \operatorname{reg}(R)\}.
 \end{aligned}$$

Now, we generalize and refine the concept of presentation of a graded module over a polynomial ring over a field, as it was introduced in (6.24). First of all we admit arbitrary Noetherian homogeneous rings with local base rings and define the notion of minimal presentation for finitely generated modules over such rings.

7.3. Exercise and Definition. A) (*Minimal Presentations of Graded Modules*) Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let M be a finitely generated graded R -module. By a *minimal free presentation* of M we mean an exact sequence of graded R -modules

$$G \xrightarrow{h} F \xrightarrow{\pi} M \rightarrow 0$$

in which the graded R -modules F and G are free and the homomorphisms h and π are minimal. Clearly if $M \neq 0$, such a minimal free presentation can always be written as this is done in (6.24)A).

B) (*Existence and Uniqueness of Minimal Presentations*) Keep the notations and hypotheses of part A). Use (7.2)B)e) and (7.2)C)a) to prove the following statements:

- Each finitely generated graded R -module M admits a minimal free resolution $G \xrightarrow{h} F \xrightarrow{l} M \rightarrow 0$.
- If $G \xrightarrow{h} F \xrightarrow{l} M \rightarrow 0$ and $G' \xrightarrow{h'} F' \xrightarrow{l'} M \rightarrow 0$ are two minimal presentations of M , there are isomorphisms u and v of graded R -modules, which appear in the following commutative diagram

$$\begin{array}{ccccccccc}
 G & \xrightarrow{h} & F & \xrightarrow{l} & M & \longrightarrow & 0 \\
 \downarrow u & & \downarrow v & & \downarrow \operatorname{id} & & \\
 G' & \xrightarrow{h'} & F' & \xrightarrow{l'} & M & \longrightarrow & 0
 \end{array}$$

Next, we extend the concept of minimal presentation to the concept of minimal resolution.

7.4. Exercise and Definition. A) (*Minimal Resolutions of Graded Modules*) Again, let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let M be a finitely generated graded R -module. By a *minimal (free) resolution* of M we mean an exact sequence of graded R -modules

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

in which all the modules F_i are free of finite rank and all the homomorphisms d_i are minimal.

B) (*Existence and Uniqueness of Minimal Resolutions*) Keep the notations and hypotheses of part A). Use (7.2)B)e) and (7.2)C)a) to prove the following claims:

a) Each finitely generated graded R -module M admits a minimal free resolution

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0.$$

b) Whenever

$$\begin{aligned} \dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0 \\ \dots \rightarrow G_{n+1} \xrightarrow{e_{n+1}} G_n \xrightarrow{e_n} G_{n-1} \rightarrow \dots \rightarrow G_1 \xrightarrow{e_1} G_0 \xrightarrow{e_0} M \rightarrow 0 \end{aligned}$$

are two minimal free resolutions of M there is family $(u_n)_{n \in \mathbb{N}_0}$ of isomorphisms of graded R -modules, which appear in the following commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_{n+1} & \xrightarrow{d_{n+1}} & F_n & \longrightarrow & \dots \longrightarrow F_0 \xrightarrow{d_0} M \longrightarrow 0 \\ & & \downarrow u_{n+1} & & \downarrow u_n & & \downarrow u_0 & \downarrow \text{id} \\ \dots & \longrightarrow & G_{n+1} & \xrightarrow{e_{n+1}} & G_n & \longrightarrow & \dots \longrightarrow G_0 \xrightarrow{e_0} M \longrightarrow 0 \end{array}$$

C) (*First Properties of Minimal Resolutions*) Keep the above notations. In particular let M be a finitely generated graded R -module with minimal resolution

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0.$$

Prove the following claims

a) $F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$ is a minimal presentation of M .

b) For each $m \in \mathbb{N}_0$ the finitely generated graded R -module $\text{Ker}(d_n) = \text{Im}(d_{n+1})$ has the minimal resolution

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_{m+2} \xrightarrow{d_{m+2}} F_{m+1} \xrightarrow{\bar{d}_m} \text{Ker}(d_m) \rightarrow 0,$$

where \bar{d}_m denotes the homomorphism given by $x \mapsto d_{m+1}(x)$.

c) If $F_n = 0$ for some $n \in \mathbb{N}_0$, then $F_m = 0$ for all $m \geq n$.

Now, we may use the concept of minimal resolution to associate certain numerical invariants to a finitely generated graded module M over a Noetherian homogeneous ring with local base ring.

7.5. Exercise and Definition. A) (*Betti Vectors and Betti Numbers*) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let M be a finitely generated graded R -module with minimal resolution

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0.$$

Let $n \in \mathbb{N}_0$. Then by (7.4)B)b) the family of non-negative integers

$$b_{n,*}(M) = b_{n,*}^R(M) := \mu_{R,*}(F_n)$$

is uniquely determined by M and by n and is called the n -th Betti vector of M . The non-negative number

$$b_n(M) = b_n^R(M) := \mu_R(F_n)$$

is called the n -th Betti number of M .

B) (*First Properties of Betti Vectors and Betti Numbers*) Keep the hypotheses and notations of part A). If $\mathbb{S} \subseteq \mathbb{Z}$ and $(\beta_i)_{i \in \mathbb{S}}$ is a family of real numbers $\beta_i \in \mathbb{R}$ we set:

$$\text{beg}((\beta_i)_{i \in \mathbb{S}}) := \inf\{i \in \mathbb{S} \mid \beta_i \neq 0\},$$

$$\text{end}((\beta_i)_{i \in \mathbb{S}}) := \sup\{i \in \mathbb{S} \mid \beta_i \neq 0\}.$$

Prove the following facts

a) For all $n \in \mathbb{N}_0$ we have

$$\text{beg}(b_{n,*}^R(M)) = \text{beg}(F_n) = \text{beg}(\text{Im}(d_n)),$$

$$\text{end}(b_{n,*}^R(M)) = \text{gendeg}(F_n) = \text{gendeg}(\text{Im}(d_n)).$$

b) If $b_{n,*}^R(M) = 0$ for some $n \in \mathbb{N}_0$, then $b_{m,*}^R(M) = 0$ for all $m \geq n$.

c) For all $n \in \mathbb{N}_0$ we have $\text{beg}(b_{n,*}^R(M)) \leq \text{beg}(b_{n+1,*}^R(M))$.

d) If the local base ring R_0 is a field, then the inequality in statement c) is strict whenever $b_{n,*}^R(M) \neq 0$.

e) For all $m, n \in \mathbb{N}_0$ we have $b_{n,*}^R(\text{Im}(d_m)) = b_{n+m,*}^R(M)$.

C) (*Homological Dimension*) Keep the previous notations and hypotheses. We define the *homological dimension* of the graded R -module M as

$$\text{hdim}(M) = \text{hdim}_R(M) := \sup\{n \in \mathbb{N}_0 \mid b_{n,*}^R(M) \neq 0\}.$$

Prove the following claims:

a) $\text{hdim}_R(M) = -\infty$ if and only if $M = 0$.

b) $\text{hdim}_R(M) = 0$ if and only if M is free.

c) For all $n \in \mathbb{N}_0$ we have $b_{n,*}^R(M) = 0$ if and only if $n > \text{hdim}_R(M)$.

d) For all $n \in \{0, 1, \dots, \text{hdim}_R(M)\}$ we have $\text{hdim}_R(\text{Im}(h_n)) = \text{hdim}_R(M) - n$.

Our next result relates the regularity of a graded module to the ends of the Betti vectors of this module.

7.6. Theorem. *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let $M \neq 0$ be a finitely generated graded R -module. Then, with $\rho := \max\{1, \text{reg}(R)\}$ we have*

a) $\text{reg}(M) \geq \sup_{n \in \mathbb{N}_0} \{\text{end}(b_{n,*}^R(M)) - n\rho\}.$

b) *If $h := \text{hdim}_R(M) < \infty$ then*

$$\text{reg}(M) \leq \text{reg}(R) + h(\rho - 1) + \sup_{n \in \mathbb{N}_0} \{\text{end}(b_{n,*}^R(M)) - n\rho\}.$$

Proof. Consider a minimal resolution

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \rightarrow 0$$

of M and consider the resulting short exact sequences of graded R -modules

a) $0 \rightarrow \text{Im}(d_{n+1}) \xrightarrow{\subseteq} F_n \xrightarrow{\bar{d}_n} \text{Im}(d_n) \rightarrow 0, \quad \forall n \in \mathbb{N}_0.$

If we apply the first inequality of (7.2)C)c) to the sequences a) we obtain

$$\text{reg}(\text{Im}(d_{n+1})) \leq \text{reg}(\text{Im}(d_n)) + \rho, \quad \forall n \in \mathbb{N}_0.$$

As $\text{Im}(d_0) = M$ it thus follows by induction that

$$\text{reg}(\text{Im}(d_n)) \leq \text{reg}(M) + n\rho, \quad \forall n \in \mathbb{N}_0.$$

As $\text{end}(b_{n,*}^R(M)) = \text{gendeg}(\text{Im}(d_n)) \leq \text{reg}(\text{Im}(d_n))$ (see (7.5)B)a) and (3.4)), we thus obtain

$$\text{end}(b_{n,*}^R(M)) \leq \text{reg}(M) + n\rho, \quad \forall n \in \mathbb{N}_0.$$

This proves statement a).

To prove statement b) we set $h := \text{hdim}_R(M)$. If $h = 0$ we have $M \cong F_0$ and hence $\text{reg}(M) = \text{reg}(F_0) = \text{gendeg}(F_0) + \text{reg}(R) = \text{end}(b_{0,*}^R(M)) + \text{reg}(R)$, and this is the requested inequality. So, let $h > 0$. Then, by induction and in view (7.5)C)d) and (7.5)B)e) we have

$$\begin{aligned} \text{reg}(\text{Im}(d_1)) &\leq \text{reg}(R) + (h - 1)(\rho - 1) + \sup_{n \in \mathbb{N}_0} \{\text{end}(b_{n,*}^R(\text{Im}(d_1))) - n\rho\} \\ &= \text{reg}(R) + (h - 1)(\rho - 1) + \sup_{n \in \mathbb{N}_0} \{\text{end}(b_{n+1,*}^R(M)) - n\rho\} \\ &= \text{reg}(R) + (h - 1)(\rho - 1) + \sup_{n \in \mathbb{N}} \{\text{end}(b_{n,*}^R(M)) - (n - 1)\rho\} \\ &= \text{reg}(R) + (h - 1)(\rho - 1) + \sup_{n \in \mathbb{N}} \{\text{end}(b_{n,*}^R(M)) - n\rho + \rho - 1\} + 1 \\ &= \text{reg}(R) + h(\rho - 1) + \sup_{n \in \mathbb{N}} \{\text{end}(b_{n,*}^R(M)) - n\rho\} + 1. \end{aligned}$$

Now, consider the above exact sequence a) with $n = 0$, keep in mind that $\text{Im}(d_0) = M$ and observe the second inequality of (7.2)C)c), in order to conclude that

$$\begin{aligned} \text{reg}(M) &\leq \max\{\text{reg}(\text{Im}(d_1)) - 1, \text{gendeg}(\text{Im}(d_0)) + \text{reg}(R)\} \\ &\leq \max\{\text{reg}(R) + h(\rho - 1) + \sup_{n \in \mathbb{N}} \{\text{end}(b_{n,*}^R(M)) - n\rho\}, \text{end}(b_{0,*}^R(M)) + \text{reg}(R)\} \end{aligned}$$

$$\begin{aligned}
&\leq \operatorname{reg}(R) + \max\{h(\rho-1) + \sup_{n \in \mathbb{N}} \{\operatorname{end}(b_{n,*}^R(M)) - n\rho\}, h(\rho-1) + \operatorname{end}(b_{0,*}^R(M))\} \\
&= \operatorname{reg}(R) + h(\rho-1) + \max\{\sup_{n \in \mathbb{N}} \{\operatorname{end}(b_{n,*}^R(M)) - n\rho\}, \operatorname{end}(b_{0,*}^R(M))\} \\
&= \operatorname{reg}(R) + h(\rho-1) + \sup_{n \in \mathbb{N}_0} \{\operatorname{end}(b_{n,*}^R(M)) - n\rho\}.
\end{aligned}$$

But this proves statement b). \square

7.7. Corollary. *Let (R_0, \mathfrak{m}_0) be a Noetherian local ring, let $r \in \mathbb{N}$ and let $M \neq 0$ be a finitely generated graded R -module over the polynomial ring $R := R_0[X_1, X_2, \dots, X_r]$. Then*

- a) $\operatorname{reg}(M) \geq \sup\{\operatorname{end}(b_{n,*}^R(M)) - n\}$.
- b) If $\operatorname{hdim}_R(M) < \infty$, we have equality in statement a).

Proof. By our hypotheses on the ring R we have $\operatorname{reg}(R) = 0$ and hence our claims are immediate by (7.6) \square

We now aim to focus to the special case, where R is a polynomial ring over a field.

We begin with a few preparations, which shall lead us to the corresponding main result, which at its turn will contain Hilbert's "Syzygiensatz".

7.8. Exercise. A) (*Lifting of Free Bases*) Let A be a ring, let X be an indeterminate, let M be an $A[X]$ -module and consider the A -module M/XM . Let \mathbb{S} be a set and let $(m_i)_{i \in \mathbb{S}}$ be a family of elements $m_i \in M$. For each element $m \in M$ we write \bar{m} for the class $m + XM \in M/XM$. Prove the following statements

- a) If $X \in \operatorname{NZD}_{A[X]}(M)$ and if the family $(\bar{m}_i)_{i \in \mathbb{S}}$ of classes $\bar{m}_i \in M/XM$ is linearly A -independent, then the family $(m_i)_{i \in \mathbb{S}}$ is linearly $A[X]$ -independent.
- b) If $M/XM = \sum_{i \in \mathbb{S}} A\bar{m}_i$ and $X(M/N) \not\subseteq M/N$ for each proper $A[X]$ -submodule $N \subsetneq M$, then $M = \sum_{i \in \mathbb{S}} A[X]m_i$.
- c) If $X \in \operatorname{NZD}_{A[X]}(M)$ and $X(M/N) \not\subseteq M/N$ for each proper $A[X]$ -submodule $N \subsetneq M$ then, the A -module M/XM is free over the basis $(\bar{m}_i)_{i \in \mathbb{S}}$ if and only if the $A[X]$ -module M is free over the basis $(m_i)_{i \in \mathbb{S}}$.

B) (*Maximal Graded CM-Modules*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. We call M a *maximal graded CM-module over R* if $\operatorname{grade}_M(R_+) = \dim(R)$ or - equivalently - if M is CM in the sense of (6.3) and $\dim_R(M) = \dim(R)$. Prove the following claims:

- a) The graded R -module M is maximally CM if and only if $H_{R_+}^i(M) = 0$ for all $i \neq \dim(R)$.
- b) If R is a domain and M is maximally CM, then M is torsion-free, or - equivalently - $\operatorname{Ass}_R(M) = \{0\}$.

- c) If $t \in \mathbb{N}_0$ and $x \in R_t \cap \text{NZD}(R) \cap \text{NZD}_R(M)$, then the graded R -module M is maximally CM if and only if the graded R/xR -module M/xM is maximally CM.

C) (*Maximal Graded CM-Modules over Polynomial Rings*) Now, let $r \in \mathbb{N}_0$, let $R = K[X_1, X_2, \dots, X_r]$ be a polynomial ring. Let $M \neq 0$ be a finitely generated graded R -module. Use the results of parts A) and B) (and in particular induction on r where this is helpful) to show that the following statements are equivalent

- (i) $H_{R_+}^i(M) = 0$ for all $i \neq r$.
- (ii) $H_{R_+}^i(M) = 0$ for all $i < r$.
- (iii) M is maximally CM.
- (iv) M is free.
- (v) There is an isomorphism of graded R -modules $M \cong \bigoplus_{n \in \mathbb{Z}} R(-n)^{\mu_{R,n}(M)}$.

7.9. Reminder and Exercise. (*Grade in Short Exact Sequences*) Let R be a Noetherian ring, let $\mathfrak{a} \subseteq R$ be an ideal and let M be a finitely generated R -module. Keep in mind that the *grade* $\text{grade}_M(\mathfrak{a})$ of \mathfrak{a} with respect to M is defined as the supremum of lengths r of M -sequences x_1, x_2, \dots, x_r in \mathfrak{a} and that (see [Br-Fu-Ro] (4.5), (4.6))

$$\text{grade}_M(\mathfrak{a}) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \neq 0\}.$$

Now, let $0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$ be a short exact sequence of finitely generated R -modules. Prove the following:

- a) If $\text{grade}_M(\mathfrak{a}) < \text{grade}_F(\mathfrak{a})$, then $\text{grade}_N(\mathfrak{a}) = \text{grade}_M(\mathfrak{a}) + 1$.
- b) If $\text{grade}_M(\mathfrak{a}) \geq \text{grade}_F(\mathfrak{a})$, then $\text{grade}_N(\mathfrak{a}) \geq \text{grade}_F(\mathfrak{a})$.

Now we may collect our previous results in order to get the second main result of this section. Statement a) of this result corresponds to Hilbert's "Syzygiensatz".

7.10. Theorem. *Let $r \in \mathbb{N}_0$ and let $M \neq 0$ be a finitely generated graded module over the polynomial ring $R = K[X_1, X_2, \dots, X_r]$. Then*

- a) $\text{hdim}_R(M) < \infty$ and $\text{hdim}_R(M) + \text{grade}_M(R_+) = r$.
- b) $\text{reg}(M) = \sup_{n \in \mathbb{N}_0} \{\text{end}(b_{n,*}^R(M)) - n\} = \max_{n=0}^{\text{hdim}_R(M)} \{\text{end}(b_{n,*}^R(M)) - n\}$.

Proof. We consider a minimal resolution

$$\dots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \dots \rightarrow F_1 \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \rightarrow M \rightarrow 0$$

of M . If we apply (7.9) to the short exact sequences

$$0 \rightarrow \text{Im}(d_{n+1}) \xrightarrow{\subseteq} F_n \rightarrow \text{Im}(d_n) \rightarrow 0,$$

observe that $\text{grade}_{F_n}(R_+) = r$, and by (7.8)C) we get

- (i) $\text{grade}_{\text{Im}(d_{n+1})}(R_+) = \text{grade}_{\text{Im}(d_n)}(R_+) + 1$, if $\text{grade}_{\text{Im}(d_n)}(R_+) < r$.
- (ii) $\text{Im}(d_n)$ is free if $\text{grade}_{\text{Im}(d_n)}(R_+) = r$.

Moreover, if $\text{Im}(d_n)$ is free for some $n \in \mathbb{N}_0$, then clearly $F_{n+1} = 0$ and hence $F_m = 0$ for all $m > n$ (see (7.4)C)c) so that $\text{hdim}_R(M) \leq n$. Now, the above statements (i) and (ii) imply statement a) of our theorem. Statement b) now follows by (7.7). \square

We now discuss the relation of the previous results to *Algebraic Vector Bundles* over projective spaces.

7.11. Remark and Exercise. A) (*Algebraic Vector Bundles*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and set $X := \text{Proj}(R)$. An *algebraic vector bundle over X* is a locally free coherent sheaf of \mathcal{O}_X -modules \mathcal{E} , so that for each point $x \in X$ there is an integer $\text{rank}_x(\mathcal{E}) \in \mathbb{N}_0$ such that

$$\mathcal{E}_x \cong \mathcal{O}^{\oplus \text{rank}_x(\mathcal{E})}.$$

The number $\text{rank}_x(\mathcal{E})$ is called the *rank of \mathcal{E} at the point $x \in X$* . If $\text{rank}_x(\mathcal{E})$ takes the same value for all points $x \in X$, we say that \mathcal{E} is an *algebraic vector bundle of constant rank*. In this situation, we denote the constant value $\text{rank}_x(\mathcal{E})$ by $\text{rank}(\mathcal{E})$ and call it the *rank of \mathcal{E}* . If r is a non-negative integer, we say that \mathcal{E} is an *algebraic vector bundle of rank r* if \mathcal{E} is an algebraic vector bundle of constant rank r . Vector bundles of rank 1 are called *line bundles*. Prove the following statements.

- a) For each $n \in \mathbb{Z}$ the sheaf $\mathcal{O}_X(n)$ of \mathcal{O}_X -modules is a line bundle over X .
- b) If the ring R is an integral domain, each algebraic vector bundle \mathcal{E} over X is of constant rank.

B) (*Direct Sums of Vector Bundles*) Keep the above notations and hypotheses. Here we also use the concept of direct sum of sheaves of \mathcal{O}_X -modules (see (3.15)D)). Prove the following facts.

- a) If the sheaves $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_t$ are algebraic vector bundles, then so is $\bigoplus_{i=1}^t \mathcal{F}_i$.
- b) In the situation of statement a) we have $\text{rank}_x(\bigoplus_{i=1}^t \mathcal{F}_i) = \sum_{i=1}^t \text{rank}_x(\mathcal{F}_i)$ for all $x \in X$.
- c) For each family $(n_i)_{i=1}^t$ of integers $n_i \in \mathbb{Z}$, the sheaf $\bigoplus_{i=1}^t \mathcal{O}_X(n_i)$ is an algebraic vector bundle of rank t over X .

C) (*Algebraic Vector Bundles over Projective Spaces*) Let $r \in \mathbb{N}_0$ and consider the projective r -space $\mathbb{P}_K^r := \text{Proj}(K[X_0, X_1, \dots, X_r])$. Verify that each algebraic vector bundle \mathcal{E} over \mathbb{P}_K^r is of constant rank. An algebraic vector bundle \mathcal{E} over \mathbb{P}_K^r is said to *split* if there is a direct sum of line bundles, or -

equivalently:

$$\mathcal{E} \cong \bigoplus_{i=1}^t \mathcal{O}_X(-a_i), \quad a_1 \leq a_2 \leq \dots \leq a_t.$$

In this latter situation, calculate the cohomological Hilbert functions (see (4.11))

$$h_{\mathcal{E}}^i : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto h_{\mathcal{E}}^i(n) = h^i(\mathbb{P}_K^r, \mathcal{E}(n))$$

for all $i \in \mathbb{N}_0$. Use this to show, that the sequence $(a_i)_{i=1}^t$ is uniquely determined by \mathcal{E} . This sequence is called the *splitting type* of \mathcal{E} .

D) (*Revisiting the Vanishing Theorem of Severi-Enriques-Zariski-Serre*) This part needs slightly more involved arguments from commutative algebra. Let $R = K \oplus R_1 \oplus R_2 \dots$ be as in part A), let $X = \text{Proj}(R)$, let M be a finitely generated graded R -module and let $\widetilde{M} =: \mathcal{F}$ be the coherent sheaf of \mathcal{O}_X -modules induced by M . Let K' be an extension field of K , consider the Noetherian homogeneous K' -algebra

$$R' := K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \dots,$$

let $X' := \text{Proj}(R')$, consider the finitely generated graded R -module

$$M' = K' \otimes_K M$$

and the induced coherent sheaf of $\mathcal{O}_{X'}$ -modules

$$\widetilde{M}' = \mathcal{F}'.$$

Observe that R' is an integral extension of R and a flat R -algebra. Prove the following statements:

- The assignment $\mathfrak{p}' \mapsto \mathfrak{p}' \cap R$ defines a surjective map $\varphi : X' \rightarrow X$ such that $\varphi^{-1}(\mathfrak{p}) = \min(\mathfrak{p}R')$ for all $\mathfrak{p} \in X$.
- $\varphi(\text{mProj}(R')) = \text{mProj}(R)$ and $\varphi^{-1}(\text{mProj}(R)) = \text{mProj}(R')$.
- If $x' \in X'$, then $\mathcal{O}_{X',x'}$ is a flat $\mathcal{O}_{X,\varphi(x')}$ -algebra with $\sqrt{\mathfrak{m}_{X,\varphi(x')} \mathcal{O}_{X',x'}} = \mathfrak{m}_{X',x'}$.
- If $x' \in X'$, then $\mathcal{F}'_{x'} \cong \mathcal{O}_{X,x'} \otimes_{\mathcal{O}_{X,\varphi(x')}} \mathcal{F}_{\varphi(x')}$.
- If $x \in X$, then $\text{depth}_{\mathcal{O}_{X',x'}}(\mathcal{F}'_{x'}) = \text{depth}_{\mathcal{O}_{X,\varphi(x')}}(\mathcal{F}_{\varphi(x')})$.
- $\delta(\mathcal{F}') = \delta(\mathcal{F})$.

Now, use the last statement and (2.4)A)b) to show that the Vanishing Theorem of Severi-Enriques-Zariski-Serre (see [Br-Fu-Ro] (12.16),(12.17)) holds over an arbitrary base field K .

E) (*Characterizations of Algebraic Vector Bundles*) Let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, which is an integral domain of strictly positive dimension and assume that the scheme $X = \text{Proj}(R)$ is *regular*, so that the local ring $\mathcal{O}_{X,x}$ of X at x is regular for all $x \in X$. Let $\mathcal{F} \neq 0$ be a coherent sheaf of \mathcal{O}_X -modules. Show (on use of the Formula of Auslander-Buchsbaum-Serre) that the following statements are equivalent:

- (i) \mathcal{F} is an algebraic vector bundle over X .
- (ii) $\delta(\mathcal{F}) = \dim(X)$.
- (iii) $H^i(X, \mathcal{F}(n)) = 0$ for all $i < \dim(X)$ and all $n \ll 0$.
- (iv) $\text{depth}_{\mathcal{O}_X}(\mathcal{F}_x) > 0$ for all closed points $x \in X$ (that is for all $x \in \text{mProj}(R)$), and $H^i(X, \mathcal{F}(n)) = 0$ for all $i \in \{1, 2, \dots, \dim(X) - 1\}$ and all $n \ll 0$.

Observe in particular that the equivalence of statements (i) and (iii) is a *Cohomological Criterion for the Coherent Sheaf \mathcal{F} to be an Algebraic Vector Bundle*.

We now easily can prove a the *Splitting Criterion of Horrocks* (see [Hor]) for vector-bundles over a projective spaces.

7.12. Theorem. *Let $r \in \mathbb{N}$, let K be a field, let $R = K[X_0, X_1, \dots, X_r]$ be a polynomial ring, consider the projective r -space $\mathbb{P}_K^r = \text{Proj}(R)$ and let $\mathcal{F} \neq 0$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}_K^r}$ -modules. Then, the following statements are equivalent:*

- (i) *There is a graded free R -module F of finite rank such that $\mathcal{F} \cong \tilde{F}$.*
- (ii) *\mathcal{F} is a splitting vector bundle over \mathbb{P}_K^r .*
- (iii) *\mathcal{F} is an algebraic vector bundle over \mathbb{P}_K^r and $H^i(\mathbb{P}_K^r, \mathcal{F}(n)) = 0$ for all $i \in \{1, 2, \dots, r - 1\}$ and all $n \in \mathbb{Z}$.*
- (iv) *$H^0(\mathbb{P}_K^r, \mathcal{F}(m)) = 0$ for all $m \ll 0$ and $H^i(\mathbb{P}_K^r, \mathcal{F}(n)) = 0$ for all $i \in \{1, 2, \dots, r - 1\}$ and all $n \in \mathbb{Z}$.*
- (v) *$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) > 0$ for all closed points $x \in \mathbb{P}_K^r$ and $H^i(\mathbb{P}_K^r, \mathcal{F}(n)) = 0$ for all $i \in \{1, 2, \dots, r - 1\}$ and all $n \in \mathbb{Z}$.*
- (vi) *The total module of global sections $\Gamma_*(\mathbb{P}_K^r, \mathcal{F})$ of \mathcal{F} (see (3.8)) is a graded free R -module of finite rank.*

Proof. By (7.10)B)c) it is immediate that statement (i) implies statement (ii). Assume that statement (ii) holds. Then, using the cohomological Hilbert function $h_{\mathcal{F}}^i : \mathbb{Z} \rightarrow \mathbb{N}_0$ as calculated in (7.11), we see immediately that statement (iii) holds. Statements (iii),(iv) and (v) are equivalent by (7.11)E).

