WHY TO BECOME A MATHEMATICIAN ?

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1. A SHORT PREVIEW

Why to become a Mathematician ?

A difficult question, which has no simple answer, indeed. To introduce the subject, we first present two typical instances of mathematical thinking:

(A) The step *from Pre-Mathematics to Mathematics* – an issue which concerns the Applications of Mathematics;

(B) The controversal relation between *True and Provable Propositions* – a theme which concerns the doing of Mathematics on its own.

Based on these two “excursions to Mathematics”, we:

(A) look at the *Relation between Science and Mathematics*,

(B) dare a glance to the *Actual State of Mathematics,*

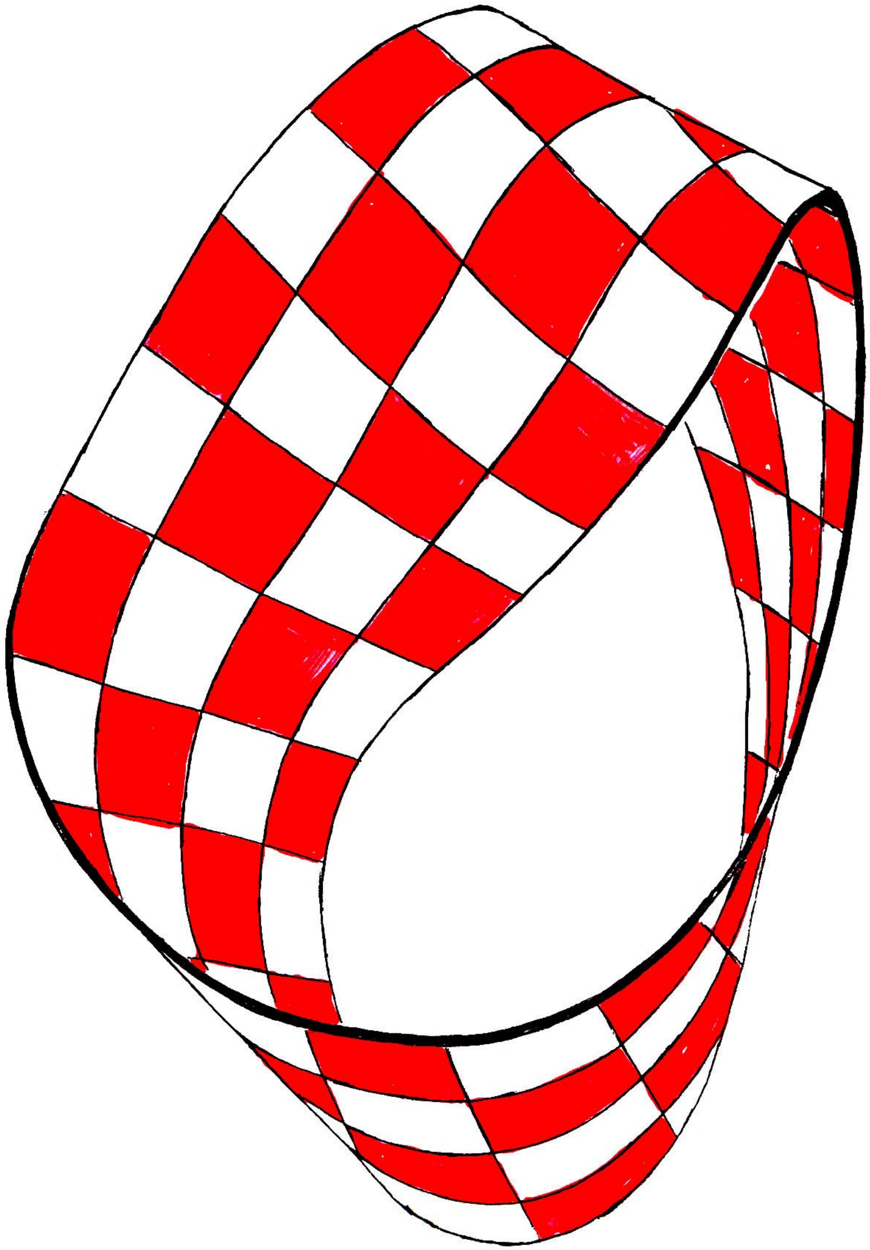
(C) state a few *Personal Requirements to become a Mathematician*

(D) and finally mention a few *Reasons to Become a Mathematician.*

2. REAL WORLD AND IMAGINATION :
FROM PRE-MATHEMATICS TO MATHEMATICS

Mathematics quite often originates from *pre-mathematical problems,* which lead to *mathematical problems.* The step from Pre-Mathematics to Mathematics is very important for all applications of Mathematics and is often called the *Modeling Process* for a problem.

Below, we illustrate this step by means of three examples.



The Moebius Strip

(Named after A. F. Möbius, 1790 – 1868 Astronomer in Leipzig, Germany)

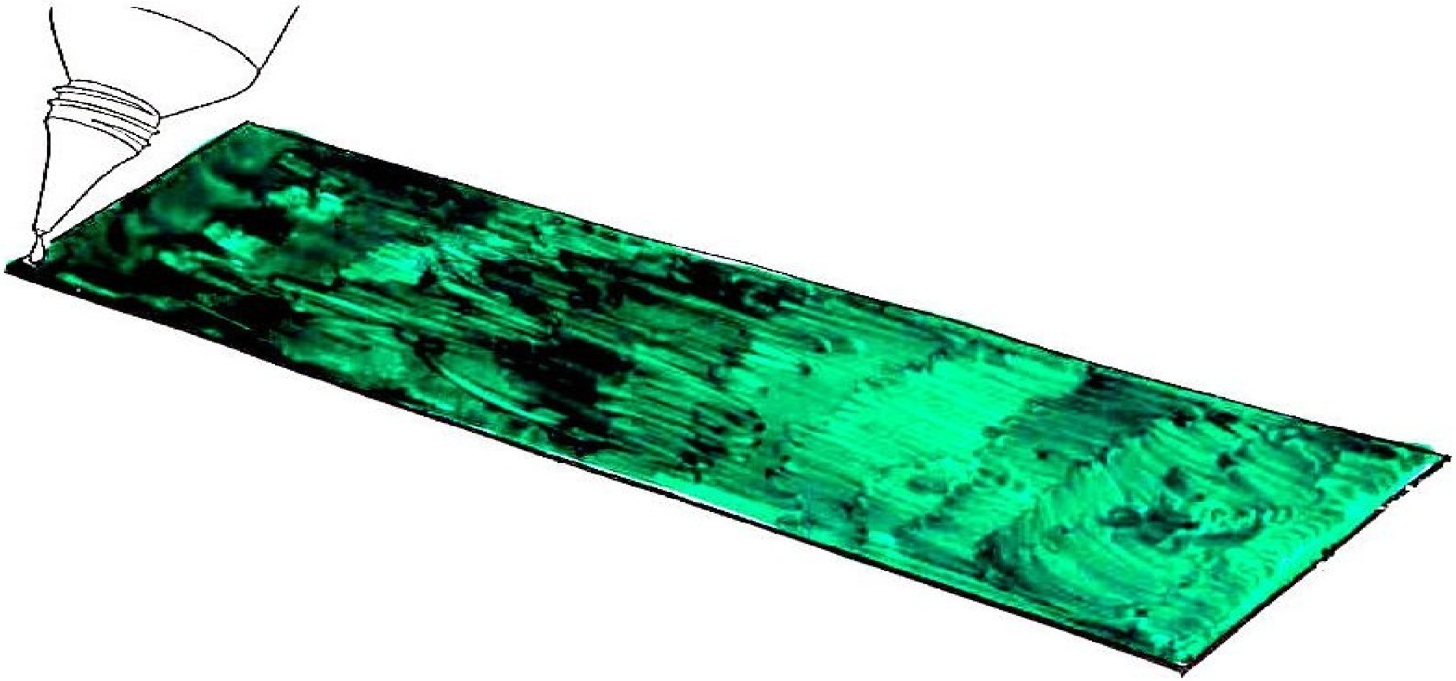
Many of you know how to make this one-sided strip out of a strip of paper.

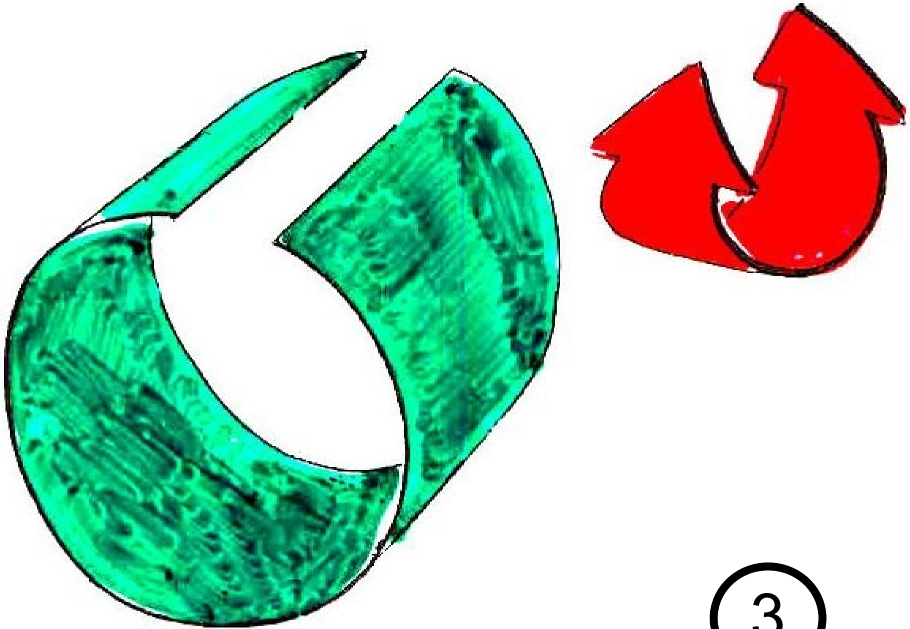
Below comes a suggestion how to proceed with the making of this strip.

Making of a Moebius Strip from a Rectangular Strip of Paper...

