Lifting the determinantal property

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Abstract. In this note we study standard and in particular good determinantal schemes. We show that there exist arithmetically Cohen-Macaulay schemes that are not standard determinantal, and whose general hyperplane section is good determinantal. We prove that if a general hyperplane section of a scheme is standard (resp. good) determinantal, then the scheme is standard (resp. good) determinantal up to flat deformation. We also study the transference of the property of being standard or good determinantal under basic double linkage.

Introduction

Standard and good determinantal schemes are a large family of projective schemes, to which belong many varieties that have been classically studied. For example rational normal scrolls, rational normal curves, and some Segre varieties are good determinantal schemes. Standard determinantal schemes are cut out by the maximal minors of a matrix of polynomials (see Definition 1.1). In particular they are arithmetically Cohen-Macaulay, and their saturated ideal is resolved by the Eagon-Northcott complex. Good determinantal schemes are standard determinantal schemes that are locally a complete intersection outside a subscheme. Ideals of minors have been the object of extensive study in commutative algebra. These families were studied from the geometric viewpoint by Kreuzer, Migliore, Nagel, and Peterson in [**13**]. In this article, they introduced the definition of standard and good determinantal schemes that we use. The relevance of standard and good determinantal schemes in the context of liaison theory became clear in [**10**], where it was shown that standard determinantal schemes belong to the Gorenstein-liaison class of a complete intersection.

In this note we study standard and good determinantal schemes and their general hyperplane sections. The property of being standard or good determinantal is preserved when taking a general hyperplane section. So we ask whether every arithmetically Cohen-Macaulay scheme whose general hyperplane section is good

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determinantal is itself good determinantal. The answer is negative. In Proposition 2.5, Example 2.6, and Proposition 2.11 we produce examples of schemes which are not standard determinantal, and whose general hyperplane section (or whose Artinian reduction) is good determinantal. We also show that a section of the schemes of Proposition 2.5 by a number of generic hyperplanes is good determinantal up to flat deformation. Then we discuss the property of being standard or good determinantal in a flat family. This is motivated by the observation that we can study flat families all of whose elements are hyperplane section of a given scheme by a hyperplane that meets it properly. We show by means of examples that we can have a flat family which contains a non standard determinantal scheme and whose general element is standard determinantal, or the other way around. In Proposition 2.15 we give sufficient conditions on a section of a scheme S by a hyperplane that meets it properly that force a general hyperplane section of S to be good determinantal. We saw that a scheme S with good determinantal general hyperplane section does not need to be good determinantal. In Theorem 2.17 we show that S is good determinantal up to flat deformation. Finally, we discuss how the property of being standard or good determinantal is preserved under basic double linkage. In Theorem 3.1 we prove that under some assumptions the property is preserved. In Example 3.3 we show that in other cases the property is not preserved. We produce a family of schemes via basic double link from the family of Proposition 2.11, and we prove that the schemes we produced are not standard determinantal, but their general hyperplane sections are good determinantal.

1. Standard and good determinantal schemes

Let S be a scheme in $\mathbb{P}^{n+1} = \mathbb{P}^{n+1}_k$, where k is an algebraically closed field. Let I_S be the saturated homogeneous ideal corresponding to S in the polynomial ring $R = k[x_0, \ldots, x_{n+1}]$. We denote by $\mathfrak{m}$ the homogeneous irrelevant maximal ideal of R, $\mathfrak{m} = (x_0, \ldots, x_{n+1})$. For a sheaf $\mathcal{F}$ we denote by

$$H^i_*(\mathcal{F}) = \bigoplus_{m\in\mathbb{Z}} H^i(\mathbb{P}^{n+1}, \mathcal{F}(m))$$

the i-th cohomology ring. We will usually be interested in the case when $\mathcal{F}$ is an ideal sheaf. Let T be a scheme that contains S. We denote by $\mathcal{I}_{S|T}$ the ideal sheaf of S restricted to T, and by $I_{S|T} = H^0_*(\mathcal{I}_{S|T})$ the ideal of S restricted to T. We often write aCM for arithmetically Cohen-Macaulay.

In this note we study schemes whose general hyperplane section is standard or good determinantal. The following definition was given in [**13**] for schemes, i.e. for saturated ideals. Here we extend it to include Artinian ideals.

Definition 1.1. An ideal $I \subseteq k[x_0, \ldots, x_{n+1}]$ of height c is *standard determinantal* if it is generated by the maximal minors of a homogeneous matrix M of polynomials of size $t \times (t + c - 1)$, for some $t \geq 1$. A matrix M with polynomial entries is *homogeneous* if its minors are homogeneous polynomials.

A *standard determinantal* scheme $S \subseteq \mathbb{P}^{n+1}$ of codimension c is a scheme whose saturated ideal I_S is standard determinantal.

A standard determinantal ideal I is *good determinantal* if after performing invertible row operations on the matrix M and then deleting a row, the ideal generated by the maximal minors of the $(t - 1) \times (t + c - 1)$ matrix obtained is standard determinantal (that is, it has height $c+1$). In particular, we formally include the possibility that $t=1$, i.e. a complete intersection is good determinantal.

A scheme S is *good determinantal* if its saturated ideal I_S is good determinantal.

Let S be a standard determinantal scheme with defining matrix $M=(F_{ij})$. We assume without loss of generality that M contains no invertible entries. Let $\mathcal{U}=(u_{ji})$ be the transposed of the matrix whose entries are the degrees of the entries of M. $\mathcal{U}$ is the *degree matrix* of S. We adopt the convention that the entries of $\mathcal{U}$ increase from right to left and from top to bottom: $u_{ji}\geq u_{lk}$ if $i\leq k$ and $j\geq l$. S can be regarded as the degeneracy locus of a degree 0 morphism

$$\varphi:\bigoplus_{i=1}^{t}R(b_i)\longrightarrow\bigoplus_{j=1}^{t+c-1}R(a_j).$$

Set $a_1\leq\ldots\leq a_{t+c-1}$ and $b_1\leq\ldots\leq b_t$. Then φ is described by the transposed of the matrix M, and $u_{ji}=a_j-b_i$.

Definition 1.2. ([**6**], Definition 1.1) A matrix $M=(F_{ij})$ is *1-generic* if the entries in each row or column are linearly independent over k.

Remark 1.3. It is shown in [**6**] that the ideal generated by the maximal minors of a 1-generic matrix defines a reduced and irreducible standard determinantal scheme. Clearly 1-genericity is preserved if we delete a row of the matrix. Therefore, the ideal of maximal minors of a 1-generic matrix defines a reduced and irreducible good determinantal scheme.

In this note we study standard and good determinantal schemes, and schemes whose hyperplane section is standard or good determinantal.

Definition 1.4. Let $S\subseteq\mathbb{P}^{n+1}$ be a projective scheme of dimension $d\geq 1$. Let H be a hyperplane. If H does not contain any component of S, we say that $S\cap H\subseteq H=\mathbb{P}^n$ is a *proper hyperplane section* of S.

Fix a geometric property $\mathcal{P}$. We say that $\mathcal{P}$ holds for a *general hyperplane section* of S if there is a nonempty open set $\mathcal{V}$ (in the $\mathbb{P}^{n+1}$ parameterizing hyperplanes in $\mathbb{P}^{n+1}$) such that $S\cap H$ has the property $\mathcal{P}$ for all $H\in\mathcal{V}$. We call $S\cap H\subseteq H=\mathbb{P}^n$ a *general hyperplane section* of S.

For a fixed scheme S, a general hyperplane section is proper. Namely, the set $\mathcal{V}$ of hyperplanes in $\mathbb{P}^{n+1}$ that do not contain any component of S is open and nonempty.

