A HYBRID EULER-HADAMARD PRODUCT AND MOMENTS OF $\zeta'(\rho)$

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Abstract. Keating and Snaith modeled the Riemann zeta-function $\zeta(s)$ by characteristic polynomials of random $N \times N$ unitary matrices, and used this to conjecture the asymptotic main term for the $2k$-th moment of $\zeta(1/2+it)$ when $k > -1/2$. However, an arithmetical factor, widely believed to be part of the leading term coefficient, had to be inserted in an *ad hoc* manner. Gonek, Hughes and Keating later developed a hybrid formula for $\zeta(s)$ that combines a truncation of its Euler product with a product over its zeros. Using it, they recovered the moment conjecture of Keating and Snaith in a way that naturally includes the arithmetical factor. Here we use the hybrid formula to recover a conjecture of Hughes, Keating and O'Connell concerning discrete moments of the derivative of the Riemann zeta-function averaged over the zeros of $\zeta(s)$, incorporating the arithmetical factor in a natural way.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. In this paper, we study discrete moments of $\zeta'(s)$ in the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k},$$

where the summation is over the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, and $N(T)$ is the usual zero counting function

$$N(T) = \sum_{0<\gamma\le T} 1 = \frac{T\mathscr{L}}{2\pi} - \frac{T}{2\pi} + O(\mathscr{L}).$$

Here and throughout the paper, we let $\mathscr{L} = \log \frac{T}{2\pi}$, and all sums involving the zeros of $\zeta(s)$ are counted with multiplicity.

The function $J_k(T)$ is defined for all $k \ge 0$, and, on the additional assumption that all the zeros are simple, for all $k \in \mathbb{R}$. Trivially, $J_0(T) = 1$, but it is still an open problem to rigorously determine the behavior of $J_k(T)$ for any other value of k. Gonek [9] proved that if the Riemann Hypothesis (RH) is true, then $J_1(T) \sim \frac{1}{12}\mathscr{L}^3$ as $T \to \infty$. Conrey and Snaith [7] conjectured the full asymptotic formula for $J_1(T)$ using the L-functions Ratios Conjecture, and Milinovich [18] proved that their formula is correct assuming RH.

For k in general, Gonek [10] and Hejhal [14] independently conjectured that

$$J_k(T) \asymp_k \mathscr{L}^{k(k+2)} \tag{1}$$

for fixed $k \in \mathbb{R}$, as $T \to \infty$. This conjecture is widely believed for non-negative values of k, but there is evidence that it is false for $k \le -3/2$. The case $k = 1$ of (1) holds on RH, of course, by the remarks above, and Ng [23] established the case $k = 2$

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assuming RH. The conjectured lower bound is known to hold for $k = -1$ under the additional condition that all the zeros of $\zeta(s)$ are simple [10, 20], and for all $k \in \mathbb{N}$ assuming the generalized Riemann Hypothesis for Dirichlet L-functions [21]. Moreover, Milinovich [19] also proved that the upper bound

$$J_k(T) \ll_{k,\varepsilon} \mathscr{L}^{k(k+2)+\varepsilon}$$

holds for all fixed $k \in \mathbb{N}$ and any $\varepsilon > 0$ on RH.

The conjecture of Gonek and Hejhal has been refined further using random matrix theory. Let U denote an $N \times N$ unitary matrix with eigenangles θ_n $(n = 1, 2, \ldots, N)$, and denote its characteristic polynomial by

$$Z(\theta) = \det\big(I - Ue^{-i\theta}\big) = \prod_{n=1}^{N} \big(1 - e^{i(\theta_n - \theta)}\big).$$

The random matrix theory model for $J_k(T)$ is

$$\int_{U(N)} \frac{1}{N} \sum_{n=1}^{N} \big|Z'(\theta_n)\big|^{2k} d\mu_N, \tag{2}$$

where the integral is over all $N \times N$ unitary matrices with respect to Haar measure. Hughes, Keating and O'Connell [15] showed that this expression is equal to

$$\frac{G^2(k+2)}{G(2k+3)} \frac{G(N)G(N+2k+2)}{NG^2(N+k+1)} \sim \frac{G^2(k+2)}{G(2k+3)} N^{k(k+2)} \tag{3}$$

for any fixed k with $\Re(k) > -3/2$, as $N \to \infty$. Here $G(k)$ is the Barnes G-function. Equating the mean densities of the zeros of $\zeta(s)$ and the eigenangles of U, that is to set

$$N \sim \mathscr{L},$$

they were led to the following conjecture.

Conjecture 1.1. (Hughes, Keating and O'Connell) *For any fixed k with $\Re(k) > -3/2$, we have*

$$J_k(T) \sim a_k \frac{G^2(k+2)}{G(2k+3)} \mathscr{L}^{k(k+2)}$$

as $T \to \infty$, where

$$a_k = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}. \tag{4}$$

We note that this agrees with the result $J_1(T) \sim \frac{1}{12}\mathscr{L}^3$ proved by Gonek [9] on RH, and also recovers a conjecture of Gonek [10, 12] in the case $k = -1$. The work of Hughes, Keating and O'Connell is closely related to the work of Keating and Snaith [17], in which they used the characteristic polynomials of large random unitary matrices to model the value distribution of the Riemann zeta-function and study the moments of $\zeta(1/2 + it)$. Evaluating the moments of $|Z(\theta)|$ over $U(N)$ with respect to Haar measure and setting $N \sim \mathscr{L}$, they made the following conjecture.

Conjecture 1.2. (Keating and Snaith) *For any fixed k with $\Re(k) > -1/2$, we have*

$$\frac{1}{T} \int_0^T \big|\zeta(\tfrac{1}{2} + it)\big|^{2k} \sim a_k \frac{G^2(k+1)}{G(2k+1)} \mathscr{L}^{k^2}$$

as $T \to \infty$*, where* a_k *is defined as in* (4)*.*

In both Conjecture 1.1 and Conjecture 1.2, the arithmetical factor a_k was inserted in an *ad hoc* manner based upon separate number theoretic considerations. This is a typical drawback of random matrix models of the Riemann zeta-function and other L-functions: they contain no arithmetical information. Moreover, there is no explanation as to why the arithmetical factor a_k is the same in both conjectures; indeed continuous averages of Dirichlet polynomials and averages of Dirichlet polynomials over the zeros of $\zeta(s)$ behave differently.

Gonek, Hughes and Keating [13] developed a new model for $\zeta(s)$ that incorporates the arithmetical information in a natural way. Their "hybrid" model is based on an approximation of the Riemann zeta-function at a height t on the critical line by a partial Euler product, $P_X(1/2+it)$, multiplied by what is essentially a partial Hadamard product, $Z_X(1/2+it)$, over the non-trivial zeros of $\zeta(s)$ close to $1/2+it$ (see the definitions of $P_X(s)$ and $Z_X(s)$ in the next section). That is, $\zeta(s)$ is represented as a product over a finite number of primes and zeros. The moments of $P_X(s)$ can be calculated rigorously and give rise to the arithmetical factor a_k, whereas the moments of the truncated Hadamard product are conjectured using random matrix theory. Under the assumption that the moments of $\zeta(s)$ split as the product of the moments of $P_X(s)$ and $Z_X(s)$, which can be proved in certain cases, they again arrived at Conjecture 1.2. An interesting feature of their approach is that the arithmetic and random matrix theory aspects are treated on an equal footing. Subsequently, the hybrid Euler-Hadamard product has been extended to various families of L-functions [3, 4, 8].

In this paper, we adapt Gonek, Hughes and Keating's model to the problem of estimating $J_k(T)$. As before, our calculations suggest that the discrete moments of the derivative of the Riemann zeta-function are asymptotic to the discrete moments of $P_X(s)$ times the discrete moments of the derivative of $Z_X(s)$. Moreover, the model explains why the same arithmetical factor a_k appears in both Conjecture 1.1 and Conjecture 1.2, above.

2. Hybrid Euler-Hadamard product and the main results

We begin by stating the hybrid Euler-Hadamard product formula of Gonek, Hughes and Keating (Theorem 1 of [13]).

Theorem 2.1. *Let* $X \geq 2$ *and* f *be a non-negative* C^∞*-function of mass* 1 *supported on* $[0,1]$*. Define*

$$U(z) = \int_0^1 f(u) E_1\big(z(u+X-1)/X\big)\,du,$$

where $E_1(z) = \int_z^\infty e^{-u}/u\,du$ *is the exponential integral. Then for* $\Re(s) = \sigma \geq 0$ *we have*

$$\zeta(s) = P_X(s) Z_X(s)\bigg(1 + O_{f,B}\bigg(\frac{X^{B+2}}{\big((|s|+1)\log X\big)^B}\bigg) + O_f(X^{-\sigma}\log X)\bigg) \tag{5}$$

for any $B > 0$*, where*

$$P_X(s) = \exp\bigg(\sum_{n\leq X} \frac{\Lambda(n)}{n^s \log n}\bigg),$$

$\Lambda(n)$ *is the von Mangoldt function, and*

$$Z_X(s)=\exp\bigg(-\sum_{\rho}U\big((s-\rho)\log X\big)\bigg).$$

As was mentioned in [13], $P_X(s)$ is roughly $\prod_{p\leq X}(1-p^{-s})^{-1}$, and $U(z)$ is roughly $E_1(z)$, which is asymptotic to $-\gamma_0-\log z$ for $|z|$ small, where γ_0 is Euler's constant. Thus, Theorem 2.1 says that $\zeta(s)$ looks roughly like

$$\prod_{p\leq X}\bigg(1-\frac{1}{p^s}\bigg)^{-1}\prod_{\substack{\rho\\ |s-\rho|\ll 1/\log X}}\big((s-\rho)e^{\gamma_0}\log X\big),$$

which is a hybrid formula in that it combines a partial Euler product and (essentially) a partial Hadamard product.

