LIFTING PROPERTIES FROM THE GENERAL HYPERPLANE SECTION OF A PROJECTIVE SCHEME

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Abstract

by

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In this dissertation, we discuss some necessary and sufficient conditions for a curve to be arithmetically Cohen-Macaulay, in terms of its general hyperplane section. We obtain a characterization of the degree matrices that can occur for points in the plane that are the general plane section of a non arithmetically Cohen-Macaulay curve of $\mathbf{P}^3$. We prove that almost all the degree matrices with positive subdiagonal that occur for the general plane section of a non arithmetically Cohen-Macaulay curve of $\mathbf{P}^3$, arise also as degree matrices of some smooth, integral, non arithmetically Cohen-Macaulay curve, and we characterize the exceptions. We give a necessary condition on the graded Betti numbers of the general plane section of an arithmetically Buchsbaum (non arithmetically Cohen-Macaulay) curve in $\mathbf{P}^n$. For curves in $\mathbf{P}^3$, we show that any set of Betti numbers that satisfy that condition can be realized as the Betti numbers of the general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. We also show that the matrices that arise as degree matrix of the general plane section of an arithmetically Buchsbaum, integral (or smooth and connected), non arithmetically Cohen-Macaulay space curve are exactly those that arise as degree matrix of the general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay space curve and have positive subdiagonal. We prove some bounds on

the dimension of the deficiency module of an arithmetically Buchsbaum curve in $\mathbf{P}^n$, in terms of entries of the lifting matrix of a general hyperplane section of the curve, and we show that they are sharp. We then turn to the question of whether it is possible to lift the property of being standard or good determinantal from the general hyperplane section of a scheme to the scheme itself. We produce examples of schemes that are not standard determinantal, but whose general hyperplane section is good determinantal. Using a result of Kleppe and Miró-Roig, we show that if one hyperplane section of a scheme of codimension 3 by a hyperplane that meets it properly is good determinantal, then a general hyperplane section of the scheme is good determinantal.

To my parents and my brother,

and to Guillermo

CONTENTS

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CHAPTER 1

INTRODUCTION

It is well known that several invariants of an arithmetically Cohen-Macaulay projective scheme, such as the degree, the h-vector, the graded Betti numbers, and many more, are preserved when we intersect the scheme with a hyperplane that meets it properly. Moreover, the intersection of an arithmetically Cohen-Macaulay scheme with a hyperplane that meets it properly is itself arithmetically Cohen-Macaulay. If we are interested in a d-dimensional, arithmetically Cohen-Macaulay scheme $V \subset \mathbf{P}^n$, we can intersect it with a hyperplane that meets it properly. Repeating the procedure d times, we get a zero-dimensional scheme $X \subset \mathbf{P}^{n-d}$. Then we can deduce the invariants of V from the invariants of X. We have then shifted our questions from our original scheme to a zero-dimensional one. In many cases, the zero-dimensional scheme will be an easier object to study.

In the general case of a scheme that is not necessarily arithmetically Cohen-Macaulay, not all the hyperplane sections have the same invariants. However, a general hyperplane H intersects V properly, and the scheme $V \cap H$ always has the same invariants. In general, though, the invariants of V are not easily deducible from those of its general hyperplane section $V \cap H$. In the case when $V \cap H$ is arithmetically Cohen-Macaulay and has dimension at least 1, however, V is forced to be arithmetically Cohen-Macaulay. In particular, we are again in the situation when we can deduce invariants of V from those of $V \cap H$.

A great deal of work has been devoted to the analysis of the case when X has dimension 0, or equivalently when V is a projective curve. Obviously we cannot expect to deduce the Cohen-Macaulayness of V from the Cohen-Macaulayness of X, without further assumptions. In fact, the general hyperplane section of a curve is a zero-dimensional scheme, so it is always arithmetically Cohen-Macaulay. A.V. Geramita and J.C. Migliore, R. Strano, R. Re, C. Huneke and B. Ulrich, found sufficient conditions on the general hyperplane section of a curve that guarantee Cohen-Macaulayness of the curve (see [22], [53], [50], [33], [45]). A brief summary and discussion of the work that has been done in the papers we just mentioned is contained in *Chapter 2*. Chapter 2 also contains background results that we will need in the following chapters. We fix some terminology and notation as well. We introduce the concept of lifting matrix of a zero-dimensional scheme $X \subset \mathbf{P}^n$ (see Definition 2.5). This is a matrix of integers, whose entries are the differences between the shifts of the last and first free module in a minimal free resolution of X.

The starting point of *Chapter 3* is a sufficient condition found by C. Huneke and B. Ulrich for V to be arithmetically Cohen-Macaulay, in terms of the graded Betti numbers of its general hyperplane section (see Theorem 2.12, Corollary 2.15 and Corollary 3.27). For example, given a curve in $\mathbf{P}^3$ the general plane section is a zero-dimensional scheme X in $\mathbf{P}^2$. The matrix of integers whose entries are the degrees of the entries of the Hilbert-Burch matrix of X is called the degree matrix of X. A sufficient condition for the curve to be arithmetically Cohen-Macaulay is that all the entries of the degree matrix of X are at least 3. The question we wish to answer is: is this condition necessary as well? That is, can we construct a non arithmetically Cohen-Macaulay curve, whose general plane section has a prescribed degree matrix, for each degree matrix that has at least one entry less than or equal to 2? In Theorem 3.3 and Theorem 3.17, we prove that the sufficient condition of Huneke and Ulrich is necessary as well. We do so by constructing a non arithmetically Cohen-Macaulay curve, whose general plane section has a prescribed degree matrix, for each degree matrix that has at least one entry smaller than or equal to 2. Each curve that we construct in Theorem 3.3 is connected and reduced, and it is the union of two arithmetically Cohen-Macaulay irreducible components. The construction of Theorem 3.3, however, requires a further assumption on one of the entries of the degree matrix of X, in case it has size bigger than 2×3. In Theorem 3.12, a similar construction produces curves, each of whose whose saturated ideal is minimally generated in low degrees. Each curve that we construct in Theorem 3.17 is a union of smooth, connected complete intersections. The construction of Theorem 3.17 works in full generality for any degree matrix that has at least one entry smaller than or equal to 2. Notice that, in this generality, one cannot expect to be able to construct reduced and irreducible curves whose general plane section has a prescribed degree matrix. In fact, the only degree matrices that can occur for the general plane section of a reduced, irreducible curve are those with positive subdiagonal. We construct reduced and irreducible curves in Chapter 4 for the degree matrices for which this is possible. Finally, we answer the question of whether it is possible to give a sufficient condition for Cohen-Macaulayness of a curve in $\mathbf{P}^3$, in terms of the h-vector of the general plane section of the curve. As one may expect, the answer to this question is negative, as we show in Proposition 3.26.

In *Chapter 4* we deal with integral (that is, reduced and irreducible) curves in $\mathbf{P}^3$. We ask whether it is possible to find a condition on the degree matrix of the general plane section of a curve, that is weaker than assuming that all the entries are bigger than or equal to 3, but still forces Cohen-Macaulayness of the curve under the hypothesis that the curve is integral. Moreover, we ask whether it is possible to give a sufficient condition for Cohen-Macaulayness of an integral curve in $\mathbf{P}^3$

in terms of the h-vector of its general plane section. We produce two families of degree matrices that do not have all the entries greater than or equal to 3, but any integral curve whose general plane section has one of those degree matrices is arithmetically Cohen-Macaulay. So we have sufficient conditions on the degree matrix of the general plane section of a curve that, together with integrality of the curve, force the curve to be arithmetically Cohen-Macaulay. They are treated in Proposition 4.4 and Proposition 4.6. From those, we are able to deduce sufficient conditions for Cohen-Macaulayness of an integral curve, in terms of the h-vector of its general plane section. In particular, the curve has the same h-vector. This is shown in Corollary 4.11. In Theorem 4.14 and Theorem 4.15 we show that, except for the two families treated in Proposition 4.4 and Proposition 4.6, the degree matrices with positive subdiagonal that correspond to points that are general plane section of a non arithmetically Cohen-Macaulay curve are the same as the degree matrices that correspond to points that are general plane section of a non arithmetically Cohen-Macaulay integral curve. Notice that the degree matrix of a zero-dimensional scheme that is general plane section of an integral curve must have positive entries on the subdiagonal. For each degree matrix that does not fall in the two categories of Proposition 4.4 and Proposition 4.6, we construct a smooth, connected, non arithmetically Cohen-Macaulay curve, whose general plane section has that degree matrix. It follows that any admissible h-vector of decreasing type, except for those treated in Corollary 4.11, can be realized as the h-vector of the general plane section of an integral, (or even smooth and connected) non arithmetically Cohen-Macaulay curve. This is proven in Corollary 4.18. Notice that any admissible h-vector of decreasing type can be realized as the h-vector of an integral, arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$, hence of its general plane section (this follows for example from [31]).

In *Chapter 5* we concentrate on arithmetically Buchsbaum curves in $\mathbf{P}^n$. We investigate whether we can give some conditions on the Betti numbers of the general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. In Proposition 5.4 we look at the lifting matrix of a zero-dimensional scheme that is the general hyperplane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. We show that at least one of the entries of such a lifting matrix must be equal to $n-1$. For the case of curves in $\mathbf{P}^3$, the lifting matrix of the general plane section coincides with its degree matrix. Therefore, the degree matrix of a zero-dimensional scheme that is the general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ has at least one entry equal to 2. In Theorem 5.6 we show that this condition is both necessary and sufficient. We do so by constructing an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ whose general plane section has a prescribed degree matrix, for any degree matrix that has at least one entry equal to 2. Then we analyze the case of integral, arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curves of $\mathbf{P}^3$. The general plane section of an integral curve is a set of points in Uniform Position, hence its degree matrix has positive subdiagonal. In Theorem 5.15, we show that for any degree matrix whose subdiagonal is positive and that has at least one entry equal to 2 we can construct a smooth, connected, arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$, whose general plane section has that degree matrix. In other words, the hypotheses that a degree matrix has positive subdiagonal and that it has at least one entry equal to 2 are necessary and sufficient in order for the degree matrix to correspond to the general plane section of some integral, (or smooth and connected) arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. We also prove some bounds for the dimension of the deficiency module of an arithmetically Buchsbaum curve: degree by degree in Proposition 5.19, and globally in Corollary 5.21. The bounds are given in terms of the entries of the degree matrix of the general plane section of the curve. In the end of Chapter 5 we prove that the bound is sharp, producing examples of arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curves of $\mathbf{P}^3$ that achieve the previously mentioned bounds, for each admissible degree matrix.

In *Chapter 6* we focus on lifting the property of being standard determinantal, or even good determinantal, from the general hyperplane section of a scheme to the scheme itself. A scheme $X \subset \mathbf{P}^n$ is called *standard determinantal* if I_X is generated by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix $A = (g_{i,j})$ representing a morphism

$$\phi : F = \bigoplus_{j=1}^{t+c-1} R(\alpha_j) \longrightarrow G = \bigoplus_{j=1}^{t} R(\beta_j)$$

of free graded R-modules. Here c is the codimension of X, $R = k[x_0, \ldots, x_n]$. The degree matrix of X is the matrix $M = (a_{i,j})$ whose entries are the degrees of the entries of A, $a_{i,j} = \beta_i - \alpha_j$. A scheme $X \subset \mathbf{P}^n$ is called *good determinantal* if it is standard determinantal and generically a complete intersection. Standard and good determinantal schemes have recently been shown to have plenty of connections to problems in liaison theory (see [35] and [43]). Moreover, since they are defined by ideals of maximal minors of a homogeneous matrix, they have been extensively studied from the algebraic point of view by many authors. Among the people that have worked on them are W. Bruns, A. Conca, D. Eisenbud, C. Huneke, J. Herzog, S. Popescu, B. Sturmfels, N. V. Trung, U. Vetter, and A. Zelevinsky (see, for example, [3], [4], [5], [10], [11], [12], [17], [18], [32], [55]). The problem of lifting the standard determinantal property has a clear answer in codimensions 1 and 2. In fact, in codimension 1 and 2 arithmetically Cohen-Macaulay schemes and standard determinantal schemes coincide, due to the Hilbert-Burch Theorem (see [4], or [14]).

Therefore, one may view of the problem of lifting the property of being standard determinantal as a generalization to higher codimension of the questions we treat in Chapters 3 and 4. So, for codimension 3 or higher, we look for a sufficient condition to lift the property of being standard or good determinantal from the general hyperplane section of a scheme to the scheme itself. This is a very interesting and hard problem, related to the study of Buchsbaum-Rim sheaves. In the papers [47] and [38], Kreuzer, Migliore, Peterson and Nagel studied the Buchsbaum-Rim sheaves, giving a new interpretation of a good determinantal scheme as the zero-locus of a regular section of the dual of some Buchsbaum-Rim sheaf. The question is also closely related to studying how the maps in the minimal free resolution of the ideal of a scheme can be lifted. In Example 6.9 and in Proposition 6.11 we produce examples of schemes of any dimension bigger than or equal to 2 and of codimension 3, such that the scheme is not standard determinantal, but its general hyperplane section is even good determinantal. This shows how the problem is interesting in any dimension, unlike the problem of lifting the property of being arithmetically Cohen-Macaulay. Moreover, we prove that if a scheme V of codimension 3 has one section $Z = V \cap H$ by a hyperplane H that meets V properly such that Z is good determinantal, then the general hyperplane section of V is good determinantal (see Theorem 6.13). This follows from the fact that the locus of good determinantal schemes with a fixed Hilbert polynomial $p(z)$ is locally closed in the Hilbert scheme that parametrizes schemes with Hilbert polynomial $p(z)$ (see [34] and the last Chapter of [35]). In analogy with the codimension 2 situation, one may wonder whether a scheme $V \subset \mathbf{P}^n$ is standard (respectively, good) determinantal given that the general hyperplane section X of V is standard (respectively, good) determinantal and the entries of the degree matrix of X are big enough. Unfortunately, this turns out not to be the case. In Proposition 2.23 we produce examples of schemes V such that the entries of the degree matrix of the general hyperplane section X can be chosen arbitrarily large, X is good determinantal and V is not even standard determinantal. We also describe a way to produce examples of schemes that are not standard determinantal, but whose general hyperplane section is good determinantal, starting from previously known examples (see Proposition 6.24 and the following example). In the end of Chapter 6 we derive an algebraic condition that is equivalent to the fact that a scheme V is good determinantal, given that a hyperplane section X is good determinantal.

The computer algebra systems CoCoA ([7]) and Macaulay 2([25]) were used during our work to perform the computations in several examples, including the examples included in this dissertation.

CHAPTER 2

PRELIMINARIES

In this chapter we state some results that we will use in the next chapters, and we establish some notation. In the first section we discuss some other authors' work, that our work is inspired and based on. Since our work makes an essential use of Liaison Theory, in the second section we make a brief summary of the definitions and the results that we need.

Throughout this dissertation, we let C be a curve in $\mathbf{P}^{n+1} = \mathbf{P}^{n+1}(k)$, where k is an algebraically closed field. We work over a field k of arbitrary characteristic, except for Chapter 4, where we require that k has characteristic 0. By *curve* we mean a non-degenerate, equidimensional, locally Cohen-Macaulay, dimension 1 subscheme of $\mathbf{P}^{n+1}$. We denote by I_C the saturated homogeneous ideal corresponding to the curve C in $S = k[x_0, x_1, \ldots, x_{n+1}]$. X denotes a zero-dimensional scheme that is a general hyperplane section of C, and I_X its homogeneous, saturated ideal in the polynomial ring $R = k[x_0, x_1, \ldots, x_n]$. Sometimes we also denote with I_X the ideal of X as a subset of $\mathbf{P}^{n+1}$; in this case I_X contains a linear form (the equation of the hyperplane containing X) and is an ideal in $S = k[x_0, x_1, \ldots, x_{n+1}]$. In general, the homogeneous saturated ideal of a scheme $V \subset \mathbf{P}^{n+1}$ is $I_V \subset S$. The definitions are analogous for subschemes of $\mathbf{P}^n$ and ideals of R. We denote by $\mathfrak{m}$ the homogeneous irrelevant maximal ideal of the polynomial ring S, $\mathfrak{m} = (x_0, x_1, \ldots, x_{n+1})$.

For a graded S-module N, we denote the d-th graded piece by N_d. $N^\vee$ denotes the k-dual of N. We let $\tilde{N}$ be the sheafification of N. $\tilde{S} = \mathcal{O}_{\mathbf{P}^{n+1}}$ is the structure sheaf of $\mathbf{P}^{n+1}$, and $\tilde{I_C} = \mathcal{I}_C \subset \mathcal{O}_{\mathbf{P}^{n+1}}$ is the ideal sheaf of C. We denote the *cohomology modules of* C by

$$H^i_*(\mathcal{I}_C) = \bigoplus_{m \in \mathbf{Z}} H^i(\mathbf{P}^{n+1}, \mathcal{I}_C(m))$$

and the dimension of their graded pieces with

$$h^i(\mathcal{I}_C(m)) = dim_k\, H^i(\mathcal{I}_C(m)).$$

The first cohomology module of a curve is also called *deficiency module* or *Hartshorne-Rao module*. We let $\mathcal{M}_C$ be the deficiency module of the curve C.

Notation 2.1 *For M a graded S-module of finite length, we will denote by $\alpha(M)$, $\alpha^+(M)$ the initial degree and final degree of the module. In symbols*

$$\alpha(M) = \min\{m \in \mathbf{Z} \mid M_m \neq 0\},$$

$$\alpha^+(M) = \max\{m \in \mathbf{Z} \mid M_m \neq 0\}.$$

A scheme $V \subset \mathbf{P}^{n+1}$ is *arithmetically Cohen-Macaulay* if S/I_V is a Cohen-Macaulay ring, that is if $dim(S/I_V) = depth(S/I_V)$. We sometimes abbreviate it by *aCM*. It is well known, that the deficiency module of a curve C is trivial if and only if C is arithmetically Cohen-Macaulay. Its deficiency module has finite length as an S-module (or equivalently, finite dimension as a k-vector space) if and only if C is locally Cohen-Macaulay and equidimensional (see [53], [30] 37.5, or [43], Theorem 1.2.5). A scheme $V \subset \mathbf{P}^{n+1}$ is *arithmetically Gorenstein* if it is arithmetically Cohen-Macaulay and the last free module in a minimal free resolution of S/I_V has rank 1, that is if S/I_V is a Gorenstein ring. We abbreviate it by *aG*. A *complete intersection* of *type* $(d_1, \ldots, d_r)$ is a scheme, whose homogeneous saturated ideal is generated by a regular sequence of forms of degrees $d_1 \leq d_2 \leq \ldots \leq d_r$.

We abbreviate it by $CI(d_1, \ldots, d_r)$, or by CI when we do not need to specify the degrees.

An *effective divisor* on a scheme $V \subset \mathbf{P}^{n+1}$ is an equidimensional, locally Cohen-Macaulay, codimension 1 subscheme of V. If $F \in S_d$ is a homogeneous form that does not vanish on any component of V, that is if F is non zero-divisor in S/I_V, then $H_F = V \cap F$ is the divisor cut out on V by F. We say that two divisors C and D are *linearly equivalent* if there exist forms F and G of the same degree such that $C + H_F = D + H_G$. $|C|$ denotes the *linear system* of effective divisors on V that are linearly equivalent to C.

We devote particular attention to space curves $C \subset \mathbf{P}^3$. In this case, I_X is a codimension 2 Cohen-Macaulay ideal of $R = k[x_0, x_1, x_2]$. By the Hilbert-Burch Theorem (see [14], Theorem 20.15) we know that it is generated by the maximal minors of a $t \times (t+1)$ homogeneous matrix $A = (F_{ij})$. Let $M = (a_{i,j})$ be the matrix whose entries are the degrees of the entries of A (M is the *degree matrix* of X). We make the convention that the entries of M decrease from right to left and from top to bottom: $a_{i,j} \leq a_{k,r}$, if $i \geq k$ and $j \leq r$. If some entry F_{ij} of A is 0, then the degree is not well defined. In this case, there exist k, l such that F_{ik}, F_{lk}, F_{lj} are all different from zero. We set $a_{ij} = a_{ik} - a_{lk} + a_{lj}$. We can assume without loss of generality that the Hilbert-Burch matrix has the property that $F_{ij} = 0$ if $a_{ij} \leq 0$, and $deg(F_{ij}) = a_{ij}$ if $a_{ij} > 0$. Note that some of the F_{ij}'s could be 0 even if $a_{ij} > 0$.

We call *homogeneous* any matrix of integers $M = (a_{i,j})$ for which $a_{i,j} + a_{r,s} = a_{i,s} + a_{r,j}$ for all $i, r = 1, \ldots, t$ and $j, s = 1, \ldots, t+1$. Notice that the degree matrix of a homogeneous matrix is homogeneous in this sense. Abusing language, we refer as *degree matrix* to any matrix of integers that is the degree matrix of some scheme in projective space.

Definition 2.2 *([38]) A* standard determinantal *scheme $V \subseteq \mathbf{P}^n$ of codimension c, is a scheme whose saturated ideal I_V is minimally generated by the maximal minors of a homogeneous matrix A with polynomial entries, of size $t \times (t+c-1)$ for some t. The matrix whose entries are the degrees of the entries of A is called the* degree matrix *of V.*

The definition of standard determinantal scheme was introduced by M. Kreuzer, J.C. Migliore, U. Nagel and C. Peterson in [38]. As we just recalled, any Cohen-Macaulay ideal of codimension 2 is standard determinantal.

Definition 2.3 *([38]) A* good determinantal *scheme is a standard determinantal scheme that is generically a complete intersection. Equivalently, a scheme $V \subseteq \mathbf{P}^n$ of codimension c is good determinantal if the following hold:*

- *there is a homogeneous matrix A with polynomial entries, of size $t \times (t+c-1)$, such that I_V is generated by the maximal minors of A*
- *the matrix that we obtain if we delete a generalized row of A defines a standard determinantal scheme.*

A generalized row *is a general linear combination of the rows of the matrix, with coefficients in the ground field k.*

We can characterize the matrices of integers that are also degree matrices of some standard/good determinantal scheme, as those that are homogeneous and whose diagonal is entirely positive.

Proposition 2.4 *Let $M = (a_{i,j})$ be a matrix of integers of size $t \times (t+c-1)$. Then M is a degree matrix if and only if it is homogeneous and $a_{h,h} > 0$ for $h = 1, \ldots, t$.*

Proof: Let us start by showing necessity, i.e. that a degree matrix has positive entries on the diagonal (it is clearly homogeneous, since its minors are homogeneous polynomials). We will prove the thesis by contradiction, showing that if $a_{h,h} \leq 0$ for any h, then the scheme X cannot be standard determinantal. So let $A = (F_{i,j})_{i=1,\dots,t;\, j=1,\dots,t+c-1}$ be the matrix defining X; I_X is minimally generated by the maximal minors of A. In particular, the determinant Δ of the submatrix $B = (F_{i,j})_{i=1,\dots,t;\, j=1,\dots,t}$ is nonzero. Assume $a_{h,h} \leq 0$ for some h, then $a_{i,j} \leq 0$ for $i \geq h$ and $j \leq h$. Hence $F_{i,j} = 0$ for $i \geq h$ and $j \leq h$. Then B contains a submatrix of zeroes of size $(t-h+1) \times h$.

We claim that $\Delta = 0$. Let us prove it by induction on the size t of B. For $t = 1$, we have $B = (0)$, B is a matrix of size 1×1. Assume now that the result holds for $t-1$ and prove it for t.

$$\Delta = \sum_{i=1}^{t} (-1)^{i+t} F_{i,t} b_{i,t}$$

where $b_{i,j} = det(B_{i,j})$ is the determinant of the submatrix $B_{i,j}$, obtained from B deleting the i-th row and the j-th column. For each i, $B_{i,t}$ is a matrix of size $(t-1) \times (t-1)$ that has a submatrix of h columns and (at least) $t-h$ rows consisting of zeroes. Thus, the induction hypothesis applies to $B_{i,t}$ for all i, giving $b_{i,t} = 0$. So $\Delta = 0$, contradicting the assumption that X is standard determinantal.

We now show that, conversely, any matrix of integers $M = (a_{i,j})$, of size $t \times (t+c-1)$ with $a_{h,h} > 0$ for all h is a degree matrix. We need to exhibit a standard determinantal scheme that has M as its degree matrix. So let

$$A = \begin{pmatrix} F_{1,1} & \cdots & F_{1,c} & 0 & 0 & \cdots \\ 0 & F_{2,2} & \cdots & F_{2,c+1} & 0 & \cdots \\ & & \ddots & & \ddots & \\ 0 & 0 & \cdots & F_{t,t} & \cdots & F_{t,t+c-1} \end{pmatrix}$$

where $F_{i,j} \in R$ are generic homogeneous polynomials of degree $deg(F_{i,j}) = a_{i,j}$. By assumption, all the degrees involved are positive. A defines a standard determinantal, reduced scheme (see [6], Proposition 2.5), whose saturated homogeneous ideal is minimally generated by the maximal minors of A. $\square$

2.1 A sufficient condition for Cohen-Macaulayness of a curve

Consider now curves embedded in a projective space of arbitrary dimension. If $C \subset \mathbf{P}^{n+1}$, the general hyperplane section is a zero-dimensional scheme $X \subset \mathbf{P}^n$. We saw that if $n = 2$, then X is standard determinantal. If $n > 2$ this is not necessarily the case. We would like to associate a matrix of integers to each zero-dimensional scheme, such that it extends the idea of degree matrix to arbitrary codimension.

Definition 2.5 *Let $X \subset \mathbf{P}^n$ be a zero-dimensional scheme with graded Betti numbers*

$$0 \longrightarrow \mathbf{F}_n = \bigoplus_{i=1}^{t} R(-m_i) \longrightarrow \mathbf{F}_{n-1} \longrightarrow \cdots \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 = \bigoplus_{j=1}^{r} R(-d_j) \longrightarrow I_X \longrightarrow 0$$

where $m_1 \geq \ldots \geq m_t$ and $d_1 \geq \ldots \geq d_r$.

We call the matrix of integers $M = (a_{ij} = m_i - d_j)$ the lifting matrix *of X.*

Notice that the lifting matrix coincides with the degree matrix of X in the case of space curves ($n = 2$). The lifting matrix will play the role of the degree matrix of X, for $n > 2$. Notice moreover, that the entries of M decrease from right to left and from top to bottom: $a_{i,j} \leq a_{k,l}$ if $i \geq k$ and $j \leq l$.

We assume that the curve $C \subset \mathbf{P}^{n+1}$ is non-degenerate. Notice that for $n = 2$, if C is a (degenerate) plane curve, then it is arithmetically Cohen-Macaulay. We can assume that the zero-dimensional scheme $X \subset \mathbf{P}^n$, general section of a non-degenerate $C \subset \mathbf{P}^{n+1}$, is non-degenerate, as the following Lemmas show.

Lemma 2.6 *(O.A. Laudal, [39], pg. 142 and 147) The general plane section of a non-degenerate space curve of degree $d \geq 3$ is non-degenerate.*

Laudal's Lemma has a more general version for curves in $\mathbf{P}^{n+1}$. See [33], or [45] Proposition 2.2 for the proof.

Lemma 2.7 *The general hyperplane section of a non-degenerate curve $C \subset \mathbf{P}^{n+1}$ of degree $d \geq n+1$ is non-degenerate.*

The case $t = 1$, $n = 2$, i.e. the case when the general plane section of $C \subset \mathbf{P}^3$ is a complete intersection, has been studied by R. Strano. He proved the following result (Theorem 6, [53]).

Theorem 2.8 *Let $C \subset \mathbf{P}^3$ be a reduced and irreducible, non-degenerate curve of degree d not lying on a quadric surface. If the general plane section X is a $CI(s,t)$, then C is a $CI(s,t)$.*

The result of Strano is sharp, in the sense that we can easily find examples of curves that are non arithmetically Cohen-Macaulay, whose general plane section is a complete intersection of a quadric and a form of degree a, for any a. We produce some examples of such curves, beginning with degree 2.

Example 2.9 *The general plane section of any reduced curve C of degree 2 is a reduced degree 2 zero-dimensional scheme, hence a complete intersection. If C is connected, then it is a plane curve, hence aCM. If C is disconnected, then it consists of two skew lines, so it's non-aCM.*

We observe that in this case, assuming that the curve is connected ensures its Cohen-Macaulayness.

The situation is different for curves of degree $2a$, for $a \geq 2$.

Example 2.10 *Consider a (general) smooth rational curve C of degree $2a$, $2 \leq a$, lying on a smooth quadric surface $\mathcal{Q} \subset \mathbf{P}^3$, e.g. the curve of parametric equations*

$$\begin{cases} x_0 = s^{2a} \\ x_1 = s^{2a-1}t \\ x_2 = st^{2a-1} \\ x_3 = t^{2a} \end{cases}$$

C is a rational, non-degenerate, smooth curve lying on the smooth quadric surface $\mathcal{Q} = x_0x_3 - x_1x_2$. In fact,the saturated ideal of C is

$$I_C = (x_0x_3 - x_1x_2,\ x_0^i x_2^{2a-1-i} - x_1^{i+1}x_3^{2a-2-i} \mid i = 0, \ldots, 2a-2).$$

C is non-aCM, since it has genus $g = 0$, hence some entry of its h-vector has to be negative. In fact, the only aCM rational curve in $\mathbf{P}^3$ is the twisted cubic (general rational curve of degree 3).

Let X be the general plane section of C. X lies on a smooth conic, and its h-polynomial is $h(z) = 1 + 2z + 2z^2 + \ldots + 2z^{a-1} + z^a$, since X has the Uniform Position Property (see [28], about the h-vector of points in the plane with the UPP). Then X is a complete intersection of type $(2, a)$.

Remark 2.11 *In some cases, it is useful to consider rational smooth curves, whose ideal is generated in small degree. If a is even, consider the curve C of parametric equations*

$$\begin{cases} x_0 = s^{2a} \\ x_1 = s^{a+1}t^{a-1} \\ x_2 = s^{a-1}t^{a+1} \\ x_3 = t^{2a} \end{cases}$$

Its saturated ideal is

$$I_C = (x_0x_3 - x_1x_2, x_0^2x_2^{a-1} - x_1^{a+1}, x_0x_2^a - x_1^a x_3, x_2^{a+1} - x_1^{a-1}x_3^2).$$

If a is odd, let C be the curve parametrized by

$$\begin{cases} x_0 = s^{2a} \\ x_1 = s^{a+2}t^{a-2} \\ x_2 = s^{a-2}t^{a+2} \\ x_3 = t^{2a} \end{cases}$$

whose saturated ideal is

$$I_C = (x_0x_3 - x_1x_2, x_0^4x_2^{a-2} - x_1^{a+2}, x_0^3x_2^{a-1} - x_1^{a+1}x_3, \ldots, x_2^{a+2} - x_1^{a-2}x_3^4).$$

In both cases, C is a rational, non-degenerate, smooth curve lying on the smooth quadric surface $\mathcal{Q} = x_0x_3 - x_1x_2$. As in Example 2.10, C is non-aCM and its general *plane section is a $CI(2,a)$. The ideal of I_C is generated in degree less than or equal to $a+1$ if a is even, and less than or equal to $a+2$ if a is odd.*

The result of Strano was extended independently by R. Re in [50] to reduced and irreducible curves in $\mathbf{P}^{n+1}$, by C. Huneke and B. Ulrich to reduced and connected curves whose general hyperplane section is arithmetically Gorenstein, and by J. Migliore in [45] to the general hypersurface section of curves that are locally Cohen-Macaulay and equidimensional.

In the rest of this section we work over an algebraically closed field k of characteristic 0. The characteristic 0 hypothesis is needed in Theorem 2.12 of Huneke and Ulrich, and in its applications (Corollary 2.15 and Corollary 3.27).

In $\mathbf{P}^2$, every arithmetically Gorenstein zero-dimensional scheme is a complete intersection. This is not the case in higher codimension, e.g. for zero-dimensional schemes in $\mathbf{P}^n$ when $n \geq 3$. The problem of finding a sufficient condition for a curve in $\mathbf{P}^n$ to be arithmetically Cohen-Macaulay, hence arithmetically Gorenstein, given that its general hyperplane section is arithmetically Gorenstein, has been solved by C. Huneke and B. Ulrich in [33]. This remarkable paper is based on the Socle Lemma (Corollary 3.11). The theorem that follows is a consequence of it.

Theorem 2.12 *(Theorem 3.16, [33])*

Let $S = k[x_0, \ldots, x_{n+1}]$, k a field of characteristic 0. Let $I_C \subset S$ be the homogeneous ideal of a reduced, connected curve $C \subset \mathbf{P}^{n+1}$. Let L be a general linear form in S and X be the corresponding general hyperplane section of C, $X \subset \mathbf{P}^n$. The homogeneous ideal of X in $R = S/(L)$ is $I_X = H^0_*(I_C + (L)/(L)) \supseteq I_C + (L)/(L)$. Let

$$0 \longrightarrow \bigoplus_{i=1}^{b_n} R(-m_{n,i}) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{b_1} R(-m_{1,i}) \longrightarrow I_X \longrightarrow 0$$

be a minimal free resolution of I_X as an R-module. If $I_X \neq I_C + (L)/(L)$, then

$$\min\{m_{n,i}\} \leq b + n$$

where $b = \min\{d \mid (I_X)_d \neq (I_C + (L)/(L))_d\}$.

Remark 2.13 *The curve* C *is arithmetically Cohen-Macaulay if and only if*

$$I_X = I_C + (L)/(L).$$

If C *is non-aCM, then there exists a minimal generator of* I_X *of degree* b*, that is not the image of any element of* I_C *under the standard projection onto the quotient* $I_C \longrightarrow I_C + (L)/(L)$*. Therefore, for some* j*,* $b = m_{1,j} \geq \min\{m_{n,i}\} - n$*.*

Remark 2.14 *It was observed by J. Migliore (see Proposition 2.2 and Theorem 2.4 of [45]) that the hypotheses that the curve* C *is reduced and connected are not necessary. In fact, he proved Theorem 2.12 for any curve* $C \subset \mathbf{P}^{n+1}$ *that is non-degenerate, locally Cohen-Macaulay and equidimensional.*

Notice moreover that the hypothesis of equidimensionality on C *cannot be weakened any further. In fact, any non-equidimensional curve is automatically non arithmetically Cohen-Macaulay. Moreover, the general hyperplane section of a curve only depends on its one-dimensional components. Summarizing, having isolated or embedded points is enough to prevent a curve from being arithmetically Cohen-Macaulay, but this "pathology" cannot be detected by looking at a general hyperplane section.*

The hypothesis that C *is locally Cohen-Macaulay is equivalent to* $\mathcal{M}_C$ *being of finite length as an* S*-module. Therefore it makes the cohomology of the curve more manageable. However, one may try to remove or weaken this hypothesis.*

Let us fix some notation. We start with an analysis of the case of space curves.

Let $C \subset \mathbf{P}^3$ be a curve, let $X \subset \mathbf{P}^2$ be its general plane section. Let A be the homogeneous matrix whose maximal minors generate I_X and M be its degree matrix. The minimal free resolution of X is

$$0 \longrightarrow \bigoplus_{i=1}^{t} R(-m_i) \xrightarrow{A'} \bigoplus_{j=1}^{t+1} R(-d_j) \longrightarrow I_X \longrightarrow 0$$

where $d_1 \geq d_2 \geq \ldots \geq d_{t+1}$ are the degrees of a minimal system of generators of I_X, $m_1 \geq m_2 \geq \ldots \geq m_t$ and A' is the transposed of A.