If the scheme S has dimension $d\geq 1$, then a general hyperplane section has dimension $d-1$. If I_S is the homogeneous saturated ideal of the scheme $S\subseteq\mathbb{P}^{n+1}$, then $I_{S\cap H|H}=H^0_*(\mathbb{P}^n,\mathcal{I}_{S\cap H})\subseteq R/(H)$ is the homogeneous saturated ideal of the general hyperplane section $S\cap H\subseteq H$. The following short exact sequence of ideal sheaves relates S to a general hyperplane section $S\cap H$

$$0\longrightarrow\mathcal{I}_S(-1)\xrightarrow{\cdot H}\mathcal{I}_S\longrightarrow\mathcal{I}_{S\cap H|H}\longrightarrow 0.$$

Taking cohomology we get the exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & I_S(-1) \arrow[r, "\cdot H"] & I_S \arrow[rr, "\pi"] \arrow[dr] & & I_{S\cap H|H} \arrow[r] & H^1_*(\mathcal{I}_S)(-1) \\
& & & I_S+(H)/(H) \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

If S is arithmetically Cohen-Macaulay, or more in general if R/I_S has depth at least 2, then $H^1_*(\mathcal{I}_S)=0$, and $I_{S\cap H|H}=I_S+(H)/(H)$. In particular, if S is arithmetically Cohen-Macaulay then a proper hyperplane section of S is also arithmetically Cohen-Macaulay. Recall that a scheme S of dimension $d\geq 0$ is arithmetically Cohen-Macaulay if and only if $H^i_*(\mathcal{I}_S)=0$ for $1\leq i\leq d$. Every zero-dimensional scheme is arithmetically Cohen-Macaulay.

If S has dimension $d=0$, then geometrically it does not make sense to take a hyperplane section. However in this case the ideal $I_S+(H)/(H)\subseteq R/(H)$ is Artinian (i.e. $R/I_S+(H)$ has Krull-dimension 0). In this case, we will abuse terminology and still call $I_S+(H)/(H)$ the ideal of a general hyperplane section of S, whenever $H\in R$ is a general linear form. The short exact sequence relating the ideals of S and of a general hyperplane section is

$$0\longrightarrow I_S(-1)\xrightarrow{\cdot H} I_S\longrightarrow I_S+(H)/(H)\longrightarrow 0.$$

We refer the interested reader to Section 1.3 of [**15**] for facts about hyperplane and hypersurface sections.

2. Lifting the determinantal property, and good determinantal schemes in flat families

In this note, we address the question of whether it is possible to lift the property of being standard or good determinantal from a general hyperplane section of a scheme to the scheme itself. For schemes of codimension 2, the Hilbert-Burch Theorem states that being standard determinantal is equivalent to being arithmetically Cohen-Macaulay. So this question is a natural generalization of the questions that were investigated by Huneke and Ulrich in [**9**], by Migliore in [**14**], and by the author in [**7**].

Before starting our discussion, we would like to observe that the good determinantal property does not behave as well as the standard determinantal property under hyperplane sections by a hyperplane that meets the scheme properly. In fact, any hyperplane section of a standard determinantal subscheme of $\mathbb{P}^{n+1}$ by a hyperplane that meets it properly is a standard determinantal subscheme of $\mathbb{P}^n$. It is not true in general that every hyperplane section of a good determinantal subscheme of $\mathbb{P}^{n+1}$ by a hyperplane that meets it properly is a good determinantal subscheme of $\mathbb{P}^n$. However, a general hyperplane section is good determinantal. Next, we see an example when this is the case. The following example was derived from Example 4.1 in [**10**].

EXAMPLE 2.1. Let $C \subseteq \mathbb{P}^4$ be a curve whose homogeneous saturated ideal is given by the maximal minors of

$$\begin{pmatrix} x_0 & x_1+x_4 & 0 & x_2 \\ 0 & x_1 & x_2 & x_0+x_1 \end{pmatrix}.$$

One can check that C is one-dimensional, hence standard determinantal. C is a cone over a zero-dimensional scheme supported on the points $[0:0:0:1]$ and $[0:1:0:-1]$. The curve C is indeed good determinantal, since deleting a generalized row we obtain the matrix of size 1×4

$$\begin{pmatrix} \alpha x_0 & (1+\alpha)x_1+\alpha x_4 & x_2 & x_0+x_1+\alpha x_2 \end{pmatrix}$$

for a generic value of α. For $\alpha \neq 0$ the entries form a regular sequence, since they are linearly independent linear forms. Therefore they define a complete intersection, that is a standard determinantal scheme, and C is good determinantal.

Let H be a general linear form. In particular we can assume that the coefficient of x_3 in the equation of H is non-zero, so that H does not contain the vertex of the cone C. Intersecting C with H we obtain a subscheme X of $\mathbb{P}^3$, whose saturated homogeneous ideal I_X is generated over $k[x_0, x_1, x_2, x_4]$ by the maximal minors of

$$\begin{pmatrix} x_0 & x_1+x_4 & 0 & x_2 \\ 0 & x_1 & x_2 & x_0+x_1 \end{pmatrix}.$$

One can show that X is good determinantal following the same steps as for C. Indeed, C is just a cone over X.

Let $H = x_4$. Intersecting C with H we obtain a subscheme Z of $\mathbb{P}^3$, whose saturated homogeneous ideal I_Z is generated over $k[x_0, \ldots, x_3]$ by the maximal minors of

$$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_1 & x_2 & x_0+x_1 \end{pmatrix}.$$

$I_Z = I_P^2$ for $P = [0:0:0:1]$, hence Z is a zero-dimensional scheme supported on the point P. Then Z is standard determinantal and a section of C by a hyperplane that meets it properly. However, Z is not good determinantal. In fact, deleting a generalized row we obtain the matrix of size 1×4

$$\begin{pmatrix} \alpha x_0 & (1+\alpha)x_1 & x_2 & x_0+x_1+\alpha x_2 \end{pmatrix}$$

whose entries generate the ideal (x_0, x_1, x_2) of codimension $3 < 4$.

Every standard determinantal scheme is arithmetically Cohen-Macaulay. Moreover, the two families coincide for schemes of codimension 1 or 2, while for codimension 3 or higher the family of arithmetically Cohen-Macaulay schemes strictly contains the family of standard determinantal schemes. From the results in [**9**] one can easily obtain a sufficient condition for a scheme $V \subseteq \mathbb{P}^{n+1}$ to be arithmetically Cohen-Macaulay in terms of the graded Betti numbers of a general hyperplane section of V. If a general hyperplane section of V is standard determinantal, the condition can be expressed in terms of the entries of its degree matrix. Notice that since the graded Betti numbers of a hyperplane section of V are the same for a general choice of the hyperplane, the degree matrix is also the same for a general choice of the hyperplane.

Lemma 2.2. *Let $V \subseteq \mathbb{P}^{n+1}$ be a projective scheme. Assume that a general hyperplane section of V is a standard determinantal subscheme of $\mathbb{P}^n$ with degree matrix*
$\mathcal{U} = (u_{ji})_{i=1,\dots,t;\ j=1,\dots,t+c-1}$. *If either $dimV \geq 2$ or*

$$u_{1,t} + \cdots + u_{c-1,t} \geq n+1$$

then V is arithmetically Cohen-Macaulay.

Proof. If $dim(V) \geq 2$ and a general hyperplane section of V is arithmetically Cohen-Macaulay, then V is arithmetically Cohen-Macaulay (see Proposition 2.1 in [**9**]). We can then reduce to the case when V is one-dimensional. Let H be a general hyperplane, and let $Z = V \cap H$. From Theorem 3.16 of [**9**] it follows that the minimum degree b of a minimal generator of $I_{Z|H}$ that is not the image of a minimal generator of I_V under the standard projection $I_V \xrightarrow{\pi} I_{Z|H}$ is

$$b \geq u_{1,1} + \cdots + u_{t,t} + u_{t+1,t} + \cdots + u_{t+c-1,t} - n =$$

$$= u_{1,t}+\cdots+u_{c-1,t}+u_{c,1}+u_{c+1,2}+\cdots+u_{t+c-1,t}-n \geq u_{c,1}+u_{c+1,2}+\cdots+u_{t+c-1,t}+1.$$

In particular, it is bigger than the maximum $u_{c,1} + u_{c+1,2} + \cdots + u_{t+c-1,t}$ of the degrees of the minimal generators of $I_{Z|H}$. Then all the minimal generators of $I_{Z|H}$ are images of the minimal generators of I_V. Hence $H^1_*(\mathcal{I}_V) = 0$, and V is arithmetically Cohen-Macaulay. □

As we mentioned, every arithmetically Cohen-Macaulay scheme of codimension 2 is standard determinantal. So Lemma 2.2 gives a sufficient condition to conclude that V is standard determinantal if $codim(V) = 2$.