(illustrated in four steps)

1





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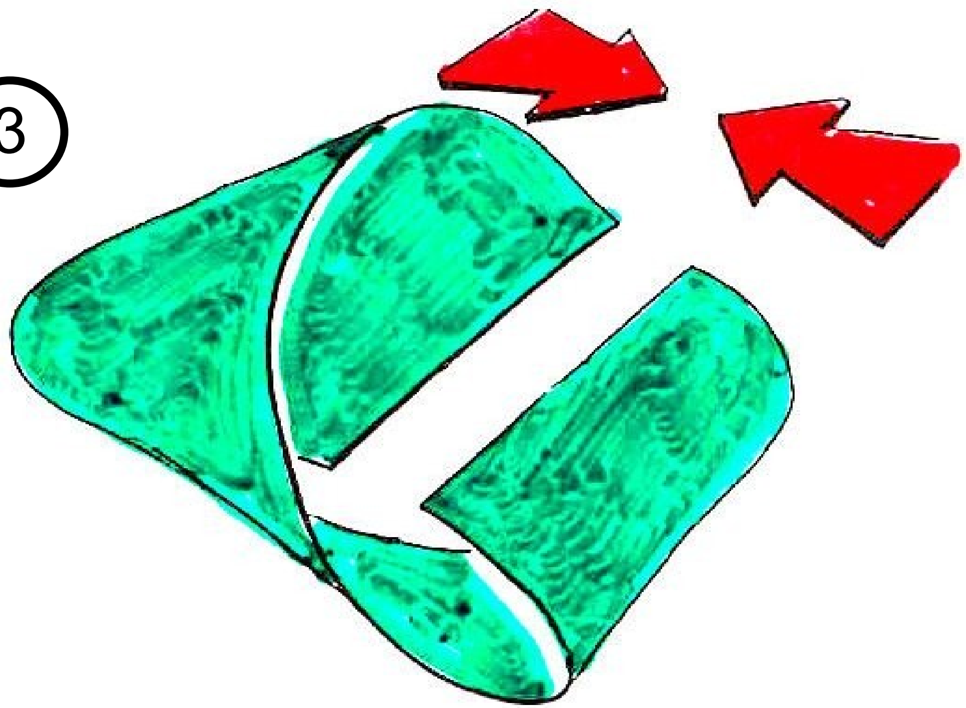

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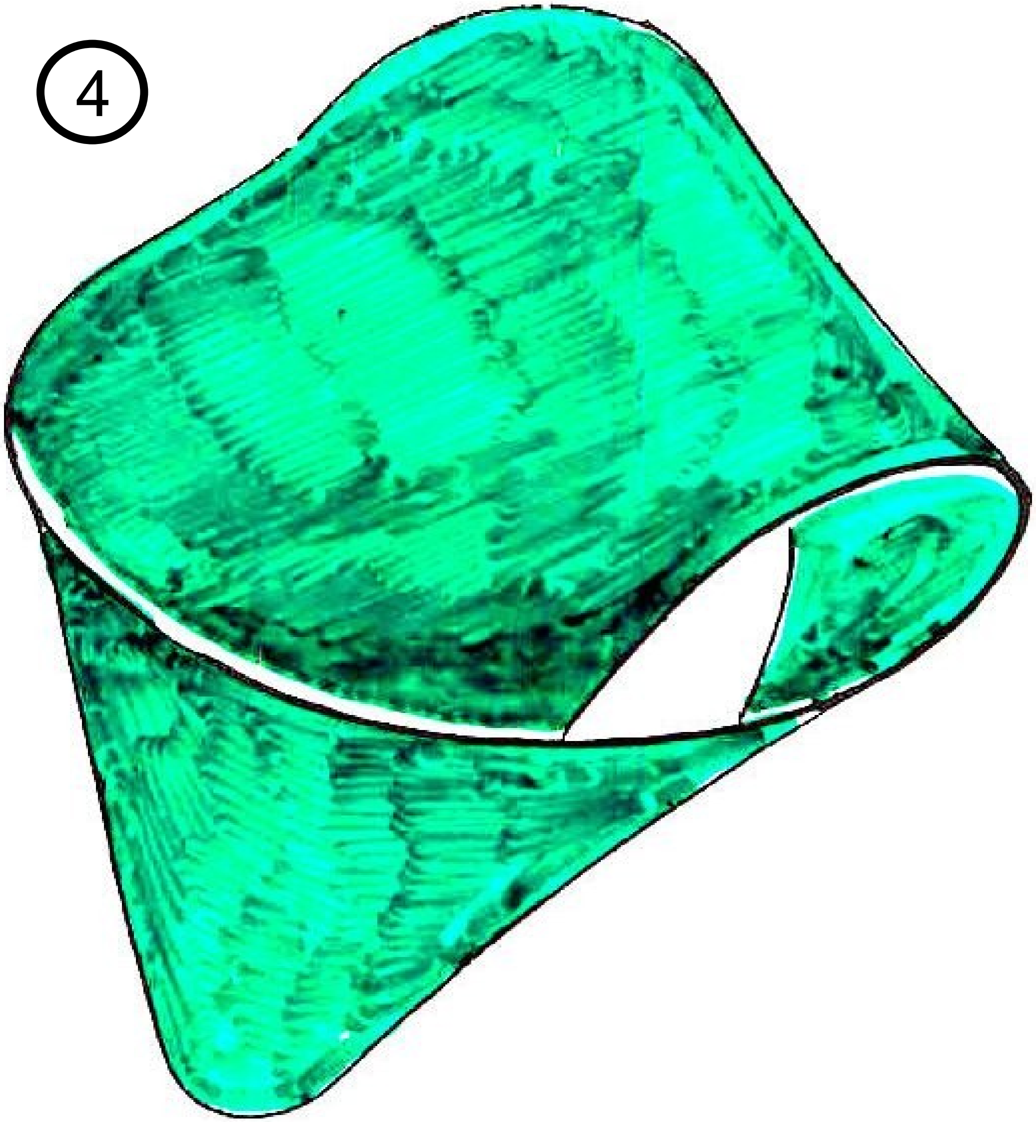
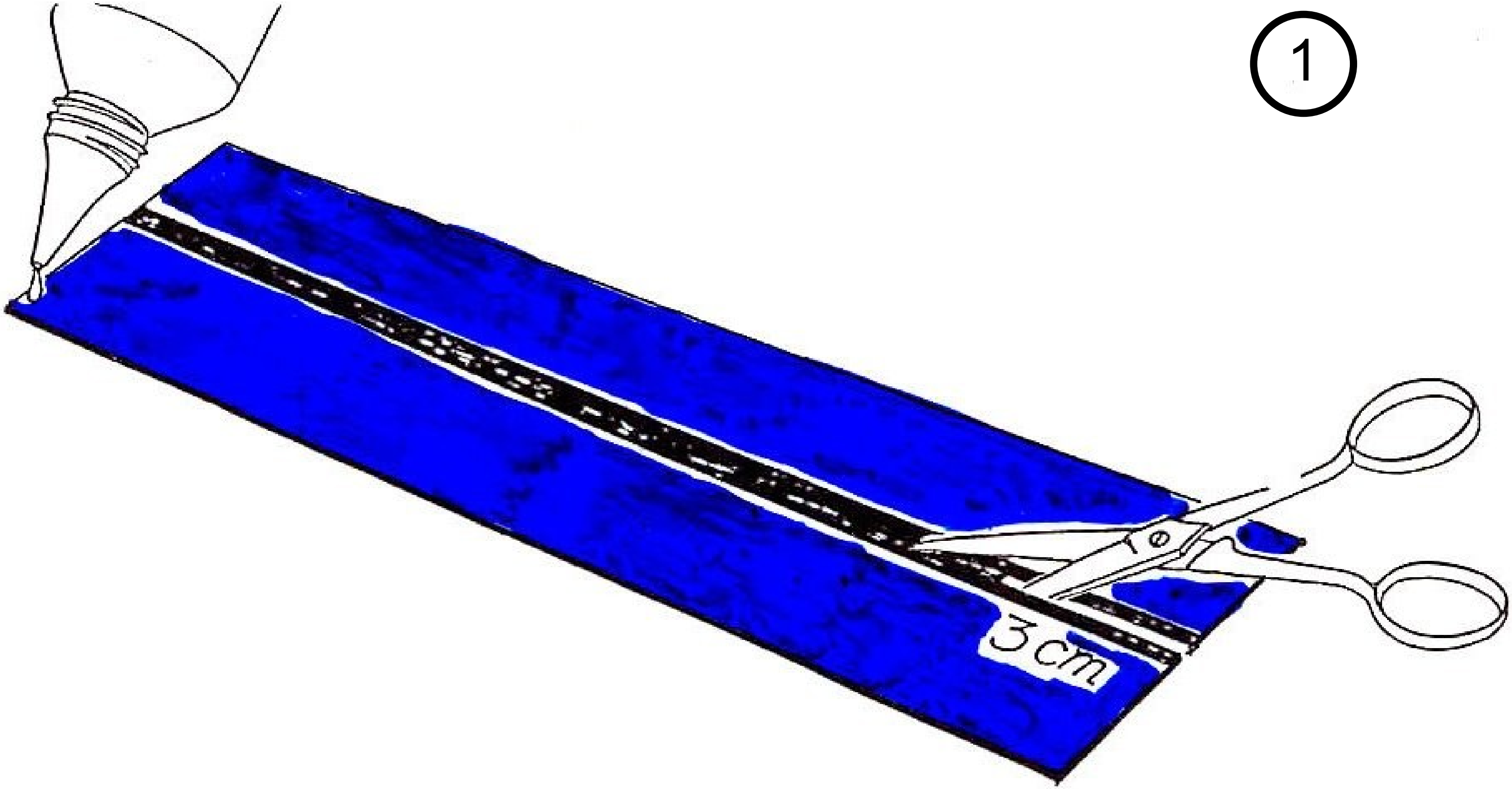


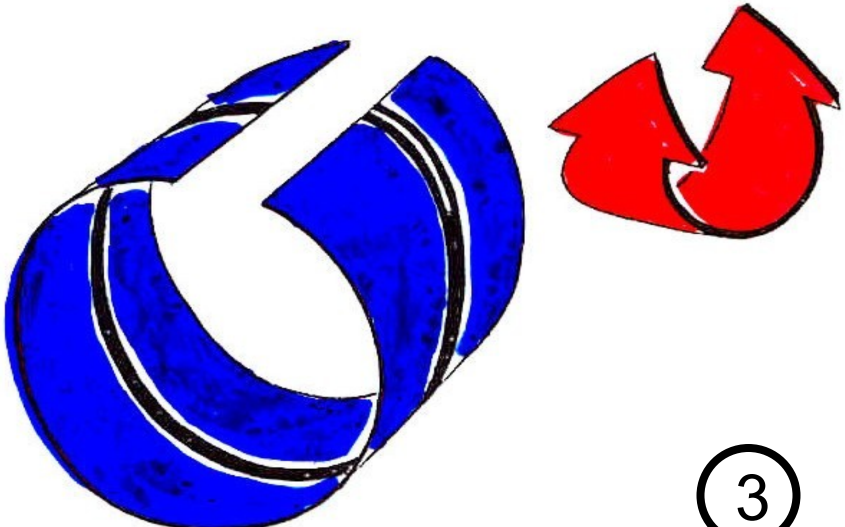
Imagination **is one Important Intellectual Capacity Needed to do Mathematics.**

So, let us ***practice*** **this Skill by Means of our Moebius Strip !**

We cut the Moebius Strip....