We note that from the series expansion of $E_1(z)$, we can interpret $\exp(-U(z))$ to be asymptotic to Cz for some constant C as $|z|\to 0$. Hence both $\zeta(s)$ and $Z_X(s)$ vanish at the zeros of the Riemann zeta-function. Using Cauchy's integral formula in a familiar way, we can differentiate both sides of (5) and maintain an asymptotic formula. In this way, assuming RH, we obtain that

$$\zeta'(\rho)=P_X(\rho)Z_X'(\rho)\bigg(1+O_{f,B}\bigg(\frac{X^{B+2}}{(|\rho|\log X)^B}\bigg)+O_f(X^{-1/2}\log X)\bigg) \tag{6}$$

for every non-trivial zero ρ of $\zeta(s)$ (since the term $P_X'(\rho)Z_X(\rho)$ vanishes).

In Section 4, we evaluate the moments of $P_X(\rho)$ rigorously and establish the following theorem.

Theorem 2.2. *Assume RH. Let $\varepsilon>0$ and X, $T\to\infty$ with $X=O((\log T)^{2-\varepsilon})$. Then for any $k\in\mathbb{R}$ we have*

$$\frac{1}{N(T)}\sum_{0<\gamma\leq T}\big|P_X(\rho)\big|^{2k}=a_k(e^{\gamma_0}\log X)^{k^2}\big(1+O_k\big((\log X)^{-1}\big)\big).$$

Heuristically, we have

$$Z_X(s)\approx\prod_{\rho}\big((s-\rho)e^{\gamma_0}\log X\big).$$

Hence

$$Z_X'(\rho)\approx(e^{\gamma_0}\log X)W_X(\tilde\rho), \tag{7}$$

where $\tilde\rho=\rho e^{\gamma_0}\log X$, and

$$W_X(\tilde\rho)=\prod_{\tilde\rho'\neq\tilde\rho}\big(\tilde\rho-\tilde\rho'\big).$$

As in the random matrix model (2) for $\zeta'(\rho)$ of Hughes, Keating and O'Connell, we model the $2k$-th moment of $W_X(\tilde\rho)$ by

$$\int_{U(N)}\frac{1}{N}\sum_{n=1}^{N}\big|Z'(\theta_n)\big|^{2k}d\mu_N.$$

Here, however, the average gap between consecutive $\tilde\rho$'s is $2\pi e^{\gamma_0}\log X/\mathscr{L}$. Therefore, equating the mean density of $\tilde\rho$ and the density of the eigenangles corresponds to the identification $N\sim\mathscr{L}/e^{\gamma_0}\log X$. Combining (3) and (7) leads to the following conjecture.

Conjecture 2.1. *Let $\varepsilon > 0$ and X, $T \to \infty$ with $X = O((\log T)^{2-\varepsilon})$. Then for any $k > -3/2$ we have*

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} \big|Z_X'(\rho)\big|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} (e^{\gamma_0} \log X)^{2k} \left(\frac{\mathscr{L}}{e^{\gamma_0}\log X}\right)^{k(k+2)}.$$

In Section 5 we shall prove the case $k = 1$ of Conjecture 2.1, assuming RH. Since, by (6),

$$\zeta'(\rho) P_X(\rho)^{-1} = Z_X'(\rho)\big(1 + o(1)\big),$$

when $\Im(\rho) = \gamma$ is large and $X = O((\log\gamma)^{2-\varepsilon})$, this amounts to proving the following result.

Theorem 2.3. *Assume RH. Let $\varepsilon > 0$ and X, $T \to \infty$ with $X = O((\log T)^{2-\varepsilon})$. Then we have*

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} \big|\zeta'(\rho)P_X(\rho)^{-1}\big|^2 \sim \frac{1}{12} \frac{\mathscr{L}^3}{e^{\gamma_0}\log X}.$$

In Section 6 we shall use the L-functions Ratios Conjectures to heuristically derive the asymptotic formula

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} \big|\zeta'(\rho)P_X(\rho)^{-1}\big|^4 \sim \frac{1}{8640} \frac{\mathscr{L}^8}{(e^{\gamma_0}\log X)^4},$$

and thus, as $1/8640 = G^2(4)/G(7)$, provide additional evidence for Conjecture 2.1 in the case $k = 2$.

Our proof of Theorem 2.3 involves replacing $P_X(\rho)^{-1}$ by a short Dirichlet polynomial and then using the method of Conrey, Ghosh and Gonek [6] to estimate the resulting mean-value. However, unlike the proof in [6], we do not need to assume the generalized Lindelöf hypothesis (GLH) for Dirichlet L-functions. We circumvent the assumption of GLH by incorporating ideas of Bui and Heath-Brown [2], who have recently proved the results in [6] assuming only RH.

Our results for the cases $k = 1$ and $k = 2$ suggest that at least when X is not too large relative to T, the $2k$-th discrete moment of $\zeta'(\rho)$ is asymptotic to the product of the discrete moments of $P_X(\rho)$ and $Z_X'(\rho)$. We believe that this is true in general, and we make the following conjecture.

Conjecture 2.2. *Let $\varepsilon > 0$ and X, $T \to \infty$ with $X = O((\log T)^{2-\varepsilon})$. Then for any $k > -3/2$ we have*

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} \big|\zeta'(\rho)\big|^{2k} \sim \left(\frac{1}{N(T)} \sum_{0<\gamma\leq T} \big|P_X(\rho)\big|^{2k}\right)\left(\frac{1}{N(T)} \sum_{0<\gamma\leq T} \big|Z_X'(\rho)\big|^{2k}\right).$$

By combining Theorem 2.2, Conjecture 2.1, and Conjecture 2.2, we recover the conjecture of Hughes, Keating and O'Connell for real values of k satisfying $k > -3/2$, and incorporate the arithmetical factor a_k in a natural way.

3. Lemmas

In order to prove Theorem 2.2, we require the following version of the Landau-Gonek explicit formula [11].

Lemma 3.1. *Let $x, T > 1$. Then we have*

$$\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(x\log(2xT)\log\log(3x)) + O\bigg(\log x \min\Big\{T, \frac{x}{\langle x\rangle}\Big\}\bigg) + O\bigg(\log(2T)\min\Big\{T, \frac{1}{\log x}\Big\}\bigg),$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself, and $\Lambda(x)$ is the generalized von Mangoldt function; that is, $\Lambda(x) = \log p$ if $x = p^k$ for a prime p and natural number k, and $\Lambda(x) = 0$ otherwise.

The next two lemmas are in [6] (see Lemma 2 and Lemma 3).

Lemma 3.2. *Suppose that $A(s) = \sum_{m=1}^{\infty} a(m)m^{-s}$, where $a(m) \ll_\varepsilon m^\varepsilon$, and $B(s) = \sum_{n\le y} b(n)n^{-s}$, where $b(n) \ll_\varepsilon n^\varepsilon$. Then we have*

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)A(s)B(1-s)ds = \sum_{n\le y}\frac{b(n)}{n}\sum_{m\le nT/2\pi} a(m)e(-m/n) + O_\varepsilon(yT^{1/2+\varepsilon}),$$

where $c = 1 + \mathscr{L}^{-1}$.

Lemma 3.3. *Suppose that $\alpha = \alpha_1 * \alpha_2$. Then we have*

$$\alpha(lm) = \sum_{\substack{l=l_1l_2\\ m=m_1m_2\\ (m_2,l_1)=1}} \alpha_1(l_1m_1)\alpha_2(l_2m_2).$$

4. Proof of Theorem 2.2

Since Theorem 2.2 holds when $k = 0$, we assume throughout this section that k is a nonzero real number. We begin by approximating $P_X(s)^k$ by a truncated Dirichlet series. Write

$$P_X(s)^k = \sum_{n=1}^{\infty}\frac{\alpha_k(n)}{n^s}. \tag{8}$$

From the definition of $P_X(s)$, we see that $\alpha_k(n)$ is multiplicative and real valued. Also, if we let

$$S(X) = \{n \in \mathbb{N} : p|n \Rightarrow p \le X\},$$

the set of X-smooth numbers, then $\alpha_k(n) = 0$ if $n \notin S(X)$. In [13] it is shown that $|\alpha_k(n)| \le d_{|k|}(n)$, and that $\alpha_k(n) = d_k(n)$ if $n \in S(\sqrt{X})$ or if n is a prime $p \le X$, where the arithmetic function $d_k(n)$ is defined in terms of the Dirichlet series

$$\zeta(s)^k = \sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}$$

for $\Re(s) > 1$ and any real number k. In [13] it is also shown (see page 518) that

$$P_X(s)^k = \sum_{\substack{n\in S(X)\\ n\le T^\vartheta}}\frac{\alpha_k(n)}{n^s} + O_{k,\varepsilon}(T^{-\varepsilon\vartheta/2}) \tag{9}$$

for any $\varepsilon, \vartheta > 0$, where ϑ will be chosen later. Using elementary inequalities, we see that

$$\begin{aligned}\Bigg|\bigg(\sum_{0<\gamma\le T}|P_X(\rho)|^{2k}\bigg)^{1/2} - \bigg(\sum_{0<\gamma\le T}\bigg|\sum_{\substack{n\in S(X)\\ n\le T^{\vartheta}}}\frac{\alpha_k(n)}{n^{\rho}}\bigg|^2\bigg)^{1/2}\Bigg| \\ \ll_{k,\varepsilon}\bigg(\sum_{0<\gamma\le T}T^{-\varepsilon\vartheta}\bigg)^{1/2} \ll_{k,\varepsilon} T^{1/2-\varepsilon\vartheta/3}.\end{aligned} \tag{10}$$

Thus, in order to establish Theorem 2.2, it suffices to estimate the second moment of the truncated Dirichlet series.