Remark 2.3. Let V be a projective scheme. If $dim(V) \geq 2$ and a general hyperplane section of V is aCM, then V is aCM. Therefore the graded Betti numbers of V coincide with the graded Betti numbers of a general hyperplane section of V (for more details see [**15**], Theorem 1.3.6). Moreover, for a scheme of codimension 2 the property of being standard determinantal can be decided by checking the graded Betti numbers. In fact, a scheme of codimension 2 is standard determinantal if and only if it is aCM, if and only if a minimal free resolution of its saturated ideal has length 2. Hence if $dim(V) \geq 2$ and $codim(V) = 2$, we can decide whether V is standard determinantal by looking at the graded Betti numbers of a general hyperplane section. However, if $codim(V) \geq 3$ then the property of being standard determinantal cannot in general be decided by looking at the graded Betti numbers. In other words, there are schemes which are not standard determinantal, but have the same graded Betti numbers as a standard determinantal scheme (see e.g. Example 2.6).

In very special cases the graded Betti numbers of a homogeneous ideal I can force the ideal to be standard determinantal, even when the codimension is 3 or higher. The next is an easy example of this phenomenon.

Example 2.4. Let $R = k[x_1, \dots, x_n]$, $\mathfrak{m} = (x_1, \dots, x_n)$. Let $I \subseteq R$ be a homogeneous ideal generated by $\binom{n+t-1}{t}$ linearly independent polynomials of degree t. Then $I_j = 0$ for all $j < t$ and $dimI_t = \binom{n+t-1}{t} = dim(\mathfrak{m}^t)_t$. Therefore $I = \mathfrak{m}^t$, so it is the ideal of maximal minors of the $t \times (t+n-1)$ matrix

$$\begin{pmatrix} x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 \\ 0 & x_1 & \cdots & x_n & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & x_1 & \cdots & x_n & 0 \\ 0 & \cdots & \cdots & 0 & x_1 & \cdots & x_n \end{pmatrix}.$$

So I is good determinantal.

The next proposition shows that this is not the case in general. We present a family of arithmetically Cohen-Macaulay schemes that are not standard determinantal, but such that the Artinian reduction of their coordinate ring is good determinantal. In particular, they have the graded Betti numbers of a standard determinantal scheme. From a more geometric point of view, it is interesting to decide whether the schemes in question have a general section which is good determinantal. In other words, whether a section of V by r generic hyperplanes is good determinantal for some $r \leq \binom{t+2}{2} - 3$. We prove that the schemes of the following proposition have a (special) $\binom{t}{2}$-th proper hyperplane section which is good determinantal.

Proposition 2.5. *Let X be a symmetric matrix of indeterminates of size $(t+1) \times (t+1)$, $t \geq 2$*

$$X = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & \cdots & x_{0,t} \\ x_{0,1} & x_{1,1} & \cdots & \cdots & x_{1,t} \\ \vdots & \vdots & & & \vdots \\ x_{0,t} & x_{1,t} & \cdots & \cdots & x_{t,t} \end{pmatrix}.$$

Let $V \subseteq \mathbb{P}^{\binom{t+2}{2}-1}$ be the scheme corresponding to the saturated ideal $I_V = I_t(X) \subseteq R = k[\, x_{i,j} \mid 0 \leq i \leq j \leq t \,]$, generated by the submaximal minors of X.

(1) *V is an arithmetically Cohen-Macaulay, integral scheme of codimension 3 which is not standard determinantal, but every Artinian reduction of its homogeneous coordinate ring is good determinantal.*
(2) *Let D be a general $\binom{t}{2}$-th hyperplane section of V. Then V has a proper $\binom{t}{2}$-th hyperplane section C that is a good determinantal scheme, and there is a flat family of schemes with fixed graded Betti numbers that contains both C and D.*

Proof. (1) The fact that V is an arithmetically Cohen-Macaulay, integral scheme of codimension 3 follows from classical results, that can be found e.g. in [**4**]. In particular a minimal free resolution of the ideal I_V is known, and the cardinality of a minimal system of generators of I_V is $m = \binom{t+2}{2}$. Then the Artinian reduction of the coordinate ring of V is good determinantal, as showed in Example 2.4. The divisor class group of V is isomorphic to $\mathbb{Z}_2$ (see [**8**]). From knowledge of the graded Betti numbers of I_V (see e.g. [**4**]), it follows that if V was standard determinantal, then its degree matrix would have size $t \times (t+2)$ and all of its entries would be equal to 1. The divisor class group of such a standard determinantal scheme is isomorphic to $\mathbb{Z}$ (see [**3**]). Therefore V is not standard determinantal. Notice that if $t=2$ then V is the Veronese surface in $\mathbb{P}^5$, which is not standard determinantal, since it is not isomorphic to a rational normal scroll surface.

(2) Consider a special $\binom{t}{2}$-th hyperplane section of V, with defining matrix of size $(t+1)\times(t+1)$

$$Y=\begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots & \cdots & x_{0,t} \\ x_{0,1} & x_{0,2} & & & x_{0,t} & x_{1,t} \\ x_{0,2} & & & x_{0,t} & x_{1,t} & \vdots \\ \vdots & & \iddots & \iddots & & \vdots \\ \vdots & x_{0,t} & x_{1,t} & & & x_{t-1,t} \\ x_{0,t} & x_{1,t} & \cdots & \cdots & x_{t-1,t} & x_{t,t} \end{pmatrix}.$$

We obtain this section intersecting with the hyperplanes $x_{i,j}-x_{0,i+j}$ for $i+j\leq t$ and $i\geq 1, j\leq t-1$ and $x_{i,j}-x_{i+j-t,t}$ for $i+j>t$ and $i\geq 1, j\leq t-1$. We take $\binom{t}{2}$ hyperplane sections by hyperplanes that meet V properly. So we obtain a scheme $C\subseteq\mathbb{P}^{2t}$ of codimension 3. C is good determinantal, with defining matrix

$$U=\begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots & \cdots & x_{0,t-1} & x_{0,t} & x_{1,t} \\ x_{0,1} & x_{0,2} & & & x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} \\ x_{0,2} & & & x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} & \vdots \\ \vdots & & \iddots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} & & & \vdots \\ x_{0,t-1} & x_{0,t} & x_{1,t} & x_{2,t} & \cdots & \cdots & \cdots & x_{t,t} \end{pmatrix}.$$

In fact, the maximal minors of U coincide with the submaximal minors of Y. Moreover, the matrix U is 1-generic. Therefore, the ideal of maximal minors of U defines a reduced and irreducible, good determinantal scheme (see also Remark 1.3).

Let D be a general $\binom{t}{2}$-th hyperplane section of V. The saturated ideal of D is the ideal $I_D=I_t(Z)$ generated by the submaximal minors of the symmetric matrix

$$Z=\begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,t-1} & x_{0,t} \\ x_{0,1} & L_{1,1} & \cdots & L_{1,t-1} & x_{1,t} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{0,t-1} & L_{1,t-1} & \cdots & L_{t-1,t-1} & x_{t-1,t} \\ x_{0,t} & x_{1,t} & \cdots & x_{t-1,t} & x_{t,t} \end{pmatrix}.$$

We can assume without loss of generality that the equations of the hyperplanes that we intersect with V are $x_{i,j}-L_{i,j}$, $i\geq 1, j\leq t-1$, where $L_{i,j}$ is a general linear form in $k[x_{0,0},\ldots,x_{0,t},x_{1,t},\ldots,x_{t,t}]$. Observe that we have a flat family of codimension 3 schemes D_s whose saturated ideal is $I_t(Z_s)$, $Z_s=sZ+(1-s)Y$. In fact, for any choice of s and for $L_{i,j}$ generic, the matrix Z_s is 1-generic. Then by Corollary 3.3 of [**6**]

$$\mathit{codim}\ I_t(Z_s)\geq 2(t+1)-1-2(t-1)=3.$$

Hence Z_s defines an aCM scheme D_s of codimension three, whose graded Betti numbers are the same as those of C and of V (this follows from [**4**], Theorem 3.5). In particular the Hilbert polynomial of D_s is the same for all s. □

The Veronese surface $V \subseteq \mathbb{P}^5$ is an example of a non standard determinantal scheme from the family of Proposition 2.5. In the next example we show that a general hyperplane section of V is a good determinantal curve.