Now, consider the total module of global sections

$$\Gamma := \Gamma_*(\mathbb{P}_K^r, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathbb{P}_K^r, \mathcal{F}(n)).$$

According to (3.11)c) we have $\mathcal{F} \cong \tilde{\Gamma}$. This shows, that statement (vi) implies statement (i). It thus remains to show that statement (iv) implies statement (vi).

Observe first that by (3.10) we have

$$H_{R_+}^i(\Gamma) = 0, \quad i = 0, 1.$$

As

$$\Gamma_n = H^0(\mathbb{P}_K^r, \mathcal{F}(n)) = 0, \quad \forall n \ll 0$$

we find some $t \in \mathbb{Z}$ such that $\Gamma = \Gamma_{\geq t}$. By (3.11)a) this means that the graded R -module Γ is finitely generated. Now, by the second part of the Serre-Grothendieck Correspondence (2.14)b) we obtain that

$$H_{R_+}^i(\Gamma)_n \cong H^{i-1}(\mathbb{P}_K^r, \mathcal{F}(n)) = 0, \quad \forall i \in \{2, 3, \dots, r\}, \forall n \in \mathbb{Z}.$$

Consequently $H_{R_+}^i(\Gamma)$ vanishes for all $i \in \{2, 3, \dots, r\}$. As we have observed above, this vanishing also holds for $i = 0, 1$. Therefore $\text{grade}_\Gamma(R_+) = r + 1$ and hence (7.10)a) implies that $\text{hdim}_R(\Gamma) = 0$, so that the finitely generated graded R -module Γ is indeed free (see (7.5)C)b). \square

As an application we now get the *Splitting Theorem of Witt-Grothendieck* (see [Gro0]) for Vector Bundles over the projective line.

7.13. Corollary. *Let K be a field. Then, each algebraic vector bundle over the projective line \mathbb{P}_K^1 splits.*

Proof. Apply (7.12) with $r = 1$. \square

8. DIAGONAL BOUNDS

Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let $X = \text{Proj}(R)$ be the induced projective scheme and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Consider the cohomological pattern

$$\mathcal{P} = \mathcal{P}(X, \mathcal{F}) = \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\}$$

of \mathcal{F} (see (2.15)). Now, for all $k \in \{0, 1, \dots, \dim(\mathcal{F})\}$ and all $r \in \mathbb{Z}$ we know that the entries $H^i(X, \mathcal{F}(n))$ of \mathcal{P} vanish right of the *diagonal* $((i, r - i))_{i=k+1}^{\dim(\mathcal{F})}$ above level k , if they vanish along this diagonal.

It is natural to ask, whether this fact finds a natural extension, which also applies to situations in which the entries of \mathcal{P} do not necessarily vanish. In this section we shall prove that this is indeed the case. More precisely, we shall compute an *upper bound* and a *right-vanishing bound* for the numbers $h^i(X, \mathcal{F}(n - r - i))$ in the range $i > k$ and $n \geq r - i$ in terms of the *r-th cohomology diagonal*

$$\text{diag}_r^{>k}(\mathcal{F}) := (h^i(X, \mathcal{F}(i - r))_{i=k+1}^{\dim(\mathcal{F})})$$

of \mathcal{F} above level k . This will tell us, that cohomology along a diagonal above a given level, bounds cohomology right of this diagonal. In particular, the cohomology diagonal of \mathcal{F} above level 0 bounds the regularity of \mathcal{F} .

We call this type of bounds *A Priori Bounds of Castelnuovo Type* as they are valid for any coherent sheaf (which is expressed by the wording "a priori") and also give upper bounds for the regularity. We also speak of *Diagonal Bounds* by the reason explained above (see [Br2],[Br4],[Br-Matt-Mi1] and [Br-Sh1]). The bounds we give in this section are not intended to be very sharp. Instead, we prefer to give bounds which may be expressed by relatively simple explicit formulas. Moreover the corresponding bounds are also valid in the more general situation in which the base ring of our Noetherian homogeneous algebra is not only a field, but just local Artinian. Readers interested in these extensions and specifications should consult the quoted references.

We also shall prove a *Left-Boundedness Result for Geometric Cohomological Hilbert Functions*. This result teaches us, that the cohomology left of a (lower partial) diagonal is bounded left of this diagonal in terms of the values attained there by the geometric cohomological Hilbert functions. But contrary to the previous bounds of Castelnuovo type, one cannot expect here a general left-vanishing bound. Nevertheless our (algebraic version of) the *Vanishing Theorem of Severi-Enriques-Zariski-Serre* (see [Br-Fu-Ro](10.17)) gives such left-vanishing bounds, but only at levels below the global subdimension. We call these ("restricted") left-vanishing bounds *A Priori Bounds of First Severi Type* (see also [Br3],[Br4], [Br-Matt-Mi1]). Above the level given by the global subdimension one cannot expect the left-vanishing of cohomology, and so left-vanishing bounds must be replaced by a conceptually new type of bound

beyond the mentioned level. We shall treat these new *A Priori Bounds of Second Severi Type* in Section 10.

But nevertheless our first Left-Boundedness Result for Geometric Cohomological Hilbert Functions enables us to look at cohomological patterns in a new way. We namely use this boundedness result to prove a *Right-Finiteness Result* for classes \mathcal{D} of pairs (X, \mathcal{F}) in which X is a projective scheme over some field K and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules of a given dimension s . This result says that if the class \mathcal{D} is of *finite cohomology* on some diagonal set $\Delta_r = \{(i, r - i) \mid i = 0, 1, \dots, s\}$, then the class \mathcal{D} is indeed of finite cohomology on each set $\mathbb{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ which is bounded to the left. We finally give an extension of this Right-Finiteness result, which allows to replace the hypotheses that the class \mathcal{D} is of finite cohomology on some diagonal set by the weaker condition, that \mathcal{D} is of finite cohomology on some *quasi-diagonal subset* $\Sigma = \{(i, n_i) \mid i = 0, 1, \dots, s\} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ with $n_s < n - s - 1 < \dots < n_0$. This is a first and not yet complete look at the general question: "What Bounds Cohomology \mathcal{D} " which will be discussed in section 10.

We now attack our task, and we do this in the ring- and module theoretic framework. To do so, we first give a number of prerequisites.

8.1. Reminder, Exercise and Definition. A) (*Geometric Cohomological Hilbert Functions*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Then, for each $i \in \mathbb{N}_0$ we may consider the i -th cohomological Hilbert function of M (see (2.4)B) and [Br-Fu-Ro](9.13))

$$h_M^i : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto h_M^i(n) = \dim_K(H_{R_+}^i(M)_n).$$

Moreover, we may define the i -th *geometric Hilbert function*

$$d_M^i : \mathbb{Z} \rightarrow \mathbb{N}_0$$

of M by

- a) $d_M^0(n) := \dim_K(M_n) - h_M^0(n) + h_M^1(n), \quad \forall n \in \mathbb{Z}$.
- b) If $i > 0$, then $d_M^i(n) := h_M^{i+1}(n), \quad \forall n \in \mathbb{Z}$.

In [Br-Fu-Ro](10.6),(10.19) we have introduced and studied these functions in the special case where K is an infinite field.

B) (*Basic Properties of Geometric Cohomological Hilbert Functions*) Let the notations and hypothesis as in part A). Prove the following statements:

- a) $d_M^0(n) = 0$ for all $n \in \mathbb{Z}$ if and only if $\dim_R(M) \leq 0$.
- b) If $\dim_R(M) > 0$, then $\dim_R(M) = \sup\{i \in \mathbb{N}_0 \mid d_M^i \neq 0\} + 1$.
- c) If $N \subseteq \Gamma_{R_+}(M)$ is a graded submodule, then $d_{M/N}^i(n) = d_M^i(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
- d) For all $i \in \mathbb{N}_0$ and for all $n, r \in \mathbb{Z}$ we have $d_{M(r)}^i(n) = d_M^i(n + r)$.

- e) If K' is an extension field of K , the finitely generated graded module $K' \otimes_K M$ over the Noetherian homogeneous K' -algebra $K' \otimes_K R$ satisfies

$$d_{K' \otimes_K M}^i(n) = d_M^i(n), \quad \forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z}.$$

- f) For all $n \in \mathbb{Z}$ we have (see (2.4)B))

$$\chi_M(n) = \sum_{i \in \mathbb{N}_0} (-1)^i d_M^i(n).$$

- g) If $X = \text{Proj}(R)$ and if $\mathcal{F} = \widetilde{M}$ is the coherent sheaf of \mathcal{O}_X -modules induced by M then

$$d_M^i(n) = h^i(X, \mathcal{F}(n)), \quad \forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z}.$$

- h) In the notations of statement g) and (2.15) we have

$$\mathcal{P}(X, \mathcal{F}) = \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid d_M^i(n) \neq 0\}.$$

C) (*Cohomology Tables*) Let the notations and hypotheses be as in parts A) and B). We define the *cohomology table of the finitely generated graded R -module M* as the family of non-negative integers

$$d_M := \left(d_M^i(n) \right)_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.$$

Let $X = \text{Proj}(R)$. Then correspondingly for each coherent sheaf of \mathcal{O}_X -modules \mathcal{F} we define the *cohomology table of the sheaf \mathcal{F}* as the family of non-negative integers

$$h_{\mathcal{F}} := \left(h^i(X, \mathcal{F}(n)) \right)_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.$$

Observe that

- a) If $\mathcal{F} = \widetilde{M}$, then $h_{\mathcal{F}} = d_M$.
 b) If $N \subseteq \Gamma_{R_+}(M)$ is a graded submodule, then $d_{M/N} = d_M$.
 c) If K' is an extension field of K , the finitely generated graded $K' \otimes_K R$ -module $K' \otimes_K M$ satisfies $d_{K' \otimes_K M} = d_M$.

D) (*Cohomology Diagonals*) Let the notations and hypotheses be as above. Then for each integer $r \in \mathbb{Z}$ we define the *r -th cohomology diagonal of the finitely generated graded R -module M* as the finite family of non-negative integers

$$\text{diag}_r(M) := \left(d_M^i(r-i) \right)_{i=0}^{\dim_R(M)-1} = d_M \upharpoonright_{\{(i,r-i) \mid i < \dim_R(M)\}}.$$

Correspondingly, we define the *r -th cohomology diagonal of the coherent sheaf of \mathcal{O}_X -modules \mathcal{F}* as the family of non-negative integers

$$\text{diag}_r(\mathcal{F}) := \left(h^i(X, \mathcal{F}(r-i)) \right)_{i=0}^{\dim(\mathcal{F})} = h_{\mathcal{F}} \upharpoonright_{\{(i,r-i) \mid i \leq \dim(\mathcal{F})\}}.$$

We usually refer to the 0-th cohomology diagonal just as the *cohomology diagonal of M (respectively of \mathcal{F})* and thus write in accordance with [Br-Fu-Ro] (10.19) B)

$$\text{diag}(M) := \text{diag}_0(M), \quad \text{diag}(\mathcal{F}) := \text{diag}_0(\mathcal{F}).$$

Observe the following facts

- a) If $\mathcal{F} = \widetilde{M}$, then $\text{diag}_r(\mathcal{F}) = \text{diag}_r(M)$.
- b) If $N \subseteq \Gamma_{R_+}(M)$ is a graded submodule, then $\text{diag}_r(M/N) = \text{diag}_r(M)$.
- c) If K' is an extension field of K , then $\text{diag}_r(K' \otimes_K M) = \text{diag}_r(M)$.
- d) $\text{diag}_r(M) = \text{diag}_0(M(r)) = \text{diag}(M(r))$.
- e) $\text{diag}_r(\mathcal{F}) = \text{diag}_0(\mathcal{F}(r)) = \text{diag}(\mathcal{F}(r))$.

E) (*Cohomology Diagonals Above a Certain Level*) Let the notations and hypotheses be as above. Let $k \in \mathbb{N}_0$. We define the *r -th cohomology diagonal of M above level k* as the family of non negative integers

$$\text{diag}_r^{>k}(M) := (d_M^i(r-i))_{i=k+1}^{\dim_R(M)-1} = d_M \upharpoonright_{\{(i,r-i) | k < i < \dim_R(M)\}} \quad .$$

Similarly, we define the *r -th cohomology diagonal of \mathcal{F} above level k* as

$$\text{diag}_r^{>k}(\mathcal{F}) := (h^i(X, \mathcal{F}(r-i)))_{i=k+1}^{\dim(\mathcal{F})} = d_{\mathcal{F}} \upharpoonright_{\{(i,r-i) | k < i \leq \dim(\mathcal{F})\}} \quad .$$

Observe the following fact:

- a) The five statements a)-e) of part D) remain valid mutatis mutandis for cohomology diagonals above a given level k .

8.2. Exercise and Definition. A) (*Diagonal Right Bounding Functions for Modules*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. For each integer $l > 1$ and each $n \in \mathbb{Z}$ we set

$$\bar{B}_M^l(n) := \sum_{i=l-1}^{\dim_R(M)-1} \binom{\dim_R(M)-l}{i-l+1} d_M^i(n-i),$$

With our usual convention that $\binom{u}{v} := 0$ for all $v \in \mathbb{N}_0$ and all $u \in \mathbb{Z}_{<v}$. Keep in mind that we also can write

$$\bar{B}_M^l(n) = \sum_{j=l}^{\dim_R(M)} \binom{\dim_R(M)-l}{j-l} h_M^j(n-j+1).$$

Observe the similarity of the *l -th diagonal right-bounding function associated with the graded R -module M*

$$\bar{B}_M^l : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto \bar{B}_M^l(n), \forall n \in \mathbb{Z}$$

with the corresponding *diagonal left-bounding functions* $B_M^k : \mathbb{Z} \rightarrow \mathbb{N}_0$ one may define for all $k < \dim_R(M)$ according to [Br-Fu-Ro](10.10)A).

B) (*First Properties of Bounding Functions*). Keep the above notations and hypotheses. Prove the following statements:

- a) For all $n, r \in \mathbb{Z}$ it holds $\bar{B}_{M(r)}^l(n) = \bar{B}_M^l(r+n)$.
- b) If K' is an extension field of K then $\bar{B}_{K' \otimes_K M}^l(n) = \bar{B}_M^l(n)$ for all $n \in \mathbb{Z}$.
- c) For all $i \in \{l, l+1, \dots, \dim_R(M)\}$ and all $n \in \mathbb{Z}$ it holds $h_M^i(n-i+1) \leq \bar{B}_M^l(n)$.
- d) If $d := \dim_R(M) > 1$ then for all $n \in \mathbb{Z}$ we have $\bar{B}_M^d(n) = h_M^d(n-d+1)$ and $\bar{B}_M^l(n) = 0$ whenever $l > d$.
- e) $\sup\{n \in \mathbb{Z} \mid \bar{B}_M^l(n) \neq 0\} + 1 = \text{reg}^l(M)$.
- f) If $N \subseteq M$ is a graded submodule with $\dim_R(M) \leq 1$ then $\bar{B}_N^l(n) = 0$ and $\bar{B}_{M/N}^l(n) = \bar{B}_M^l(n)$ for all $n \in \mathbb{Z}$.
- g) If $N \subseteq \Gamma_{R_+}(M)$ is a graded submodule, then $\bar{B}_{M/N}^l(n) = \bar{B}_M^l(n) \quad \forall n \in \mathbb{Z}$.

The following Lemma is rather similar to [Br-Fu-Ro](10.10)

8.3. Lemma. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, Let M be a finitely generated graded R -module, let $x \in R_1 \cap \text{NZD}_R(M)$ and let $l > 1$ be an integer. Then*

$$\bar{B}_{M/xM}^l(n) \leq \bar{B}_M^l(n), \quad \forall n \in \mathbb{Z}.$$

Proof. Let $d = \dim_R(M)$. If $d \leq 2$, we have $\dim_R(M/xM) \leq 1$ and hence $\bar{B}_{M/xM}^l(n) = 0$ for all $n \in \mathbb{Z}$. So, let $d > 2$. Then $\dim_R(M/xM) = d-1$. Moreover, if we apply cohomology to the short exact sequence of graded R -modules $0 \rightarrow M(-1) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ we obtain that $h_{M/xM}^j(n-j+1) \leq h_M^j(n-j+1) + h_M^{j+1}(n-j)$ for all $j \in \mathbb{N}$ and all $n \in \mathbb{Z}$. Consequently

$$d_{M/xM}^i(n-i) \leq d_M^i(n-i) + d_M^{i+1}(n-(i+1)), \quad \forall n \in \mathbb{Z}.$$

Therefore, on use of the Pascal formula we get

$$\begin{aligned} \bar{B}_{M/xM}^l(n) &= \sum_{i=l-1}^{d-2} \binom{d-l-1}{i-l+1} d_{M/xM}^i(n-i) \leq \\ &\leq \sum_{i=l-1}^{d-2} \binom{d-l-1}{i-l+1} (d_M^i(n-i) + d_M^{i+1}(n-(i+1))) = \\ &= d_M^{l-1}(n-(l-1)) + d_M^{d-1}(n-(d-1)) + \\ &\sum_{i=l}^{d-2} \left(\binom{d-l-1}{i-l} + \binom{d-l-1}{i-l+1} \right) d_M^i(n-i) = \\ &= \sum_{i=l-1}^{d-1} \binom{d-l}{i-l+1} d_M^i(n-i) = \bar{B}_M^l(n). \end{aligned}$$

□

8.4. **Exercise.** Let K be an algebraically closed field and let V, W be two K -vector spaces such that $0 < \dim_K(V) < \infty$. Let $f, g : V \rightarrow W$ be two K -linear maps such that the map $\alpha f + \beta g : V \rightarrow W$ is surjective for all pairs $(\alpha, \beta) \in K^2 \setminus \{(0, 0)\}$. Show that $\dim_K(W) < \dim_K(V)$. (Hint: See [Br-Fu-Ro](10.7).)

8.5. **Theorem.** Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module of dimension $d > 0$. Then for each $r \in \mathbb{Z}$ and each integer $l \in \{2, 3, \dots, d\}$ we have

- a) $\text{reg}^l(M) \leq r + (2\bar{B}_M^l(r))^{2^{d-l}}$.
b) $\bar{B}_M^l(n) \leq \frac{1}{2}(2\bar{B}_M^l(r))^{2^{d-l}}$ for all $n \geq r$.

Proof. By (8.2)B)b) we may assume as usually that K is algebraically closed. Moreover by replacing M by $M(r)$ we may assume that $r = 0$ (see (3.3)B)b) and (8.2)B)a)). Now, let

$$\mathcal{P} := \text{Ass}_R(M) \cap (\text{mProj}(R) \cup \{R_+\}), \quad \mathfrak{a} := \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}, \quad \bar{M} = M/\Gamma_{\mathfrak{a}}(M).$$

Then once more by [Br-Bo-Ro](1.9),(10.3)C) we have (see also in the proof of (4.7))

$$\text{Ass}_R(\Gamma_{R_+}(M)) = \mathcal{P}, \quad \dim(R/\mathfrak{p}) > 1, \forall \mathfrak{p} \in \text{Ass}_R(\bar{M}).$$

In particular $\dim_R(\bar{M}) = d$, $\dim_R(\Gamma_{R_+}(M)) \leq 1$ and hence $\bar{B}_{\bar{M}}^l(n) = \bar{B}_M^l(n)$ for all $n \in \mathbb{Z}$ (see (8.2)B)f)). So, we may replace M by \bar{M} and hence assume in addition, that $\dim(R/\mathfrak{p}) > 1$ for all $\mathfrak{p} \in \text{Ass}_R(M)$. Consequently by [Br-Fu-Ro](10.5) there is a K -vector space $L \subseteq R_1$ with $\dim_K(L) = 2$ and $L \setminus \{0\} \subseteq \text{NZD}_R(M)$. So, if f, g form a K -basis of L , we have

$$\alpha f + \beta g \in \text{NZD}_R(M), \quad \forall (\alpha, \beta) \in K^2 \setminus \{(0, 0)\}.$$

If we set $x = \alpha f + \beta g$ and apply cohomology to the exact sequence of graded R -modules $0 \rightarrow M(-1) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ we get exact sequences of K -vector spaces

$$(i) \quad H_{R_+}^i(M)_{n-1} \xrightarrow{\alpha f + \beta g} H_{R_+}^i(M)_n \rightarrow H_{R_+}^i(M/(\alpha f + \beta g)M)_n$$

for all $i \in \mathbb{N}_0$, all $n \in \mathbb{Z}$ and all $(\alpha, \beta) \in K^2 \setminus \{(0, 0)\}$.

Now, we proceed by induction on $d - l$. If $d - l = 0$ we have $\bar{B}_M^l(n) = \bar{B}_M^d(n) = h_M^d(n - d + 1)$ for all $n \in \mathbb{Z}$ (see (8.2)B)d)). For all $(\alpha, \beta) \in K^2 \setminus \{(0, 0)\}$ the element $\alpha f + \beta g \in R_1$ belongs to $\text{NZD}_R(M)$. Therefore $\dim_R(M/(\alpha f + \alpha g)M) < d$, whence $H_{R_+}^d(M/(\alpha f + \beta g)M) = 0$. So, if we apply the sequences (i) with $i = d$, we get epimorphisms

$$H_{R_+}^d(M)_{n-1-d+1} \xrightarrow{\alpha f + \beta g} H_{R_+}^d(M)_{n-d+1} \rightarrow 0, \quad \forall (\alpha, \beta) \in K^2 \setminus \{(0, 0)\}, \forall n \in \mathbb{Z}.$$

Thus, for all $n \in \mathbb{Z}$ we get by (8.3) that

$$\bar{B}_M^d(n) = h_M^d(n-d+1) \leq \max\{h_M^d(n-1-d+1)-1, 0\} = \max\{\bar{B}_M^d(n-1)-1, 0\}.$$

Therefore $\bar{B}_M^d(n) \leq \bar{B}_M^d(0)$ for all $n \geq 0$ and $\bar{B}_M^d(n) = 0$ for all $n \geq \bar{B}_M^d(0)$. This proves (more than) our claim if $d-l=0$.

Now, let $d-l > 0$. Choose $(\alpha, \beta) \in K^2 \setminus \{(0, 0)\}$. Then in particular $\dim_R(M/(\alpha f + \beta g)M) = d-1$. So, by induction and in view of (8.2)B)e) and (8.3) we have

$$(ii) \quad \bar{B}_{M/(\alpha f + \beta g)M}^l(n) \leq \frac{1}{2}(2\bar{B}_M^l(0))^{2^{d-l-1}}, \quad \forall n \in \mathbb{Z}.$$

$$(iii) \quad \bar{B}_{M/(\alpha f + \beta g)M}^l(n) = 0, \quad \forall n \geq (2\bar{B}_M^l(0))^{2^{d-l-1}}.$$

Now, the sequences (i) yield that

$$h_M^j(n-j+1) \leq h_M^j(n-1-j+1) + h_{M/(\alpha f + \beta g)M}^j(n-j+1), \quad \forall j \in \mathbb{N}, \forall n \in \mathbb{Z}.$$

According to (8.2)A) this implies that

$$\bar{B}_M^l(n) \leq \bar{B}_M^l(n-1) + \bar{B}_{M/(\alpha f + \beta g)M}^l(n), \quad \forall n \in \mathbb{Z},$$

whence, by induction on n :

$$\bar{B}_M^l(n) \leq \bar{B}_M^l(0) + \sum_{k=1}^n \bar{B}_{M/(\alpha f + \beta g)M}^l(k), \quad \forall n \in \mathbb{N}_0.$$

But now, the above statements (ii) and (iii) imply that

$$\begin{aligned} \bar{B}_M^l(n) &\leq \bar{B}_M^l(0) + \frac{1}{2}(2\bar{B}_M^l(0))^{2^{d-l-1}} \max\{0, (2\bar{B}_M^l(0))^{2^{d-l-1}} - 1\} \leq \\ &\frac{1}{2}[(\bar{B}_M^l(0))^{2^{d-l-1}}]^2 = \frac{1}{2}(\bar{B}_M^l(0))^{2^{d-l}}. \end{aligned}$$

This proves our statement b).

It remains to show statement a). By (8.2)B)e) this comes up to show that

$$\bar{B}_M^l(n) = 0, \quad \forall n \geq (2\bar{B}_M^l(0))^{2^{d-l}}.$$

In order to do so, we choose any $n \geq (2\bar{B}_M^l(0))^{2^{d-l-1}}$, so that by statement (iii) we have $\bar{B}_{M/(\alpha f + \beta g)M}^l(n) = 0$ and hence (see (8.2)B)c))

$$h_{M/(\alpha f + \beta g)M}^i(n-i+1) = 0, \quad \forall i \in \{l, l+1, \dots, d\}.$$

So, by the exact sequences (i) we get an epimorphism

$$H_{R_+}^i(M)_{n-i} \xrightarrow{\alpha f + \beta g} H_{R_+}^i(M)_{n-i+1} \rightarrow 0$$

for all $i \in \{l, l+1, \dots, d\}$ and all pairs $(\alpha, \beta) \in K^2 \setminus \{(0, 0)\}$. By (8.4) this allows to conclude that

$$h_M^i(n-i+1) \leq \max\{h_M^i(n-i)-1, 0\}, \quad \forall i \in \{l, l+1, \dots, d\}.$$

From this we obtain by (8.2)A) that

$$\bar{B}_M^l(n) \leq \max\{\bar{B}_M^l(n-1) - 1, 0\}, \quad \forall n \geq (2\bar{B}_M^l(0))^{2^{d-l-1}}.$$

It follows that

$$\bar{B}_M^l(n) = 0, \quad \forall n \geq \bar{B}_M^l((2\bar{B}_M^l(0))^{2^{d-l-1}}) + (2\bar{B}_M^l(0))^{2^{d-l-1}} =: B.$$

By statement b) we have

$$\bar{B}_M^l((2\bar{B}_M^l(0))^{2^{d-l-1}}) \leq \frac{1}{2}(2\bar{B}_M^l(0))^{2^{d-l}}.$$

As in addition

$$(B_M^l(0))^{2^{d-l-1}} \leq \frac{1}{2}(2\bar{B}_M^l(0))^{2^{d-l}}$$

it follows that $B \leq (2\bar{B}_M^l(0))^{2^{d-l}}$. This proves our claim. \square

8.6. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module of dimension $d > 0$. Then, for all $l \in \{2, 3, \dots\}$ we have*

$$\text{a) } \text{reg}^l(M) \leq (2 \sum_{j=l}^d \binom{d-l}{j-l} h_M^j(1-j))^{2^{d-l}}.$$

$$\text{b) } \sum_{j=1}^d \binom{d-l}{j-l} h_M^j(n-j-1) \leq \frac{1}{2} (2 \sum_{j=l}^d \binom{d-l}{j-l} h_M^j(1-j))^{2^{d-l}}, \quad \forall n \in \mathbb{N}_0.$$

Proof. Apply 8.4 with $r = 0$ and observe the second equality in (8.2)A). \square

In order to express the previous results in sheaf theoretic terms, we extend the notion of regularity of a coherent sheaf over a projective scheme as it was introduced in (3.6). We also give a sheaf-theoretic version of the bounding functions defined in (8.2).