Assuming RH, $1-\rho=\overline{\rho}$ for any non-trivial zero ρ of $\zeta(s)$. Therefore

$$\begin{aligned}\sum_{0<\gamma\le T}\bigg|\sum_{\substack{n\in S(X)\\ n\le T^{\vartheta}}}\frac{\alpha_k(n)}{n^{\rho}}\bigg|^2 &= \sum_{\substack{mn\in S(X)\\ m,n\le T^{\vartheta}}}\frac{\alpha_k(m)\alpha_k(n)}{n}\sum_{0<\gamma\le T}\Big(\frac{m}{n}\Big)^{-\rho} \\ &= M+E_1+E_2,\end{aligned}$$

say, where M, E_1, and E_2 are the sums representing the contributions from the terms $m=n$, $m<n$, and $m>n$, respectively. Since $1-\rho=\overline{\rho}$, we see that $E_2=\overline{E_1}$. Thus, it suffices to estimate E_1 and M. From Lemma 3.1, we deduce that E_1 equals

$$\begin{aligned}-\frac{T}{2\pi}\sum_{\substack{mn\in S(X)\\ m<n\le T^{\vartheta}}}\frac{\alpha_k(m)\alpha_k(n)}{n}\Lambda\Big(\frac{n}{m}\Big) + O\bigg(\mathscr{L}\log\mathscr{L}\sum_{m<n\le T^{\vartheta}}\frac{d_{|k|}(m)d_{|k|}(n)}{m}\bigg) \\ + O\bigg(\mathscr{L}\sum_{m<n\le T^{\vartheta}}\frac{d_{|k|}(m)d_{|k|}(n)}{m\langle n/m\rangle}\bigg) + O\bigg(\mathscr{L}\sum_{m<n\le T^{\vartheta}}\frac{d_{|k|}(m)d_{|k|}(n)}{n\log n/m}\bigg).\end{aligned}$$

We denote these four terms by E_{11}, E_{12}, E_{13}, and E_{14}, respectively. Now

$$\begin{aligned}E_{11} &\ll T\sum_{mn\in S(X)}\frac{d_{|k|}(m)d_{|k|}(n)}{n}\Lambda\Big(\frac{n}{m}\Big) \\ &\ll T\sum_{p\le X}\sum_{r\ge1}\frac{\log p}{p^r}\sum_{m\in S(X)}\frac{d_{|k|}(m)d_{|k|}(mp^r)}{m} \\ &\ll T\sum_{p\le X}\sum_{r\ge1}\frac{d_{|k|}(p^r)\log p}{p^r}\sum_{m\in S(X)}\frac{d_{|k|}(m)^2}{m}.\end{aligned}$$

Since the innermost sum over m is $\ll \prod_{p\le X}(1-1/p)^{-k^2} \ll_k (\log X)^{k^2}$, it follows that

$$E_{11} \ll_k T(\log X)^{k^2}\sum_{p\le X}\frac{\log p}{p} \ll_k T(\log X)^{k^2+1}.$$

Trivially we have that

$$E_{12} \ll_{k,\varepsilon} T^{\vartheta+\varepsilon}$$

for any $\varepsilon > 0$. To estimate E_{13}, we write $n = um + v$ where $|v/m| \leq 1/2$. We observe that $\langle n/m \rangle = |v/m|$ if u is a prime power and $v \neq 0$, otherwise $\langle n/m \rangle \geq 1/2$. Hence

$$E_{13} \ll_{k,\varepsilon} T^{\varepsilon}\bigg(\sum_{um \ll T^{\vartheta}} \sum_{1 \leq v \leq m/2} \frac{d_{|k|}(m)}{v} + \sum_{m,n \leq T^{\vartheta}} \frac{d_{|k|}(m) d_{|k|}(n)}{m}\bigg) \ll_{k,\varepsilon} T^{\vartheta+\varepsilon}.$$

For E_{14}, we note that $\log \frac{n}{m} \geq \log \frac{n}{n-1} \gg 1/n$. Therefore

$$E_{14} \ll_{\varepsilon} T^{\varepsilon} \sum_{m,n \leq T^{\vartheta}} d_{|k|}(m) d_{|k|}(n) \ll_{k,\varepsilon} T^{2\vartheta+\varepsilon}.$$

Combining the above estimates, we have shown that

$$E_1 + E_2 \ll_{k,\varepsilon} T(\log X)^{k^2+1} + T^{2\vartheta+\varepsilon}. \tag{11}$$

For the evaluation of M, we appeal to Lemma 3.2 of [13] and its proof, and get

$$\begin{aligned} M &= N(T) \sum_{\substack{n \in S(X) \\ n \leq T^{\vartheta}}} \frac{\alpha_k(n)^2}{n} \\ &= N(T) a_k (e^{\gamma_0} \log X)^{k^2} \big(1 + O_k\big((\log X)^{-1}\big)\big). \end{aligned} \tag{12}$$

Theorem 2.2 now follows from (10), (11), and (12) by choosing any $\vartheta < 1/2$.

Remark. The above proof illustrates why the arithmetical factor a_k is the same in both Conjecture 1.1 and Conjecture 1.2, and this arises from a combination of two different phenomena. First of all, while $\zeta'(s)$ is approximated by $P_X'(s)Z_X(s) + P_X(s)Z_X'(s)$, as we noted above $\zeta'(\rho)$ is approximated by $P_X(\rho)Z_X'(\rho)$. Consequently, the arithmetical factor a_k arises solely from moments of the truncated Euler product $P_X(s)$, and not from the moments of its derivative $P_X'(s)$. Moreover, as is the case with continuous moments of $P_X(s)$, there is no off-diagonal contribution to the main term of these moments. For a "typical" Dirichlet polynomial we expect an additional main term contribution from the sum corresponding to E_{11} in the above proof. However, in the present case, the arithmetic nature of the coefficients $\alpha_k(n)$ (i.e. supported on X-smooth numbers with $X = O((\log T)^{2-\varepsilon})$) implies that the term E_{11} contributes an amount which is an error term.

5. Proof of Theorem 2.3

5.1. **Initial setup.** Using the expression in (9) with $k = -1$, we have

$$\sum_{0 < \gamma \leq T} \big|\zeta'(\rho) P_X(\rho)^{-1}\big|^2 = \sum_{\substack{mn \in S(X) \\ m,n \leq T^{\vartheta}}} \frac{\alpha_{-1}(m)\alpha_{-1}(n)}{\sqrt{mn}} I(m,n) + O_{\varepsilon}\big(T^{1-\varepsilon\vartheta/3}\big), \tag{13}$$

where

$$I(m,n) = \sum_{0 < \gamma \leq T} \big|\zeta'(\rho)\big|^2 \Big(\frac{m}{n}\Big)^{-i\gamma}.$$

Throughout the proof of Theorem 2.3, we shall repeatedly use the estimate $|\alpha_{-1}(n)| \leq d(n)$, where $d(n)$ is the divisor function.

We differentiate both sides of the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s)$$

to obtain

$$\zeta'(s) = -\chi(s)\Big(\zeta'(1-s) - \frac{\chi'}{\chi}(s)\zeta(1-s)\Big). \tag{14}$$

It follows that $\zeta'(1-\rho) = -\chi(1-\rho)\zeta'(\rho)$. Thus, assuming RH and using Cauchy's theorem, we get

$$\begin{aligned} I(m,n) &= -\sum_{0<\gamma\le T} \chi(1-\rho)\zeta'(\rho)^2\Big(\frac{m}{n}\Big)^{-i\gamma} \\ &= -\frac{1}{2\pi i}\int_{\mathscr{C}} \chi(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)^2\Big(\frac{m}{n}\Big)^{-s+1/2} ds, \end{aligned}$$

where $\mathscr{C}$ is the positively oriented rectangle with vertices at $1-c+i$, $c+i$, $c+iT$ and $1-c+iT$. Here $c = 1+\mathscr{L}^{-1}$ and T is chosen so that the distance from T to the nearest ordinate of a zero is $\gg \mathscr{L}^{-1}$.

By standard estimates, for s on $\mathscr{C}$ we have $\zeta'(s)/\zeta(s) \ll \mathscr{L}^2$, $\zeta'(s) \ll T^{(1-\sigma)/2}\mathscr{L}$, and $\chi(1-s) \ll T^{\sigma-1/2}$. Hence, the contribution from the horizontal segments of $\mathscr{C}$ is

$$\ll_\varepsilon (m+n)(mn)^{-1/2}T^{1/2+\varepsilon}.$$

We denote the contributions from the right-hand and left-hand edges of $\mathscr{C}$ by $I_R(m,n)$ and $I_L(m,n)$, respectively. Thus,

$$I_R(m,n) = -\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)\frac{\zeta'}{\zeta}(s)\zeta'(s)^2\Big(\frac{m}{n}\Big)^{-s+1/2} ds, \tag{15}$$

and $I_L(m,n)$ is the same except that the integral is from $1-c+iT$ to $1-c+i$. Logarithmically differentiating the functional equation, we have

$$\frac{\zeta'}{\zeta}(1-s) = \frac{\chi'}{\chi}(1-s) - \frac{\zeta'}{\zeta}(s). \tag{16}$$

Using (14) twice and substituting $1-s$ for s, we see that

$$\begin{aligned} I_L(m,n) &= -\frac{1}{2\pi i}\int_{c-i}^{c-iT} \chi(1-s)\Big(\frac{\chi'}{\chi}(1-s) - \frac{\zeta'}{\zeta}(s)\Big) \\ &\qquad\qquad \times\Big(\zeta'(s) - \frac{\chi'}{\chi}(1-s)\zeta(s)\Big)^2\Big(\frac{m}{n}\Big)^{s-1/2} ds \\ &= \overline{I_R(n,m)} + \overline{I'(m,n)} + \overline{I''(m,n)}, \end{aligned}$$

where

$$I'(m,n) = \frac{1}{2\pi i}\int_{c+i}^{c+iT} \frac{\chi'}{\chi}(1-s)^3\zeta(s)\zeta(1-s)\Big(\frac{m}{n}\Big)^{s-1/2} ds$$

and

$$I''(m,n) = \frac{-3}{2\pi i}\int_{c+i}^{c+iT} \frac{\chi'}{\chi}(1-s)\zeta'(s)\zeta'(1-s)\Big(\frac{m}{n}\Big)^{s-1/2} ds.$$

Thus,

$$I(m,n) = I_R(m,n) + \overline{I_R(n,m)} + \overline{I'(m,n)} + \overline{I''(m,n)} + O_\varepsilon\big((m+n)(mn)^{-1/2}T^{1/2+\varepsilon}\big).$$

We shall write the sum on the right-hand side of (13) as

$$\sum_{\substack{mn\in S(X)\\ m,n\le T^{\vartheta}}}\frac{\alpha_{-1}(m)\alpha_{-1}(n)}{\sqrt{mn}}I(m,n)=J_1+J_2+J_3+J_4+J_5 \tag{17}$$

corresponding to this decomposition of $I(m,n)$.