EXAMPLE 2.6. The Veronese surface $V \subseteq \mathbb{P}^5$ is an example from the family of Proposition 2.5, for $t = 2$. Its homogeneous saturated ideal is the ideal

$$I_V = I_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

$I_V \subseteq S = k[x_0, \ldots, x_5]$. Its general hyperplane section is a reduced and irreducible arithmetically Cohen-Macaulay curve $C \subseteq \mathbb{P}^4$ of degree 4, hence a rational normal curve. In particular, a general hyperplane section of V is good determinantal, with defining matrix equal to (after a change of coordinates and invertible row and column operations)

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

Kleppe, Migliore, Miró-Roig, Nagel and Peterson proved that under certain assumptions the closure of the locus of good determinantal schemes with a fixed degree matrix M is an irreducible component in the corresponding Hilbert scheme (see chapters 9 and 10 of [**10**] and the paper [**11**]). Clearly, standard determinantal schemes with the same degree matrix $\mathcal{U}$ belong to the closure of the locus of good determinantal ones. It is natural to ask whether a general $\binom{t}{2}$-th hyperplane section of a scheme V as in Proposition 2.5 is standard (or good) determinantal. The following example shows that this is in general not the case.

EXAMPLE 2.7. Let $V \subseteq \mathbb{P}^9$ be the scheme whose saturated homogeneous ideal I_V is generated by the submaximal minors of the matrix

$$X = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} \\ x_{0,1} & x_{1,1} & x_{1,2} & x_{1,3} \\ x_{0,2} & x_{1,2} & x_{2,2} & x_{2,3} \\ x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}.$$

In Proposition 2.5 we showed that V has a 3-rd hyperplane section $C \subseteq \mathbb{P}^6$ that is good determinantal. More precisely, the ideal I_C is generated by the maximal minors of the matrix

$$Y = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{1,3} \\ x_{0,1} & x_{0,2} & x_{0,3} & x_{1,3} & x_{2,3} \\ x_{0,2} & x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}.$$

The homogeneous saturated ideal of a general 3-rd hyperplane section of V is generated by the maximal minors of the matrix

$$Z = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} \\ x_{0,1} & L_{1,1} & L_{1,2} & x_{1,3} \\ x_{0,2} & L_{1,2} & L_{2,2} & x_{2,3} \\ x_{0,3} & x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}$$

where $L_{1,1}, L_{1,2}, L_{2,2} \in k[x_{0,0}, x_{0,1}, x_{0,2}, x_{0,3}, x_{1,3}, x_{2,3}, x_{3,3}]$ are general linear forms. Let $I(s) = I_3(Z_s)$ be the ideal generated by the submaximal minors of the matrix

$Z_s = sZ + (1-s)Y$. Then one can check that for a generic value of s the cardinality of a minimal system of generators of $I(s)^2$ is $\mu(I(s)^2) = 55$ (we used the computer algebra software CoCoA [**5**]). If $I(s)$ defines a standard determinantal scheme, then it follows by knowledge of the graded Betti numbers of $I(s)$ that it must be associated to a matrix of linear forms of size 3×5. In that case we have 5 linearly independent Plücker relations, which implies that $\mu(I(s)^2) \leq 50$. Therefore $I(s)$ cannot define a standard determinantal scheme. Hence V has a good determinantal 3-rd hyperplane section by hyperplanes that meet it properly, while its general 3-rd hyperplane section is not standard determinantal.

The last family of examples that we wish to study consists of non standard determinantal curves, whose general hyperplane section is good determinantal (see Proposition 2.11). The result of the next lemma is not new. For completeness we give a simple algebraic proof of it.

LEMMA 2.8. *Let $I \subseteq k[x_0, \ldots, x_n]$ be the ideal generated by all the squarefree monomials of degree d. Then I is a good determinantal ideal.*

PROOF. Let A be a matrix of size $d \times (n+1)$ with entries in k such that all the maximal minors of A are nonzero, $A = (a_{i,j})_{1 \leq i \leq d;\, 0 \leq j \leq n}$. Consider the matrix M that we obtain from A by multiplying each entry in the j-th column by x_j, $M = (a_{i,j}x_j)_{1 \leq i \leq d;\, 0 \leq j \leq n}$. The minor involving columns $0 \leq j_1 < \ldots < j_d \leq n$ is $\alpha_{j_1,\ldots,j_d} x_{j_1} \cdot \ldots \cdot x_{j_d}$, where $\alpha_{j_1,\ldots,j_d}$ is the determinant of the submatrix of A consisting of the columns $j_1, \ldots, j_d$. If $d = 1$ then I is a complete intersection, hence good determinantal. If $d \geq 2$ the height of I is $n + 2 - d$, then I is standard determinantal. In particular, $k[x_0, \ldots, x_n]/I$ is Cohen-Macaulay. If we delete a generalized row of M, up to nonzero scalar multiples the $(d-1) \times (d-1)$ minors of the remaining rows are all the squarefree monomials of degree $d-1$ in $k[x_0, \ldots, x_n]$. Since they generate a standard determinantal ideal, I is good determinantal. $\square$

REMARK 2.9. In order for the result of Lemma 2.8 to hold we do not even need the ground field k to have infinite cardinality. However we need to have enough scalars in k so that we can find a matrix A of size $d \times (n+1)$ with entries in k such that all the $d \times d$ minors of A are nonzero. If $|k| \geq d + 1$ we can let A be the Vandermonde matrix in $\alpha_1, \ldots, \alpha_d$, distinct elements in k^*, i.e. $a_{ij} = \alpha_i^{j-1}$.

The next proposition is a straightforward consequence of Lemma 2.8.

PROPOSITION 2.10. $n+1$ *generic points in $\mathbb{P}^n$ are a good determinantal scheme.*

PROOF. Observe that $n+1$ generic points in $\mathbb{P}^n$ can be mapped via a change of coordinates to the $n+1$ coordinate points. The saturated ideal of the $n+1$ coordinate points in $\mathbb{P}^n$ is generated by the squarefree monomials of degree 2 in $x_0, \ldots, x_n$. Therefore it is a good determinantal scheme by Lemma 2.8. $\square$

Let $C \subseteq \mathbb{P}^{n+1}$ be a nondegenerate, reduced and irreducible curve of degree $n+1$. Then C is a rational normal curve, in particular it is good determinantal. In the next proposition we produce a nondegenerate, arithmetically Cohen-Macaulay, reduced curve of degree $n+1$ in $\mathbb{P}^{n+1}$ that is not standard determinantal and whose general hyperplane section is good determinantal. The curve is necessarily reducible, because of what we just observed.

Proposition 2.11. *Let $C_1 \subseteq \mathbb{P}^{n+1}$ be a cone over n generic points in $\mathbb{P}^n$. Let $C_2 \subseteq \mathbb{P}^{n+1}$ be a generic line through a point in C_1. Let $C = C_1 \cup C_2$. Then C is not standard determinantal, and a general hyperplane section of C is good determinantal.*

Proof. Without loss of generality we can let the n generic points in $\mathbb{P}^n$ be all the coordinate points except for $[1 : 0 : \ldots : 0]$. Then the saturated ideal of $C_1 \subseteq \mathbb{P}^{n+1}$ is

$$I_{C_1} = \bigcap_{i=1}^{n} (x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_0) + \sum_{1 \le i < j \le n} (x_i x_j).$$

We can also assume that $I_{C_2} = (x_2, \ldots, x_{n+1})$. Then the saturated ideal of C is

$$I_C = I_{C_1} \cap I_{C_2} = x_0(x_2, \ldots, x_{n+1}) + \sum_{1 \le i < j \le n} (x_i x_j).$$

Since $I_{C_1} + I_{C_2} = (x_0, x_2, \ldots, x_{n+1}) = I_P$ where P is the point $[0 : 1 : 0 : \ldots : 0]$, it follows that C is arithmetically Cohen-Macaulay. This follows from computing cohomology from the short exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{I}_{C_1} \oplus \mathcal{I}_{C_2} \longrightarrow \mathcal{I}_P \longrightarrow 0.$$

In fact we obtain the long exact sequence

$$0 \longrightarrow I_C \longrightarrow I_{C_1} \oplus I_{C_2} \longrightarrow I_{C_1} + I_{C_2} = I_P \xrightarrow{0} H^1_*(\mathcal{I}_C) \longrightarrow H^1_*(\mathcal{I}_{C_1}) \oplus H^1_*(\mathcal{I}_{C_2}) = 0$$

where vanishing of the last module follows from the observation that C_1 and C_2 are aCM. The curve C has degree $n + 1$, and its general hyperplane section consists of $n + 1$ generic points in $\mathbb{P}^n$ by construction. Therefore a general hyperplane section of C is good determinantal by Proposition 2.10.