8.7. Exercise and Definition. A) (*Regularity of Sheaves Above a Certain Level*) Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring, set $X := \text{Proj}(R)$, let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules and let $k \in \mathbb{N}_0$. We define the *regularity of the coherent sheaf of \mathcal{O}_X -modules \mathcal{F} above level k* by

$$\text{reg}^k(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{F}(r-i)) = 0, \quad \forall i > k\}.$$

Prove the following statements:

$$\text{a) } \text{reg}^0(\mathcal{F}) = \text{reg}(\mathcal{F}).$$

$$\text{b) } \text{For all } k, l \in \mathbb{N}_0 \text{ with } k \leq l \text{ we have } \text{reg}^l(\mathcal{F}) \leq \text{reg}^k(\mathcal{F}).$$

$$\text{c) } \text{For all } k \in \mathbb{N}_0 \text{ and all } r \in \mathbb{Z} \text{ we have } \text{reg}^k(\mathcal{F}(r)) = \text{reg}^k(\mathcal{F}) - r.$$

$$\text{d) } \text{If } M \text{ is a finitely generated graded } R\text{-module with } \widetilde{M} = \mathcal{F}, \text{ then } \text{reg}^k(\mathcal{F}) = \text{reg}^{k+2}(M) \text{ for all } k \in \mathbb{N}_0.$$

B) (*Diagonal Right Bounding Functions for Sheaves*) Now, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, set $X :=$

$\text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. For each $k \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we set

$$\tilde{B}_{\mathcal{F}}^k(n) := \sum_{i=k+1}^{\dim(\mathcal{F})} \binom{\dim(\mathcal{F}) - k - 1}{i - k - 1} h^i(X, \mathcal{F}(n - i))$$

and consider the corresponding k -th diagonal right-bounding function associated to the coherent sheaf \mathcal{F}

$$\tilde{B}_{\mathcal{F}}^k : \mathbb{Z} \rightarrow \mathbb{N}_0, \quad n \mapsto \tilde{B}_{\mathcal{F}}^k(n), \forall n \in \mathbb{Z}.$$

Prove the following facts

- For all $n, r \in \mathbb{Z}$ it holds $\tilde{B}_{\mathcal{F}(r)}^k(n) = \tilde{B}_{\mathcal{F}}(r + n)$.
- For all $i \in \{k+1, k+2, \dots, \dim(\mathcal{F})\}$ and all $n \in \mathbb{Z}$ it holds $h^i(X, \mathcal{F}(n - i)) \leq \tilde{B}_{\mathcal{F}}^k(n)$.
- If $d := \dim(\mathcal{F}) > 0$ then for all $n \in \mathbb{Z}$ we have $\tilde{B}_{\mathcal{F}}^{d-1}(n) = h^d(X, \mathcal{F}(n - d))$ and $\tilde{B}_{\mathcal{F}}^k(n) = 0$ whenever $k \geq d$.
- For all $k \in \mathbb{N}_0$ we have $\text{reg}^k(\mathcal{F}) = \inf\{r \in \mathbb{Z} \mid \tilde{B}_{\mathcal{F}}^k(r) = 0\}$.
- If M is a finitely generated graded R -module with $\mathcal{F} = \widetilde{M}$, then $\tilde{B}_{\mathcal{F}}^k(n) = \bar{B}_M^{k+2}(n)$ for all $k \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.

8.8. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, set $X := \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules of dimension $s \geq 0$. Then for each $k \in \{0, 1, \dots, s - 1\}$ and all $r \in \mathbb{Z}$ we have*

- $\text{reg}^k(\mathcal{F}) \leq r + (\tilde{B}_{\mathcal{F}}^k(r))^{2^{s-k-1}}$.
- $\tilde{B}_{\mathcal{F}}^k(n) \leq \frac{1}{2} (2\tilde{B}_{\mathcal{F}}^k(r))^{2^{s-k-1}}$ for all $n \geq r$.

Proof. Let M be a finitely generated graded R -module with $\mathcal{F} = \widetilde{M}$, observe that $\dim_R(M) = s + 1$, keep in mind (8.7)A)d),B)e) and apply (8.5). \square

8.9. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, set $X := \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules of dimension $s \geq 0$. Then, for each $k \in \{0, 1, \dots, s - 1\}$ we have*

- $\text{reg}^k(\mathcal{F}) \leq (2 \sum_{i=k+1}^s \binom{s-k-1}{i-k-1} h^i(X, \mathcal{F}(-i)))^{2^{d-k-1}}$.
- $\sum_{i=k+1}^s \binom{s-k-1}{i-k-1} h^i(X, \mathcal{F}(n - i)) \leq \frac{1}{2} (2 \sum_{i=k+1}^s \binom{s-k-1}{s-i-1} h^i(X, \mathcal{F}(-i)))^{2^{s-k-1}}$ for all $n \in \mathbb{N}_0$.

Proof. Apply (8.8) with $r = 0$. \square

8.10. Corollary. *Let K be field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, set $X := \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules of dimension $s \geq 0$. Then the 0-th cohomology diagonal $\text{diag}_0^{>0}(\mathcal{F})$*

above level 0 bounds the regularity of \mathcal{F} . More precisely

$$\text{reg}(\mathcal{F}) \leq \left(2 \sum_{i=1}^s \binom{s-1}{i-1} h^i(X, \mathcal{F}(-i))\right)^{2^{s-1}}.$$

Proof. Apply (8.9)a) with $k = 0$. □

In the previous results, we only did use information on cohomology diagonals above level 0. So, it is natural to ask, whether we could draw further reaching conclusions if we knew the full cohomology diagonal. We shall answer this question affirmatively in a way which will lead us to look at our cohomological patterns from a new point of view. To do so, we first prove a Left-Bounding Result for Geometric Cohomological Hilbert Functions, which holds in the range "left of a diagonal below a certain level". We begin with an auxiliary result.

8.11. Lemma. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module and let $x \in R_1 \cap \text{NZD}_R(M)$. Then, with the notational convention that $d_N^j(n) = 0$ for all $j < 0$, all finitely generated graded R -modules N and all $n \in \mathbb{Z}$ we have for all $i, k \in \mathbb{N}_0$:*

- a) $d_M^i(m-1) \leq d_M^i(m) + d_{M/xM}^{i-1}(m)$, for all $m \in \mathbb{Z}$.
- b) $d_M^i(n) \leq d_M^i(-i) + \sum_{n < m \leq -i} d_{M/xM}^{i-1}(m)$, for all $n \leq -i$.
- c) $\sum_{l=0}^{i-1} \binom{i-1}{l} d_{M/xM}^{l+k}(-l) \leq \sum_{j=0}^i \binom{i}{j} d_M^{j+k}(-j)$.

Proof. "a)": First, we apply cohomology to the short exact sequence of graded R -modules

$$0 \rightarrow M(-1) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$$

to conclude that

$$h_M^{i+1}(m-1) \leq h_M^{i+1}(m) + h_{M/xM}^i(m), \quad \forall m \in \mathbb{Z}.$$

If $i > 2$, this proves statement a). If $i = 2$ we get

$$d_M^1(m-1) \leq d_M^1(m) + h_{M/xM}^1(m), \quad \forall m \in \mathbb{Z}.$$

As $h_{M/xM}^0(m) \leq \dim_K((M/xM)_m)$ we have for all $m \in \mathbb{Z}$ the inequality

$$h_{M/xM}^1(m) \leq \dim_K((M/xM)_m) + h_{M/xM}^1(m) - h_{M/xM}^0(m) = d_{M/xM}^0(m).$$

This proves statement a) if $i = 1$. So, let $i = 0$. Then by [Br-Fu-Ro](10.8)a) we have $d_M^0(m-1) \leq d_M^0(m)$ for all $m \in \mathbb{Z}$. This proves statement a) in this case.

"b)": This follows immediately from statement a).

"c)": By statement a) we have

$$d_{M/xM}^{l+k}(-l) \leq d_M^{l+k+1}(-l-1) + d_M^{l+k}(-l), \quad \forall l \in \mathbb{N}_0.$$

Therefore on use of the Pascal formula

$$\begin{aligned} \sum_{l=0}^{i-1} \binom{i-1}{l} d_{M/xM}^{l+k}(-l) &\leq \sum_{l=0}^{i-1} \binom{i-1}{l} [d_M^{l+k+1}(-l-1) + d_M^{l+k}(-l)] = \\ d_M^k(0) + \sum_{j=1}^{i-1} \left[\binom{i-1}{j-1} + \binom{i-1}{j} \right] d_M^{j+k}(-j) + d_M^{j+k}(-i) &= \sum_{j=0}^i d_M^{j+k}(-j). \end{aligned}$$

□

Now we are ready to prove the announced *Left-Bounding Result for Geometric Cohomological Hilbert Functions*. It gives an upper bound on the geometric cohomological Hilbert functions left of a given diagonal.

8.12. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be finitely generated graded R -module. Then, for all $i \in \mathbb{N}_0$ we have*

$$d_M^i(n) \leq \sum_{j=0}^i \binom{-n-j-1}{i-j} \left[\sum_{l=0}^{i-j} \binom{i-1}{l} d_M^{i-l}(l-i) \right], \quad \forall n \leq -i.$$

Proof. We proceed by induction on i . If $i = 0$ we may conclude as $d_M^0(n) \leq d_M^i(0)$ for all $n \leq 0$ (see either (8.2)d) or [Br-Fu-Ro](10.8a)). So let $i > 0$. As usually we can assume that K is infinite and that $\Gamma_{R_+}(M) = 0$, so that there is some element $x \in R_1 \cap \text{NZD}_R(M)$. Then, according to (8.11b) we have

$$d_M^i(n) \leq d_M^i(-i) + \sum_{n < m \leq -i} d_{M/xM}^{i-1}(m), \quad \forall n \leq -i.$$

By induction we also have

$$d_{M/xM}^{i-1}(m) \leq \sum_{j=0}^{i-1} \binom{-m-j-1}{i-1-j} \left[\sum_{l=0}^{i-1-j} \binom{i-l-1}{l} d_{M/xM}^{i-1-l}(l-i+1) \right], \quad \forall m \leq 0.$$

So, combining both inequalities we can say

(i) For all $n \leq -i$ it holds

$$\begin{aligned} d_M^i(n) &\leq d_M^i(-i) + \\ &\sum_{n < m \leq -i} \left\{ \sum_{j=0}^{i-1} \binom{-m-j-1}{i-1-j} \left[\sum_{l=0}^{i-1-j} \binom{i-l-1}{l} d_{M/xM}^{i-1-l}(l-i+1) \right] \right\}. \end{aligned}$$

Now, by (8.11c) (applied to $M(-j)$ with $k = j$) we may write

$$\sum_{l=0}^{i-1-j} \binom{i-1-j}{l} d_{M/xM}^{i-1-l}(l-i+1) =$$

$$\begin{aligned}
& \sum_{l=0}^{i-j-1} \binom{i-j-1}{i-j-1-l} d_{M/xM}^{(i-j-1)+j}(-(i-j-1-l)-j) = \\
& \sum_{h=0}^{i-j-1} \binom{i-j-1}{h} d_{M^{(-j)}/xM^{(-j)}}^{h+j}(h) \leq \sum_{g=0}^{i-j} \binom{i-j}{g} d_{M^{(-j)}}^{g+j}(-g) = \\
& \sum_{g=0}^{i-j} \binom{i-j}{i-j-g} d_M^{i-(i-j-g)}((i-j-g)-i) = \sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i).
\end{aligned}$$

Now, on use of the above inequality (i) and as

$$\sum_{n < m \leq -i} \binom{-m-j-1}{i-1-j} = \binom{-n-j-1}{i-j}$$

we obtain

$$\begin{aligned}
d_M^i(n) & \leq d_M^i(-i) + \sum_{n < m \leq -i} \left\{ \sum_{j=0}^{i-1} \binom{-m-j-1}{i-1-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i) \right] \right\} = \\
& d_M^i(-i) + \left\{ \sum_{j=0}^{i-1} \sum_{n < m \leq -i} \binom{-m-j-1}{i-1-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i) \right] \right\} = \\
& d_M^i(-i) + \left\{ \sum_{j=0}^{i-1} \binom{-n-j-1}{i-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i) \right] \right\} = \\
& \sum_{j=0}^i \binom{-n-j-1}{i-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i) \right].
\end{aligned}$$

□

The previous bounding results taken for its own occurs indeed not to be very appealing. But its consequences give a hint to consider cohomology tables from a new point of view. To make this precise, we first introduce some notations.

8.13. Notation. Let $d \in \mathbb{N}$. By \mathcal{M}^d we denote the class of all pairs (R, M) in which $R = K \oplus R_1 \oplus R_2 \dots$ is a Noetherian homogeneous K -algebra over some field K and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a finitely generated graded R -module such that $\dim_R(M) = d$.

Correspondingly let $s \in \mathbb{N}_0$ and let \mathcal{S}^s the class of all pairs (X, \mathcal{F}) in which X is a projective scheme over some field K and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules with $\dim(\mathcal{F}) = s$. Observe that

$$\mathcal{S}^s = \{(\text{Proj}(R), \widetilde{M}) \mid (R, M) \in \mathcal{M}^{s+1}\}.$$

Now, we define a concept, which will play an important role later on in these lectures: the concept of subclass $\mathcal{C} \subseteq \mathcal{M}^d$ (or $\subseteq \mathcal{S}^s$) which is of finite cohomology on a subset $\mathcal{S} \subseteq \{0, 1, \dots, d-1\} \times \mathbb{Z}$ (respectively $\subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$).

8.14. **Exercise and Definition.** A) (*Classes of Finite Cohomology*) Let $d \in \mathbb{N}$ and let $\mathbb{S} \subseteq \{0, 1, \dots, d-1\} \times \mathbb{Z}$ be a set. A subclass $\mathcal{C} \subseteq \mathcal{M}^d$ is said to be (*a class*) of finite cohomology on \mathbb{S} if the set of families

$$\{(d_M^i(n))_{(i,n) \in \mathbb{S}} \mid (R, M) \in \mathcal{C}\} = \{d_M \upharpoonright_{\mathbb{S}} \mid (R, M) \in \mathcal{C}\}.$$

is finite. We say that the class $\mathcal{C} \subseteq \mathcal{M}^d$ is of finite cohomology (*at all*) if it is of finite cohomology on the set $\{0, 1, \dots, d-1\} \times \mathbb{Z}$.

The notion of subclass $\mathcal{D} \subseteq \mathcal{S}^s$ of finite cohomology (*on a set* $\mathbb{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$) is defined similarly.

B) (*First Properties of Classes of Finite Cohomology*) Let $r \in \mathbb{N}_0$, let

$$\mathcal{C}, \mathcal{C}_i, \mathcal{D} \subseteq \mathcal{M}^d \quad (\text{or } \subseteq \mathcal{S}^s), \quad (i = 1, 2, \dots, r)$$

be subclasses and let

$$\mathbb{S}, \mathbb{S}_i, \mathbb{T} \subseteq \{0, 1, \dots, d-1\} \times \mathbb{Z} \quad (\text{or } \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}), \quad (i = 1, 2, \dots, r)$$

be subsets. Observe the following easy facts

- a) \mathcal{C} is of finite cohomology on \emptyset .
- b) If \mathcal{C} is finite, it is of finite cohomology on \mathbb{S} .
- c) If $\mathcal{C} \subseteq \mathcal{D}$ and \mathcal{D} is of finite cohomology on \mathbb{S} , then \mathcal{C} is of finite cohomology on \mathbb{S} .
- d) If \mathcal{C} is of finite cohomology on \mathbb{S} and if $\mathbb{T} \subseteq \mathbb{S}$, then \mathcal{C} is of finite cohomology on \mathbb{T} .
- e) If \mathcal{C}_i is of finite cohomology on \mathbb{S} for all $i \in \{1, 2, \dots, r\}$, then $\cup_{i=1}^r \mathcal{C}_i$ is of finite cohomology on \mathbb{S} .
- f) If \mathcal{C} is of finite cohomology on \mathbb{S}_i for all $i \in \{1, 2, \dots, r\}$, then \mathcal{C} is of finite cohomology on $\cup_{i=1}^r \mathbb{S}_i$.

C) (*An Example*) Let the notations and hypotheses as in be as in parts A) and B), choose $(R, M) \in \mathcal{M}^d$ and let $\mathcal{C} := \{(R, M^{\oplus r}) \mid r \in \mathbb{N}\}$. Show that

$$\{0, 1, \dots, d-1\} \times \mathbb{Z} \setminus \mathcal{P}(\text{Proj}(R), \widetilde{M})$$

is the unique maximal subset $\mathbb{S} \subseteq \{0, 1, \dots, d-1\} \times \mathbb{Z}$ on which the class \mathcal{C} is of finite cohomology.

Now, we can prove the following *Right-Finiteness Result for Classes* $\mathcal{C} \subseteq \mathcal{M}^d$, a module-theoretic formulation of the corresponding announced sheaf-theoretic finiteness result for classes $\mathcal{D} \subseteq \mathcal{S}^s$.

8.15. **Proposition.** *Let $d \in \mathbb{N}$, $r \in \mathbb{Z}$ and let $\mathcal{C} \subseteq \mathcal{M}^d$ be a subclass which is of finite cohomology on the diagonal set $\Delta_r := \{(i, r-i) \mid i = 0, 1, \dots, d-1\}$. Then, for each $t \in \mathbb{Z}$ the class \mathcal{C} is of finite cohomology on the set*

$$\mathbb{S} := \{0, 1, \dots, d-1\} \times \mathbb{Z}_{\geq t}.$$

Proof. In view of (8.14)B)d) we may assume that $r-d \geq t$, so that in particular $r-i \geq t$ for all $i \in \{0, 1, \dots, d-1\}$. Now, for each integer $i \in \{0, 1, \dots, d-1\}$ and each integer $n \leq r-i$ we may use (8.12) to see that

$$\begin{aligned} d_M^i(n) &= d_{M(r)}^i(n-r) \leq \sum_{j=0}^i \binom{n-j-1}{i-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_{M(r)}^{i-l}(l-i) \right] = \\ &= \sum_{j=0}^i \binom{n-j-1}{i-j} \left[\sum_{k=j}^i \binom{i-j}{i-k} d_{M(r)}^k(-k) \right] = \\ &= \sum_{j=0}^i \binom{n-j-1}{i-j} \left[\sum_{k=j}^i \binom{i-j}{i-k} d_M^k(r-k) \right]. \end{aligned}$$

So, the class \mathcal{C} is of bounded cohomology on the set

$$\mathbb{S}_1 := \{(i, n) \mid i \in \{0, 1, \dots, d-1\}, \quad n \in \{t, t+1, \dots, i-r\}\}.$$

If we apply (8.5) with $l=2$ and bear in mind the definition of the numbers $\bar{B}_M^2(n)$ (see (8.2)A)) we see immediately that the class \mathcal{C} is of finite cohomology on the set

$$\mathbb{S}_2 := \{(i, n) \mid i \in \{1, 2, \dots, d-1\}, \quad n \in \mathbb{Z}_{\geq r-i}\}.$$

Now, set

$$\mathbb{S}_3 := \{0\} \times \mathbb{Z}_{>r}.$$

Then clearly $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \cup \mathbb{S}_3$. So, by (8.14)B)f) it remains to show that the class \mathcal{C} is bounded on the set \mathbb{S}_3 .

To do so. let $(R, M) \in \mathcal{C}$. Then the Hilbert polynomial $P_M \in \mathbb{Q}[X]$ is of degree $d-1$ and moreover we have (see (8.1)B)b),f))

$$P_M(n) = \chi_M(n) = \sum_{i=0}^d (-1)^i d_M^i(n), \quad \forall n \in \mathbb{Z}.$$

In particular, the polynomial P_M is determined by the finite family

$$F_M := (d_M^i(n))_{(i,n) \in \{0,1,\dots,d-1\} \times \{r-d, r-d+1, \dots, r\}}.$$

As $\{0, 1, \dots, d-1\} \times \{r-d, r-d+1, \dots, r\} \subseteq \mathbb{S}_1 \cup \mathbb{S}_2$ and as \mathcal{C} is of finite cohomology on the set $\mathbb{S}_1 \cup \mathbb{S}_2$, the set $\{F_M \mid (R, M) \in \mathcal{C}\}$ is finite. Therefore the set of Hilbert polynomials P_M with $(R, M) \in \mathcal{C}$ is finite, thus:

$$(i) \quad \#\{P_M \mid (R, M) \in \mathcal{C}\} < \infty.$$

Moreover, as the class \mathcal{C} is of finite cohomology on the set \mathbb{S}_2 , and as $d_M^i(n) = h_M^{i+1}(n) = 0$ for all $i > 0$ and all $n \gg 0$, there is some integer $s > r$ such that

$$(ii) \quad d_M^i(n) = 0, \quad \forall (R, M) \in \mathcal{C}, \forall i \in \{1, 2, \dots, d-1\}, \forall n > s.$$

In addition by our above description of the Hilbert polynomial P_M we have

$$(iii) \quad d_M^0(n) = P_M(n) - \sum_{i=1}^{d-1} (-1)^i d_M^i(n), \quad \forall n \in \mathbb{Z}.$$

As \mathcal{C} is of finite cohomology on the set \mathbb{S}_2 , statement (i) and (ii) imply that the class \mathcal{C} is of finite cohomology on the set $\{0\} \times \{r+1, r+2, \dots, s\} \subseteq \mathbb{S}_3$. It thus remains to show that \mathcal{C} is of finite cohomology on the set

$$\mathbb{S}_3 \setminus [\{0\} \times \{r+1, r+2, \dots, s\}] = \{0\} \times \mathbb{Z}_{>s}.$$

But this follows easily by statements (i),(ii) and (iii). \square

In sheaf-theoretic terms, the previous right-finiteness result takes the form of the following *Right-Finiteness Result for Classes $\mathcal{D} \subseteq \mathcal{S}^s$* .

8.16. Corollary. *Let $s \in \mathbb{N}_0$, let $r \in \mathbb{Z}$ and let $\mathcal{D} \subseteq \mathcal{S}^s$ be a subclass which is of finite cohomology on the diagonal subset $\Delta_r := \{(i, r-i) \mid i = 0, 1, \dots, s\}$ of $\{0, 1, \dots, s\} \times \mathbb{Z}$. Then, for each $t \in \mathbb{Z}$ the class \mathcal{D} is of finite cohomology on the set*

$$\mathbb{S} := \{0, 1, \dots, s\} \times \mathbb{Z}_{\geq t}.$$

Proof. This follows immediately by (8.15) and the last observation made in (8.13). \square

So, we can say, that a subclass $\mathcal{D} \subseteq \mathcal{S}^s$ which is of finite cohomology on a diagonal set $\Delta_r \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ is also of finite cohomology on each left bounded subset \mathbb{S} of $\{0, 1, \dots, s\} \times \mathbb{Z}$. It is natural to ask, whether the same conclusion holds if the diagonal subset Δ_r is replaced by a set subject to weaker conditions. This is indeed true, as we are going to show now. We begin with the following auxiliary result.

8.17. Lemma. *Let $d \in \mathbb{N}$, let $(n_i)_{i=0}^{d-1}$ be a sequence of integers such that $n_{d-1} < n_{d-2} < \dots < n_0$ and let $\mathcal{C} \subseteq \mathcal{M}^d$ be a subclass which is of finite cohomology on the subset $\Sigma := \{(i, n_i) \mid i = 0, 1, \dots, d-1\}$ of $\{0, 1, \dots, d-1\} \times \mathbb{Z}$. Then, the class \mathcal{C} is of finite cohomology on the diagonal set*

$$\Delta = \Delta_{d+n_{d-1}} := \{(i, d+n_{d-1}-i) \mid i = 0, 1, \dots, d-1\}.$$

Proof. We proceed by induction on

$$\delta = \delta(\Sigma) := n_0 - n_{d-1} \quad (\geq d).$$

If $\delta = d$ we clearly have $\Sigma = \Delta$ and our claim is clear.

So, let $\delta > d$. Then, there is some $i \in \{0, 1, \dots, d-2\}$ such that $n_i - n_{i+1} > 1$. We chose i minimal with this property, write $i = i(\Sigma)$ if necessary, and proceed by induction on $i = i(\Sigma)$.

Assume first, that $i = 0$. Then $n_1 + 1 < n_0$ and it follows by (8.12) applied with $i = 0$ that $d_M^0(n_1 + 1) = d_{M(n_0)}^0(n_1 + 1 - n_0) \leq d_{M(n_0)}^0(0) = d_M^0(n_0)$. But this implies that the class \mathcal{C} is of finite cohomology on the set $\Sigma' := \{(0, n_1 + 1)\} \cup \{(j, n_j) \mid j = 1, 2, \dots, d-1\}$. But for this set we also have $\delta(\Sigma') < \delta(\Sigma) = \delta$. Now, by induction the class \mathcal{C} is of finite cohomology on the set Δ .

Now, let $i > 0$. Then clearly $n_i - 1 - n_0 = -i - 1$ and \mathcal{C} is of finite cohomology on the non-empty set

$$\{(i - l, n_0 + l - i) \mid l = 0, 1, \dots, i\} = \{(k, n_k) \mid k = 0, 1, \dots, i\} \subseteq \Sigma.$$

So, there is some $h \in \mathbb{N}_0$ such that $d_{M(n_0)}^{i-l}(l - i) = d_M^{i-l}(n_0 + l - i) \leq h$ for all $l \in \{0, 1, \dots, i\}$ and all pairs $(R, M) \in \mathcal{C}$. By (8.12) it follows that there is some $h' \in \mathbb{N}_0$ such that

$$d_M^i(n_i - 1) = d_{M(n_0)}^i(n_i - 1 - n_0) = d_{M(n_0)}^i(-i - 1) \leq h', \quad \forall (R, M) \in \mathcal{C}.$$

From this we obtain that the class \mathcal{C} is of finite cohomology on the set

$$\Sigma'' := \{(j, n_j) \mid j = 0, 1, \dots, i-1\} \cup \{(i, n_i-1)\} \cup \{(k, n_k) \mid k = i+1, i+2, \dots, d-1\}.$$

As now $i(\Sigma'') = i(\Sigma) - 1 = i - 1$, we may conclude by induction. \square

To formulate the announced finiteness result, we introduce a further notion.

8.18. Definition. Let $t \in \mathbb{N}_0$. A set $\Sigma \subseteq \{0, 1, \dots, t\} \times \mathbb{Z}$ is said to be a *quasi-diagonal subset* if there are integers $n_t < n_{t-1} < \dots < n_0$ such that

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \dots, t\}.$$

Observe that diagonal subsets are quasidiagonal.

Now, we may prove the module-theoretic version of the announced extension of our Right-Finiteness Result.

8.19. Proposition. Let $d \in \mathbb{N}$ and let $\mathcal{C} \subseteq \mathcal{M}^d$ be a subclass which is of finite cohomology on some quasi-diagonal subset $\Sigma \subseteq \{0, 1, \dots, d-1\} \times \mathbb{Z}$. Then, for each $t \in \mathbb{Z}$ the class \mathcal{C} is of finite cohomology on the set $\{0, 1, \dots, d-1\} \times \mathbb{Z}_{\geq t}$.

Proof. this is immediate by (8.15) and (8.17). \square

Finally, in sheaf-theoretic terms we now can say:

8.20. Corollary. Let $s \in \mathbb{N}_0$ and let $\mathcal{D} \subseteq \mathcal{S}^s$ be a subclass which is of finite cohomology on some quasi-diagonal subset $\Sigma \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$. Then, for each $t \in \mathbb{Z}$ the class \mathcal{D} is of finite cohomology on the set $\{0, 1, \dots, s\} \times \mathbb{Z}_{\geq t}$.

Proof. This follows once more by mere translation from (8.20). \square

So, we finally indeed can say, that any subclass \mathcal{D} of \mathcal{S}^s which is of finite cohomology on some quasi-diagonal subset Σ of $\{0, 1, \dots, s\} \times \mathbb{Z}$ is of finite cohomology on any left-bounded subset \mathcal{S} of $\{0, 1, \dots, s\} \times \mathbb{Z}$.

9. MODULES OF DEFICIENCY

In this section we introduce an important tool for the treatment of local cohomology modules, the so called *Modules of Deficiency*. We restrict ourselves to do this in the special framework which is most relevant for these lectures - namely over Noetherian homogeneous algebras over fields. In this special case, the functor of taking *Graded Matlis Duals* luckily coincides with the functor of taking *Graded Duals* with respect to the base field of our Noetherian homogeneous ring. In our construction, this will allow us to shortcut the theory of *Graded Gorenstein Rings* and to define the requested deficiency modules simply as graded duals of local cohomology modules with respect to the irrelevant ideal. So, we invest in our definition a fact which, in a more general situation corresponds to the *Graded Local Duality Theorem*. This simplification comes for free, as our rings are graded homomorphic images of polynomial rings over fields and hence a fortiori of Gorenstein rings.

On the other hand as we renounced to define our modules of deficiency in the usual way by means of certain Ext-modules, we now are left with the task to prove that these modules are finitely generated. We shall do this in two steps. First we compute the modules of deficiency of a polynomial ring over a field. In a second step, which is incorporated in the proof of our *Main Theorem on Modules of Deficiency* (9.7) we use an induction argument to show the requested finiteness result in general. In order to be able to perform efficient homological arguments, we actually shall introduce the *Functors of Deficiency* as the composition of local cohomology functors with respect to the the irrelevant ideal and the graded duality functor. As this latter functor behaves well in the subcategory of *Graded Modules with Finite Components*, we get the expected Graded Local Duality. In our Main Theorem (9.7) we shall collect all the relevant properties of deficiency modules.