5.2. **The evaluation of J_3, J_4 and J_5.** The term J_5 is easy to handle since

$$J_5\ll_{\varepsilon}T^{1/2+\varepsilon}\sum_{m,n\le T^{\vartheta}}\frac{d(m)d(n)(m+n)}{mn}\ll_{\varepsilon}T^{1/2+\vartheta+\varepsilon}. \tag{18}$$

To estimate J_3 and J_4, we move the line of integration in both $I'(m,n)$ and $I''(m,n)$ to the $\frac{1}{2}$-line. As in (18), this produces an error of size $O_{\varepsilon}(T^{1/2+\vartheta+\varepsilon})$. Therefore

$$J_3=\frac{1}{2\pi}\int_1^T\frac{\chi'}{\chi}\big(\tfrac12+it\big)^3\big|\zeta(\tfrac12+it)\big|^2\Big|\sum_{\substack{n\in S(X)\\ n\le T^{\vartheta}}}\frac{\alpha_{-1}(n)}{n^{1/2+it}}\Big|^2dt+O_{\varepsilon}(T^{1/2+\vartheta+\varepsilon}) \tag{19}$$

and

$$J_4=-\frac{3}{2\pi}\int_1^T\frac{\chi'}{\chi}\big(\tfrac12+it\big)\big|\zeta'(\tfrac12+it)\big|^2\Big|\sum_{\substack{n\in S(X)\\ n\le T^{\vartheta}}}\frac{\alpha_{-1}(n)}{n^{1/2+it}}\Big|^2dt+O_{\varepsilon}(T^{1/2+\vartheta+\varepsilon}). \tag{20}$$

Let

$$J_3'=\int_1^T\big|\zeta(\tfrac12+it)\big|^2\Big|\sum_{\substack{n\in S(X)\\ n\le T^{\vartheta}}}\frac{\alpha_{-1}(n)}{n^{1/2+it}}\Big|^2dt \tag{21}$$

and

$$J_4'=\int_1^T\big|\zeta'(\tfrac12+it)\big|^2\Big|\sum_{\substack{n\in S(X)\\ n\le T^{\vartheta}}}\frac{\alpha_{-1}(n)}{n^{1/2+it}}\Big|^2dt. \tag{22}$$

If $\vartheta<\frac12$, then the integral in (21) is of the form evaluated in [1], while the integral in (22) is almost of this form, but not quite. However, with obvious changes to the argument in [1] that we will not carry out here, one may show that

$$\begin{aligned}&\int_1^T\zeta(\tfrac12+it+\alpha)\zeta(\tfrac12-it+\beta)\Big|\sum_{n\le N}\frac{a(n)}{n^{1/2+it}}\Big|^2dt\\ &=\sum_{m,n\le N}\frac{a(m)\overline{a(n)}(m,n)^{1+\alpha+\beta}}{mn}\\ &\qquad\times\int_1^T\bigg(m^{-\beta}n^{-\alpha}\zeta(1+\alpha+\beta)+\Big(\frac{t(m,n)^2}{2\pi}\Big)^{-\alpha-\beta}m^{\alpha}n^{\beta}\zeta(1-\alpha-\beta)\bigg)dt\\ &\quad+O_B(T\mathscr{L}^{-B})+O_{\varepsilon}(N^2T^{\varepsilon}),\end{aligned} \tag{23}$$

uniformly for $\alpha, \beta \ll \mathscr{L}^{-1}$ and for any $B > 0$. We use (23) to estimate both J_3' and J_4'. Applying it first to (21), we find that

$$\begin{aligned} J_3' &= T \sum_{\substack{mn\in S(X)\\ m,n\le T^{\vartheta}}} \frac{\alpha_{-1}(m)\alpha_{-1}(n)(m,n)}{mn}\bigg(\log\frac{T(m,n)^2}{2\pi mn} + 2\gamma_0 - 1\bigg) \\ &\qquad + O_B(T\mathscr{L}^{-B}) + O_\varepsilon(T^{2\vartheta+\varepsilon}) \\ &= T\mathscr{L} \sum_{\substack{mn\in S(X)\\ m,n\le T^{\vartheta}}} \frac{\alpha_{-1}(m)\alpha_{-1}(n)(m,n)}{mn} + O\bigg(T \sum_{lmn\in S(X)} \frac{d(lm)d(ln)\log mn}{lmn}\bigg) \\ &\qquad + O_B(T\mathscr{L}^{-B}) + O_\varepsilon(T^{2\vartheta+\varepsilon}). \end{aligned} \tag{24}$$

The double sum in the main term of (24) has been evaluated by Gonek, Hughes and Keating (see equations (34)–(38) in [13]). The analysis in [13] implies that

$$T\mathscr{L} \sum_{\substack{mn\in S(X)\\ m,n\le T^{\vartheta}}} \frac{\alpha_{-1}(m)\alpha_{-1}(n)(m,n)}{mn} = \frac{T\mathscr{L}}{e^{\gamma_0}\log X}\big(1 + O\big((\log X)^{-1}\big)\big).$$

The sum in the first big-O term of (24) is

$$\sum_{lmn\in S(X)} \frac{d(lm)d(ln)\log mn}{lmn} \ll \sum_{l\in S(X)} \frac{d(l)^2}{l}\bigg(\sum_{n\in S(X)} \frac{d(n)\log n}{n}\bigg)^2.$$

Writing

$$f(\sigma) = \sum_{n\in S(X)} \frac{d(n)}{n^\sigma} = \prod_{p\le X}\left(1 - \frac{1}{p^\sigma}\right)^{-2},$$

we see that

$$\sum_{n\in S(X)} \frac{d(n)\log n}{n} = -f'(1) = 2f(1)\sum_{p\le X}\frac{\log p}{p-1} \ll (\log X)^3. \tag{25}$$

Hence the first big-O term in (24) is $\ll (\log X)^{10}$. Thus, we have shown that

$$J_3' = \frac{T\mathscr{L}}{e^{\gamma_0}\log X}\big(1 + O\big((\log X)^{-1}\big)\big) + O_\varepsilon(T^{2\vartheta+\varepsilon}). \tag{26}$$

Similarly, applying (23) to (22), we obtain

$$\begin{aligned} J_4' &= \frac{T\mathscr{L}^3}{3} \sum_{\substack{mn\in S(X)\\ m,n\le T^{\vartheta}}} \frac{\alpha_{-1}(m)\alpha_{-1}(n)(m,n)}{mn} \\ &\qquad + O\bigg(T\mathscr{L}^2 \sum_{lmn\in S(X)} \frac{d(lm)d(ln)\log mn}{lmn}\bigg) + O_B(T\mathscr{L}^{-B}) + O_\varepsilon(T^{2\vartheta+\varepsilon}) \\ &= \frac{T\mathscr{L}^3}{3e^{\gamma_0}\log X}\big(1 + O\big((\log X)^{-1}\big)\big) + O_\varepsilon(T^{2\vartheta+\varepsilon}). \end{aligned} \tag{27}$$

To obtain the estimates for (19) and (20) from (26) and (27), we use the well known approximation

$$\frac{\chi'}{\chi}(\tfrac{1}{2}+it) = -\log\frac{t}{2\pi} + O(t^{-1}) \quad (\text{for } t \geq 1) \tag{28}$$

and integration by parts. In this way we deduce that

$$J_3 = -\frac{T\mathscr{L}}{2\pi}\frac{\mathscr{L}^3}{e^{\gamma_0}\log X}\big(1+O\big((\log X)^{-1}\big)\big) + O_\varepsilon(T^{2\vartheta+\varepsilon}) \tag{29}$$

and

$$J_4 = \frac{T\mathscr{L}}{2\pi}\frac{\mathscr{L}^3}{e^{\gamma_0}\log X}\big(1+O\big((\log X)^{-1}\big)\big) + O_\varepsilon(T^{2\vartheta+\varepsilon}). \tag{30}$$

5.3. **The evaluation of J_1 and J_2.** Note that J_1+J_2 equals

$$-2\,\Re\bigg\{\frac{1}{2\pi i}\int_{c+i}^{c+iT}\chi(1-s)\bigg(\frac{\zeta'}{\zeta}(s)\zeta'(s)^2\sum_{\substack{m\in S(X)\\ m\leq T^\vartheta}}\frac{\alpha_{-1}(m)}{m^s}\bigg)\bigg(\sum_{\substack{n\in S(X)\\ n\leq T^\vartheta}}\frac{\alpha_{-1}(n)}{n^{1-s}}\bigg)ds\bigg\}.$$