We now study the last morphism in a minimal free resolution of I_C, in order to show that C is not standard determinantal. Let $I = x_0(x_2, \ldots, x_{n+1})$, $J = \sum_{1 \le i < j \le n} (x_i x_j)$. Then clearly $I_C = I + J$. So we have the short exact sequence

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0. \tag{2.1}$$

Let

$$0 \longrightarrow \mathbb{F}_n \longrightarrow \mathbb{F}_{n-1} \longrightarrow \cdots \longrightarrow \mathbb{F}_1 \longrightarrow I \longrightarrow 0 \tag{2.2}$$

be a minimal free resolution of I. Then $\mathbb{F}_n = R(-n-1)$ and $\mathbb{F}_{n-1} = R(-n)^n$. The last morphism in (2.2) is

$$R(-n-1) \overset{(x_2, -x_3, x_4, \ldots, (-1)^{n+1} x_{n+1})}{\longrightarrow} R(-n)^n. \tag{2.3}$$

Let

$$0 \longrightarrow \mathbb{G}_{n-1} \xrightarrow{M} \mathbb{G}_{n-2} \longrightarrow \cdots \longrightarrow \mathbb{G}_1 \longrightarrow J \longrightarrow 0 \tag{2.4}$$

be a minimal free resolution of J. The ideal J is a lexsegment squarefree monomial ideal, hence morphisms in a minimal free resolution are explicitly computed in [**1**], Theorem 2.1. It turns out that $\mathbb{G}_{n-1} = R(-n)^{n-1}$, $\mathbb{G}_{n-2} = R(-n+1)^{n(n-2)}$, and the matrix M describing the last morphism in (2.4) has size $n(n-2) \times (n-1)$ and is of the form

$$M = \begin{pmatrix} c_1 & \ldots & c_{n-1} \end{pmatrix}$$

where each c_i is a column with exactly $n-1$ nonzero entries (all the indeterminates but x_i). Finally, $I \cap J = x_0 J$, so the minimal free resolution (2.4) twisted by -1 is a minimal free resolution of $I \cap J$

$$0 \longrightarrow \mathbb{G}_{n-1}(-1) \xrightarrow{M} \mathbb{G}_{n-2}(-1) \longrightarrow \cdots \longrightarrow \mathbb{G}_1(-1) \longrightarrow I \cap J \longrightarrow 0. \tag{2.5}$$

Using the mapping cone construction on the short exact sequence (2.1), one can write the last matrix in a minimal free resolution of $I + J = I_C$:

$$\begin{pmatrix} x_2 & 0 & \dots & \dots & 0 \\ -x_3 & -x_1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ (-1)^{n+1}x_{n+1} & 0 & \dots & 0 & -x_1 \\ 0 & x_0 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & x_0 \\ 0 & & & & \\ 0 & & M & & \\ 0 & & & & \end{pmatrix}. \tag{2.6}$$

The matrix corresponds to a morphism

$$R(-n-1)^n = \mathbb{F}_n \oplus \mathbb{G}_{n-1}(-1) \longrightarrow \mathbb{F}_{n-1} \oplus \mathbb{G}_{n-1} \oplus \mathbb{G}_{n-2}(-1) = R(-n)^{n^2-1},$$

where the block consisting of the first $n-1$ rows and the first column comes from the last map in a minimal free resolution of I, i.e. (2.3). The block consisting of the last $n(n-2)$ rows and the last $n-1$ columns comes from the last map M in a minimal free resolution of $I \cap J$. The block consisting of the first $2n-1$ rows and last $n-1$ columns comes from the morphism $\mathbb{G}_{n-1}(-1) \longrightarrow \mathbb{F}_{n-1} \oplus \mathbb{G}_{n-1}$ induced by the diagonal morphism $I \cap J \longrightarrow I \oplus J$. The rows $2, \dots, n$ have $-x_1$ on the diagonal and zeroes anywhere else, while the rows $n+1, \dots, 2n-1$ have x_0 on the diagonal and zeroes anywhere else. This corresponds to the fact that each minimal generator of $I \cap J$ is of the form x_0 multiplied by a minimal generator of J, which is also equal to x_1 multiplied by a minimal generator of I. The indeterminate x_{n+1} appears in the matrix (2.6) only in one position. From this observation and from the form of M it is easy to see that the ideal of 2×2 minors of the matrix (2.6) is $(x_0, \dots, x_n)^2 + x_{n+1}(x_0, \dots, x_n)$.

Suppose by contradiction that the curve C is standard determinantal. Then there exist linear forms $L_0, \dots, L_{2n+1} \in k[x_0, \dots, x_{n+1}]$ such that I_C is the ideal of 2×2 minors of the matrix

$$\begin{pmatrix} L_0 & \dots & L_n \\ L_{n+1} & \dots & L_{2n+1} \end{pmatrix}.$$

The Eagon-Northcott complex is a minimal free resolution of the ideal I_C. The last matrix in the complex has a block form, where the basic block is given by the two column vectors

$$U = \begin{pmatrix} -L_{n+1} \\ L_{n+2} \\ \vdots \\ (-1)^{n+1} L_{2n+1} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} L_0 \\ -L_1 \\ \vdots \\ (-1)^n L_n \end{pmatrix}.$$

The matrix has the form

$$\begin{pmatrix} U & V & 0 & 0 & \dots & 0 \\ 0 & U & V & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & U & V & 0 \\ 0 & \dots & 0 & 0 & U & V \end{pmatrix}. \tag{2.7}$$

In particular the ideal of 2×2 minors of the matrix (2.7) is

$$(L_0, \dots, L_{2n+1})^2 = (x_0, \dots, x_n)^2 + x_{n+1}(x_0, \dots, x_n).$$

Taking radicals we obtain

$$(L_0, \dots, L_{2n+1}) \subseteq \sqrt{(L_0, \dots, L_{2n+1})^2} = (x_0, \dots, x_n)$$

hence $x_{n+1} \notin (L_0, \dots, L_{2n+1})$, which is a contradiction. □

In Proposition 2.5, Example 2.6, and Proposition 2.11 we discussed some examples of "pathological" behavior connected with lifting the property of being standard or good determinantal. The schemes we studied are all defined by minors of matrices with linear entries. In analogy with the question of lifting the property of being arithmetically Cohen-Macaulay (see Lemma 2.2), one could ask the following.

Question 2.12. Assume that $char(k) = 0$ and let $V \subseteq \mathbb{P}^{n+1}_k$ be an aCM scheme. Let $C \subseteq \mathbb{P}^n$ be a general hyperplane section of V, and assume that C is standard/good determinantal. Does there exist an N such that if all the entries of the degree matrix of C are at least N, then V is standard/good determinantal?

The next example illustrates the necessity of requiring that a general hyperplane section of the scheme is standard (or good) determinantal, as opposed to requiring that a hyperplane section by a hyperplane that meets the scheme properly is standard (or good) determinantal. Notice that the entries of the degree matrix M in the next example can be taken arbitrarily large.