The result that follows was observed by J. Migliore in [45] (Proposition 2.2 and Remark 2.3). It is an easy consequence of Theorem 2.12.

Corollary 2.15 *Let $C \subset \mathbf{P}^3$ be a curve, whose general plane section $X \subset \mathbf{P}^2$ has degree matrix $M = (a_{i,j})_{i=1,\ldots,t;j=1,\ldots,t+1}$. If $a_{t,1} \geq 3$, then C is arithmetically Cohen-Macaulay.*

Proof: Let L be the equation of the plane of $\mathbf{P}^3$ in which X is contained. L is unique by non-degeneracy of X and C. Assume by contradiction that C is non-aCM and let b be the minimum degree in which the ideal $I_X \subset S/(L)$ differs from $I_C + (L)/(L) \subset S/(L)$, as in the statement of Theorem 2.12. By Theorem 2.12 we have that

$$b \geq min\{m_i\} - 2 = m_t - 2 = d_1 + a_{t,1} - 2 \geq d_1 + 1.$$

Hence all the minimal generators of I_X come from images of the minimal generators of I_C under the standard projection. Then C is arithmetically Cohen-Macaulay, contradicting our assumption. □

In Chapter 3, we are going to show that the sufficient condition of Corollary 2.15 is the best possible. For each degree matrix M that has at least one entry smaller than or equal to 2, we construct a reduced and connected curve that is non arithmetically Cohen-Macaulay and whose general plane section has degree matrix M.

2.2 A few results from Liaison Theory

Many of the arguments in the next chapters will be constructive. The main tool that we use is Liaison Theory. Liaison is the study of properties shared by two schemes whose union is a complete intersection or, more generally, an arithmetically

Gorenstein scheme (see [43] or [46] for a complete treatment of Liaison). Although Liaison is a very interesting subject in its own right, here it is used mainly as a tool. It is a very useful method of proof, allowing us to shift a problem from one scheme to another one that we understand better. Moreover, it is extremely useful for constructing families of examples.

Definition 2.16 *Let $C, D \subset \mathbf{P}^n$ be non-empty, locally Cohen-Macaulay, equidimensional schemes. E is* directly G-linked *to D by an arithmetically Gorenstein scheme E if $I_E \subset I_C \cap I_D$ and we have $I_E : I_C = I_D$ and $I_E : I_D = I_C$. In this case we say that D is the* residual *to C in E.*

Remark 2.17 *Let $C, D, E \subset \mathbf{P}^n$ be as in the definition above. If C and D have no common components, then they are directly G-linked if and only if $C \cup D = E$.*

Definition 2.18 Gorenstein liaison *or* G-liaison *is the study of the equivalence relation generated by direct G-linkage. More precisely, two schemes $C, D \subset \mathbf{P}^n$ are* G-linked *if there exist schemes $C_1, \ldots, C_m$ such that $C_1 = C$, $C_m = D$, and C_i is directly G-linked to C_{i+1} for $i = 1, \ldots, m-1$. We also say that C and D are in the same* G-linkage class*.*

The definitions of *direct CI-linkage* and *CI-linkage* are analogous, replacing arithmetically Gorenstein with complete intersection. Many properties are invariant in a liaison class, or can be deduced for a given scheme from properties of a directly linked scheme. We list here some results in this direction. The first is the Hartshorne-Schenzel Theorem, that relates the cohomology modules of schemes that are directly linked.

Theorem 2.19 *([43], Theorem 5.3.1) Let $C, D \subset \mathbf{P}^{n+1}$ be locally Cohen-Macaulay schemes of codimension c, and assume that C and D are directly G-linked via an arithmetically Gorenstein scheme E. Let*

$$0 \longrightarrow S(-t) \longrightarrow \mathbf{F}_{c-1} \longrightarrow \cdots \longrightarrow \mathbf{F}_1 \longrightarrow I_E \longrightarrow 0$$

be a minimal free resolution of I_E as an S-module. Then

$$H_*^{n-c+i}(\mathcal{I}_C) \cong H_*^i(I_D)^\vee(n+1-t).$$

Corollary 2.20 *Let $C, D \subset \mathbf{P}^3$ be curves, and assume that C and D are directly G-linked via a complete intersection curve E of type (a, b). Then*

$$\mathcal{M}_C \cong \mathcal{M}_D^\vee(4-a-b).$$

Notice that, in particular, the property of being arithmetically Cohen-Macaulay, or arithmetically Buchsbaum (see Definition 5.1), is invariant in a linkage class.

We can also relate the graded Betti numbers of two directly linked, locally Cohen-Macaulay schemes. Even in the case that C and D are arithmetically Cohen-Macaulay and we start from a minimal free resolution of C, the free resolution that we obtain for the linked scheme D is not necessarily minimal.

Proposition 2.21 *([43], Proposition 5.2.10) Let $C, D \subset \mathbf{P}^{n+1}$ be schemes of codimension c, that are directly G-linked via the arithmetically Gorenstein scheme E. Let*

$$0 \longrightarrow S(-t) \longrightarrow \mathbf{G}_{c-1} \longrightarrow \cdots \longrightarrow \mathbf{G}_1 \longrightarrow I_E \longrightarrow 0$$

be a minimal free resolution of I_E and let

$$0 \longrightarrow \mathbf{F}_c \longrightarrow \mathbf{F}_{c-1} \longrightarrow \cdots \longrightarrow \mathbf{F}_1 \longrightarrow I_C \longrightarrow 0$$

be a locally free resolution of I_C. Then I_D has a locally free resolution

$$0 \longrightarrow \mathbf{G}_1^\vee(-t) \longrightarrow \mathbf{F}_1^\vee(-t) \oplus \mathbf{G}_2^\vee(-t) \longrightarrow \cdots \longrightarrow \mathbf{G}_{c-1}^\vee(-t) \oplus \mathbf{F}_c^\vee(-t) \longrightarrow I_D \longrightarrow 0.$$

From Proposition 2.21, one can easily deduce the following two corollaries.

Corollary 2.22 *([43], Corollary 5.2.12) Let $C, D \subset \mathbf{P}^{n+1}$ be directly G-linked via the arithmetically Gorenstein scheme E. Then C is locally Cohen-Macaulay if and only if D is locally Cohen-Macaulay.*

Corollary 2.23 *([43], Corollary 5.2.13) Let $C, D \subset \mathbf{P}^{n+1}$ be directly G-linked via the arithmetically Gorenstein scheme E. If one (hence both) of C and D is locally Cohen-Macaulay, then*

$$deg(C) + deg(D) = deg(E).$$

We now present an important construction, introduced by R. Lazarsfeld and P. Rao in [40], and it is inspired by some constructions of P. Schwartau (see [52]). The construction was later generalized by A.V. Geramita and J. Migliore to Gorenstein Linkage. It appears in [35] in the generality that we need. It will be crucial in the next chapters.

Proposition 2.24 *(Proposition 5.4.5 in [43]) Let $S \subset \mathbf{P}^{n+1}$ be an arithmetically Cohen-Macaulay, generically Gorenstein scheme. Let D be a divisor on S without embedded components, and let F be a form of degree d that does not vanish on any component of S. Let C be the divisor on S defined as a scheme by the ideal $I_S + F \cdot I_D$. The following hold:*

- *as sets $C = D \cup (S \cap F)$,*
- *we have a short exact sequence*

$$0 \longrightarrow I_S(-d) \longrightarrow I_D(-d) \oplus I_S \longrightarrow I_C \longrightarrow 0,$$

- *$I_S + F \cdot I_D$ is a saturated ideal, i.e. $I_C = I_S + F \cdot I_D$,*
- *$H^i_*(\mathcal{I}_C) \cong H^i(\mathcal{I}_D)(-d)$ for $1 \leq i \leq dim(C)$,*
- *C is in the same liaison class of D, and is G-linked to D in two steps.*

Definition 2.25 *In the situation of Proposition 2.24, we say that C is a* Basic Double Link *or* Basic Double G-Link *of D on S.*

Basic Double Links preserve the property of being arithmetically Cohen-Macaulay, or arithmetically Buchsbaum (see Definition 5.1).

CHAPTER 3

NON ARITHMETICALLY COHEN-MACAULAY CURVES

In this chapter, we deal with curves $C \subset \mathbf{P}^3$ and their general hyperplane sections $X \subset \mathbf{P}^2$. In the previous chapter we presented a sufficient condition for the Cohen-Macaulayness of a curve C in terms of the lifting matrix of its general hyperplane section. In particular, Corollary 2.15 proves that $C \subset \mathbf{P}^3$ is arithmetically Cohen-Macaulay whenever all the entries of the degree matrix of its general plane section X are greater than or equal to 3.

The goal of this chapter is to show that this sufficient condition is the best possible. We do this by constructing a non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$ whose general plane section $X \subset \mathbf{P}^2$ has degree matrix M, for each M that has at least one entry smaller than or equal to 2. Moreover, we show that the curve C can be taken reduced and connected, except for the case when the general plane section is a complete intersection of a line and a plane conic. The case when the general plane section of C is a complete intersection was treated in Chapter 2. Therefore, in this chapter we concentrate on curves whose general plane section has degree matrix of size $t \times (t+1)$, for $t \geq 2$.

The constructions are explained in detail in the proofs of Theorem 3.3, Theorem 3.12, and Theorem 3.17. In Remark 3.5 we compute the deficiency modules of the non arithmetically Cohen-Macaulay curves that we construct in Theorem 3.3; in Remark 3.6 we present a variation of the construction of Theorem 3.3 and we compute the deficiency modules of the curves that one can produce in this way. In Example 3.8 we describe the construction of Theorem 3.3 for the case of degree matrices of size 2×3. The construction turns out to be very simple in this case, and once again we can compute the deficiency modules of the curves that we obtain.

We start by analyzing the degree matrices that correspond to generic points.

Example 3.1 *(Degree matrix of three generic points)*

Consider the degree matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

M is the degree matrix of three generic points in $\mathbf{P}^2$. We claim that a connected, reduced cubic curve $C \subset \mathbf{P}^3$ is arithmetically Cohen-Macaulay. In fact, up to isomorphism, the only integral, non-degenerate cubic curve in $\mathbf{P}^3$ is the twisted cubic, which is arithmetically Cohen-Macaulay. Any reduced, reducible, connected cubic curve is the union of a line and a plane conic (possibly reducible), meeting in a point. The curves cannot lie on the same plane, since we are assuming that all of the curves that we deal with are non-degenerate. Each of these curves is aCM. So it is not possible to find a connected, reduced, non-aCM curve $C \subset \mathbf{P}^3$, whose general plane section has degree matrix M.

Dropping the requirement that the curve is connected, we can take C to be the union of three skew lines in $\mathbf{P}^3$, or the disjoint union of a line and a plane conic. The curve C is smooth, disconnected and not arithmetically Cohen-Macaulay.

We also have a non-reduced curve: a fat line, whose ideal is given by $(L_1, L_2)^2$, where L_1, L_2 are linearly independent linear forms. A fat line is a degree 3, non-degenerate aCM curve. Its general plane section is a fat point, whose degree matrix is M.

For this particular matrix M then, requiring that C is connected forces Cohen-Macaulayness of the curve. Notice that Cohen-Macaulayness of the curve in this case does not follow from Theorem 2.12.

Example 3.2 *(Generic points)*

Let X consist of d generic points in $\mathbf{P}^2$. The h-vector of X is

$$h(z) = 1 + 2z + \ldots + nz^{n-1} + \left(d - \binom{n+1}{2}\right) z^n$$

where $n = \max \{ \, i \mid \binom{i+1}{2} \leq d \, \}$. Let $m = d - \binom{n+1}{2}$.

The initial degree of the saturated ideal I_X is $\alpha(I_X) = n$, and its minimal free resolution is

$$0 \longrightarrow R(-n-2)^m \oplus R(-n-1)^{n-2m} \longrightarrow R(-n)^{n+1-m} \longrightarrow I_X \longrightarrow 0$$

if $0 \leq m \leq \left[\frac{n}{2}\right]$, or

$$0 \longrightarrow R(-n-2)^m \longrightarrow R(-n)^{n+1-m} \oplus R(-n-1)^{2m-n} \longrightarrow I_X \longrightarrow 0$$

if $\left[\frac{n}{2}\right] \leq m \leq n$.

Here $\left[\frac{n}{2}\right] = \max\{z \in \mathbf{Z} \mid 2z \leq n\}$. The degree matrix of X is then

$$M = \underbrace{\begin{pmatrix} \left.\begin{matrix} 2 & \cdots & \cdots & 2 \\ \vdots & & & \vdots \\ 2 & \cdots & \cdots & 2 \end{matrix}\right\} m \\ \left.\begin{matrix} 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{matrix}\right\} n-2m \end{pmatrix}}_{n+1-m}$$

or respectively

$$M = \left.\begin{pmatrix} \underbrace{\begin{matrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{matrix}}_{2m-n} & \underbrace{\begin{matrix} 2 & \cdots & 2 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 2 & \cdots & 2 \end{matrix}}_{n+1-m} \end{pmatrix} \quad \right\} m$$

Claim. The general plane section of a general rational curve of degree d in $\mathbf{P}^3$ is a generic set of d points in the plane.

Let us consider a generic zero-dimensional scheme X of degree d in the plane. We only need to consider the case $d \geq 4$, since for $d = 1, 2, 3$ a general rational *curve of degree d is respectively a line, a smooth plane conic, and a twisted cubic. In all of those cases we know that the general plane section consists of generic points. Notice that for $d \leq 3$ a general rational curve is arithmetically Cohen-Macaulay. Moreover, a general rational curve is smooth and connected for any degree d.*

By a result of Ballico and Migliore (Theorem 1.6 of [2]), we know that there exists a smooth rational curve of degree d, that has X as a proper section. Then, a general (smooth) rational curve C, of degree d also has a generic zero-dimensional scheme of degree d as its proper section. By upper-semicontinuity, we can then conclude that a general hyperplane section of C is a generic zero-dimensional scheme of degree d.

In particular, for all the degree matrices M that correspond to d generic points in the plane, $d \geq 4$, we can find a smooth, non-aCM rational curve, whose general plane section has degree matrix M. A smooth curve of degree d and genus $g = 0$, with h-vector $(1, h_1, \ldots, h_s)$, $h_s \neq 0$, has $0 = g = h_2 + 2h_3 + \ldots + (s-1)h_s$. Then it cannot be aCM unless $s = 1$, since for an aCM curve $h_i > 0$ for all $i = 1, \ldots, s$. If instead $s = 1$, C has degree $d = h_0 + h_1 \leq 3$ and it is aCM as we already observed.

We are now going to analyze the general case. We start from matrices of size 2×3 or, more generally, matrices of any size with an assumption on one of the entries. See Example 3.1 and Remark 3.11 for the necessity of the assumption that M is not a 2×3 matrix with all the entries equal to 1.

Theorem 3.3 *Let $M = (a_{i,j})$ be a degree matrix of size $t \times (t+1)$ such that $a_{r,r-1} \leq 2$, for some r. Assume M is not a 2×3 matrix with all the entries equal to 1. Then there exists a reduced, connected, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$ whose general plane section $X \subset \mathbf{P}^2$ has degree matrix M.*

Proof: Consider the two submatrices of $M = (a_{i,j})_{i=1,\ldots,t;j=1,\ldots,t+1}$,

$$L_1 = (a_{i,j})_{i=1,\ldots,r-1;j=1,\ldots,r-1} \quad \text{and} \quad N = (a_{i,j})_{i=r,\ldots,t;j=r,\ldots,t+1}$$

where r is an integer $2 \leq r \leq t$, such that $a_{r,r-1} \leq 2$. Let $a = a_{1,1} + a_{r,t+1} + a_{r,r} + a_{r+1,r+1} + \ldots + a_{t,t} - a_{r,1}$ and let L be the matrix obtained by adding to L_1 the column vector

$$(a, a - a_{1,r-1} + a_{2,r-1}, a - a_{1,r-1} + a_{3,r-1}, \ldots, a - a_{1,r-1} + a_{r-1,r-1})^t$$

as the r-th column.

The entries on the diagonal of M are positive by Proposition 2.4. Notice that all the entries on the diagonal on L are positive, since they coincide with the first $r-1$ entries of the diagonal of M. Moreover,

$$a - a_{1,r-1} = a_{1,1} + a_{r,t+1} + a_{r,r} + a_{r+1,r+1} + \ldots + a_{t,t} - a_{r,1} - a_{1,r-1} =$$

$$a_{r,t+1} + a_{r,r} + a_{r+1,r+1} + \ldots + a_{t,t} - a_{r,r-1} \geq a_{r,r} + a_{r+1,r+1} + \ldots + a_{t,t} > 0,$$

by Proposition 2.4. So $a > a_{1,r-1}$ and L is a degree matrix, with the convention on the order of the entries that we put in the definition (the entries decrease from top to bottom and from right to left). The entries on the diagonal of N are also positive, since they are a subset of the entries on the diagonal of M. Then, both L and N are degree matrices. Notice also that a scheme with degree matrix L has exactly one minimal generator in minimum degree $a_{1,1}+\ldots+a_{r-1,r-1}$, and the other minimal generators in degrees $a - a_{i,r-1} + a_{1,1} + \ldots + a_{r-1,r-1}$ for $i = 1, \ldots, r-1$.

Let us consider two reduced, connected, arithmetically Cohen-Macaulay curves $C_1, C_2 \subset \mathbf{P}^3$, with degree matrices N, L respectively. Let C_1, C_2 be generic through a fixed (common) point P. We can assume that a generic curve with a prescribed degree matrix is reduced, by [19] or by Proposition 2.5 in [6]. Moreover, we can assume that C_1 and C_2 are connected curves, since for any degree matrix there is a connected, arithmetically Cohen-Macaulay curve that has that degree matrix. For example, for any given degree matrix we can take the curve to be the cone over the zero-dimensional scheme constructed as in [19] or in Proposition 2.5 of [6]. Such a curve is reduced and connected. If we assume that the entries on the subdiagonal of M are positive (except possibly for $a_{r,r-1}$), then so are the entries on the subdiagonals of L and N. In this situation, by a result of T. Sauer (see [51]) we can assume that C_1 and C_2 are smooth and connected.

Let $C = C_1 \cup C_2$ be the union of the two curves. C is reduced, non-degenerate and connected by construction. It has two irreducible components, both of them smooth if the subdiagonal of M is positive. Moreover, the ideal $I_{C_1} + I_{C_2}$ is not saturated, since its saturation is the homogeneous ideal of a point. Looking at the short exact sequence

$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow I_{C_1} + I_{C_2} \longrightarrow 0$$

we have that $\mathcal{M}_C = H^0_*(\mathcal{I}_{C_1} + \mathcal{I}_{C_2})/(I_{C_1} + I_{C_2}) \neq 0$, so C is not arithmetically Cohen-Macaulay.

Taking a general plane section of C, we obtain a zero-dimensional scheme $X \subset \mathbf{P}^2$ with saturated homogeneous ideal I_X. As a scheme, $X = X_1 \cup X_2$, where X_1, X_2 are general plane sections of C_1, C_2 respectively. Let

$$0 \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 \longrightarrow I_{X_1} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbf{G}_2 \longrightarrow \mathbf{G}_1 \longrightarrow I_{X_2} \longrightarrow 0$$

be minimal free resolutions of X_1 and X_2, respectively.

Let F be a generator of minimum degree in a minimal system of generators of I_{X_2}, and let $d = deg(F) = a_{1,1} + a_{2,2} + \ldots + a_{r-1,r-1}$ (notice that $d > 0$, since the entries on the diagonal of M are positive). By genericity of our choice of C_1 and C_2, we can assume that F is non-zerodivisor modulo I_{X_1}. Consider now the ideal $I_{X_1} + (F)$. It is an Artinian ideal of $R = k[x_0, x_1, x_2]$, with minimal free resolution

$$0 \longrightarrow \mathbf{F}_2(-d) \longrightarrow \mathbf{F}_2 \oplus \mathbf{F}_1(-d) \longrightarrow \mathbf{F}_1 \oplus R(-d) \longrightarrow I_{X_1} + (F) \longrightarrow 0 \qquad (3.1)$$

and socle in degree $s = a_{1,1} + a_{2,2} + \ldots + a_{t,t} + a_{r,t+1} - 3$.

Except for F, all the minimal generators of I_{X_2}, have degrees greater than or equal to

$$d-a_{1,r-1}+a = 2a_{1,1}+a_{2,2}+\ldots+a_{r-1,r-1}-a_{1,r-1}+a_{r,t+1}+a_{r,r}+a_{r+1,r+1}+\ldots+a_{t,t}-a_{r,1} =$$

$$= a_{1,1}+\ldots+a_{t,t}-a_{r,r-1}+a_{r,t+1} = s+3-a_{r,r-1} \geq s+1,$$

by assumption that $a_{r,r-1} \leq 2$. Since s is the socle degree of the quotient ring $R/I_{X_1}+(F)$, all the minimal generators of I_{X_2} are equal to zero modulo $I_{X_1}+(F)$. Therefore

$$I_{X_1}+(F) = I_{X_1}+I_{X_2}.$$

Let

$$0 \longrightarrow \mathbf{H}_2 \longrightarrow \mathbf{H}_1 \longrightarrow I_X \longrightarrow 0$$

be a minimal free resolution of I_X. Applying the mapping cone construction to the short exact sequence

$$0 \longrightarrow I_X \longrightarrow I_{X_1} \oplus I_{X_2} \longrightarrow I_{X_1}+I_{X_2} = I_{X_1}+(F) \longrightarrow 0$$

we get the following free resolution for $I_{X_1}+(F)$

$$0 \longrightarrow \mathbf{H}_2 \longrightarrow \mathbf{H}_1 \oplus \mathbf{G}_2 \oplus \mathbf{F}_2 \longrightarrow \mathbf{G}_1 \oplus \mathbf{F}_1 \longrightarrow I_{X_1}+(F) \longrightarrow 0. \qquad (3.2)$$

Comparing (3.1) and (3.2) gives

$$\mathbf{H}_2 = \mathbf{G}_2 \oplus \mathbf{F}_2(-d) \oplus \mathbf{F}, \qquad \mathbf{H}_1 = \mathbf{G}_1' \oplus \mathbf{F}_1(-d) \oplus \mathbf{F}$$

for some free R-module $\mathbf{F}$. Moreover, $\mathbf{G}_1 = \mathbf{G}_1' \oplus R(-d)$. This follows from the fact that there can be no cancellation between $\mathbf{G}_1'$ and $\mathbf{F}_2$ in the resolution of $I_{X_1}+(F)$ obtained via the mapping cone. In fact, the two free modules come from the same minimal free resolution (the one of $I_{X_1} \oplus I_{X_2}$). Moreover, the shifts of the free summand of $\mathbf{G}_2$ are all different from the shifts of the free summands of $\mathbf{F}_1(-d)$. In fact, the smallest shift among the free summands in $\mathbf{G}_2$ is

$$d+a+a_{r-1,r-1}-a_{1,r-1} = d+a_{t,r+1}+a_{r,r}+\ldots+a_{t,t}-a_{r,1}+a_{r-1,1} >$$

$$d + a_{t,r+1} + a_{r+1,r+1} + \ldots + a_{t,t},$$

that is the highest shift among the free summands of $\mathbf{F}_1(-d)$.

The free summands $\mathbf{F}$ cannot cancel with each other in the minimal free resolution of $I_{X_1} + (F)$, because they both come from the minimal free resolution of I_X, hence the map between them is not an isomorphism on any free submodule (the map is left unchanged under the mapping cone). Then $\mathbf{F} = 0$, since (3.2) must equal (3.1) after splitting. We obtain the following minimal free resolution for I_X

$$0 \longrightarrow \mathbf{G}_2 \oplus \mathbf{F}_2(-d) \longrightarrow \mathbf{G}_1' \oplus \mathbf{F}_1(-d) \longrightarrow I_X \longrightarrow 0.$$

The degree matrix of X is then $(b_{i,j})$, where

$$b_{i,j} = a_{i,j} \text{ for } 1 \leq i \leq r-1, 1 \leq j \leq r-1 \text{ and } r \leq i \leq t, r \leq j \leq t+1.$$

Moreover,

$$b_{r,1} = d + (\text{maximum shift in } \mathbf{F}_2) - (\text{maximum shift in } \mathbf{G}_1').$$

Then

$$b_{r,1} = d + (a_{r,r} + \ldots + a_{t,t} + a_{r,t+1}) - (d - a_{1,1} + a) = a_{r,1}.$$

Notice that, since M is homogeneous, all of its entries are determined by L_1, N and $a_{r,1}$. This proves that M is the degree matrix of X. $\square$

We now illustrate the construction of Theorem 3.3 in an example.

Example 3.4 *Consider the degree matrix*

$$M = \begin{pmatrix} 3 & 3 & 5 & 6 & 7 \\ 2 & 2 & 4 & 5 & 6 \\ 0 & 0 & 2 & 3 & 4 \\ -1 & -1 & 1 & 2 & 3 \end{pmatrix}.$$

Let $r = 3$, $a_{3,2} = 0 \leq 2$. *Then* $a = 3 + 4 + 2 + 2 - 0 = 11$. *Let*

$$L = \begin{pmatrix} 3 & 3 & 11 \\ 2 & 2 & 10 \end{pmatrix}$$

and
$$N = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$
Let C_1 be a curve with degree matrix N, C_2 be a curve with degree matrix L, both generic through the point $P = [0:0:0:1]$. $C = C_1 \cup C_2$ is reduced, connected and non arithmetically Cohen-Macaulay. Let X be a general plane section of C, X_1 be a general plane section of C_1, and X_2 be a general plane section of C_2. $X = X_1 \cup X_2$. Using a computer algebra system, one can compute the minimal free resolution of C to be
$$0 \longrightarrow \begin{array}{c} S^5(-22) \oplus S^3(-23) \\ \oplus \\ S^2(-21) \end{array} \longrightarrow \begin{array}{c} S^9(-21) \oplus S^3(-22) \\ S(-19) \oplus S^7(-20) \\ S(-12) \oplus S(-13) \end{array} \longrightarrow$$
$$\longrightarrow \begin{array}{c} S^3(-19) \oplus S^2(-20) \oplus S(-21) \\ S^2(-17) \oplus S^2(-18) \\ S(-9) \oplus S(-10) \oplus S(-11) \end{array} \longrightarrow I_C \longrightarrow 0.$$
The minimal free resolution of I_X is
$$0 \longrightarrow \begin{array}{c} R(-16) \oplus R(-15) \\ \oplus \\ R(-13) \oplus R(-12) \end{array} \longrightarrow \begin{array}{c} R(-13)^2 \oplus R(-11) \\ \oplus \\ R(-10) \oplus R(-9) \end{array} \longrightarrow I_X \longrightarrow 0.$$
Then the degree matrix of the general plane section X of C is
$$M = \begin{pmatrix} 3 & 3 & 5 & 6 & 7 \\ 2 & 2 & 4 & 5 & 6 \\ 0 & 0 & 2 & 3 & 4 \\ -1 & -1 & 1 & 2 & 3 \end{pmatrix}$$
as we expected.

Remark 3.5 *We can easily compute the deficiency module for the curves constructed in Theorem 3.3. In fact, $C = C_1 \cup C_2$ with C_1 and C_2 arithmetically Cohen-Macaulay curves meeting in exactly one point P. So we have the exact sequence*
$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow I_P \longrightarrow \mathcal{M}_C \longrightarrow \mathcal{M}_{C_1} \oplus \mathcal{M}_{C_2} = 0$$
that together with the short exact sequence
$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow I_{C_1} + I_{C_2} \longrightarrow 0$$

gives the isomorphism

$$\mathcal{M}_C \cong I_P/(I_{C_1} + I_{C_2}).$$

In particular, $\alpha(\mathcal{M}_C) = 1$*. Notice that the curves that we construct in Theorem 3.3 are almost never arithmetically Buchsbaum (see Chapter 5 for the definition of arithmetically Buchsbaum).*

Remark 3.6 *Instead of taking* C_1 *and* C_2 *generic through the same point, we could just take them generic, therefore disjoint, with the same degree matrices as in the proof of Theorem 3.3. In this case we get a non-degenerate, reduced, non-aCM, disconnected curve* $C = C_1 \cup C_2$ *with two connected components. The general plane section of* C *has degree matrix* M*. This can be proved in the same way as Theorem 3.3. The ideal* $I_{X_1} + I_{X_2}$ *is not saturated, and its saturation is the ring* R*.*

Again, if the entries on the subdiagonal of M *are positive except possibly for* $a_{r,r-1}$*, we can take* C_1 *and* C_2 *to be smooth and connected. In this case,* $C = C_1 \cup C_2$ *is a non-degenerate, smooth, non-aCM, disconnected curve with two smooth connected components, whose general plane section has degree matrix* M*.*

In the case that C_1 *and* C_2 *are disjoint, we can compute the deficiency module* $\mathcal{M}_C$ *of* C *following the same procedure of Remark 3.5. We have the exact sequence*

$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow R \longrightarrow \mathcal{M}_C \longrightarrow \mathcal{M}_{C_1} \oplus \mathcal{M}_{C_2} = 0$$

that combined with the short exact sequence

$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow I_{C_1} + I_{C_2} \longrightarrow 0$$

gives the isomorphism

$$\mathcal{M}_C \cong R/(I_{C_1} + I_{C_2}).$$

In particular, $\alpha(\mathcal{M}_C) = 0$*. Notice that the curves that we construct are almost never arithmetically Buchsbaum (see Chapter 5 for the definition).*

Let us see what happens if we apply the alternative construction of Remark 3.6 to the degree matrix of Example 3.4.

Example 3.7 *Consider the degree matrix*

$$M = \begin{pmatrix} 3 & 3 & 5 & 6 & 7 \\ 2 & 2 & 4 & 5 & 6 \\ 0 & 0 & 2 & 3 & 4 \\ -1 & -1 & 1 & 2 & 3 \end{pmatrix}.$$

Let $r = 3$*,* $a_{3,2} = 0 < 2$*. Then* $a = 3 + 4 + 2 + 2 - 0 = 11$*. Let*

$$L = \begin{pmatrix} 3 & 3 & 11 \\ 2 & 2 & 10 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$

Let C_1 *be a curve with degree matrix* N*,* C_2 *be a curve with degree matrix* L*, both generic.* $C = C_1 \cup C_2$ *is reduced, disconnected and non arithmetically Cohen-Macaulay. Let* X *be a general plane section of* C*,* X_1 *be a general plane section of* C_1*, and* X_2 *be a general plane section of* C_2*.* $X = X_1 \cup X_2$*. Using a computer algebra system, one can compute the ideals of* C_1 *and* C_2 *as ideals of maximal minors of two matrices of homogeneous polynomials of the degrees prescribed in* L *and* M*. The minimal free resolution of* C *is*

$$0 \longrightarrow \begin{array}{c} S(-24) \\ \oplus \\ S(-22) \oplus S^2(-23) \end{array} \longrightarrow \begin{array}{c} S^4(-21) \oplus S(-22) \\ S(-19) \oplus S^4(-20) \\ S(-12) \oplus S(-13) \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{c} S^2(-18) \oplus S^2(-19) \\ S(-11) \oplus S^2(-17) \\ S(-9) \oplus S(-10) \end{array} \longrightarrow I_C \longrightarrow 0.$$

As expected, the minimal free resolution of the curve that we construct here is different from the one of Example 3.4. One can compute the minimal free resolution of I_X*, that turns out to be*

$$0 \longrightarrow \begin{array}{c} R(-16) \oplus R(-15) \\ \oplus \\ R(-13) \oplus R(-12) \end{array} \longrightarrow \begin{array}{c} R(-13)^2 \oplus R(-11) \\ \oplus \\ R(-10) \oplus R(-9) \end{array} \longrightarrow I_X \longrightarrow 0\ .$$

Then the degree matrix of the general plane section X of C is

$$M = \begin{pmatrix} 3 & 3 & 5 & 6 & 7 \\ 2 & 2 & 4 & 5 & 6 \\ 0 & 0 & 2 & 3 & 4 \\ -1 & -1 & 1 & 2 & 3 \end{pmatrix}$$

as we expected.

The construction of Theorem 3.3 is very simple in the case of matrices of size 2×3. In this case, one can take $r = 2$, and the condition $a_{r,r-1} = a_{2,1} \leq 2$ is satisfied by assumption.

Example 3.8 *Consider a degree matrix*

$$M = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

In order for M to be a degree matrix, all the entries have to be positive, except possibly for d. We are under the assumption that d ≤ 2. Following the proof of Theorem 3.3, let $C = CI(a, b+f) \cup CI(e, f) \subset \mathbf{P}^3$, where the Complete Intersections are generic through a common point P. Then C is a non-aCM, connected, reduced, non-degenerate space curve. C is smooth outside of P, and its general plane section X has degree matrix M. Moreover, up to a change of coordinates, the deficiency module is $\mathcal{M}_D \cong (x_0, x_1, x_2)/(F_1, F_2, G_1, G_2)$, where (F_1, F_2) and (G_1, G_2) are the ideals of two generic complete intersections of type $(a, b+f)$ and (e, f) through the point $[0 : 0 : 0 : 1]$. Equivalently, we can let F_1, F_2, G_1, G_2 be generic elements the ideal (x_0, x_1, x_2), of degrees $a, b+f, e, f$.

Let $D = CI(a, b+f) \cup CI(e, f) \subset \mathbf{P}^3$ where the complete intersections are generic, hence disjoint. Then D is a non-aCM, reduced space curve, with two smooth connected components. The general plane section X of D has degree matrix M. Moreover, the deficiency module is $\mathcal{M}_D \cong k[x_0, x_1, x_2, x_3]/CI(a, e, f, b+f)$.

Notice that in both cases the initial degree of the ideal I_C of C is the same as the initial degree of I_X, and the highest degree for a minimal generator of I_C is $b+2f$.

Remark 3.9 *In Theorem 3.3 we assume that* $a_{r,r-1} \leq 2$ *for some* r*. Notice that, for any choice of* $r \geq 2$ *for which* $a_{r,r-1} \leq 2$*, our construction yields a reduced, non-aCM curve, whose general plane section has degree matrix* M*. The curves that we get for two different choices of* r *are not projectively isomorphic, since their connected components are not (their connected components do not even have the same degree matrices).*

The assumption that $a_{r,r-1} \leq 2$ in Theorem 3.3 is essential. In fact, if $a_{r,r-1} \geq 3$ for all r, the degree matrix that we obtain following the procedure of Theorem 3.3 is not the required one. In particular, its size is in general strictly bigger than $t \times (t+1)$. In the following example, we show how the construction of Theorem 3.3 does not yield a curve whose general plane section has the desired degree matrix, in the case that the the hypothesis $a_{r,r-1} \leq 2$ is not satisfied.