(as indicated below in four steps)





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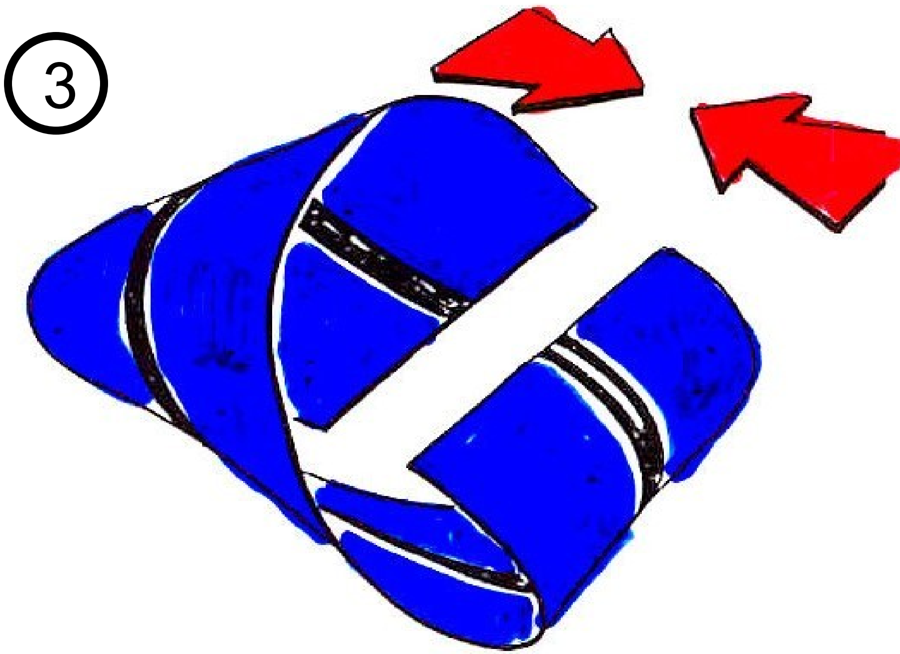

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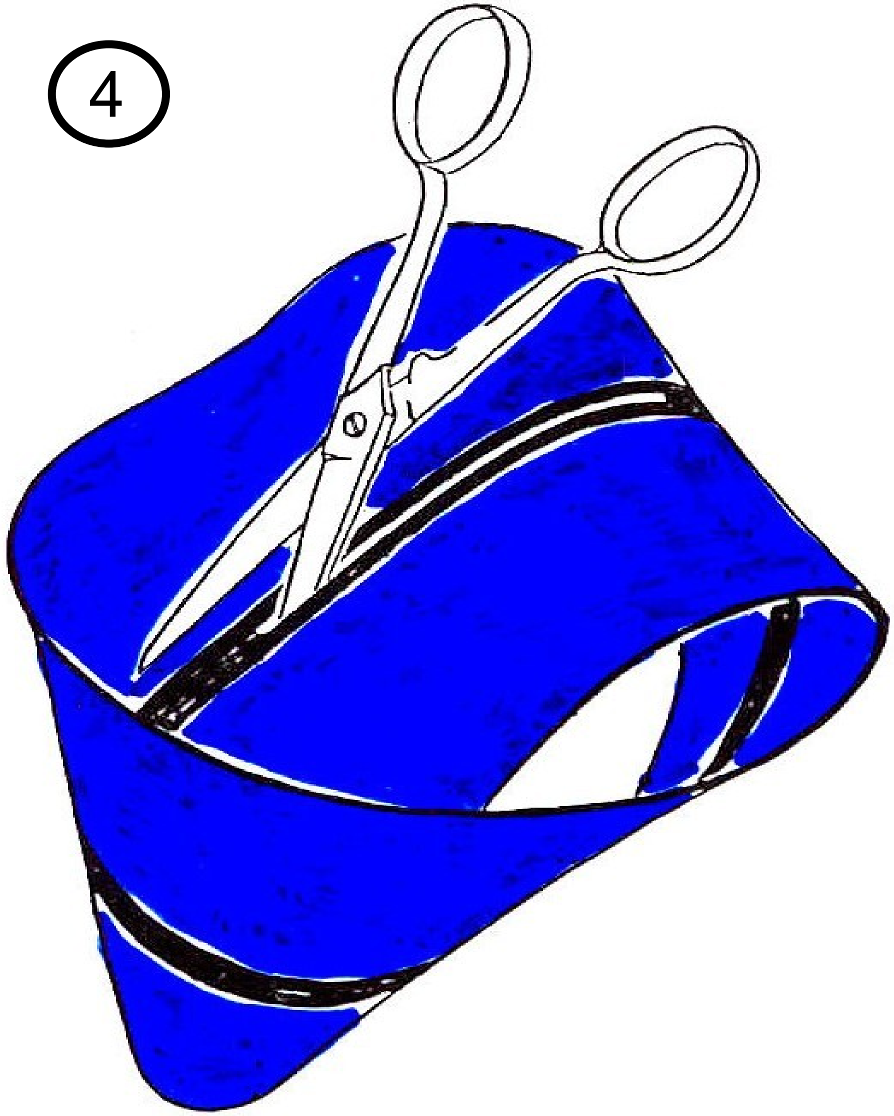

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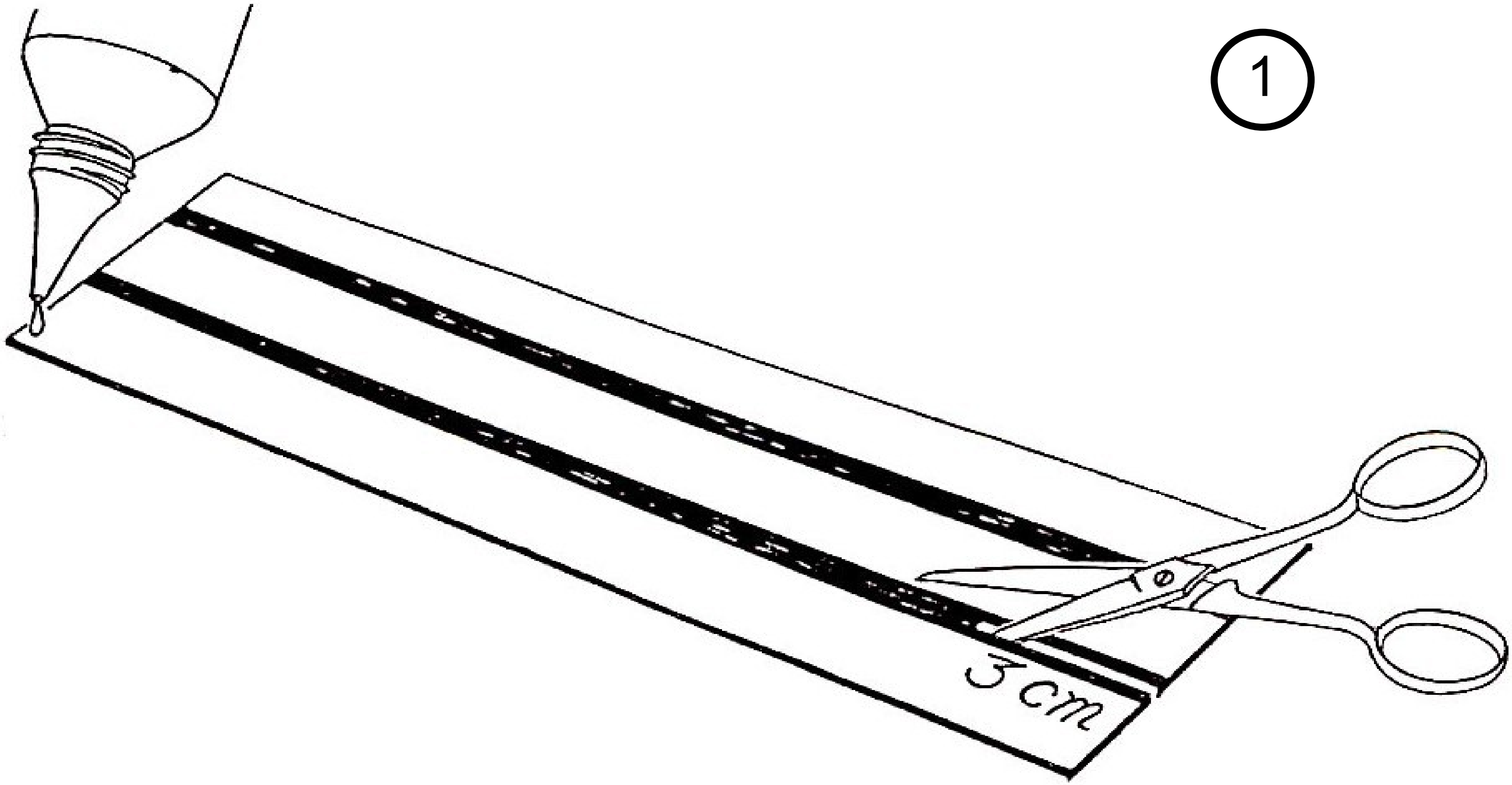
To Practice your Imagination:

Guess first what comes out – then start cutting !

Now, a more Challenging Task:



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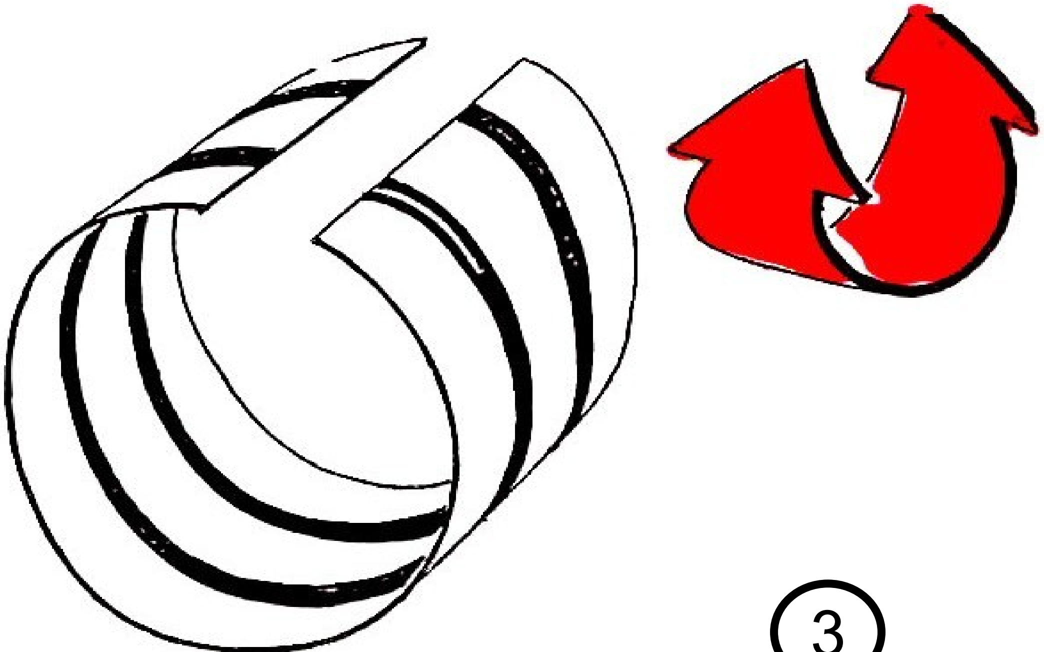