As an application, we shall be able to introduce the concept of *Cohomological Hilbert Polynomial* and the notion of *Cohomological Postulation Number* of a finitely generated graded module over a Noetherian homogeneous K -algebra. The latter invariant finds a lower bound in terms of the regularity of deficiency modules, and this shall us lead to the investigation of our next section. We also compute the *top local cohomology module of a polynomial ring over a field* in an example and exercise.

We also introduce the *canonical module* of a finitely generated graded module M over a Noetherian homogeneous algebra over a field K as the highest non-vanishing deficiency module. We then prove a few properties about these modules. The most basic of these says, that that the grade of the canonical module of a finitely generated graded module M is at least as big as the minimum of 2 and the dimension of M . We then derive a *Structure Theorem for Canonical Modules* and show that the canonical module of a CM-module is again CM.

Finally we devote an extended remark to the link between our "narrow-gauge" way of approaching the theory of deficiency module and the "standard-gauge" way which relies on (the graded form) of Grothendieck's *Local Duality Theorem*. At the very end of this section we shortly shall discuss in an exercise and remark the fact that our local cohomology modules are *Artinian*.

We now begin with a number of fairly general preparations, which shall pave the way to define and to study modules of deficiency.

9.1. Construction and Exercise. A) (*Graded Dual Modules*) For the moment let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be an arbitrary graded ring and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R -module. We consider the R_0 -module

$$\mathrm{Hom}_{R_0}(M, R_0)$$

of all R_0 -linear maps $h : M \rightarrow R_0$. By means of the scalar multiplication defined by

$$xh := h \circ x\mathrm{Id}_M, \quad \forall x \in R, \forall h \in \mathrm{Hom}_{R_0}(M, R_0)$$

the R_0 -module $\mathrm{Hom}_{R_0}(M, R_0)$ is turned into an R -module. We consider the subset

$$D(M) := \{h \in \mathrm{Hom}_{R_0}(M, R_0) \mid \#\{n \in \mathbb{Z} \mid h(M_n) \neq 0\} < \infty\}$$

of $\mathrm{Hom}_{R_0}(M, R_0)$ consisting of all R_0 -linear maps $h : M \rightarrow R_0$ which vanish on almost all graded components of M . Moreover, for each $t \in \mathbb{Z}$ we define the subset

$$D(M)_t := \{h \in \mathrm{Hom}_{R_0}(M, R_0) \mid h(M_n) = 0, \forall n \neq -t\}$$

of $D(M)$ consisting of all R_0 -linear maps $h : M \rightarrow R_0$ which vanish on all graded components of M in degrees different from $-t$. Prove the following statements:

- a) $D(M) \subseteq \mathrm{Hom}_{R_0}(M, R_0)$ is an R -submodule.
- b) For all $t \in \mathbb{Z}$ the set $D(M)_t \subseteq D(M)$ is an R_0 -submodule.
- c) The family $(D(M)_t)_{t \in \mathbb{Z}}$ of R_0 -submodules $D(M)_t \subseteq D(M)$ defines a grading on the R -module $D(M)$.
- d) For all $t \in \mathbb{Z}$ there is an isomorphism of R_0 -modules

$$\tau_t^M : \mathrm{Hom}_{R_0}(M_{-t}, R_0) \xrightarrow{\cong} D(M)_t$$

given by

$$\tau_t^M(h)(m) := h(m_{-t}), \quad \forall h \in \mathrm{Hom}_{R_0}(M_{-t}, R_0), \forall m := (m_n)_{n \in \mathbb{Z}} \in M = \bigoplus_{n \in \mathbb{Z}} M_n.$$

- e) For all $r, t \in \mathbb{Z}$ we have $D(M(r))_t = D(M)_{t-r}$.

From now on, we always furnish the R -module $D(M)$ with the grading mentioned in statement c), hence write

$$D(M) = \bigoplus_{t \in \mathbb{Z}} D(M)_t,$$

and call $D(M)$ the *graded (R_0) -dual of M* . Observe that by statement e) we have

$$f) \quad D(M(r)) = D(M)(-r), \quad \forall r \in \mathbb{Z}.$$

B) (*The Graded Duality Functor*) Keep the notations and hypotheses of part A) and let $h : M \rightarrow N$ be a homomorphism of graded R -modules. Show that there is a homomorphism of graded R -modules

$$D(h) : D(N) \rightarrow D(M), \quad f \mapsto f \circ h, \quad \forall f \in D(N).$$

The homomorphisms of graded R -modules $D(h)$ is called the *graded (R_0) -dual of h* . Prove the following claims:

- a) $D(\text{Id}_M) = \text{Id}_{D(M)}$.
- b) If $h : M \rightarrow N$ and $g : N \rightarrow P$ are homomorphisms of graded R -modules, then $D(g \circ h) = D(h) \circ D(g)$.
- c) If $h, l : M \rightarrow N$ are homomorphisms of graded R -modules, then $D(h + l) = D(h) + D(l)$.
- d) For all $r \in \mathbb{Z}$, all $x \in R_r$ and each homomorphism of graded R -modules $h : M \rightarrow N$ the homomorphism of graded R -modules $xh : M \rightarrow N(r)$ satisfies $D(xh) = xD(h) : D(N)(-r) \rightarrow D(M)$ (see [Br-Fu-Ro](8.5E))
- e) If $L \xrightarrow{u} M \xrightarrow{v} N \rightarrow 0$ is an exact sequence of graded R -modules, we have an exact sequence of graded R -modules

$$0 \rightarrow D(N) \xrightarrow{D(v)} D(M) \xrightarrow{D(u)} D(L).$$

So, similar as in [Br-Fu-Ro](8.6A),D) we can say that we have a *contravariant, R -linear, left exact functor of graded R -modules*

$$D(\bullet) : (M \xrightarrow{h} N) \mapsto (D(N) \xrightarrow{D(h)} D(M)),$$

the *functor of taking graded (R_0) -duals* or the *graded duality functor (with respect to R_0)*.

C) (*First Properties of Graded Duality Functors*) Keep the notations and hypotheses of parts A) and B). Show the following

- a) For all $t \in \mathbb{Z}$ there is a natural equivalence of contravariant functors from graded R -modules to R_0 -modules

$$\tau_t^M : \text{Hom}_{R_0}(\bullet_{-t}, R_0) \xrightarrow{\cong} D(\bullet)_t : M \mapsto (\text{Hom}_{R_0}(M_{-t}, R_0) \xrightarrow{\tau_t^M} D(M)_t),$$

where τ_t^M is defined as in statement A)d).

- b) There is a natural transformation of covariant functors of graded R -modules

$$\gamma : \bullet \rightarrow D(D(\bullet)) : M \mapsto (M \xrightarrow{\gamma^M} D(D(M))),$$

where the homomorphism $\gamma^M : M \rightarrow D(D(M))$ is given by

$$\gamma^M(m)(f) = f(m), \quad \forall m \in M, \forall f \in D(M).$$

D) (*Base Ring Independence of Graded Duals*) Keep the notations of part A) and assume that $\mathfrak{a} \subsetneq R$ is a proper graded ideal such that $\mathfrak{a}_0 = 0$ and $\mathfrak{a}M = 0$. We identify $R_0 = (R/\mathfrak{a})_0$. Show the following facts.

- The R_0 -module $\text{Hom}_{R_0}(M, R_0)$ stays the same, if consider M as an R/\mathfrak{a} -module.
- The R -module $\text{Hom}_{R_0}(M, R_0)$ is annihilated by \mathfrak{a} and its structure as an R/\mathfrak{a} -module coincides with the structure inherited from the R/\mathfrak{a} -module M .
- The graded R -module $D(M)$ satisfies $\mathfrak{a}D(M) = 0$ and is independent on whether we consider M as an R -module or an R/\mathfrak{a} -module.

We now shall begin to focus to the special case, where the graded ring R is a Noetherian homogeneous algebra over a field. In order to do so, we first of all recall a few general facts about *duals of vector spaces*.

9.2. **Reminder and Exercise.** A) (*Duality Functors for Vector Spaces*) Let K be a field. Keep in mind that the assignment

$$(V \xrightarrow{h} W) \rightsquigarrow (W^\vee = \text{Hom}_K(W, K) \xrightarrow{h^\vee = \text{Hom}_K(h, K)} V^\vee = \text{Hom}_K(V, K))$$

with

$$h^\vee(f) = \text{Hom}_K(h, K)(f) := f \circ h, \quad \forall f \in W^\vee = \text{Hom}_K(W, K)$$

defines a *contravariant, linear, exact functor of K -vector spaces*

$$\bullet^\vee = \text{Hom}_K(\bullet, K),$$

the *functor of taking K -duals* or the *duality functor for K -vector spaces*.

B) (*Dualizing and Finite Direct Sums*) Let $r \in \mathbb{N}$ and let $V_\bullet = (V_i)_{i=1}^r$ be a family of K -vector spaces. Check that there is an isomorphism of K -vector spaces

$$\nu^{V_\bullet} = \nu : \bigoplus_{i=1}^r (V_i^\vee) \xrightarrow{\cong} \left(\bigoplus_{i=1}^r V_i \right)^\vee$$

given by

$$\nu(u_1, u_2, \dots, u_r)(v_1, v_2, \dots, v_r) = \sum_{i=1}^r u_i(v_i), \quad u_i \in V_i^\vee, v_i \in V_i, (i = 1, 2, \dots, r).$$

Formulate and prove the fact that this isomorphism is *natural* in the obvious sense, so that the duality functor \bullet^\vee *commutes with finite direct sums*. Show that for a K -vector space V we can say:

- $V \cong V^\vee$ if and only if $\dim_K(V) < \infty$.
- If $\dim_K(V) < \infty$, then $\dim_K(V^\vee) = \dim_K(V)$.

C) (*Biduals*) We now are concerned with the *covariant linear exact functor* of K -vector spaces

$$\bullet^{\vee\vee} := (\bullet^{\vee})^{\vee}$$

of taking *biduals*. Check that for each K -vector space V there is a K -linear map

$$\gamma^V = \gamma : V \rightarrow V^{\vee\vee} : \quad \gamma(v)(f) = f(v), \quad \forall v \in V, \forall f \in V^{\vee}$$

and prove the following statements:

- For each K -vector space V , the linear map $\gamma^V : V \rightarrow V^{\vee\vee}$ is injective.
- The assignment $\gamma : V \mapsto (V \xrightarrow{\gamma^V} V^{\vee\vee})$ is a natural transformation of covariant functors of K -vector spaces $\gamma : \bullet \rightarrow \bullet^{\vee\vee}$.
- If $\dim_K(V) < \infty$, then the map $\gamma^V : V \rightarrow V^{\vee\vee}$ is an isomorphism.

D) (*Dualizing and Diagonals*) Let $r \in \mathbb{N}$, fix a K -vector space V and consider the r -th diagonal map on V , that is the injective K -linear map

$$\delta = \delta_r^V : V \rightarrow V^{\oplus r}, \quad v \mapsto (v, v, \dots, v), \forall v \in V,$$

and the surjective K -linear map

$$\delta^{\vee} = (\delta_r^V)^{\vee} : (V^{\oplus r})^{\vee} \rightarrow V^{\vee}.$$

Show that there is a commutative diagram

$$\begin{array}{ccc} (V^{\vee})^{\oplus r} & \xrightarrow[\cong]{\nu} & (V^{\oplus r})^{\vee} \\ & \searrow \beta & \swarrow \delta^{\vee} \\ & & V^{\vee} \end{array}$$

in which $\nu = \nu^{(V, V, \dots, V)}$ is the natural isomorphism defined for the finite family (V, V, \dots, V) of r copies of V in part B) and β is defined by the assignment $(u_1, u_2, \dots, u_r) \mapsto \sum_{i=1}^r u_i$.

E) (*Duality and Kernels*) Fix two K -vector spaces V and W , let $r \in \mathbb{N}$ and fix a finite family of K -linear maps

$$h_{\bullet} := (h_k)_{k=1}^r, \quad h_k \in \text{Hom}_K(V, W), \forall k \in \{1, 2, \dots, r\}.$$

Moreover consider the composition of the diagonal map δ_r^V of part D) with the direct sum of the maps h_k , that is the map

$$\sigma^{h_{\bullet}} = \sigma := (\oplus_{k=1}^r h_k) \circ \delta_r^V : V \rightarrow W^{\oplus r}, \quad v \mapsto (h_k(v))_{k=1}^r.$$

and its dual

$$(\sigma^{h_{\bullet}})^{\vee} = \sigma^{\vee} : (W^{\oplus r})^{\vee} \rightarrow V^{\vee}.$$

Prove the following statements:

- $\text{Ker}(\sigma) = \bigcap_{k=1}^r \text{Ker}(h_k)$.
- $\text{Im}(\sigma^{\vee}) = \sum_{k=1}^r \text{Im}(h_k^{\vee})$.

c) There is a short exact sequence of K -vector spaces

$$0 \rightarrow \sum_{k=1}^r \operatorname{Im}(h_k^\vee) \xrightarrow{\text{incl}} V^\vee \xrightarrow{\text{incl}^\vee} \left(\bigcap_{k=1}^r \operatorname{Ker}(h_k) \right)^\vee \rightarrow 0.$$

d) There is an isomorphism of K -vector spaces

$$V^\vee / \sum_{k=1}^r \operatorname{Im}(h_k^\vee) \xrightarrow{\cong} \left(\bigcap_{k=1}^r \operatorname{Ker}(h_k) \right)^\vee,$$

given by

$$u + \sum_{k=1}^r \operatorname{Im}(h_k^\vee) \mapsto u \upharpoonright_{\bigcap_{k=1}^r \operatorname{Ker}(h_k)}.$$

We now use the previous reminder, to establish a few basic facts about graded duals over graded K -algebras.

9.3. Exercise and Definition. A) (*Graded Duals over K -Algebras*) Let K be a field and let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded K -algebra, so that $R_0 = K$. We now reconsider the covariant, linear, left exact functor of graded R -modules $D(\bullet)$ introduced in (9.1). Use (9.1)B)a) and (9.2)A) to show:

a) For each homomorphism $h : M \rightarrow N$ of graded R -modules and all $t \in \mathbb{Z}$ we have the commutative diagram of K -linear maps

$$\begin{array}{ccc} N_{-t}^\vee & \xrightarrow[\cong]{\tau_t^N} & D(N)_t \\ \downarrow h_{-t}^\vee & & \downarrow D(h)_t \\ M_{-t}^\vee & \xrightarrow[\cong]{\tau_t^M} & D(M)_t \end{array}$$

where the maps τ_t^M and τ_t^N are defined according to (9.1)C)a).

b) The contravariant linear functor $D(\bullet)$ of graded R -modules is exact.

B) (*Modules with Finite Components*) We say that a graded R -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ has *finite components* if

$$\dim_K(M_n) < \infty, \quad \forall n \in \mathbb{Z}.$$

We denote the class of graded R -modules with finite components by \mathbb{F}_R . Use A)a), (9.2)B,b) and (9.2)C)c) to prove the following statements:

- If $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ is an exact sequence of graded R -modules, we have $M \in \mathbb{F}_R$ if and only if $N, P \in \mathbb{F}_R$.
- If $r \in \mathbb{N}$ and $M^{(1)}, M^{(2)}, \dots, M^{(r)} \in \mathbb{F}_R$, then $\bigoplus_{i=1}^r M^{(i)} \in \mathbb{F}_R$.
- If $M \in \mathbb{F}_R$, then $\dim_K(D(M)_t) = \dim_K(M_{-t})$ for all $t \in \mathbb{Z}$.
- If $M \in \mathbb{F}_R$, then $D(M) \in \mathbb{F}_R$.
- If $M \in \mathbb{F}_R$, the canonical map $\gamma^M : M \rightarrow D(D(M))$ (see (9.1)C)b)) is an isomorphism of graded R -modules.

f) If $M \in \mathbb{F}_R$, then $D(M) = 0$ if and only if $M = 0$.

C) (*Equihomogeneous Ideals*) Keep the above notations and hypotheses. An ideal $\mathfrak{a} \subseteq R$ is said to be *equihomogeneous* if it is generated by homogeneous elements of the same degree. We now are interested in finitely generated equihomogeneous ideals. So, let $s \in \mathbb{Z}$, let $r \in \mathbb{N}$, let $x_1, x_2, \dots, x_r \in R_s$, let M be a graded R -module and consider the multiplication maps given by these elements, that is the homomorphisms of graded R -modules

$$x_i = x_i \text{Id}_M : M \rightarrow M(s), \quad m \mapsto x_i m, (i = 1, 2, \dots, r).$$

Use (9.2)E) to show the following facts:

- a) $(0 :_M \langle x_1, x_2, \dots, x_r \rangle)_{-t} = \bigcap_{i=1}^r \text{Ker}(x_i \upharpoonright_{M_{-t}})$ for all $t \in \mathbb{Z}$.
 b) There is an isomorphism of graded R -modules

$$D(M) / \langle x_1, x_2, \dots, x_r \rangle D(M) \xrightarrow{\cong} D(0 :_M \langle x_1, x_2, \dots, x_r \rangle)$$

defined by

$$u + \langle x_1, x_2, \dots, x_r \rangle D(M) \mapsto u \upharpoonright_{(0 :_M \langle x_1, x_2, \dots, x_r \rangle)}, \quad \forall u \in D(M).$$

Now, we definitively shall consider the situation in which the graded ring R is a Noetherian homogeneous algebra over a field. In this situation we introduce a new class of functors, which we call *deficiency functors* and which are obtained by composing the local cohomology functors with respect to the irrelevant ideal R_+ of R with the graded duality functor .

9.4. Exercise and Definition. A) (*Deficiency Functors and -Modules*) Let K be a field and let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra. For each $i \in \mathbb{N}_0$ we define the *i -th deficiency functor* $K^i = K^i(\bullet)$ (over R) as the contravariant linear functor of graded R -modules obtained by composing the graded local cohomology functor ${}^*H_{R_+}^i(\bullet)$ with the graded duality functor $D = D(\bullet)$, thus the functor of graded R -modules given by the assignment

$$(M \xrightarrow{h} N) \mapsto (K^i(M) = D(H_{R_+}^i(N)) \xrightarrow{K^i(h)=D(H_{R_+}^i(h))} D(H_{R_+}^i(M)) = K^i(M)).$$

For each graded R -module M , the graded R -module $K^i(M)$ is called the *i -th deficiency module of M* .

B) (*First Properties of Deficiency Functors*) Keep the notations and hypotheses of part A). Let $i \in \mathbb{N}_0$. Prove the following facts:

- a) (*Duals of Deficiency Modules*) There is a natural transformation of covariant functors of graded R -modules

$$\kappa^i : H_{R_+}^i(\bullet) \rightarrow D(K^i(\bullet)) : M \mapsto (H_{R_+}^i(M) \xrightarrow{\kappa^i, M := \gamma^{H_{R_+}^i(M)}} D(K^i(M))),$$

where the homomorphism

$$\gamma^{H_{R_+}^i(M)} : H_{R_+}^i(M) \rightarrow D(D(H_{R_+}^i(M))) = D(K^i(M))$$

is defined according to (9.1C)b).

- b) (*Base Ring Independence of Deficiency Modules*) If M is a graded R -module, $\mathfrak{a} \subsetneq R$ is a proper graded ideal with $\mathfrak{a}M = 0$ we have $\mathfrak{a}K^i(M) = 0$ and (up to an isomorphism of graded R -modules) the module $K^i(M)$ remains the same if we consider M as a graded R/\mathfrak{a} -module.

C) (*Deficiency Modules of Finitely Generated Modules*) Let the notations be as in parts A) and B) and assume that the graded R -module M is finitely generated. Prove the following facts:

- a) $H_{R_+}^i(M)$ and $K^i(M)$ are graded modules with finite components, and hence belong to the class \mathbb{F}_R (see (9.3B)).
 b) $\dim_K(K^i(M))_n = h_M^i(-n)$ for all $n \in \mathbb{Z}$.
 c) $\text{beg}(K^i(M)) = -\text{end}(H_{R_+}^i(M)) > -\infty$.
 d) The natural homomorphism of graded R -modules of (9.4B)a) becomes an isomorphism

$$\kappa^{i,M} : H_{R_+}^i(M) \xrightarrow{\cong} D(K^i(M)).$$

D) (*The Deficiency Sequence*) Keep the above notations and hypothesis and let

$$\mathbb{S} : 0 \rightarrow N \xrightarrow{h} M \xrightarrow{l} P \rightarrow 0$$

be an exact sequence of graded R -modules. We form the *exact graded cohomology sequence with respect to R_+ and associated to \mathbb{S}* (see [Br-Fu-Ro](8.26)A))

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{R_+}^0(N) & \xrightarrow{H_{R_+}^0(h)} & H_{R_+}^0(M) & \xrightarrow{H_{R_+}^0(l)} & H_{R_+}^0(P) \\ & & \xrightarrow{\delta_{\mathbb{S}}^{0,R_+}} & H_{R_+}^1(N) & \xrightarrow{H_{R_+}^1(h)} & H_{R_+}^1(M) & \xrightarrow{H_{R_+}^1(l)} \dots \\ & & & & & \dots & \longrightarrow H_{R_+}^{i-1}(P) \\ & & \xrightarrow{\delta_{\mathbb{S}}^{i-1,R_+}} & H_{R_+}^i(N) & \xrightarrow{H_{R_+}^i(h)} & H_{R_+}^i(M) & \xrightarrow{H_{R_+}^i(l)} H_{R_+}^i(P) \\ & & \xrightarrow{\delta_{\mathbb{S}}^{i,R_+}} & H_{R_+}^{i+1}(N) & \xrightarrow{H_{R_+}^{i+1}(h)} & H_{R_+}^{i+1}(M) & \longrightarrow \dots \end{array}$$

Then, we apply the contravariant linear exact functor of graded R -modules $D(\bullet)$ to this sequence, write

$$\varepsilon_{\mathbb{S}}^i := D(\delta_{\mathbb{S}}^{i,R_+}), \quad \forall i \in \mathbb{N}_0$$

and thus end up with an exact sequence of graded R -modules

$$\begin{array}{ccccccc}
\dots & \longrightarrow & K^{i+1}(M) & \xrightarrow{K^{i+1}(h)} & K^{i+1}(N) & \xrightarrow{\varepsilon_{\mathbb{S}}^i} & \longrightarrow \\
K^i(P) & \xrightarrow{K^i(l)} & K^i(M) & \xrightarrow{K^i(h)} & K^i(N) & \xrightarrow{\varepsilon_{\mathbb{S}}^{i-1}} & \longrightarrow \\
K^{i-1}(P) & \longrightarrow & \dots & & & & \\
\dots & \longrightarrow & K^1(M) & \xrightarrow{K^1(h)} & K^1(N) & \xrightarrow{\varepsilon_{\mathbb{S}}^0} & \longrightarrow \\
K^0(P) & \xrightarrow{K^0(l)} & K^0(M) & \xrightarrow{K^0(h)} & K^0(N) & \longrightarrow & 0
\end{array}$$

We call this sequence the *deficiency sequence associated to \mathbb{S}* . Formulate and prove the fact, that the deficiency sequence is *natural*.

E) (*Socles of Local Cohomology Modules*) Let R be as above. For any graded R -module U one defines the *socle of U* as the graded submodule

$$\text{soc}(U) := (0 :_U R_+) \subseteq U.$$

Observe that $R_+ \text{soc}(U) = 0$, so that $\text{soc}(U)$ is a vector space over $R/R_+ \cong K$ and the R -submodules of $\text{soc}(U)$ are the same as the K -vector subspaces.

Now, let M be a finitely generated graded R -module and chose elements $x_1, x_2, \dots, x_r \in R_1$ such that

$$\langle x_1, x_2, \dots, x_r \rangle = R_+.$$

Let $i \in \mathbb{N}_0$ and use the developments of (9.3)C) to prove the following statements:

a) There is an isomorphism of graded R -modules

$$K^i(M)/R_+K^i(M) \xrightarrow{\cong} D(\text{soc}(H_{R_+}^i(M))).$$

b) $K^i(M)$ is finitely generated if and only if $\text{end}(H_{R_+}^i(M)) < \infty$ and $\text{soc}(H_{R_+}^i(M))$ is finitely generated.

c) If the equivalent conditions of statement b) are satisfied, then

$$\mu_{R,*}(K^i(M)) = b_{0,*}^R(K^i(M)) = (\dim_K(\text{soc}(H_{R_+}^i(M))_{-n})_{n \in \mathbb{Z}}).$$

F) (*Canonical Modules*) Keep the above notations and hypotheses. Let M be a finitely generated graded R -module. Prove that

a) $\sup\{i \in \mathbb{N}_0 \mid K^i(M) \neq 0\} = \dim_R(M)$.

The highest order non-vanishing deficiency module of M is called the *canonical module* of M and denoted by $K(M)$, thus

$$K(M) := \begin{cases} K^{\dim_R(M)} & , \quad M \neq 0 \\ 0 & , \quad M = 0 \end{cases}$$

In the next exercise we prepare some arguments which will be used repeatedly later.

9.5. Exercise. A) (*Deficiency Modules and Torsion*). Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a graded R -module. Prove the following statements

- a) If M is R_+ -torsion, then $K^i(M) = 0$ for all $i \in \mathbb{N}$.
- b) If M is finitely generated, then $K^0(M)$ is R_+ -torsion, finitely generated and satisfies $\dim_K(K^0(M)) = \dim_K(H_{R_+}^0(M)) < \infty$.
- c) If $N \subseteq M$ is a graded submodule which is R_+ -torsion and $p : M \rightarrow M/N$ is the canonical homomorphism, then the induced homomorphism $K^i(p) : K^i(M/N) \rightarrow K^i(M)$ is an isomorphism if $i > 0$ and a monomorphism if $i = 0$.

B) (*Deficiency Modules and Non-Zero Divisors*) Let the notations and hypotheses be as in part A). Let $t \in \mathbb{N}$ and let $x \in R_t \cap \text{NZD}_R(M)$. If we form the deficiency sequence associated to the short exact sequence of graded R -modules

$$\mathbb{S} : 0 \rightarrow M(-t) \xrightarrow{x} M \xrightarrow{p} M/xM \rightarrow 0$$

and write $\varepsilon_{M,x}^i := \varepsilon_{\mathbb{S}}^i$ for all $i \in \mathbb{N}_0$ (see (9.3D)), we can say:

- a) For each $i \in \mathbb{N}_0$ there is an exact sequence of graded R -modules

$$K^{i+1}(M) \xrightarrow{x} K^{i+1}(M)(t) \xrightarrow{\varepsilon_{M,x}^i} K^i(M/xM) \xrightarrow{K^i(p)} K^i(M) \xrightarrow{x} K^i(M)(t).$$

Consequently

- b) For each $i \in \mathbb{N}_0$ there is a short exact sequence of graded R -modules

$$0 \rightarrow (K^{i+1}(M)/xK^{i+1}(M))(t) \rightarrow K^i(M/xM) \rightarrow (0 :_{K^i(M)} x) \rightarrow 0.$$

Now we are ready to formulate and to prove our first result on the structure of deficiency modules.

9.6. Proposition. *Let K be a field, let $r \in \mathbb{N}_0$ and let $R := K[X_1, X_2, \dots, X_r]$ be a polynomial ring.*

- a) *If $i \neq r$, then $K^i(R) = 0$.*
- b) *$K^r(R) \cong R(-r)$.*

Proof. As R is CM, we have $H_{R_+}^i(R) = 0$ for all $i \neq r$. So, statement a) follows from (9.4)C)b).

We prove statement b) by induction on r . If $r = 0$, we have $R = K = H_{R_+}^0(R)$. If we apply (9.4)C)b) with $i = 0$ it follows that $K^0(M) = K = R = R(-0)$.