Example 3.10 *Let*

$$M = \begin{pmatrix} 3 & 4 & 4 & 5 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{pmatrix}$$

and let $r = 3$*. Notice that* $a_{3,2} = 3 \not\leq 2$*. Let*

$$L = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 3 & 7 \end{pmatrix}, \qquad N = (3, 4)$$

and let C_1, C_2 *be aCM, smooth, generic curves through a common point, with degree matrices* N, L *respectively. Let* X_1, X_2 *be the general plane sections of* C_1, C_2*, respectively. The minimal free resolution of* $I = I_{X_1} + I_{X_2}$ *turns out to be*

$$0 \longrightarrow R(-12)^3 \longrightarrow \begin{matrix} R(-7) \oplus R(-9) \\ \oplus \\ R(-10) \oplus R(-11)^3 \end{matrix} \longrightarrow \begin{matrix} R(-3) \oplus R(-4) \\ \oplus \\ R(-6) \oplus R(-10) \end{matrix} \longrightarrow I \longrightarrow 0$$

hence the degree matrix of the general plane section of $C = C_1 \cup C_2$ *is*

$$M' = \begin{pmatrix} 3 & 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}$$

and not the required matrix M. The problem comes from the fact that the socle of $I_{X_1} + (F)$ has final degree 10*, and I_{X_2} has a minimal generator in degree* 10*.*

Remark 3.11 *In the statement of Theorem 3.3, we pointed out that the construction does not work for the matrix*

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

that we analyzed in Example 3.1. In fact, for this matrix our construction yields curves $C_1 =$ a plane conic and $C_2 =$ a line, meeting in a point. In this case, $I_{C_1} + I_{C_2}$ is saturated and $C = C_1 \cup C_2$ is arithmetically Cohen Macaulay.

Notice that, if we take a generic (disjoint) union of a line and a conic, we get a non-degenerate, smooth, non-aCM, disconnected curve, whose general plane section consists of three generic points and has degree matrix M.

We now present an alternative construction for the degree matrices of size 2×3. The advantage with respect to the construction of Theorem 3.3 is that the saturated ideal of the curves that we obtain in the following theorem are minimally generated in low degree. This will be useful in the next chapter. In fact, in the proof of Theorem 4.15 we must start from a curve whose minimal generators have small degrees in order to construct smooth and connected, non Cohen-Macaulay curves with a prescribed degree matrix.

Theorem 3.12 *Let M be a degree matrix of size 2×3,*

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

and assume that $a_{2,1} \leq 2$. Then there exists a reduced, connected, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$, whose general plane section $X \subset \mathbf{P}^2$ has degree matrix M, and such that the saturated ideal I_C of C is minimally generated in degree smaller than or equal to $a_{1,2} + a_{2,3} + 1$.

Proof: Let C_1 be a generic complete intersection of type $(a_{2,2}, a_{2,3})$, and $I_{C_1} \subset S = k[x_0, \ldots, x_3]$ the saturated ideal of C_1. Let G be a generic form of degree $a_{1,1}$. Then $I = I_{C_1} + (G)$ is the saturated ideal of a generic complete intersection of type $(a_{1,1}, a_{2,2}, a_{2,3})$. Therefore the scheme Z associated to I is a zero-dimensional scheme, consisting of $a_{1,1} \cdot a_{2,2} \cdot a_{2,3}$ distinct points. Let P be one of the points of Z, and let $X = Z - P$ be the complement of P in Z. Notice that X is linked to P via the complete intersection Z, therefore by Proposition 2.21 one gets a free resolution of I_X of the form

$$0 \longrightarrow S(-a_{1,1} - a_{2,2} - a_{2,3} + 1)^3 \longrightarrow \begin{array}{c} S(-a_{1,1} - a_{2,2} - a_{2,3} + 2)^3 \oplus \\ S(-a_{1,1} - a_{2,2}) \oplus \\ S(-a_{2,2} - a_{2,3}) \oplus S(-a_{1,1} - a_{2,3}) \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{c} S(-a_{1,1} - a_{2,2} - a_{2,3} + 3) \oplus \\ S(-a_{1,1}) \oplus S(-a_{2,2}) \oplus \\ S(-a_{2,3}) \end{array} \longrightarrow I_X \longrightarrow 0.$$

The resolution is not a priori minimal.

The socle of the complete intersection Z is concentrated in degree $a_{1,1} + a_{2,2} + a_{2,3} - 3 \leq a_{1,2} + a_{2,3} - 1$, since $a_{2,1} \leq 2$ by assumption. Therefore, the Hilbert function of Z in degree $a_{1,2} + a_{2,3}$ is

$$H_Z(a_{1,2} + a_{2,3}) = deg(Z).$$

The Hilbert function of X in the same degree is

$$H_X(a_{1,2} + a_{2,3}) \leq deg(X) = deg(Z) - 1.$$

Then there is a surface F of degree $a_{1,2}+a_{2,3}$ that contains X but does not contain Z, so it contains X and not P. Let the surface F be generic, with this property. Let the curve C_2 be the scheme-theoretic intersection of F and G. C_2 is a complete intersection curve of type $(a_{1,1}, a_{1,2}+a_{2,3})$. By the construction, $C_1 \cap C_2 = X$. Let C be the union of the two complete intersection curves, $C = C_1 \cup C_2$. The curve C is reduced and connected, and it has two irreducible components. Its general plane section is the union of a $CI(a_{1,1}, a_{1,2}+a_{2,3})$ and a $CI(a_{2,2}, a_{2,3})$. The same argument as in the proof of Theorem 3.3 applies, showing that the general plane section of C has degree matrix M.

We need to show that C is not arithmetically Cohen-Macaulay. From the long exact sequence

$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow I_X \longrightarrow M_C \longrightarrow 0$$

we see that the deficiency module of C is

$$\mathcal{M}_C \cong I_X/(I_{C_1} + I_{C_2}).$$

Then C is arithmetically Cohen-Macaulay if and only if $I_{C_1} + I_{C_2} = I_X$, if and only if $I_{C_1} + I_{C_2}$ is saturated.

In order to show that the ideal $I_{C_1} + I_{C_2}$ is not saturated, we compute a free resolution of it. Multiplication by F in S/I yields the long exact sequence

$$0 \longrightarrow (I : F)/I(-a_{1,2} - a_{2,3}) \longrightarrow S/I(-a_{1,2} - a_{2,3}) \longrightarrow S/I \longrightarrow S/(I + (F)) \longrightarrow 0.$$

$I : F = I : (I + (F))$, and since $I + (F) = I_{C_1} + I_{C_2}$, then $I : F = I : (I_{C_1} + I_{C_2})$. The saturation of $I_{C_1} + I_{C_2}$ is I_X, since $C_1 \cap C_2 = X$. Therefore

$$I : F = I : (I_{C_1} + I_{C_2}) = I : I_X = I_P.$$

The last equality follows from the fact that P is the residual to X in the complete intersection Z, whose homogeneous saturated ideal is I. Then

$$I : F = I_P \quad \text{and} \quad I_{C_1} + I_{C_2} = I + (F).$$

These equalities give the short exact sequence

$$0 \longrightarrow S/I_P(-a_{1,2} - a_{2,3}) \longrightarrow S/I \longrightarrow S/(I_{C_1} + I_{C_2}) \longrightarrow 0.$$

Using the mapping cone construction, we obtain a free resolution for $I_{C_1}+I_{C_2}$

$$0 \longrightarrow S(-a_{1,2}-a_{2,3}-3) \longrightarrow \begin{array}{c} S(-a_{1,2}-a_{2,3}-2)^3 \\ \oplus \\ S(-a_{1,1}-a_{2,2}-a_{2,3}) \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{c} S(-a_{1,2}-a_{2,3}-1)^3 \oplus \\ S(-a_{1,1}-a_{2,2}) \oplus \\ S(-a_{2,2}-a_{2,3}) \oplus \\ S(-a_{1,1}-a_{2,3}) \end{array} \longrightarrow \begin{array}{c} S(-a_{1,1}) \oplus \\ S(-a_{2,2}) \oplus \\ S(-a_{2,3}) \oplus \\ S(-a_{1,2}-a_{2,3}) \end{array} \longrightarrow I_{C_1}+I_{C_2} \longrightarrow 0.$$

The resolution is not minimal a priori, however no cancellation can take place between the last free module and the following one, because $a_{1,2}+a_{2,3}+3 > a_{1,1}+a_{2,2}+a_{2,3}$, since $a_{2,1} < 3$. This proves that the ideal $I_{C_1}+I_{C_2}$ is not saturated, therefore C is not arithmetically Cohen-Macaulay.

Consider the short exact sequence

$$0 \longrightarrow I_{C_1}+I_{C_2} \longrightarrow I_X \longrightarrow \mathcal{M}_C \longrightarrow 0.$$

The mapping cone procedure produces a free resolution of $\mathcal{M}_C$ of the form

$$0 \longrightarrow S(-a_{1,2}-a_{2,3}-3) \longrightarrow \begin{array}{c} S(-a_{1,2}-a_{2,3}-2)^3 \\ \oplus \\ S(-a_{1,1}-a_{2,2}-a_{2,3}) \end{array} \longrightarrow$$

$$\rightarrow \begin{array}{c} S(-a_{1,2}-a_{2,3}-1)^3 \oplus \\ S(-a_{1,1}-a_{2,2}) \oplus S(-a_{1,1}-a_{2,3}) \\ S(-a_{2,2}-a_{2,3}) \\ S(-a_{1,1}-a_{2,2}-a_{2,3}+1)^3 \end{array} \rightarrow \begin{array}{c} S(-a_{1,2}-a_{2,3}) \oplus S(-a_{1,1}) \oplus S(-a_{2,2}) \\ S(-a_{2,2}-a_{2,3}) \oplus S(-a_{1,1}-a_{2,3}) \\ S(-a_{2,3}) \oplus S(-a_{1,1}-a_{2,2}) \\ S(-a_{1,1}-a_{2,2}-a_{2,3}+2)^3 \end{array} \rightarrow$$

$$\longrightarrow \begin{array}{c} S(-a_{1,1}-a_{2,2}-a_{2,3}+3) \oplus \\ S(-a_{2,3}) \oplus \\ S(-a_{1,1}) \oplus S(-a_{2,2}) \end{array} \longrightarrow \mathcal{M}_C \longrightarrow 0.$$

The free summands $S(-a_{1,1}) \oplus S(-a_{2,2}) \oplus S(-a_{2,3})$ in the first free module of the resolution of $\mathcal{M}_C$ come from the free resolution of I_X. Since the minimal generators of $I_{C_1}+I_{C_2}$ in those degrees coincide with the minimal generators of I_X, the free summands that did not cancel in the minimal free resolution of I_X do cancel in the minimal free resolution of $\mathcal{M}_C$ with the corresponding free summands in the second free module (coming from the free resolution of $I_{C_1}+I_{C_2}$). Therefore the first free module in the minimal free resolution of $\mathcal{M}_C$ is simply $S(-a_{1,1} - a_{2,2} - a_{2,3} + 3)$. This proves that the initial degree of $\mathcal{M}_C$ is

$$\alpha(M_C) = a_{1,1} + a_{2,2} + a_{2,3} - 3.$$

From the shifts in the free resolution of $\mathcal{M}_C$, one can also deduce an upper bound for the Castelnuovo-Mumford regularity of $\mathcal{M}_C$:

$$reg(\mathcal{M}_C) \leq a_{1,2} + a_{2,3} - 1.$$

The saturated ideal of the general hyperplane section of C has no minimal generators in degree greater than or equal to $a_{1,2} + a_{2,3} + 1$, and the last non-zero component of the deficiency module of C occurs in degree

$$\alpha^+(\mathcal{M}_C) \leq a_{1,2} + a_{2,3} - 1.$$

Therefore, by Lemma 3.12 in [21], it follows that the ideal I_C is minimally generated in degree smaller than or equal to $a_{1,2} + a_{2,3} + 1$. $\square$

Remark 3.13 *In the proof of Theorem 3.12 we construct a curve C whose general plane section has a given degree matrix, and we compute a free resolution of the deficiency module $\mathcal{M}_C$ of C. Moreover, we prove that*

$$\alpha(M_C) = a_{1,1} + a_{2,2} + a_{2,3} - 3, \qquad \alpha^+(\mathcal{M}_C) \leq a_{1,2} + a_{2,3} - 1$$

and that the Castelnuovo-Mumford regularity of $\mathcal{M}_C$ is bounded by

$$reg(\mathcal{M}_C) \leq a_{1,2} + a_{2,3} - 1.$$

Remark 3.14 *The saturated ideal of the general plane section X of the curve C has a minimal generator in degree $a_{1,2}+a_{2,3}$. Therefore the ideal of any curve that has X as a general plane section necessarily has a minimal generator in degree $a_{1,2} + a_{2,3}$ or higher.*

We now perform the construction of Theorem 3.12 in an example.

Example 3.15 *Let M be the degree matrix*

$$\begin{pmatrix} 1 & 4 & 4 \\ 0 & 3 & 3 \end{pmatrix}$$

and let $C_1 = CI(3,3) \subset \mathbf{P}^3$ be a generic complete intersection of two surfaces of degree 3. Let G be a generic linear form and let Z be the complete intersection of C_1 with the plane of equation G. Z is a $CI(1,3,3)$. Let X consist of 8 points of Z and let P be the residual point. The minimal free resolution of X is

$$0 \longrightarrow S^2(-6) \longrightarrow S^2(-4) \oplus S^3(-5) \longrightarrow S(-1) \oplus S^2(-3) \oplus S(-4) \longrightarrow I_X \longrightarrow 0.$$

With a computer algebra system, one can verify that a generic surface F of degree 7 that contains X does not contain P. Let C_2 be the curve with saturated ideal (F,G), let $C = C_1 \cup C_2$. The minimal free resolution of $I_{C_1} + I_{C_2}$ is

$$0 \longrightarrow S(-10) \longrightarrow \begin{matrix} S(-7) \\ \oplus \\ S^3(-9) \end{matrix} \longrightarrow \begin{matrix} S^2(-4)\oplus \\ S(-6)\oplus \\ S^3(-8) \end{matrix} \longrightarrow \begin{matrix} S(-1)\oplus \\ S^2(-3)\oplus \\ S(-7) \end{matrix} \longrightarrow I_{C_1} + I_{C_2} \longrightarrow 0$$

while the minimal free resolution of C is

$$0 \longrightarrow S(-10) \longrightarrow S(-7) \oplus S^3(-9) \longrightarrow S^2(-4) \oplus S^2(-8) \longrightarrow I_C \longrightarrow 0.$$

The ideal of C has two minimal generators in degree 4, whose images in the ideal $I_{C\cap H}$ of a general plane section of C are minimal generators. It also has two minimal generators in degree $8 = 4+3+1$.

For the arguments that follow, we need to show the existence of smooth surfaces containing the curves constructed in Theorem 3.12.

Lemma 3.16 *Let C be a curve constructed as in Theorem 3.12. For each $d \geq a_{1,2} + a_{2,3} + 1$ there is a smooth surface of degree d containing C.*

Proof: Consider the linear system Δ of surfaces of $\mathbf{P}^3$ of degree d containing C. $C = C_1 \cup C_2$ is a union of 2 complete intersection curves. Let $Sing(C) = X \cup Y$ be the singular locus of C. C is singular at the points where the two components intersect, and possibly at some other zero-dimensional subset $Y \subset C_2$. The general element of Δ is basepoint-free outside of C, hence smooth outside of C by Bertini's Theorem. Consider now a point $P \in C$. We want to show that the general element of Δ is smooth at P. By Corollary 2.10 in [26], it is enough to exhibit two elements of Δ meeting transversally at P. Since C is smooth outside of $Sing(C)$, for each point $P \notin Sing(C)$ we have two minimal generators of I_C, call them F and G, meeting transversally at P. The degree of each of them is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P.

In order to complete the proof, we need to check that the points of $Sing(C)$ are not fixed singular points for Δ. So it is enough to find a surface for each $P \in Sing(C)$ that contains C and is non-singular at P. For each point $Q \in Y$ we have a smooth surface G containing C_2. Taking the union of G with a smooth surface of C_1 of appropriate degree (C_1 is smooth, so we can always find such a surface) that does not contain $Q \notin C_1$ gives a surface that is smooth at Q and contains C.

Let $Q \in X$. We need to find a surface containing C that is smooth at Q. Let $F_1, \ldots, F_n$ be a minimal system of generators of I_C. $d_i := deg(F_i) \leq d$ for all i. Some of the minimal generators of I_C are smooth at Q (the ones of degrees $a_{1,1}, a_{2,2}, a_{2,3}$ are smooth by genericity). Assume that $F_1, \ldots, F_r$ are smooth at Q. Then let $T = G_1F_1 + \ldots + G_rF_r$ where each G_i is a generic polynomial of degree $d - d_i$. The surface defined by T contains C by construction. In order to check that T is smooth at Q, it suffices to show that not all the partial derivatives of T vanish at Q. Denote the derivative of F_i with respect to x_j by $F_{i,j}$. Some of the partial derivatives of F_i do not vanish at Q. For example, assume that $F_{1,2}(Q) \neq 0$. Then the partial derivative of T with respect to x_2 evaluated at Q is $T_2(Q) = G_{1,2}(Q)F_1(Q) + \ldots + G_{r,2}(Q)F_r(Q) + G_1(Q)F_{1,2}(Q) + \ldots + G_r(Q)F_{r,2}(Q) =$

$G_1(Q)F_{1,2}(Q)+\ldots+G_r(Q)F_{r,2}(Q)$. By genericity of $G_1,\ldots,G_r$ we can assume that none of them vanishes at Q and that $G_1(Q)F_{1,2}(Q)+\ldots+G_r(Q)F_{r,2}(Q)\neq 0$. This shows smoothness of T at Q, and therefore concludes the proof. $\square$

The following Theorem gives us a general construction for curves in $\mathbf{P}^3$. For any degree matrix M such that one of its entries is smaller than or equal to 2, we construct an example of a reduced, connected, non-aCM curve, whose general plane section has degree matrix M. Notice that not all the degree matrices can correspond to points that are general plane section of an integral curve. In particular, none of the curves that we construct in the proof of the following Theorem is integral. In the next chapter, we deal with degree matrices of points that can lift to an integral curve.

Theorem 3.17 *Let $M=(a_{i,j})$ be a degree matrix of size $t\times(t+1)$ such that $a_{t,1}\leq 2$. Assume M is not a 2×3 matrix with all the entries equal to 1. Then there exists a reduced, connected, non-aCM curve $C\subset\mathbf{P}^3$ whose general plane section $X\subset\mathbf{P}^2$ has degree matrix M.*

Proof: We proceed by induction on the size t of M. We include in the induction hypothesis that $\alpha(I_C)=\alpha(I_X)$. The thesis is true for $t=2$, as shown in Theorem 3.3 and in Example 3.8. In fact, the curve that we construct in the proof of Theorem 3.3 is $C=CI(a_{1,1},a_{1,2}+a_{2,3})\cup CI(a_{2,2},a_{2,3})$, then $\alpha(I_C)=a_{1,1}+a_{2,2}$.

Let $M=(a_{i,j})_{i=1,\ldots t;\, j=1,\ldots t+1}$ be a degree matrix with $a_{t,1}\leq 2$. Assume that $a_{t-1,1}\leq 2$ and let $N=(a_{i,j})_{i=1,\ldots t-1;\, j=1,\ldots t}$ be the submatrix of M consisting of the first $t-1$ rows and the first t columns. The entries on the diagonal of N agree with the first $t-1$ entries on the diagonal of M, so they are positive. Then N is a degree matrix. By the induction hypothesis, we have a non-aCM, reduced, connected curve $D\subset\mathbf{P}^3$, whose general plane section $Y\subset\mathbf{P}^2$ has degree matrix N. Moreover, $\alpha(I_D)=\alpha(I_Y)=a_{1,1}+\ldots+a_{t-1,t-1}$. Let S be a surface of degree

$s = a_{1,1} + \ldots + a_{t,t}$ containing D. Such an S exists, since $s = \alpha(I_D) + a_{t,t} > \alpha(I_D)$. Moreover, S can be chosen such that its image in I_Y is not a minimal generator, since $deg(S) > \alpha(I_Y)$. Perform a basic double link on S, with a generic surface F of degree $a_{t,t+1} > 0$, that meets D in (at least) a point. Let $C = D \cup (S \cap F)$. Then C is reduced and connected, and $\mathcal{M}_C \cong \mathcal{M}_D(-a_{t,t+1}) \neq 0$, so C is non-aCM. Moreover C is non-degenerate, since D is non-degenerate. By genericity of our choices, D and $S \cap F$ meet transversally at each of their points of intersection, and each of their points of intersection is a smooth point on both D and $S \cap F$. We have the short exact sequence (see Proposition 2.24)

$$0 \longrightarrow R(-s - a_{t,t+1}) \longrightarrow I_Y(-a_{t,t+1}) \oplus R(-s) \longrightarrow I_X \longrightarrow 0.$$

Then, using the mapping cone construction, a free resolution of I_X is given by

$$0 \longrightarrow \begin{array}{c} R(-s - a_{t,t+1}) \\ \oplus \\ \mathbf{F}_2(-a_{t,t+1}) \end{array} \longrightarrow \begin{array}{c} R(-s) \\ \oplus \\ \mathbf{F}_1(-a_{t,t+1}) \end{array} \longrightarrow I_X \longrightarrow 0.$$

Notice that $R(-s-a_{t,t+1})$ cannot cancel with any of the free summands of $\mathbf{F}_1(-a_{t,t+1})$, since the image of S in I_Y is not a minimal generator. Moreover, none of the shifts appearing in $\mathbf{F}_2(-a_{t,t+1})$ can be equal to s, since $a_{i,t+1} > 0$ for all i (if $a_{i,t+1} \leq 0$ for some i, then $a_{i,j} \leq 0$ for all j, and this is not possible for a degree matrix). This shows that the resolution is minimal. Then the degree matrix of X is M, as required. Notice that by construction $\alpha(I_C) \leq s = \alpha(I_X)$, so $\alpha(I_X) \leq \alpha(I_C) = \alpha(I_X)$.

The case when $a_{t-1,1} \geq 3$ and $a_{t,t-1} > 0$ is analogous. Let N be the submatrix of M consisting of the last $t-1$ rows and the first t columns, $N = (a_{i,j})_{i=2,\ldots t;\, j=1,\ldots t}$. Notice that since $a_{t-1,1} > 0$, then $a_{i,1} > 0$ for all $i \neq t$, hence $a_{i,i-1} > 0$ for $i = 2, \ldots, t-1$. $a_{t,t-1} > 0$ by assumption. The entries on the diagonal of N are $a_{i,i-1} > 0$ for $i = 2, \ldots, t$, so N is a degree matrix. By the induction hypothesis, there is a non-aCM, reduced, connected curve $D \subset \mathbf{P}^3$, whose general plane section $Y \subset \mathbf{P}^2$ has degree matrix N. Moreover, $\alpha(I_D) = \alpha(I_Y) = a_{2,1} + \ldots + a_{t,t-1}$. Let S be a surface of degree $s = a_{1,1} + \ldots + a_{t,t}$ containing D. Such an S exists, since

$s = \alpha(I_D) + a_{1,t} > \alpha(I_D)$. Moreover, S can be chosen such that its image in I_Y is not a minimal generator, since $deg(S) > \alpha(I_Y)$. Perform a basic double link on S, with a generic surface F of degree $a_{1,t+1} > 0$, that meets D in (at least) a point. Let $C = D \cup (S \cap F)$. Then C is reduced and connected, and $\mathcal{M}_C \cong \mathcal{M}_D(-a_{1,t+1}) \neq 0$, so C is non-aCM. The curve C is non-degenerate, since D is non-degenerate. By genericity of our choices, D and $S \cap F$ meet transversally at each of their points of intersection, and each of their points of intersection is a smooth point on both D and $S \cap F$. We have the short exact sequence (see Proposition 2.24)

$$0 \longrightarrow R(-s-a_{1,t+1}) \longrightarrow I_Y(-a_{1,t+1}) \oplus R(-s) \longrightarrow I_X \longrightarrow 0.$$

Using the mapping cone construction, a free resolution of I_X is given by

$$0 \longrightarrow \begin{array}{c} R(-s-a_{1,t+1}) \\ \oplus \\ \mathbf{F}_2(-a_{1,t+1}) \end{array} \longrightarrow \begin{array}{c} R(-s) \\ \oplus \\ \mathbf{F}_1(-a_{1,t+1}) \end{array} \longrightarrow I_X \longrightarrow 0.$$

Notice that $R(-s-a_{1,t+1})$ cannot cancel with any of the free summands of $\mathbf{F}_1(-a_{1,t+1})$, since the image of S in I_Y is not a minimal generator. Moreover, none of the shifts appearing in $\mathbf{F}_2(-a_{1,t+1})$ can be equal to s, since $a_{i,t+1} > 0$ for all i (if $a_{i,t+1} \leq 0$ for some i, then all the entries of the i-th row of M would be less than or equal to zero, contradicting the fact that M is a degree matrix). This shows that the resolution is minimal. Then the degree matrix of X is M, as required. By construction $\alpha(I_C) \leq s = \alpha(I_X)$, so $\alpha(I_C) = \alpha(I_X)$.

The case when $a_{t-1,1} \geq 3$ and $a_{t,t-1} \leq 0$ is similar. Let N be the submatrix of M consisting of the last $t-1$ rows and the last t columns, $N = (a_{i,j})_{i=2,\ldots t;\, j=2,\ldots t+1}$. N is a degree matrix, since the entries on its diagonal are $a_{i,i} > 0$ for $i = 2, \ldots, t$. Notice that $a_{t,2} \leq a_{t,t-1} \leq 0 < 2$. By the induction hypothesis, there is a non-aCM, reduced, connected curve $D \subset \mathbf{P}^3$, whose general plane section $Y \subset \mathbf{P}^2$ has degree matrix N. Moreover, $\alpha(I_D) = \alpha(I_Y) = a_{2,2} + \ldots + a_{t,t}$. Let S be a surface of degree $s = a_{1,2}+\ldots+a_{t,t+1}$ containing D. Such an S exists, since $s = \alpha(I_D)+a_{1,t+1} > \alpha(I_D)$. Moreover, S can be chosen such that its image in I_Y is not a minimal generator, since $deg(S) > \alpha(I_Y)$. Perform a basic double link on S, with a generic surface F of degree $a_{1,1} > 0$, that meets D in (at least) a point. Let $C = D \cup (S \cap F)$. Then C is reduced and connected, and $\mathcal{M}_C \cong \mathcal{M}_D(-a_{1,1}) \neq 0$, so C is non-aCM. The curve C is non-degenerate, since D is non-degenerate. By genericity of our choices, D and $S \cap F$ meet transversally at each of their points of intersection, and each of their points of intersection is a smooth point on both D and $S \cap F$. We have the short exact sequence (see Proposition 2.24)

$$0 \longrightarrow R(-s - a_{1,1}) \longrightarrow I_Y(-a_{1,1}) \oplus R(-s) \longrightarrow I_X \longrightarrow 0.$$

Then, using the mapping cone construction, a free resolution of I_X is given by

$$0 \longrightarrow \begin{array}{c} R(-s-a_{1,1}) \\ \oplus \\ \mathbf{F}_2(-a_{1,1}) \end{array} \longrightarrow \begin{array}{c} R(-s) \\ \oplus \\ \mathbf{F}_1(-a_{1,1}) \end{array} \longrightarrow I_X \longrightarrow 0.$$

Notice that $R(-s-a_{1,1})$ cannot cancel with any of the free summands of $\mathbf{F}_1(-a_{1,1})$, since the image of S in I_Y is not a minimal generator. Moreover, there can be no splitting between $\mathbf{F}_2(-a_{1,1})$ and $R(-s)$. Indeed, if there was such a splitting, $s = a_{1,1}+a_{2,2}+\ldots+a_{t,t}+a_{i,t+1}$ for some i. But $a_{i,1} > 0$ for all $i \leq t-1$ by assumption, so $a_{t,1} = 0$ and $R(-s)$ would split with $R(-a_{1,1}-a_{2,2}-\ldots-a_{t,t}-a_{t,t+1})$. Then there would be a minimal generator of the first syzygy module of the minimal generators of I_Y equal to S in the t-th component, and we can exclude this since S is not a minimal generator of I_Y. This shows that the resolution is minimal. Then the degree matrix of X is M, as required. Moreover, $\alpha(I_C) \leq \alpha(I_D) + a_{1,1} = \alpha(I_X)$, so $\alpha(I_C) = \alpha(I_X)$. $\square$

Remark 3.18 *The curve C that we constructed in Theorem 3.17 is a union of t complete intersections. More precisely, if $a_{k,l} \leq 2$, $a_{k-1,l} > 0$ and $a_{k,l+1} > 0$, then C can be built following the inductive procedure we showed, starting from the submatrix*

$$\begin{pmatrix} a_{k-1,l} & a_{k-1,l+1} & a_{k-1,l+2} \\ a_{k,l} & a_{k,l+1} & a_{k,l+2} \end{pmatrix}.$$

Notice that one can always find such k, l. Moreover, one can assume that $l \leq k-1$, since the entries on the diagonal of M are positive.

If $l \leq k-2$, then C can be taken to be the union

$$C = CI(a_{k-1,l}, a_{k-1,l+1} + a_{k,l+2}) \cup CI(a_{k,l+1}, a_{k,l+2}) \cup$$

$$CI(a_{k-2,l} + a_{k-1,l+1} + a_{k,l+2}, a_{k-2,l-1}) \cup \ldots \cup CI(a_{k-l,2} + \ldots + a_{k,l+2}, a_{k-l,1}) \cup$$

$$CI(a_{k-l-1,1} + \ldots + a_{k,l+2}, a_{k-l-1,l+3}) \cup \ldots \cup CI(a_{1,1} + \ldots + a_{k,k}, a_{1,k+1}) \cup$$

$$CI(a_{1,1} + \ldots + a_{k+1,k+1}, a_{k+1,k+2}) \cup \ldots \cup CI(a_{1,1} + \ldots + a_{t,t}, a_{t,t+1}).$$

If $l = k-1$, then C can be taken to be the union

$$C = CI(a_{k-1,k-1}, a_{k-1,k} + a_{k,k+1}) \cup CI(a_{k,k}, a_{k,k+1}) \cup$$

$$CI(a_{k-2,k-1} + a_{k-1,k} + a_{k,k+1}, a_{k-2,k-2}) \cup \ldots \cup CI(a_{1,2} + \ldots + a_{k,k+1}, a_{1,1}) \cup$$

$$CI(a_{1,1} + \ldots + a_{k+1,k+1}, a_{k+1,k+2}) \cup \ldots \cup CI(a_{1,1} + \ldots + a_{t,t}, a_{t,t+1}).$$

Clearly there are other ways to perform the basic double links other than the examples that we present here. Different sequences of basic double links yield curves that are not projectively isomorphic, since they are unions of complete intersections of different types. Therefore, following the construction of Theorem 3.17, one can produce different curves from the examples that we just gave.

One can easily show by induction that

$$\mathcal{M}_C \cong (L_1, L_2, L_3)/(F_1, F_2, G_1, G_2)(-a)$$

as an S-module, where

$$a = a_{k-2,l-1} + \ldots + a_{k-l,1} + a_{k-l-1,l+3} + \ldots + a_{1,k+1} + a_{k+1,k+2} + \ldots + a_{t,t+1}$$

if $l \leq k-2$ and

$$a = a_{1,1} + \ldots + a_{k-2,k-2} + a_{k+1,k+2} + \ldots + a_{t,t+1}$$

if $l = k-1$.

Here F_1, F_2 *and* G_1, G_2 *are two regular sequences with* F_1, F_2, G_1, G_2 *generic of degrees* $a_{k-1,l}, a_{k-1,l+1} + a_{k,l+2}, a_{k,l+1}, a_{k,l+2}$ *passing through a common point, that is the common zero of the linear forms* L_1, L_2, L_3 *(see also Remark 3.5). In particular,*

$$\alpha(\mathcal{M}_C) = a_{k-2,l-1} + \ldots + a_{k-l,1} + a_{k-l-1,l+3} + \ldots + a_{1,k+1} + a_{k+1,k+2} + \ldots + a_{t,t+1} + 1$$

if $l \leq k-2$ *and*

$$\alpha(\mathcal{M}_C) = a_{1,1} + \ldots + a_{k-2,k-2} + a_{k+1,k+2} + \ldots + a_{t,t+1} + 1$$

if $l = k-1$.

Remark 3.19 *If we don't require connectedness of* C*, we can perform the construction of Theorem 3.17 in such a way that we have a surface* S *containing* C *of degree* $a_{1,1} + \ldots + a_{t,t} + a$*, for each* $a > 0$*.* S *can be taken smooth on the complement of a zero-dimensional subset of* C*, and such that its image in* I_X *is a multiple of a minimal generator of minimal degree* $a_{1,1} + \ldots + a_{t,t}$ *by a form of degree* $a > 0$*.*

Proof: The degree matrix of the general plane section X of C is $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots,t+1}$. We proceed by induction on the number of rows of M.

For $t = 2$, let $C = CI(F,G) \cup CI(H,J)$ be the disjoint union of two generic, smooth, integral complete intersections. $deg\ F = a_{1,1}$, $deg\ G = a_{1,2} + a_{2,3}$, $deg\ H = a_{2,2}$, $deg\ J = a_{2,3}$. Then C is smooth and contained in the surface of equation $T = FH$. T has degree $a_{1,1} + a_{2,2}$, and its image in I_X is a minimal generator. Let S be the union of T with a generic surface U of degree a. The singular locus of T is $F \cap H$, so it is disjoint from C. Let $Sing(S)$ denote the singular locus of S. $Sing(S) \cap C \subseteq U \cap C$, so it is a zero-dimensional subset of C, by generality of U. The image of S in I_X is a multiple of the minimal generator T of minimal degree by the form U of degree $a > 0$.

Proceeding by induction on t, let $C = D \cup C_t$ be a basic double link of D on a surface S_1 of degree $a_{1,1} + \ldots + a_{t,t}$, with a general form of degree $a_{t,t+1}$. By the induction hypothesis applied to D, we can choose S_1 smooth on the support of D, except possibly for a zero-dimensional subset. By generality of our choice of the form of degree $a_{t,t+1}$, we can also assume that the surface individuated by this form does not pass through any of the singular points of S_1 contained in D. Let X, Y be the general plane sections of C, D respectively. We can assume that the image of S_1 in I_Y is a multiple of a minimal generator of minimal degree by a form of degree $a_{t,t} > 0$. The image of S_1 in I_X is a minimal generator, by construction. Let $S = S_1 \cup U$, U a generic surface of degree a. By generality of U, we can assume that U doesn't pass through any of the points of $D \cap C_t$ and that $U \cap C$ is zero-dimensional. $Sing(S) = Sing(S_1) \cup (S_1 \cap U)$, so $Sing(S) \cap C = (Sing(S_1) \cap D) \cup (Sing(S_1) \cap C_t) \cup (S_1 \cap U \cap C)$. $Sing(S_1) \cap D$ is zero-dimensional by assumption, $Sing(S_1) \cap C_t$ is zero-dimensional, since $Sing(S_1) \cap C_t \cap D$ is empty by assumption. $S_1 \cap U \cap C$ is zero-dimensional, since $U \cap C$ is. The image of S in I_X is a multiple of the minimal generator of minimal degree image of S_1 by a form of degree $a > 0$ (the image of U). This is the proof, in the case $a_{t-1,1} \leq 2$. The proof in the other cases (see the proof of Theorem 3.17) are exactly the same: only the degrees of S_1 and U change. □

Remark 3.20 *The space curve C that we constructed in Remark 3.19 is reduced and non-degenerate, non-aCM, and it has two connected components. We can take the complete intersections that constitute C to be smooth, so that C has singularities only at the points of intersections of its irreducible components.*

Remark 3.21 *I_C as constructed in Theorem 3.17 or in Remark 3.19 is minimally generated in degree less than or equal to*

$$a_{k-1,l+1}+2a_{k,l+2}+a_{k-2,l-1}+\ldots+a_{k-l,1}+a_{k-l-1,l+3}+\ldots+a_{1,k+1}+a_{k+1,k+2}+\ldots+a_{t,t+1} =$$

$$a_{1,2} + \ldots + a_{t,t+1} + a_{l-1,1} - a_{l-1,l} + a_{k,l+2}$$

if $l \leq k-2$, and in degree less than or equal to

$$a_{k-1,k} + 2a_{k,k+1} + a_{k-2,k-2} + \ldots + a_{1,1} + a_{k+1,k+2} + \ldots + a_{t,t+1} =$$

$$a_{1,2} + \ldots + a_{t,t+1} + a_{k,1} - a_{k,k-1} + a_{k,k+1}$$

if $l = k-1$. *Notice that the curve* C *actually has a minimal generator in that degree. Here* $a_{1,2} + \ldots + a_{t,t+1}$ *is the highest degree of a minimal generator of the ideal* I_X.