By Lemma 3.2, we find that

$$J_1+J_2 = -2\,\Re\bigg\{\sum_{\substack{n\in S(X)\\ n\leq T^\vartheta}}\frac{\alpha_{-1}(n)}{n}\sum_{m\leq nT/2\pi}a(m)e(-m/n)\bigg\} + O_\varepsilon(T^{1/2+\vartheta+\varepsilon}),$$

where the arithmetic function $a(m)$ is defined by

$$\frac{\zeta'}{\zeta}(s)\zeta'(s)^2\sum_{\substack{m\in S(X)\\ m\leq T^\vartheta}}\frac{\alpha_{-1}(m)}{m^s} = \sum_{m=1}^\infty\frac{a(m)}{m^s} \tag{31}$$

for $\Re(s)>1$. By the work of Conrey, Ghosh and Gonek (see Sections 5 and 6 and (8.2) of [6]), and of Bui and Heath-Brown [2], we have

$$J_1+J_2 = M_R + E_R + O_\varepsilon(T^{1/2+\vartheta+\varepsilon}),$$

where

$$M_R = -2\sum_{\substack{ln\in S(X)\\ ln\leq T^\vartheta}}\frac{\alpha_{-1}(ln)}{ln}\frac{\mu(n)}{\varphi(n)}\sum_{\substack{m\leq nT/2\pi\\ (m,n)=1}}a(lm) \tag{32}$$

and

$$E_R \ll_{c,B,\varepsilon} T\exp\big(-c\sqrt{\log T}\big) + T\mathscr{L}^{-B} + T^{5/6+\vartheta/3+\varepsilon} \tag{33}$$

for some absolute constant $c>0$, and for any $B>0$.

Write

$$\bigg(-\frac{\zeta'}{\zeta}(s)\bigg)^j = \sum_{m=1}^\infty\frac{\Lambda_j(m)}{m^s} \quad\text{and}\quad -\frac{\zeta'}{\zeta}(s)\zeta'(s)^2 = \sum_{m=1}^\infty\frac{g(m)}{m^s}$$

for $\Re(s)>1$. From (31) and Lemma 3.3, we see that

$$a(lm) = -\sum_{\substack{l=l_1l_2\\ m=m_1m_2\\ (m_2,l_1)=1}}g(l_1m_1)\alpha_{-1}(l_2m_2),$$

and thus

$$\sum_{\substack{m\le nT/2\pi\\(m,n)=1}} a(lm) = -\sum_{l=l_1l_2}\ \sum_{\substack{l_2m_2\in S(X)\\ l_2m_2\le T^{\vartheta}\\ (m_2,l_1n)=1}} \alpha_{-1}(l_2m_2) \sum_{\substack{m_1\le nT/2\pi m_2\\ (m_1,n)=1}} g(l_1m_1).$$

The innermost sum on the right-hand side has been evaluated by Conrey, Ghosh and Gonek. By Lemma A of [6], the sum over m_1 is

$$\begin{aligned} &= \frac{nT}{2\pi m_2}\frac{\varphi(n)^2}{n^2}\sum_{j=0}^{3}\binom{3}{j}\beta_j(l_1)\delta(l_1)\frac{(\log nT/2\pi m_2)^{j+1}}{(j+1)!} + O\big(nT\mathscr{L}^3 d(l_1)/m_2\big)\\ &= \frac{T\mathscr{L}^4}{48\pi}\frac{\varphi(n)^2\delta(l_1)}{m_2n} + O\big(T\mathscr{L}^3\varphi(n)d(l_1)(\log l_1n)/m_2\big), \end{aligned} \tag{34}$$

where $\delta(l)=\prod_{p|l}(2-1/p)$ and $\beta_j(l)=\sum_{d|l}\Lambda_{3-j}(d)/\delta(d)$. We insert this estimate into (32). The contribution of the big-O term in the last line of (34) to (32) is

$$\ll T\mathscr{L}^3 \sum_{l_1l_2mn\in S(X)} \frac{d(l_1l_2n)d(l_2m)d(l_1)\log(l_1n)}{l_1l_2mn} \ll T\mathscr{L}^3\bigg(\sum_{n\in S(X)}\frac{d(n)^2\log n}{n}\bigg)^4.$$

By the same method we used to obtain the estimate in (25), the sum over n on the right-hand side is $\ll (\log X)^5$. Thus, the contribution from the big-O term is $O(T\mathscr{L}^3(\log X)^{20})$. We therefore have that

$$M_R = \frac{T\mathscr{L}^4}{24\pi}\sum_{\substack{l_1l_2n\in S(X)\\ l_1l_2n\le T^{\vartheta}}}\ \sum_{\substack{m\in S(X)\\ l_2m\le T^{\vartheta}\\ (m,l_1n)=1}} \frac{\alpha_{-1}(l_2m)\alpha_{-1}(l_1l_2n)\mu(n)\varphi(n)\delta(l_1)}{l_1l_2mn^2} + O(T\mathscr{L}^3(\log X)^{20}).$$

Next we show that we may extend the sums to all products $l_1l_2mn\in S(X)$ with $(m,l_1n)=1$ with an acceptable error term. This follows from "Rankin's trick", for we have

$$\begin{aligned} \sum_{\substack{l_1l_2mn\in S(X)\\ l_1l_2mn>T^{\vartheta}}} \frac{d(l_2m)d(l_1l_2n)d(l_1)}{l_1l_2mn} &\ll \sum_{l_1l_2mn\in S(X)} \frac{d(l_2m)d(l_1l_2n)d(l_1)}{l_1l_2mn}\bigg(\frac{l_1l_2mn}{T^{\vartheta}}\bigg)^{1/4}\\ &\ll T^{-\vartheta/4}\bigg(\sum_{n\in S(X)}\frac{d(n)^2}{n^{3/4}}\bigg)^4 \ll T^{-\vartheta/4}\prod_{p\le X}\bigg(1-\frac{1}{p^{3/4}}\bigg)^{-16}\\ &\ll T^{-\vartheta/4}e^{100X^{1/4}/\log X} \ll T^{-\vartheta/5} \end{aligned}$$

since $X=O((\log T)^{2-\varepsilon})$. Hence, writing n for l_1n and l for l_2, we have

$$M_R = \frac{T\mathscr{L}^4}{24\pi}\sum_{\substack{lmn\in S(X)\\ (m,n)=1}} \frac{\alpha_{-1}(lm)\alpha_{-1}(ln)g(n)}{lmn} + O(T\mathscr{L}^3(\log X)^{20}), \tag{35}$$

where

$$g(n)=\sum_{d|n}\frac{\mu(d)\varphi(d)\delta(n/d)}{d}.$$

Let $P=\prod_{p\le X}p$. Since $\alpha_{-1}(n)=0$ if n is not a cube-free integer, we can restrict the summation over l to summation over $l=u_1u_2^2$, where $u_1|P$, and $u_2|(P/u_1)$. The summation over m and n can also be restricted to $(m,u_2)=(n,u_2)=1$, since otherwise $\alpha_{-1}(lm)\alpha_{-1}(ln)=0$. Thus, apart from the big-O term in (35), we see that M_R equals

$$\frac{T\mathscr{L}^4}{24\pi}\sum_{u_1|P}\frac{1}{u_1}\sum_{u_2|(P/u_1)}\frac{\alpha_{-1}(u_2^2)^2}{u_2^2}\sum_{\substack{m\in S(X)\\(m,u_2)=1}}\sum_{\substack{n\in S(X)\\(n,u_2m)=1}}\frac{\alpha_{-1}(u_1m)\alpha_{-1}(u_1n)g(n)}{mn}.$$

Arguing similarly, we see that if $r=(u_1,m)$ and $m=rm_1$, then we can assume that $(r,m_1)=1$ so that $(u_1,m_1)=1$. Consequently, the summation over m can be replaced by

$$\sum_{r|u_1}\sum_{\substack{m_1\in S(X)\\(m_1,u_1u_2)=1}}.$$

Similarly, for $s=(u_1,n)$ and $n=sn_1$, we can sum over $(u_1,n_1)=1$. The condition $(m,n)=1$ is equivalent to $(m_1,n_1)=(m_1,s)=(r,n_1)=(r,s)=1$. Now, $(r,s)=1$ if and only if $s|(u_1/r)$. Also, $(m_1,s)=1$ and $(n_1,r)=1$ are implied by $(m_1n_1,u_1)=1$. Thus, M_R equals

$$\frac{T\mathscr{L}^4}{24\pi}\sum_{u_1|P}\frac{1}{u_1}\sum_{u_2|(P/u_1)}\frac{\alpha_{-1}(u_2^2)^2}{u_2^2}\sum_{r|u_1}\sum_{\substack{m_1\in S(X)\\(m_1,u_1u_2)=1}}$$
$$\sum_{s|(u_1/r)}\sum_{\substack{n_1\in S(X)\\(n_1,u_1u_2m_1)=1}}\frac{\alpha_{-1}(u_1rm_1)\alpha_{-1}(u_1sn_1)g(sn_1)}{rsm_1n_1}$$
$$=\frac{T\mathscr{L}^4}{24\pi}\sum_{u_1|P}\frac{\alpha_{-1}(u_1)^2}{u_1}\sum_{u_2|(P/u_1)}\frac{\alpha_{-1}(u_2^2)^2}{u_2^2}\sum_{r|u_1}\frac{\alpha_{-1}(r^2)}{\alpha_{-1}(r)r}\sum_{s|(u_1/r)}\frac{\alpha_{-1}(s^2)g(s)}{\alpha_{-1}(s)s}$$
$$\sum_{\substack{m_1\in S(X)\\(m_1,u_1u_2)=1}}\frac{\alpha_{-1}(m_1)}{m_1}\sum_{\substack{n_1\in S(X)\\(n_1,u_1u_2m_1)=1}}\frac{\alpha_{-1}(n_1)g(n_1)}{n_1}.$$