So let $r > 0$. We consider the polynomial ring

$$R' := K[X_1, X_2, \dots, X_{r-1}].$$

By induction we have $K^{r-1}(R') \cong R'(-r+1)$. Observe that there is an isomorphism of graded R -modules $R' \cong R/X_r R$. So, by the Base Ring Independence of Deficiency Modules (9.4)B)b) we get an isomorphism of graded R -modules

$$K^{r-1}(R/X_r R) \cong (R/X_r R)(-r+1).$$

If we apply the short exact sequence (9.5)B)b) with $i = r-1$, $x = X_r$, $M = R$ and keep in mind that $K^{r-1}(R) = 0$ we therefore get isomorphisms of graded R -modules

$$K^r(R)/X_r K^r(R) \cong K^{r-1}(R/X_r R)(-1) \cong (R/X_r R)(-r).$$

As a consequence

$$\begin{aligned} K^r(R)/(R_+)K^r(R) &\cong R/(X_r R)(-r)/(R_+)(R/X_r R)(-r) \cong \\ &\cong ((R/X_r R)/(R_+)(R/X_r R))(-r) \cong (R/R_+)(-r). \end{aligned}$$

This shows that $K^r(R)/(R_+)K^r(R)$ is generated by a single element of degree r , hence an element of the form $a + (R_+)K^r(R)$, with $a \in K(R)_r$. Consequently $K^r(R) = aR + (R_+)K^r(R)$. As $\text{beg}(K^r(M)) = -\text{end}(H_{R_+}^r(R)) > -\infty$ (see (9.4)C)b)), the Graded Nakayama Lemma implies that $K^r(R) = Ra$. So, there is an epimorphism of graded R -modules

$$R(-r) \xrightarrow{\pi} K^r(R) \rightarrow 0, \quad f \mapsto fa.$$

As $p_R(X) = \binom{X+r-1}{r-1}$ and $R_{-n} = H_{R_+}^i(M)_{-n} = 0$ for all $n > 0$, we have

$$(-1)^{r-1} h_R^r(-n) = \chi_R(-n) = p_R(-n) = \binom{-n+r-1}{r-1}$$

for all $n > 0$ and hence

$$h_R^r(-n) = \binom{n-1}{r-1} = \dim_K(R_{n-r}), \quad \forall n \geq r.$$

So, by (9.4)C)b) we end up with

$$\dim_K(K^r(M)_n) = \dim_K(R_{n-r}) = \dim_K(R(-r)_n), \quad \forall n \geq r.$$

This proves, that the epimorphism π is indeed an isomorphism. \square

Now, we are ready to prove the following *Main Theorem on Deficiency Modules*.

9.7. Theorem. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module and let $i \in \mathbb{N}_0$. Then*

- a) $K^i(M)$ is a finitely generated graded R -module.
- b) $\mu_{R,*}(K^i(M)) = b_{0,*}^R(K^i(M)) = (\dim_K(\text{soc}(H_{R_+}^i(M))_{-n}))_{n \in \mathbb{Z}}$.
- c) $\dim_K(K^i(M)_n) = h_M^i(-n)$ for all $n \in \mathbb{Z}$.
- d) $\text{beg}(K^i(M)) = -\text{end}(H_{R_+}^i(M)) > -\infty$.
- e) $K^i(M) = 0$ for all $i > \dim_R(M)$.
- f) $\dim_R(K^i(M)) \leq i$ for all $i \leq \dim_R(M)$ with equality if $i = \dim_R(M)$.

Proof. "a)": We find a polynomial ring $S = K[X_1, X_2, \dots, X_r]$ and a proper graded ideal $\mathfrak{a} \subsetneq S$ such that $R = S/\mathfrak{a}$. According to the Base Ring Independence of Deficiency Modules (9.4)B)b) we may consider M as a graded S -module and hence assume that $R = K[X_1, X_2, \dots, X_r]$. If $M = 0$ we have $K^i(M) = 0$. So, let $M \neq 0$. We show by induction on $h := \text{hdim}(M)$ that $K^i(M)$ is finitely generated. If $h = 0$ we have an isomorphism of graded R -modules

$$M \cong \bigoplus_{k=1}^s R(-a_k), \quad a_k \in \mathbb{Z}, \forall k \in \{1, 2, \dots, s\}, \quad a_1 \leq a_2 \leq \dots \leq a_s.$$

So, by (9.6) and the additivity of the contravariant functor of graded R -modules $K^i(\bullet)$ we get $K^i(M) = 0$ if $i \neq r$ and $K^r(M) \cong \bigoplus_{k=1}^s R(-r + a_k)$.

Now, let $h > 0$ and consider a minimal presentation

$$\mathbb{S} : 0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0, \quad F = \bigoplus_{k=1}^s R(-a_k)$$

of M . As $\text{hdim}(N) = \text{hdim}(M) - 1 = h - 1$, by induction $K^j(N)$ is finitely generated for all $j \in \mathbb{N}_0$. By the case $h = 0$ we have $K^j(F) = 0$ for all $j \neq r$ and $K^r(F)$ is a graded free R -module of finite rank. So, the deficiency sequence (9.4)D) associated to \mathbb{S} gives rise to isomorphisms of graded R -modules

$$K^{j+1}(N) \cong K^j(M), \quad \forall j \in \{0, 1, \dots, r-2\},$$

an epimorphism of graded R -modules

$$K^r(N) \rightarrow K^{r-1}(M) \rightarrow 0,$$

and a short exact sequence of graded R -modules

$$K^{r+1}(N) \rightarrow K^r(M) \rightarrow K^r(F).$$

Hence, $K^i(M)$ is finitely generated if $i \leq r$. As $K^i(M) = 0$ if $i > \dim_R(M)$ (see (9.4)F)a) and as $\dim_R(M) \leq r$, we get our claim.

"b)": This follows from statement a) and (9.4)E)c).

"c)": This is nothing else than (9.4)C)b).

"d)": This is a restatement of (9.4)C)c).

"e)": This is clear by (9.4)F)a).

"f)": Let $M \neq 0$. We prove by induction on i that $\dim_R(K^i(M)) \leq i$. The case $i = 0$ is immediate by (9.5)A)b). So, let $i > 0$. By (9.5)A)c) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $R_+ \notin \text{Ass}_{R_+}(M)$. So, by the Homogeneous Prime Avoidance Lemma there is some $t \in \mathbb{N}$ and some $x \in R_t$ which avoids all members of $\text{Ass}_R(M)$. Therefore $x \in R_t \cap \text{NZD}_R(M)$ and thus by (9.5)B)b) we get a short exact sequence of graded R -modules

$$(i) \quad 0 \rightarrow (K^i(M)/xK^i(M))(t) \rightarrow K^{i-1}(M/xM) \rightarrow (0 :_{K^{i-1}(M)} x) \rightarrow 0.$$

By induction, we have $\dim_R(K^{i-1}(M/xM)) \leq i - 1$. Therefore

$$\dim_R(K^i(M)/xK^i(M)) = \dim_R((K^i(M)/xK^i(M))(t)) \leq i - 1$$

and hence $\dim_R(K^i(M)) \leq \dim_R(K^i(M)/xK^i(M)) + 1 \leq i - 1 + 1 = i$.

It remains to show, that $\dim_R(K^d(M)) \geq d := \dim_R(M)$. We do this by induction d . The case $d = 0$ follows easily from (9.5)A)b). So, let $d > 0$. Then $\dim_R(M/\Gamma_{R_+}(M)) = d$ and as previously we may assume that $R_+ \notin \text{Ass}_R(M)$. Now, by the Homogeneous Prime Avoidance Lemma there is some $t \in \mathbb{N}$ and some $x \in R_t$ such that

$$x \notin \bigcup_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p}, \quad \mathcal{S} := \text{Ass}_R(M) \bigcup ((\text{Ass}_R(K_d(M)) \cup \text{Ass}_R(K^{d-1}(M)) \setminus \{R_+\}).$$

In particular x is a non-zero divisor with respect to M and filter-regular with respect to $K^d(M)$ and $K^{d-1}(M)$. Now, we may write down the sequence (i) with $i = d$ and get the short exact sequence of graded R -modules

$$(ii) \quad 0 \rightarrow (K^d(M)/xK^d(M))(t) \rightarrow K^{d-1}(M/xM) \rightarrow (0 :_{K^{d-1}(M)} x) \rightarrow 0$$

in which $(0 :_{K^{d-1}(M)} x)$ is R_+ -torsion and hence of dimension ≤ 0 . by the filter-regularity of x with respect to $K^{d-1}(M)$. As x is a non-zero divisor with respect to M we have $\dim_R(M/xM) = d - 1$. So, by induction we have $\dim_R(K^{d-1}(M/xM)) \geq d - 1$.

Our next aim is to show that $\dim_R(K^d(M)) > 0$. Indeed, assuming that $\dim(K^d(M)) \leq 0$, the sequence (ii) would imply that $\dim_R(K^{d-1}(M/xM)) \leq 0$ and hence that $d = 1$, so that the Hilbert polynomial P_M of M would be of degree 0, whence $h_M^1(n) = \chi_M(n) = P_M(n) \neq 0$ for all $n \ll 0$. Consequently by (9.4)C)b) we would have $K^1(M)_n \neq 0$ for all $n \gg 0$, which contradicts the assumption that $\dim_R(K^1(M)) \leq 0$.

Now, as $\dim_R(K^d(M)) > 0$ the element x is a parameter with respect to $K^r(M)$, whence

$$\dim_R(K^r(M)) = \dim_R(K^r(M)/xK^r(M)) + 1 = \dim_R(T) + 1,$$

where $T := (K^r(M)/xK^r(M))(t)$. As $K^r(M) \neq 0$ is finitely generated and $x \in R_+$ we have $K^r(M)/xK^r(M) \neq 0$ and hence $T \neq 0$. So, by the sequence (ii) and bearing in mind that $\dim_R(K^{d-1}(M/xM)) = 0$ we obtain $\dim_R(T) = \dim_R(K^{d-1}(M/xM)) \geq d - 1$ and hence that $\dim_R(K^d(M)) \geq d$. \square

9.8. Remark and Definition. A) (*Cohomological Hilbert Polynomials*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Fix $i \in \mathbb{N}_0$ and consider the Hilbert polynomial $P_{K^i(M)}$ of the finitely generated graded R -module $K^i(M)$. Then, by the definition of $P_{K^i(M)}$ and by (9.10)c) we have

$$h_M^i(n) = \dim_K(K^i(M)_{-n}) = P_{K^i(M)}(-n), \quad \forall n \ll 0.$$

If we set

$$p_M^i(X) := P_{K^i(M)}(-X)$$

we thus have

$$h_M^i(n) = p_M^i(n), \quad \forall n \ll 0.$$

The polynomial $p_M^i \in \mathbb{Q}[X]$ is called the *i-th cohomological Hilbert polynomial of M*.

B) (*First Properties of Cohomological Hilbert Polynomials*) Let the notations and hypotheses be as in part A). Prove the following facts:

- $\deg(p_M^i) \leq i - 1$ with equality if $i = \dim_K(M) > 0$.
- $p_{M(r)}^i(X) = p_M^i(r + X)$ for all $r \in \mathbb{Z}$.
- $P_M(X) = \sum_{i=1}^{\dim_R(M)-1} (-1)^{i-1} p_M^i(-X) = \sum_{n \in \mathbb{N}} (-1)^{i-1} p_M^i(-X)$.

C) (*Cohomological Postulation Numbers*) Let the notions and hypotheses be as in parts A) and B). Then clearly

$$\nu_M^i := \inf\{n \in \mathbb{Z} \mid p_M^i(n) \neq h_M^i(n)\} \in \mathbb{Z} \cup \{\infty\}.$$

The number ν_M^i is called the *i-th cohomological postulation number of M*. Prove the following statements:

- $\nu_M^i = \infty$ if and only if $H_{R_+}^i(M) = 0$.
- If $\nu_M^i < \infty$, then $\nu_M^i \leq \text{end}(H_{R_+}^i(M))$.
- $\nu_{M(r)}^i = \nu_M^i - r$ for all $r \in \mathbb{Z}$.
- $\nu_M^i \geq -\text{reg}(K^i(M))$.

9.9. Examples and Exercises. A) (*Homogeneous Gorenstein Algebras*) Let K be a field and let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra. Assume in addition, that R is CM. Prove that the following statements are equivalent;

- $K(R)$ is a cyclic R -module.
- $D(R) \cong H_{R_+}^d(R)(t)$ for some $t \in \mathbb{Z}$.
- $\text{soc}(H_{R_+}^d(M)) \cong (R/R_+)(t)$ for some $t \in \mathbb{Z}$.

If the Noetherian homogeneous CM-algebra R satisfies these equivalent conditions, it is called a *homogeneous Gorenstein algebra*. Prove the following facts:

- a) If $d > 0$ and $x \in R_+$ is a homogeneous non-zero divisor in R , then R is Gorenstein if and only if R/xR is.
- b) If R is Gorenstein, then $H_{R_+}^d(R)$ is $*$ injective.
- c) Each polynomial ring $K[X_1, X_2, \dots, X_r]$ is Gorenstein.

B) (*Top Local Cohomology of Polynomial Rings*) Let $r \in \mathbb{N}$, consider the polynomial ring

$$R := K[X_1, X_2, \dots, X_r]$$

and the *Laurent algebra*

$$L := K[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_r, X_r^{-1}],$$

furnished with its natural grading, so that

$$L_n = \bigoplus_{\nu_1, \dots, \nu_r \in \mathbb{Z}: \nu_1 + \dots + \nu_r = n} KX_1^{\nu_1} \dots X_r^{\nu_r}, \quad \forall n \in \mathbb{Z}.$$

Moreover consider the graded R -submodule

$$W := \bigoplus_{(\nu_1, \dots, \nu_r) \in \mathbb{Z}^r \setminus \mathbb{Z}_{<0}^r} KX_1^{\nu_1} \dots X_r^{\nu_r} \subseteq L$$

and the graded R -module

$$K[X_1^-, X_2^-, \dots, X_r^-] = R^- := L/W.$$

For each Laurent polynomial $l \in L$ let $l^- := l + W \in R_-$. Prove the following facts:

- a) For all $n \in \mathbb{Z}$ we have $W_n = \bigoplus_{\nu_1, \dots, \nu_r \in \mathbb{Z}_{<0}: \nu_1 + \dots + \nu_r = n} K(X_1^{\nu_1} \dots X_r^{\nu_r})^-$.
- b) $\text{end}(R^-) = -1$ and $\dim_K(R_n^-) = \dim_K(R_{-n-1}) = \binom{-n+r-2}{r-1}$ for all $n < 0$.
- c) $(0 :_{R_-} \langle X_1, X_2, \dots, X_r \rangle) = R_{-1}^-$.
- d) $D(R^-) = D(R^-)_1 R$.
- e) There is an isomorphism of graded R -modules $D(R^-) \cong R(-1)$.
- f) There is an isomorphism of graded R -modules

$$H_{\langle X_1, X_2, \dots, X_r \rangle}^r(K[X_1, X_2, \dots, X_r]) \cong K[X_1^-, X_2^-, \dots, X_r^-].$$

Next, we aim to prove a few basic results canonical modules. We begin with a statement on the *Grade of Canonical Modules*. This result already hints an important property of the operation of taking canonical modules: namely its "improving effect on grade".

9.10. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Then*

$$\text{grade}_{K(M)}(R_+) \geq \min\{2, \dim_R(M)\}.$$

Proof. Let $d := \dim_R(M)$. If $d \leq 0$ our claim is obvious. So, let $d > 0$ and set $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\dim_R(\bar{M}) = d$ and hence $K(\bar{M}) = K^d(\bar{M}) \cong K^d(M) = K(M)$ (see (9.5)A)c). This allows us to replace M by \bar{M} and hence to assume that $\Gamma_{R_+}(M) = 0$. So, once more by the Homogeneous Prime Avoidance Lemma we find some $t \in \mathbb{N}$ and some $x \in R_t \cap \text{NZD}_R(M)$. Now, by the exact sequence (9.5)B)b), applied with $i = d$, we get an epimorphism

$$K^d(M/xM) \rightarrow (0 :_{K^d(M)} x) \rightarrow 0.$$

As $x \in R_+ \cap \text{NZD}_R(M)$ we also have $\dim_R(M/xM) = d - 1$ and hence $K^d(M/xM) = 0$ (see ((9.4)F)). It follows that $(0 :_{K^d(M)} x) = 0$ and hence $x \in \text{NZD}_R(K^d(M))$. Thus, if $d = 1$, we get our claim. So, let $d > 1$. Another use of the sequence (9.5)B)b), this time applied with $i = d - 1$, yields a monomorphism

$$0 \rightarrow (K^d(M)/xK^d(M))(t) \rightarrow K^{d-1}(M/xM).$$

As $\dim_R(M/xM) = d - 1 > 0$ we have $K^{d-1}(M/xM) = K(M/xM)$ and hence by induction we get $\text{grade}_{K^{d-1}(M/xM)}(R_+) > 0$, hence $\Gamma_{R_+}(M/xM) = 0$. Now, the above monomorphism shows that $\Gamma_{R_+}((K^d(M)/xK^d(M))(t)) = 0$ and hence $\Gamma_{R_+}(K^d(M)/xK^d(M)) = 0$, so that $\text{grade}_{K^d(M)/xK^d(M)}(R_+) \geq 1$. As $x \in R_+ \cap \text{NZD}_R(K^d(M))$ it follows that $\text{grade}_{K^d(M)}(R_+) \geq 2$ and this proves our claim. \square

The previous result tells us, that in certain cases the grade of a module may go up if one passes to the canonical module. This hints, that the formation of canonical modules has a "smoothing effect". Our next result is a *Structure Theorem for Canonical Modules* which supports this observation. Its first statement says that canonical modules are *unmixed* as usually CM-modules are. The second statement says, that the canonical module of a graded module is not affected if one replaces the original module by its unmixed part. The third statement says, that the canonical module satisfies a strong version of the *second Serre property* S_2 .

We first give a few preparations which are related to the notion of unmixedness.

9.11. Exercise and Definition. A) (*Unmixed Graded Modules*). Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. We say that M is *unmixed* if $\dim(R/\mathfrak{p}) = \dim_R(M)$ for all $\mathfrak{p} \in \text{Ass}_R(M)$. Keep in mind (or reprove) the following fact:

a) If M is CM, then M is unmixed.

B) (*Unmixed Parts*) Keep the notations and hypotheses of part A). We introduce the following notation;

$$\text{Ass}_R^{[0]}(M) := \{\mathfrak{p} \in \text{Ass}_R(M) \mid \dim(R/\mathfrak{p}) = \dim_R(M)\},$$

$$\mathfrak{a}^{[0,M]} := \bigcap_{\mathfrak{p} \in \text{Ass}_R(M) \setminus \text{Ass}_R^{[0]}(M)} \mathfrak{p},$$

with the convention that $\mathfrak{a}^{[0,M]} = R$ if $\text{Ass}_R^{[0]}(M) = \text{Ass}_R(M)$. Prove the following statements:

- $\Gamma_{\mathfrak{a}^{[0,M]}}(M) \subseteq M$ is the largest graded submodule whose dimension is strictly less than the dimension of M .
- $M^{[0]} := M/\Gamma_{\mathfrak{a}^{[0,M]}}(M)$ is unmixed with $\text{Ass}_R(M^{[0]}) = \text{Ass}_R^{[0]}(M)$.
- If $p : M \rightarrow M^{[0]}$ is the canonical epimorphism and $q : M \rightarrow \bar{M}$ is a further epimorphism of graded R -modules such that \bar{M} is unmixed with $\dim_R(\bar{M}) = \dim_R(M)$, there is a unique homomorphism of graded R -modules s , which occurs in the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{p} & M^{[0]} \\ & \searrow q & \swarrow s \\ & & \bar{M} \end{array}$$

So, $p : M \rightarrow M^{[0]}$ is characterized as the largest unmixed quotient of M which has the same dimension as M . Therefore, the graded R -module $M^{[0]} = M/\Gamma_{\mathfrak{a}^{[0,M]}}(M)$ is called the *unmixed part* of M .

Now, we are ready to prove the announced structure result for canonical modules.

9.12. Theorem. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Then*

- $\text{Ass}_R(K(M)) = \{\mathfrak{p} \in \text{Ass}_R(M) \mid \dim(R/\mathfrak{p}) = \dim_R(M)\}$. In particular $K(M)$ is unmixed.
- The canonical epimorphism $p : M \rightarrow M^{[0]}$ induces an isomorphism of graded R -modules

$$K^{\dim_R(M)} : K(M^{[0]}) \rightarrow K(M).$$

- For all $\mathfrak{q} \in {}^*\text{Spec}(R)$ with $\text{depth}_{R_{\mathfrak{q}}}(K(M)_{\mathfrak{q}}) = 1$ it holds

$$\dim(R/\mathfrak{q}) = \dim_R(M) - 1.$$

Proof. "a)": Let $d := \dim_R(M)$. If $d = -\infty$, we have $M = K(M) = 0$, and our claim is obvious. If $d = 0$ we have $\dim_R(K(M)) = 0$ and hence $\text{Ass}_R(K(M)) =$

$R_+ = \text{Ass}_R(M)$, and our claim is again clear. So, let $d > 0$ and let us proceed by induction. As usually, we first may assume that $\text{grade}_M(R_+) > 0$.

Let $\mathfrak{p} \in \text{Ass}_R(M)$ with $\dim(R/\mathfrak{p}) = d$. We aim to show that $\mathfrak{p} \in \text{Ass}_R(K(M))$. By our choice of \mathfrak{p} we find an integer s and an element $m \in M_s$ such that $\mathfrak{p} = (0 :_R m)$ and so multiplication by m yields an exact sequence of graded R -modules $0 \rightarrow R/\mathfrak{p} \rightarrow M(s) \rightarrow N \rightarrow 0$. Applying the associated deficiency sequence (9.4)D) we get an exact sequence of graded R -modules

$$K^d(M(s)) \rightarrow K^d(R/\mathfrak{p}) \rightarrow K^{d-1}(N).$$

By the Base Ring Independence of Deficiency Modules (see (9.4)B)b)) we may consider $K^d(R/\mathfrak{p})$ as a graded R/\mathfrak{p} -module and by (9.7)f) this R/\mathfrak{p} -module has dimension d . Therefore the zero ideal in R/\mathfrak{p} is associated to $K^d(R/\mathfrak{p})$, whence $\mathfrak{p} \in \text{Ass}_R(K^d(R/\mathfrak{p})) \subseteq \text{Supp}_R(K^d(R/\mathfrak{p}))$. Moreover by (7.9)f) we have $\dim_R(K^{d-1}(N)) \leq d - 1$ and hence $\mathfrak{p} \notin \text{Supp}_R(K^{d-1}(N))$. If we localize the above exact sequence at \mathfrak{p} we thus get an epimorphism of $R_{\mathfrak{p}}$ -modules $(K^d(M(s)))_{\mathfrak{p}} \rightarrow K^d(R/\mathfrak{p})_{\mathfrak{p}}$ in which the target does not vanish. This shows that $\mathfrak{p} \in \text{Supp}_R(K^d(M(s))) = \text{Supp}_R(K^d(M))$. As $\dim(R/\mathfrak{p}) = d = \dim_R(K^d(M))$ it follows that \mathfrak{p} is a minimal member of $\text{Supp}_R(K^d(M))$ and hence $\mathfrak{p} \in \text{Ass}_R(K^d(M))$.

Now, let $\mathfrak{p} \in \text{Ass}_R(K^d(M))$. By the linearity of the functor $K^d(\bullet)$ we have $(0 :_R M)K^d(M) = 0$, so that $\mathfrak{p} \in \text{Var}(0 :_R M) = \text{Supp}_R(M)$. It thus remains to show that $\dim(R/\mathfrak{p}) \geq d$. By (9.10) we have $\text{grade}_M(R_+) > 0$. By assumption we also have $\text{grade}_M(R_+) > 0$. So, by the Homogeneous Prime Avoidance Lemma as usually we find some $t \in \mathbb{N}$ and some $x \in R_t \cap \text{NZD}_R(K^d(M))$. As $\mathfrak{p}, Rx \subseteq R_+$ we find a minimal prime ideal \mathfrak{q} of $\mathfrak{p} + Rx$. According to the Non-Zero Divisor Lemma of Matsumura we have $\mathfrak{q} \in \text{Ass}_R(K^d(M)/xK^d(M))$, hence $\mathfrak{q} \in \text{Ass}_R((K^d(M)/xK^d(M))(t))$. If we use the sequence (9.5)B)b) with $i = d - 1$ we obtain a monomorphism $(K^d(M)/xK^d(M))(t) \hookrightarrow K^{d-1}(M/xM)$ so that finally $\mathfrak{q} \in \text{Ass}_R(K^{d-1}(M/xM))$. By our choice of x we also have $\dim_R(M/xM) = d - 1$ so that $K^{d-1}(M/xM) = K(M/xM)$. Hence by induction we have $\dim(R/\mathfrak{q}) = d - 1$, and as $\mathfrak{p} \subsetneq \mathfrak{q}$ we get $\dim(R/\mathfrak{p}) \geq d$ as requested.

"b)": Keep all notations introduced in the proof of statement a). If $d \leq 0$ we have $M^{[0]} = M$ and our claim is obvious. If $d > 0$, we form the deficiency sequence associated to the short exact sequence of graded R -modules

$$0 \rightarrow \Gamma_{\mathfrak{a}^{[0],M]}(M) \xrightarrow{\subseteq} M \xrightarrow{p} M^{[0]} \rightarrow 0$$

(see (9.4)D)). Bearing in mind that $\dim_R(\Gamma_{\mathfrak{a}^{[0],M]}(M)) < d$ and hence $K^i(M) = 0$ for all $i \geq d$ (see (9.4)F)), we get an isomorphism of graded R -modules $K^d(p) : K^d(M^{[0]}) \rightarrow K^d(M)$, and this is precisely our claim.

"c)": By statement b) we may replace M by $M^{[0]}$ and hence assume that M is unmixed. So, by statement a) we have $\text{Ass}_R(K(M)) = \text{Ass}_R(M)$ with

$\dim(R/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \text{Ass}_R(M)$. Now, let

$$\mathfrak{q} \in {}^*\text{Spec}(R) \text{ such that } \text{depth}_{R_{\mathfrak{q}}}(K(M)_{\mathfrak{q}}) = 1.$$

Then clearly $\mathfrak{q} \in \text{Supp}_R(K(M)) \setminus \text{Ass}_R(K(M))$, so that $\mathfrak{q} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R(K(M)) = \text{Ass}_R(M) =: \mathcal{S}$. Consequently by the Homogeneous Prime Avoidance Lemma there is some $t \in \mathbb{N}$ and some $x \in \mathfrak{q}_t \setminus \bigcup_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p}$. Therefore $x \in \mathfrak{q}_t \cap \text{NZD}_R(K(M)) = \mathfrak{q}_t \cap \text{NZD}_R(M)$. As $\text{depth}_{R_{\mathfrak{q}}}(K(M)_{\mathfrak{q}}) = 1$, it follows that

$$\mathfrak{q} \in \text{Ass}_R(K(M)/xK(M)) = \text{Ass}_R((K^d(M)/xK^d(M))(t)).$$

By the exact sequence (9.5)B)b) we have a monomorphism of graded R -modules

$$(K^d(M)/xK^d(M))(t) \hookrightarrow K^{d-1}(M/xM),$$

so that $\mathfrak{q} \in \text{Ass}_R(K^{d-1}(M/xM))$. As $\dim_R(M/xM) = d-1$ and $K^{d-1}(M/xM) = K(M/xM)$ it follows by statement a) that $\dim(R/\mathfrak{q}) = d-1$. \square

Our next result says that the canonical module of a CM-module is again a CM-module.