One can easily show it proceeding by induction on t, *and using the short exact sequence*

$$0 \longrightarrow S(-s-t) \longrightarrow I_D(-t) \oplus S(-s) \longrightarrow I_C \longrightarrow 0$$

connecting the ideal of a scheme D *with the ideal of its basic double link* C *on a surface* S *of degree* s, *with a form* F *of degree* t *(see Proposition 2.24). The case of a degree matrix of size* 2×3 *is examined in Example 3.8, and can be used as the basis of the induction.*

Another way to prove the upper bound on the degrees of the minimal generators of C *is the following. It is easy to see that the bound holds in the examples constructed in Remark 3.18. One can then check that the order in which the basic double links are performed does not change the highest degree of a minimal generator of the ideal of the curve.*

We can compute the deficiency module of the curves constructed in Remark 3.19. The reasoning is similar to Remark 3.18.

Remark 3.22 *The curves constructed as in Remark 3.19 have*

$$\mathcal{M}_C \cong S/(F_1, F_2, G_1, G_2)(-a)$$

as an S*-module, where*

$$a = a_{k-2,l-1} + \ldots + a_{k-l,1} + a_{k-l-1,l+3} + \ldots + a_{1,k+1} + a_{k+1,k+2} + \ldots + a_{t,t+1}.$$

Here F_1, F_2, G_1, G_2 *is a regular sequence, with* $F_1, F_2, G_1,$ *and* G_2 *generic forms of degrees* $a_{k-1,l}, a_{k-1,l+1} + a_{k,l+2}, a_{k,l+1},$ *and* $a_{k,l+2}$ *respectively (see also Example 3.8). In particular,*

$$\alpha(\mathcal{M}_C) = a_{k-2,l-1} + \ldots + a_{k-l,1} + a_{k-l-1,l+3} + \ldots + a_{1,k+1} + a_{k+1,k+2} + \ldots + a_{t,t+1}.$$

Notice that, as for all the constructions presented up to this point, the curves that we obtain are almost never arithmetically Buchsbaum.

We now find smooth surfaces that contain the curves constructed in Remark 3.19. They will be used in Chapter 4.

Lemma 3.23 *Let $C \subset \mathbf{P}^3$ be a curve as constructed in Remark 3.19. Assume that the saturated ideal of C is minimally generated in degree smaller than or equal to d. Then there is a smooth surface of degree d containing C.*

Proof: Consider the linear system Δ of surfaces of $\mathbf{P}^3$ of degree d containing C. $C = C_1 \cup C_2 \cup \ldots \cup C_t$ is a reduced union of t complete intersection curves. Let $Sing(C) = \cup_{i<j} C_i \cap C_j$ be the singular locus of C (see Remark 3.20 about how the singular locus of C looks like). The general element of Δ is basepoint-free outside of C, hence smooth outside of C by Bertini's Theorem. Consider now a point $P \in C$. We claim that the general element of Δ is smooth at P. By Corollary 2.10 in [26], it suffices to exhibit two elements of Δ meeting transversally at P. Since C is smooth outside of $Sing(C)$, for each point $P \notin Sing(C)$ we have two minimal generators of I_C, call them F and G, meeting transversally at P. The degree of each is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P.

In order to complete the proof, we need to check that the points of $Sing(C)$ are not fixed singular points for Δ. So it is enough to find a surface for each $P \in Sing(C)$ that contains C and is non-singular at P. Each singular point of C is the intersection of two irreducible components of the curve, $P \in C_i \cap C_j$ for some $1 \leq i < j \leq t$. We can assume, by generality of our choices, that i, j are determined by P, i.e. we can assume that there are exactly two irreducible components of C meeting at P. Without loss of generality, we can then assume that $j = t$ and that $C = D \cup C_t$, where $P \notin Sing(D)$. As we saw in Remark 3.19, we can perform the basic double link in such a way that the surface S_1 of degree $a_{1,1} + \ldots + a_{t,t}$ that we perform the link on is smooth on D outside of a zero-dimensional subscheme. Moreover, we can assume that the singular locus of S_1 does not contain any of the points of $D \cap C_t$. In particular, S_1 is smooth at P and contains C. Notice that $d \geq a_{1,1} + \ldots + a_{t,t} = \alpha(I_C)$. Add to S_1 a generic surface of degree $d - a_{1,1} - \ldots - a_{t,t}$ to obtain a surface containing C and smooth at P. Notice that a generic surface of degree d containing C is also integral, since it is smooth and connected. □

Remark 3.24 *If we start the construction of Theorem 3.17 from one of the curves constructed in Theorem 3.12, we obtain a curve C whose saturated ideal I_C is generated in degree smaller than or equal to*

$$a_{k-1,l+1}+a_{k,l+2}+1+a_{k-2,l-1}+\ldots+a_{k-l,1}+a_{k-l-1,l+3}+\ldots+a_{1,k+1}+a_{k+1,k+2}+\ldots+a_{t,t+1} =$$

$$a_{1,2} + \ldots + a_{t,t+1} + a_{l-1,1} - a_{l-1,l} + 1$$

if $l \leq k-2$, and in degree less than or equal to

$$a_{k-1,k} + a_{k,k+1} + 1 + a_{k-2,k-2} + \ldots + a_{1,1} + a_{k+1,k+2} + \ldots + a_{t,t+1} =$$

$$a_{1,2} + \ldots + a_{t,t+1} + a_{k,1} - a_{k,k-1} + 1$$

if $l = k-1$.

The curve C is a union of t complete intersections. The same considerations as in Remark 3.18 hold. Using Remark 3.13 and the Hartshorne-Schenzel Theorem, one can compute explicitely the initial and final degrees of the deficiency module of the curve, in terms of the entries of the degree matrix M of its general plane section.

Remark 3.25 *Using Lemma 3.16 and following the proof of Remark 3.19, one can see that the curve C constructed as in Remark 3.24 is contained in a smooth surface of degree d, for all $d \geq a_{1,2} + a_{2,3} + 1$.*

One could also ask whether it is possible to give a sufficient condition for Cohen-Macaulayness of $C \subset \mathbf{P}^3$, in terms of the entries of the h-vector of its general plane section $X \subset \mathbf{P}^2$. It is easy to see that this is not possible, as the next proposition shows.

Proposition 3.26 *Let $h(z) = 1 + h_1 z + \ldots + h_s z^s$, $h_s \neq 0$ be the h-vector of some zero-dimensional scheme in $\mathbf{P}^2$. Then there exists a non-aCM, reduced curve $C \subset \mathbf{P}^3$, whose general plane section $X \subset \mathbf{P}^2$ has h-vector $h(z)$. The curve C can be taken connected, unless $h(z) = 1 + 2z$.*

Proof: To any h-vector $h(z)$, we can uniquely associate a degree matrix M with no entries equal to 0, such that if $X \subset \mathbf{P}^2$ is a zero-dimensional scheme with degree matrix M, then the h-vector of X is $h(z)$. If M has one entry less than or equal to 2 and is not a 2×3 matrix with all its entries equal to 1, by Theorem 3.17 we can find a non-aCM, reduced, connected curve $C \subset \mathbf{P}^3$, whose general plane section $X \subset \mathbf{P}^2$ has degree matrix M, hence h-vector $h(z)$.

If M is the degree matrix of size 2×3 with all entries equal to 1, i.e. if the h-vector is $h(z) = 1 + 2z$, let C be the disjoint union of a reduced plane conic and a line.

If $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots,t+1}$ has $a_{t,1} \geq 3$, let $N = (b_{i,j})_{i=1,\ldots,t+1;\, j=1,\ldots,t+2}$ be the degree matrix with entries $b_{i,j} = a_{i,j-1}$ for $i = 1, \ldots, t$, $j = 2, \ldots, t+2$, $b_{t+1,1} = 0$, $b_{t+1,2} = 2$. N is determined by these entries, under the assumption that it is homogeneous. $b_{i,j} > 0$ for $(i,j) \neq (t+1, 1)$, so N is a degree matrix. Moreover, the h-vector of a zero-dimensional scheme that has degree matrix N is again $h(z)$. Then, by Theorem 3.17, there exists a non-aCM, reduced, connected curve $C \subset \mathbf{P}^3$, whose general plane section $X \subset \mathbf{P}^2$ has degree matrix N, hence h-vector $h(z)$. $\square$

Let us now look at the general case of a curve $C \subset \mathbf{P}^{n+1}$, whose general hyperplane section is the zero-dimensional scheme $X \subset \mathbf{P}^n$. With the notation of Definition 2.5, let $M = (a_{ij})$ be the lifting matrix of X.

We have a sufficient condition for the Cohen-Macaulayness of C, analogous to the case $n = 2$. The result easily follows from Theorem 2.12.

Corollary 3.27 *Assume that* $char(k) = 0$. *Let* $C \subset \mathbf{P}^{n+1}$ *be a curve, whose general hyperplane section* $X \subset \mathbf{P}^n$ *has lifting matrix* $M = (a_{ij})$. *If* $a_{t,1} \geq n + 1$, *then* C *is arithmetically Cohen-Macaulay.*

Proof: With the notation of Theorem 2.12, if C is not arithmetically Cohen-Macaulay, we have

$$b \geq m_t - n \geq d_1 + 1.$$

Then all the minimal generators of I_X lift to I_C, so C is arithmetically Cohen-Macaulay. This contradicts our assumptions. $\square$

CHAPTER 4

INTEGRAL AND SMOOTH CURVES

Throughout the chapter, we work over an algebraically closed field k of characteristic 0. We concentrate on integral (reduced and irreducible), locally Cohen-Macaulay, equidimensional, non-degenerate curves $C \subset \mathbf{P}^3$. Under the assumption of integrality of the curve, we wish to investigate whether one can give a condition on the degree matrix of X that forces C to be arithmetically Cohen-Macaulay, and is weaker than the sufficient condition found in Corollary 2.15.

In this chapter, we completely characterize the graded Betti numbers that occur for points in Uniform Position in $\mathbf{P}^2$, that arise as the general plane section of a reduced and irreducible (or just as well smooth and connected) non arithmetically Cohen-Macaulay curve of $\mathbf{P}^3$. First of all, in Proposition 4.4 and Proposition 4.6 we exhibit two families of degree matrices with at least one entry smaller than 3 such that any integral curve whose general plane section has one of those degree matrices is arithmetically Cohen-Macaulay. In other words, we find hypotheses on the degree matrix of the general plane section of a curve C that, together with reducedness and irreducibility of the curve, force C to be arithmetically Cohen-Macaulay. In Corollary 4.11, we deduce sufficient conditions for Cohen-Macaulayness of an integral curve in terms of the h-vector of its general plane section. Then we show that, with the exception of the two families mentioned above, each degree matrix with positive subdiagonal that corresponds to a collection of points that is a general plane section of a non arithmetically Cohen-Macaulay curve also corresponds to points that are a general plane section of a non arithmetically Cohen-Macaulay, reduced and irreducible curve. Notice that the degree matrix of a general plane section of an integral curve must have positive entries on the subdiagonal, as shown in Theorem 4.1 below. For each degree matrix that does not belong to any of the two families mentioned above, we construct a smooth, connected, non arithmetically Cohen-Macaulay curve whose general plane section has that degree matrix (see Theorem 4.14 and Theorem 4.15).

The main tools that we use in our constructions come from linkage theory. In particular, we perform basic double links and direct links on smooth surfaces. Bertini's Theorem is used to show that the curves that we construct are smooth.

The problem of characterizing the degree matrices that occur for the general plane section of an integral, non-aCM curve of $\mathbf{P}^3$ is connected to many other interesting questions. In fact, characterizing thosee degree matrices is the same as characterizing the possible graded Betti numbers for a collection of points with the Uniform Position Property, that arise as general plane section of non-aCM integral curves. There have been many attempts to characterise the h-vectors and the graded Betti numbers of zero-dimensional schemes with the Uniform Position Property. E. Ballico, L. Chiantini, S. Diaz, D. Eisenbud, A. V. Geramita, M. Green, J. Harris, M. Kreuzer, R. Maggioni, J. Migliore, U. Nagel, F. Orecchia, A. Ragusa, L. Robbiano, T. Sauer, G. Valla, and K. Yaganawa are among the people that worked on these problems (see for example [1], [8], [13], [15], [16], [20], [23], [28], [37], [41], [53]). J. Harris has shown in [28] that the h-vectors of zero-dimensional schemes of codimension 2 in Uniform Position are of decreasing type. The question is much harder in codimension 3 or higher. However, one may hope to characterize the h-vectors that occur for points in Uniform Position that have additional properties.

As we mentioned in Chapter 2, there is a characterization of the matrices of integers that occur as degree matrices of a zero-dimensional scheme in $\mathbf{P}^2$ that is the plane section of an integral space curve by a plane that meets it properly. We call such a matrix an *integral degree matrix.* Integral degree matrices have been characterized in [8], [53], [31] and [21]. In our language, they prove the following result:

Theorem 4.1 *Let $M = (a_{i,j})$ be a homogeneous matrix of integers of size $t \times (t+1)$. Then M is an integral degree matrix if and only if $a_{i,i-1} > 0$ for $i = 2, \dots, t$.*

We start our investigation from an example.

Example 4.2 *Consider the following degree matrix*

$$M = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix}.$$

The matrix M corresponds to some zero-dimensional scheme X of degree 15. The minimal free resolution of the saturated ideal of X is

$$0 \longrightarrow R(-7)^2 \longrightarrow R(-6) \oplus R(-4)^2 \longrightarrow I_X \longrightarrow 0.$$

The construction of Theorem 3.3 produces an example of a reduced, connected space curve that is non Cohen-Macaulay, and such that its general plane section has degree matrix M.

So one can let X be a general plane section of the curve $A = CI(1,6) \cup CI(3,3)$, where the two complete intersections are generic, through a common point. Following the construction of Remark 3.6, one can also let X be a general plane section of $B = CI(1,6) \cup CI(3,3)$, where the complete intersections are generic, hence disjoint.

Assume now that $C \subset \mathbf{P}^3$ is a reduced, irreducible curve whose general plane section X has degree matrix M. By Theorem 2.12, the minimal degree of an element *of I_X that is not the image of some element of I_C under the standard projection map is $b \geq 7-2=5$. Then the two minimal generators of I_X of degree 4 are the images of two minimal generators F, G of I_C. Moreover, F and G are both irreducible forms, since C is integral. Hence they form a regular sequence in $S = k[x_0, x_1, x_2, x_3]$. Let E be the curve with saturated ideal $I_E = (F, G) \subset S$. E is a complete intersection and it contains C, hence C is linked via E to a curve D. From Corollary 2.23, it follows that D has degree $deg(D) = deg(E) - deg(C) = 1$. Then D is a line. In particular, D is aCM. Since the property of being aCM is an invariant of the CI-linkage class of a scheme, C is aCM as well.*

The example we just saw inspires the following observation.

Lemma 4.3 *Let $C \subset \mathbf{P}^3$ be a curve whose general plane section X has degree matrix $M = (a_{i,j})$ of size $t \times (t+1)$. Assume that $a_{t,j} \geq 3$. Then the $t+2-j$ minimal generators of lowest degrees of I_X are images of the $t+2-j$ minimal generator of lowest degrees of I_C.*

Proof: This follows directly from Theorem 2.12. Let $d_j, \ldots, d_{t+1}$ be the degrees of the $t+2-j$ minimal generators of lowest degrees of I_X, $d_{t+1} \leq \ldots \leq d_j$ (here we follow the notation of Theorem 2.12; notice that some of the degrees may be repeated). The lowest shift in the last free module of the minimal free resolution of I_Z is $d_{t+1} + a_{t,t+1} = d_j + a_{t,j}$.

If $a_{t,j} \geq 3$, by Theorem 2.12 it follows that the minimum degree of a polynomial in I_X that is not the image of an element of I_C under the standard projection map is $b \geq d_j + a_{t,j} - 2 > d_j$. Therefore the $t+2-j$ minimal generators of lowest degrees of I_X are images of minimal generators of I_C. □

We now state the first condition that forces an integral curve $C \subset \mathbf{P}^3$ to be arithmetically Cohen-Macaulay. The condition is given in terms of the entries of the degree matrix of the general plane section of C. The proof is a generalization of the argument of Example 4.2.

Proposition 4.4 *Let $C \subset \mathbf{P}^3$ be a reduced, irreducible curve, whose general plane section X has degree matrix $M = (a_{i,j})$ of size 2×3. Assume that $a_{2,2} \geq 3$ and that $a_{1,1}, a_{2,1} \neq 2$. Then C is aCM.*

Proof: Since $a_{2,2} \geq 3$, it follows from Lemma 4.3 that the two generators of minimal degrees of I_X lift to two minimal generators of I_C. Call them F and G. Following the strategy of Example 4.2, we notice that F, G form a regular sequence in S, since they are irreducible polynomials. Let E be the complete intersection with homogeneous saturated ideal $I_E = (F, G) \subset S$. Let D be the residual curve to C in E. Taking general plane sections the link is preserved, so that the general plane section Y of D has degree matrix $(a_{1,1}, a_{2,1})$. By the result of Strano mentioned above (Theorem 6, [53]), D is aCM. Then C is aCM as well, since the property of being aCM is an invariant of the CI-linkage class of a scheme (see Theorem 2.19). □

In what follows, we make extensive use of Bertini's Theorem. For our convenience, we recall it here in the form we need it. See [29], Corollary 3.10.9 and Remark 3.10.10 for a proof.

Theorem 4.5 *(Bertini) Let S be an integral (respectively, smooth) projective scheme of dimension at least 2 over an algebraically closed field of characteristic 0. Let δ be a basepoint-free linear system on S. Then a generic element of δ is an integral (respectively, smooth) subscheme of S.*

Using Bertini's Theorem, we can find another family of degree matrices M such that every integral space curve C whose general plane section X has degree matrix M is arithmetically Cohen-Macaulay.

Proposition 4.6 *Let $C \subset \mathbf{P}^3$ be a reduced, irreducible curve whose general plane section X has degree matrix $M = (a_{i,j})$ of size 2×3. Assume that $a_{1,1}, a_{2,3} \geq 3$ and that $a_{2,1}, a_{2,2} \neq 2$. Then C is arithmetically Cohen-Macaulay.*

Proof: Since $a_{2,3} \geq 3$, by Lemma 4.3 the generator of minimal degree of I_X lifts to a minimal generator of I_C. Then, I_X and I_C have the same initial degree $\alpha = a_{1,1}+a_{2,2}$. Let T be a surface of degree α containing C. T is integral, since C is integral.

Consider the linear system Σ_d of the curves cut out on T outside of C by the surfaces of degree d containing C. For $d \gg 0$, namely for d greater than or equal to the largest degree of a minimal generator of I_C, the linear system Σ_d is basepoint-free. Let D be a generic element of Σ_d. D is an integral curve by Bertini's Theorem. Notice that D is CI-linked to C by construction. Let Y be the general plane section of D. Then Y is CI-linked to the general plane section X of C via a $CI(\alpha, d)$. The degree matrix of X is M by assumption. Hence a minimal free resolution for I_Y is (see Proposition 2.21)

$$0 \longrightarrow \begin{array}{c} R(-d-a_{1,1}+a_{1,3}) \\ \oplus \\ R(-d-a_{2,2}+a_{2,3}) \end{array} \longrightarrow \begin{array}{c} R(-a_{1,1}-a_{2,2}) \oplus R(-d+a_{1,3}) \\ \oplus \\ R(-d+a_{2,3}) \end{array} \longrightarrow I_Y \longrightarrow 0$$

since the form of degree α is a minimal generator of I_X, while the form of degree d isn't. Then the degree matrix of I_Y is

$$N = \begin{pmatrix} a_{2,2} & a_{1,2} & d-a_{1,1}-a_{2,3} \\ a_{2,1} & a_{1,1} & d-a_{2,2}-a_{1,3} \end{pmatrix}.$$

Since we are taking $d \gg 0$, we can assume that $d - a_{2,2} - a_{1,3} \geq a_{1,1}$. Notice that this again guarantees minimality of the resolution of I_Y above. By assumption we have $a_{1,1} \geq 3$ and $a_{2,1}, a_{2,2} \neq 2$, so we can apply Proposition 4.4 to conclude that D is arithmetically Cohen-Macaulay. Then C is arithmetically Cohen-Macaulay as well.

□

Remark 4.7 *The result of Proposition 4.6 is relevant when $a_{2,1} \leq 2$. In fact, if $a_{2,1} \geq 3$ Cohen-Macaulayness of the curve already follows from Corollary 2.15. Then assuming $a_{2,1} \neq 2$ is equivalent to $a_{2,1} = 1$, since $a_{2,1}$ is positive.*

From Proposition 4.4 and Proposition 4.6, we can derive some conditions on the h-vector of X that force C to be arithmetically Cohen-Macaulay.

For what follows, we derive a formula for the h-vector of a zero-dimensional scheme $X \subset \mathbf{P}^2$ in terms of the entries of the degree matrix of X.

Lemma 4.8 *Let $X \subset \mathbf{P}^n$ be an arithmetically Cohen-Macaulay scheme of codimension 2. Let $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots,t+1}$ be its degree matrix. Then the h-vector of X is*

$$h(z) = \sum_{i=1}^{t} z^{a_{1,1}+\ldots+a_{i-1,i-1}}(1+z+\ldots+z^{a_{i,i}-1})(1+z+\ldots+z^{a_{i+1,i+1}+\ldots+a_{t,t}+a_{i,t+1}}).$$

Proof: The minimal free resolution of X is

$$0 \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 \longrightarrow I_X \longrightarrow 0$$

where

$$\mathbf{F}_2 = \bigoplus_{i=1}^{t} R(-a_{1,1} - \ldots - a_{t,t} - a_{i,t+1}),$$

$$\mathbf{F}_1 = \bigoplus_{j=1}^{t} R(-a_{1,1} - \ldots - a_{t,t} + a_{j,j} - a_{j,t+1}) \oplus R(-a_{1,1} - \ldots - a_{t,t}),$$

and I_X is an ideal in $R = k[x_0, \ldots, x_n]$. Then the h-vector of X is

$$h(z) = \frac{1 - \sum_{i=1}^{t} z^{a_{1,1}+\ldots+a_{t,t}-a_{i,i}+a_{i,t+1}} - z^{a_{1,1}+\ldots+a_{t,t}} + \sum_{i=1}^{t} z^{a_{1,1}+\ldots+a_{t,t}+a_{i,t+1}}}{(1-z)^2}.$$

By means of computations, we get

$$1 - z^{a_{1,1}+\ldots+a_{t,t}} + \sum_{i=1}^{t}(z^{a_{1,1}+\ldots+a_{t,t}+a_{i,t+1}} - z^{a_{1,1}+\ldots+a_{t,t}-a_{i,i}+a_{i,t+1}}) =$$

$$= (1-z)[(1+z+\ldots+z^{a_{1,1}+\ldots+a_{t,t}-1}) - \sum_{i=1}^{t} z^{a_{1,1}+\ldots+a_{t,t}-a_{i,i}+a_{i,t+1}}(1+z+\ldots+z^{a_{i,i}-1})] =$$

$$= (1-z)^2 \sum_{i=1}^{t} z^{a_{1,1}+\ldots+a_{i-1,i-1}}(1+z+\ldots+z^{a_{i,i}-1})(1+z+\ldots+z^{a_{i+1,i+1}+\ldots+a_{t,t}+a_{i,t+1}}).$$

□

Remark 4.9 *The degree matrix of a scheme X as in Lemma 4.8 determines the h-vector of X, while the h-vector of X determines the degree matrix only under the hypothesis that all the entries of the degree matrix of X are positive.*

Remark 4.10 *From Lemma 4.8, we can easily see that the h-vector of a general plane section X of an integral curve is of decreasing type.*

In fact, the h-vector of X can be formally written as a sum of certain shifts of the h-vectors $h_i(z)$ of t complete intersections of type $(a_{i,i}, a_{i+1,i+1}+\ldots+a_{t,t}+a_{i,t+1}+1)$, for $i = 1, \ldots, t$. The h-vector $h_i(z)$ has increasing coefficients in degrees $1, \ldots, a_{i,i} - 1$. The coefficients are then constant until degree $a_{i+1,i+1} + \ldots + a_{t,t} + a_{i,t+1}$, and then they are decreasing. Looking at $k_i(z) = z^{a_{1,1}+\ldots+a_{i-1,i-1}} h_i(z)$, we have that the coefficients start decreasing in degree $f_i = a_{1,1}+\ldots+a_{i-1,i-1}+a_{i+1,i+1}+\ldots+a_{t,t}+a_{i,t+1}$ and the last nonzero coefficient appears in degree $e_i = f_i + a_{i,i} - 1$.

Under the assumption that the degree matrix M is integral, we have $a_{i+1,i} > 0$ for all i, that gives $e_{i+1} - f_i = f_{i+1} + a_{i+1,i+1} - 1 - f_i = a_{i+1,i} - 1 \geq 0$, so each $k_{i+1}(z)$ does not end on the flat (constant) part of $k_i(z)$.

Therefore, the h-vector of X is of decreasing type. Moreover, $h_j - h_{j+1} \geq 2$ for all j such that $f_i \leq j \leq e_{i+1}$ for some i, and only for those j's.

We are now ready to derive some sufficient conditions for an integral curve C to be arithmetically Cohen-Macaulay, in terms of the h-vector of its general plane section X.

Corollary 4.11 *Let $C \subset \mathbf{P}^3$ be a reduced, irreducible curve, whose general plane section X has h-vector $h = 1 + h_1 z + \ldots + h_s z^s$, $h_s \neq 0$. Let*

$$u = \max\{i \mid h_i = i+1\}, \quad v = \max\{i \mid h_i = u+1\},$$

$$w = \min\{i \mid v \leq i \leq s-1, \ h_i - h_{i+1} \neq 1\}.$$

If $\{i \mid v \leq i \leq s-1, \ h_i - h_{i+1} \neq 1\} = \{w\}$ and either

$$s = u + v - 1, \quad u + v - w \neq 2, \quad \textit{and } w - v \geq 2$$

or

$$s = u + v - 1, \;\; v \geq 6, \;\; w - u \geq 3, \;\; \textit{and} \;\; w \neq v + 1$$

then C is arithmetically Cohen-Macaulay.

Proof: Let $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots,t+1}$ be the degree matrix of X. For $i = 1, \ldots, t$ let

$$h_i(z) = (1 + z + \ldots + z^{a_{i,i}-1})(1 + z + \ldots + z^{a_{i+1,i+1}+\ldots+a_{t,t}+a_{i,t+1}}) \qquad (4.1)$$

be the h-vector of a complete intersection of type $(a_{i,i}, a_{i+1,i+1} + \ldots + a_{t,t} + a_{i,t+1} + 1)$. By Lemma 4.8, we can think of $h(z)$ as the sum of t h-vectors of Complete Intersections:

$$h(z) = \sum_{i=1}^{t} z^{a_{1,1}+\ldots+a_{i-1,i-1}}(1 + z + \ldots + z^{a_{i,i}-1}h_i(z).$$

By assumption,

$$\{i \mid v \leq i \leq s-1, \; h_i - h_{i+1} \neq 1\} = \{w\},$$

so $h_i - h_{i+1} = 1$ for $v \leq i \leq s-1$, $i \neq w$, so the h-vector has only one jump of more than 1, once it starts decreasing. Therefore, it has to be the sum of only two h-vectors h_i, that is $t = 2$. The degree matrix of X has then size 2×3. X is the general plane section of an integral curve C (so it has UPP, see [28] about the general plane section of an integral curve and its h-vector). Then M is integral, in particular $a_{2,1} > 0$. All the entries of M are positive, so the h-vector of X determines the degree matrix. From equation (4.1), we can compute

$$u = a_{1,1} + a_{2,2} - 1, \qquad v = a_{1,1} + a_{2,3},$$

$$w = a_{1,1} + a_{2,2} + a_{2,3} - a_{2,1} \quad \text{and} \quad s = 2a_{1,1} + a_{2,2} + a_{2,3} - 2. \qquad (4.2)$$

The assumption that $\{i \mid v \leq i \leq s-1, \; h_i - h_{i+1} \neq 1\} = \{w\}$ forces $a_{2,1} = 1$: in fact, $h_i = h_{i+1} - 2$ for $w = a_{1,1} + a_{2,2} + a_{2,3} - a_{2,1} \leq i \leq a_{1,1} + a_{2,2} + a_{2,3} - 1$. Solving the equations (4.2) gives

$$s = u + v - 1$$

and

$$a_{1,1} = u+v-w, \quad a_{2,2} = w-v+1, \quad a_{2,3} = w-u,$$

so, in terms of u, v, w, the degree matrix of X has the following form

$$M = \begin{pmatrix} u+v-w & u & v-1 \\ 1 & w-v+1 & w-u \end{pmatrix}.$$

By Proposition 4.4, C is arithmetically Cohen-Macaulay if $u+v-w \neq 2$ and $w-v \geq 2$. By Proposition 4.6, C is arithmetically Cohen-Macaulay if $u+v-w \geq 3$, $w-u \geq 3$ and $w-v+1 \neq 2$, or equivalently if $w-u \geq 3$, $v \geq 6$ and $w \neq v+1$. $\square$

For any degree matrix that has at least one entry smaller than 3 and does not fall in one of the two classes of examples of Proposition 4.4 and Proposition 4.6, we can produce a smooth, connected, non arithmetically Cohen-Macaulay curve whose general plane section has degree matrix M. In particular, we can construct such a curve for any degree matrix of size $t \times (t+1)$, for $t \geq 3$.

The following lemmas will be needed for the construction of a smooth, connected curve whose general plane section has a prescribed degree matrix.

Lemma 4.12 *Let $C \subset \mathbf{P}^3$ be a smooth space curve, whose ideal is minimally generated in degree smaller than or equal to d. Then there is a smooth surface of degree d containing C.*

Proof: Consider the linear system Δ of surfaces of $\mathbf{P}^3$ of degree d, containing C. The general element of Δ is basepoint-free outside of C, hence smooth outside of C by Bertini's Theorem. Consider now a point $P \in C$. By Corollary 2.10 in [26], it's enough to exhibit two elements of Δ meeting transversally at P. Since C is smooth, for each point we have two minimal generators of I_C, say F and G, meeting transversally at P. The degree of each of them is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P. $\square$

Lemma 4.13 *Let C be a curve, lying on an integral surface $S \subset \mathbf{P}^3$. Let M be the degree matrix of the general plane section of C, and assume that all the entries of M are different from zero. Let D be a generic element of the linear system $|C|$. Then the degree matrix of the general plane section of D is M.*

Proof: Let $X, Y \subset \mathbf{P}^2$ be the general plane sections of C, D respectively. We have $\mathcal{I}_X \cong \mathcal{I}_Y$ as $\mathcal{K}$-modules, where $\mathcal{K}$ the sheaf of total quotients of the structure sheaf of $S \cap H$ for a general plane $H = \mathbf{P}^2$. X and Y have the same Hilbert function. In fact, $H^0(\mathcal{I}_X(m))$ and $H^0(\mathcal{I}_Y(m))$ have the same dimension as $H^0(\mathcal{K})$-vector spaces for all m. Therefore they have the same dimension as k-vector spaces. Notice that $H^0(\mathcal{K})$ is a field, since $S \cap H$ is an integral curve by Bertini's Theorem. Consider the linear system $|X|$ of effective divisors on $S \cap H$ that are linearly equivalent to X. Then Y is a generic element of $|X|$. Since the degree matrix of X has no entries equal to zero, neither does the degree matrix of Y, by upper-semicontinuity. Then $X, Y \subset \mathbf{P}^2$ have the same Hilbert series and both of their degree matrices have only nonzero entries. Then they have the same degree matrix. $\square$

We are now ready to construct a smooth, connected, non-aCM curve, whose general plane section has a prescribed degree matrix. We can perform the construction for each integral degree matrix such that at least one of the entries is smaller than 3, and it does not fall in the classes of examples covered by Proposition 4.4 and Proposition 4.6. We exclude from our analysis the degree matrix of size 2×3 with all the entries equal to 1. In fact, as we saw in Example 3.1, any reduced and connected curve whose general plane section has that degree matrix is arithmetically Cohen-Macaulay.

Let us start with an analysis of the degree matrices of size 2×3. Notice that for 2×3 matrices being integral is equivalent to having positive entries, since $a_{2,1} > 0$.

Theorem 4.14 *Let*

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$

be a degree matrix with positive entries, such that $a_{2,1} \leq 2$. Suppose that the entries of M are not all equal to 1, and that they do not satisfy the hypothesis of either Proposition 4.4, or Proposition 4.6. Then there exists a connected, smooth, non arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$, whose general plane section has degree matrix M.

Proof: We perform different constructions, depending on the entries of the degree matrix M.

Case 1. Assume that $a_{2,1} = 2$.

In this case

$$M = \begin{pmatrix} a_{1,1} & a_{1,1} + a_{2,2} - 2 & a_{1,1} + a_{2,3} - 2 \\ 2 & a_{2,2} & a_{2,3} \end{pmatrix}.$$

Let D be a general rational smooth curve of degree $2a_{1,1}$, lying on a smooth quadric surface. Taking D as in Remark 2.11, we may assume that the saturated ideal I_D is generated in degree less than or equal to $a_{1,1} + 2$. The general plane section of D is a complete intersection of type $(2, a_{1,1})$. Let F be the equation of a smooth surface of degree $a_{1,1} + a_{2,2}$ containing D. Such an F exists, by Lemma 4.12. Consider the linear system of curves cut out on F outside of D by surfaces of degree $a_{1,1} + a_{2,3}$ containing D. The linear system is basepoint-free, since $a_{1,1} + a_{2,3} \geq a_{1,1} + 2$, that is the highest degree of a minimal generator of the ideal of D. By Bertini's Theorem, the general element C is a smooth, connected curve (since S is integral, C is integral and smooth). By construction, C is linked to D via a $CI(a_{1,1} + a_{2,2}, a_{1,1} + a_{2,3})$. Then by Proposition 2.21, the general plane section X of C has a free resolution

$$0 \to \begin{array}{c} R(-a_{1,1} - a_{2,2} - a_{2,3}) \\ \oplus \\ R(-2a_{1,1} - a_{2,2} - a_{2,3} + 2) \end{array} \to \begin{array}{c} R(-a_{1,1} - a_{2,2} - a_{2,3} + 2) \\ \oplus \\ R(-a_{1,1} - a_{2,2}) \oplus R(-a_{1,1} - a_{2,3}) \end{array} \to I_X \to 0.$$

No cancellation can occur, since all entries of M are positive. Hence the free resolution is minimal. So the general plane section X of C has degree matrix M.

Case 2. Assume that $a_{2,1} = 1$ and $a_{2,2} = 2$.