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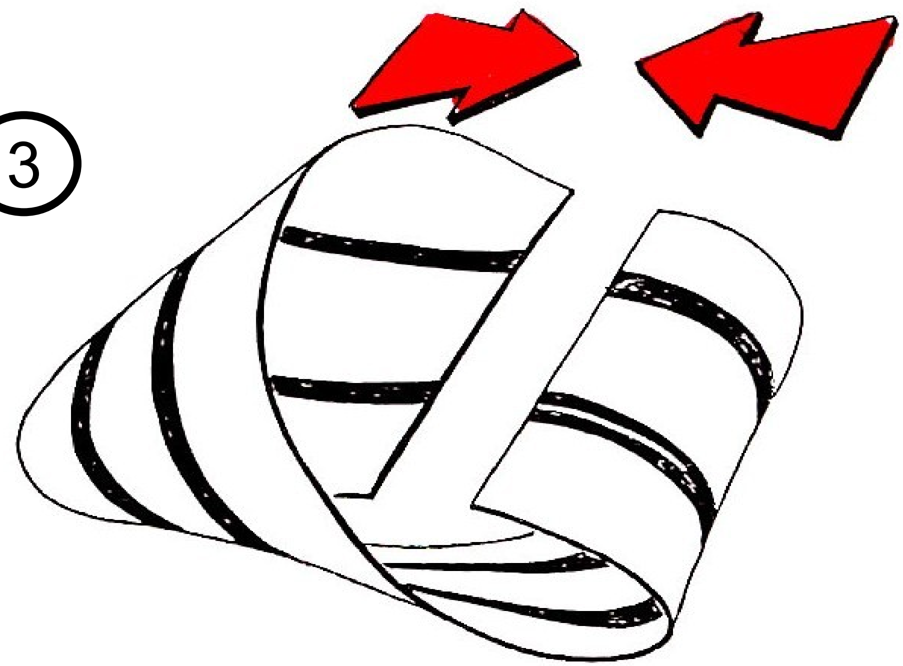
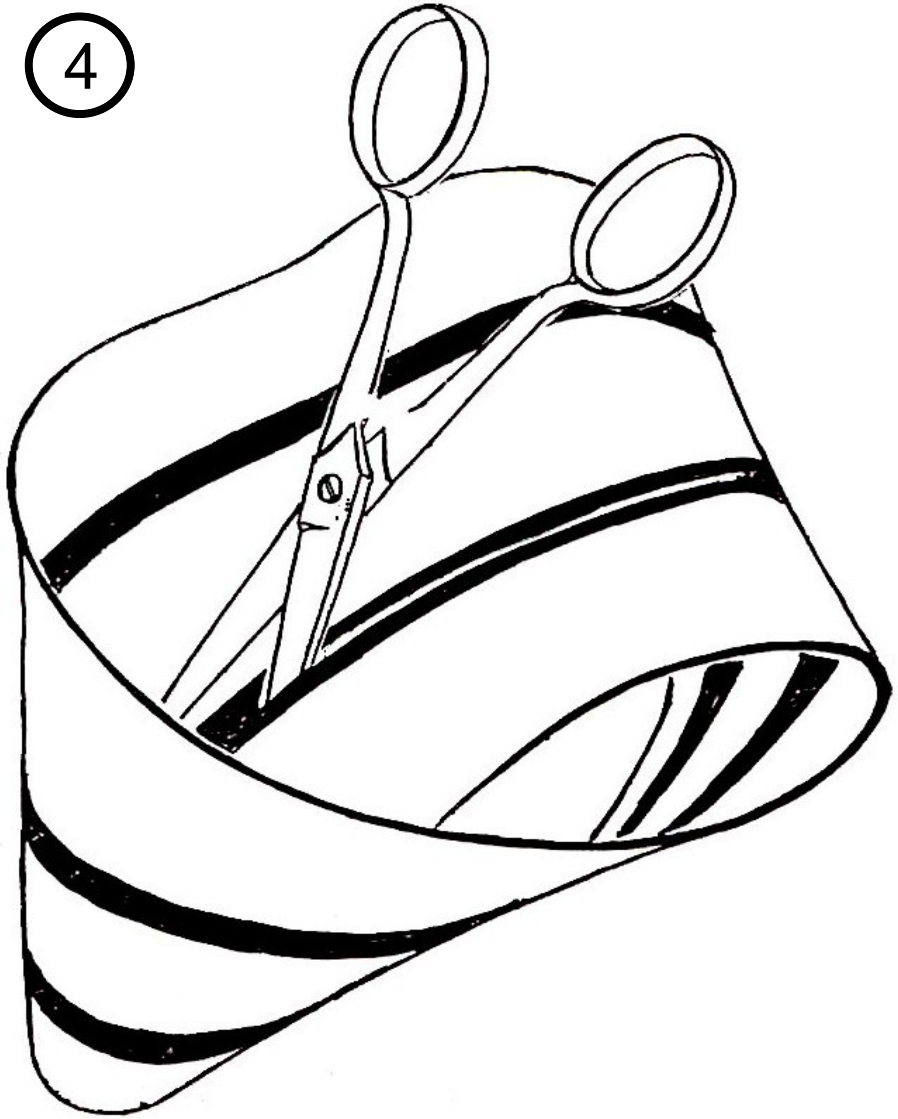


④



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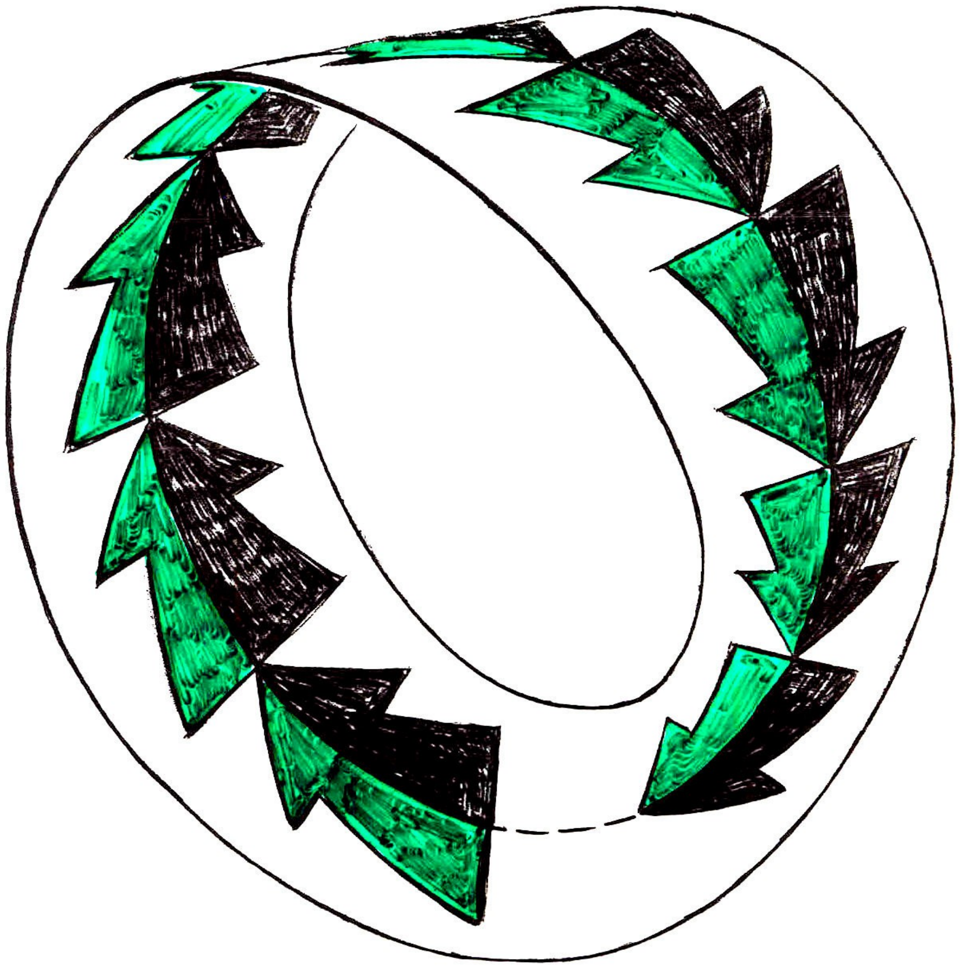
Here, too: First guess, then cut !

The Moebius Strip is One-Sided !

Each try, to paint the „two sides“ of the Moebius strip with different colours leads us into troubes...

The Moebius Strip is not Orientable... and this is in Fact a Mathematical Property:

There is a closed path on the Moebius strip such that „a small geometric figure returns with reverse orientation, if moved along this path“....



To describe this observation in strict terms, one would indeed have to use Mathematics – namely Differential Geometry.

An Ordinary Band is Orientable...

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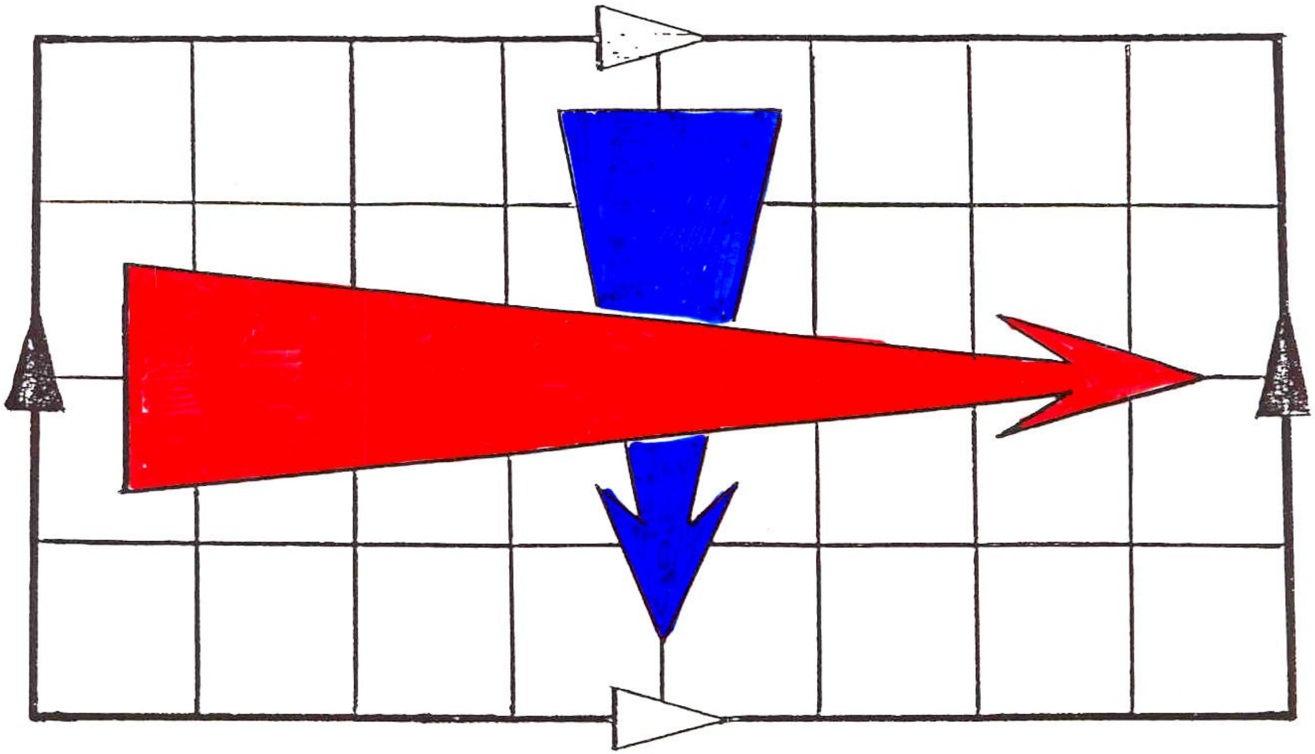
Leaving the Realm of Paper, Cutting and Gluing: Proceeding only by Geometric Imagination

Geometric Imagination **allows to describe procedures to make surfaces, which cannot be made by bending and gluing a strip or a more complicated shaped piece of paper.**

We now describe an example of an imaginative process, which leads to such a surface.