Since m_1 and n_1 make no contribution unless they are cube-free, this last expression is equal to

$$\frac{T\mathscr{L}^4}{24\pi}\sum_{u_1|P}\frac{\alpha_{-1}(u_1)^2}{u_1}\sum_{u_2|(P/u_1)}\frac{\alpha_{-1}(u_2^2)^2}{u_2^2}\sum_{r|u_1}\frac{\alpha_{-1}(r^2)}{\alpha_{-1}(r)r}\sum_{s|(u_1/r)}\frac{\alpha_{-1}(s^2)g(s)}{\alpha_{-1}(s)s}$$
$$\sum_{m_1|(P/u_1u_2)^2}\frac{\alpha_{-1}(m_1)}{m_1}\sum_{n_1|(P/u_1u_2m_1)^2}\frac{\alpha_{-1}(n_1)g(n_1)}{n_1}. \tag{36}$$

Next we define the following multiplicative functions:

$$T_1(n)=\sum_{d|n}\frac{\alpha_{-1}(d)g(d)}{d},\quad T_2(n)=\sum_{d|n}\frac{\alpha_{-1}(d)}{dT_1(d^2)},$$
$$T_3(n)=\sum_{d|n}\frac{\alpha_{-1}(d^2)g(d)}{\alpha_{-1}(d)d},\quad T_4(n)=\sum_{d|n}\frac{\alpha_{-1}(d^2)}{\alpha_{-1}(d)dT_3(d)},$$
$$T_5(n)=\sum_{d|n}\frac{\alpha_{-1}(d^2)^2}{d^2T_1(d^2)T_2(d^2)}\quad\text{and}\quad T_6(n)=\sum_{d|n}\frac{\alpha_{-1}(d)^2T_3(d)T_4(d)}{dT_1(d^2)T_2(d^2)T_5(d)}.$$

The sum over n_1 in (36) equals

$$T_1\big((P/u_1u_2m_1)^2\big) = \frac{T_1(P^2)}{T_1(u_1^2)T_1(u_2^2)T_1(m_1^2)},$$

and therefore the double summation over m_1 and n_1 in (36) is equal to

$$\frac{T_1(P^2)}{T_1(u_1^2)T_1(u_2^2)}T_2\big((P/u_1u_2)^2\big) = \frac{T_1(P^2)T_2(P^2)}{T_1(u_1^2)T_2(u_1^2)T_1(u_2^2)T_2(u_2^2)}.$$

Similarly, the summation over r and s in (36) is

$$T_3(u_1)T_4(u_1).$$

It follows that

$$\begin{aligned} M_R &= \frac{T\mathscr{L}^4}{24\pi}T_1(P^2)T_2(P^2)\sum_{u_1|P}\frac{\alpha_{-1}(u_1)^2T_3(u_1)T_4(u_1)}{u_1T_1(u_1^2)T_2(u_1^2)}\sum_{u_2|(P/u_1)}\frac{\alpha_{-1}(u_2^2)^2}{u_2^2T_1(u_1^2)T_2(u_1^2)} \\ &= \frac{T\mathscr{L}^4}{24\pi}T_1(P^2)T_2(P^2)T_5(P)T_6(P) \\ &= \frac{T\mathscr{L}^4}{24\pi}\prod_{p\le X}\left(T_1(p^2)T_2(p^2)T_5(p) + \frac{\alpha_{-1}(p)^2T_3(p)T_4(p)}{p}\right). \end{aligned}$$

To simplify this expression, first note that

$$g(p) = 1 \qquad \text{and} \qquad g(p^2) = \frac{2}{p} - \frac{1}{p^2}.$$

Moreover, $\alpha_{-1}(p) = -1$ for all $p \le X$, so

$$\frac{\alpha_{-1}(p)^2T_3(p)T_4(p)}{p} = \frac{T_3(p)}{p} - \frac{\alpha_{-1}(p^2)}{p^2} = \frac{1}{p} - \frac{2\alpha_{-1}(p^2)}{p^2},$$

and

$$\begin{aligned} T_1(p^2)T_2(p^2)T_5(p) &= T_1(p^2)T_2(p^2) + \frac{\alpha_{-1}(p^2)^2}{p^2} \\ &= T_1(p^2) - \frac{1}{p} + \frac{\alpha_{-1}(p^2)}{p^2} + \frac{\alpha_{-1}(p^2)^2}{p^2} \\ &= 1 - \frac{2}{p} + \frac{\alpha_{-1}(p^2)\big(1 + g(p^2) + \alpha_{-1}(p^2)\big)}{p^2}. \end{aligned}$$

Since we also have that $\alpha_{-1}(p^2) = 0$ if $p \le \sqrt{X}$, we see that

$$\begin{aligned} &T_1(p^2)T_2(p^2)T_5(p) + \frac{\alpha_{-1}(p)^2T_3(p)T_4(p)}{p} \\ &\qquad = \begin{cases} 1 - 1/p, & \text{if } p \le \sqrt{X}, \\ 1 - 1/p + O(1/p^2), & \text{if } \sqrt{X} < p \le X. \end{cases} \end{aligned}$$

Collecting these estimates, we now have, apart from the big-O term in (35), that

$$\begin{aligned} M_R &= \frac{T\mathscr{L}^4}{24\pi}\prod_{p\le\sqrt{X}}\left(1 - \frac{1}{p}\right)\prod_{\sqrt{X}<p\le X}\left(1 - \frac{1}{p} + O(1/p^2)\right) \\ &= \frac{T\mathscr{L}}{2\pi}\frac{\mathscr{L}^3}{12e^{\gamma_0}\log X}\big(1 + O\big((\log X)^{-1}\big)\big). \end{aligned}$$

Combining this expression with (13), (17), (18), (29), (30), (33), and (35), we obtain

$$\frac{1}{N(T)}\sum_{0<\gamma\le T}\big|\zeta'(\rho)P_X(\rho)^{-1}\big|^2=\frac{\mathscr{L}^3}{12e^{\gamma_0}\log X}\big(1+O\big((\log X)^{-1}\big)\big)$$
$$+O_c\big(\exp\big(-c\sqrt{\log T}\big)\big)+O_B(\mathscr{L}^{-B})$$
$$+O_\varepsilon\big(T^{-1/2+\vartheta+\varepsilon}+T^{-1+2\vartheta+\varepsilon}+T^{-1/6+\vartheta/3+\varepsilon}\big).$$

Theorem 2.3 now follows by choosing any $\vartheta<1/2$.

6. The twisted moment conjectures

In this section, we use a modification of the recipe in [5, 7] to formulate a conjecture for the discrete moments of $Z'_X(\rho)$. We start by considering the twisted $2k$-th moment of the derivative of the Riemann zeta-function, that is

$$I_{2k}(m,n)=\sum_{0<\gamma\le T}\big|\zeta'(\rho)\big|^{2k}\Big(\frac{m}{n}\Big)^{-i\gamma}.$$

We assume RH and, for simplicity, we assume that $(m,n)=1$. Using Cauchy's theorem, we may write this sum as a contour integral; namely

$$I_{2k}(m,n)=\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{\zeta'(s)}{\zeta(s)}\zeta'(s)^k\zeta'(1-s)^k\Big(\frac{m}{n}\Big)^{-s+1/2}ds,$$

with the contour $\mathscr{C}$ running from $1-c+i$ to $c+i$, $c+iT$ and $1-c+iT$, where as before $c=1+\mathscr{L}^{-1}$. Using standard estimates for the integrand, we can show that the contribution from the horizontal segments of the contour is negligible. Therefore, it suffices to estimate the right-hand and left-hand portions of the contour, $I_{2k,R}(m,n)$ and $I_{2k,L}(m,n)$, say. We first examine the integral from $c+i$ to $c+iT$, which is

$$\begin{aligned}I_{2k,R}(m,n)&=\frac{1}{2\pi}\int_1^T\frac{\zeta'(c+it)}{\zeta(c+it)}\zeta'(c+it)^k\zeta'(1-c-it)^k\Big(\frac{m}{n}\Big)^{-c-it+1/2}dt\\&=\frac{d}{d\alpha_1}\cdots\frac{d}{d\alpha_{k+1}}\frac{d}{d\beta_1}\cdots\frac{d}{d\beta_k}\frac{1}{2\pi}\int_1^T\frac{\zeta(c+it+\alpha_{k+1})}{\zeta(c+it)}\\&\qquad\times\prod_{j=1}^k\Big(\zeta(c+it+\alpha_j)\zeta(1-c-it+\beta_j)\Big)\Big(\frac{m}{n}\Big)^{-c-it+1/2}dt\Big|_{\underline{\alpha}=\underline{\beta}=0}.\end{aligned}$$

Following the recipe outlined in [5, 7], we replace each of the zeta-functions in the numerator by

$$\zeta(s)\sim\sum_{n\le\sqrt{t/2\pi}}\frac{1}{n^s}+\chi(s)\sum_{n\le\sqrt{t/2\pi}}\frac{1}{n^{1-s}},$$

and we replace the zeta-function in the denominator by

$$\frac{1}{\zeta(s)}=\sum_{n=1}^\infty\frac{\mu(n)}{n^s}.$$

Multiplying out the various sums, we obtain 2^{2k+1} terms in the integrand. We note that Stirling's formula for the Gamma function implies that

$$\chi(s+\alpha)\chi(1-s+\beta) = \left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\left(1+O\left(\frac{1}{|t|+1}\right)\right) \tag{37}$$

as $t \to \infty$. We only keep the terms with the same number of χ factors coming from $\zeta(s)$ and from $\zeta(1-s)$. Consider the term coming from the product of the first term of each approximate functional equation, namely

$$\sum_{\substack{a_1,\ldots,a_{k+2}\\ b_1,\ldots,b_k}} \frac{\mu(a_{k+2})}{a_1^{\alpha_1}\ldots a_{k+1}^{\alpha_{k+1}} b_1^{1+\beta_1}\ldots b_k^{1+\beta_k}} \left(\frac{a_1\ldots a_{k+2}}{b_1\ldots b_k}\right)^{-c-it}\left(\frac{m}{n}\right)^{-c-it+1/2}.$$

