CASTELNUOVO- MUMFORD REGULARITY - A SURVEY

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The Historic Roots

Castelnuovo-Mumford regularity has two historic roots:

A geometric result of 1893 by Castelnuovo on linear systems on projecive curves.

An algebraic result - Hilbert's Syzygy Theorem of 1910.

NB: Only Serre's Sheaf Cohomology Theory for algebraic varieties of 1955 and Grothendieck's Local Cohomology Theory of 1962 furnished the appropriate tools to link the geometric and the algebraic root of Castelnuovo-Mumford regularity...

NB: Hilbert's Theorem initiated a controversy among the algebraists of his time, which culminated in the „Problem of the Finitely may Steps“ which was solved in 1923-1926 by Hentzelt-Noether and Hermann.

1. The Bounding Result of Castelnuovo

CASTELNUOVO'S BOUNDING RESULT (1893):
Let C be a smooth complete curve of degree d in the complex projective 3-space P^3. Then, for all n > d-3 and each surface X of degree n in P^3, the complete linear system which is cut out on C by X is spanned by n-forms. Moreover, if C is rational, this bound is sharp.

IN MODERN TERMS (AS OBSERVED BY MUMFORD, 1966):
The first Serre cohomolgy group H^1(P^3,J(n)) of P^3 with coefficients in the n-fold twist J(n) of the sheaf of vanishing ideals J of C equals 0 for all n > d-3. For rational curves, this bound is sharp.

2. The Syzygy Theorem of Hilbert

NOTATION: Let K be a field, let r be a positive integer and let

$$S := K[X_1, \ldots, X_r]$$

be the polynomial ring over K in the r indeterminates $X_1, \ldots, X_r$. Let I be a homogeneous ideal of S.

HILBERT'S SYZYGY THEOREM (1910):

The ideal I has a minimal free resolution. This resolution is unique (up to isomorphism of complexes) and its length cannot exceed the number r of indeterminates.

NB: We can replace I by any finitely generated S-module.

NOTATION: For a finitely generated graded S-module M let

$$d_i(M) := d(F_i)$$

denote the generating degree of the i-th free module F_i in the minimal free resolution on M.

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3. The Problem of the Finitely Many Steps

PROBLEM OF THE FINITELY MANY STEPS (~ 1910): Does the generating degree $d(I)$ of the ideal I bound the generating degrees $d_i(I) = d(F_i)$ of all free modules F_i which occur in the minimal free resolution of I ?

NB: An affirmative answer guarantees that the minimal free resolution of I may be computed in „fintely many steps“ from a system of generators of I.

THE BOUNDING RESULT OF HENTZELT-NOETHER (1923) AND HERMANN (1926): The generating degree $d(I)$ of I bounds all the degrees $d_i(I)$.
So, the “Problem of the Finitely Many Steps“ is answered affirmatively.

The Definition of Castelnuovo Regularity in Terms of Sheaf Cohomology

In 1966, Mumford defined the notion of Castelnuovo regularity for a coherent sheaf of ideals over a projective space and established a fundamental upper bound for this invariant.

NB: Mumford's bound can be made explicit and thus opened a new view to Grothendieck's construction of the Hilbert scheme of 1962.

NB: Mumfords bounding result made come up again an old question of Algebraic Geometry, which was hidden already behind the „Problem of the Finitely Many Steps“, namely:

„What can be computed in Algebraic Geometry ?“

- the title of a joint paper by Bayer and Mumford of 1991.

4. The Definition given by Mumford

MUMFORD'S DEFINITION OF REGULARITY (1966):

Let J be a coherent sheaf of ideals over projective s-space P^s. Then, the Castelnuovo regularity of J is defined as

$$\mathrm{reg}(J) := \min\{n : H^i(P^s, J(n-i)) = 0 \text{ for all } i > 0\},$$

where H^i(P^s,J(n-i)) denotes the i-th Serre cohomology group of P^s with coefficients in the (n-i)-th twist of J. The Castelnuovo regularity of a closed subscheme X (thus i. p. of a projective variety) in P^s is the regularity of its sheaf J_X of vanishing ideals: reg(X) : = reg(J_X).

MUMFORD'S OBSERVATION: Castelnuovo's Bounding Result says, that the a smooth complete curve C of degree d in complex projective three space P^3 satisfies reg(C) < d, and this bound is sharp if C is rational.