We Produce a New Surface by Though...

The opposite edges of the rectangle are to be glued as indicated below



Hint: perform first the gluing step indicated by blue colour !

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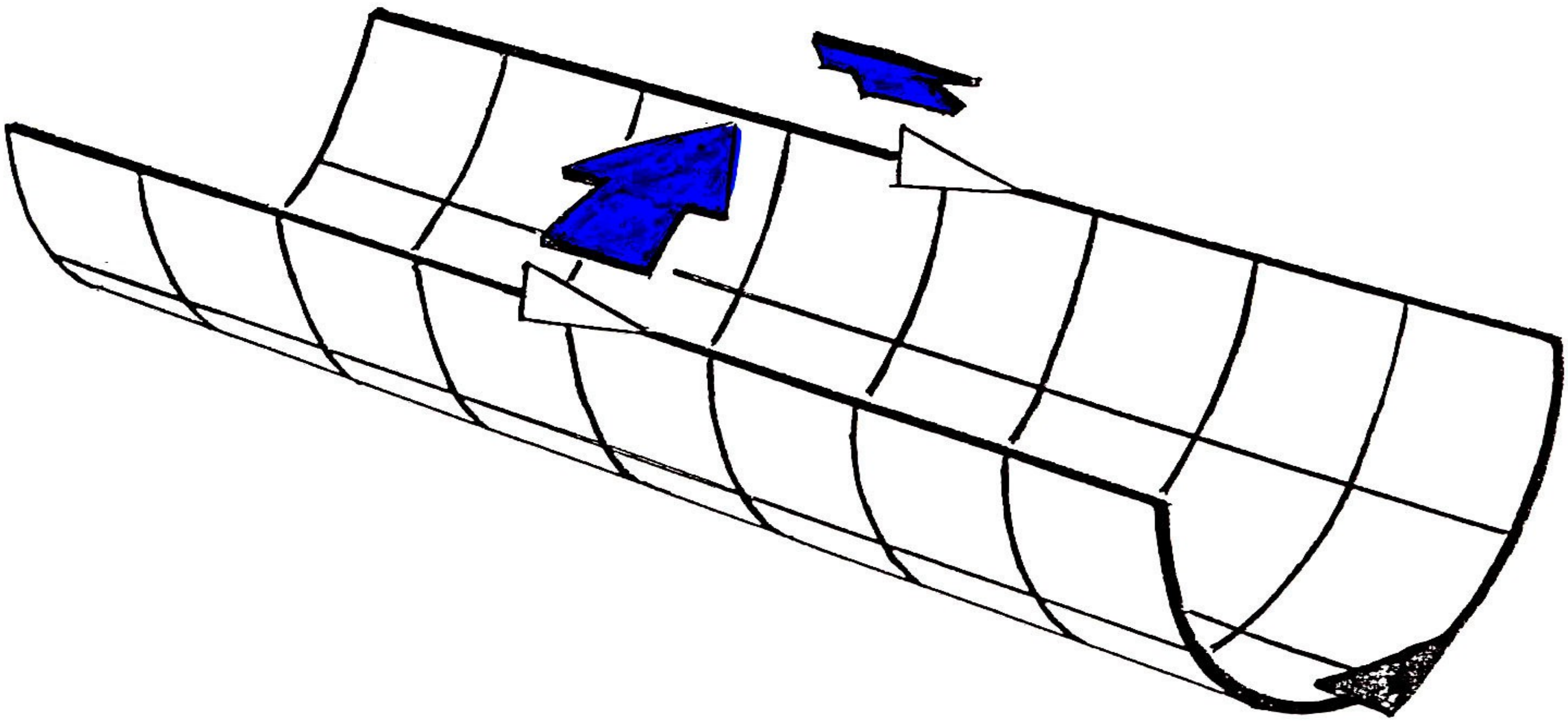
1

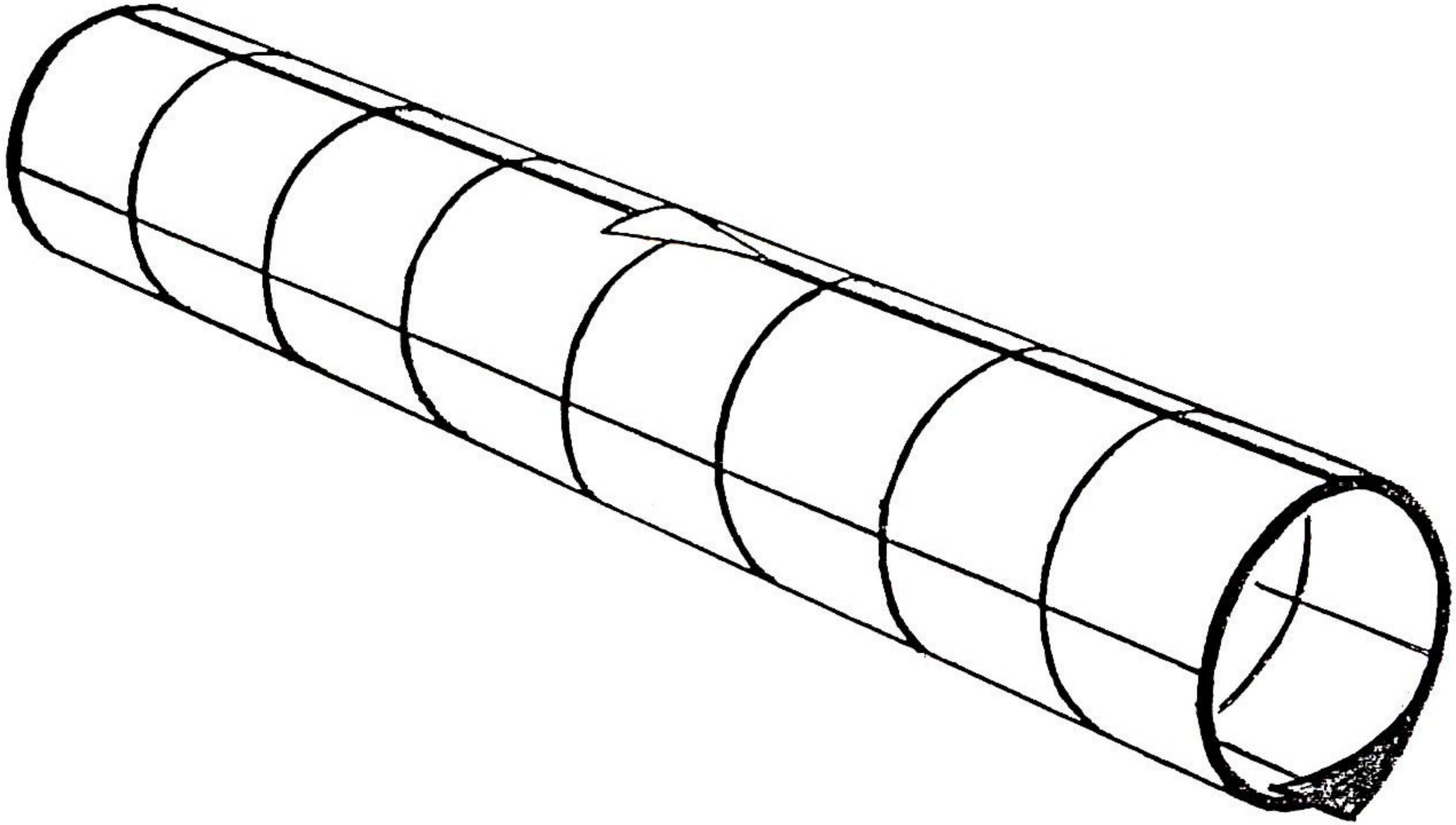
We first get ... a half tube...



Can you already guess what comes out ?

2

The next step leads to ... a tube...