9.13. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module which is CM. Then the canonical module $K(M)$ of M is CM, too.*

Proof. Let $d := \dim_R(M)$. If $d \leq 2$, we may conclude by (9.10). So, let $d > 2$. Then M and $K^d(M)$ are both of grade ≥ 2 , and so clearly there is some $t \in \mathbb{N}$ and some $x \in R_t \cap \text{NZD}_R(M) \cap \text{NZD}_R(K^d(M))$. By (9.5)B)b), applied with $i = d-1$, we get an exact sequence of graded R -modules

$$0 \rightarrow (K^d(M)/xK^d(M))(t) \rightarrow K^{d-1}(M/xM) \rightarrow (0 :_{K^{d-1}(M)} x) \rightarrow 0.$$

As M is CM of dimension d we have $H_{R_+}^{d-1}(M) = 0$ and hence $K_{d-1}(M) = D(H_{R_+}^{d-1}(M)) = 0$. So, the above sequence yields an isomorphism of graded R -modules

$$(K^d(M)/xK^d(M))(t) \cong K^{d-1}(M/xM).$$

As $x \in R_+ \cap \text{NZD}_R(M)$, the R -module M/xM is CM of dimension $d-1$. By induction and by the above isomorphism it thus follows that

$$\text{grade}_{(K^d(M)/xK^d(M))(t)}(R_+) = d-1$$

and hence $\text{grade}_{K(M)/xK^d(M)}(R_+) = d-1$. As $x \in R_+ \cap \text{NZD}_R(K^d(M))$ this implies that $\text{grade}_{K^d(M)}(R_+) = d$. As $\dim(K^d(M)) = d$, this proves our claim. \square

As the more experienced readers may have observed, our approach to deficiency modules is not the standard one, which reveals itself from Grothendieck's *Local Duality Theorem* in its graded form. In the following remark, we sketch the relation between the standard point of view and the approach we have chosen in these lectures.

9.14. **Remark.** A) (*Deficiency Modules over Noetherian Local Rings*) Usually deficiency modules are introduced in the situation, where M is a finitely generated module over a Noetherian local ring (R, \mathfrak{m}) which at its turn is a homomorphic image of a Noetherian local Gorenstein ring (R', \mathfrak{m}') . Then, for $i \in \mathbb{N}_0$ one defines the i -th deficiency module of M to be the finitely generated graded module

$$K^i(M) := \text{Ext}_{R'}^{\dim(R')-i}(M, R'),$$

furnished with its natural structure as an R -module. One can show, that up to isomorphism, this module is indeed independent of the choice of the local Gorenstein ring R' , as long as R is a quotient of R' . An extended study of this modules and their structure may be found in [Sc1]. A particularly interesting special case is again the *canonical module*

$$K(M) := K^{\dim_R(M)}(M)$$

of M . Even in the special situation where $M = R$ is a CM-ring, the canonical module $K(R)$ is an interesting object. A classical introduction to this subject may be found in [Her-Kun]. We refer the reader also to [Br-Sh1], [Bru-Her] or [E1].

B) (*Matlis Duals*) Keep all notations and hypotheses of part A). Let E denote the *injective hull* of the R -module R/\mathfrak{m} and consider the contravariant linear exact functor

$$\text{Hom}_R(\bullet, E) := D(\bullet) : (M \xrightarrow{h} N) \mapsto (D(N) \xrightarrow{D(h)} D(M))$$

of taking *Matlis Duals*. This functor is of basic importance in commutative algebra. In certain cases, taking *Matlis biduals* $D(D(M))$ gives back the original module M , as stated by the so called *Matlis Duality Theorem*. We refer the interested reader to [Br-Sh1].

C) (*Local Duality*) Keep all hypotheses and notations of parts A) and B). Then, the *Local Duality Theorem* of Grothendieck [Gro2] says that for each finitely generated R -module M and each $i \in \mathbb{N}_0$ there is an isomorphism of R -modules

$$H_{\mathfrak{m}}^i(M) \cong \text{Hom}_R(\text{Ext}_{R'}^{\dim(R')-i}(M, R'), E) = D(K^i(M)).$$

Moreover, if the local ring R is \mathfrak{m} -adically complete, then there are isomorphisms (see [?] (3.5.8) for example)

$$K^i(M) \cong D(H_{\mathfrak{m}}^i(M)).$$

We also refer the reader to [Br-Sh1], [Bru-Her] or [E1]. The particularity of these result is that they describe local cohomology modules as Matlis duals of certain finitely generated R -modules – and vice versa, if R is \mathfrak{m} -adically complete.

D) (*Graded Deficiency Modules*) Now, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring (R_0, \mathfrak{m}_0) and let $R' \rightarrow R$ be a surjective homomorphism of graded rings such that $R' = \bigoplus_{n \in \mathbb{N}_0} R'_n$ is a Noetherian homogeneous Gorenstein ring with local base ring (R'_0, \mathfrak{m}'_0) . Then, for each

graded R -module M and each $i \in \mathbb{N}_0$ one may define the i -th *graded deficiency module* of M as the graded module

$$K^i(M); = {}^*\text{Ext}_{R'}^{\dim(R')-i}(M, R'(-\dim(R'))),$$

where ${}^*\text{Ext}_{R'}^j(\bullet, \bullet)$ denotes the j -th Ext bifunctor in the category of graded R' -modules. Also here one shows, that up to isomorphism of graded R -modules, the module $K^i(M)$ does not depend on the chosen surjective homomorphism $R' \rightarrow R$ of graded rings, as long as R' is Gorenstein. Keep in mind that for all $j \in \mathbb{N}_0$ and any finitely generated graded R -module M the covariant linear functor $\text{Ext}_{R'}^j(M, \bullet)$ has the $*$ restriction property (see [Br-Fu-Ro] (8.9), [Br-Sh1] (12.2.7)), so that for each graded R' -module N and each $j \in \mathbb{N}_0$ there is a "natural" isomorphism of R -modules ${}^*\text{Ext}_{R'}^j(M, N) \cong \text{Ext}_{R'}^j(M, N)$. Thus in particular, for all $i \in \mathbb{N}_0$ and any finitely generated graded R -module M we also may write

$$K^i(M) = \text{Ext}_{R'}^{\dim(R')-i}(M, R'(-\dim(R'))),$$

where the right hand side R -module is furnished with the grading resulting from the $*$ restriction property.

E) (*Graded Matlis Duals*) Keep the hypotheses and notations of part D) and let $\mathfrak{m} := \mathfrak{m}_0 + R_1$ denote the *graded maximal ideal* of R . Let *E denote the $*$ injective hull of the graded R -module R/\mathfrak{m} and consider the contravariant linear exact functor of graded R -modules given by

$${}^*D(\bullet) : (M \xrightarrow{h} N) \rightsquigarrow ({}^*\text{Hom}_R(N, {}^*E) \xrightarrow{{}^*\text{Hom}_R(h, {}^*E)} {}^*\text{Hom}_R(M, {}^*E)),$$

the functor of taking *graded Matlis duals*. Here again, if the graded R -module M is finitely generated, we may write

$${}^*D(M) = \text{Hom}_R(M, {}^*E).$$

Moreover, if the base ring $R_0 = K$ is a field, the *graded Matlis dual* ${}^*D(M)$ of the graded R -module M luckily coincides with the graded dual $D(M)$ of M as it was introduced in (9.1) (see [Br-Sh1](13.3.5)). This means that in this particular situation, we can identify the two duality functors and write

$${}^*D(\bullet) = D(\bullet).$$

F) (*Graded Local Duality*) Keep the notations and hypotheses of parts D) and E). Then the *Graded Local Duality Theorem* says that for each finitely generated graded R -module M and each $i \in \mathbb{N}_0$ there are isomorphisms of graded R -modules (see [Br-Sh1] (13.4.3) for example):

$$H_{\mathfrak{m}}^i(M) \cong {}^*\text{Hom}_R(M, {}^*\text{Ext}_{R'}^{\dim(R')-i}(M, R'(-\dim(R')))) = {}^*D(K^i(M)).$$

Moreover, if the local base ring R_0 is \mathfrak{m}_0 -adically complete, then there are isomorphisms (see [?] (3.6.19) for example)

$$K^i(M) \cong {}^*D(H_{\mathfrak{m}}^i(M)).$$

This result translates the meaning of ordinary local duality over local rings to the graded context: The i -th local cohomology module $H_{\mathfrak{m}}^i(M)$ of the finitely generated graded R -module M is the graded Matlis dual of the finitely generated graded R -module $K^i(M)$ and that the graded deficiency module $K^i(M)$ at its turn is the graded Matlis dual of the local cohomology module $H_{\mathfrak{m}}^i(M)$.

G) (*Duality over Homogeneous K -Algebras*) Let the notations and hypotheses be as in parts D),E) and F). Assume in addition, that the base ring R_0 is a field K so that $R = K \oplus R_1 \oplus R_2 \dots$. Observe that in this case we have $\mathfrak{m} = R_+$. Now, let M be a finitely generated graded R -module. Then on use of the identification suggested at the end of part E) we get isomorphisms of graded R -modules

$$\begin{aligned} H_{R_+}^i(M) &\cong D(K^i(M)), \\ K^i(M) &\cong D(H_{R_+}^i(M)). \end{aligned}$$

In (9.4) we took the latter of these two isomorphisms (which holds for finitely generated graded modules M) to define the notion of deficiency module and of deficiency functor for arbitrary graded R -modules. This definition prevented us from introducing the whole (co-)homological machinery which is needed to install the Graded Local Duality Theorem. But on the other hand, our approach covers only the special case of Noetherian homogeneous algebras over a field. For the purpose of these lectures, we decided to consider this narrow-gauge track to the subject as being adequate.

Let us conclude this section which another theme neglected up to now in these lectures.

9.15. Exercise and Remark. A) (*Graded Noetherian and Graded Artinian Modules*) Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring and let M be a graded R -module. We say that M is **Noetherian* or *graded Noetherian*, if each ascending sequence $(N^{(i)})_{i \in \mathbb{N}_0}$ of graded submodules $N^{(i)} \subseteq M$ becomes stationary. Correspondingly we say that M is **Artinian* or *graded Artinian* if each descending sequence $(N^{(i)})_{i \in \mathbb{N}_0}$ of graded submodules $N^{(i)} \subseteq M$ becomes stationary. Observe the following facts:

- a) M is **Noetherian* if and only if all graded submodules of M are finitely generated.
- b) The properties of being **Noetherian* and **Artinian* are inherited by graded subquotients.
- c) If M is Noetherian, it is **Noetherian*, and if R is Noetherian, the converse is true also.
- d) If M is Artinian, it is **Artinian*.
- e) If R is positively graded and M is **Noetherian*, then $\text{beg}(M) > -\infty$ and M_n is a Noetherian R_0 -module for all $n \in \mathbb{Z}$.
- f) If R is positively graded and M is **Artinian*, then $\text{end}(M) < \infty$ and M_n is an Artinian R_0 -module for all $n \in \mathbb{Z}$.

B) (*Graded Noetherian and Graded Artinian Modules over K -Algebras*) Keep the notations and hypotheses of part A). Assume in addition that $R_0 = K$ is a field. Use what is said in (9.3) to prove the following:

$M \in \mathbb{F}_R$ is *Noetherian if and only if $D(M)$ is *Artinian.

$M \in \mathbb{F}_R$ is *Artinian if and only if $D(M)$ is *Noetherian.

C) (*Graded Noetherian and Graded Artinian Modules over Homogeneous K -Algebras*) Keep the notations of part B) but assume that R is positively graded. Prove the following:

a) If M is *Noetherian or *Artinian, then $M \in \mathbb{F}_R$.

b) M is *Noetherian if and only if $D(M)$ is *Artinian.

c) M is *Artinian if and only if $D(M)$ is *Noetherian.

D) (*Local Cohomology modules over Noetherian Homogeneous K -Algebras*) Let R be as in statement C) but in addition Noetherian and homogeneous. Prove the following:

a) If M is a finitely generated graded R -module, then the local cohomology module $H_{R_+}^i(M)$ is *Artinian for each $i \in \mathbb{N}_0$.

b) In the situation of statement a), the module $H_{R_+}^i(M)$ is indeed Artinian.

10. REGULARITY OF MODULES OF DEFICIENCY

Already in Mumford's Lecture Note [Mu1] the study of the regularity of deficiency modules is called to be of basic significance. In this section, we are precisely concerned with this issue. Our main result will say that the regularity of the deficiency modules of a given finitely generated graded module over a Noetherian homogeneous K -algebra is bounded in terms of the cohomology diagonal of M and the beginning of M . We rephrase this a bit more precisely: Let $d \in \mathbb{N}$ and let $i \in \mathbb{N}_0$. Then, there is a function

$$G_d^i : \mathbb{N}_0^d \times \mathbb{Z} \rightarrow \mathbb{Z}$$

such that for each field K each Noetherian homogeneous K -algebra $R = K \oplus R_1 \oplus R_2 \dots$ and each a finitely generated graded R -module M with $\dim_R(M) \leq d$ we have the estimate

$$\operatorname{reg}(K^i(M)) \leq G_d^i(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)).$$

With this result we will have reached the climax of our course. Indeed, the result taken for its own seems to have a very technical flavour and it may not be evident at once, why this estimate should be the ultimate peak in our climbing tour. But we shall be able to draw some conclusions from it, which show that it has indeed far reaching consequences.

Instead of starting to dwell on these consequences, we now immediately begin with our last "tour de force" and meet the technical preparations which will help to bring us to the last peak we are heading for - in the hope that we shall get recompensation by the view from the top.

10.1. Lemma. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module and let $x \in R_1$ be a filter-regular element with respect to M . Then*

$$\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM) \leq \operatorname{reg}(M).$$

Proof. We have two short exact sequences of graded R -modules

$$0 \rightarrow (0 :_M x) \rightarrow M \rightarrow M/(0 :_M x) \rightarrow 0,$$

$$0 \rightarrow (M/(0 :_M x))(-1) \rightarrow M \rightarrow M/xM \rightarrow 0.$$

As $(0 :_M x)$ is R_+ -torsion we get an isomorphism of graded R -modules

$$H_{R_+}^1(M) \cong H_{R_+}^1(M/(0 :_M x)),$$

so that $\operatorname{reg}^1(M/(0 :_M x)) = \operatorname{reg}^1(M)$. Now, by (3.3)B)b) and with (3.3)C)b), applied to the second exact sequence it follows that

$$\begin{aligned} \operatorname{reg}^1(M) &= \operatorname{reg}^1(M/(0 :_M x)) = \operatorname{reg}((M/(0 :_M x))(-1)) - 1 \leq \\ &\leq \max\{\operatorname{reg}^1(M), \operatorname{reg}(M/xM) + 1\} - 1, \end{aligned}$$

whence $\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM)$.

If we apply (3.3)C)d) and (3.3)B)b) to the second sequence we get

$$\begin{aligned} \operatorname{reg}(M/xM) &\leq \max\{\operatorname{reg}^1((M/(0 :_M x))(-1)) - 1, \operatorname{reg}(M)\} = \\ &= \max\{\operatorname{reg}^1(M), \operatorname{reg}(M)\} = \operatorname{reg}(M), \end{aligned}$$

whence $\operatorname{reg}(M/xM) \leq \operatorname{reg}(M)$. \square

The next result also has the flavour of a lemma. But as it has so many nice uses we decided to honour it by calling it a proposition.

10.2. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module, let $x \in R_1$ be filter-regular with respect to M and let $m \in \mathbb{Z}$ be such that $\operatorname{reg}(M/xM) \leq m$ and $\operatorname{gendeg}((0 :_M x)) \leq m$. Then*

$$\operatorname{reg}(M) \leq m + h_M^0(m).$$

Proof. By (10.1) we have $\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM) \leq m$. So, it remains to show that

$$\operatorname{end}(H_{R_+}^0(M)) \leq m + h_M^0(m).$$

The short exact sequence of graded R -modules

$$0 \rightarrow (M/(0 :_M x))(-1) \rightarrow M \rightarrow M/xM \rightarrow 0$$

induces exact sequences of K -vector spaces

$$\begin{aligned} 0 \rightarrow H_{R_+}^0(M/(0 :_M x))_n \rightarrow H_{R_+}^0(M)_{n+1} \rightarrow \\ \rightarrow H_{R_+}^0(M/xM)_{n+1} \rightarrow H_{R_+}^1(M/(0 :_M x))_n \end{aligned}$$

for all $n \in \mathbb{Z}$. As $H_{R_+}^0(M/xM)_{n+1} = 0$ for all $n \geq m$, we therefore obtain

$$H_{R_+}^0(M/(0 :_M x))_n \cong H_{R_+}^0(M)_{n+1}, \quad \forall n \geq m.$$

The short exact sequence of graded R -modules

$$0 \rightarrow (0 :_M x) \rightarrow M \rightarrow M/(0 :_M x) \rightarrow 0$$

and the facts that

$$H_{R_+}^0((0 :_M x)) = (0 :_M x), \quad H_{R_+}^1((0 :_M x)) = 0$$

induces short exact sequences of K -vector spaces

$$0 \rightarrow (0 :_M x)_n \rightarrow H_{R_+}^0(M)_n \rightarrow H_{R_+}^0(M/(0 :_M x))_n \rightarrow 0, \quad \forall n \in \mathbb{Z}.$$

So, for all $n \geq M$ we get an exact sequence of K -vector spaces

$$0 \rightarrow (0 :_M x)_n \rightarrow H_{R_+}^0(M)_n \xrightarrow{\pi_n} H_{R_+}^0(M)_{n+1} \rightarrow 0.$$

To prove our claim we may assume that $\operatorname{end}(H_{R_+}^0(M)) > m$. As

$$\operatorname{end}((0 :_M x)) = \operatorname{end}(H_{R_+}^0(M)), \quad \operatorname{gendeg}((0 :_M x)) \leq m$$

it follows that

$$(0 :_M x)_n \neq 0, \quad \forall n \in \{m, m+1, \dots, \operatorname{end}(H_{R_+}^0(M))\}.$$

Hence for all these values of n the homomorphism π_n is surjective but not injective. Therefore

$$h_M^0(n) > h_M^0(n+1), \quad \forall n \in \{m, m+1, \dots, \text{end}(H_{R_+}^0(M))\}.$$

So, in the range $n \geq m$ the function $n \mapsto h_M^0(n)$ is strictly decreasing until it reaches the value 0. Therefore $h_M^0(n) = 0$ for all $n > m + h_M^0(m)$. This proves our claim. \square

The following result is a "graded version" of a corresponding "local" result shown in [Sc2], Proposition 2.4. It tells us, that the graded short exact sequences of (9.5)B)b) also exist if the occurring homogeneous element x is only filter-regular with respect to M . As one sees immediately, the statement we are heading for is an easy consequence of (9.5)B)b) in the case $i > 0$, whereas in the case $i = 0$ some extra work is needed. Indeed, we would not use this result in the present general form to prove our main result, as the attentive reader will observe later. But we decided to present this result for the fun of its own.

10.3. Proposition. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module, let $t \in \mathbb{N}$ and let $x \in R_t$ be filter-regular with respect to M . Then for each $i \in \mathbb{N}_0$ there is an exact sequence of graded R -modules*

$$0 \rightarrow (K^{i+1}(M)/xK^{i+1}(M))(t) \rightarrow K^i(M/xM) \rightarrow (0 :_{K^i(M)} x) \rightarrow 0.$$

Proof. Let $\bar{M} := M/H_{R_+}^0(M)$. Then $x \in R_t \cap \text{NZD}_R(\bar{M})$ and $K^i(\bar{M}) \cong K^i(M)$ for all $i \in \mathbb{N}$ and $K^0(\bar{M}) = 0$ (see (9.5)A)c),b)). So, in view of (9.5)B)b) we get the requested short exact sequences for all $i \in \mathbb{N}$. It remains to treat the case $i = 0$. First of all by (9.5)B)b) and the previous observations on the module $K^i(\bar{M})$ we get an isomorphism of graded R -modules

$$(i) \quad (K^1(M)/xK^1(M))(t) \cong K^0(\bar{M}/x\bar{M}).$$

Observe that the canonical epimorphism of graded R -modules $p : M/xM \rightarrow \bar{M}/x\bar{M}$ satisfies

$$\begin{aligned} \text{Ker}(p) &= (H_{R_+}^0(M) + xM)/xM \cong \\ &H_{R_+}^0(M)/(xM \cap H_{R_+}^0(M)) = H_{R_+}^0(M)/x(H_{R_+}^0(M) :_M x). \end{aligned}$$

As x is filter-regular with respect to M we have $(H_{R_+}^0(M) :_M x) = H_{R_+}^0(M)$, so that finally $\text{Ker}(p) = H_{R_+}^0(M)/xH_{R_+}^0(M)$. Therefore, we end up with the short exact sequence of graded R -modules

$$0 \rightarrow H_{R_+}^0(M)/xH_{R_+}^0(M) \rightarrow M/xM \rightarrow \bar{M}/x\bar{M} \rightarrow 0.$$

As $H_{R_+}^0(M)/xH_{R_+}^0(M)$ is R_+ -torsion, we have $K^1(H_{R_+}^0(M)/xH_{R_+}^0(M)) = 0$. If we form the deficiency sequence associated to the above short exact sequence, we thus get an exact sequence of graded R -modules

$$0 \rightarrow K^0(\bar{M}/x\bar{M}) \rightarrow K^0(M/xM) \rightarrow K^0(H_{R_+}^0(M)/xH_{R_+}^0(M)) \rightarrow 0.$$

In view of the previously observed isomorphism (i) it thus remains to show that there is an isomorphism of graded R -modules

$$K^0(H_{R_+}^0(M)/xH_{R_+}^0(M)) \cong (0 :_{K^0(M)} x).$$

By (9.4)C)d) we have a natural isomorphism $H_{R_+}^0(M) \cong D(K^0(M))$ so that in view of (9.3)C)b) we obtain isomorphisms of graded R -modules

$$H_{R_+}^0(M)/xH_{R_+}^0(M) \cong D(K^0(M))/xD(K^0(M)) \cong D(0 :_{K^0(M)} x).$$

Therefore, by (9.4)B)a) it we get isomorphisms of graded R -modules

$$\begin{aligned} K^0(H_{R_+}^0(M)/xH_{R_+}^0(M)) &\cong D(H_{R_+}^0(H_{R_+}^0(M)/xH_{R_+}^0(M))) \cong \\ &D(H_{R_+}^0(M)/xH_{R_+}^0(M)) \cong D(D(0 :_{K^0(M)} x)) \cong (0 :_{K^0(M)} x). \end{aligned}$$

□

10.4. Lemma. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module. Then, for all $i \in \mathbb{N}_0$ and all $n \geq i$ we have*

$$\dim_K(K^{i+1}(M)_n) \leq \sum_{j=0}^i \binom{n-j-1}{i-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i) \right].$$

Proof. Observe that for $i \in \mathbb{N}_0$ and all $n \geq i$ we have $-n \leq -i$ and hence (see (8.12))

$$d_M^i(-n) \leq \sum_{j=0}^i \binom{n-j-1}{i-j} \left[\sum_{l=0}^{i-j} \binom{i-j}{l} d_M^{i-l}(l-i) \right].$$

If $i > 0$, then

$$d_M^i(-n) = h_M^{i+1}(-n).$$

Moreover

$$h_M^1(-n) \leq \dim_K(M_{-n}) - h_M^0(-n) + h_M^1(-n) = d_M^0(-n).$$

As $h_M^{i+1}(-n) = \dim_K(K^{i+1}(M)_n)$ (see (9.4)C)b)) our claim follows. □

Now, we define the bounding functions $G_d^i : \mathbb{N}_0 \times \mathbb{Z} \rightarrow \mathbb{Z}$, which were mentioned already at the beginning of this section.

10.5. Definition. *(A Class of Bounding Functions)* For all $d \in \mathbb{N}$ and all $i \in \{0, 1, \dots, d\}$ we define the functions

$$G_d^i : \mathbb{N}_0^d \times \mathbb{Z} \rightarrow \mathbb{Z}$$

recursively as follows. In the case $i = 0$ we define

$$(i) \quad G_d^0(x_0, x_1, \dots, x_{d-1}, y) := -y.$$

In the case $i = 1$ we set:

$$(ii) \quad G_d^1(x_0, y) := y - 1;$$

$$(iii) \quad G_d^1(x_0, x_1, \dots, x_{d-1}, y) := \max\{0, 1 - y\} + \sum_{i=0}^{d-2} \binom{d-1}{i} x_{d-i-2}, \text{ for } d \geq 2.$$

In the case $i = d = 2$ we define

$$(iv) \quad G_2^2(x_0, x_1, y) := G_2^1(x_0, x_1, y) + 2.$$

Now, assume that $d \geq 3$ and that the functions G_{d-1}^{i-1}, G_{d-1}^i and G_d^{i-1} are already defined. In order to define the function G_d^i we first intermediately introduce the following notation:

$$(v) \quad m_i := \max\{G_{d-1}^{i-1}(x_0 + x_1, \dots, x_{d-2} + x_{d-1}, y), G_{d-1}^i(x_0, \dots, x_{d-1}, y) + 1\} + 1.$$

$$(vi) \quad n_i := G_{d-1}^i(x_0 + x_1, \dots, x_{d-2} + x_{d-1}, y),$$

$$(vii) \quad t_i := \max\{m_i, n_i\},$$

$$(viii) \quad \Delta_{ij} := \sum_{l=0}^{i-j-1} \binom{i-j-1}{l} x_{i-l-1}.$$

Using these notational conventions, we define

$$(ix) \quad G_d^i(x_0, \dots, x_{d-1}, y) := t_i + \sum_{j=0}^{i-1} \Delta_{ij}, \quad \forall i \in \{2, 3, \dots, d-1\}.$$

Finally, if $d \geq 3$ and G_{d-1}^{d-1} and G_d^{d-1} are already defined, we set (see (v))

$$(x) \quad G_d^d(x_0, \dots, x_{d-1}, y) := m_d.$$

In order to prove our main result, we need a few more preparations. The following three exercises are devoted to these.

10.6. Exercise. A) (*Monotonicity of the Bounding Functions G_d^i*) Let $d \in \mathbb{N}_0$, let $i \in \{0, 1, \dots, d-1\}$ and let

$$(x_0, x_1, \dots, x_{d-1}, y), \quad (x'_0, x'_1, \dots, x'_{d-1}, y') \in \mathbb{N}_0^d \times \mathbb{Z}$$

such that

$$x_i \leq x'_i, \quad \forall i \in \{0, 1, \dots, d-1\}, \quad y' \leq y.$$

Prove by induction on i and d , that under these circumstances we have

$$G_d^i(x_0, x_1, \dots, x_{d-1}, y) \leq G_d^i(x'_0, x'_1, \dots, x'_{d-1}, y').$$

(B) (*Two Further Properties*) Let the notations be as in (10.5). Use induction on i to show the following statements

$$a) \quad \min\{m_i, t_i\} \geq i.$$

b) If $i \leq s \leq d$ and $(x_0, x_1, \dots, x_{s-1}, y) \in \mathbb{N}^s \times \mathbb{Z}$, then

$$G_s^i(x_0, x_1, \dots, x_{s-1}, y) \leq G_d^i(x_0, x_1, \dots, x_{s-1}, 0, \dots, 0, y).$$

10.7. Exercise. A) (*Dual Vector Spaces and Base Field Extensions*) Let K be a field and let K' be an extension field of K . We identify $K' = K' \otimes_K K$. Verify that there is a natural transformation of contravariant linear exact functors of K' -vector spaces

$$\iota : K' \otimes_K \text{Hom}_K(\bullet, K) \rightarrow \text{Hom}_{K'}(K' \otimes_K \bullet, K'),$$

such that for all K -vector spaces V , all $c' \in K'$ and all $h \in \text{Hom}_K(V, K)$ we have

$$\iota_V(c' \otimes h) = c' \text{Id}_{K'} \otimes_K h.$$

Prove the following:

- a) If V is a K -vector space of finite dimension, then the above natural transformation yields an isomorphism of K' -vector spaces

$$\iota_V : K' \otimes_K \text{Hom}_K(V, K) \xrightarrow{\cong} \text{Hom}_{K'}(K' \otimes_K V, K').$$

B) (*Graded Duals and Base Field Extensions*) Keep the above notations and hypotheses. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded K -algebra, so that $R_0 = K$. Consider the graded K' -algebra $R' := K' \otimes_K R = \bigoplus_{n \in \mathbb{Z}} (K' \otimes_K R_n)$. For each graded R -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ we furnish $K' \otimes_K M \cong \bigoplus_{n \in \mathbb{Z}} (K' \otimes_K M_n)$ with its canonical structure as a graded R' -module.