Averaging over t, only the diagonal terms $a_1\ldots a_{k+2}m = b_1\ldots b_k n$ are retained and we obtain

$$\int_1^T \sum_{am=bn} \frac{A_{\underline{\alpha}}(a)B_{\underline{\beta}}(b)}{\sqrt{ab}}dt, \tag{38}$$

where

$$A_{\underline{\alpha}}(a) = \sum_{a_1\ldots a_{k+2}=a} \frac{\mu(a_{k+2})}{a_1^{\alpha_1}\ldots a_{k+1}^{\alpha_{k+1}}},$$

and

$$B_{\underline{\beta}}(b) = \sum_{b_1\ldots b_k=b} \frac{1}{b_1^{\beta_1}\ldots b_k^{\beta_k}}.$$

Since $(m,n)=1$, the only solutions of $am=bn$ are $a=un$ and $b=um$. Thus, since $A_{\underline{\alpha}}(a)$ and $B_{\underline{\beta}}(b)$ are multiplicative functions, the integral in (38) equals

$$\frac{1}{\sqrt{mn}}\int_1^T \sum_{u=1}^\infty \frac{A_{\underline{\alpha}}(un)B_{\underline{\beta}}(um)}{u}dt = \frac{1}{\sqrt{mn}}\int_1^T \sum_{u=1}^\infty \frac{A_{\underline{\alpha}}(u)B_{\underline{\beta}}(u)}{u}$$
$$\times \prod_{\substack{p^{m_p}||m\\ p^{n_p}||n}} \left(\frac{\sum_{j=0}^\infty A_{\underline{\alpha}}(p^{j+n_p})B_{\underline{\beta}}(p^j)/p^j}{\sum_{j=0}^\infty A_{\underline{\alpha}}(p^j)B_{\underline{\beta}}(p^j)/p^j}\frac{\sum_{j=0}^\infty A_{\underline{\alpha}}(p^j)B_{\underline{\beta}}(p^{j+m_p})/p^j}{\sum_{j=0}^\infty A_{\underline{\alpha}}(p^j)B_{\underline{\beta}}(p^j)/p^j}\right)dt.$$

We denote the integrand on the right-hand side of the above equation by $T_{\underline{\alpha},\underline{\beta}}(m,n)$, and we denote the product over primes in this integrand by $C_{\underline{\alpha},\underline{\beta}}(m,n)$. Now the sum over u in $T_{\underline{\alpha},\underline{\beta}}(m,n)$ is

$$\sum_{u=1}^\infty \frac{A_{\underline{\alpha}}(u)B_{\underline{\beta}}(u)}{u} = \prod_p \left(\sum_{\sum_{j=1}^{k+2}a_j=\sum_{j=1}^k b_j} \frac{\mu(p^{a_{k+2}})}{p^{\sum_{j=1}^{k+1}(1/2+\alpha_j)a_j + a_{k+2}/2 + \sum_{j=1}^k(1/2+\beta_j)b_j}}\right).$$

Taking out the divergent terms from the above formula in the form of zeta-functions, the integrand $T_{\underline{\alpha},\underline{\beta}}(m,n)$ equals

$$\frac{\prod_{\substack{1\leq i\leq k+1\\ 1\leq j\leq k}}\zeta(1+\alpha_i+\beta_j)}{\prod_{1\leq j\leq k}\zeta(1+\beta_j)}\prod_p\Bigg(\prod_{\substack{1\leq i\leq k+1\\ 1\leq j\leq k}}\Big(1-\frac{1}{p^{1+\alpha_i+\beta_j}}\Big)\prod_{1\leq j\leq k}\Big(1-\frac{1}{p^{1+\beta_j}}\Big)^{-1}$$
$$\times\sum_{\sum_{j=1}^{k+2}a_j=\sum_{j=1}^{k}b_j}\frac{\mu(p^{a_{k+2}})}{p^{\sum_{j=1}^{k+1}(1/2+\alpha_j)a_j+a_{k+2}/2+\sum_{j=1}^{k}(1/2+\beta_j)b_j}}\Bigg)C_{\underline{\alpha},\underline{\beta}}(m,n).$$

We handle the other terms which arise from multiplying out the approximate functional equations in a similar manner, but we also take into account the asymptotic formula (37). Adding the resulting terms, we obtain

$$I_{2k,R}(m,n)=\frac{d}{d\alpha_1}\ldots\frac{d}{d\alpha_{k+1}}\frac{d}{d\beta_1}\ldots\frac{d}{d\beta_k}\frac{1}{2\pi\sqrt{mn}}$$
$$\times\int_1^T\sum_{0\leq j\leq k}\sum_{\substack{P\subset\{\alpha_1,\ldots,\alpha_{k+1}\}\\ Q\subset\{\beta_1,\ldots,\beta_k\}\\ |P|=|Q|=j}}T_{\underline{\alpha}_P,\underline{\beta}_Q}(m,n)\Big(\frac{t}{2\pi}\Big)^{-P-Q}dt\Big|_{\underline{\alpha}=\underline{\beta}=0}+O_{k,\varepsilon}(T^{1/2+\varepsilon}),$$

where if $P=\{\alpha_{u_1},\ldots,\alpha_{u_j}\}$ and $Q=\{\beta_{v_1},\ldots,\beta_{v_j}\}$ with $u_1<\ldots<u_j$ and $v_1<\ldots<v_j$, then $(\underline{\alpha}_P,\underline{\beta}_Q)$ is the $(2k{+}1)$-tuple obtained from

$$(\alpha_1,\ldots,\alpha_{k+1},\beta_1,\ldots,\beta_k)$$

by replacing α_{u_r} with $-\beta_{v_r}$ and replacing β_{v_r} with $-\alpha_{u_r}$ for all $1\leq r\leq j$. Here $(t/2\pi)^{-P-Q}$ stands for

$$(t/2\pi)^{-\sum_{x\in P}x-\sum_{y\in Q}y}.$$

There is a concise way to write these $\binom{2k+1}{k}$ terms as a contour integral (see [5]), namely $I_{2k,R}(m,n)$ equals

$$\frac{d}{d\alpha_1}\ldots\frac{d}{d\alpha_{k+1}}\frac{d}{d\beta_1}\ldots\frac{d}{d\beta_k}\frac{1}{2\pi\sqrt{mn}}\int_1^T\Big(\frac{t}{2\pi}\Big)^{\frac{-\sum_j\alpha_j-\sum_j\beta_j}{2}}\frac{1}{(k+1)!k!(2\pi i)^{2k+1}}$$
$$\times\oint\ldots\oint\Big(\frac{t}{2\pi}\Big)^{\frac{\sum_j s_j-\sum_j z_j}{2}}\frac{T_{\underline{s},-\underline{z}}(m,n)\Delta(s_1,\ldots,s_{k+1},z_1,\ldots,z_k)^2}{\prod_{i,j}(s_i-\alpha_j)\prod_{i,j}(s_i+\beta_j)\prod_{i,j}(z_i-\alpha_j)\prod_{i,j}(z_i+\beta_j)}$$
$$\times ds_1\ldots ds_{k+1}dz_1\ldots dz_k dt\Big|_{\underline{\alpha}=\underline{\beta}=0}+O_{k,\varepsilon}(T^{1/2+\varepsilon}),$$

where $\Delta(.)$ is the Vandermonde function and the paths of integration are small circles around the poles α_j and $-\beta_j$. We observe that

$$\frac{d}{d\alpha}\frac{e^{-a\alpha}}{\prod_{j=1}^n(z_j-\alpha)}\Big|_{\alpha=0}=\frac{1}{\prod_{j=1}^n z_j}\Big(\sum_{j=1}^n\frac{1}{z_j}-a\Big)\tag{39}$$

and

$$\frac{d}{d\beta}\frac{e^{-a\beta}}{\prod_{j=1}^n(z_j+\beta)}\Big|_{\beta=0}=\frac{1}{\prod_{j=1}^n z_j}\Big(-\sum_{j=1}^n\frac{1}{z_j}-a\Big).\tag{40}$$

Thus

$$I_{2k,R}(m,n)=\frac{1}{2\pi\sqrt{mn}(k+1)!k!(2\pi i)^{2k+1}}\int_1^T\oint\dots\oint\left(\frac{t}{2\pi}\right)^{\frac{\sum_j s_j-\sum_j z_j}{2}}$$
$$\times\frac{T_{\underline{s},-\underline{z}}(m,n)\Delta(s_1,\dots,s_{k+1},z_1,\dots,z_k)^2}{(\prod_{j=1}^{k+1}s_j\prod_{j=1}^k z_j)^{2k+1}}\Bigg(-\frac{\mathscr{L}}{2}+\sum_{j=1}^{k+1}\frac{1}{s_j}+\sum_{j=1}^{k}\frac{1}{z_j}\Bigg)^{k+1}$$
$$\times\Bigg(-\frac{\mathscr{L}}{2}-\sum_{j=1}^{k+1}\frac{1}{s_j}-\sum_{j=1}^{k}\frac{1}{z_j}\Bigg)^{k}ds_1\dots ds_{k+1}dz_1\dots dz_k dt+O_{k,\varepsilon}(T^{1/2+\varepsilon}).$$

The contribution from the left-hand side of the contour of integration is

$$I_{2k,L}(m,n)=-\frac{1}{2\pi}\int_1^T\frac{\zeta'(1-c+it)}{\zeta(1-c+it)}\zeta'(1-c+it)^k\zeta'(c-it)^k\left(\frac{m}{n}\right)^{c-it-1/2}dt.$$