Example 2.13. Let k have arbitrary characteristic. Let $V \subseteq \mathbb{P}^5_k$ be the scheme corresponding to the saturated ideal

$$I_V = I_2 \begin{pmatrix} x_0^n & x_1^n & x_2^n \\ x_1^n & x_3^n & x_4^n \\ x_2^n & x_4^n & x_5^n \end{pmatrix} \subseteq k[x_0, \dots, x_5].$$

The ideal I_V is saturated and has height 3, hence it defines a surface $V \subseteq \mathbb{P}^5$. Since $\mathrm{ht} I_V = 3$, a minimal free resolution of I_V can be obtained from a minimal free resolution of the Veronese surface by substituting x_i by x_i^n for $i = 0, \dots, 5$. This follows from Theorem 3.5 in [**4**]. One can check that V is not standard determinantal by a similar argument to that used for the Veronese surface in Proposition 2.6.

Let us intersect V with a hyperplane H of equation $x_3 - x_2 = 0$. The scheme $D = V \cap H$ is arithmetically Cohen-Macaulay, and its saturated ideal I_D is generated by the submaximal minors of the matrix

$$\begin{pmatrix} x_0^n & x_1^n & x_2^n \\ x_1^n & x_2^n & x_4^n \\ x_2^n & x_4^n & x_5^n \end{pmatrix}.$$

Consider a rational normal curve C whose saturated ideal I_C is generated by the submaximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

C is good determinantal, hence the Eagon-Northcott complex is a minimal free resolution of I_C. Since $\mathrm{ht} I_D = \mathrm{ht} I_C = 3$ and I_D is obtained from I_C by replacing each occurrence of x_i by x_i^n, it follows from Theorem 3.5 in [**4**] that we can obtain a minimal free resolution of I_D from a minimal free resolution of I_C by replacing each occurrence of x_i by x_i^n. D is good determinantal, since C is.

We now present an easy example that shows how the closure of the locus of good determinantal schemes in the Hilbert scheme can contain also schemes that are not standard determinantal (or not even arithmetically Cohen-Macaulay).

Example 2.14. Consider the Hilbert scheme $\mathbb{H}$ parameterizing curves of degree 9 and genus 10 in $\mathbb{P}^3$. Let D be the locus of $\mathbb{H}$ whose points correspond to a $CI(3,3)$. Let E be the locus of $\mathbb{H}$ whose points correspond to curves of type $(3,6)$ on a smooth quadric surface. The elements of E are non-aCM. In fact, up to linear equivalence, a curve of type $(3,6)$ is $C = C_1 \cup C_2$ where C_1 consists of 3 skew lines and C_2 consists of 6 skew lines. Moreover, each line of C_1 intersects each line of C_2, so $C_1 \cap C_2$ consists of 18 distinct points. Let $I_C \subset R = k[x_0, x_1, x_2, x_3]$ be the ideal corresponding to C. The minimal free resolution of I_C as an R-module is

$$0 \longrightarrow R^2(-8) \longrightarrow R^6(-7) \longrightarrow R^4(-6) \oplus R(-2) \longrightarrow I_C \longrightarrow 0.$$

In particular, C is non-aCM.

By the uppersemicontinuity principle, no point of the closure of E can be aCM, so E is closed. But since $\mathbb{H}$ is connected, the closure of D needs to intersect E, therefore there is a point in the closure of D that corresponds to a non-aCM curve. Notice that aCM schemes and standard determinantal schemes coincide in the codimension 2 case. So this shows that the closure of the locus of good determinantal schemes in the Hilbert scheme can contain also schemes that are not standard determinantal (and not even arithmetically Cohen-Macaulay).

Examples 2.7 and 2.14 show that we can have a flat family which contains a non standard determinantal scheme and whose general element is standard determinantal, or the other way around. Notice however that while all the schemes in the flat family of Example 2.7 are arithmetically Cohen-Macaulay, the non standard determinantal element in the flat family of Example 2.14 is not aCM.

In Example 2.7 we exhibit an arithmetically Cohen-Macaulay scheme that has a proper 3-rd hyperplane section which is good determinantal, but whose general 3-rd hyperplane section is not good determinantal. Under some assumptions we can conclude that if a scheme V has a good determinantal section by a hyperplane that meets V properly, then a general hyperplane section of V is good determinantal. In the sequel, we will see that this forces V to be good determinantal up to flat deformation (see Theorem 2.17).

Let $S \subseteq \mathbb{P}^{n+1}$ be a scheme of dimension $d \geq 2$ and let $C \subseteq \mathbb{P}^n$ be a general hyperplane section of S. Notice that since we are working with schemes of dimension greater than or equal to 1, it is not restrictive to assume that S is arithmetically Cohen-Macaulay. In fact, C aCM forces S to be aCM. Sufficient conditions for the unobstructedness of C are discussed in the last two chapters of [**10**].

PROPOSITION 2.15. *Let k have characteristic zero. Let $S \subseteq \mathbb{P}^{n+1}$ be an aCM scheme and let $C \subseteq \mathbb{P}^n$ be a hyperplane section of S by a hyperplane that meets S properly. Assume that C is good determinantal, and let $\mathcal{U} = (u_{ij})$ be the degree matrix of C. Let p be the Hilbert polynomial of S and C, and let $Hilb^p(\mathbb{P}^n)$ be the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial p. Assume that C belongs to the interior of the locus of good determinantal schemes with degree matrix $\mathcal{U}$ in $Hilb^p(\mathbb{P}^n)$, and that it is unobstructed. Assume moreover that one of the following holds:*

- *S has codimension 3, and $n \geq 5$;*
- *S has codimension 3, $n \geq 4$, $u_{i,i-\min\{2,t\}} \geq 0$ for $\min\{2,t\} \leq i \leq t$, and $u_{t,t+1} > u_{t,t} + u_{1,t-1}$;*
- *S has codimension 3, $n = 4$, and $u_{t,0} > u_{t,1} + u_{t,2}$;*
- *S has codimension 4, $n \geq 6$, and $u_{i,i-\min\{3,t\}} \geq 0$ for $\min\{3,t\} \leq i \leq t$;*
- *S has codimension 4, $n \geq 5$, $u_{i,i-\min\{3,t\}} \geq 0$ for $\min\{3,t\} \leq i \leq t$, and $u_{t,t+2} > u_{t,t} + u_{1,t-1}$;*
- *S has codimension $c \geq 5$, $n \geq c+1$, $u_{i,i-\min\{3,t\}} \geq 0$ for $\min\{3,t\} \leq i \leq t$, and $u_{t,t+j-2} > \sum_{k=t}^{t+j-4} u_{t,k} - \sum_{k=0}^{j-5} u_{t,k} + u_{1,t-1}$ for $5 \leq j \leq c$.*

Then a general hyperplane section of S is good determinantal with degree matrix $\mathcal{U}$.

PROOF. Let H be a hyperplane that meets S properly and let $C = S \cap H$. Let D be a general section of S. Then we have a flat family of subschemes $D_s \subseteq \mathbb{P}^n$ such that for all s D_s is a section of S by a hyperplane that meets it properly, $D_0 = C$ and $D_1 = D$. Consider the Hilbert scheme $Hilb^p(\mathbb{P}^n)$, where p is the Hilbert polynomial of C. Under our assumptions, Proposition 10.7 in [**10**] and the results in Section 4 of [**11**], we have that $dim_C Hilb^p(\mathbb{P}^n) = dim W$, where $W \subseteq Hilb^p(\mathbb{P}^n)$ is the locus of good determinantal schemes whose degree matrix is the same as the one of C. Moreover, W is irreducible, therefore its closure is an irreducible component of $Hilb^p(\mathbb{P}^n)$. Since C is a smooth point of $Hilb^p(\mathbb{P}^n)$, we have that the irreducible component of $Hilb^p(\mathbb{P}^n)$ containing C contains D as well. Since C belongs to the interior of W, then D_s belongs to W for a generic value of s. Therefore a general hyperplane section of S is good determinantal with degree matrix $\mathcal{U}$. □

EXAMPLE 2.16. Let $S \subseteq \mathbb{P}^7$ be a fourfold and let H be a hyperplane that meets S properly. Let $C \subseteq H = \mathbb{P}^6$ be a threefold whose saturated ideal is generated by the maximal minors of a generic matrix of linear forms of size 3×5. C is a smooth scroll over $\mathbb{P}^2$, and it has Hilbert polynomial $p(t) = \frac{5}{3}t^3 + 4t^2 + \frac{10}{3}t + 1$. It follows from Proposition 5.4 of [**2**] that the Hilbert scheme $Hilb^p(\mathbb{P}^6)$ has an irreducible component $\mathcal{H}$ of dimension 72, whose general element is good determinantal and defined by the maximal minors of a 3×5 matrix of linear forms. C is unobstructed and it belongs to the interior of the locus of good determinantal schemes as above (whose closure is $\mathcal{H}$). Then by Proposition 2.15 a general hyperplane section of S is good determinantal and defined by the maximal minors of a 3×5 matrix of linear forms.