The degree matrix M is of the form

$$M = \begin{pmatrix} a_{1,1} & a_{1,1}+1 & a_{1,1}+a_{2,3}-1 \\ 1 & 2 & a_{2,3} \end{pmatrix}.$$

Let D be the union of two skew lines, and perform a basic double link using generic polynomials $F \in I_D$ and $G \in S = k[x_0, x_1, x_2, x_3]$, of degrees $a_{1,1}+2$ and $a_{1,1}+a_{2,3}-1$, respectively. We obtain a curve $C = D \cup (F \cap G)$ whose general plane section has degree matrix M. In fact, we have the short exact sequence (see [43], Theorem 3.2.3 and Remark 3.2.4 b)

$$0 \longrightarrow R(-2a_{1,1}-a_{2,3}-1) \longrightarrow I_{D\cap H}(-a_{1,1}-a_{2,3}+1) \oplus R(-a_{1,1}-2) \longrightarrow I_{C\cap H} \longrightarrow 0$$

for H a general plane in $\mathbf{P}^2$, $R = k[x_0, x_1, x_2] \cong S/(H)$.

The surface defined by F is smooth by genericity of our choice, and the linear system $|C|$ of curves on F that are linearly equivalent to C is basepoint-free. In fact, the linear system $|D|$ on F is itself basepoint-free: let P be a point of D and let U be a generic surface of degree d, containing D. For $d \gg 0$, U is smooth and meets F transversally. Let $U \cap F = D \cup D'$. By genericity, we can assume that $P \notin D'$. Let T be the equation of a generic surface of the same degree d, containing the curve D'. $F \cap T = D' \cup E'$. By the genericity assumption the surface T, hence the curve E', does not pass through P and the divisor $D-(U\cap F)+(T\cap F) = D-D-D'+D'+E' = E'$ is linearly equivalent to D. Hence, $|D|$ is basepoint-free.

By Bertini's Theorem, $|C|$ contains a smooth, connected, non-aCM curve, whose general plane section has degree matrix M by Lemma 4.13.

Case 3. Assume that $a_{2,1} = 1$ and $a_{1,1} = 2$.

In this case, the degree matrix is

$$M = \begin{pmatrix} 2 & a_{1,2} & a_{1,3} \\ 1 & a_{1,2}-1 & a_{1,3}-1 \end{pmatrix}.$$

Let D be two skew lines. Its general plane section consists of two distinct points, hence it has degree matrix $(1, 2)$. Let U be a smooth surface of degree $a_{1,2}+1 \geq 3$

containing D. Let C be the general element of the linear system cut out on U, outside of D, by the surfaces of degree $a_{1,3}+1 \geq 3$. The linear system is basepoint-free outside of D, since the ideal I_D is generated entirely in degree 2. The general element of the linear system links D to the curve C, that is smooth and connected by Bertini's Theorem. Moreover, C is not arithmetically Cohen-Macaulay, since D is not.

The general plane sections X, Y of C, D are CI-linked via a complete intersection of type $(a_{1,2}+1, a_{1,3}+1)$. By Proposition 2.21, we have the following free resolution for the ideal of X

$$0 \longrightarrow \begin{array}{c} R(-a_{1,2}-a_{1,3}) \\ \oplus \\ R(-a_{1,2}-a_{1,3}-1) \end{array} \longrightarrow \begin{array}{c} R(-a_{1,2}-1) \oplus R(-a_{1,3}-1) \\ \oplus \\ R(-a_{1,2}-a_{1,3}+1) \end{array} \longrightarrow I_X \longrightarrow 0.$$

No cancellation can occur in the free resolution of X, since none of the entries of M is zero. So the degree matrix of the general plane section X of C is M.

Case 4. Assume that $a_{2,1}=1$ and $a_{1,1}=1$.

By Proposition 4.4 we assume $a_{2,2} \leq 2$. Hence we can assume $a_{2,2}=1$, since the situation when $a_{2,2}=2$ is treated in Case 2. The degree matrix is then of the form

$$M = \begin{pmatrix} 1 & 1 & a \\ 1 & 1 & a \end{pmatrix}$$

for some $a \geq 2$. For $a=1$, assuming C integral or even C reduced and connected forces C to be aCM (see Example 3.1). If $a=2$, we can let C be a general smooth rational curve of degree 5. Its general plane section consists of 5 generic points in $\mathbf{P}^2$, as we showed in Example 3.2. Hence it has degree matrix M.

For any $a \geq 2$, let D consist of $2a-1$ skew lines lying on a smooth quadric surface Q. The general plane section Y of D has degree matrix

$$N = \begin{pmatrix} 1 & 1 & a-1 \\ 1 & 1 & a-1 \end{pmatrix}$$

and I_D is minimally generated in degrees $2, a$. Let E be the complete intersection whose saturated ideal is $I_E = (Q, F)$. Here F is the equation of a generic surface of degree $2a$ containing D. Let F vary among all the surfaces of degreee $2a$ containing D and consider the linear system of curves that are residual to D in the complete intersection E. The linear system is basepoint-free, so that Bertini's Theorem applies. So the residual C to D in E, for a generic F, is smooth and connected. C is non-aCM, since it is CI-linked to D via E.

Applying Proposition 2.21 to the general sections X, Y of C, D, we get the following free resolution for X

$$0 \longrightarrow R(-a-2)^2 \longrightarrow R(-a-1)^2 \oplus R(-2) \longrightarrow I_X \longrightarrow 0.$$

No cancellation can occur, therefore the free resolution is minimal. In conclusion, the curve C is smooth, connected, non-aCM and its general plane section has degree matrix M.

Case 5. Assume that $a_{2,1} = 1$ and $a_{1,1} \geq 3$.

We may assume that $a_{2,2} = 1$, since $a_{2,2} = 2$ is treated in Case 2. By Proposition 4.6 we may also assume that $a_{2,3} \leq 2$.

The proof in the case $a_{2,3} = 2$ is analogous to the proof of Case 2: start with D equal to two skew lines and perform a basic double link using generic forms $F \in I_D$ and $G \in S$, of degrees $a_{1,1}+1, a_{1,2}$ respectively. We obtain a curve $C = D \cup (F \cap G)$, whose general plane section has degree matrix M. We have the short exact sequence (see [43], Theorem 3.2.3 and Remark 3.2.4 b)

$$0 \longrightarrow R(-a_{1,1} - a_{1,2} - 1) \longrightarrow I_{D\cap H}(-a_{1,2}) \oplus R(-a_{1,1} - 1) \longrightarrow I_{C\cap H} \longrightarrow 0$$

for H a general plane in $\mathbf{P}^2$, $R = k[x_0, x_1, x_2] \cong S/(H)$. The surface defined by F is smooth by genericity of our choice, and the linear system $|C|$ of curves on F that are linearly equivalent to C is basepoint-free. In fact, the linear system $|D|$ on F is itself basepoint-free (as we have seen in Case 2). By Bertini's Theorem, $|C|$ contains a smooth, connected, non-aCM curve, whose general plane section has degree matrix M by Lemma 4.13.

Assume now that $a_{2,3} = 1$. The degree matrix is

$$M = \begin{pmatrix} a & a & a \\ 1 & 1 & 1 \end{pmatrix}.$$

Let D consist of $2a+1$ skew lines on a smooth quadric surface. The ideal I_D is generated in degrees $2, a+1$, and the degree matrix of a general plane section Y of D is

$$N = \begin{pmatrix} 1 & 1 & a \\ 1 & 1 & a \end{pmatrix}.$$

Let E be a generic complete intersection of two surfaces of degrees $a+1, a+2$, containing D. The image in I_Y of the surface of degree $a+1$ is a minimal generator. Let C be the residual curve to D in E. By Lemma 4.12, we can assume that both surfaces are smooth and connected, since the ideal of D is minimally generated in degree smaller than or equal to $a+1$. Moreover, the linear system of curves that we obtain fixing one of the surfaces and letting the other one vary is basepoint-free. Then C is smooth and connected by Bertini's Theorem. C is non-aCM since it's CI-linked to D non-aCM.

Applying Proposition 2.21 to the general sections X, Y of C, D, we have that the minimal free resolution of X is

$$0 \longrightarrow R(-2a-1) \oplus R(-a-2) \longrightarrow R(-a-1)^3 \longrightarrow I_X \longrightarrow 0,$$

so X has degree matrix M. $\square$

We are now ready to prove the main result of the chapter. Let M be an integral degree matrix of size at least 3×4, that has at least one entry smaller than or equal to 2. For nay such M, we construct a smooth, connected, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$, whose general plane section has degree matrix M.

Theorem 4.15 *Let $M = (a_{i,j})$ be an integral degree matrix of size $t \times (t+1)$, such that $a_{t,1} \leq 2$, $t \geq 3$. Then there exists a smooth, connected non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$, whose general plane section $X \subset \mathbf{P}^2$ has degree matrix M.*

Proof: For some (k,l), $a_{k,l} \leq 2$ and $a_{k,l+1} > 0, a_{k-1,l} > 0$. Fix one of such pairs (k,l), and assume that $1 \leq l \leq k-2$ and $3 \leq k \leq t$. Notice that we can find such a pair (k,l), since M has positive subdiagonal by assumption.

Let N be the transpose about the anti-diagonal of the first $t-1$ columns of M.

$$N = \left(\begin{array}{cccc} a_{t,t-1} & \cdots & \cdots & a_{1,t-1} \\ \vdots & & & \vdots \\ a_{t,1} & \cdots & \cdots & a_{1,1} \end{array} \right).$$

Then N is a degree matrix, since $a_{2,1}, \ldots, a_{t,t-1} > 0$ by assumption. Further N has one entry smaller than 3 since $a_{t,1} \leq 2$. Let D be the curve constructed as in Theorem 3.17 (particularly as seen in Remark 3.24), starting from the submatrix

$$L = \left(\begin{array}{ccc} a_{k,l+1} & a_{k-1,l+1} & a_{k-2,l+1} \\ a_{k,l} & a_{k-1,l} & a_{k-2,l} \end{array} \right).$$

Here l and k are the pair of integers chosen in the beginning. Since $l+1 \leq t-1$ and $k-2 \geq 1$, L is a submatrix of N.

The curve D is non-degenerate, reduced, it has two connected components, and it has singularities only in the intersections of its irreducible components, as we saw in Remark 3.20. It is non arithmetically Cohen-Macaulay, and its general plane section has degree matrix N.

In case N is a 2×3 matrix whose entries are all equal to 1, we can still let D be the generic union of a line and a smooth plane conic. The general plane section of D consists of three non-collinear points, hence it has degree matrix N. In this case, D is non-degenerate, smooth, disconnected and non arithmetically Cohen-Macaulay. It saturated ideal is generated in degree 2.

The highest degree of a minimal generator of the ideal of D is $a_{t-1,t-1} + \ldots + a_{1,1} + a_{t,k} - a_{k,k} + 1$, as we showed in Remark 3.24. Since $a_{t,k} \leq a_{k,k}$ and $1 \leq a_{t,t}$, then $a_{1,1} + \ldots + a_{t,t} \geq a_{t-1,t-1} + \ldots + a_{1,1} + a_{t,k} - a_{k,k} + 1$. From Remark 3.25, there exists a smooth surface U of degree $a_{1,1} + \ldots + a_{t,t}$ containing D. Let T be a generic surface of degree $a_{1,1} + \ldots a_{t-1,t-1} + a_{t,t+1}$. Abusing notation, we refer to both the surface and its equation as U, or T, respectively. Then $I_E = (U,T)$ is the saturated ideal of a complete intersection E, containing D. Let C be the residual curve to D in E. By Bertini's Theorem, C is smooth and connected. In fact, it is the general element of the linear system of curves cut out on the smooth surface U, outside of D, by surfaces of degree $a_{1,1} + \ldots a_{t-1,t-1} + a_{t,t+1}$. The linear system is basepoint-free, since

$$a_{1,1} + \ldots + a_{t-1,t-1} + a_{t,t+1} \geq a_{1,1} + \ldots + a_{t-1,t-1} + a_{t,k} - a_{k,k} + 1$$

that is bigger than or equal to the highest degree of a minimal generator of I_D. The following Claim concludes the proof.

Claim: M is the degree matrix of the general plane section of C.

Let $X \subset \mathbf{P}^2$ be the general plane section of C. By construction, X is CI-linked to the general plane section Y of D via a $CI(a_{1,1}+\ldots+a_{t,t}, a_{1,1}+\ldots a_{t-1,t-1}+a_{t,t+1})$. The minimal free resolution of I_Y is

$$0 \to \bigoplus_{i=1}^{t-1} R(-\sum_{j=1}^{t-1} a_{t-j,t-j} - a_{t,i}) \to \bigoplus_{i=0}^{t-1} R(-\sum_{j=1}^{i} a_{t+1-j,t-j} - \sum_{j=i+1}^{t-1} a_{t-j,t-j}) \to I_Y \to 0.$$

By Proposition 2.21, the minimal free resolution of I_X is of the form

$$0 \longrightarrow \bigoplus_{i=1}^{t-1} R(-\sum_{j=1}^{t} a_{j,j} - a_{t,i}) \longrightarrow \bigoplus_{i=0}^{t} R(-\sum_{j=1}^{i} a_{j,j} - \sum_{j=i+1}^{t} a_{j,j+1}) \longrightarrow I_X \longrightarrow 0.$$

This proves that degree matrix of X is M: no cancellation can occur in the free resolution of X. In fact, no cancellation occurs between the shifts corresponding to the submatrix N. The entries in the last two columns of M are positive, since $a_{t,t} > 0$, therefore no cancellation can occur there either. $\square$

Remark 4.16 *Assume that for a given degree matrix M, we can find a pair (k,l) as in the proof of Theorem 4.15, and such that $a_{k,t} \geq a_{k-2,l}$. Then we can perform the construction of the theorem starting from the curves constructed in Theorem 3.17, or in Remark 3.19. The curves that we produce in this way are not projectively isomorphic to the curves that we produce in Theorem 4.15. In general, their deficiency module has smaller initial degree and higher final degree.*

Similarly, starting from different curves or performing the links in a different order, one can produce several curves that are not projectively isomorphic, whose general plane section has the same degree matrix M. Two curves constructed starting from the same curve and performing the links in a different order have isomorphic deficiency modules, therefore each of them can be deformed into the other.

We now illustrate the construction of Theorem 4.15 in an example.

Example 4.17 *Consider the degree matrix*

$$M = \begin{pmatrix} 3 & 4 & 4 & 4 \\ 3 & 4 & 4 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

M has positive subdiagonal. Following the notation of Theorem 4.15, let $k = 3$ and $l = 1$. Start from the submatrix

$$N = \begin{pmatrix} 1 & 4 & 4 \\ 0 & 3 & 3 \end{pmatrix}$$

and let $D = CI(3,3) \cup CI(1,7)$ be the union of two complete intersections meeting transversally in 8 points. See also Example 3.15 about the construction and the invariants of this curve. We saw that the general plane section of D has degree matrix N, and that D is minimally generated by forms of degree 4 and 8. One can check with a computer algebra system that a generic surface S of degree 8 containing D is smooth (we proved this in Lemma 3.16). Performing a link via a complete intersection of S and a generic surface T of degree 8, we obtain a smooth curve C with minimal free resolution

$$0 \longrightarrow S(-12) \longrightarrow \begin{array}{c} S(-12)^2 \oplus S(-11)^3 \\ \oplus \\ S(-9) \end{array} \longrightarrow \begin{array}{c} S(-11) \\ \oplus \\ S(-8)^3 \oplus S(-10)^3 \end{array} \longrightarrow I_C \longrightarrow 0.$$

The general plane section of C has degree matrix M.

As we mentioned earlier, the h-vectors of zero-dimensional schemes of $\mathbf{P}^2$ that occur as the general plane section of some connected, smooth curve $C \subset \mathbf{P}^3$ have been characterized in [28], [27], [51], [41], and [23]. They are the ones of *decreasing type*, i.e. the h-vectors $h(z) = 1+h_1z+\ldots+h_sz^s$, $h_s \neq 0$, for which $h_i > h_{i+1}$ implies $h_{i+1} > h_{i+2}$, for $i \leq s-2$. The results we mentioned, together with Corollary 4.11, Theorem 4.14 and Theorem 4.15, give the following result.

Corollary 4.18 *Let $h(z) = 1 + h_1z + \ldots + h_sz^s$, $h_s \neq 0$, be the h-vector of some zero-dimensional scheme $X \subset \mathbf{P}^2$. $h(z)$ occurs as the h-vector of the general plane section of some smooth, connected, non-aCM curve $C \subset \mathbf{P}^3$ if and only if it is of decreasing type and it is different from the h-vector of a $CI(a, b)$, $a \neq 2$, $b \geq a$, from the h-vectors of Corollary 4.11 and from $1 + 2z$.*

Proof: If $h(z)$ is the h-vector of the general plane section of some integral, smooth, non-aCM curve $C \subset \mathbf{P}^3$, then it is of decreasing type, as shown in [28]. Moreover, it's different from the h-vector of a $CI(a, b)$, $a \neq 2$, $b \geq a$ and from the h-vectors of Corollary 4.11. In fact, a zero-dimensional scheme that has the h-vector of a $CI(a, b)$ is a $CI(a, b)$, and if $a \neq 2$, $2 \neq b \geq a$. Then C is arithmetically Cohen-Macaulay by Theorem 2.8. If the general plane section of an integral C is a $CI(1, 2)$, then C is arithmetically Cohen-Macaulay. If the general plane section of C has one of the h-vectors of Corollary 4.11, then C has to be arithmetically Cohen-Macaulay.

Conversely, let $h(z)$ be an h-vector of decreasing type, different from the h-vector of a $CI(a, b)$, $a \neq 2$, $b \geq a$ and from the h-vectors of Corollary 4.11. To any h-vector $h(z)$, we can uniquely associate a degree matrix M with no entries equal to 0, such that if $X \subset \mathbf{P}^2$ is a zero-dimensional scheme with degree matrix M, then the h-vector of X is $h(z)$. Under our assumptions, M can be either one of the following:

- $M = (2, a)$ for some $a \geq 2$,
- M is a matrix of size 2×3, with positive entries (since M is the degree matrix of points in Uniform Position), that does not satisfy the hypothesis of either Proposition 4.4, or Proposition 4.6, and such that not all of its entries are equal to 1 (since $h(z) \neq 1 + 2z$),

- M is integral and has size $t \times (t+1)$, for some $t \geq 3$.

If $M = (2, a)$ for some $a \geq 2$, let C be a generic, smooth, rational curve on a smooth quartic surface, as in Example 2.10 or Remark 2.11. C is non-aCM and its general plane section has degree matrix M, hence h-vector $h(z)$.

If M is a degree matrix of size 2×3 with positive entries, such that $a_{2,1} \leq 2$, then by Theorem 4.14 there exists a smooth, integral, non-aCM curve C whose general plane section has degree matrix M, hence h-vector $h(z)$.

If M has size bigger than or equal to 3×4, and $a_{t,1} \leq 2$, then by Theorem 4.15 there exists a smooth, integral, non-aCM curve C whose general plane section has degree matrix M, hence h-vector $h(z)$.

If $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots,t+1}$ has $a_{t,1} \geq 3$, let $N = (b_{i,j})_{i=1,\ldots,t+1;\, j=1,\ldots,t+2}$ be the degree matrix with entries $b_{i,j} = a_{i,j-1}$ for $i = 1, \ldots, t$, $j = 2, \ldots, t+2$, $b_{t+1,1} = 0$, $b_{t+1,2} = 2$. N is determined by these entries, under the assumption that it is homogeneous. $b_{i,j} > 0$ for $(i,j) \neq (t+1, 1)$, so N is an integral degree matrix. Moreover, the h-vector of a zero-dimensional scheme that has degree matrix N is $h(z)$. Then, by Theorem 4.15, there exists a non-aCM, reduced, connected curve $C \subset \mathbf{P}^3$, whose general plane section $X \subset \mathbf{P}^2$ has degree matrix M, hence h-vector $h(z)$. $\square$

CHAPTER 5

ARITHMETICALLY BUCHSBAUM CURVES

In this chapter we work over a field k of arbitrary characteristic. We want to investigate the relations between an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve and its general hyperplane section. In particular, we address the question of which graded Betti numbers can correspond to points that are the general hyperplane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. This gives as a consequence an easy sufficient condition for the Cohen-Macaulayness of an arithmetically Buchsbaum curve, in terms of the graded Betti numbers of its general hyperplane section.

In the first section, we give an explicit characterization of the degree matrices that correspond to points that are a general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ (see Theorem 5.6). In Proposition 5.4 we find a necessary condition on the lifting matrix of points in $\mathbf{P}^n$ that are a general hyperplane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve (see Definition 2.5 for the definition of lifting matrix). In Theorem 5.15 we characterize the integral matrices that occur as the degree matrix of the general plane section of some arithmetically Buchsbaum, non arithmetically Cohen-Macaulay, integral curve of $\mathbf{P}^3$.

In the second section, we prove some upper bounds on the dimension as k-vector spaces of the graded components of the deficiency module $\mathcal{M}_C$ of a Buchsbaum curve C (in Proposition 5.21), and on the initial and final degree of $\mathcal{M}_C$ (in Proposition 5.19). The bounds are given in terms of the entries of the lifting matrix of a general plane section X of C. In Theorem 5.31 we prove the sharpness of the bounds of Proposition 5.19 and of Proposition 5.21. For each degree matrix that satisfies the necessary and sufficient conditions of Theorem 5.6, we construct an example of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ whose deficiency module achieves all the bounds in all degrees.

Definition 5.1 *Let $C \subset \mathbf{P}^{n+1}$ be a curve. C is* arithmetically Buchsbaum*, or briefly* Buchsbaum*, if its deficiency module $\mathcal{M}_C$ is annihilated by the irrelevant maximal ideal $\mathfrak{m} = (x_0, \ldots, x_{n+1})$ of S, i.e. if its coordinate ring is Buchsbaum.*

For an introduction to Buchsbaum curves and their properties, or Buchsbaum rings, see Chapter 3 of [43], or the book [54]. For results about arithmetically Buchsbaum curves and their general hyperplane section, especially in the case of space curves, see the papers [21] and [22].

5.1 The lifting matrix of the general hyperplane section

We begin our study with some preliminary observations on the deficiency module of a Buchsbaum curve. Since the curves that we examine are locally Cohen-Macaulay and equidimensional, their deficiency modules have finite length.

C denotes an arithmetically Buchsbaum curve in $\mathbf{P}^{n+1}$ and $\mathcal{M}_C$ its deficiency module. $X \subset \mathbf{P}^n$ is a general hyperplane section of C, by a hyperplane of equation $L = 0$.

The deficiency module of an arithmetically Buchsbaum curve has a simple structure. However it captures precisely the difference between the graded Betti numbers of the curve and those of its general hyperplane section.

Proposition 5.2 *Let $C \subset \mathbf{P}^{n+1}$ be a Buchsbaum curve and let $X \subset \mathbf{P}^n$ be its hyperplane section, by a general hyperplane of equation $L = 0$. Then*

$$\mathcal{M}_C = I_X/(I_C + (L))(1) \subseteq soc\ H^1_*(\mathcal{I}_X)$$

where $soc\ H^1_*(\mathcal{I}_X)$ denotes the socle of $H^1_*(\mathcal{I}_X)$.

Proof: Look at the short exact sequence of ideal sheaves

$$0 \longrightarrow \mathcal{I}_C(-1) \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Taking global sections, we get the standard long exact sequence of cohomology modules

$$0 \longrightarrow I_C(-1) \xrightarrow{\cdot L} I_C \longrightarrow I_X \longrightarrow \mathcal{M}_C(-1) \xrightarrow{0} \mathcal{M}_C \longrightarrow H^1_*(\mathcal{I}_X) \longrightarrow \cdots .$$

The map $I_C(-1) \longrightarrow I_C$ is multiplication by L. The map $\mathcal{M}_C(-1) \longrightarrow \mathcal{M}_C$ is again multiplication by L, hence the zero map, by the assumption that C is Buchsbaum.

From the long exact sequence above, we can conclude that:

- $\mathcal{M}_C(-1) = Ker(\mathcal{M}_C(-1) \xrightarrow{0} \mathcal{M}_C) = Coker(I_C \longrightarrow I_X) = I_X/(I_C + (L))$
- $\mathcal{M}_C = soc\ \mathcal{M}_C \subseteq soc\ H^1_*(\mathcal{I}_X)$.

Putting these two observations together gives the thesis. □

From Proposition 5.2, we can easily derive an upper bound on the dimension as a k-vector space of the deficiency module of the curve C.

Corollary 5.3 *Let $C \subset \mathbf{P}^{n+1}$ be a Buchsbaum curve and $X \subset \mathbf{P}^n$ its general hyperplane section. Let $\mathcal{M}_C$ be the deficiency module of C and let*

$$0 \longrightarrow \mathbf{F}_n = \bigoplus_{i=1}^{t} R(-m_i) \longrightarrow \mathbf{F}_{n-1} \longrightarrow \cdots \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 \longrightarrow I_X \longrightarrow 0$$

be the minimal free resolution of I_X. Then

$$dim_k\ \mathcal{M}_C \leq t = rk(\mathbf{F}_n).$$

Consider a zero-dimensional scheme $X \subset \mathbf{P}^n$ that is a general hyperplane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^{n+1}$. We are now going to derive a necessary condition on the entries of the lifting matrix of X.

Proposition 5.4 *Let $X \subset \mathbf{P}^n$ be the general hyperplane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^{n+1}$.*
Let $M = (a_{ij})_{i=1,\dots,t;\, j=1,\dots,r}$ be the lifting matrix of X. Then $a_{ij} = n$, for some i, j.

Proof: By Proposition 5.2, the deficiency module $\mathcal{M}_C$ of C is

$$\mathcal{M}_C = I_X/(I_C + (L)/(L))(1) \subseteq soc\ H^1_*(\mathcal{I}_X)$$

where L is a general linear form, and $soc\ H^1_*(\mathcal{I}_X)$ denotes the socle of the module $H^1_*(\mathcal{I}_X)$. Since C is non-aCM,

$$\mathcal{M}_C = I_X/(I_C + (L)/(L))(1) \neq 0$$

and the initial degree of the deficiency module of C is $\alpha(\mathcal{M}_C) = d_j - 1$ for some $j = 1, \dots, r$. Moreover,

$$M_C \subseteq soc\ H^1_*(\mathcal{I}_X) = \bigoplus_{i=1}^{t} k(-m_i + n + 1).$$

Then $\alpha(\mathcal{M}_C) = m_i - n - 1$ for some $i = 1, \dots, t$. Hence $d_j = m_i - n$ for some i, j. For those i and j we have

$$a_{ij} = m_i - d_j = n.$$

□

We now quote a result of A.V. Geramita and J. Migliore that gives a bound on the degrees of a minimal generating system for $C \subset \mathbf{P}^3$, in terms of the degrees of the minimal generators of the saturated ideal of the general plane section X. We are going to use this result in the proof of the next theorem.

Proposition 5.5 *(Corollary 2.5, [22]) Let $C \subset \mathbf{P}^3$ be an arithmetically Buchsbaum curve, let $X \subset \mathbf{P}^2$ be its general plane section. If I_X is generated in degree less than or equal to d, then I_C is generated in degree less than or equal to $d+1$.*

We can now give a characterization of the matrices with integer entries that occur as degree matrix of the general plane section of an arithmetically Buchsbaum, non-aCM curve $C \subset \mathbf{P}^3$. This is a refinement of Proposition 5.4 in the case when $n = 2$, since if $X \subseteq \mathbf{P}^2$ the lifting matrix of X coincides with its degree matrix (see Definition 2.5 and the following observations).

Theorem 5.6 *Let $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots t+1}$ be a degree matrix. Then M is the degree matrix of the general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$ if and only if $a_{i,j} = 2$, for some i, j.*

For any such M, C can be chosen such that if the ideal I_C is minimally generated in degree less than or equal to d, then C lies on a smooth surface of degree d.

Proof: Assume that $M = (a_{i,j})$ is the degree matrix of some zero-dimensional scheme $X \subset \mathbf{P}^2$ that is the general plane section of an arithmetically Buchsbaum curve $C \subset \mathbf{P}^3$. Proposition 5.4 proves that $a_{i,j} = 2$ for some i, j.

Conversely, we are going to show that $a_{i,j} = 2$ for some i, j is sufficient in order for $M = (a_{i,j})$ to occur as the degree matrix of the general plane section of some arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$. We proceed by induction on the size t of M. For each M, we are going to construct a curve in the linkage class of two skew lines, i.e. a curve whose deficiency module is one-dimensional as a k-vector space.

If $t = 1$, then either $M = (1, 2)$ or $M = (2, a)$ for $a \geq 2$. If $M = (1, 2)$, let C be two skew lines: its general plane section consists of two distinct points, hence a $CI(1, 2)$ as desired. C lies on a smooth quadric surface. Since S is smooth and its ideal is generated in degree 2, by Lemma 4.12 it lies on a smooth surface of degree d

for any $d \geq 2$. If $M = (2, a)$, let D consist of two skew lines, $D \subset CI(2, a+1)$. We can let the surface of degree 2 be smooth, and the surface of degree $a+1$ generic. Let C be the residual curve to D in the link. By Bertini's Theorem, C is smooth and connected. Moreover, the general plane section X of C is linked to a $CI(1, 2)$ via a $CI(2, a+1)$. Using Proposition 2.21 in [43], the minimal free resolution of X is

$$0 \longrightarrow R(-a-2) \longrightarrow R(-a-1) \oplus R(-2) \longrightarrow I_X \longrightarrow 0$$

so X is a $CI(2, a)$. Notice that, in this case, $\mathcal{M}_C = k(1-a)$ and the module lies in the highest degree possible for a fixed a. The ideal I_C is generated in degree less than or equal to $a+1$, so C lies on a smooth surface of degree d for any $d \geq a+1$ by Lemma 4.12.

Let $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots,t+1}$ and assume that $a_{i,j} = 2$ for some $2 \leq j \leq t$. Let

$$N = \begin{pmatrix} a_{t,t} & \cdots & \cdots & a_{1,t} \\ \vdots & & & \vdots \\ a_{t,2} & \cdots & \cdots & a_{1,2} \end{pmatrix}.$$

N is the transpose about the anti-diagonal of the submatrix obtained by deleting the first and last columns of M. Notice that N is a degree matrix. By the induction hypothesis, there is an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section Y has degree matrix N. The saturated ideal I_D of D is generated in degree less than or equal to $a_{1,2} + \ldots + a_{t-1,t} + 1$, by Proposition 5.5. $\alpha(I_D) \leq a_{2,2} + \ldots + a_{t,t} + 1$. So we can find a complete intersection of forms of degrees $a_{1,1} + \ldots + a_{t,t}, a_{1,t+1} + a_{2,2} + \ldots + a_{t,t}$ containing D. Both the surfaces that cut out the complete intersection can be chosen in such a way that their images in I_Y are not minimal generators. Let C be the residual of D in the $CI(a_{1,1} + \ldots + a_{t,t}, a_{1,t+1} + a_{2,2} + \ldots + a_{t,t})$. By the Hartshorne-Schenzel Theorem, C is in the linkage class of two skew lines, as D is. Since Y has degree matrix N, using Proposition 2.21, we see that X has degree matrix M. The surface of degree $a_{1,t+1} + a_{2,2} + \ldots + a_{t,t}$ can be taken smooth, by induction hypothesis applied to D.

The ideal of C is generated in degree less than or equal to $a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, by Proposition 5.5. Let $d \geq a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, and consider the linear system Δ_d of surfaces of degree d containing C. We want to show that the general element is smooth. By Bertini's Theorem, it is smooth outside of C. Consider now a point $P \in C$. By Corollary 2.10 in [26], it's enough to exhibit two elements of Δ_d meeting transversally at P. If C is smooth at P, we have two minimal generators of I_C, call them F and G, meeting transversally at P. The degree of each of them is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P. Finally, we need to check that the singular points of C are not fixed singular points for Δ_d. So it is enough to find a surface for each of those points that contains C and is non-singular at P. This follows from the fact that we have a smooth surface containing C of degree $a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}<d$. Add generic planes as needed to get a surface that is non-singular at P and contains C.

Consider now the case $a_{i,1}=2$ for some $i \neq 1$. Let

$$N = \begin{pmatrix} a_{t,t} & \cdots & \cdots & a_{1,t} \\ \vdots & & & \vdots \\ a_{t,3} & \cdots & \cdots & a_{1,3} \\ a_{t,1} & \cdots & \cdots & a_{1,1} \end{pmatrix},$$

N is the transpose about the anti-diagonal of the matrix obtained deleting the second and last column of M. Notice that N is a degree matrix, since $a_{2,1} \geq a_{i,1} > 0$. By the induction hypothesis, there is an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section Y has degree matrix N. The saturated ideal I_D of D is generated in degree less then or equal to $a_{1,1}+a_{2,3}+\ldots+a_{t-1,t}+1$, by Proposition 5.5. $\alpha(I_D) \leq a_{t,t}+\ldots+a_{3,3}+a_{2,1}+1$, so we can find a complete intersection of forms of degrees $a_{t,t}+\ldots+a_{3,3}+a_{2,1}+a_{1,2}, a_{t,t}+\ldots+a_{3,3}+a_{2,1}+a_{1,t+1}$ containing D. Both the surfaces that cut out the complete intersection can be chosen in such a way that their images in I_Y are not minimal generators. Let C be the residual of D in the $CI(a_{t,t}+\ldots+a_{3,3}+a_{2,1}+a_{1,2}, a_{t,t}+\ldots+a_{3,3}+a_{2,1}+a_{1,t+1})$. C is in the linkage class of two skew lines, by the Hartshorne-

Schenzel Theorem. Since Y has degree matrix N, using Proposition 2.21, we see that X has degree matrix M. The surface of degree $a_{t,t}+\ldots+a_{3,3}+a_{2,1}+a_{1,t+1}$ can be taken smooth, by induction hypothesis applied to D. The ideal of C is generated in degree less than or equal to $a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, by Proposition 5.5. Let $d \geq a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, and consider the linear system Δ_d of surfaces of degree d containing C. We want to show that the general element is smooth. By Bertini's Theorem, it is smooth outside of C. Consider now a point $P \in C$. By Corollary 2.10 in [26], it's enough to exhibit two elements of Δ_d meeting transversally at P. If C is smooth at P, we have two minimal generators of I_C, call them F and G, meeting transversally at P. The degree of each of them is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P. Finally, we need to check that the singular points of C are not fixed singular points for Δ_d. So it is enough to find a surface for each of those points that contains C and is non-singular at P. This follows from the fact that we have a smooth surface containing C of degree $a_{t,t}+\ldots+a_{3,3}+a_{2,1}+a_{1,t+1} \leq d$. Add generic planes as needed to get a surface that is non-singular at P and contains C.