6. The Hilbert Scheme of Grothendieck

GROTHENDIECK'S CONSTRUCTION AND DEFINITION OF THE HILBERT SCHEME (1962): Let p be a polynomial of degree s. Then, there is a projective scheme Hilb_p (called the Hilbert scheme of p), which parametrizes all coherent sheaves J of ideals over P^s which satisfy p_J = p.

NB: An essential step in Grothendieck's construction is to show that there is some m = m(p) such that H^i(P^s, J(n)) = 0 for all i>0, all n>m and all J with p_J = p.

NB: Grothendiek's proof of this fact does not hint a possbilty to compute such a bound m(p). (The proof of) Mumford's Bounding Result allows to compute such a bound. So, Mumford's approach emphasizes the computational aspect of Grothendieck's construction...

7. Generalizing Castelnuovo's Bound

THE BOUND OF GRUSON-LAZARSFELD-PESKINE (1983): An irreducible projective curve C of degree d in P^s satisfies

$$\mathrm{reg}(C) < d - s + 3.$$

THE CONJECTURE OF EISENBUD-GOTO (1984): An irreducible projective variety X of degree d in P^s satisfies

$$\mathrm{reg}(X) < d - s + \dim(X) + 2.$$

THE BOUNDS OF PINKHAM AND LAZARSFELD (1986, 1987): If X is a smooth surface of degree d in the complex projective space P^s, the Eisenbud-Goto Conjecture holds.

NB: The Eisenbud-Goto Conjecture is still open for arbitrary projective surfaces. However, it has been established in the last years for many classes of projective varieties.

Joining the Two Roots

In 1982 Ooishi used Grothendieck's Local Cohomology to define the notion of Castelnuovo-Mumford regularity for finitely generated graded modules – an extension of Mumford's concept of Castelnuovo regularity. As shown by Eisenbud-Goto in 1984, the Castelnuovo-Mumford regularity and the degrees of syzygies of a finitely generated graded module are closely related.
So, the geometric and the algebraic root of our theory are now brought together !

NB: At the same time, Computational Algebra and Compuational Algebraic Geometry became new and quickly growing theories, making both gain from the fact, that now highly powered computers were available to handle successfully complex symbolic algorithms.
The basic standard task in this field is yet the computation of minimal free resolutions. This turns Castelnuovo-Mumford regularity into the ultimate measure of complexity for algebraic-geometric computations.

8. The Use of Local Cohomology

OOISHI'S DEFINITION (1982): Let M be a finetely generated graded module over the polynomial ring S. Then, the Castelnuovo-Mumford regularity of M is defined as

$$\mathrm{reg}(M) := \max\{\mathrm{end}(H^i(M)) : i = 0,1, \ldots\},$$

where $H^i(M)$ denotes the (graded !) i-th local cohomology module of M (with respect to the homogeneous maximal ideal of S) and $\mathrm{end}(H^i(M))$ denotes the largest degree in which the graded S-module $H^i(M)$ does not vanish.

NB: The Serre-Grothendieck Correspondence between sheaf cohomology and local cohomology yields, that for a homogeneous saturated ideal I of S and its induced sheaf of ideals J over $P^{(r-1)}$ we have reg(J) = reg (I). As each sheaf of ideals over a projective space is induced by a homogeneous saturated ideal in a polynomial ring, Ooishis Definition is more general than Mumford's.

9. Relating Regularity to Syzygies

A THEOREM OF EISENBUD-GOTO (1984): Let M be a finitely generated graded S-module. Then it holds

$$\mathrm{reg}(M) = \max\{d_i(M) - i : i = 0, \ldots , r\}.$$

COROLLARY: The cohomological invariant reg(M) governs the computational complexity of the minimal free resolution of M.

COHOMOLOGICAL FORMULATION OF THE PROBLEM OF THE FINITELY MANY STEPS: Let I be a homogeneous ideal of the polynomial ring S. Is reg(I) bounded in terms of the generating degree d(I) of I ?

NB: By the Bounding Result of Henzelt-Noether and Hermann, and the above theorem of Eisenbud-Goto this holds. But can one explicitly compute a satisfactory upper bound for reg(I) in terms of r and d := d(I) ?

10. The Pace for a Bound

THE BOUND OF HERMANN (1926): Working through the arguments of Hermann's proof, one obtains

$$\mathrm{reg}(I) < 1 + (2d)^{2^{(r-1)r}}.$$

THE BOUND OF BAYER-MUMFORD (1991):

$$\mathrm{reg}(I) < 1 + (2d)^{(r-1)!}.$$

THE BOUND OF GALIGO (1979), GIUSTI (1984) AND CAVIGLIA-SBARRA (2005):

$$\mathrm{reg}(I) < 1 + (2d)^{2^{r-1}}.$$

NB: Galligo and Giusti established only the case $\mathrm{Char}(K) = 0$.