We saw that a scheme with good determinantal general hyperplane section does not need to be good determinantal. However, it is good determinantal up to flat deformation.

THEOREM 2.17. *Let $S \subseteq \mathbb{P}^n$, be an aCM scheme and let C be a proper hyperplane section of S. Then one can find a flat family T_s whose elements all have C as a proper hyperplane section, and such that $T_1 = S$ and T_0 is a cone over C. In particular, if C is standard (resp. good) determinantal, one can find a flat family T_s whose elements all have C as a proper hyperplane section, and such that $T_1 = S$ and T_0 is standard (resp. good) determinantal.*

PROOF. By assumption $C = S \cap H$ for some hyperplane H that meets S properly. With no loss of generality, we can assume that H is the hyperplane of equation $x_{n+1} = 0$. Let $C \subseteq \mathbb{P}^n = H \subseteq \mathbb{P}^{n+1}$. Let C' be the cone over C, so that H intersects C' properly and $C' \cap H = C$. Then if I_S has a minimal system of generators $F_1, \ldots, F_m$, then $I_{C'}$ has $F_1(x_0, \ldots, x_n, 0), \ldots, F_m(x_0, \ldots, x_n, 0)$ as a minimal system of generators. Consider the flat family T_s of schemes with homogeneous saturated ideal

$$I_{T_s} = (F_1(x_0, \ldots, x_n, sx_{n+1}), \ldots, F_m(x_0, \ldots, x_n, sx_{n+1})).$$

Then $S_0 = C'$ and $S_1 = S$. The graded Betti numbers are constant in the family, since the graded Betti numbers of C' and S coincide by assumption, and for $s \neq 0$ T_s and S only differ by a change of coordinates. Moreover, $T_s \cap H = C$ since for all s

$$I_{T_s} + (x_{n+1})/(x_{n+1}) =$$
$$(F_1(x_0, \ldots, x_n, 0), \ldots, F_m(x_0, \ldots, x_n, 0), x_{n+1})/(x_{n+1}) = I_{C|H}.$$

□

In Theorem 2.17 we cannot conclude that S belongs to the closure of the locus of the Hilbert scheme consisting of good determinantal schemes. This is connected to the fact that we cannot prove that a general element of the flat family that we construct is good determinantal. Indeed this is not necessarily the case, as the next example shows.

EXAMPLE 2.18. Let $V \subseteq \mathbb{P}^5$ be the Veronese variety, let $C' \subseteq \mathbb{P}^5$ be a cone over a rational normal quartic curve in $\mathbb{P}^4$. Let

$$M_s = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & (1-s)x_2 + sx_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

and let T_s be the surface in $\mathbb{P}^5$ with saturated ideal $I_{T_s} = I_2(M_s)$. Then $S_0 = C'$ while $T_s \cong V$ for $s \neq 0$. So the general element of the flat family is not standard determinantal. Moreover, a dimension count shows that a generic good determinantal scheme belongs to a different component of the Hilbert scheme from the one containing V. In fact, the dimension of the Hilbert scheme at V is 27, while the dimension of the component which is the closure of the locus of good determinantal schemes is 29 (the latter can be computed using the formulas in [**11**]). In particular C' is not unobstructed (notice that unobstructedness results such as Corollary 10.15 in [**10**] do not apply to this setting, since C' is not a Cartier divisor of the scheme defined by the matrix obtained by deleting a column of M_0). Notice moreover that a general hyperplane section of T_s is a rational quartic curve in $\mathbb{P}^4$ for all s.

3. The determinantal property via basic double linkage

In this section we show how to produce a standard or good determinantal scheme by basic double link from another determinantal scheme. We also show how to produce a non standard determinantal scheme by basic double link from a non standard determinantal scheme. Putting these together, one can start from a scheme which is non standard determinantal and whose general hyperplane section is standard determinantal, and produce another scheme with the same property. We refer the reader to Proposition 5.4.6 in [**15**] for the definition and facts about basic double links.

Theorem 3.1. *Let $C \subseteq S \subseteq \mathbb{P}^n$ be standard determinantal schemes, such that C has codimension 1 in S. Assume that for a suitable choice of defining matrices M and N for C and S, either M is obtained from N by deleting a row or N is obtained from M by deleting a column. Then a basic double link D of C on S is standard determinantal. Moreover, if C is good determinantal then a basic double link D of C on S via a generic hypersurface is good determinantal. In this sense, the property of being standard/good determinantal is preserved under basic double linkage.*

Proof. Let $C \subseteq S \subseteq \mathbb{P}^n$ be standard (resp. good) determinantal schemes, where the saturated ideal of C is generated by the maximal minors of a $t \times (t+c)$ matrix M. $I_C = I_t(M)$ and C is standard determinantal, i.e. it has codimension $c+1$. Assume that the matrix N defining S is obtained from the one of C by adding a row, $I_S = I_{t+1}(N)$. S has codimension c by assumption. Notice that S is good determinantal by construction, in particular it is generically complete intersection (see [**13**], Remark 3.5). $I_{t+1}(N) \subseteq I_t(M)$, so $S \supseteq C$. Let D be a basic double link of C on S, $D = C \cup (S \cap F)$ for some hypersurface F that meets S properly. If C is good determinantal, then after applying generic invertible row operations to M we have a submatrix $M' \subseteq M$ whose maximal minors define a standard determinantal scheme U. M' is obtained from M by deleting a row. If we apply the same row operations to N and delete the corresponding row, we obtain $N' \subseteq N$. The ideal of maximal minors of N' defines a scheme V which is standard determinantal of codimension $c+1$ (as N is the defining matrix of a good determinantal scheme). We assume that F meets V properly as well. Notice that this holds for a generic choice of F. The saturated ideal of D is then

$$I_D = I_S + F \cdot I_C$$

(see Proposition 5.4.5 in [**15**]), so it is minimally generated by the maximal minors of the matrix obtained by adding to N a column vector, whose entries are all equal to 0, except for an entry equal to F. In other words, let $M = (m_{i,j})_{i=1,\dots,t;\ j=1,\dots,t+c}$ and $N = (n_{i,j})_{i=1,\dots,t;\ j=1,\dots,t+c}$, with $n_{i,j} = m_{i,j}$ for $i \leq k-1$, $n_{i,j} = m_{i-1,j}$ for $i \geq k+1$ (inserting a row in position k). If $deg(n_{k,l-1}) \leq deg(F) \leq deg(n_{k,l})$, then the defining matrix of D is $O = (o_{i,j})$ with $o_{i,j} = n_{i,j}$ for $j \leq l$, $o_{k,l} = F$, $o_{i,l} = 0$

for $i \neq k$ and $o_{i,j} = n_{i,j-1}$ for $j \geq l+1$. Therefore $M \subseteq N \subseteq O$, N is obtained from O by removing a column and M is obtained from N by removing a row. If C is good determinantal, then we have $M' \subseteq M$ whose maximal minors define the standard determinantal scheme U. M' is obtained from M by deleting a row. If we apply the same row and column operations to O and delete the corresponding row, we obtain $O' \subseteq O$. The ideal of maximal minors of O' defines a scheme which is a basic double link of U on V. In fact

$$I_t(O') = I_t(N') + F \cdot I_{t-1}(M') = I_V + F \cdot I_U.$$

Recall that by assumption F meets V properly. In particular, $I_t(O')$ defines a standard determinantal scheme of codimension $c+2$. This proves that D is good determinantal.