Assume now that $a_{1,1}=2$, i.e. $i=j=1$. Let

$$N=\begin{pmatrix} a_{1,1} & \cdots & \cdots & a_{1,t} \\ \vdots & & & \vdots \\ a_{t-1,1} & \cdots & \cdots & a_{t-1,t} \end{pmatrix},$$

be the submatrix of M, consisting of the first $t-1$ rows and first t columns. By the induction hypothesis, there is an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section Y has degree matrix N. The saturated ideal I_D of D is generated in degree less than or equal to $a_{1,2}+\ldots+a_{t-1,t}+1$, by Proposition 5.5. $\alpha(I_D) \leq a_{1,1}+\ldots+a_{t-1,t-1}+1$, so we can find a surface S of degree $s=a_{1,1}+\ldots+a_{t,t}$, containing D. The surface can be chosen such that its image in I_Y is not a minimal generator. Perform a basic double link of degrees $s, a_{t,t+1}$. Let C be the curve obtained in the $BDL(a_{1,1}+\ldots+a_{t,t}, a_{t,t+1})$. Let the surface F of degree $a_{t,t+1}$ be generic. C is in the linkage class of two skew lines, as D is. Since Y has degree matrix N, using Proposition 2.24, we see that X has degree matrix M. No cancelation can occur, since the image of S in I_Y is not a minimal generator, and by genericity of F. The ideal of C is generated in degree less than or equal to $a_{1,t+1} + a_{2,2} + \ldots + a_{t,t} + 1$, by Proposition 5.5. Let $d \geq a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, and consider the linear system Δ_d of surfaces of degree d containing C. We want to show that the general element is smooth. By Bertini's Theorem, it is smooth outside of C. Consider now a point $P \in C$. By Corollary 2.10 in [26], it's enough to exhibit two elements of Δ_d meeting transversally at P. If C is smooth at P, we have two minimal generators of I_C, call them F and G, meeting transversally at P. The degree of each of them is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P. Finally, we need to check that the singular points of C are not fixed singular points for Δ_d. So it is enough to find a surface for each of those points that contains C and is non-singular at P. By the induction hypothesis, we can find a smooth surface T of degree $a_{1,1} + a_{2,3} + \ldots + a_{t-1,t} + 1$ containing D. By genericity, we can assume that the surface F used in the construction of C is smooth. $T \cup F$ is a surface of degree $a_{1,1} + a_{2,3} + \ldots + a_{t,t+1} + 1 = a_{1,t+1} + a_{2,1} + a_{3,3} + \ldots + a_{t,t} < d$. Add generic planes as needed to get a surface that is non-singular at each point of C, except for the points of intersection of D and $S \cap F$. The surfaces S and $T \cup F$ meet transversally, so those can't be fixed singular points of Δ_d either.

Finally, let $j = t+1$, i.e. $a_{i,t+1} = 2$ for some i. Let

$$N = \begin{pmatrix} a_{t,t+1} & \cdots & \cdots & a_{1,t+1} \\ a_{t,t-1} & \cdots & \cdots & a_{1,t-1} \\ \vdots & & & \vdots \\ a_{t,2} & \cdots & \cdots & a_{1,2} \end{pmatrix},$$

N is the transpose about the anti-diagonal of the matrix obtained deleting the first and t-th columns of M. By the induction hypothesis, there is an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section Y has degree matrix N. The saturated ideal I_D of D is generated in degree less than or equal to $a_{1,2}+\ldots+a_{t-2,t-1}+a_{t-1,t+1}+1$, by Proposition 5.5. Moreover, $\alpha(I_D) \leq a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{2,2}+1$, so we can find a complete intersection of forms of degrees $a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{1,1}, a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{2,2}+a_{1,t}$ containing Both the surfaces that cut out the complete intersection can be chosen in such a way that their images in I_Y are not minimal generators. Let C be the residual curve to D in the $CI(a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{1,1}, a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{2,2}+a_{1,t})$. By the Hartshorne-Schenzel Theorem, C is in the linkage class of two skew lines, as D is. Since Y has degree matrix N, using Proposition 2.21, we see that X has degree matrix M. The surface of degree $a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{2,2}+a_{1,t}$ can be taken smooth, by induction hypothesis applied to D. The ideal of C is generated in degree less than or equal to $a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, by Proposition 5.5. Let $d \geq a_{1,t+1}+a_{2,2}+\ldots+a_{t,t}+1$, and consider the linear system Δ_d of surfaces of degree d containing C. We wish to show that the general element is smooth. By Bertini's Theorem, it is smooth outside of C. Consider now a point $P \in C$. By Corollary 2.10 in [26], it's enough to exhibit two elements of Δ_d meeting transversally at P. If C is smooth at P, we have two minimal generators of I_C, call them F and G, meeting transversally at P. The degree of each of them is at most d. Add generic planes as needed, to obtain surfaces of degree d that meet transversally at P. Finally, we need to check that the singular points of C are not fixed singular points for Δ_d. So it is enough to find a surface for each of those points that contains C and is non-singular at P. This follows from the fact that we have a smooth surface containing C of degree $a_{t,t+1}+a_{t-1,t-1}+\ldots+a_{2,2}+a_{1,t} < d$. Add generic planes as needed to get a surface that is non-singular at P and contains C. □

Let us observe a few consequences of the theorem we just proved.

Remark 5.7 *The proof of Theorem 5.6 shows that the following facts about a degree matrix $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots t+1}$ are equivalent:*

- $a_{i,j} = 2$ *for some* i, j;

- *there exists a zero-dimensional scheme $X \subset \mathbf{P}^2$ and a Buchsbaum, non arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^3$ such that X is the general plane section of C and M is the degree matrix of X;*

- *there exists a zero-dimensional scheme $X \subset \mathbf{P}^2$ and a Buchsbaum curve $C \subset \mathbf{P}^3$ in the linkage class of two skew lines such that X is the general plane section of C and M is the degree matrix of X.*

Remark 5.8 *Introducing a minor modification in the proof, we can show that we can always construct a curve C whose deficiency module is $\mathcal{M}_C = k(-d_m+1)$, where $m = \min\{\, j \mid a_{i,j} = 2,$ for some $i\}$. Notice that this is the highest possible degree in which the deficiency module can appear, given the lifting matrix of the general plane section X of C. See the next section for a discussion about $\mathcal{M}_C$ and bounds on the dimension of the deficiency module in each degree.*

Remark 5.9 *From Theorem 5.6 it also follows that $d = \binom{n}{2}$ generic points in $\mathbf{P}^2$ cannot be the general plane section of an arithmetically Buchsbaum curve for any n, unless the curve is arithmetically Cohen-Macaulay. This was observed by A.V. Geramita and J. Migliore in [21], Proposition 4.9.*

Theorem 5.6 extends a result by A.V. Geramita and J. Migliore. In [21], they prove the following.

Proposition 5.10 *([21], Proposition 4.7) Let $C \subset \mathbf{P}^3$ be an arithmetically Buchsbaum curve lying on no quadric surface. Let X be a general plane section of C. Assume that $\alpha(I_C) = \alpha(I_X)$ and that X is a complete intersection. Then C is a complete intersection.*

After analyzing the case of Buchsbaum curves in general, we are now going to restrict our attention to integral (that is, reduced and irreducible), arithmetically Buchsbaum curves in $\mathbf{P}^3$. We are interested in characterizing the degree matrices that can occur for a general plane section of an integral, arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve of $\mathbf{P}^3$. For the rest of the section, we assume that the ground field k has characteristic 0.

For the purpose of our investigation, we only need to look at degree matrices that satisfy certain conditions.

Notation 5.11 *Let $C \subset \mathbf{P}^3$ be an integral, arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. Let $X \subset \mathbf{P}^2$ be its general plane section. Let $M = (a_{i,j})_{i=1,\dots,t;\, j=1,\dots,t+1}$ be the degree matrix of X. M is then an integral matrix, i.e. $a_{i+1,i} > 0$ for all i (see also the beginning of Chapter 4). By Theorem 5.6 we have that $a_{i,j} = 2$ for some i, j.*

Remark 5.12 *In Chapter 4, we described some classes of degree matrices M of size 2×3 such that, if the general plane section of $C \subset \mathbf{P}^3$ integral has degree matrix M, then C is forced to be arithmetically Cohen-Macaulay (see Propositions 4.4 and 4.6). Notice that all of those matrices have no entry equal to 2, so they cannot occur as the degree matrix of the general plane section of some arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve. Therefore, we expect be able to realize all the matrices of Notation 5.11 as degree matrices of a general plane section of some reduced and irreducible, Buchsbaum, non-aCM curve in $\mathbf{P}^3$.*

We can give a characterization of the matrices M that occur as the degree matrix of the general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay integral curve $C \subset \mathbf{P}^3$. Because of what we already saw in Chapter 4 and in Theorem 5.6, they need to satisfy the conditions of Notation 5.11. We are going to show that all the degree matrices that can possibly occur for the general plane section of an integral, arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve, do in fact occur. Moreover, for each of these matrices the curve C can be taken smooth and connected. We treat separately the case $t = 1$, that is the case when X is a complete intersection.

Remark 5.13 *Any integral curve $C \subset \mathbf{P}^3$ of degree 2 is a plane conic. So there cannot be any integral arithmetically Buchsbaum curve that is non-aCM and whose general plane section is a $CI(1, 2)$.*

Proposition 5.14 *Let $M = (a, b)$, $b \geq a > 0$. M is the degree matrix of the general plane section of some smooth, integral, arithmetically Buchsbaum, non-aCM curve $C \subset \mathbf{P}^3$ if and only if $a = 2$.*

Proof: Assume that M is the degree matrix of the general plane section of some smooth, integral, Buchsbaum, non-aCM curve $C \subset \mathbf{P}^3$. We saw in Proposition 5.4 that M needs to contain a 2. The Remark above shows that $a \neq 1$, so $a = 2$.

Conversely, let $M = (2, b)$, $b \geq 2$. We want to construct a smooth, connected, arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve C, whose general plane section has degree matrix M. Let D be two skew lines, and let Q be a smooth quadric surface containing D. Notice that the image of Q in the saturated ideal of a general plane section of D is a minimal generator. Consider the linear system of curves cut out on Q, outside of D, by surfaces of degree $b+1$ containing D. It is basepoint-free, since I_D is generated in degree $2 < b+1$. By Bertini's Theorem (see Theorem 4.5), the general element C of the linear system is smooth and connected. C is in the linkage class of two skew lines by construction, and its general plane section has degree matrix M, by Proposition 2.21. $\square$

The following Theorem characterizes the integral matrices that occur as the degree matrix of the general plane section of some arithmetically Buchsbaum, non arithmetically Cohen-Macaulay, integral curve of $\mathbf{P}^3$.

Theorem 5.15 *Let* $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots t+1}$, $t \geq 2$ *be an integral degree matrix. Then* M *is the degree matrix of a general plane section of an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay, integral curve* $C \subset \mathbf{P}^3$ *if and only if* $a_{i,j} = 2$*, for some* i, j*. Moreover, for such a matrix* M *the curve* C *can be chosen to be smooth and connected.*

Proof: Necessity of the hypothesis $a_{i,j} = 2$ has been proven in Theorem 5.6.

Let M be an integral degree matrix such that $a_{i,j} = 2$, for some i, j. We are going to construct an smooth, connected Buchsbaum curve C in the linkage class of two skew lines, such that its general plane section X has degree matrix M. We start from degree matrices of size 2×3. Notice that in this case, all the entries of the matrix M are positive. We have the following possibilities for M.

Case 1. Let

$$M = \begin{pmatrix} 2 & a & b \\ 1 & a-1 & b-1 \end{pmatrix}$$

and let D be two skew lines. Let Y be a general plane section of D, $Y = CI(1, 2)$. $D \subset CI(a + 1, b + 1)$, the surface of degree $a + 1 \geq 3$ can be taken smooth by Lemma 4.12. Choosing a generic surface of degree $b + 1$, we have that the residual to D in the complete intersection is smooth and connected by Bertini's Theorem. In fact, the linear system of curves cut out outside of D by surfaces of degree $b + 1$ containing D is basepoint-free. Notice that the images in I_Y of the equations of the surfaces of degrees $a + 1, b + 1$ are not minimal generators. Let C be the residual curve to D in the CI. The general plane section of C has degree matrix M by Proposition 2.21.

Case 2. Let

$$M = \begin{pmatrix} a & b & c \\ 2 & b+2-a & c+2-a \end{pmatrix}$$

and let D be the residual to two skew lines in a generic $CI(2, a+1)$. The ideal of D is generated in degree less than or equal to $a + 1$ and its general plane section is $Y = CI(2, a)$ (see the proof of Theorem 5.6 for more details). $D \subset CI(b+2, c+2)$, where the surface of degree $b+2 \geq a+2$ can be taken smooth. Choosing a generic surface of degree $c+2$, we have that the residual to D in the complete intersection is smooth and integral by Bertini's Theorem. This follows from the observation that the linear system of curves cut out outside of D by surfaces of degree $c+2$ containing D is basepoint-free, because the ideal I_D is generated in degree less than or equal to $a+1 < c+2$. Notice that the images in I_Y of the equations of the surfaces of degrees $b+2, c+2$ are not minimal generators. Let C be the residual curve to D in the complete intersection. The general plane section of C has degree matrix M by Proposition 2.21.

Case 3. Let

$$M = \begin{pmatrix} 1 & 2 & a \\ 1 & 2 & a \end{pmatrix}$$

and let D be two skew lines. D is contained in a smooth, connected surface of degree 3, call it S. Perform a basic double link on S, using a general surface of degree a, let $C = D \cup CI(3, a)$. The general plane section of C has degree matrix M by Proposition 2.24. The linear system of curves on S that are linearly equivalent to C is basepoint-free (in fact, the linear system $|D|$ is itself basepoint-free, as shown in Theorem 4.14), so the general element of $|C|$ is smooth and connected. By Lemma 4.13, its general plane section has degree matrix M.

Case 4. Let

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

and let D be two skew lines. D is contained in a smooth, connected surface of degree 3, call it S. Perform a basic double link on S, using a general plane, let $C = D \cup CI(1, 3)$. The general plane section of C has degree matrix M by Proposition 2.24. The linear system of curves on S that are linearly equivalent to C is basepoint-free (in fact, the linear system $|D|$ is itself basepoint-free, as in the proof of Theorem 4.14), so the general element of $|C|$ is smooth and connected. By Lemma 4.13, its general plane section has degree matrix M. This concludes the proof of the case $t = 2$.

Assume now that $t \geq 3$ and that $j \leq t-1$. Consider the submatrix

$$N = \begin{pmatrix} a_{t,t-1} & \cdots & \cdots & a_{1,t-1} \\ \vdots & & & \vdots \\ a_{t,1} & \cdots & \cdots & a_{1,1} \end{pmatrix},$$

N is the transpose about the anti-diagonal of the first $t-1$ columns of M. By Theorem 5.6, we have an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section has degree matrix N. By Proposition 5.5 it follows that the ideal of D is generated in degree less than or equal to $a_{1,1} + \ldots + a_{t-1,t-1} + 1$. So by Theorem 5.6 there is a smooth surface S of degree $a_{1,1} + \ldots + a_{t,t}$ containing D. Consider the linear system of curves cut out on S, outside of D, by surfaces of degree $a_{1,1} + \ldots + a_{t-1,t-1} + a_{t,t+1}$ containing D. The linear system is basepoint-free, so by Bertini's Theorem, the general element C is smooth and connected. The general plane section of C has degree matrix M by Proposition 2.21.

The cases when $j = t, t+1$ can be proved in an analogous way. If $j = t$, start from the degree matrix

$$N = \begin{pmatrix} a_{t,t} & \cdots & \cdots & a_{1,t} \\ a_{t,t-2} & \cdots & \cdots & a_{1,t-2} \\ \vdots & & & \vdots \\ a_{t,1} & \cdots & \cdots & a_{1,1} \end{pmatrix},$$

N is the transpose about the anti-diagonal of the submatrix of M obtained by deleting columns $t-1$ and $t+1$. By Theorem 5.6, we have an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section has degree matrix N. By Proposition 5.5 it follows that the ideal of D is generated in degree less than or equal to $a_{1,1} + \ldots + a_{t-2,t-2} + a_{t-1,t} + 1$. So by Theorem 5.6 there is a smooth surface S of degree $a_{1,1} + \ldots + a_{t,t}$ containing D. Consider the linear system of curves cut out on S, outside of D, by surfaces of degree $a_{1,1} + \ldots + a_{t-1,t-1} + a_{t,t+1}$ containing D. The linear system is basepoint-free, so by Bertini's Theorem, the general element C is smooth and connected. The general plane section of C has degree matrix M by Proposition 2.21.

If $j = t+1$, start from the degree matrix

$$N = \begin{pmatrix} a_{t,t+1} & \cdots & \cdots & a_{1,t+1} \\ a_{t,t-2} & \cdots & \cdots & a_{1,t-2} \\ \vdots & & & \vdots \\ a_{t,1} & \cdots & \cdots & a_{1,1} \end{pmatrix},$$

N is the transpose about the anti-diagonal of the submatrix of M obtained by deleting columns $t-1$ and t. By Theorem 5.6, we have an arithmetically Buchsbaum curve D in the linkage class of two skew lines, whose general plane section has degree matrix N. By Proposition 5.5 it follows that the ideal of D is generated in degree less than or equal to $a_{1,1} + \ldots + a_{t-2,t-2} + a_{t-1,t+1} + 1$. So by Theorem 5.6 there is a smooth surface S of degree $a_{1,1} + \ldots + a_{t-1,t-1} + a_{t,t+1}$ containing D. Consider the linear system of curves cut out on S, outside of D, by surfaces of degree $a_{1,1} + \ldots + a_{t-2,t-2} + a_{t-1,t} + a_{t,t+1}$ containing D. The linear system is basepoint-free, so by Bertini's Theorem, the general element C is smooth and connected. The general plane section of C has degree matrix M by Proposition 2.21. $\square$

Remark 5.16 *In the proof of Theorem 5.15, we showed that the following facts about an* integral *degree matrix* $M = (a_{i,j})_{i=1,\ldots,t;\, j=1,\ldots t+1}$ *are equivalent:*

- $a_{i,j} = 2$ *for some* i, j*;*
- *there exists a zero-dimensional scheme* $X \subset \mathbf{P}^2$ *and an integral Buchsbaum, non arithmetically Cohen-Macaulay curve* $C \subset \mathbf{P}^3$ *such that* X *is the general plane section of* C *and* M *is the degree matrix of* X*;*
- *there exists a zero-dimensional scheme* $X \subset \mathbf{P}^2$ *and a smooth, connected Buchsbaum, non arithmetically Cohen-Macaulay curve* $C \subset \mathbf{P}^3$ *such that* X *is the general plane section of* C *and* M *is the degree matrix of* X*;*
- *there exists a zero-dimensional scheme* $X \subset \mathbf{P}^2$ *and an integral Buchsbaum curve* $C \subset \mathbf{P}^3$ *in the linkage class of two skew lines such that* X *is the general plane section of* C *and* M *is the degree matrix of* X*;*

- *there exists a zero-dimensional scheme $X \subset \mathbf{P}^2$ and a smooth, connected Buchsbaum curve $C \subset \mathbf{P}^3$ in the linkage class of two skew lines such that X is the general plane section of C and M is the degree matrix of X.*

Remark 5.17 *G. Paxia and A. Ragusa proved in [49] that any integral, arithmetically Buchsbaum curve $C \subset \mathbf{P}^3$ can be deformed to a smooth, connected curve. The deformation, moreover, preserves the cohomology, hence the deficiency module, of the curve. Their proof relies heavily on works of M. Martin-Deschamps and D. Perrin (see [42]), and of S. Nollet (see [48]).*

Their result is related to some of the implications of Remark 5.16. In fact, we show that the existence of an integral, arithmetically Buchsbaum curve, whose general plane section has a prescribed degree matrix is equivalent to the existence of a smooth, connected, arithmetically Buchsbaum curve, whose general plane section has that same degree matrix. The result of G. Paxia and A. Ragusa does not imply the results of Theorem 5.15 and of Remark 5.16. In fact, deforming an integral, arithmetically Buchsbaum curve to a smooth, connected one does not in general preserve the degree matrix of the general plane section. In particular, the way the deformation is done in [49] implies that if the general plane section X of an integral, Buchsbaum curve C has a minimal free resolution

$$0 \longrightarrow \mathbf{F}_2 \oplus \mathbf{F} \longrightarrow \mathbf{F}_1 \oplus \mathbf{F} \longrightarrow I_X \longrightarrow 0$$

where $\mathbf{F}_2$ and $\mathbf{F}_1$ are free R-modules without any (abstractly) isomorphic free summand, then for most curves C the minimal free resolution of the general plane section Y of the smooth, connected deformation D of C is

$$0 \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 \longrightarrow I_Y \longrightarrow 0.$$

However, the result of Paxia and Ragusa is stronger than our result in the case of curves whose general section X has a degree matrix that does not contain any zeroes. In this case, in fact, their deformation preserves the degree matrix of the general plane section.

On the other hand, our result is stronger than that of G. Paxia and A. Ragusa in the case of arithmetically Buchsbaum curves whose deficiency module is a one-dimensional k-vector space. In fact, for each integral degree matrix we constructed a smooth, connected curve with a one-dimensional deficiency module. Then there is a flat deformation between every integral curve with the same deficiency module and the smooth, connected curve that we constructed.

5.2 The deficiency module

We now turn to the study of the deficiency module of a Buchsbaum curve. We work over a field k of arbitrary characteristic. Throughout this section, we assume only that the curve $C \subset \mathbf{P}^{n+1}$ is arithmetically Buchsbaum (therefore locally Cohen-Macaulay), non arithmetically Cohen-Macaulay, equidimensional and non-degenerate. The notation will be as follows.

Notation 5.18 *Let $C \subset \mathbf{P}^{n+1}$ be an arithmetically Buchsbaum curve, and $X \subset \mathbf{P}^n$ a general hyperplane section of C. Let the minimal free resolution of X be*

$$0 \longrightarrow \mathbf{F}_n = \bigoplus_{i=1}^{t} R(-m_i) \longrightarrow \mathbf{F}_{n-1} \longrightarrow \cdots \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 = \bigoplus_{j=1}^{r} R(-d_j) \longrightarrow I_X \longrightarrow 0$$

where $m_1 \geq \ldots \geq m_t$ and $d_1 \geq \ldots \geq d_r$. The lifting matrix of X is

$$M = (a_{ij})_{i=1,\ldots,t;\, j=1,\ldots r}, \quad \textit{where } \ a_{i,j} = m_i - d_j.$$

Using the results of Proposition 5.2, some easy bounds for the initial and final degrees of $\mathcal{M}_C$ in terms of the entries of the lifting matrix of X can be derived.

Proposition 5.19 *Let $C \subset \mathbf{P}^{n+1}$ be an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve, let $X \subset \mathbf{P}^n$ be a general plane section of C. Let $M = (a_{ij})_{i=1,\ldots,t;\, j=1,\ldots r}$ be the lifting matrix of X. Then*

$$\alpha(\mathcal{M}_C) \geq \max\{m_t - n - 1,\ \alpha(I_X) - 1\}$$

and

$$\alpha(\mathcal{M}_C)^+ \leq m_1 - n - 1.$$

Proof: Following Notation 5.18,

$$soc\ H^1_*(\mathcal{I}_X) = \bigoplus_{i=1}^{t} k(-m_i + n + 1).$$

So $\alpha(\mathcal{M}_C) \geq m_t - n - 1$ and $\alpha(\mathcal{M}_C)^+ \leq m_1 - n - 1$. Moreover,

$$\mathcal{M}_C = I_X/(I_C + (L))(1)$$

that gives $\alpha(\mathcal{M}_C) \geq d_r - 1 = \alpha(I_X) - 1$. □

Following the same principle, we can give a more precise estimate of what the initial degree of the deficiency module of C can be.

Remark 5.20 *Since $\mathcal{M}_C = I_X/(I_C+(L))(1)$, then $d_m - 1 \leq \alpha(\mathcal{M}_C) \leq d_l - 1$ where $m = \max\{\, j \mid a_{i,j} = 2,$ for some $i\}$ and $l = \min\{\, j \mid a_{i,j} = 2,$ for some $i\}$.*

From Proposition 5.2, we can also deduce an upper bound on the dimension of $\mathcal{M}_C$ in each degree, hence an upper bound on the dimension of $\mathcal{M}_C$ as a k-vector space.

Proposition 5.21 *Following Notation 5.11, let*

$$J = \{\, j \mid d_j = m_{k(j)} - n \ \ \textit{for some } k(j)\}$$

and for each $j \in J$ let $\mu(j)$ be the number of minimal generators of I_X of degree d_j. Then, for $i \in \mathbf{Z}$, the dimension of the i-th graded component of $\mathcal{M}_C$ is

$$dim_k(\mathcal{M}_C)_i = 0 \quad \textit{if } \ i \neq d_j - 1 \textit{ for all } j \in J$$

and for $j \in J$

$$dim_k(\mathcal{M}_C)_{d_j-1} \leq \min\{dim\ soc\ H^1_*(\mathcal{I}_X)_{d_j-1}, \mu(j)\}.$$

Then

$$dim_k(\mathcal{M}_C) \leq \sum_{j\in J} \min\{dim\ soc\ H^1_*(\mathcal{I}_X)_{d_j-1}, \mu(j)\}.$$

Proof: First we observe that the set of all degrees i where we can possibly have $dim(\mathcal{M}_C)_i \neq 0$ is $\{d_j - 1 \mid j \in J\}$. In fact, by Proposition 5.2

$$\mathcal{M}_C = I_X/(I_C + (L))(1) \subseteq soc\ H^1_*(\mathcal{I}_X).$$

In particular, $(\mathcal{M}_C)_i$ can be non-zero only for $i \in \{m_1 - n - 1, \ldots, m_t - n - 1\}$, since those are the degrees in which $soc\ H^1_*(\mathcal{I}_X)$ is non-zero. Each minimal generator of $\mathcal{M}_C$ is a minimal generator of $I_X/(I_C+(L))(1)$. Therefore, each minimal generator of $\mathcal{M}_C$ has degree $d_j - 1$ for some j. Since by assumption the structure of $\mathcal{M}_C$ as an S-module is trivial, a minimal system of generators of $\mathcal{M}_C$ as an S-module is also a basis as a k-vector space. Then the set of all possible degrees where the deficiency module can possibly be non-zero is $\{d_j - 1 \mid j \in J\}$. Moreover, in each degree $i = d_j - 1$ where $dim(\mathcal{M}_C)_i$ can be non-zero we have

$$dim(\mathcal{M}_C)_{d_j-1} \leq \min\{dim\ soc\ H^1_*(\mathcal{I}_X)_{d_j-1}, \mu(j)\}.$$

□

Remark 5.22 *Notice that $\mu(j)$ is the number of columns that are equal to the j-th column. Moreover, $dim\ soc\ H^1_*(\mathcal{I}_X)_{d_j-1} = dim\ soc\ H^1_*(\mathcal{I}_X)_{m_{k(j)}-n-1}$ is the number of rows that are equal to the $k(j)$-th row.*

Definition 5.23 *Let M be a lifting matrix. By a* **block** *of entries equal to n we mean a group of entries of M such that:*

- *$a_{i,j} = n$ for $i_1 \leq i \leq i_2$ and $j_1 \leq j \leq j_2$, and*
- *$a_{i,j} \neq n$ if either $i = i_1 - 1$ and $j_1 \leq j \leq j_2$, or $i = i_2 + 1$ and $j_1 \leq j \leq j_2$, or $j = j_1 - 1$ and $i_1 \leq i \leq i_2$, or $j > j_2$ and $i_1 \leq i \leq i_2$.*

Remark 5.24 *The proof of Proposition 5.21 also shows that each block of n's in the lifting matrix corresponds to a degree in which the deficiency module of C is possibly non-zero.*

From our observations, we can easily derive a criterion for lifting minimal generators from the saturated ideal I_X of a general hyperplane section X, to the saturated ideal I_C of the curve C. Notice that this sufficient condition is weaker than the sufficient condition of Lemma 4.3, for curves that are not necessarily Buchsbaum.

Corollary 5.25 *Let C be an arithmetically Buchsbaum, non arithmetically Cohen-Macaulay curve, let X be its general hyperplane section, and let M be the lifting matrix of X. If for some j we have $a_{ij} \neq n$ for all i, then the minimal generators of degree d_j of I_X lift to I_C. In particular, if $a_{1,j} < n$ then the minimal generators of degrees $d_1, \ldots, d_j$ of I_X lift to I_C.*

Proof: Let

$$0 \longrightarrow \bigoplus_{i=1}^{t} R(-m_i) \longrightarrow \ldots \longrightarrow \bigoplus_{j=1}^{r} R(-d_j) \longrightarrow I_X \longrightarrow 0$$

be the minimal free resolution of I_X. $d_j = m_i - a_{ij}$, then $d_j \neq m_i - n$ if and only if $a_{ij} \neq n$. Fix a j such that $a_{ij} \neq n$ for all i. Then $d_j \neq m_i - n$ for all i, so $(\mathcal{M}_C)_{d_j - 1} = 0$ by Proposition 5.21. Therefore all the minimal generators of degree d_j of I_X lift to I_C. This proves the first part of the statement.

Assume now that $a_{1,j} < n$ for some j. Then $a_{1,l} < n$ for $l \leq j$. In particular, $a_{il} \neq n$ for all i and for all $l \leq j$. Then the minimal generators of degrees $d_1, \ldots, d_j$ of I_X lift to I_C. $\square$

Remark 5.26 *In the case of points in $\mathbf{P}^2$, assuming $a_{1,j} < 2$ is equivalent to assuming $a_{1,j} = 1$. In fact $a_{1,j} \leq 0$ implies $a_{i,j} \leq 0$ for all i, and the Hilbert-Burch matrix of an arithmetically Cohen-Macaulay scheme of codimension 2 cannot have a column of zeroes.*

Remark 5.27 *Corollary 5.25 clarifies how, if $X \subset \mathbf{P}^2$ is a generic zero-dimensional scheme of degree $d = \binom{n}{2}$ for some n, any arithmetically Buchsbaum curve of $\mathbf{P}^3$ that has X as its general plane section needs to be arithmetically Cohen-Macaulay. All the entries of the degree matrix of X are equal to 1, therefore all the minimal generators of I_X lift to I_C.*

In the next example, we provide some evidence that the bounds on the dimension of $\mathcal{M}_C$ of Proposition 5.19 and Proposition 5.21 are sharp. In Remark 5.8 we saw that the upper bound on the final degree $\alpha^+(\mathcal{M}_C)$ is sharp for curves that belong to the linkage class of two skew lines. In the example below we look at space curves whose deficiency module is concentrated in one degree. We focus on curves in $\mathbf{P}^3$ that are minimal for their Liaison class, showing that $\mathcal{M}_C = soc\, H^1_*(\mathcal{I}_X)$, therefore all the bounds of Proposition 5.21 and Proposition 5.19 are attained.

Example 5.28 *Let $M_n = K^n(-2n+2)$ be a deficiency module. Let C_n be a minimal curve for the Liaison class corresponding to the deficiency module $\mathcal{M}_n$ (see [43] for definition and facts about minimal curves). We can construct such a C_n starting from two skew lines and using Liaison Addition, as discussed in [43], Section 3.3. Let $S = k[x_0, x_1, x_2, x_3]$ and $R = k[x_0, x_1, x_2]$. It is easy to show by induction on n that the minimal free resolution of C_n is*

$$0 \longrightarrow S(-2n-2)^n \longrightarrow S(-2n-1)^{4n} \longrightarrow S(-2n)^{3n+1} \longrightarrow I_{C_n} \longrightarrow 0.$$

Analogously, since the minimal free resolution of the general plane section X_1 of $C_1 =$ two skew lines is

$$0 \longrightarrow R(-3) \longrightarrow R(-2) \oplus R(-1) \longrightarrow I_{X_1} \longrightarrow 0,$$

using the short exact sequence (see [52], or [43] Section 3.2 for a description of Liaison Addition and details on these techniques)

$$0 \longrightarrow R(-2n) \longrightarrow I_{X_1}(-2n+2) \oplus I_{X_{n-1}}(-2) \longrightarrow I_{X_n} \longrightarrow 0$$

by induction on n we can compute the minimal free resolution of I_{X_n}, that turns out to be

$$0 \longrightarrow R(-2n-1)^n \longrightarrow R(-2n) \oplus R(-2n+1)^n \longrightarrow I_{X_n} \longrightarrow 0.$$

Therefore, the degree matrix of the general plane section X_n of C_n is

$$\left.\begin{pmatrix} 1 & \underbrace{\begin{matrix} 2 & \cdots & 2 \\ \vdots & & \vdots \\ 2 & \cdots & 2 \end{matrix}}_{n} \\ \vdots & \\ 1 & \end{pmatrix}\right\} n$$

In this family of examples, $\mathcal{M}_C = soc\ H^1_*(\mathcal{I}_X) = k^n(-2n-1+3)$, so equality is attained in Proposition 5.2, Proposition 5.19 and Corollary 5.21.

Only one of the minimal generators of I_X lifts to I_C: the one of maximum degree d_1, corresponding to $a_{1,1} = 1$, as shown in Corollary 5.25.

Remark 5.29 *Notice that if $a_{1,t+1} = 2$, then the deficiency module $\mathcal{M}_C$ must be concentrated in degree $a_{1,1} + \ldots + a_{t,t} - 1$. The degree matrix M has the special form*

$$\begin{pmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & 2 & \cdots & 2 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}$$

where possibly the block of zeroes does not appear. Then the deficiency module is concentrated in degree $a_{1,1} + \ldots + a_{t,t} + 2 - 3$. This is for example the case for generic points in $\mathbf{P}^2$ whose degree d is not a binomial coefficient ($d \neq \binom{n}{2}$ for all n).

In particular, all the minimal generators of I_X that are not in the initial degree lift to I_C.

Therefore, for generic points in $\mathbf{P}^2$ we have the following.

Corollary 5.30 *Let $C \subset \mathbf{P}^3$ be an arithmetically Buchsbaum curve of degree d. Assume that $d \neq \binom{n}{2}$ for all n, and that the general plane section of C consists of* *generic points. Then the deficiency module of C must be concentrated in a single degree.*

We now show that the bounds on the dimension of $\mathcal{M}_C$ of Proposition 5.19 and Proposition 5.21 are sharp in the case of curves in $\mathbf{P}^3$ and points in $\mathbf{P}^2$.

Theorem 5.31 *Let M be a degree matrix with at least one entry equal to* 2*. Then there exists an arithmetically Buchsbaum curve $C \subset \mathbf{P}^3$ whose general plane section has degree matrix M, and such that the dimension of the deficiency module $\mathcal{M}_C$ in each degree achieves the bound of Proposition 5.21. Moreover, $\mathcal{M}_C$ achieves the bounds for the initial and final degree of Proposition 5.19.*

Proof: In Remark 5.24, we noticed that the number of non-zero components of the deficiency module is bounded above by the number of blocks of 2's in the degree matrix M. Notice that if the dimension of $\mathcal{M}_C$ as a k-vector space is the maximum possible, according to Proposition 5.21, then the dimension of $(\mathcal{M}_C)_i$ for each i is the maximum possible. Moreover, in this situation, all the graded components of $\mathcal{M}_C$ that can possibly be non-zero are actually different from zero. Hence the bounds of Proposition 5.19 on the initial and final degree of $\mathcal{M}_C$ are also attained. Therefore, in order to prove that the bounds of Proposition 5.21 in every degree and the bounds of Proposition 5.19 are sharp, it is enough to construct a curve whose deficiency module has maximum possible dimension globally. We indicate by $\delta(M)$ the maximum possible dimension for $\mathcal{M}_C$. Notice that $\delta(M)$ depends on the entries of the degree matrix M. We prove the thesis by induction on $\delta(M)$. Following the notation of Proposition 5.21, we let

$$J = \{\, j \mid d_j = m_{k(j)} - n \ \text{ for some } k(j)\}$$

and

$$\delta(M) = \sum_{j \in J} \min\{\lambda(j), \mu(j)\}.$$

Here $\lambda(j)$ is the number of rows that equal the $k(j)$-th row, and $\mu(j)$ is the number of columns that equal the j-th column (the entries on the intersection of these rows and columns form a block of 2's inside M, by our choice of $k(j)$).