By the functional equation for $\zeta'(s)/\zeta(s)$ in (16), we have

$$\frac{\zeta'(1-c+it)}{\zeta(1-c+it)}=\frac{\chi'(1-c+it)}{\chi(1-c+it)}-\frac{\zeta'(c-it)}{\zeta(c-it)}.$$

Thus,

$$I_{2k,L}(m,n)=-\frac{1}{2\pi i}\int_{1-c+i}^{1-c+iT}\frac{\chi'(s)}{\chi(s)}\zeta'(s)^k\zeta'(1-s)^k\left(\frac{m}{n}\right)^{-s+1/2}ds$$
$$+\frac{1}{2\pi}\int_1^T\frac{\zeta'(c-it)}{\zeta(c-it)}\zeta'(c-it)^k\zeta'(1-c+it)^k\left(\frac{m}{n}\right)^{c-it-1/2}dt.$$

We note that the second term on the right-hand side is equal to $\overline{I_{2k,R}(n,m)}$. To handle the first term, we may first shift the line of integration to the $\frac{1}{2}$-line with a negligible error. Then, using the approximation for $\chi'(s)/\chi(s)$ in (28), we find that this term is roughly equal to

$$\frac{\mathscr{L}}{2\pi}\int_1^T\zeta'(\tfrac12+it)^k\zeta'(\tfrac12-it)^k\left(\frac{m}{n}\right)^{-it}dt=\frac{d}{d\alpha_1}\cdots\frac{d}{d\alpha_k}\frac{d}{d\beta_1}\cdots\frac{d}{d\beta_k}\frac{\mathscr{L}}{2\pi}$$
$$\times\int_1^T\prod_{j=1}^k\Big(\zeta(\tfrac12+it+\alpha_j)\zeta(\tfrac12-it+\beta_j)\Big)\left(\frac{m}{n}\right)^{-it}dt\Bigg|_{\underline{\alpha}=\underline{\beta}=0}.$$

Hughes and Young [16] have conjectured that this integral equals

$$\frac{1}{\sqrt{mn}}\int_1^T\Bigg(\sum_{0\le j\le k}\sum_{\substack{P\subset\{\alpha_1,\dots,\alpha_k\}\\ Q\subset\{\beta_1,\dots,\beta_k\}\\ |P|=|Q|=j}}S_{\underline{\alpha}_P,\underline{\beta}_Q}(m,n)\left(\frac{t}{2\pi}\right)^{-P-Q}\Bigg)dt+O_{k,\varepsilon}(T^{1/2+\varepsilon}),$$

where

$$S_{\underline{\alpha}_P,\underline{\beta}_Q}(m,n)=\prod_{1\le i,j\le k}\zeta(1+\alpha_i+\beta_j)$$
$$\times\prod_p\Bigg(\prod_{1\le i,j\le k}\left(1-\frac{1}{p^{1+\alpha_i+\beta_j}}\right)\sum_{\sum_{j=1}^k a_j=\sum_{j=1}^k b_j}\frac{1}{p^{\sum_{j=1}^k(1/2+\alpha_j)a_j+(1/2+\beta_j)b_j}}\Bigg)D_{\underline{\alpha},\underline{\beta}}(m,n),$$

with

$$D_{\underline{\alpha},\underline{\beta}}(m,n)=\prod_{\substack{p^{m_p}||m\\ p^{n_p}||n}}\bigg(\frac{\sum_{j=0}^{\infty}B_{\underline{\alpha}}(p^{j+m_p})B_{\underline{\beta}}(p^j)/p^j}{\sum_{j=0}^{\infty}B_{\underline{\alpha}}(p^j)B_{\underline{\beta}}(p^j)/p^j}\times\frac{\sum_{j=0}^{\infty}B_{\underline{\alpha}(p^j)B_{\underline{\beta}}}(p^{j+n_p})/p^j}{\sum_{j=0}^{\infty}B_{\underline{\alpha}}(p^j)B_{\underline{\beta}}(p^j)/p^j}\bigg).$$

This expression can be treated as before, that is, by expressing it as a contour integral, and using (39) and (40). In this way, we obtain the following conjecture.

Conjecture 6.1. *Suppose $m,n\in\mathbb{N}$ with $(m,n)=1$, and $mn\ll_\varepsilon T^{1/2-\varepsilon}$. Then we have*

$$\begin{aligned}I_{2k}(m,n) \;=\; & \frac{1}{2\pi\sqrt{mn}(k+1)!k!(2\pi i)^{2k+1}}\int_1^T\oint\cdots\oint\Big(\frac{t}{2\pi}\Big)^{\frac{\sum_j s_j-\sum_j z_j}{2}}\\
&\times\frac{\big(T_{\underline{s},-\underline{z}}(m,n)+T_{\underline{s},-\underline{z}}(n,m)\big)\Delta(s_1,\ldots,s_{k+1},z_1,\ldots,z_k)^2}{(\prod_{j=1}^{k+1}s_j\prod_{j=1}^{k}z_j)^{2k+1}}\\
&\times\Big(-\frac{\mathscr{L}}{2}+\sum_{j=1}^{k+1}\frac{1}{s_j}+\sum_{j=1}^{k}\frac{1}{z_j}\Big)^{k+1}\\
&\times\Big(-\frac{\mathscr{L}}{2}-\sum_{j=1}^{k+1}\frac{1}{s_j}-\sum_{j=1}^{k}\frac{1}{z_j}\Big)^{k}ds_1\ldots ds_{k+1}dz_1\ldots dz_kdt\\
&+\frac{\mathscr{L}}{2\pi\sqrt{mn}(k!)^2(2\pi i)^{2k}}\int_1^T\oint\cdots\oint\Big(\frac{t}{2\pi}\Big)^{\frac{\sum_j s_j-\sum_j z_j}{2}}\\
&\times\frac{S_{\underline{s},-\underline{z}}(m,n)\Delta(s_1,\ldots,s_k,z_1,\ldots,z_k)^2}{(\prod_{j=1}^{k}s_jz_j)^{2k}}\\
&\times\Big(-\frac{\mathscr{L}}{2}+\sum_{j=1}^{k}\Big(\frac{1}{s_j}+\frac{1}{z_j}\Big)\Big)^{k}\\
&\times\Big(-\frac{\mathscr{L}}{2}-\sum_{j=1}^{k}\Big(\frac{1}{s_j}+\frac{1}{z_j}\Big)\Big)^{k}ds_1\ldots ds_kdz_1\ldots dz_kdt+O_{k,\varepsilon}(T^{1/2+\varepsilon}).\end{aligned}$$

We now use Conjecture 6.1 to give another heuristic argument for Conjectures 2.1 and 2.2. Since high moments have much more complicated arithmetic contributions, we shall only treat the case $k=2$. Conjecture 6.1 asserts that $I_4(m,n)$ is asymptotic to $T\mathcal{P}(\mathscr{L})/\sqrt{mn}$, where $\mathcal{P}(x)$ is a polynomial of degree 9 with coefficients depending on m and n. We wish to extract the leading term from this expression. To do this we compute the residues at $s_1=s_2=s_3=z_1=z_2=0$ of the contour integrals. In this way, we find that

$$I_4(m,n)=\frac{T\mathscr{L}}{2\pi}\frac{\mathscr{L}^8}{8640\zeta(2)}\frac{\delta(m)\delta(n)}{\sqrt{mn}}+O\big((mn)^{-1/2}d(m)d(n)T\mathscr{L}^8\big),\tag{41}$$

where

$$\delta(n)=\prod_{p^{n_p}||n}\Big(1+n_p\frac{1-1/p}{1+1/p}\Big).$$

Using the expression in (9) with $k=-2$, we have

$$\sum_{0<\gamma\leq T}\left|\zeta'(\rho)P_X(\rho)^{-1}\right|^4=\sum_{\substack{mn\in S(X)\\ m,n\leq T^{\vartheta}}}\frac{\alpha_{-2}(m)\alpha_{-2}(n)}{\sqrt{mn}}\sum_{0<\gamma\leq T}\left|\zeta'(\rho)\right|^4\left(\frac{m}{n}\right)^{-i\gamma}+O_{\varepsilon}(T^{1-\varepsilon\vartheta/3}). \tag{42}$$

It follows from (41) that the sum over m and n here equals

$$\frac{T\mathscr{L}}{2\pi}\frac{\mathscr{L}^8}{8640\zeta(2)}\sum_{\substack{mn\in S(X)\\ m,n\leq T^{\vartheta}}}\frac{\alpha_{-2}(m)\alpha_{-2}(n)\delta\big(m/(m,n)\big)\delta\big(n/(m,n)\big)(m,n)}{mn}$$
$$+O\bigg(T\mathscr{L}^8\sum_{mn\in S(X)}\frac{d(m)^2d(n)^2(m,n)}{mn}\bigg). \tag{43}$$

The big-O term is

$$\ll T\mathscr{L}^8\sum_{l\in S(X)}\frac{d(l)^4}{l}\bigg(\sum_{m\in S(X)}\frac{d(m)^2}{m}\bigg)^2\ll T\mathscr{L}^8(\log X)^{24},$$

while the sum over m and n in the main term has been evaluated by Gonek, Hughes and Keating (see pp. 534, 538 of [13]) and is

$$\sim\frac{\pi^2}{6}(e^{\gamma_0}\log X)^{-4}.$$

Thus, combining with (42), (43), and choosing ϑ sufficiently small, we obtain

$$\frac{1}{N(T)}\sum_{0<\gamma\leq T}\left|\zeta'(\rho)P_X(\rho)^{-1}\right|^4\sim\frac{1}{8640}\frac{\mathscr{L}^8}{(e^{\gamma_0}\log X)^4}.$$

This heuristic argument provides further evidence for Conjecture 2.1 and Conjecture 2.2 in the case $k=2$.

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