Assume now the matrix N that defines S is obtained from M by deleting the k-th column. $I_S = I_t(N)$ and S has codimension c by assumption. Notice that all the minimal generators of I_S are also minimal generators of I_C. Moreover, S is good determinantal (as shown in [**10**], Theorem 3.6). $I_t(N) \subseteq I_t(M)$, so $S \supseteq C$. Let D be a basic double link of C on S, $D = C \cup (S \cap F)$ for some hypersurface F that meets S properly. The saturated ideal of D is $I_D = I_S + F \cdot I_C$ (see Proposition 5.4.5 in [**15**]), so it is minimally generated by the maximal minors of the matrix O obtained by adding to N the k-th column of M, after multiplying all of the entries by F. If C is good determinantal, then after applying generic invertible row operations to M we have a submatrix $M' \subseteq M$ whose maximal minors define a standard determinantal scheme U. M' is obtained from M by deleting a row. If we apply the same row operations to the matrix O and delete the corresponding row, we obtain $O' \subseteq O$. The ideal of maximal minors of O' defines a scheme which is a basic double link of U, in particular it has codimension c hence it is standard determinantal. Therefore D is good determinantal. □

We now give an example of how one can systematically produce families of schemes which are not standard determinantal. This can be achieved by taking a basic double link of a scheme C which is not standard determinantal on a standard determinantal scheme S. Of course one needs to check that the result is not standard determinantal, since clearly Theorem 3.1 does not guarantee it. Let H be a hyperplane that meets C, D and S properly. In order to guarantee that the basic double link $D \cap H$ of $C \cap H$ on $S \cap H$ is standard determinantal, we can lift a basic double link of the type described in Theorem 3.1 from $C \cap H$ to C.

Example 3.2. Let $V \subseteq \mathbb{P}^5$ be the Veronese surface

$$I_V = I_2 \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \subseteq k[x_0, \ldots, x_5].$$

Let $S \subseteq \mathbb{P}^5$ be the threefold defined by

$$I_S = I_3 \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_5 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_0 \end{pmatrix}.$$

Then S is good determinantal and contains V. Let F be a general linear form. Then a basic double link $W = V \cup (S \cap F)$ of V on S is not standard determinantal. This can be checked by computing the cardinality of a minimal system of generators of I_W^2 and counting Plücker relations (as done in Example 2.7).

Let $C \subseteq \mathbb{P}^4$ be a smooth rational normal curve

$$I_C = I_2 \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \subseteq k[x_0, \ldots, x_4].$$

Let $H \subseteq \mathbb{P}^5$ by the hyperplane of equation $x_2 - x_5 = 0$. Then H meets V properly and $C = V \cap H \cong \mathbb{P}^4$. Moreover, if $T = S \cap H$, then $D = W \cap H = C \cup (T \cap F)$ is a basic double link of C on T. D is good determinantal by Theorem 3.1. The saturated ideal of D is

$$I_D = I_3 \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & 0 \\ x_1 & x_2 & x_3 & x_4 & 0 \\ x_2 & x_3 & x_4 & x_0 & \overline{F} \end{pmatrix}$$

where we denote by $\overline{F}$ the equation of F restricted to H.

Next we show in an example how one can use a similar construction to produce a scheme that is not standard determinantal and whose general hyperplane section is good determinantal.

Example 3.3. Consider the curve $C \subseteq \mathbb{P}^{n+1}$ of Proposition 2.11. We use the same notation as in the proof of the proposition. We saw that

$$I_C = x_0(x_2, \ldots, x_{n+1}) + \sum_{1 \leq i < j \leq n} (x_i x_j).$$

Let $S \supseteq C$ be the surface cut out by all the squarefree monomials of degree 2 in $x_0, x_2, \ldots, x_n$. S is good determinantal by Lemma 2.8. Let L be a hyperplane that meets S properly, let $D = C \cup (S \cap L)$. To simplify the computation, we let $L = x_1$. D is a basic double link of C on S, and has saturated ideal

$$I_D = x_1 I_C + I_S = (x_0 x_1 x_{n+1}) + x_1^2(x_2, \ldots, x_n) + \sum_{i,j \in \{0,2,\ldots,n\},\ i<j} (x_i x_j).$$

We now sketch the proof that D is not standard determinantal. In order to show it, we proceed as in the proof of Proposition 2.11 and examine the last matrix in a minimal free resolution of the ideal of D. We can follows the same steps as in the proof of Proposition 2.11, taking into account the fact that the minimal generators $x_0 x_{n+1}$ and $x_1(x_2, \ldots, x_n)$ are replaced by their multiples by x_1. Therefore we can write the last matrix in a minimal free resolution of I_D as

$$\begin{pmatrix} x_2 & 0 & \ldots & \ldots & \ldots & 0 \\ -x_3 & -x_1^2 & 0 & \ldots & \ldots & 0 \\ \vdots & 0 & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ (-1)^n x_n & 0 & \ldots & 0 & -x_1^2 & 0 \\ (-1)^{n+1} x_1 x_{n+1} & 0 & \ldots & 0 & 0 & -x_1^2 \\ 0 & x_0 & 0 & \ldots & \ldots & 0 \\ \vdots & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 0 & x_0 \\ 0 & & & & & \\ 0 & & & M' & & \\ 0 & & & & & \end{pmatrix}. \tag{3.1}$$

The matrix M' is obtained from the matrix M in (2.6) by replacing each occurrence of x_1 by x_1^2. Then one checks that the ideal of 2×2 minors of the matrix (3.1) is monomial, and it does not contain any pure power of x_{n+1}. However, it contains all the monomials of degree 2 in $x_0, x_2, \ldots, x_n$, as well as x_1^4, $x_1^3x_{n+1}$ and $x_1^2x_i$ for $i = 0, 2, \ldots, n$. As in Proposition 2.11, one can write down the last matrix in a minimal free resolution of the ideal of maximal minors of a $2\times(n+1)$ matrix of indeterminates $z_0, \ldots, z_{2n+1}$. The matrix has been explicitly described in (2.7). One can check that the ideal of 2×2 minors of (2.7) is $(z_0, \ldots, z_{2n+1})^2$. Therefore, we conclude that D is not standard determinantal by a specialization argument as in Proposition 2.11. If I_D is the ideal of 2×2 minors of a $2\times(n+1)$ matrix of linear forms, then the entries of the matrix do not involve x_{n+1}, which is a contradiction.

We show that a general hyperplane section of D is good determinantal. Let $H \subseteq \mathbb{P}^{n+1}$ be a general hyperplane of equation $x_{n+1}-h$. Let $X = C\cap H$, $Y = D\cap H$, and $E = S\cap H$. Then X, Y are zero-dimensional subschemes of $H \cong \mathbb{P}^n$, $X, Y \subseteq E$. $x_0, \ldots, x_n$ are coordinates on H and

$$I_{X|H} = x_0(x_2, \ldots, x_n, h) + \sum_{1\leq i<j\leq n}(x_ix_j) =$$

$$x_0(y, x_2, \ldots, x_n) + y(x_2, \ldots, x_n) + \sum_{2\leq i<j\leq n}(x_ix_j).$$

Here $y = \alpha x_0 + \beta x_1$ for generic $\alpha, \beta \in k^*$. Then $I_{X|H}$ is generated by the squarefree monomials of degree 2 in $x_0, y, x_2, \ldots, x_n$, hence it corresponds to $n+1$ generic points in $\mathbb{P}^n$. So $I_{X|H}$ is the ideal of maximal minors of the matrix

$$\begin{pmatrix} x_0 & y & x_2 & \ldots & x_n \\ x_0 & \gamma y & \gamma^2 x_2 & \ldots & \gamma^n x_n \end{pmatrix}$$

for $\gamma \in k^*$ generic. The ideal $I_{S|H}$ is generated by the maximal minors of

$$\begin{pmatrix} x_0 & x_2 & \ldots & x_n \\ x_0 & \gamma^2 x_2 & \ldots & \gamma^n x_n \end{pmatrix}.$$

Therefore Theorem 3.1 applies, and $Y \subseteq H \cong \mathbb{P}^n$ is good determinantal with defining matrix

$$\begin{pmatrix} x_0 & yx_1 & x_2 & \ldots & x_n \\ x_0 & \gamma y x_1 & \gamma^2 x_2 & \ldots & \gamma^n x_n \end{pmatrix}.$$

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