We consider the functor D of taking graded duals of graded R -modules and the functor D' of taking graded duals of graded R' -modules as introduced in (9.1). Show that the natural transformation ι of part A) gives rise to a natural transformation of functors from graded R -modules to graded R' -modules

$$\omega : K' \otimes_K D(\bullet) \rightarrow D'(K' \otimes_K \bullet)$$

such that for each integer $t \in \mathbb{Z}$ and each graded R -module M we have

$$(\omega_M)_t = (\iota_M) \upharpoonright_{K' \otimes_K D(M)_t}.$$

Prove the following statement.

- a) For each graded R -module M and each $t \in \mathbb{Z}$ we have the commutative diagram

$$\begin{array}{ccc} K' \otimes_K \text{Hom}_K(M_{-t}, K) & \xrightarrow[\cong]{K' \otimes_K \tau_t^M} & K' \otimes_K D(M)_t \\ \iota_{M_{-t}} \downarrow & & (\omega_M)_t \downarrow \\ \text{Hom}_{K'}(K' \otimes_K M_{-t}, K') & \xrightarrow[\cong]{\tau_t^{K' \otimes_K M}} & D'(K' \otimes_K M)_t \end{array}$$

where the maps τ_t^M and $\tau_t^{K' \otimes_K M}$ are the natural isomorphisms defined according to (9.1)A)d),C)a). Use this to show

- b) If the graded R -module has finite components (and hence belongs to the class \mathbb{F}_R of (9.3)B)) the natural transformation ω yields an isomorphism of graded R' -modules

$$\omega_M : K' \otimes_K D(M) \xrightarrow{\cong} D'(K' \otimes_K M).$$

C) (*Modules of Deficiency and Base Field Extensions*) Let K and K' be as above and assume this time that $R = K \oplus R_1 \oplus R_2 \dots$ is a Noetherian and

homogeneous K -algebra. Keep in mind, that then R' is a Noetherian homogeneous K' -algebra. Let $i \in \mathbb{N}_0$. For any graded R -module M we may identify $K' \otimes_K K^i(M) = K' \otimes_K D(H_{R_+}^i(M))$, $D'(H_{R_+}^i(K' \otimes_K M)) = K^i(K' \otimes_K M)$ and consider the homomorphisms of graded R' -modules

$$\psi_M^i : K' \otimes_K K^i(M) \rightarrow K^i(K' \otimes_K M)$$

given as the composition

$$K' \otimes_K D(H_{R_+}^i(M)) \xrightarrow{\omega_{R_+}^i(M)} D'(K' \otimes_K H_{R_+}^i(M)) \xrightarrow{D'(*\tau_M^{i,R_+,K' \otimes_K \bullet})^{-1}} K^i(K' \otimes_K M)$$

where

$$*\tau_M^{i,R_+,K' \otimes_K \bullet} : K' \otimes_K H_{R_+}^i(M) \xrightarrow{\cong} H_{R_+}^i(K' \otimes_K M)$$

is the natural isomorphism of (1.15)B),C). Observe that in this way we get a natural transformation of functors from graded R -modules to graded R' -modules

$$\psi^i : K' \otimes_K K^i(\bullet) \rightarrow K^i(K' \otimes_K \bullet)$$

given by

$$M \rightsquigarrow \psi_M^i : K' \otimes_K K^i(M) \rightarrow K^i(K' \otimes_K M).$$

Use what was established in part B) to show the following *Base Change Property of Modules of Deficiency*:

- a) If M is a finitely generated graded R -module, the natural transformation ψ^i yields an isomorphism of graded R' -modules

$$\psi_M^i : K' \otimes_K K^i(M) \xrightarrow{\cong} K^i(K' \otimes_K M).$$

10.8. Exercise. Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let M be a finitely generated graded R -module, let $t \in \mathbb{N}$ and let $x \in R_t$ be filter-regular with respect to M . Show that for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we have the inequality

$$d_{M/xM}^i(n) \leq d_M^i(n) + d_M^{i+1}(n-t).$$

Now, we are ready to formulate and to prove the announced main result.

10.9. Theorem. Let $d \in \mathbb{N}$, let $i \in \{0, 1, \dots, d\}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module with $\dim_R(M) = d$. Then

$$\text{reg}(K^i(M)) \leq G_d^i(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \text{beg}(M)).$$

Proof. We proceed by induction on i . By (9.5)A)b) we have $\dim_R(K^0(M)) \leq 0$. So, in view of (9.4)C)b) we get

$$\text{reg}(K^0(M)) = \text{end}(K^0(M)) = -\text{beg}(H_{R_+}^0(M)) \leq -\text{beg}(M) =$$

$$G_d^0(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \text{beg}(M)).$$

This clearly proves the case $i = 0$.

So let $i > 0$. Let K' be an infinite extension field of K , consider the Noetherian homogeneous K' -algebra $R' := K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \dots$ and the finitely generated graded R' -module $M' := K' \otimes_K M = \bigoplus_{n \in \mathbb{Z}} K' \otimes_K M_n$. Then clearly $\text{beg}(M') = \text{beg}(M)$, $\dim_{R'}(M') = d$ (see (2.4)C)b) and moreover $d_{M'}^j(n) = d_M^j(n)$ for all $j \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (see (8.1)C)e)). In addition we have an isomorphism of graded R' -modules $K^i(M') \cong K' \otimes_K K^i(M)$ (see (10.7)C)a)) so that $\text{reg}(K^i(M')) = \text{reg}(K^i(M))$ (see (3.3)B)h)). This allows to replace R and M respectively by R' and M' and hence to assume that K is infinite.

Let $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\dim_R(\bar{M}) = d$, $d_{\bar{M}}^j(n) = d_M^j(n)$ for all $j \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (see (8.1)B)c)). In addition clearly $\text{beg} M \leq \text{beg}(\bar{M})$, whence by (10.6)A) we get

$$\begin{aligned} & G_d^i(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \text{beg}(\bar{M})) \leq \\ & \leq G_d^i(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \text{beg}(M)). \end{aligned}$$

As moreover we have an isomorphism of graded R -modules $K^i(\bar{M}) \cong K^i(M)$ (see (9.5)A)c)), we thus may replace M by \bar{M} and hence assume that $\Gamma_{R_+}(M) = 0$. Therefore we find some element $x \in R_1 \cap \text{NZD}_R(M)$. By Homogeneous Prime Avoidance we may assume in addition, that x is filter-regular with respect to the modules $K^0(M), K^1(M), \dots, K^d(M)$. In particular, by (10.3) (indeed even by (9.5)B)b)) there is an exact sequence of graded R -modules

$$\text{a) } 0 \rightarrow (K^{j+1}(M)/xK^{j+1}(M))(+1) \rightarrow K^j(M/xM) \rightarrow (0 :_{K^j(M)} x) \rightarrow 0,$$

for all $j \in \mathbb{N}_0$. Since $H_{R_+}^0(M) = 0$ we have $K^0(M) = 0$ (see (9.7)c) for example), so that the sequence a) gives rise to an isomorphism of graded R -modules

$$\text{b) } (K^1(M)/xK^1(M))(+1) \cong K^0(M/xM).$$

As $\dim_R(K^0(M/xM)) \leq 0$ (see (9.7)f)), the above isomorphism shows that $K^1(M)/xK^1(M)$ is R_+ -torsion, so that (see (9.7)c))

$$\begin{aligned} \text{reg}(K^1(M)/xK^1(M)) &= \text{reg}(K^0(M/xM)) + 1 = \text{end}(K^0(M/xM)) + 1 = \\ & 1 - \text{beg}(H_{R_+}^0(M/xM)) \leq 1 - \text{beg}(M/xM) \leq 1 - \text{beg}(M). \end{aligned}$$

It follows that

$$\text{c) } \text{reg}(K^1(M)/xK^1(M)) \leq 1 - \text{beg}(M).$$

We first assume that $d = 1$. Then clearly $i = 1$, whence $K^i(M) = K^1(M) = K(M)$ so that by (9.10) we get $\text{grade}_{K^1(M)}(R_+) = 1$ hence $H_{R_+}^0(K^1(M)) = 0$, so that $\text{reg}(K^1(M)) = \text{reg}^1(K^1(M))$. It follows that (see (10.1))

$$\text{reg}(K^1(M)) \leq \text{reg}(K^1(M)/xK^1(M)) \leq 1 - \text{beg}(M) = G_1^1(d_M^0(0), \text{beg}(M)).$$

This proves our claim if $d = 1$.

So, assume from now on, that $d \geq 2$. We first treat the case $i = 1$. To do so, we consider the sequence a) for $j = 1$, hence

$$d) 0 \rightarrow (K^2(M)/xK^2(M))(+1) \rightarrow K^1(M/xM) \rightarrow (0 :_{K^1(M)} x) \rightarrow 0.$$

If $d = 2$, we have $\dim_R(M/xM) = 1$ and so by the already treated case $d = 1$ we get

$$\operatorname{reg}(K^1(M/xM)) \leq 1 - \operatorname{beg}(M/xM) \leq 1 - \operatorname{beg}(M).$$

Consequently by (3.4) we have

$$\begin{aligned} \operatorname{gendeg}((0 :_{K^1(M)} x)) &\leq \operatorname{gendeg}(K^1(M/xM)) \leq \\ &\leq \operatorname{reg}(K^1(M/xM)) \leq 1 - \operatorname{beg}(M). \end{aligned}$$

Assume first that $m_0 := 1 - \operatorname{beg}(M) \leq 0$. Then, by (10.2) (applied with $m = 0$) we obtain (see (9.7)c))

$$\operatorname{reg}(K^1(M)) \leq 0 + h_{K^1(M)}^0(0) \leq \dim_K(K^1(M)_0) = h_M^1(0) \leq d_M^0(0).$$

Now, assume that $m_0 := 1 - \operatorname{beg}(M) > 0$. Then $d_M^0(-m_0) \leq d_M^0(0)$ (see (8.11)b)). So by statement c), by (10.2) and by (9.7)c) we get

$$\begin{aligned} \operatorname{reg}(K^1(M)) &\leq m_0 + h_{K^1(M)}^0(m_0) \leq m_0 + \dim_K(K^1(M)_{m_0}) = \\ &= 1 - \operatorname{beg}(M) + h_M^1(-m_0) \leq 1 - \operatorname{beg}(M) + d_M^0(-m_0) \leq 1 - \operatorname{beg}(M) + d_M^0(0). \end{aligned}$$

Therefore, bearing in mind (10.5)(iii) we finally obtain

$$\begin{aligned} \operatorname{reg}(K^1(M)) &\leq \max\{d_M^0(0), 1 - \operatorname{beg}(M) + d_M^0(0)\} \leq \\ &\leq \max\{0, 1 - \operatorname{beg}(M)\} + d_M^0(0) = G_2^1(d_M^0(0), d_M^1(-1), \operatorname{beg}(M)). \end{aligned}$$

This proves the case in which $d = 2$ and $i = 1$.

Now, let $d \geq 3$, but still let $i = 1$. Then, by induction on d we may write (see (10.5)(iii))

$$\begin{aligned} \operatorname{reg}(K^1(M/xM)) &\leq G_{d-1}^1(d_{M/xM}^0(0), \dots, d_{M/xM}^{d-2}(2-d), \operatorname{beg}(M/xM)) = \\ &= \max\{0, 1 - \operatorname{beg}(M/xM)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} d_{M/xM}^{d-i-3}(i+3-d). \end{aligned}$$

According to (10.8) we have

$$d_{M/xM}^{d-i-3}(i+3-d) \leq d_M^{d-i-3}(i+3-d) + d_M^{d-i-2}(i+2-d), \quad \forall i \in \{0, 1, \dots, d-3\}.$$

Therefore we obtain

$$\begin{aligned} &\operatorname{reg}(K^1(M/xM)) \leq \\ &\leq \max\{0, 1 - \operatorname{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} [d_M^{d-i-3}(i+3-d) + d_M^{d-i-2}(i+2-d)] =: t_0. \end{aligned}$$

By the exact sequence d) and (3.4) we now get

$$\operatorname{gendeg}((0 :_{K^1(M)} x)) \leq \operatorname{reg}(K^1(M/xM)) \leq t_0.$$

By the above inequality c) and the definition of t_0 we have

$$\operatorname{reg}(K^1(M)/xK^1(M)) \leq t_0.$$

As $t_0 \geq 0$ we also have $d_M^0(-t_0) \leq d_M^0(0)$. So, by (10.2) and (9.7)c) we obtain the inequalities

$$\begin{aligned} \operatorname{reg}(K^1(M)) &\leq t_0 + h_{K^1(M)}^0(t_0) \leq t_0 + \dim_K(K^1(M)_{t_0}) = \\ &= t_0 + h_M^1(-t_0) \leq t_0 + d_M^0(-t_0) \leq t_0 + d_M^0(0) = \\ &= \max\{0, 1 - \operatorname{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} [d_M^{d-i-3}(i+3-d) + d_M^{d-i-2}(i+2-d)] + d_M^0(0) = \\ &= \max\{0, 1 - \operatorname{beg}(M)\} + d_M^{d-2}(2-d) + \sum_{i=1}^{d-3} \left[\binom{d-2}{i-1} + \binom{d-2}{i} \right] d_M^{d-i-2}(i+2-d) + \\ &\quad + (d-2)d_M^0(0) + d_M^0(0) = \\ &= \max\{0, 1 - \operatorname{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-1}{i} d_M^{d-i-2}(i+2-d) + (d-1)d_M^0(0) = \\ &= \max\{0, 1 - \operatorname{beg}(M)\} + \sum_{i=0}^{d-2} \binom{d-1}{i} d_M^{d-i-2}(i+2-d). \end{aligned}$$

In view of (10.5)(iii) this means that

$$\operatorname{reg}(K^1(M)) \leq G_d^1(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)).$$

So, we have settled the case $i = 1$ for all $d \in \mathbb{N}$.

We now attack the cases with $i \geq 2$. We begin with the case in which $d = 2$ and hence $i = 2$. In view of the exact sequence d) we obtain (see (3.3)B)b),C)a)

$$\operatorname{reg}(K^2(M)/xK^2(M)) \leq \max\{\operatorname{reg}(K^1(M/xM)), \operatorname{reg}((0 :_{K^1(M)} x)) + 1\} + 1.$$

Observe that $\dim_R(M/xM) = 1$, so that by what we know from the already treated case $i = d = 1$ we get

$$\begin{aligned} \operatorname{reg}(K^1(M/xM)) &\leq G_1^1(d_{M/xM}^0(0), \operatorname{beg}(M/xM)) = \\ &\operatorname{beg}(M/xM) - 1 \leq \operatorname{beg}(M) - 1. \end{aligned}$$

As x is filter-regular with respect to $K^1(M)$, we have $(0 :_{K^1(M)} x) \subseteq H_{R_+}^0(M)$, so that

$$\operatorname{reg}((0 :_{K^1(M)} x)) = \operatorname{end}((0 :_{K^1(M)} x)) \leq \operatorname{end}(H_{R_+}^0(K^1(M))) \leq \operatorname{reg}(K^1(M)).$$

By what we know from the already treated case with $i = 1$ and $d = 2$ we have $\operatorname{reg}(K^1(M)) \leq G_2^1(d_M^0(0), d_M^1(-1), \operatorname{beg}(M)) = \max\{0, 1 - \operatorname{beg}(M)\} + d_M^0(0)$.

Therefore we get

$$\begin{aligned} \operatorname{reg}(K^2(M)/xK^2(M)) &\leq \max\{1 - \operatorname{beg}(M), \max\{0, 1 - \operatorname{beg}(M)\} + d_M^0(0) + 1\} + 1 \\ &\leq \max\{0, 1 - \operatorname{beg}(M)\} + d_M^0(0) + 2. \end{aligned}$$

As $\text{grade}_{K^2(M)}(R_+) = \text{grade}_{K(M)}(R_+) \geq \min\{2, d\} = 2 = d$ (see (9.10)) we have $\text{grade}_{K^2(M)}(R_+) = 2$, whence $H_{R_+}^j(K^2(M)) = 0$ for $j = 0, 1$. This means, that $\text{reg}(K^2(M)) = \text{reg}^1(K^2(M))$. So by (10.1) we obtain

$$\begin{aligned} \text{reg}(K^2(M)) &\leq \text{reg}(K^2(M)/xK^2(M)) \leq \\ &\leq \max\{0, 1 - \text{beg}(M)\} + d_M^0(0) + 2 = G_2^2(d_M^0(0), d_M^1(-1), \text{beg}(M)). \end{aligned}$$

This completes our proof in the cases with $i \geq 2$ and $d = 2$.

So, let $d > 2$ and $i \geq 2$. By (10.8) we have

$$d_{M/xM}^j(-j) \leq d_M^j(-j) + d_M^{j+1}(-j-1), \quad \forall j \in \mathbb{N}_0.$$

Let $k \in \{0, 1, \dots, d-1\}$. Then, by induction on d and in view of (10.6)A) we have

$$\begin{aligned} \text{reg}(K^k(M/xM)) &\leq G_{d-1}^k(d_{M/xM}^0(0), \dots, d_{M/xM}^{d-2}(2-d), \text{beg}(M/xM)) \leq \\ &\leq G_{d-1}^k(d_M^0(0) + d_M^1(-1), \dots, d_M^{d-2}(2-d) + d_M^{d-1}(1-d), \text{beg}(M)) =: n_k. \end{aligned}$$

Therefore

$$\text{e) } \text{reg}(K^k(M/xM)) \leq n_k \text{ for all } k \in \{0, 1, \dots, d-1\}.$$

Clearly, by induction on i we have

$$\text{f) } \text{reg}(K^{i-1}(M)) \leq G_d^{i-1}(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \text{beg}(M)) =: v_{i-1}.$$

If we apply the exact sequence a) with $j = i-1$ we get (see (3.3)B)b), C)a))

$$\text{reg}(K^i(M)/xK^i(M)) \leq \max\{\text{reg}(K^{i-1}(M/xM)), \text{reg}((0 :_{K^{i-1}(M)} x)) + 1\} + 1.$$

By the inequality e) we have

$$\text{reg}(K^{i-1}(M/xM)) \leq n_{i-1}.$$

Moreover, as x is filter-regular with respect to $K^{i-1}(M)$ we have once more $\text{reg}((0 :_{K^{i-1}(M)} x)) \leq \text{end}(H_{R_+}^0(K^{i-1}(M))) \leq \text{reg}(K^{i-1}(M))$, so that by the inequality f) we have

$$\text{reg}((0 :_{K^{i-1}(M)} x)) \leq v_{i-1}.$$

Thus, gathering together we we obtain

$$\text{g) } \text{reg}(K^i(M)/xK^i(M)) \leq \max\{n_{i-1}, v_{i-1} + 1\} + 1 =: m_i.$$

Assume first, that $2 \leq i \leq d-1$. Observe that by (10.6)B)a) we have

$$t_i := \max\{m_i, n_i\} \geq i.$$

Moreover, if we apply the sequence a) with $j = i$ and keep in mind the inequality e) we get (see also (3.4))

$$\text{gendeg}((0 :_{K^i(M)} x)) \leq \text{reg}(K^i(M/xM)) \leq n_i.$$

So, by (10.2), applied to the graded R -module $K^i(M)$ with $m := t_i$ and with (10.4) applied with $n = t_i$ and with $i - 1$ instead of i we obtain

$$\begin{aligned} \operatorname{reg}(K^i(M)) &\leq t_i + h_{K^i(M)}^0(t_i) \leq t_i + \dim_K(K^i(M)_{t_i}) \leq \\ &\leq t_i + \sum_{j=0}^{i-1} \binom{t_i - j - 1}{i - j - 1} \left[\sum_{l=0}^{i-j-1} \binom{i - j - 1}{l} d_M^{i-l-1} (l - i + 1) \right]. \end{aligned}$$

In view of (10.5)(viii),(ix) this means that

$$\operatorname{reg}(K^i(M)) \leq G_d^i(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)).$$

This completes our proof in the cases with $i \leq d - 1$.

It remains to treat the cases with $i = d > 2$. Observe that by (9.10) we have $\operatorname{grade}_{K^d(M)}(R_+) = 2$, so that again $\operatorname{reg}(K^d(M)) = \operatorname{reg}^1(K^d(M))$. Keep in mind, that x is filter-regular with respect to $K^d(M)$. So, if we apply (10.1) to this latter module and bear in mind the previous inequality g) we obtain

$$\operatorname{reg}(K^d(M)) \leq \operatorname{reg}(K^d(M)/xK^d(M)) \leq m_d.$$

In view of (10.5)(x) this means that

$$\operatorname{reg}(K^d(M)) \leq G_d^d(d_M^0(0), d_M^1(-1), \dots, d_M^{d-1}(1-d), \operatorname{beg}(M)).$$

This completes our proof. □

Finally, we now have reached the last peak of the mountain we have attacked and shall do just one last tiny step forward on the top platform.

10.10. Corollary. *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, let $d \in \mathbb{N}$, let $x_0, x_1, \dots, x_{d-1} \in \mathbb{N}_0$ and let $y \in \mathbb{Z}$. Then for each finitely generated graded R -module M which satisfies the conditions*

$$\dim_R(M) \leq d, \quad d_M^j(-j) \leq x_j \quad \forall j \in \{0, 1, \dots, d-1\}, \quad \operatorname{beg}(M) \geq y$$

it holds

$$\operatorname{reg}(K^i(M)) \leq G_d^i(x_0, x_1, \dots, x_{d-1}, y), \forall i \in \{0, 1, \dots, d\}.$$

Proof. If $M=0$, our claim is obvious. If $\dim_R(M) = 0$ we have $M = H_{R_+}^0(M)$, $K^i(M) = 0$ for all $i > 0$ (see (9.7)e)) and $\dim_R(K^0(M)) = 0$ (see (9.7)f)). Therefore we can say that (see (9.7)c) and (10.5)(i))

$$\begin{aligned} \operatorname{reg}(K^0(M)) &= \operatorname{end}(K^0(M)) = -\operatorname{beg}(H_{R_+}^0(M)) = \operatorname{beg}(M) \leq \\ &\leq -y = G_1^0(x_0, x_1, \dots, x_{d-1}, y). \end{aligned}$$

So, we may assume from now on, that $\dim_R(M) > 0$. But in this situation we may conclude by (10.9) and (10.6)A),B). □

As a first and immediate application we now get a lower bound on the cohomological postulation numbers

$$\nu_M^i := \inf\{n \in \mathbb{Z} \mid p_M^i(n) \neq h_M^i(n)\}$$

of a finitely generated graded module M over a Noetherian homogeneous K -algebra R , as they were introduced in (9.8)C).

10.11. Corollary. *Let $d \in \mathbb{N}$, let $i \in \{0, 1, \dots, d-1\}$, let $x_0, x_1, \dots, x_{d-1} \in \mathbb{N}_0$, let $y \in \mathbb{Z}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra and let M be a finitely generated graded R -module which satisfies the conditions*

$$\dim_R(M) \leq d, \quad d_M^j(-j) \leq x_j \quad \forall i \in \{0, 1, \dots, d-1\}, \quad \text{beg}(M) \geq y.$$

Then

$$\nu_M^i \geq -G_d^i(x_0, x_1, \dots, x_{d-1}, y).$$

Proof. This is immediate by (10.10) and (9.8)C)d). \square

In order to be able to deduce some further conclusions from the last bounding result, we now introduce some appropriate notions.

10.12. Exercise and Definition. A) (*Cohomological Serre Polynomials of Coherent Sheaves*) Let K be a field, let $R = K \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous K -algebra, set $X := \text{Proj}(R)$ and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Moreover let M be a finitely generated graded R -module such that $\mathcal{F} = \widetilde{M}$. Let $i \in \mathbb{N}_0$. Show that for the $(i+1)$ -st cohomological Hilbert polynomial p_M^{i+1} of M (see (9.8)A)) we have (see also (4.11)A))

$$a) \quad p_M^{i+1}(n) = h^i(X, \mathcal{F}(n)) = h_{\mathcal{F}}^i(n) \text{ for all } n \ll 0.$$

In particular, we can say, that for each $i \in \mathbb{N}_0$ there is a unique polynomial

$$p_{\mathcal{F}}^i = p_{\mathcal{F}}^i(X) \in \mathbb{Q}[X],$$

characterized by the property that

$$h^i(X, \mathcal{F}(n)) = p_{\mathcal{F}}^i(n), \quad \forall n \ll 0.$$

This numerical polynomial $p_{\mathcal{F}}^i \in \mathbb{Q}[X]$ is called the *i -th cohomological Serre polynomial of \mathcal{F}* . Prove that in the above notations we have

$$b) \quad p_{\mathcal{F}}^i = p_M^{i+1}.$$

$$c) \quad \deg(p_{\mathcal{F}}^i) \leq i, \text{ with equality if } i = \dim(\mathcal{F}) \geq 0.$$

$$d) \quad \text{If } i > \dim(\mathcal{F}), \text{ then } p_{\mathcal{F}}^i = 0.$$

$$e) \quad p_{\mathcal{F}(r)}^i(X) = p_{\mathcal{F}}^i(X+r) \text{ for all } r \in \mathbb{Z}.$$

f) The Serre polynomial $P_{\mathcal{F}}$ of \mathcal{F} (see (4.11)B)) satisfies

$$P_{\mathcal{F}} = \sum_{i=0}^{\dim(\mathcal{F})} (-1)^i p_{\mathcal{F}}^i = \sum_{i \in \mathbb{N}_0} (-1)^i p_{\mathcal{F}}^i.$$

B) (*Cohomological Postulation Numbers of Coherent Sheaves*) Let the notations be as in part A). Then clearly

$$\nu_{\mathcal{F}}^i := \inf\{n \in \mathbb{Z} \mid p_{\mathcal{F}}^i(n) \neq h^i(X, \mathcal{F}(n))\} \in \mathbb{Z} \cup \{\infty\}.$$

The number $\nu_{\mathcal{F}}^i$ is called the *i-th cohomological postulation number of \mathcal{F}* . Prove the following statements, in which ν_M^{i+1} denotes the $(i+1)$ -th cohomological postulation number of the module M (see (9.8C))

- a) If $i \in \mathbb{N}$, then $\nu_{\mathcal{F}}^i = \nu_M^{i+1}$.
- b) $\nu_{\mathcal{F}}^0 \geq \min\{\nu_M^1, \text{beg}(M)\}$.
- c) If $i = \dim(\mathcal{F}) \geq 0$, then $\nu_{\mathcal{F}}^i \in \mathbb{Z}$.
- d) If $i > \dim(\mathcal{F})$, then $\nu_{\mathcal{F}}^i = \infty$.
- e) $\nu_{\mathcal{F}(r)}^i = \nu_{\mathcal{F}}^i - r$ for all $r \in \mathbb{Z}$.

To simplify the notational form of our next result, we prefer to introduce the following bounding functions.

10.13. Notation. Let $s \in \mathbb{N}_0$ and let $i \in \{0, 1, \dots, s\}$. We then define the *bounding function*

$$L_s^i : \mathbb{N}_0^{s+1} \rightarrow \mathbb{Z}$$

by the prescription

$$L_s^i(x_0, x_1, \dots, x_s) := -G_{s+1}^{i+1}(x_0, x_1, \dots, x_s, 0), \quad \forall x_0, x_1, \dots, x_s \in \mathbb{N}_0,$$

where the function

$$G_{s+1}^{i+1} : \mathbb{N}_0^s \times \mathbb{Z} \rightarrow \mathbb{Z}$$

is defined according to (10.5).

Now, we are ready to formulate and to prove our first main application of (10.10), which says that the *cohomology diagonal of a coherent sheaf \mathcal{F} over a projective K -scheme X bounds the cohomological postulation numbers of the sheaf \mathcal{F}* .