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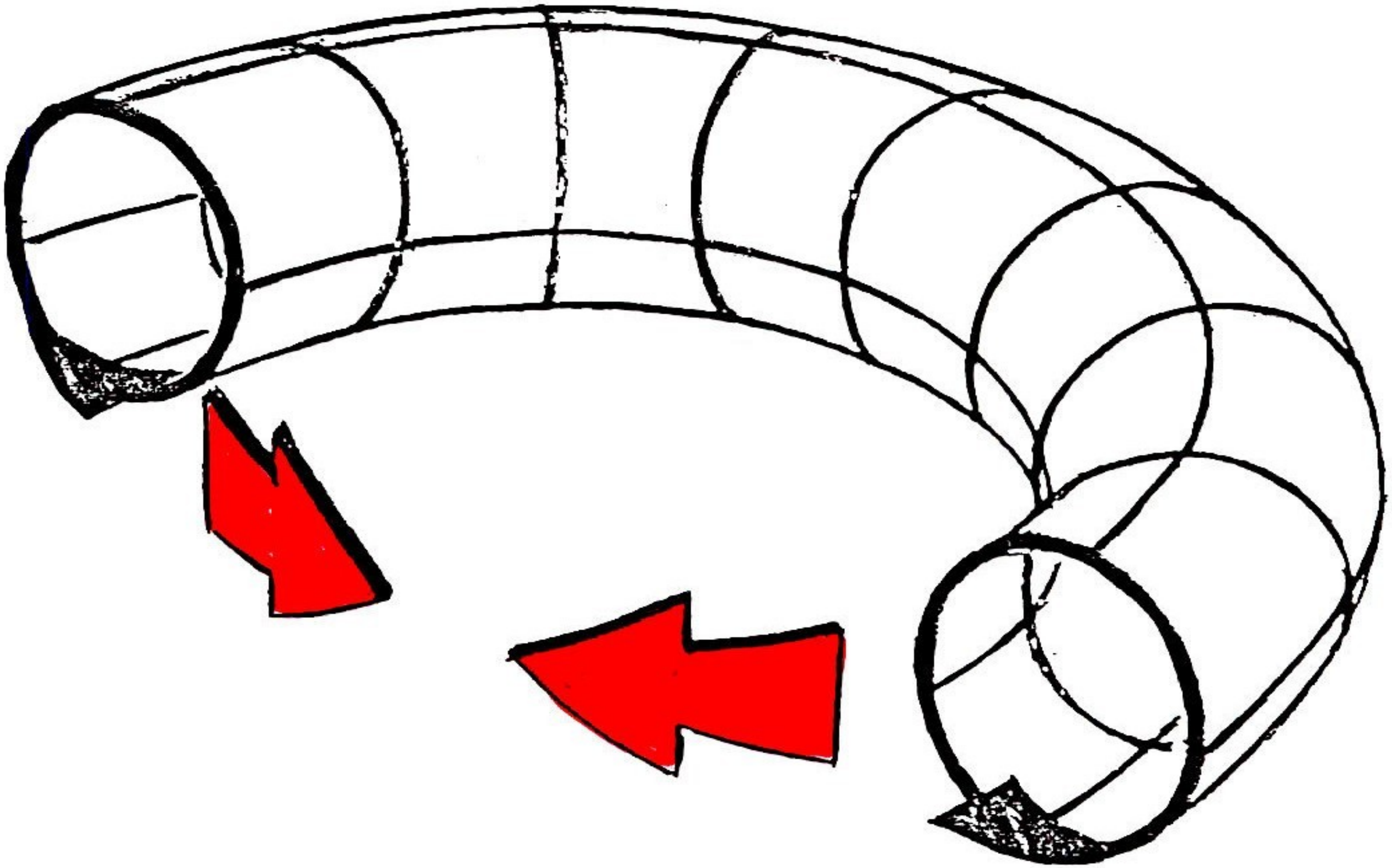
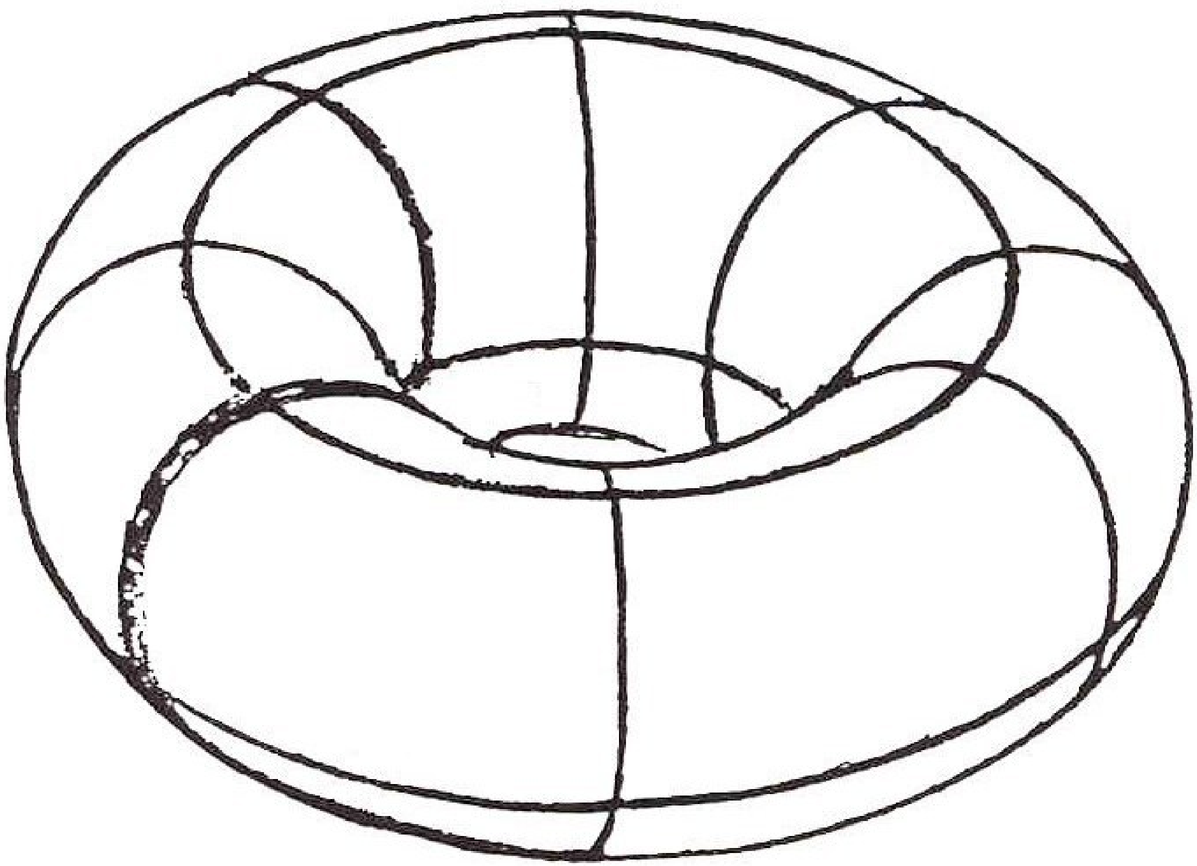
Then we get ... a bended tube...



④ and, finally...a „swimming tube” or „donought“...



... usually called ***Torus*** **in Mathematics.**

When Imagination Comes to its End

Geometric imagination has its limits. **To a great deal, these limits are given by the individuals capacity of imagination. These individual limits may be partly overcome by *practicing*.**

But, there seems to exist a more basic and universal limit:

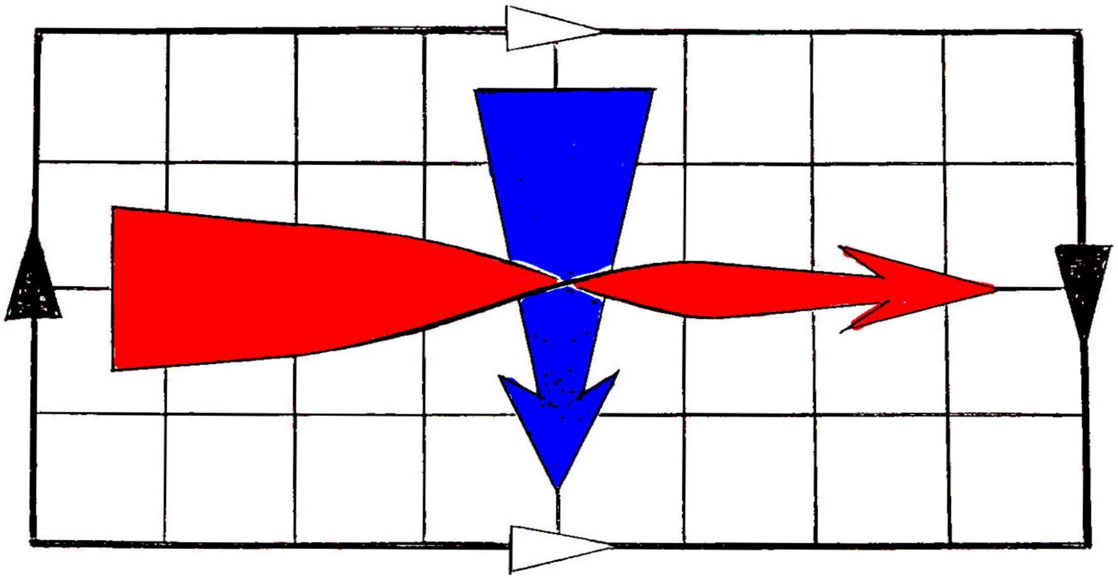
Dimensionality !

Beyond three-space, a direct visualization of geometric objects is not possible. But there are various indirect ways, to overcome this limit, at least to a certain extend.

Below, we present an example of this.

Leaving the Realm of Geometric Imagination: A Glance Beyond 3-Space...

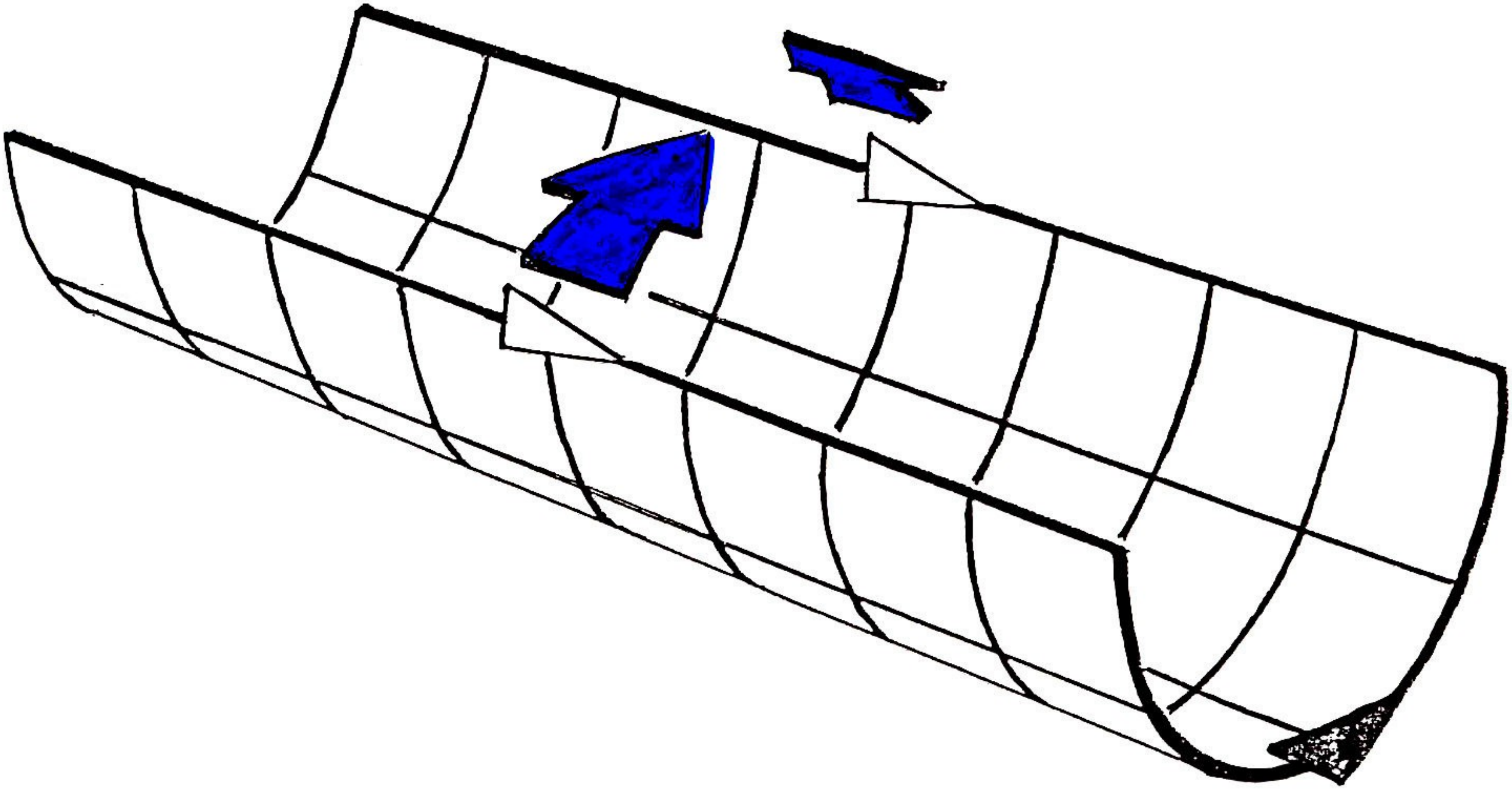
We glue the opposite edges of a rectangle as indicated below