MAYR-MEYER (1982): For each $r > 3$ there is an ideal I such that with $c := 2^{0.2}$ one has

$$\mathrm{reg}(I) > 1 + (2d)^{c^{r-1}}.$$

QUESTION: Can we replace 2^{r-1} by C^{r-1} with $c<C<2$?

11. Computational Algebra

THE REGULARITY TEST OF BAYER-STILLMAN (1987): The Castelnuovo-Mumford regularity reg(I) of the homogenous ideal I of S is the maximum degree of all reduced Gröbner bases of I and this maximum is attained for some standard term order on S.

NB: This results relates Castelnuovo-Mumford regularity to Gröbner bases. These bases are the fundamental tool for all algorithms of Computational Algebra in polynomial algebras and related objects. These algorithms are implemented in high power programs like MACAULAY, SINGULAR , COCOA and a number of commercial programs. They allow to compute minimal free resolutions, Ext-, Deficiency-, Tor-modules and other satellite objects, but also the Hilbert functions and -polynomials and other numerical invariants of finitely generated graded modules over polynomial rings.

NB: All modules are given by their presentation matrix, hence a matrix whose entries are homogeneous polynomials.

Two Recent Bounding Results

In his opening address to a workshop on Castelnuovo-Mumford regularity in 2007 at the Max Planck Institute for Applied Mathematics Leipzig (Germany) E. Zeidler – mathematical physisist and at that time director of the institution – said:
„Mathemathical physicists like Algebraic Geometry because it produces so many invariants – and one of the most important among these is Castelnuovo-Mumford regularity.“

In 2009 a problem related to characteristic varieties of D-modules came up in the research group of Mathematical Physics of our Institute in Zürich, which could be solved if a particular bounding result for the Castelnuovo-Mumford regularity were available. We could establish this particular result, and in doing so were lead to prove a number of other bounding results for the Castelnuovo-Mumford regularity *). We shall present two of these.

*) M. Brodmann, C.H.Linh, M.-H. Seiler: *Castelnuovo-Mumford regularity of annihilators, Ext- and Tor- modules;* to appear in: „Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of His 65th Birthday“ (I. Peeva Editor), Chapter 6, 25 pp.; Springer Science & Business Media, New York 2013.

12. A Hint from Mathematical Pysics

A BOUNDING RESULT FOR THE REGULARITY OF ANNIHILATORS (#- LINH-SEILER, 2012):
Let M be a finitely generated graded S-module generated by homogeneous elements of degree 0.
Then, the Castelnuovo-Mumford regularity reg(Ann(M)) of the annihilator ideal Ann(M) of M is bounded by the Hilbert function h_M of M.
NB: The bound is given explicitly.

COROLLARY: Let Char(K) = 0 and let W be a D-module over the r-th Weyl algebra A_r = K[X_1, ... , X_r, D_1,... , D_r].
Let V(W) in P^(2r-1) be the characteristic variety of W. Let F be an admissible filtration of W. Then, the degree of the polynomials needed to define the set V(W) is bounded in terms of the Hilbert function h_(W,F) of W with respect to F.

NB: This helps to solve a problem from Mathematical Physics related to certain systems of PDE's.

13. A Regularity Bound for Tor-Modules

A BOUNDING RESULT FOR THE REGULARITY OF CERTAIN TOR-MODULES: (# - LINH-SEILER 2012):
Let M and N be finitely generated graded S-modules whose induced coherent sheaves over P^s have only finitely many singular points in common. Then

$$\mathrm{reg}(\mathrm{Tor}_k(M,N)) < \mathrm{reg}(M) + \mathrm{reg}(N) + k + 1 \text{ for all } k = 0,1, \ldots .$$

COROLLARY: If M and N are the total modules sections of vector bundles, then the above inequality holds.

NB: The above result follows from a more general bounding result, in which S is replaced by an arbitrary homogeneous K-algebra R whose underlying projective scheme X has only finitely many singular points. The condition on the singularities of the sheaves induced by M and N has to be replaced by the condition that $\mathrm{Tor}_1(M,N)$ has at most Krull dimension 1. Moreover, at least one of M or N should be of finite projective dimension p. Then, in the above estimate one has to add

$$p\,(k+1)\,\mathrm{reg}(R)$$

to the right hand side of the inequality.