10.14. Theorem. *Let $s \in \mathbb{N}_0$, let $i \in \{0, 1, \dots, s\}$, let $x_0, x_1, \dots, x_s \in \mathbb{N}_0$, let X be a projective scheme over some field K and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules which satisfies the conditions*

$$\dim(\mathcal{F}) \leq s, \quad h^i(X, \mathcal{F}(-j)) \leq x_j \quad \forall j \in \{0, 1, \dots, s\}.$$

Then

$$\nu_{\mathcal{F}}^i \geq L_s^i(x_0, x_1, \dots, x_s).$$

Proof. We write $X = \text{Proj}(R)$, where $R = K \oplus R_1 \oplus R_2 \dots$ is a Noetherian homogeneous K -algebra. We may choose some finitely generated graded R -module M such that $\mathcal{F} = \widetilde{M}$. As $\widetilde{M} = \widetilde{M}_{\geq 0}$ we may replace M by $M_{\geq 0}$ and hence assume that

$$\text{beg}(M) \geq 0.$$

Observe that

$$\dim_R(M) \leq \dim(\mathcal{F}) + 1 \leq s + 1.$$

Moreover, in view of (8.1)B)g) we have

$$d_M^j(-j) = h^j(X, \mathcal{F}(-j)) \leq x_j, \quad \forall j \in \{0, 1, \dots, s\}.$$

So, we may apply (10.11) with $y = 0$ and with $i + 1$ instead of i and obtain

$$\nu_M^{i+1} \geq -G_{s+1}^{i+1}(x_0, x_1, \dots, x_s, 0) = L_s^i(x_0, x_1, \dots, x_s).$$

By (10.12)B)a) we have in addition that $\nu_{\mathcal{F}}^i = \nu_M^{i+1}$ provided that $i > 0$. In these cases we therefore have our claim. So, it remains to consider the case $i = 0$. By (10.12)B)b) and the previous estimate we have

$$\nu_{\mathcal{F}}^0 \geq \min\{\nu_M^1, 0\} \geq \min\{L_s^0(x_0, x_1, \dots, x_s), 0\}.$$

According to (10.5)(iii) we have

$$L_s^0(x_0, x_1, \dots, x_s) = -G_{s+1}^1(x_0, x_1, \dots, x_s, 0) < 0,$$

so that indeed $\nu_{\mathcal{F}}^0 \geq L_s^0(x_0, x_1, \dots, x_s)$, as requested. \square

Now, it is at about the time, to comment on the previous results. We try to present the special flavour or "spirit" of our basic bounding result (10.9) and its corollary (10.10). Then we discuss the "historic background" of the bounding result (10.14). We also tie the link to our quantitative version [Br-Fu-Ro](10.17) of the Vanishing Theorem of Severi-Enriques-Zariski-Serre.

10.15. Exercise and Remark. A) (*Around Regularity of Modules of Deficiency*) The bounding result (10.9) is given in a more general form in [Br-Ja-Li1], namely for finitely generated graded modules over Noetherian homogeneous rings with Artinian local base ring. Here we did restrict ourselves to modules over Noetherian homogeneous K -algebras, as we did develop the theory of modules of deficiency only in this special context in our course. The bounding result (10.9) and its corollary (10.10) are results in the spirit of the first bounding result given expressis verbis for regularities: Mumford's bounding result in [Mu1], which we presented in an extended form in section 4 of these lectures, (see (4.7), (4.8), (4.12), (4.13), (4.14)C)(i) and (4.14)D)). The typical property of Mumford's original result is the fact that it gives an explicit and universal upper bound on the regularity of a coherent sheaf of ideals over a projective space over a field in terms of the Serre coefficients of this sheaf. Clearly our extensions of Mumford's result given in section 4 have the same characteristic property.

Let us recall once more, that Mumford's result did open a new view to Hilbert schemes: instead of using non-constructive compactness and semicontinuity arguments to show the boundedness of regularities and cohomologies of the class of ideals parametrized by a given Hilbert scheme, one now had an explicit and algorithmic "a priori bound" at hands. It would be rather surprising, if Mumford's result (and related results of the same type, as found in for example in Kleiman's contribution to [Gro4] for example) would not have been one of

the driving forces for the revival of *Computational Algebraic Geometry* around the year 1980 (see [B-Mu] for example).

Let us admit, that we dare compare (10.9) with Mumford's previously quoted bounding result. Namely: our bounding result answers a "classical" question (suggested by Mumford [Mu1], indeed), and it does this in an explicit and algorithmic way by giving universal upper bounds on the regularities of the deficiency modules of a graded module - only in terms of the cohomology diagonal and the beginning of this module. Clearly, we also aim to remain modest and dare not think at all, that our bounding results (10.9) and (10.10) open a new view to some basic objects of algebraic geometry, as this was the case for Mumford's result. One should not forget, that meanwhile 44 years have passed - and algorithmic results have become a common issue in Algebraic Geometry. To convince the reader, that our bounding results might have some significance beyond themselves, we shall use them to study classes of finite cohomology, as they were introduced and treated towards the end of section 8.

B) (*Around Cohomological Postulation Numbers*) First, let us briefly recall the "history" of our bounding theorem (10.14). In [Matt] it was shown, that the cohomological postulation numbers of a coherent sheaf \mathcal{F} over a projective scheme X over a field K are bounded merely in terms of the cohomology diagonal and the cohomological Serre polynomials of \mathcal{F} . In [Br-Matt-Mi2] we did show that the same holds for coherent sheaves over projective schemes over Artinian rings.

In [L] and [Br-L2] it is shown by a completely different method, that the cohomological postulation numbers of a coherent sheaf \mathcal{F} over a projective scheme X over a field K are bounded in terms of the cohomology diagonal of \mathcal{F} . Due to the method used in those papers, the attained bounds are essentially weaker than those given in (10.14). The bounding result (10.14) actually is given in a more general setting in [Br-Ja-Li1]: this bound again holds for all coherent sheaves over projective schemes over local Artinian rings. Let us just mention, that unlike to our first approach practiced in [L] and [Br-L2] the bounding result (10.9) (resp. its generalization to projective schemes over local Artinian rings) is the basic tools to obtain (10.14) (resp. its corresponding generalization). This points out once more the significance of (10.9).

C) (*Revisiting once more the Vanishing Theorem of Severi-Enriques-Zariski-Serre*) Now, let K be (an algebraically closed) field, let X be a projective scheme over K and let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Use [Br-Fu-Ro] (10.17) to show the following statements, in which

$$\delta(\mathcal{F}) := \inf\{\text{depth}_{\mathcal{O}_{X,x}} \mid x \in \text{m}X\}$$

denotes the subdepth of \mathcal{F} (see [Br-Fu-Ro](12.15)B)).

a) $\delta(\mathcal{F}) = \inf\{i \in \mathbb{N}_0 \mid p_{\mathcal{F}}^i \neq 0\}$.

b) For all $i < \delta(\mathcal{F})$ it holds

$$\nu_{\mathcal{F}}^i > \left(2 \sum_{j=0}^{\delta(\mathcal{F})-1} \binom{\delta(\mathcal{F})-1}{i} h^j(X, \mathcal{F}(-j)) \right)^{2^{\delta(\mathcal{F})-1}}.$$

(You may indeed use (7.11)D) to eliminate the condition that K is algebraically closed). These statements obviously are the *Sheaf Theoretic Formulation of the Quantitative Version of the Vanishing Theorem of Severi-Enriques Zariski-Serre* given in [Br-Fu-Ro](10.17). Observe that the above statements a) and b) are a substitute for the bounding result (10.14), but only available in the range $i < \delta(\mathcal{F})$. Observe also, that the bound of statement b), in the range it applies at all, is sharper than the corresponding bound of (10.14). Nevertheless, (10.14) gives indeed an extension of the bounding result formulated in statements a) and b) beyond the critical level $i = \delta(\mathcal{F}) - 1$. Therefore we may consider our bounding result (10.14) as an *Ultimate Quantitative Version of the Vanishing Theorem of Severi-Enriques-Zariski-Serre*.

We now give a last application of (10.9). It will be devoted to classes of finite cohomology as they were introduced in (8.14). We namely shall establish an essential improvement of our earlier result (8.20). This application should help to illuminate the significance of the bounding result (10.9), as we hope to make clear in our final discussion. For the occurring notations and notions occurring in this result, the reader should consult (8.13), (8.14)A) and (8.18).

10.16. Theorem. *Let $s \in \mathbb{N}_0$, let $\Sigma \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ be a quasi-diagonal subset and let $\mathcal{D} \subseteq \mathcal{S}^s$ be a subclass which is of finite cohomology on Σ . Then the class \mathcal{D} is of finite cohomology at all.*

Proof. According to (8.20) the class \mathcal{D} is of finite cohomology on the set $\{0, 1, \dots, s\} \times \mathbb{Z}_{\geq -s}$. In particular there are numbers $x_0, x_1, \dots, x_s \in \mathbb{N}_0$ such that

$$h^j(X, \mathcal{F}(-j)) \leq x_j, \quad \forall j \in \{0, 1, \dots, s\}, \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

Now, in the notations of (10.13) let

$$l := \min\{L_s^i(x_0, x_1, \dots, x_s) \mid i = 0, 1, \dots, s\}.$$

Then, according to (10.14) we have

$$\nu_{\mathcal{F}}^i \geq l, \quad \forall i \in \{0, 1, \dots, s\}, \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

Next, fix some integer $t < l - d$. Let $(X, \mathcal{F}) \in \mathcal{D}$. Then for each $n < l$ we have $h^i(X, \mathcal{F}(n)) = p_{\mathcal{F}}^i(n)$. As $\deg(p_{\mathcal{F}}^i) \leq i \leq s$ (see (10.12)A)c)) it follows that the family $(h^i(X, \mathcal{F}(n)))_{n < l}$ is uniquely determined by its finite subfamily

$$F_{\mathcal{F}} := (h^i(X, \mathcal{F}(n)))_{t \leq n < l}.$$

According to (8.20) the class \mathcal{D} is of finite cohomology on the set $\{0, 1, \dots, s\} \times \mathbb{Z}_{\geq t}$. This clearly shows that the set of finite families

$$\{F_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite. By the previous observation this implies that the class \mathcal{D} is of finite cohomology on the set $\{0, 1, \dots, s\} \times \mathbb{Z}_{<l}$. As \mathcal{D} is of finite cohomology on $\{0, 1, \dots, s\} \times \mathbb{Z}_{\geq t}$ and as $t < l$ it follows that \mathcal{D} is of finite cohomology on $\{0, 1, \dots, s\} \times \mathbb{Z}$ (see (8.14)Bf)). This proves our claim. \square

We now give a number of applications of the previous result, which generalize what is known in the theory of *Hilbert schemes*: The sheaves of ideals parametrized by a given Hilbert scheme form a class of finite cohomology. We give these applications in the spirit of what we said towards the end of (10.15)A), namely in order to illustrate the ease and the great generality of conclusions that may be drawn from our Bounding Theorem (10.9). We begin with linking classes of finite cohomology to classes of bounded regularity.

10.17. Exercise and Remark. A) (*Specifying classes of Finite Cohomology*) Let $s \in \mathbb{N}_0$ and let $\mathcal{D} \subseteq \mathcal{S}^s$ be a subclass. Fix a quasi-diagonal subset

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \dots, s\} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}, \quad n_s < n_{s-1} < \dots < n_0.$$

Then our previous result says that the class \mathcal{D} is of finite cohomology if and only if the set

$$\{h^i(X, \mathcal{F}(n_i)) \mid (X, \mathcal{F}) \in \mathcal{D}\} = \{h_{\mathcal{F}}^i(n_i) \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite for all $i \in \{0, 1, \dots, s\}$. So, the $s + 1$ numerical invariants $h_{\mathcal{F}}^i(n_i)$ with $i = 0, 1, \dots, s$ may be used to specify subclasses $\mathcal{D} \subseteq \mathcal{S}^s$ of finite cohomology. Indeed, specifying classes of finite cohomology by subjecting numerical invariants to some conditions, is a basic issue. In this spirit we suggest to prove the following statement as an exercise.

- a) The class $\mathcal{D} \subseteq \mathcal{S}^s$ is of finite cohomology if and only if there are integers $r \in \mathbb{Z}$ and $h \in \mathbb{N}_0$ such that $\text{reg}(\mathcal{F}) \leq r$ and $h^0(X, \mathcal{F}(r)) \leq h$ for all pairs $(X, \mathcal{F}) \in \mathcal{D}$.

We say that the class $\mathcal{D} \subseteq \mathcal{S}^s$ is of *bounded regularity* if the set of integers

$$\{\text{reg}(\mathcal{F}) \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

has an upper bound. Prove the following statement.

- b) The class $\mathcal{D} \subseteq \mathcal{S}^s$ is of finite cohomology if and only if it is of bounded regularity and the set of Serre polynomials $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{S}^s\}$ is finite.

B) (*Regularity and Classes of Subschemes and Quotient Sheaves*) Let $s \in \mathbb{N}_0$. we consider the class

$$\mathcal{S}^{\leq s} := \cup_{i=0}^s \mathcal{S}^i$$

of all pairs (X, \mathcal{F}) in which X is a projective scheme over some field K and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules with $\dim(\mathcal{F}) \leq s$. The notions of *subclass* $\mathcal{D} \subseteq \mathcal{S}^{\leq s}$ of *finite cohomology* and of *bounded regularity* are defined in the obvious way. Now, let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq s}$. We say that \mathcal{D} is a *class of subschemes with*

respect to \mathcal{C} if for all pairs $(X, \mathcal{F}) \in \mathcal{D}$ there is a monomorphism of sheaves $0 \rightarrow \mathcal{F} \xrightarrow{h} \mathcal{G}$ with $(X, \mathcal{G}) \in \mathcal{C}$. Prove the following statement

- a) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq s}$ be such that \mathcal{C} is of finite cohomology and \mathcal{D} is a class of subsheaves with respect to \mathcal{C} . Then the class \mathcal{D} is of finite cohomology if and only if it is of bounded regularity.

If X is a projective scheme over some field K and \mathcal{F}, \mathcal{G} are two coherent sheaves of \mathcal{O}_X -modules we say that \mathcal{F} is a *quotient of \mathcal{G}* if there is an epimorphism of sheaves $\mathcal{G} \xrightarrow{h} \mathcal{F} \rightarrow 0$. Accordingly we say that \mathcal{D} is a *class of quotient sheaves with respect to \mathcal{C}* if for each pair $(X, \mathcal{F}) \in \mathcal{D}$ there is a pair $(X, \mathcal{G}) \in \mathcal{C}$ such that \mathcal{F} is a quotient of \mathcal{G} . Prove the following statement.

- b) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq s}$ be such that \mathcal{C} is of finite cohomology and \mathcal{D} is a class of quotient sheaves with respect to \mathcal{C} . Then the class \mathcal{D} is of finite cohomology if and only if it is of bounded regularity.

C) (*Serre Polynomials and Classes of Subsheaves and Quotient Sheaves*) This part generalizes what was said above about Hilbert schemes. Keep the notations and hypotheses of part B). Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^s$ be subclasses. Prove the following statement

- a) Let \mathcal{D} be a class of subsheaves (resp. of quotient sheaves) with respect to \mathcal{C} and assume that \mathcal{C} is of finite cohomology. Then the class \mathcal{D} is of finite cohomology if and only if the set of Serre polynomials $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite

The following special case of the previous statement covers most closely our previous observation on Hilbert schemes: Fix a pair $(X, \mathcal{G}) \in \mathcal{S}^{\leq s}$ and let \mathcal{D} be a class of subsheaves or of quotient sheaves of \mathcal{G} . Show that the following statements are equivalent.

- (i) \mathcal{D} is a class of finite cohomology.
(ii) \mathcal{D} is a class of bounded regularity.
(iii) The set $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

Now we give another remark, which concerns *sets which bound cohomology*.

10.18. **Exercise and Remark.** A) (*Subsets which Bound Cohomology*) Let the notations and hypotheses as in (10.17). We say that *the subset*

$$\mathbb{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$$

bounds cohomology if each class $\mathcal{D} \subseteq \mathcal{S}^{\leq s}$ which is of finite cohomology on \mathbb{S} is of finite cohomology at all. According to (10.16) we can say

- a) If the set $\mathbb{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ contains a quasi-diagonal subset Σ , then \mathbb{S} bounds cohomology.

This result is shown in greater generality in [Br-Ja-Li2]. It namely holds even if $\mathcal{S}^{\leq s}$ is replaced by the class of pairs (X, \mathcal{F}) in which X is a projective scheme over some Artinian ring and \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules. It is natural to ask whether the condition to contain a quasi-diagonal subset is also necessary for a subset $\mathcal{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ to bound cohomology (for the class $\mathcal{S}^{\leq s}$). This is indeed true, as is shown in [Br-Ja-Li2], too.

B) (*Bounding Sets for Classes of Vector Bundles*) It is natural to ask, whether for appropriate subclasses of $\mathcal{D} \subseteq \mathcal{S}^{\leq s}$ there are more sets which bound cohomology than those specified above. A particularly interesting setting for this question is given as follows: Let K be a field and let

$$\mathcal{V} \subseteq \mathcal{S}^{\leq s}$$

be the family of all algebraic vector bundles over the projective space $\mathbb{P}_K^s = \text{Proj}(K[X_0, X_1, \dots, X_s])$ and let $\mathcal{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$. We say that \mathcal{S} *bounds cohomology of vector bundles (over) \mathbb{P}_K^s* , if each subclass $\mathcal{D} \subseteq \mathcal{V}$ which is of finite cohomology on \mathcal{S} is of finite cohomology at all. We do not know yet a precise combinatorial characterization of those subsets \mathcal{S} which bound cohomology of vector bundles. What is shown in the Master thesis [Ke] is the following special result

- a) If the sets $\mathcal{S} \cap (s \times \mathbb{Z}_{<0})$ and $\mathcal{S} \cap (0 \times \mathbb{Z}_{>0})$ are both finite, then the set \mathcal{S} bounds cohomology of vector bundles if and only if it contains a quasi-diagonal subset of $\{0, 1, \dots, s\} \times \mathbb{Z}$.

Clearly this means in particular;

- b) A finite subset $\mathcal{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ bounds cohomology of vector bundles if and only if it contains a quasi-diagonal subset.

So, here is a problem;

- c) Is there a (necessarily infinite) set $\mathcal{S} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ which contains no quasi-diagonal subset $\Sigma \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}$ but which does bound cohomology of vector bundles?

C) (*Counting Cohomology Tables*) Keep the notations of part A). Fix an arbitrary (quasi-)diagonal subset

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \dots, s\} \subseteq \{0, 1, \dots, s\} \times \mathbb{Z}, \quad (n_s < n_{s-1} < \dots < n_0).$$

and fix a family of non-negative integers

$$\bar{h} := (h^i)_{i=0}^s.$$

Then clearly we know by (10.17), that the number of cohomology tables

$$N_{\Sigma, \bar{h}} := \#\{h_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{S}^{\leq s} : h^i(X, \mathcal{F}(n_i)) = h_i, \quad i = 0, 1, \dots, s\}$$

is finite. Going tediously through our arguments on could indeed get out some upper bound for this number, at least in the case where Σ is the standard diagonal subset $\{(i, -i) \mid i = 0, 1, \dots, s\}$. So, one could get stuck to the idea

of counting all possible cohomology tables with a given standard cohomology diagonal, or at least to bound their number in a satisfactory way. Clearly, one cannot expect, that a bound which is obtained on use of the arguments of our proof will be satisfactory. The enormous discrepancy between the expected and the actual number of cohomology tables is made evident in the Master thesis [Cat].

So roughly speaking, our bounding results are not appropriate to perform quantitative arguments in the sense of counting cohomology tables in a satisfactory way. On the other hand our bounding results furnish at least the equivalence of the following statements, which also follows from the properties of cohomological patterns (see (2.15) and (2.16)) - and whose proof we suggest as an exercise.

- (i) $\mathcal{F} = 0$.
- (ii) $h^i(X, \mathcal{F}(-i)) = 0$ for all $i \in \{0, 1, \dots, s\}$.
- (iii) There is some $t \in \mathbb{Z}$ such that $H^i(X, \mathcal{F}(t-i)) = 0$ for all $i \in \{0, 1, \dots, s\}$.
- (iv) $h^i = 0$ for all $i \in \{0, 1, \dots, s\}$.

Finally, let us have another short glance from the top we have reached and look back to the landscape we were traveling through at an early stage of our excursion. We content ourselves to look back just to one quite fascinating side valley at which we had only a very short and limited look when we came across its entrance. We even missed to mention the name of the wide landscape hidden behind the narrow entrance to this valley: *asymptotic behaviour of cohomology*.

10.19. Remark. A) (*Revisiting Cohomological Tameness*) The existence of cohomological Serre polynomials clearly tells us, that the cohomological pattern (see (2.17))

$$\mathcal{P} = \mathcal{P}(X, \mathcal{F}) = \{(i, n) \mid h^i(X, \mathcal{F}(n))\}$$

of a coherent sheaf \mathcal{F} over a projective scheme X over a field K is always *tame* in the sense of (2.17). We mentioned already that in the case, where the base field K is replaced by an arbitrary Noetherian ring R_0 (even if this latter has very nice properties) \mathcal{F} need not be cohomologically tame (see (2.11)C)). Let us mention here once more, that cohomological Hilbert (resp. Serre) polynomials always exist in the case where $\dim(R_0) = 0$, so that also in this case the sheaf \mathcal{F} is cohomologically tame. But clearly tameness is only a very weak consequence of the existence of cohomological Hilbert (resp. Serre) polynomials.

B) (*Asymptotic Behaviour of Cohomology*) (See [Br6]) Let $R = R_0 \oplus R_1 \oplus R_2 \dots$ be a Noetherian homogeneous ring, let $X = \text{Proj}(R)$ let M be a finitely generated graded R -module and let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf of \mathcal{O}_X -modules induced by M . It is natural to ask “how the R_0 modules $H_{R_+}^i(M)_n$ (or equivalently: the R_0 -modules $H^i(X, \mathcal{F}(n))$) behave if $n \rightarrow -\infty$ ”, that is to ask for *the asymptotic behaviour* of these modules for $n \rightarrow -\infty$. In particular,

one may ask whether certain invariants of these modules ultimately become constant or- equivalently-are *asymptotically stable* for $n \rightarrow -\infty$. In the case of numerical invariants one could ask whether these are presented by a polynomial for all $n \ll 0$ or-equivalently-whether they are *anti-polynomial*.

A very weak form of asymptotic stability for $n \rightarrow -\infty$ is tameness. A very satisfactory form of asymptotic stability is the anti-polynomiality of the cohomological Hilbert-functions h_M^i (resp. $h_{\mathcal{F}}^i$). Keep in mind that tameness fails in general if $\dim(R_0) \geq 3$ whereas in the case $\dim(R_0) = 0$ the cohomological Hilbert functions are indeed anti-polynomial. So, if $\dim(R_0)$ increases, the asymptotic behaviour of cohomology quickly becomes more and more unstable. In between the two extrema of tameness and anti-polynomiality one has the important issues of *asymptotic stability of associated primes* which says that the set $\text{Ass}_{R_0}(H_{R_+}^i(M)_n)$ ultimately stabilizes if $n \rightarrow -\infty$. This nice behaviour is always given if either:

a) (R_0, \mathfrak{m}_0) is local and of dimension one (see [Br-Fu-T]),

or

b) essentially of finite type over a field and of dimension ≤ 2 (see [Br-Fu-Lim] and [Br7]).

In geometric terms, one may draw the following conclusion from this (see [Br7]):

c) If $X \xrightarrow{\pi} X_0$ is a proper morphism such that X_0 is essentially of finite type over a field, \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules and \mathcal{L} is an ample invertible sheaf of \mathcal{O}_X -modules, then for each $i \in \mathbb{N}_0$ the set

$$\{x_0 \in \text{Ass}_{X_0}(\mathcal{R}^i(\pi^*)(\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{F})) \mid \dim(\mathcal{O}_{X_0, x_0}) \leq 2\}$$

of points $x_0 \in X_0$ of codimension ≤ 2 and associated to the i -th direct image sheaf of the n -th \mathcal{L} -twist of \mathcal{F} with respect to π ultimately becomes constant if $n \rightarrow -\infty$.

In fact, one may say even more (see [Bä-Br]), namely:

d) In the notations and hypotheses of statement a), for each $x_0 \in X_0$ with $\dim(\mathcal{O}_{X_0, x_0}) \leq 2$, the number

$$\text{depth}_{\mathcal{O}_{X_0, x_0}}(\mathcal{R}^i(\pi^*)(\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{F}))$$

ultimately becomes constant if $n \rightarrow -\infty$

If either (R_0, \mathfrak{m}_0) is local and of dimension 1 or at specific levels i , quite a lot can be said on the anti-polynomiality of numerical invariants of the R_0 -modules $H_{R_+}^i(M)_n$. We do not spell out the corresponding statements here. Instead we just mention the references [Br-Fu-T], [Br-Ro], [Br-Ro-Sa] and [Br-Ku-Ro].

11. BIBLIOGRAPHICAL HINTS

We append to these notes a rough classification of the references occurring in our bibliography. We do this in the hope that interested readers get help and encouragement to penetrate further into the subject or to clarify the background which we considered as known in our lectures. The reader should be aware of the fact, that our bibliography is far from covering the topics we list below. The quoted Diploma-, Master-, and PhD thesis written at the University of Zürich are available on request in form of PDF.

1. General Commutative Algebra, Homological Algebra and Algebraic Geometry:

[Br0], [Br-Bo-Ro], [Bru-Her], [E1], [Ev-Gri], [Gro-D], [Gro5], [H1], [Kun1], [Mat], [Rot], [Sh].

2. General Local Cohomology and Sheaf Cohomology:

[Br-Fu-Ro], [Br-Sh1], [Gro-D], [Gro2], [H1], [Se].

3. Structure, Vanishing and Bounding Results for Local Cohomology and Sheaf Cohomology:

[A-Br], [B-Mu], [Br1], [Br2], [Br3], [Br4], [Br8], [Br-He], [Br-Ja-Li1], [Br-Ja-Li2], [Br-K-Sh], [Br-L1], [Br-L2], [Br-Matt-Mi1], [Br-Matt-Mi2], [Br-N], [Br-Sh1], [Br-Sh2], [Br-Sh3], [Cat], [Ch2], [En], [Fa1], [Fa2], [Fu2], [Gro4], [H1], [H2], [K], [Ke], [Kl], [Ko], [L], [M], [Matt], [Mi-N-P], [Mu2], [R], [Ro], [Se], [Sev], [Si], [Tru], [Z].

4. Castelnuovo-Mumford regularity and its Historic Background:

[B-Mu], [B-St], [Bäc], [Be1], [Be2], [Br2], [Br4], [Br5], [Br8], [Br-Gö], [Br-Ja-Li1], [Br-Matt-Mi1], [Br-Sh1], [Br-Vo], [Bu], [C], [Cav], [Cav-Sb], [Ch1], [Ch4], [Ch-DA], [Ch-D'C], [Ch-F], [Ch-F-N], [Ch-Ha-Ho], [Ch-Mi-Tr], [Ch-MS], [Ch-Ph], [Ch-U], [E-G], [G], [Gi], [Go1], [Gru-La-P], [Hen-Noe], [Herm], [Hi1], [Hi2], [Ho], [Ho-Hy], [La], [Mas-W], [Ma-Me], [Mu1], [O], [Pi], [Ros-Tr-V], [Sei].

5. Hilbert Schemes:

[Fu2], [Go1], [Go2], [Gro6], [H1], [H3], [Mal], [P], [Pe-St].

5. Vector Bundles and their Cohomology:

[A-Br], [Br4], [Cat], [El-Fo], [En], [Ev-Gri], [Gr-Ri], [Gro0], [Gro4], [H1], [Hor], [Ke], [Ko], [Matt], [Mu1], [Mu2], [Se], [Sev], [Z].

6. Cohomology Tables, Cohomological Patterns, Tameness and Asymptotic Behaviour of Cohomology:

[Bä-Br], [Br4], [Br6], [Br7], [Br-Fu-Lim], [Br-Fu-T], [Br-He], [Br-K-Sh], [Br-Ku-Ro], [Br-Ro], [Br-Ro-Sa], [Cat], [Ch-Cu-Her-Sr], [K], [Ke], [Lim1], [Lim2], [Lim3], [Mat], [M], [Mi-N-P], [Rott-Seg], [Si].

7. Deficiency and Canonical Modules:

[Br8], [Br-Sh1], [Bru-Her], [Her-Kun], [Sc1], [Sc2].

8. Related Work on Projective Varieties:

[A-Br], [Be1], [Be2], [Br1], [Br4], [Br-Sc1], [Br-Sc3], [Br-Sc3], [Br-Vo], [C], [Ch3], [En], [Gru-La-P], [H1], [Ko], [La], [Mat], [Mi-N-P], [Mu2], [Pi], [Se], [Sev], [Z].

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