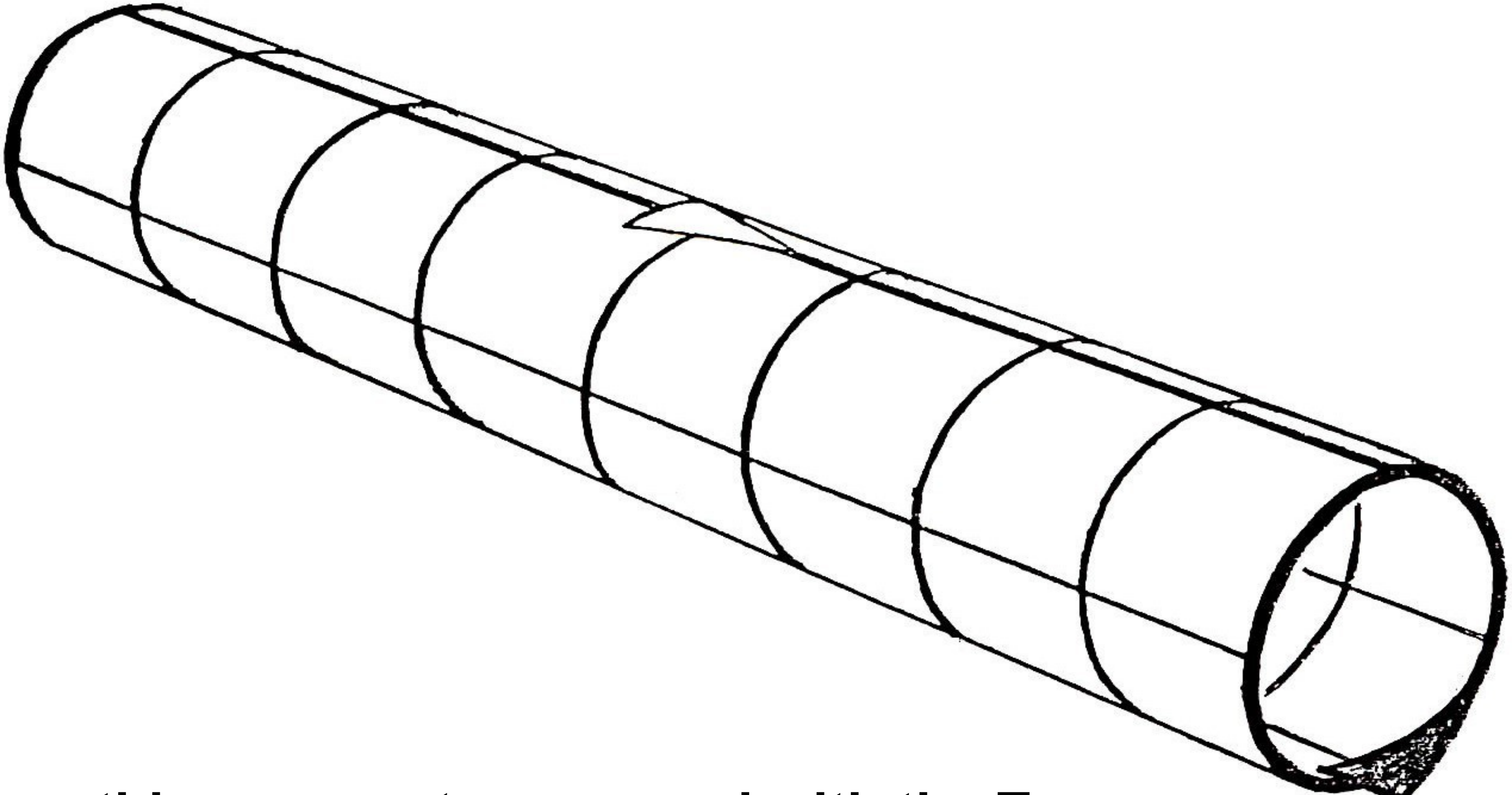
Observe the difference to the Torus case: One of the shorter edges is reversed before it is glued to its opponent !

Hint: perform first the gluing indicated in blue colour !

1 The first step leds to a half tube: nothing new...

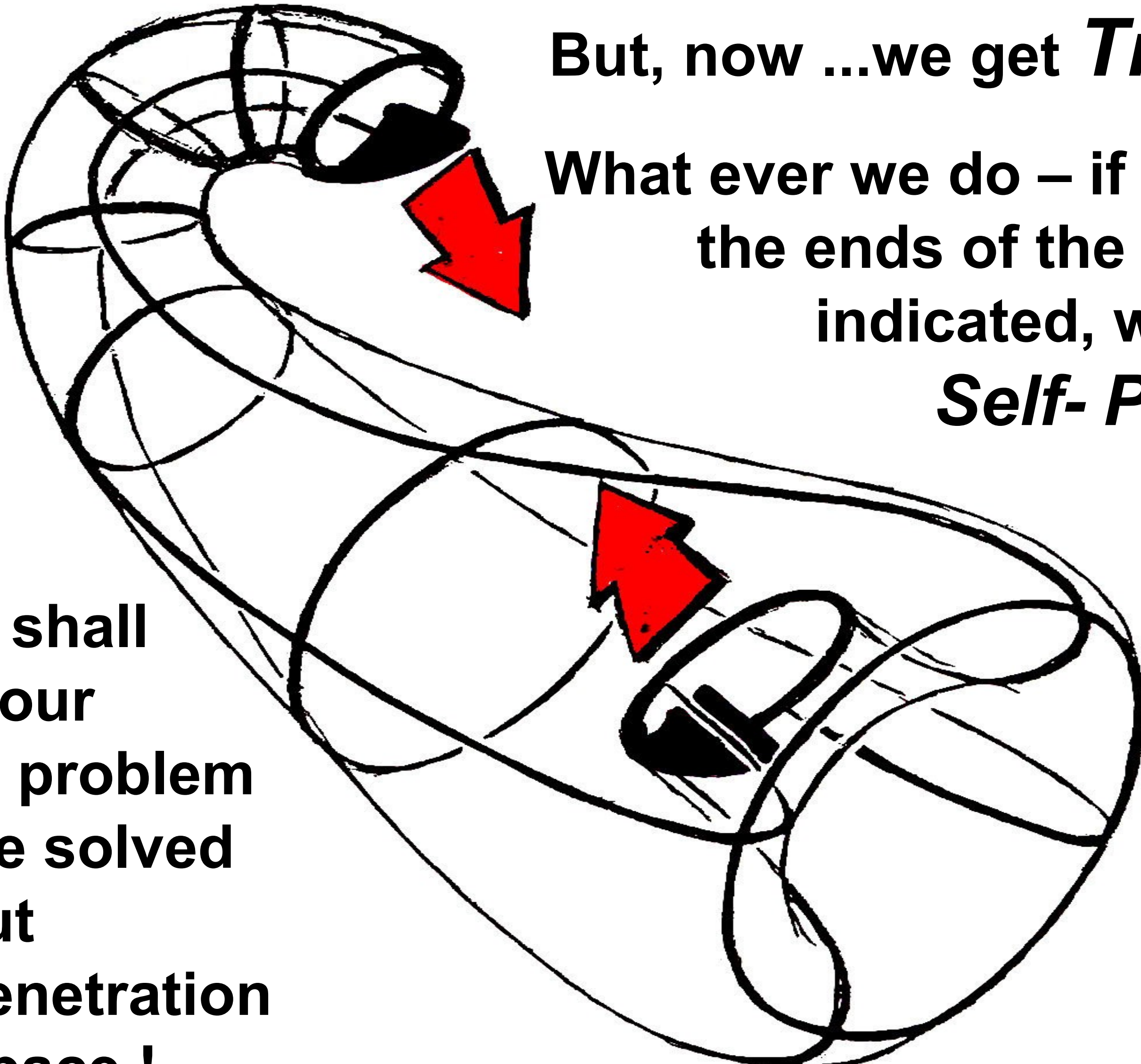


② The next step leaves us with...a tube...hence



... nothing new yet, compared with the Torus...

3



But, now ...we get *Troubles !!*

What ever we do – if we want to glue the ends of the bended tube as indicated, we must admit a *Self- Penetration* of our tube!

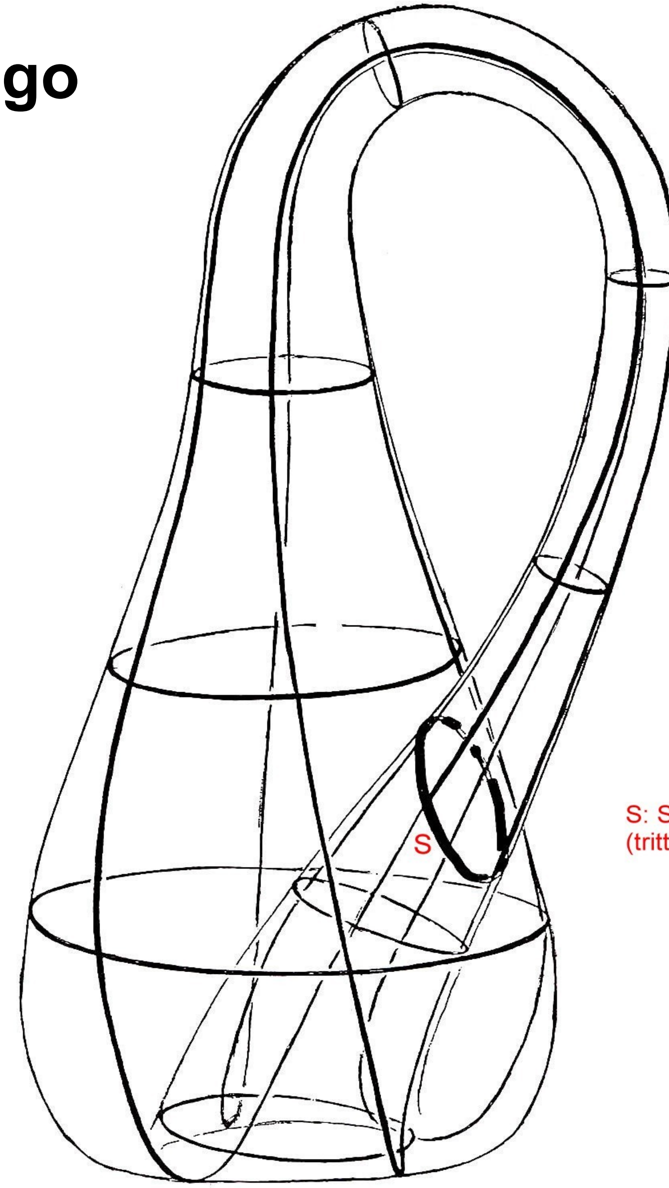
As we shall learn, our gluing problem may be solved without self-penetration in 4-Space !

(4) But for the moment, we go on in 3-Space...