If $\delta(M) = 1$, we can let C be the curve that we constructed in Theorem 5.6. These curves are all in the Linkage class of two skew lines, hence they have $\delta(M) = 1$.

So assume that we know the thesis for $\delta(M)-1$, and prove it for $\delta(M)$. Let (i,j) be such that $a_{i,j} = 2$, and assume that $j \leq i$. Let N be the submatrix of M obtained by deleting the i-th row and the j-th column of M. The entries on the diagonal of N are $a_{1,1}, \ldots, a_{j-1,j-1}, a_{j,j+1}, \ldots, a_{i-1,i}, a_{i+1,i+1}, \ldots, a_{t,t}$. They are positive, so N is a degree matrix with $\delta(N) = \delta(M) - 1$. By induction hypothesis we have an arithmetically Buchsbaum curve D with $dim(\mathcal{M}_D) = \delta(N)$, whose general plane section Y has degree matrix N. Let E be two skew lines. Let $Z = CI(1,2)$ be a general plane section of E. Using Liaison Addition, we look at $I_C = FI_E + QI_D$ where Q is a minimal generator of I_E and F is a form of degree

$$a = a_{1,1} + \ldots + a_{j-1,j-1} + a_{j,j+1} + \ldots + a_{i-1,i} + a_{i+1,i+1} + \ldots + a_{t,t} - 1 + a_{i,t+1}$$

in the ideal of I_D. Notice that

$$a - (a_{1,1} + \ldots + a_{j-1,j-1} + a_{j,j+1} + \ldots + a_{i-1,i} + a_{i+1,i+1} + \ldots + a_{t,t} + 1) =$$

$$a_{i,t+1} - 2 \geq 0.$$

Therefore

$$\alpha(I_D) \leq a_{1,1} + \ldots + a_{j-1,j-1} + a_{j,j+1} + \ldots + a_{i-1,i} + a_{i+1,i+1} + \ldots + a_{t,t} + 1 \leq a,$$

and we can find a form F as claimed. By Theorem 3.2.3 in [43] we have that:

- as sets, $C = D \cup E \cup CI(2,a)$ and
- $\mathcal{M}_C \cong \mathcal{M}_D(-2) \oplus \mathcal{M}_E(-a)$.

In particular, C is an arithmetically Buchsbaum curve and

$$dim(\mathcal{M}_C) = \delta(N) + 1 = \delta(M).$$

We still need to prove that the general plane section of C has degree matrix M. Let X be a general plane section of C. Then $I_X = FI_Z + QI_Y$, and we have the short exact sequence

$$0 \longrightarrow R(-a-2) \longrightarrow I_Y(-2) \oplus I_Z(-a) \longrightarrow I_X \longrightarrow 0.$$

Using the mapping cone argument, we obtain a free resolution for I_X of the form

$$0 \longrightarrow \begin{array}{c} R(-2-a) \oplus R(-3-a) \\ \oplus \\ \mathbf{F}_2(-2) \end{array} \longrightarrow \begin{array}{c} \mathbf{F}_1(-2) \\ \oplus \\ R(-2-a) \oplus R(-1-a) \end{array} \longrightarrow I_X \longrightarrow 0 \tag{5.1}$$

where

$$0 \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 \longrightarrow I_Y \longrightarrow 0$$

is a minimal free resolution for I_Y. Since the image of Q in I_Z is a minimal generator, the free summands $R(-2-a)$ cancel in (5.1). No other cancellation may occur, because all the other free summands come from the same minimal free resolution (the one of $I_Y(-2) \oplus I_Z(-a)$), then the maps between them are left unchanged under the mapping cone. Then X has minimal free resolution

$$0 \longrightarrow R(-3-a) \oplus \mathbf{F}_2(-2) \longrightarrow R(-1-a) \oplus \mathbf{F}_1(-2) \longrightarrow I_X \longrightarrow 0$$

and its degree matrix has size $t \times (t+1)$, and entries as follows. N is a submatrix of it, coming from the submap $\mathbf{F}_2(-2) \longrightarrow \mathbf{F}_1(-2)$. To obtain the degree matrix of X from N, we add a row and a column correponding to the map $R(-3-a) \longrightarrow R(-1-a) \oplus \mathbf{F}_1(-2)$ for the row, and $R(-3-a) \oplus \mathbf{F}_2(-2) \longrightarrow R(-1-a)$ for the column. Then the entry in intersection between the row and the column is $3 + a - (1 + a) = 2$. By homogeneity, the other entries on the row and column that we are adding are determined by only one of them. For example, the highest entry in the row is

$$3 + a - (a_{1,1} + \ldots + a_{j-1,j-1} + a_{j,j+1} + \ldots + a_{i-1,i} + a_{i+1,i+1} + \ldots + a_{t,t} + 2) = a_{i,t+1},$$

which coincides with the highest entry in the i-th row of M. This proves that the degree matrix of X is M.

We now examine the case when $a_{i,j} = 2$, for some $j > i$. Pick the maximum i and the minimum j for which $a_{i,j} = 2$. We can also assume that $a_{k,l} \neq 2$ for $k \leq l$. Proceed by induction on the size t of M. If $t = 1$ the only possibility is $M = (1, 2)$ and we can take C to be two skew lines. Consider the matrix M of size $t \times (t+1)$, and let N be the submatrix consisting of the last $t - 1$ rows and last t columns

$$N = \begin{pmatrix} a_{2,2} & a_{2,3} & \cdots & a_{2,t+1} \\ \vdots & \vdots & & \vdots \\ a_{t,2} & a_{t,3} & \cdots & a_{t,t+1} \end{pmatrix}.$$

Let D be an arithmetically Buchsbaum curve, whose general plane section Y has degree matrix N and whose deficiency module has dimension $\delta(N)$. Induction hypothesis on t gives the existence of D. If $\delta(N) = \delta(M)$, let S be a surface of degree $a_{1,2} + \ldots + a_{t,t+1}$ containing D. Such an S exists since $a_{1,2} + \ldots + a_{t,t+1} \geq 1 + a_{2,2} + \ldots + a_{t,t} \geq \alpha(I_D)$, by Proposition 5.5. Let T be a generic surface of degree $a_{1,1}$. Then $C = D \cup (S \cap T)$ is bilinked to D, therefore $dim(\mathcal{M}_C) = dim(\mathcal{M}_D) = \delta(M)$. The general plane section of C has degree matrix M, by Proposition 2.21. No cancelation occurs by genericity of the choice of T. If $\delta(N) = \delta(M) - 1$, then we can let $a_{i,j} = a_{1,j} = 2$ for some $j \geq 2$. By the induction hypothesis, we have an arithmetically Buchsbaum curve D, whose general plane section Y has degree matrix N, and such that $dim(\mathcal{M}_D) = \delta(N)$. Let E be a curve in the linkage class of two skew lines with general plane section $Z = CI(2, a_{1,2})$. Existence of E follows from Theorem 5.6. Using Liaison Addition, let $I_C = FI_E + GI_D$ where G is an element of I_E of degree 2 and F is a form of degree $a = a_{2,3} + \ldots + a_{t,t+1}$ in the ideal of I_D. F can be chosen such that its image in I_Y is a minimal generator, since the first column of N has no entry equal to 2 (see also Corollary 5.25). By Theorem 3.2.3 in [43] we have that:

- as sets, $C = D \cup E \cup CI(2, a)$ and
- $\mathcal{M}_C \cong \mathcal{M}_D(-2) \oplus \mathcal{M}_E(-a)$.

In particular, C is an arithmetically Buchsbaum curve and

$$dim(\mathcal{M}_C) = \delta(N) + 1 = \delta(M).$$

We still need to prove that the general plane section of C has degree matrix M. Let X be a general plane section of C. Then $I_X = FI_Z + GI_Y$, and we have the short exact sequence

$$0 \longrightarrow R(-a-2) \longrightarrow I_Y(-2) \oplus I_Z(-a) \longrightarrow I_X \longrightarrow 0.$$

Using the mapping cone argument, we obtain a free resolution for I_X of the form

$$0 \longrightarrow \begin{array}{c} R(-a-2) \oplus R(-2-a_{2,2}-a) \\ \oplus \\ \mathbf{F}_2(-2) \end{array} \longrightarrow \begin{array}{c} \mathbf{F}_1(-2) \\ \oplus \\ R(-a_{2,2}-a) \oplus R(-2-a) \end{array} \longrightarrow I_X \longrightarrow 0 \tag{5.2}$$

where

$$0 \longrightarrow \mathbf{F}_2 \longrightarrow \mathbf{F}_1 \longrightarrow I_Y \longrightarrow 0$$

is a minimal free resolution for I_Y. Since the image of F in I_Y is a minimal generator, the free summand $R(-2-a)$ cancels with a free summand of $\mathbf{F}(-2)$ in (5.2). No other cancelation can take place, because all the other free summands come from the same minimal free resolution (the one of $I_Y(-2) \oplus I_Z(-a)$), then the maps between them are left unchanged under the mapping cone. Let $\mathbf{F}_1 = \mathbf{F}_1' \oplus R(-a)$. Then X has minimal free resolution

$$0 \longrightarrow R(-2-a_{1,2}-a) \oplus \mathbf{F}_2(-2) \longrightarrow R(-a_{1,2}-a) \oplus R(-2-a) \oplus \mathbf{F}_1'(-2) \longrightarrow I_X \longrightarrow 0.$$

The degree matrix of X has size $t \times (t+1)$, and entries as follows. The last $t-1$ columns of N are coontained in it, since they come from the submap $\mathbf{F}_2(-2) \longrightarrow$

$\mathbf{F}_1'(-2)$. To obtain the degree matrix of X from this, we add a row and two columns correponding to the maps $R(-2-a_{1,2}-a) \longrightarrow R(-a_{1,2}-a) \oplus R(-2-a) \oplus \mathbf{F}_1'(-2)$ for the row, and $R(-2-a_{1,2}-a) \oplus \mathbf{F}_2(-2) \longrightarrow R(-a_{1,2}-a) \oplus R(-2-a)$ for the column. Then the entries in intersection between the row and the columns are $2+a_{1,2}+a-a_{1,2}-a=2$, and $2+a_{1,2}+a-2-a=a_{1,2}$. By homogeneity, the other entries on the row and columns that we are adding are determined by only one of them. For example, the highest entry in the row is

$$2+a_{1,2}+a-(a_{2,2}+\ldots+a_{t,t}+2) = a_{1,2}+a_{3,3}+\ldots+a_{t,t}+a_{2,t+1}-(a_{2,2}+\ldots+a_{t,t}) = a_{1,t+1},$$

which coincides with the highest entry in the first row of M. This proves that the degree matrix of X is M. $\square$

Remark 5.32 *For each degree matrix M containing at least an entry equal to 2, one can construct an arithmetically Buchsbaum curve C whose general plane section has degree matrix M and whose deficiency module has dimension d for each $1 \leq d \leq \delta(M)$. This can be done starting from the curves that we constructed in Theorem 5.6, then using liaison addition (possibly more than once) in an analogous way to how we did in the proof of Theorem 5.31.*

CHAPTER 6

LIFTING THE DETERMINANTAL PROPERTY

In this chapter, we investigate the question of whether it is possible to lift the property of being standard or good determinantal from the general hyperplane section of a scheme to the scheme itself. For schemes of codimension 2, the Hilbert-Burch Theorem states that being standard determinantal is equivalent to being Cohen-Macaulay. So this question can be regarded as a natural generalization of the questions that we investigated in the previous chapters. Naturally, we expect that we must impose some conditions either on the general hyperplane section of the scheme (in the spirit of the previous chapters) or on the scheme itself. In codimension 3 or higher the question immediately appears to be much more complicated than in the codimension 2 situation.

6.1 First results and examples

In this section, we consider a scheme $V \subset \mathbf{P}^{n+1}$ with general hyperplane section $X \subset \mathbf{P}^n$. We assume that the general hyperplane section X is standard/good determinantal, and we deduce a simple condition that forces the scheme V to have the same graded Betti numbers as a standard/good determinantal scheme. We also construct examples of arithmetically Cohen-Macaulay schemes of codimension 3 that are not standard determinantal, but whose general hyperplane section is good determinantal. Moreover, using a result of R. M. Miró-Roig and J. Kleppe, we show that if one hyperplane section of V by a hyperplane that meets it properly is good determinantal, then a general hyperplane section is also good determinantal. In this sense, the property of being good determinantal for a hyperplane section is an open property. We start by observing a series of problems that one has to face while studying determinantal schemes, that do not appear in the codimension 2 (arithmetically Cohen-Macaulay) situation.

Remark 6.1 *For a scheme of codimension* 2*, the property of being standard determinantal can be decided by checking the graded Betti numbers. However, for a scheme of codimension* 3 *or higher this is no longer the case. In fact, one must check that the maps in the minimal free resolution of the saturated ideal of the scheme are of Eagon-Northcott type. That is, one needs to check that some Eagon-Northcott complex gives a minimal free resolution for the ideal in question. This corresponds to the fact that there are schemes that have the same graded Betti numbers of the standard determinantal schemes, but that have different algebraic structure. See Example 6.9 for an example of this.*

From the results in the previous chapters, one can easily obtain a sufficient condition for a scheme $V \subset \mathbf{P}^{n+1}$ to be arithmetically Cohen-Macaulay. If the general hyperplane section $X \subset \mathbf{P}^n$ of V is standard determinantal, the condition can be expressed in terms of the entries of its degree matrix.

Corollary 6.2 *Let* $V \subset \mathbf{P}^{n+1}$ *be a scheme with general hyperplane section* $X \subset \mathbf{P}^n$*. If* X *is standard determinantal with degree matrix* $M = (a_{ij})_{i=1,\ldots,t;\, j=1,\ldots,t+c-1}$ *and*

$$a_{t,1} + \cdots + a_{t,c-1} \geq n + 1,$$

then V *is arithmetically Cohen Macaulay. In particular,* V *has the same graded Betti numbers as* X*.*

Proof: If $dim(X) \geq 1$ there is nothing to prove. We can then reduce to the case when X is zero-dimensional. From Theorem 2.12, it follows that the minimum degree of a minimal generator of I_X that is not the image of a minimal generators of I_V under the standard projection map is

$$b \geq a_{1,1} + \cdots + a_{t,t} + a_{t,t+1} + \cdots + a_{t,t+c-1} - n =$$

$$= a_{t,1} + \cdots + a_{t,c-1} + a_{1,c} + a_{2,c+1} + \cdots + a_{t,t+c-1} \geq a_{1,c} + a_{2,c+1} + \cdots + a_{t,t+c-1} + 1.$$

In particular, it is bigger than the maximum $a_{1,c}+a_{2,c+1}+\cdots+a_{t,t+c-1}$ of the degrees of the minimal generators of I_X. Then all the minimal generators of I_X are images of the minimal generators of I_V, and V is arithmetically Cohen-Macaulay. $\square$

In some very special cases, the graded Betti numbers of a homogeneous ideal I can force the ideal to be standard determinantal, even when the codimension is 3 or higher.

Example 6.3 *Let $R = k[x_0, x_1, \ldots, x_n]$, $\mathfrak{m} = (x_0, x_1, \ldots, x_n)$, $t > n$. Let $I \subset R$ be a homogeneous ideal, minimally generated by $\binom{t+n}{n}$ forms of degree t. Then $I_j = 0$ for all $j < t$ and $dim(I)_t = \binom{t+n}{n} = dim(\mathfrak{m}^t)_t$. Therefore $I = \mathfrak{m}^t$, so I is the ideal of maximal minors of the matrix*

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 \\ 0 & x_0 & x_1 & \cdots & x_n & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & x_0 & x_1 & \cdots & x_n & 0 \\ 0 & \cdots & \cdots & 0 & x_0 & x_1 & \cdots & x_n \end{pmatrix}.$$

Then I is good determinantal.

Example 6.4 *Any $X \subset \mathbf{P}^n$ zero-dimensional scheme that has the same graded Betti numbers as $\binom{t+1}{2}$ generic points (for $t > n$) has an Artinian reduction that is good determinantal, as we saw in the previous example. However, we expect that not all of these schemes are standard determinantal.*

It is worth mentioning that the good determinantal property does not behave as well as the standard determinantal property under hyperplane sections by a hyperplane that meets the scheme properly.

Remark 6.5 *Any hyperplane section of a standard determinantal subscheme of* $\mathbf{P}^{n+1}$ *by a hyperplane that meets it properly is a standard determinantal subscheme of* $\mathbf{P}^n$*. It is not true in general that any hyperplane section of a good determinantal subscheme of* $\mathbf{P}^{n+1}$ *by a hyperplane that meets it properly is a good determinantal subscheme of* $\mathbf{P}^n$*. However, a general hyperplane section is good determinantal.*

Next, we see an example when this is the case. The example of the standard determinantal scheme supported on a point that is not good determinantal is Example 4.1 in [35].

Example 6.6 *Let* $C \subset \mathbf{P}^4$ *be a curve whose homogeneous saturated ideal is given by the maximal minors of*

$$\begin{pmatrix} x_0 & x_1+x_4 & 0 & x_2 \\ 0 & x_1 & x_2 & x_0+x_1 \end{pmatrix}.$$

One can check that C *is one-dimensional, hence standard determinantal, computing the minimal free resolution of* I_C *over* $S = k[x_0, \ldots, x_4]$*. The curve* C *is indeed good determinantal, since deleting a generalized row we obtain the matrix of size* 1×4

$$\begin{pmatrix} x_0 & x_1+\alpha x_4 & x_2 & x_0+x_1+\alpha x_2 \end{pmatrix}$$

for a generic value of α*. For* $\alpha \neq 0$ *the entries form a regular sequence, since they are linearly independent linear forms. Therefore they define a complete intersection, that is a standard determinantal scheme, and* C *is good determinantal.*

Let H *be a general linear form. In particular, we can assume that the coefficient of* x_3 *in the equation of* H *is non-zero. Intersecting* C *with* H *we obtain a subscheme* X *of* $\mathbf{P}^3$*, whose saturated homogeneous ideal* I_X *is generated over*

$R = k[x_0, \ldots, x_3]$ by the maximal minors of

$$\begin{pmatrix} x_0 & x_1 + x_4 & 0 & x_2 \\ 0 & x_1 & x_2 & x_0 + x_1 \end{pmatrix}.$$

One can show that X is good determinantal following the same steps as for C.

Let $H = x_4$. Intersecting C with H we obtain a subscheme Z of $\mathbf{P}^3$, whose saturated homogeneous ideal I_Z is generated over $R = k[x_0, \ldots, x_3]$ by the maximal minors of

$$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_1 & x_2 & x_0 + x_1 \end{pmatrix}.$$

$I_Z = I_P^2$ for $P = [0 : 0 : 0 : 1]$, hence Z is a zero-dimensional scheme supported on the point P. Then Z is standard determinantal and a section of C by a hyperplane that meets it properly. However, Z is not good determinantal. In fact, deleting a generalized row we obtain the matrix of size 1×4

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_0 + x_1 + \alpha x_2 \end{pmatrix}$$

whose entries generate the ideal (x_0, x_1, x_2) of codimension $3 < 4$. Then, deleting a generalized row we obtain a scheme that is not standard determinantal, so Z is not good determinantal.

The next proposition gives an example of an ideal that is not standard determinantal, but that has the same graded Betti numbers as some standard determinantal ideal.

Proposition 6.7 *Let X be a symmetric matrix of indeterminates, of size $(t+1) \times (t+1)$, $t \geq 2$*

$$X = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & \cdots & x_{0,t} \\ x_{0,1} & x_{1,1} & \cdots & \cdots & x_{1,t} \\ \vdots & \vdots & & & \vdots \\ x_{0,t} & x_{1,t} & \cdots & \cdots & x_{t,t} \end{pmatrix}.$$

Let $V \subset \mathbf{P}^n$, $n = \binom{t+2}{2}$, be the scheme corresponding to the saturated ideal $I_V = I_t(X) \subset R = k[\ x_{i,j} \mid 0 \leq i \leq j \leq t\ ]$, generated by the submaximal minors of X. We have the following facts about V:

1. *V is an arithmetically Cohen-Macaulay, integral scheme of codimension* 3

2. *the Artinian reduction of R/I_V is isomorphic to $k[y_0, y_1, y_2]/(y_0, y_1, y_2)^t$, so the minimal free resolution of I_V as an R-module is of the form*

$$0 \longrightarrow R(-t-2)^{\binom{t+1}{2}} \longrightarrow R(-t-1)^{3\binom{t+1}{2}-\binom{t}{2}} \longrightarrow R(-t)^{\binom{t+2}{2}} \longrightarrow I_V \longrightarrow 0$$

3. *I_V is of linear type; in particular the cardinality of a minimal generating system for $(I_V)^r$ is $\binom{m+r-1}{r}$, where $m = \binom{t+2}{2}$ is the cardinality of a minimal system of generators of I_V*

4. *I_V is not standard determinantal*

Proof: 1. and 2. are proved in Aldo Conca's Ph.D. Thesis (see [9], Theorem 4.4.14 and Example 4.5.8). He also shows that the Poincaré Series of R/I_V is

$$P_V(z) = \frac{1 + 3z + \ldots + \binom{i+2}{2}z^i + \ldots + \binom{t+1}{2}z^{t-1}}{(1-z)^{n-3}}.$$

In particular, the cardinality of a minimal system of generators of I_V is $m = \binom{t+2}{2}$.

3. The proof that I_V is of linear type can be found in [36], Proposition 2.10. This implies that the fiber cone of R/I_V is a polynomial ring. Hence the cardinality of a minimal system of generators of $(I_V)^r$ is $\binom{m+r-1}{r}$. Notice that this is the maximum possible cardinality for a minimal system of generators of I_V, given that m is the number of minimal generators of I_V.

4. If I_V was standard determinantal, its degree matrix would have size $t \times (t+2)$ all of its entries would be equal to 1. The number of Plücker relations for a matrix of size $t \times (t+2)$ is $\binom{t+2}{4}$. So the cardinality of a minimal system of generators for I_V^2 would be less than or equal to $\binom{m+1}{2} - \binom{t+2}{4}$, $m = \binom{t+2}{2}$. But this contradicts the fact that I_V^2 has the maximum possible number of minimal generators, $\binom{m+1}{2}$.

Another way to see that V is not standard determinantal is the following. If it were, it would be a rational normal scroll. But the Picard group of V is isomorphic to $\mathbf{Z}_2$ (see [24]), while the Picard group of a rational normal scroll is isomorphic to $\mathbf{Z}$. □

Remark 6.8 *Notice that the Artinian reduction of the coordinate ring of V is good determinantal, as we showed in Example 6.3.*

In the next example, we consider the Veronese surface $V \subset \mathbf{P}^5$. We observe that V is not standard determinantal, but its general hyperplane section is good determinantal.

Example 6.9 *The Veronese surface $V \subset \mathbf{P}^5$ is an example from the family of Proposition 6.7, for $t = 2$. In fact, its homogeneous saturated ideal is the ideal*

$$I_V = I_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

$I_V \subset R = k[x_0, \ldots, x_5]$. Its general hyperplane section is a reduced and irreducible aritmetically Cohen-Macaulay curve $C \subset \mathbf{P}^4$, of degree 4, hence a rational normal curve. In particular, the general hyperplane section of V is good determinantal, with defining matrix equal, after a change of coordinates and elementary row and column operations, to the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

Remark 6.10 *The matrix whose submaximal minors define the ideal of the general hyperplane section of the Veronese surface is*

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & L & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

where $L \in k[x_0, x_1, x_2, x_4, x_5]$ is a general linear form. One can check that it is not true that after a change of coordinates and elementary row and column operations the matrix M can be reduced to the matrix

$$N = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

whose submaximal minors coincide with the maximal minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix}.$$

So the matrices M and N cannot be reduced to each other via a change of coordinates and elementary row and column operations. However, the submaximal minors of M and N generate the same ideal (possibly after a change of coordinates).

The Veronese surface is an example of an arithmetically Cohen-Macaulay, smooth and connected scheme that is not standard determinantal, but whose general hyperplane section is standard (and even good) determinantal. We are now going to see that this is the case for all the schemes of Proposition 6.7.

Proposition 6.11 *Let $V \subset \mathbf{P}^n$, $n = \binom{t+2}{2}$ be the scheme associated to the saturated homogeneous ideal $I_V = I_t(X)$, as in Proposition 6.7. A general $\binom{t}{2}$-th hyperplane section of V is a good determinantal scheme.*

Proof: Consider first a special $\binom{t}{2}$-th hyperplane section of V, whose saturated ideal is generated by the submaximal minors of the homogeneous matrix

$$Y = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,t} \\ x_{0,1} & x_{0,2} & & x_{0,t} & x_{1,t} \\ x_{0,2} & & x_{0,t} & x_{1,t} & \vdots \\ \vdots & x_{0,t} & x_{1,t} & & x_{t-1,t} \\ x_{0,t} & x_{1,t} & \cdots & x_{t-1,t} & x_{t,t} \end{pmatrix}.$$

We obtain this section by intersecting with the hyperplanes $x_{i,j} - x_{0,i+j}$ for $i+j \leq t$ and $i \geq 1, j \leq t-1$ and $x_{i,j} - x_{i+j-t,t}$ for $i+j > t$ and $i \geq 1, j \leq t-1$. We take $\binom{t}{2}$ hyperplane sections, by hyperplanes that meet V properly, so we obtain a scheme $C \subset \mathbf{P}^{2t+1}$ of codimension 3. C is good determinantal, with defining matrix

$$U = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,t-1} & x_{0,t} & x_{1,t} \\ x_{0,1} & x_{0,2} & & x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} \\ x_{0,2} & & x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} & \vdots \\ \vdots & x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} & & \vdots \\ x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} & \cdots & \cdots & x_{t,t} \end{pmatrix}.$$

The maximal minors of U coincide with the submaximal minors of Y. Let D be the general $\binom{t}{2}$-th hyperplane section of V. The saturated ideal of D is the ideal

$I_D = I_t(Z)$ generated by the submaximal minors of the symmetric matrix

$$Z = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,t-1} & x_{0,t} \\ x_{0,1} & L_{1,1} & \cdots & L_{1,t-1} & x_{1,t} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{0,t-1} & L_{1,t-1} & \cdots & L_{t-1,t-1} & x_{t-1,t} \\ x_{0,t} & x_{1,t} & \cdots & x_{t-1,t} & x_{t,t} \end{pmatrix}.$$

We can assume without loss of generality that the equations of the hyperplanes that we intersect with V are $x_{i,j} - L_{i,j}$, $i \geq 1, j \leq t-1$, where $L_{i,j}$ is a general linear form in $k[x_{0,0}, \ldots, x_{0,t}, x_{1,1}, \ldots, x_{1,t}]$. We want to show that D is standard determinantal. Observe that D can be deformed to C. In fact, we have a flat family of curves D_s, whose saturated ideal is $I_t(Z_s)$, $Z_s = sZ + (1-s)Y$, for generic values of the parameter s. Applying Proposition 10.7 in [35], we have that C is unobstructed and that $dim_C Hilb^p(\mathbf{P}^{2t+1}) = dim W$, where $Hilb^p(\mathbf{P}^{2t+1})$ is the Hilbert scheme parametrizing subschemes of $\mathbf{P}^{2t+1}$ whose Hilbert polynomial is the same as the Hilbert polynomial p of C and $W \subseteq Hilb^p(\mathbf{P}^{2t+1})$ is the locus of good determinantal schemes, whose degree matrix is the same as the one of C. Since C is a smooth point of $Hilb^p(\mathbf{P}^{2t+1})$, we have that the irreducible component of $Hilb^p(\mathbf{P}^{2t+1})$ containing C, contains D as well. Since W is an open subset of the irreducible component of the Hilbert scheme that contains it, a generic D belongs to W. $\square$

The key point is a result of Kleppe, Migliore, Miró-Roig, Nagel and Peterson that states that the locus of good determinantal schemes with a fixed degree matrix M is locally closed in the corresponding Hilbert scheme (see [35], Chapters 9 and 10). The standard determinantal schemes with a fixed degree matrix M belong to the closure of the locus of the good determinantal schemes with the same degree matrix M. We present an easy example that shows how the closure of the locus of good determinantal schemes in the Hilbert scheme can contain also schemes that are not standard determinantal (or not even arithmetically Cohen-Macaulay).

Example 6.12 *Consider the Hilbert scheme parametrising curves of degree* 9 *and genus* 10 *in* $\mathbf{P}^3$. *This is the Hilbert scheme* H *whose points correspond to subschemes of* $\mathbf{P}^3$ *with Hilbert polynomial* $p(z) = 9z - 9$.

Let D *be the locus of* H *whose points correspond to complete intersections of type* $(3, 3)$. *Let* E *be the locus of* H *whose points correspond to curves of type* $(3, 6)$ *on a smooth surface. The elements of* E *are non-aCM. In fact, up to linear equivalence, a curve of type* $(3, 6)$ *is* $C = C_1 \cup C_2$ *where* C_1 *consists of* 3 *skew lines and* C_2 *consists of* 6 *skew lines. Moreover, each line of* C_1 *intersects each line of* C_2, *so* $C_1 \cap C_2$ *consists of* 18 *distinct points. Let* $I_C \subset R = k[x_0, x_1, x_2, x_3]$ *be the ideal corresponding to* C. *One can check that the minimal free resolution of* I_C *as an* R*-module is*

$$0 \longrightarrow R^2(-8) \longrightarrow R^6(-7) \longrightarrow R^4(-6) \oplus R(-2) \longrightarrow I_C \longrightarrow 0.$$

In particular, C *is non-aCM.*

By the uppersemicontinuity principle, no point of the closure of E *can be aCM, so* E *is closed. But since* H *is connected, the closure of* D *needs to intersect* E, *therefore there is a point in the closure of* D *that corresponds to a non-aCM curve. Notice that non-aCM schemes and standard determinantal schemes coincide in the codimension* 2 *case. So this shows that the closure of the locus of good determinantal schemes in the Hilbert scheme can contain also schemes that are not standard determinantal (and not even arithmetically Cohen-Macaulay).*

The following Theorem states that having a good determinantal hyperplane section is an "open condition", for an arithmetically Cohen-Macaulay scheme of codimension 3 and dimension at least 3. Notice that since we are working with schemes of positive dimension, it is not restrictive to assume that S is arithmetically Cohen-Macaulay. In fact, C aCM forces S to be aCM as well.

Theorem 6.13 *Let $S \subset \mathbf{P}^{n+1}$, be an arithmetically Cohen-Macaulay scheme of codimension 3, $n \geq 5$. If a proper hyperplane section C of S is good determinantal, then a general hyperplane section of S is good determinantal.*

Proof: Let H be a hyperplane that meets S properly and let $C = S \cap H$. Let D be a general section of S by a hyperplane L. Assume without loss of generality that H is the hyperplane of equation $x_{n+1} = 0$, and that the coefficient of x_{n+1} in the equation of L is different from zero. Look at the family of schemes

$$C_s = S \cap H_s, \quad H_s = sL + (1-s)H.$$

For generic values of the parameter s, C_s is a codimension c subscheme of the hyperplane H_s, and the coefficient of x_{n+1} in the equation of H_s is nonzero. Moreover, $C_0 = C$ and $C_1 = D$. Consider the changes of coordinates

$$\begin{array}{rcl} \varphi_s : \mathbf{P}^{n+1} & \longrightarrow & \mathbf{P}^{n+1} \\ x_0 & \longmapsto & x_0 \\ \vdots & & \vdots \\ x_n & \longmapsto & x_n \\ H_s & \longmapsto & x_{n+1} \end{array}$$

Notice that $\varphi_s(H_s) = H$ for all s. Consider the family of curves $D_s := \varphi_s(C_s) \subset H = \mathbf{P}^n$. We identify C with $\varphi_0(C)$ and D with $\varphi_1(D)$, thinking of them as codimension 3 subschemes of $\mathbf{P}^n$. D_s is a flat family of subschemes of $\mathbf{P}^n$ containing C, D. Observe that all the curves in the flat family have the same Hilbert polynomial, in fact the same graded Betti numbers, because they are sections of the aCM scheme S by hyperplanes that meet it properly.

C has dimension $n - 3 \geq 2$, by assumption. Look at the Hilbert scheme $Hilb^p(\mathbf{P}^n)$, where p is the Hilbert polynomial of C. Applying Proposition 10.7 in [35], we have that C is unobstructed and that $dim_C Hilb^p(\mathbf{P}^n) = dim W$, where $W \subseteq Hilb^p(\mathbf{P}^n)$ is the locus of good determinantal schemes, whose degree matrix is the same as the one of C. Since C is a smooth point of $Hilb^p(\mathbf{P}^n)$, we have that the irreducible component of $Hilb^p(\mathbf{P}^n)$ containing C contains D as well. Since W is a locally closed subset of the Hilbert scheme, a generic D belongs to W. Therefore, a general hyperplane section D of S is good determinantal. $\square$

Under some extra assumptions, one can prove an analogous result for the general hyperplane section of a surface $S \subset \mathbf{P}^5$.

Theorem 6.14 *Let $S \subset \mathbf{P}^5$, be an arithmetically Cohen-Macaulay surface. Assume that a proper hyperplane section C of S is good determinantal, with defining matrix A and degree matrix $M = (a_{i,j})_{i=1,\dots,t;\, j=1,\dots t+2}$. Let B be the matrix obtained deleting the k-th column of A. Let T be the surface whose saturated ideal is the ideal of maximal minors of B. If*

$$H^1(\mathcal{I}_T/\mathcal{I}_T^2(-a_{1,1} - \ldots - a_{k-1,k-1} - a_{k,k+1} - \ldots - a_{t,t+1} - a_{1,t+2} + a_{1,k})) = 0$$

then a general hyperplane section of S is good determinantal.

Proof: The proof is the exact analogue of the proof of Theorem 6.13. In order to apply Proposition 10.7 in [35] to a curve, we need the extra hypothesis on the vanishing of the cohomology. $\square$

Remark 6.15 *Theorem 6.14 gives us another way to show that a general hyperplane section of the Veronese surface is a good determinantal curve. In fact, we know that one hyperplane section of it is rational normal quartic curve in $\mathbf{P}^4$, then it is good determinantal with defining matrix*

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

Let $T \subset \mathbf{P}^4$ be the surface with defining matrix

$$B = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

obtained deleting the last column of A. The ideal of maximal minors of B is $I_T \subset R = k[x_0, \dots, x_4]$. Since none of the minimal generators of I_T involves x_4, T is a *cone over a twisted cubic of* $\mathbf{P}^3$*. So one can easily show that both* R/I_T *and* R/I_T^2 *are Cohen-Macaulay. In particular,* $H^1_*(\mathcal{I}_T) = 0$ *and* $H^2_*(\mathcal{I}_T^2) = 0$*. From the long exact cohomology sequence*

$$\cdots \longrightarrow H^1_*(\mathcal{I}_T) \longrightarrow H^1_*(\mathcal{I}_T/\mathcal{I}_T^2) \longrightarrow H^2_*(\mathcal{I}_T^2) \longrightarrow \cdots$$

we see that $H^1_*(\mathcal{I}_T/\mathcal{I}_T^2) = 0$*, then the hypotheses of Theorem 6.14 are verified. So a general hyperplane section of the Veronese surface is good determinantal.*

In Corollary 2.15, we saw that if all the entries of the degree matrix of a general plane section of some curve $C \subset \mathbf{P}^3$ are at least 3, we can lift the property of being arithmetically Cohen-Macaulay from the general plane section of C to the curve itself. In codimension 2, being arithmetically Cohen-Macaulay is equivalent to being standard determinantal. In analogy with the codimension 2 case, one could ask the following.