... and obtain a so-called *model* of a closed surface which is „embedded into 4-Space“:

The Klein Bottle

...named after: Felix Klein
(1849 – 1924)
Mathematician in Göttingen,Germany



Modeling Three-Dimensional Space

To model a plane or a three-dimensional space, we introduce a system of coordinates. Then each point of our plane is presented by a pair of numbers (x,y) and each point in our three-dimensional space is presented by a triplet (x,y,z) of numbers. So, the *model of a plane* is the set

{(x,y) : x and y are real numbers},

where as the *model of a three-dimensional space* is the set

{(x,y,z): x,y and z are real numbers}.

... and Four-Dimensional Space

Now, by analogy we say that the set

{(x,y,z,w): x,y,z and w are real numbers}

is *the (model of the) four-dimensional space*.

In the last step we did an important thing:

We extended the idea of space to a situation which goes beyond our practical experience !

Mathematical Modeling of the Moebius Strip

(A) A rectangular strip is mathematically modeled by a rectangle in the plane, for example

S:={(u,v) : -pi < u < pi and -1 < v <1} (with pi = 3.14...).

(B) Our gluing process of the *Moebius strip* M can be mathematically modeled by a so called P*arametrization Map*

f:S--->> M: f(u,v) = (x(u,v), y(u,v), z(u,v)),

which assigns to each point (u,v) of the rectangle S a point f(u,v) in three-space. The three *coordinate functions* of f are given by:

x(u,v) = cos(u)[1+vcos(u/2)], y(u,v) = sin(u)[1+vcos(u/2)],

z(u,v) = vsin(u/2).

Modelling the Klein Bottle

(C) The *Klein bottle* is modeled by a parametrization map g(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v)) which assigns to each point (u,v) in S a point in four-space. This time, the four coordinate functions x(u,v), y(u,v), z(u,v) and w(u,v) of g are more complicated...

(D) Our *model surface of the Klein bottle* in three-space is the image of the actual Klein bottle under the *projection map* which assigns to the point (x,y,z,w) in four-space the point (x,y,z) in three-space. So, the model surface is something like a "three-dimensional photograph" of the actual surface in four-space.

Conclusion

Mathematics studies only mathematical models of real objects.

This point of view has *advantages* but also some *disadvantages:*

(A) *Advantages:* - Limits caused by material or imaginative constraints can be overcome.

- Results are formulated in mathematical terms and hence not ambiguous.

- Computer assisted research is made possible.

(B) *Disadvantages: -* Loss of non-modelled properties.

- Found results are primarily only true in the model.

- Mathematical skills are required to handle the model.

Applying Mathematics* concerns the *relation between the real object and its mathematical model*, doing *Mathematics happens* exclusively *inside the mathematical model.

3. TRUTH VERSUS CORRECTNESS: TRUE AND PROVABLE PROPOSITIONS

It is a common idea, that within Mathematics the truth of a statement can be *decided with absolute reliability.* The method to decide on the truth or the falseness of a statement is by *proof*. Our second excursion will teach us, that this view of Mathematics has its limits. It will illustrate, that truth and provability of mathematical statements need not be the same:

There are true propositions which do not admit a proof.

This observation invites us to deeper thoughts about the *limits of mathematical reasoning* and of *scientific thinking* at all.

Richard's Antinomy

(found by the French Mathematician J.A. Richard, (1862-1956))

Assume that we write down all propositions A(x) which say something on positive integers x -which can be true or false if one makes a particular choice of x. Examples for A(x) are:

“ x is an odd number ”, “ x is a prime number ” , “x is the third power of an even integer “, “x is square-free”, “ the square of x is the sum of the square of two positive integers “, “ 2x > x^2 “, “ x + 7x^3 is an odd number “, “ x^2 -3x = 1”, “ x is an even number bigger than 3 and the sum of two square numbers “ ...

Enumerating Arithmetic Propositions

Observe, that all propositions A(x) are written down with finitely many symbols chosen from a finite list, say : “ A, a, B, b, C, c, ..., X, x, Y, y, Z, z, (,), [,], {, }, =, > , < , + , - , ..., ^” and the list of all positive integers “1, 2, ...”.

This allows, to make a complete index of these propositions, hence to enumerate them all:

“ A_1(x), A_2(x), A_3(x) ,..., A_n(x) , ...”

In mathematical terms: the set of all our propositions A(x) is *countable.*

The number n is called the ***Richard Number*** of A_n(x).

Richard's Antinomy (1905)

Now, consider the proposition $\neg A\_x(x)$, that is the *Negation* of the proposition $A\_x(x)$, so that for each x we can say:

$$\neg A\_x(x) \text{ is true if and only if } A\_x(x) \text{ is wrong.}$$

Let m be the Richard number of $\neg A\_x(x)$, hence:

$$\neg A\_x(x) = A\_m(x).$$

Choosing x = m we get:

$$\neg A\_m(m) = A\_m(m).$$

Apparent Conclusion: $A\_m(m)$ *is neither true nor false, it is self-contradictory and hence an Antinomy.*

To avoid this antinomy, Arithmetic Propositions must be written down in a Formal Language (for Example Predicate Calculus).

Gödel's Undecidability Theorem (1931)

The Austrian-American mathematician *K.F. Gödel* (1906 – 1978) set the vague idea of Richard right by one of the most fundamental results on the Foundations of Mathematics.

He used the formal language of *Principia Arithmetica* (suggested by D. Hilbert) which formalizes *Elementary Arithmetics.*

He then introduced a method of enumeration of propositions and formal proofs which can be expressed within the formal system, the *Gödel Enumeration*.

Gödel's Undecidable Proposition

On use of his enumeration, Gödel could construct in the formal system Principia Arithmetica a proposition U(x) saying: " the proposition with number x cannot be formally proved".

Let m be the Gödel number of the proposition U(x).

Assume that U(m) is wrong. Then U(m) can be proved. But then U(m) must be true – a contradiction. So, U(m) is true. But then, U(m) cannot be formally proved. So:

U(m) is a true proposition, which cannot be formally proved.

Conclusions

Gödels construction says that in Elementary Arithmetic there are propositions on whose truth or falseness one cannot decide by a formal proof.

This problem can only be overcome, if one admits a formal language, which allows to speak on arbitrary sets of integers.
But one cannot decide by means of an algorithm, whether two arbitrary sets of integers are equal or not, hence indeed:

Elementary Arithmetic is Undecidable !

The property of undecidability is in fact inherited by all mathematical theories in which the *Principle of Complete Induction* is used. So:

Many Mathematical Theories are Undecidable !

Hence, in many theories we can say:

There are True Propositions which which cannot be Proved !

Therefore, as *formal proving is an algorithm*:

Truth Cannot be Verified in General by Means of an Algorithm !

4. MATHEMATICS IN SCIENCE AND ENGINEERING

The relation between Mathematics on the one hand and Engineering or Science on the other hand is essentially governed by the relation between

Real Objects* and their *Mathematical Models.

Mathematicians mostly work within the models which arise from real world problems, and they do no practical work on the underlying real objects. So:

Mathematicians are the Poets of Science and Engineering.

They namely “write poems on real live”. Their poems are written in a language which has to observe a very strict grammar - the mathematical rules. A good poet knows and cares about the reality he is writing about. Therefore:

Mathematicians must be able and willing to understand the practical consequences of the results they have found !

Applications of Mathematics: Examples

One hardly needs to convince people, that "the Poetry of Mathematics" has many *practical applications*. However we point out some of our favorite examples:

1) ***Physics:*** **Vector bundles, Lie groups and Lie algebras, systems of differential equations, algebras of differential operators, algebraic geometry, commutative algebra.**
2) ***Medical Engineering (Tomography):*** **Radon-Nikodym and further integral transformations.**
3) ***Cryptography and Coding:*** **Finite fields, arithmetic geometry notably elliptic curves or Shimura curves.**

4) ***Aerodynamics (Airplanes, Cars...)*****: Advanced numerical modeling.**

5) ***Electrical Engineering*****: Fourier, Laplace, Z-transformations and related methods.**

6) ***Finance (Derivatives and Stock Market)*****: Stochastic differential equations.**

7) ***Epidemology*****: Stochastics, differential equations.**

Thus:

A Great Variety of Mathematical Methods find their Applications !

5. MATHEMATICS TODAY: A GLANCE AT THE ACTUAL STATE

It is a general belief, even in educated persons, sometimes even in scientists, that Mathematics has found its final form and does not develop anymore. Many people are not aware of the fact that Mathematics goes far beyond what they have learned in School and that Mathematics is a field with highly active contemporary research.

Another idea is, that only applicable Methods of Mathematics are a justified, but the doing of Mathematics for its own purpose is an activity without any further impact in real life...

But, as we shall see below:

Both of these ideas depict a wrong image of Mathematics !

Facts About Contemporary Mathematics

1) Mathematics is a very extended field which at about 80 specified major subbranches. Thus:

Mathematics is a very broad field with many domains of specialization !

2) Mathematical research is a very active field. At about 50'000 research articles per year are published in more than 300 international mathematical periodicals with peer review.

3) A considerable number of new Mathematical theories were developed during the past few decades and some of these helped to solve long standing open problems. Prominent examples are:

The proof of the Mordell Conjecture (Faltings, 1983), of Fermat's Last Theorem (Taylor and Wiles, 1993) and of the Poincaré Conjecture (Perelman, 2009). So:

Mathematics is a strongly growing field with great research output !

4) A number of very advanced theories of "Pure Mathematics" found unexpected applications. Theories, which 40 years ago were an exclusive domain of pure mathematicians, meanwhile became important tools for engineers and scientists. Hence:

There is a shrinking gap between Pure and Applied Mathematics !

6. REQUIREMENTS TO BECOME A MATHEMATICIAN

1) ***Willingness to Work Hard:*** **Learning and doing Mathematics needs much practice and training of intellectual capacities !**

2) ***Imagination:*** **To find Mathematical ideas very often needs imagination !**

3) ***Ability to Love Abstract Thinking:*** **Mathematics is performed in mathematical models. To work in this abstract world with success and pleasure, you most bring it to live by loving it !**

4) *Intellectual Curiosity:* To learn and to do Mathematics, you must be attracted by open problems and interested in knowing or finding their solution !

5) *Sense for Reasoning and Logical Arguments:* An obvious requirement !

6) *Ability to Communicate:* Mathematics lives from sharing ideas !

7) *Common Sense:* Prevents you to lose your way in an abstract world of human imagination and enables you to understand the consequences of your findings.

7. REASONS TO BECOME A MATHEMATICIAN

1) *Mathematics is an interesting and challenging activity, confronts you with the boundaries of human thinking and gives you fascinating insights.*

2) *Mathematics is the most exact of all Sciences and thus shapes your thinking.*

3) *Mathematics can be done "by pen and paper" and you can do it everywhere.*

4) *Mathematics is a very strong and versatile "Key Technology" in Science and Engineering and so allows you to do things of great practical meaning.*

5) ***Mathematics is the "Franca Lingua" of Science and hence gives you access to a worldwide community which shares your interests and ideas.***

6) ***Mathematics is the root of long term Scientific and Technological development, and hence it allows you to contribute to the social development of your country.***

7) ***Mathematics is a subject very interesting to teach, and in doing so you contribute a lot to the scientific and social standard of Society.***

SO, WHY NOT BECOME A MATHEMATICIAN ?

Thank You !