Question 6.16 *Let* $V \subset \mathbf{P}^{n+1}$ *be an aCM scheme and let* $X \subset \mathbf{P}^n$ *be its general hyperplane section. Assume that* X *is standard/good determinantal. Does there exist an* N *such that if all the entries of the degree matrix of* X *are at least* N*, then* V *is standard/good determinantal?*

Unfortunately, the next proposition shows that one cannot hope to obtain a result in those lines. In fact, the entries of the degree matrix M in the proposition can be taken arbitrarily large.

Proposition 6.17 *Let* $M = (a_{ij})_{i=1,\dots,t;\, j=1,\dots,t+2}$ *be a degree matrix with the property that*

$$a_{ij} = a_{kl} \ \text{ if } \ i+j = k+l.$$

Then there exist schemes $X \subset \mathbf{P}^{2t}$ *and* $V \subset \mathbf{P}^{\binom{t}{2}+2t}$ *such that:*

- X *is the general* $\binom{t}{2}$*-th hyperplane section of* V*,*

- *X is good determinantal with degree matrix M, and*

- *V is not standard determinantal.*

Proof: Let $A = (x_{i+j-2}^{a_{ij}})_{i=1,\ldots,t;\, j=1,\ldots,t+2}$, $t \geq 3$. So A is a homogeneous matrix with entries in $R = k[x_0, \ldots, x_{2t}]$. The degree matrix of A is M. Let $X \subset \mathbf{P}^{2t}$ be the scheme whose homogeneous ideal is generated by the maximal minors of A, $I_X = I_t(A)$. X is a scheme of dimension $2t-3$ and codimension 3, and it is clearly good determinantal. I_X coincides with the ideal of submaximal minors of the symmetric matrix $N = (x_{i+j-2}^{a_{ij}})_{i=1,\ldots,t+1;\, i=1,\ldots,t+1}$.

Let $B = (F_{ij})$ where

$$F_{ij} = \begin{cases} x_{i+j-2}^{a_{ij}} & \text{if } i = 1, t+1 \text{ or } j = 1, t+1, \\ y_{ij}^{a_{ij}} & \text{if } i \leq j, \\ y_{ji}^{a_{ij}} & \text{if } i > j. \end{cases}$$

B is a homogeneous matrix with entries in $S = R[y_{ij}]$. Let $V \subset \mathbf{P}^{\binom{t}{2}+2t}$ be the scheme associated to the ideal of submaximal minors of B, $I_V = I_t(B)$. V is not standard determinantal. This can be proved in an analogous way to the proof of part 4 of Proposition 6.7, verifying that the number of minimal generators of I_V^2 is too large for V to be standard determinantal.

Intersect V with $\binom{t}{2}$ hyperplanes of equation $y_{ij} - x_{i+j-2}$ for all i, j. We obtain the scheme X, so V has a section that is good determinantal. Counting dimensions, one can check that X is a section of V by a sequence of hyperplanes that intersect the scheme properly. By Theorem 6.13 we conclude that a general $\binom{t}{2}$-th hyperplane section of V is good determinantal. $\square$

We now see that, by linking the Veronese surface, one can construct a family of non-standard determinantal surfaces in $\mathbf{P}^5$, whose general hyperplane section is a standard determinantal curve. The Lemmas below closely follow the results in Chapter 3 of [35].

Lemma 6.18 *Let $C \subset S \subset \mathbf{P}^n$ be standard determinantal schemes. Assume that the saturated ideal of C is generated by the maximal minors of a $t \times (t+c)$ matrix, and that the matrix defining S is obtained from the one of C by adding a row. Let D be a basic double G-link of C on S. Then D is standard determinantal.*

Proof: Let M be the homogeneous matrix associated to C. The matrix M has size $t \times (t+c)$. Let $I_C = I_t(M)$. The scheme C is standard determinantal, i.e. it has codimension $c+1$. Let N be a matrix obtained by adding a row to M, in such a way that $I_S = I_{t+1}(N)$. Since S is standard determinantal, it has codimension c. Notice that S is good determinantal by construction, in particular it is generically a complete intersection (see [35], Remark 3.5). Let D be a basic double G-link of C on S, $D = C \cup (S \cap F)$ for some hypersurface F that meets S properly. The saturated ideal of D is $I_D = I_S + F \cdot I_C$ (see Proposition 2.24), so it is minimally generated by the maximal minors of the matrix obtained by adding to N a column vector, whose entries are all equal to 0, except for an F in the last entry.

In other words, let $M = (m_{i,j})_{i=1,\dots,t;\, j=1,\dots,t+c}$ and $N = (n_{i,j})_{i=1,\dots,t;\, j=1,\dots,t+c}$, with $n_{i,j} = m_{i,j}$ for $i \leq k-1$, $n_{i,j} = m_{i-1,j}$ for $i \geq k+1$ (inserting a row in position k). If $deg(n_{k,l-1}) \leq deg(F) \leq deg(u_{k,l})$, then the defining matrix of D is $O = (o_{i,j})$ with $o_{i,j} = n_{i,j}$ for $j \leq l$, $o_{k,l} = F$, $o_{i,l} = 0$ for $i \neq k$ and $o_{i,j} = n_{i,j-1}$ for $j \geq l+1$. $\square$

There is another way that we can preserve the standard determinantal property of a scheme C, performing a basic double G-link on a standard determinantal scheme.

Lemma 6.19 *Let $C \subset S \subset \mathbf{P}^n$ be standard determinantal schemes. Assume that the saturated ideal of C is generated by the maximal minors of a $t \times (t+c)$ matrix, and the matrix defining S is obtained from the one defining C by deleting a column. Let D be a basic double G-link of C on S. Then D is standard determinantal.*

Proof: Let M be the homogeneous matrix associated to C. The matrix M has size $t \times (t+c)$. Let $I_C = I_t(M)$. The scheme C is standard determinantal, i.e. it has codimension $c+1$. Let N be a matrix obtained by deleting the k-th column of M, in such a way that $I_S = I_t(N)$. Since S is standard determinantal, it has codimension c. Notice that all of the minimal generators of I_S are also minimal generators of I_C. Let D be a basic double G-link of C on S, $D = C \cup (S \cap F)$ for some hypersurface F that meets S properly. The saturated ideal of D is $I_D = I_S + F \cdot I_C$ (see Proposition 2.24), so it is minimally generated by the maximal minors of the matrix obtained by adding to N a column whose entries are the entries of the k-th column of M multiplied by F. $\square$

Remark 6.20 *Notice that, in the situation of Lemma 6.18 and Lemma 6.19, if C is good determinantal then D is good determinantal as well (and viceversa).*

We can summarize the results of the two Lemmas in the following statement.

Proposition 6.21 *Let $C \subset S$ be standard determinantal schemes, such that C has codimension one in S. Assume that, for a suitable choice of defining matrices M and N for C and S, either M is a submatrix of N, or viceversa. Then any basic double G-link D of C on S is standard determinantal. Moreover, if C is good determinantal then D is good determinantal. In this sense, the property of being standard/good determinantal is preserved under basic double G-linkage.*

Remark 6.22 *A basic double G-link of a standard determinantal scheme on a standard determinantal scheme is not always standard determinantal. The assumptions about the defining matrices in Lemma 6.18 and Lemma 6.19 come from the necessity of having some "compatibility" between the maps in the minimal free resolutions of C and S, in order to be able to control the maps in the minimal free resolution of the basic double G-link D.*

We now see an example of a standard determinantal scheme whose basic double G-link on a standard determinantal scheme is not standard determinantal.

Example 6.23 *Let $C \subset \mathbf{P}^4$ be the good determinantal scheme corresponding to the ideal of maximal minors of*

$$\begin{pmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix}$$

and let $P = [0:0:0:0:1]$ be a point on C. Clearly P is good determinantal, since it is a complete intersection. Let $F = x_0^2$; one can easily check that F meets C properly. Then the minimal free resolution of the scheme $Z = P \cup (C \cap F)$ is

$$0 \longrightarrow \begin{matrix} R^3(-10) \\ \oplus \\ R(-6) \end{matrix} \to \begin{matrix} R^{11}(-8) \\ \oplus \\ R^4(-5) \end{matrix} \longrightarrow \begin{matrix} R^{14}(-6) \\ \oplus \\ R^6(-4) \end{matrix} \longrightarrow \begin{matrix} R^6(-4) \\ \oplus \\ R^4(-3) \end{matrix} \longrightarrow I_Z \longrightarrow 0.$$

If Z was standard determinantal, from the degrees of the minimal generators of I_Z we see that the degree matrix of Z has to be one of the following:

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

But a standard determinantal scheme with degree matrix L has codimension $5-3+1=3$, while Z has codimension 4. Therefore, if Z is standard determinantal, then it has degree matrix M. But the graded Betti numbers of a standard determinantal scheme X with degree matrix M are

$$0 \longrightarrow R^4(-9) \longrightarrow \begin{matrix} R^3(-8) \\ \oplus \\ R^{12}(-7) \end{matrix} \longrightarrow \begin{matrix} R^8(-6) \\ \oplus \\ R^{12}(-5) \end{matrix} \longrightarrow \begin{matrix} R^6(-4) \\ \oplus \\ R^4(-3) \end{matrix} \longrightarrow I_X \longrightarrow 0.$$

Therefore Z is not standard determinantal.

We now want to show that a correspondent result holds for some non standard determinantal schemes: their basic double G-link on a standard determinantal scheme is not standard determinantal. This will give us a way to produce more examples of schemes that are not standard determinantal, but whose general plane section is good determinantal.

Proposition 6.24 *Let $C \subset S \subset \mathbf{P}^n$ be projective schemes. Assume that C has codimension 1 in S. Assume that both $(I_C)^2$ and $(I_S)^2$ have the maximum possible number of minimal generators (for a given number of minimal generators for I_C and I_S), and that $I_C \cdot I_S$ is minimally generated by all the products of a minimal generator of I_C and one of I_S. Let I_D be a basic double G-link of C on S. Then $(I_D)^2$ has the maximum possible number of minimal generators $\binom{m+1}{2}$, where m is the cardinality of a minimal generating set for I_D.*

Proof: Let $D = C \cup (F \cap S)$, for F a hypersurface of degree d that meets S properly. By Proposition 2.24, we have the following short exact sequence

$$0 \longrightarrow I_S(-d) \longrightarrow I_S \oplus I_C(-d) \longrightarrow I_D \longrightarrow 0.$$

Let s, u be the cardinalities of minimal system of generators of I_S and I_C, respectively. Then the cardinality r of a minimal system of generators of I_D is less than or equal to $s + u$. We will show that the cardinality of a minimal system of generators of I_D^2 is $\binom{r+1}{2}$, in particular this forces r to be the exact number of minimal generators of I_D. Moreover, D can't be standard determinantal. In fact, if it had a defining matrix of size $t \times (t + c - 1)$, the cardinality of a minimal system of generators of I_D^2 would be less than or equal to $\binom{r+1}{2} - \binom{t+2}{4}$ (see also Proposition 6.7, 4.).

$$I_D^2 = (I_S + F \cdot I_C)^2 = I_S^2 + F \cdot I_S \cdot I_C + F^2 \cdot I_C^2.$$

Let F_i, $i = 1, \ldots, s$ be a minimal system of generators for I_S and G_j, $j = 1, \ldots, u$ be a minimal system of generators of I_C. Clearly, the set F_iF_k with $i \le k$, $F^2G_jG_l$ with $j \le l$ and FG_iG_j generates I_D^2. We want to show that this system of generators is minimal, so we need to check that none of the following can happen:

1. $F_aF_b = \sum_{i \le j,\ (i,j) \ne (a,b)} \alpha_{ij} F_iF_j + \sum_{k \le l} \beta_{kl} F^2 G_k G_l + \sum_{m,n} \gamma_{mn} F F_m G_n$
2. $FF_aG_b = \sum_{i \le j} \alpha_{ij} F_iF_j + \sum_{k \le l} \beta_{kl} F^2 G_k G_l + \sum_{m,n,\ (m,n) \ne (a,b)} \gamma_{mn} F F_m G_n$
3. $F^2G_aG_b = \sum_{i \le j} \alpha_{ij} F_iF_j + \sum_{k \le l,\ (k,l) \ne (a,b)} \beta_{kl} F^2 G_k G_l + \sum_{m,n} \gamma_{mn} F F_m G_n$

If (1) happens, then

$$F_aF_b - \sum_{i\leq j,\ (i,j)\neq(a,b)} \alpha_{ij}F_iF_j = F\cdot(\sum_{k\leq l}\beta_{kl}FG_kG_l + \sum_{m,n}\gamma_{mn}F_mG_n),$$

so

$$\sum_{k\leq l}\beta_{kl}FG_kG_l + \sum_{m,n}\gamma_{mn}F_mG_n \in I_S^2 : F = I_S^2.$$

Therefore

$$\sum_{k\leq l}\beta_{kl}FG_kG_l + \sum_{m,n}\gamma_{mn}F_mG_n = \sum_{i\leq j}\delta_{ij}F_iF_j$$

and

$$(1-\delta_{ab}F)F_aF_b = \sum_{i\leq j,\ (i,j)\neq(a,b)}(\alpha_{ij}+\delta_{ij}F)F_iF_j.$$

If $\delta_{ab} = 0$, this contradicts the minimality of F_iF_k with $i \leq k$ as a system of generators for I_S^2. So $\delta_{ab} \neq 0$, and by taking the homogeneous components of the equality above, we again get an expression for F_aF_b in terms of the other minimal generators of I_S^2. This is a contradiction.

If (2) happens, then

$$F\cdot(F_aG_b - \sum_{m,n,\ (m,n)\neq(a,b)}\gamma_{mn}F_mG_n + \sum_{k\leq l}\beta_{kl}FG_kG_l) = \sum_{i\leq j}\alpha_{ij}F_iF_j,$$

so

$$F_aG_b - \sum_{m,n,\ (m,n)\neq(a,b)}\gamma_{mn}F_mG_n + \sum_{k\leq l}\beta_{kl}FG_kG_l \in I_S^2 : F = I_S^2.$$

Therefore

$$F_aG_b - \sum_{m,n,\ (m,n)\neq(a,b)}\gamma_{mn}F_mG_n + \sum_{k\leq l}\beta_{kl}FG_kG_l = \sum_{i\leq j}\delta_{ij}F_iF_j$$

or equivalently

$$F\cdot(\sum_{k\leq l}\beta_{kl}G_kG_l) = \sum_{i\leq j}\delta_{ij}F_iF_j - F_aG_b + \sum_{m,n,\ (m,n)\neq(a,b)}\gamma_{mn}F_mG_n$$

and

$$\sum_{k\leq l}\beta_{kl}G_kG_l \in I_SI_C : F = I_SI_C.$$

Then we have

$$\sum_{k\le l}\beta_{kl}G_kG_l=\sum_{m,n}\eta_{mn}F_mG_n$$

and

$$(1-\eta_{ab}F)F_aG_b-\sum_{m,n,\ (m,n)\neq(a,b)}(\gamma_{mn}-\eta_{mn}F)F_mG_n=\sum_{i\le j}\delta_{ij}F_iF_j.$$

This contradicts the assumption that $\{F_iG_j\}$ is a minimal system of generators for I_SI_C modulo I_S^2. One can argue analogously to how we did in (1).

If (3) happens, then

$$F\cdot(FG_aG_b-\sum_{k\le l,\ (k,l)\neq(a,b)}\beta_{kl}FG_kG_l+\sum_{m,n}\gamma_{mn}F_mG_n)=\sum_{i\le j}\alpha_{ij}F_iF_j$$

so

$$FG_aG_b-\sum_{k\le l,\ (k,l)\neq(a,b)}\beta_{kl}FG_kG_l+\sum_{m,n}\gamma_{mn}F_mG_n\in I_S^2:F=I_S^2.$$

Therefore

$$FG_aG_b-\sum_{k\le l,\ (k,l)\neq(a,b)}\beta_{kl}FG_kG_l+\sum_{m,n}\gamma_{mn}F_mG_n=\sum_{i\le j}\delta_{ij}F_iF_j$$

or equivalently

$$F\cdot(G_aG_b-\sum_{k\le l,\ (k,l)\neq(a,b)}\beta_{kl}G_kG_l)=\sum_{i\le j}\delta_{ij}F_iF_j-\sum_{m,n}\gamma_{mn}F_mG_n$$

and

$$G_aG_b-\sum_{k\le l,\ (k,l)\neq(a,b)}\beta_{kl}G_kG_l\in I_SI_C:F=I_SI_C.$$

But this contradicts the assumption that $\{G_iG_j\}$ is a minimal system of generators for I_C^2 modulo $I_S^2+I_SI_C$. One can argue analogously to how we did in (1).

So the minimal number of generators of I_D^2 is

$$\mu(I_D^2)=\mu(I_S^2)+\mu(I_C^2)+\mu(I_CI_S)=\binom{s+1}{2}+\binom{u+1}{2}+su=\binom{s+u+1}{2}.$$

$\square$

As we already mentioned, we can use this result to produce more examples of schemes that are not standard determinantal but whose general hyperplane section is good determinantal. We are now going to see an example of use of Proposition 6.24.

Example 6.25 *Let $V \subset \mathbf{P}^5$ be the Veronese surface. $I_V \subset S = k[x_0, \dots, x_5]$ is the ideal generated by the 2×2 minors of the matrix*

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

$V \subset S$ where S is the arithmetically Cohen-Macaulay threefold, whose saturated ideal is generated by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_3 & x_4 & x_5 \\ x_2 & x_4 & x_5 & x_0 \end{pmatrix}.$$

Let L be the equation of a general hyperplane, and let W be the basic double link of V cut out on S by L. As schemes, $W = V \cup (S \cap L)$ and $I_W = LI_V + I_S$. Using CoCoA or Macaulay 2 (see [7] and [25]), one can verify easily that the hypotheses of Proposition 6.24 are verified. Then, we can compute the graded Betti numbers of W, that turn out to be

$$0 \longrightarrow S(-5)^6 \longrightarrow S(-4)^{15} \longrightarrow S(-3)^{10} \longrightarrow I_W \longrightarrow 0.$$

Therefore W has the same graded Betti numbers of a standard determinantal scheme with degree matrix of size 3×5 with all the entries equal to 1. In particular, the Artinian reduction of S/I_W is isomorphic to $k[x_0, x_1, x_2]/(x_0, x_1, x_2)^3$. However, the ideal I_W^2 has $55 = \binom{11}{2}$ minimal generators, that is the maximum possible number, given that I_W has 10 minimal generators. The square of the ideal of a standard determinantal scheme with degree matrix of size 3×5 with all the entries equal to 1 has 50 minimal generators (since there are 5 independent Plücker relations). Therefore, W is not standard determinantal. Notice moreover that the ideal of W

is generated by the submaximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_3 & x_4 & x_5 \\ x_2 & x_4 & x_5 & x_0 \\ 0 & 0 & 0 & L \end{pmatrix}.$$

The general hyperplane section D of W is a basic double link of a rational normal quartic curve of $\mathbf{P}^4$. Since the hypotheses of Proposition 6.21 are satisfied, we can conclude that D is good determinantal.

Iterating the same procedure, one can construct a surface $U \subset \mathbf{P}^5$ that is not standard determinantal, but whose general hyperplane section is a good determinantal curve. The Artinian reduction of S/I_U is isomorphic to $k[x_0, x_1, x_2]/(x_0, x_1, x_2)^4$.

6.2 An algebraic equivalent to being good determinantal

We are now going to derive an algebraic condition that is equivalent to the fact that a scheme V is good determinantal, given that a hyperplane section X is good determinantal. The notation is the following.

Notation 6.26 *$V \subset \mathbf{P}^{n+1}$ is an arithmetically Cohen-Macaulay scheme, $H = x_{n+1}$ a hyperplane that meets V properly. Let $S = k[x_0, \ldots, x_{n+1}]$ be the coordinate ring of $\mathbf{P}^{n+1}$, $R = k[x_0, \ldots, x_n]$ be the coordinate ring of $\mathbf{P}^n$. $X = V \cap H$ is a good determinantal scheme. We do not assume that V is good determinantal.*

The saturated ideal $I_X \subset R$ is the ideal of maximal minors of a homogeneous matrix ϕ. Abusing notation, we indicate both a map and the matrix that represents it with the same letter. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & B \longrightarrow & \mathbf{F} \xrightarrow{\phi} & \mathbf{G} \longrightarrow & Coker\phi \longrightarrow 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 \longrightarrow & B' \longrightarrow & \mathbf{F} \xrightarrow{\phi'} & \mathbf{G}' \longrightarrow & Coker\phi' \longrightarrow 0. \end{array} \tag{6.1}$$

Here $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{G}'$ are free R-modules. The matrix representing ϕ' is obtained from the matrix representing the map ϕ by deleting a generalised row (see [35]). The map

$B \longrightarrow B'$ is the inclusion in the short exact sequence

$$0 \longrightarrow B \longrightarrow B' \xrightarrow{b} I_X \longrightarrow 0. \tag{6.2}$$

The map $Coker\phi \longrightarrow Coker\phi'$ fits into the short exact sequence

$$0 \longrightarrow R/I_X \longrightarrow Coker\phi \longrightarrow Coker\phi' \longrightarrow 0.$$

Let $v_1, \dots, v_m$ be a minimal system of generators of B' as an R-module. $e_1, \dots, e_t$ is the standard basis of $\mathbf{F}$ as an R-module.

Let $I_V \subset S$ be the homogeneous saturated ideal of V. When V is good determinantal, I_V is the ideal of maximal minors of a homogeneous matrix ψ. Moreover, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & C \longrightarrow & \mathbf{L} \xrightarrow{\psi} & \mathbf{H} \longrightarrow & Coker\psi \longrightarrow 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 \longrightarrow & C' \longrightarrow & \mathbf{L} \xrightarrow{\psi'} & \mathbf{H}' \longrightarrow & Coker\psi' \longrightarrow 0. \end{array} \tag{6.3}$$

$\mathbf{L}$, $\mathbf{H}$ and $\mathbf{H}'$ are free S-modules. The matrix representing ψ' is obtained from the matrix representing the map ψ by deleting a generalised row. The map $C \longrightarrow C'$ is the inclusion in the short exact sequence

$$0 \longrightarrow C \longrightarrow C' \xrightarrow{c} I_V \longrightarrow 0. \tag{6.4}$$

The map $Coker\psi \longrightarrow Coker\psi'$ fits into the short exact sequence

$$0 \longrightarrow R/I_V \longrightarrow Coker\psi \longrightarrow Coker\psi' \longrightarrow 0.$$

Diagram (6.1) is obtained from diagram (6.3), tensoring over S with R. Notice that R is a flat S-module, therefore tensoring over S with R preserves exactness. All the quotient maps $\mathbf{L} \longrightarrow \mathbf{F}$, $\mathbf{H} \longrightarrow \mathbf{G}$, $\mathbf{H}' \longrightarrow \mathbf{G}'$, $C \longrightarrow B$, $C' \longrightarrow B'$, $Coker\psi \longrightarrow Coker\phi$, $Coker\psi' \longrightarrow Coker\phi'$, and $I_V \longrightarrow I_X$ are denoted with π. $\varepsilon_1, \dots, \varepsilon_t$ is the standard basis of $\mathbf{L}$ as an S-module.

The Lemma that follows is needed in the proof of the next proposition. It easily follows from a graded version of Nakayama's Lemma.

Lemma 6.27 *Let M, N be finitely generated S-modules, $f : M \longrightarrow N$ an S-module homomorphism. Let $g : M \otimes_S R \longrightarrow N \otimes_S R$ be the map induced by f when we tensor over S with R, $g = f \otimes_S 1_R$. If g is onto, then f is onto.*

Proof: Consider the exact sequence

$$M \xrightarrow{f} N \longrightarrow Coker f \longrightarrow 0$$

and tensor over S with R. We obtain the exact sequence

$$M \otimes_S R \xrightarrow{g} N \otimes_S R \longrightarrow (Coker f) \otimes_S R \longrightarrow 0.$$

Since g is onto by assumption, $(Coker f) \otimes_S R = 0$. Then $Coker f = HCoker f$, therefore $Coker f = 0$ by Nakayama's Lemma. $\square$

The next proposition gives a first equivalent condition to the good determinantal property for a scheme V.

Proposition 6.28 *V is good determinantal if and only if all of the following are true:*

1. *there exist C, C' finitely generated graded S-modules such that $B = C \otimes_S R$, $B' = C' \otimes_S R$, and we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \xrightarrow{c} & I_V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & B' & \xrightarrow{b} & I_X & \longrightarrow & 0 \end{array}$$

   whose rows are short exact sequences and whose vertical maps are induced by tensoring over S with R

2. *let $\mathbf{L}, \mathbf{H}'$ be free S-modules such that $\mathbf{F} \cong \mathbf{L} \otimes_S R$ and $\mathbf{G}' \cong \mathbf{H}' \otimes_S R$. There exist elements $h_1, \ldots, h_t \in \mathbf{H}'$ such that $\pi(h_i) = \phi'(e_i) \in \mathbf{G}'$ for all $i = 1, \ldots, t$, and $C' \subseteq Syz(h_1, \ldots, h_t) \subseteq \mathbf{L}$.*

Proof: Assume that V and $X = V \cap H$ are both good determinantal. Then (1) is automatically verified. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & C' & \longrightarrow & \mathbf{L} & \xrightarrow{\psi'} & \mathbf{H}' & \longrightarrow & Coker\psi' & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & B' & \longrightarrow & \mathbf{F} & \xrightarrow{\phi'} & \mathbf{G}' & \longrightarrow & Coker\phi' & \longrightarrow 0 \end{array}$$

and let $h_i = \psi'(\varepsilon_i)$ for all $i = 1, \ldots, t$. Then $\pi(h_i) = \pi(\psi'(\varepsilon_i)) = \phi'(\pi(\varepsilon_i)) = \phi'(e_i) \in \mathbf{G}'$. For all $a = (a_1, \ldots, a_t) \in C'$, $\psi'(a) = 0 = h_1 a_1 + \cdots + h_t a_t$, therefore $a \in Syz(h_1, \ldots, h_t)$. Then (2) is verified as well.

Conversely, assume that (1) and (2) are satisfied. Since X is good determinantal, we have an exact sequence

$$0 \longrightarrow B' \longrightarrow \mathbf{F} \xrightarrow{\phi'} \mathbf{G}' \longrightarrow Coker\phi' \longrightarrow 0$$

or equivalently a short exact sequence

$$0 \longrightarrow \mathbf{F}/B' \xrightarrow{\phi'} \mathbf{G}' \longrightarrow Coker\phi' \longrightarrow 0$$

with $\mathbf{F}, \mathbf{G}'$ finitely generated free R-modules.

By hypothesis (1) it follows that V is the zero locus of the section c of the dual of the sheaf $\mathcal{C}' = \tilde{C}'$. By Theorem 3.1 in [38], it suffices to show that C' is a first Buchsbaum-Rim sheaf. In other words, it suffices to find a map ψ' that makes the following diagram commute

$$\begin{array}{ccc} \mathbf{L}/C' & \xrightarrow{\psi'} & \mathbf{H}' \\ \downarrow & & \downarrow \\ \mathbf{F}/B' & \xrightarrow{\phi'} & \mathbf{G}'. \end{array}$$

The vertical maps are induced by tensoring over S by R. If ψ' exists, the quotient map $\pi : Im\psi' \longrightarrow Im\phi'$ is onto by commutativity of the diagram. We claim that ψ' is injective. In fact tensoring the short exact sequence

$$0 \longrightarrow Ker\psi' \longrightarrow \mathbf{L}/C' \longrightarrow Im\psi' \longrightarrow 0$$

over S by R we obtain the short exact sequence

$$0 \longrightarrow Ker\psi' \otimes_S R \longrightarrow \mathbf{F}/B' \longrightarrow Im\psi' \otimes_S R \cong Im\phi' \longrightarrow 0.$$

Therefore $Ker\psi' \otimes_S R = 0$, so $Ker\psi' = x_{n+1}Ker\psi'$ and $Ker\psi' = 0$ by Nakayama's Lemma.

Let $e_1, \ldots, e_m$ be a basis of $\mathbf{F}$ as an R-module, $\varepsilon_1, \ldots, \varepsilon_m$ a basis of $\mathbf{L}$ as an S-module. By assumption (2) we can choose elements $h_1, \ldots, h_t \in \mathbf{H}'$ such that $\pi(h_i) = \phi'(e_i) \in \mathbf{G}'$ for $i = 1, \ldots, t$. This defines a map $\psi' : \mathbf{L} \longrightarrow \mathbf{H}'$, $\psi'(\varepsilon_i) = h_i$. The map makes the diagram commute since $C' \subseteq Syz(h_1, \ldots, h_t)$, so $\psi'(C') = 0$. $\square$

Remark 6.29 *Notice that under our assumptions $C' \subseteq Syz(h_1, \ldots, h_t) \subseteq \mathbf{L}$ is equivalent to $C' = Syz(h_1, \ldots, h_t)$.*

The Theorem below follows the Notation 6.26. It gives an algebraic restatement of the good determinantal property for the scheme V.

Theorem 6.30 *The scheme V is good determinantal if and only if hypothesis (2) of Proposition 6.28 is satisfied and there exists a system of generators $g_1, \ldots, g_m$ of I_V such that*

$$Syz(v_1 + x_{n+1}w_1, \ldots, v_m + x_{n+1}w_m) \subseteq Syz(g_1, \ldots, g_m)$$

for some $w_1, \ldots, w_m \in \mathbf{L}$ with $deg(w_i) = deg(v_i) - 1$ for $i = 1, \ldots, m$.

Proof: Let $v_1, \ldots, v_m$ be a minimal system of generators of B' as an R-module. V is good determinantal if and only if the following are true:

1. there exists a first Buchsbaum-Rim S-module C' with a minimal system of generators of the form $v_1 + x_{n+1}w_1, \ldots, v_m + x_{n+1}w_m$,

2. there exists a map c that makes the following diagram commute

$$\begin{array}{ccc} C' & \xrightarrow{c} & I_V \\ \downarrow & & \downarrow \\ B' & \xrightarrow{b} & I_X \end{array}$$

where the vertical maps are both given by π. Notice that if c exists it is surjective by Lemma 6.27. Moreover, letting $C = Kerc$ and restricting π to C, we have an induced projection map $\pi : C \longrightarrow B$ that makes the diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & I_V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & I_X & \longrightarrow & 0. \end{array}$$

Both the rows of the diagram are short exact sequences, and the second is obtained from the first by tensoring over S with R.

Let us consider the short exact sequence

$$0 \longrightarrow I_V(-1) \xrightarrow{\cdot x_{n+1}} I_V \xrightarrow{\pi} I_X \longrightarrow 0.$$

Applying the functor $Hom_S(C', -)$ to it, we get the long exact sequence

$$0 \longrightarrow Hom_S(C', I_V)(-1) \xrightarrow{\cdot x_{n+1}} Hom_S(C', I_V) \xrightarrow{\pi \odot -} Hom_S(C', I_X) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} Ext^1_S(C', I_V)(-1) \longrightarrow \cdots$$

where $\pi \odot -$ denotes composition with π on the left, and δ denotes the connecting map. Given (1), (2) is true if and anly if $b \odot \pi \in Im(\pi \odot -)$, if and only if $\delta(b \odot \pi) = 0 \in Ext^1_S(C', I_V)$. Let

$$0 \longrightarrow M \longrightarrow L_0 \longrightarrow C' \longrightarrow 0$$

be a short exact sequence where Ł$_0$ is a free S-module of rank m with basis $e_1, \ldots, e_m$. The map $d_0 : L_0 \longrightarrow C'$ is defined by $d_0(e_i) \mapsto v_i + x_{n+1}w_i$, and $M = Kerd_0 \subset L_0$. Clearly $M = Syz(v_1 + x_{n+1}w_1, \ldots, v_m + x_{n+1}w_m)$.

Since $b(v_i)$ are a system of generators for I_X, we can choose a system of generators $g_1, \ldots, g_m$ for I_V such that $\pi(g_i) = b(v_i) \in I_X$. This defines a map $\varphi : L_0 \longrightarrow I_V$. We can restrict the map to M and have $\varphi : M \longrightarrow I_V$. The map φ fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & L_0 & \xrightarrow{d_0} & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_V & \xrightarrow{i_1} & P & \xrightarrow{d_0 \odot \pi_2} & C' & \longrightarrow & 0. \end{array}$$

where $P = (I_V \oplus L_0)/ < (-\varphi(m), m) \mid m \in M >$, and the vertical maps are φ, i_2, id from left to right.

$\delta(b \odot \pi) = 0 \in Ext^1_S(C', I_V)$ if and only if the second row of the diagram splits, if and only if $\varphi = 0$ on M. In fact, if $\varphi(m) = 0$ for all $m \in M$ then $< (-\varphi(m), m) | m \in M > = < (0, m) | m \in M > = 0 \oplus M$, hence

$$P = (I_V \oplus L_0)/(0 \oplus M) \cong I_V \oplus C'$$

and the maps in the second row are exactly inclusion in the first component and projection onto the second component. Conversely, if the second row splits we have a surjective map $j : P \longrightarrow I_V$ (that we can assume to be given by projection on the first component) such that its composition with i_1 is the identity. If $a = \varphi(m) \in I_V$ for some $m \in M$, then $a = j(i_1(a)) = j(a, 0) = j(0, m) = 0$. Then $\varphi = 0$ is the zero map on M.

But $\varphi = 0$ on M is equivalent to the existence of $w_1, \dots, w_m \in \mathbf{L}$ such that for some system of generators $g_1, \dots, g_m$ of I_V we have

$$M = Syz(v_1 + x_{n+1}w_1, \dots, v_m + x_{n+1}w_m) \subseteq Syz(g_1, \dots, g_m).$$

So conditions (1) and (2) are equivalent to the existence of $w_1, \dots, w_m \in \mathbf{L}$ such that C' is a first Buchsbaum-Rim module and for some system of generators $g_1, \dots, g_m$ of I_V we have

$$M = Syz(v_1 + x_{n+1}w_1, \dots, v_m + x_{n+1}w_m) \subseteq Syz(g_1, \dots, g_m).$$

However, V is good determinantal if and only if hypotheses (1) and (2) of Proposition 6.28 are satisfied.

Assuming V good determinantal, hypothesis (2) of Proposition 6.28 is satisfied and there exist $w_1, \dots, w_m \in \mathbf{L}$ such that for some system of generators $g_1, \dots, g_m$ of I_V we have

$$M = Syz(v_1 + x_{n+1}w_1, \dots, v_m + x_{n+1}w_m) \subseteq Syz(g_1, \dots, g_m).$$

This follows from what we just saw.

Assuming that hypothesis (2) of Proposition 6.28 is satisfied and that there exist $w_1, \ldots, w_m \in \mathbf{L}$ such that for some system of generators $g_1, \ldots, g_m$ of I_V we have

$$M = Syz(v_1 + x_{n+1}w_1, \ldots, v_m + x_{n+1}w_m) \subseteq Syz(g_1, \ldots, g_m),$$

we can show that hypothesis (1) of Proposition 6.28 is satisfied as well. In fact, we can let $C' =< v_1 + x_{n+1}w_1, \ldots, v_m + x_{n+1}w_m >$ and by what we just saw we can construct a map c lifting b. Therefore we let $C = Kerc$. As we remarked in condition (2) in the beginning of the proof, this choice of C, C' and c verify all the conditions of hypothesis (1) of Proposition 6.28. $\square$

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