# ON BALAZARD, SAIAS, AND YOR'S EQUIVALENCE TO THE RIEMANN HYPOTHESIS

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Abstract. Balazard, Saias, and Yor proved that the Riemann Hypothesis is equivalent to a certain weighted integral of the logarithm of the Riemann zeta-function along the critical line equaling zero. Assuming the Riemann Hypothesis, we investigate the rate at which a truncated version of this integral tends to zero, answering a question of Borwein, Bradley, and Crandall and disproving a conjecture of the same authors. A simple modification of our techniques gives a new proof of a classical Omega theorem for the function $S(t)$ in the theory of the Riemann zeta-function.

## 1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. In [1], Balazard, Saias, and Yor gave an elegant proof of the formula

$$\int_{\Re(s)=1/2} \frac{\log|\zeta(s)|}{|s|^2}|ds| = 2\pi \sum_{\beta>1/2} \log\left|\frac{\rho}{1-\rho}\right|, \tag{1.1}$$

where the sum runs over the nontrivial zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with real part strictly greater than $1/2$. Since the Riemann Hypothesis (RH) states that $\beta=1/2$ for all the nontrivial zeros of $\zeta(s)$, it follows that RH is equivalent to the expression

$$\int_{\Re(s)=1/2} \frac{\log|\zeta(s)|}{|s|^2}|ds| = 0. \tag{1.2}$$

This equivalence led Borwein, Bradley, and Crandall [2] to study the function

$$I(T) = \int_{-T}^{T} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}\, dt.$$

Since by (1.2), RH is equivalent to the assertion that $I(T)\to 0$ as $T\to\infty$, they asked the following question: What are the admissible positive values of $\alpha$ such that $I(T)=O(T^{-\alpha})$ as $T\to\infty$ on RH? Based upon numerical evidence, they conjectured that $I(T)=O(T^{-2})$.

In this note, we answer their question and disprove their conjecture by showing that $I(T)=O(T^{-\alpha})$ for any fixed positive $\alpha<2$ as $T\to\infty$, but that $I(T)\neq O(T^{-2})$. Precisely, we prove the following theorem.

**Theorem 1.1.** *Assume RH. Then we have*

$$I(T) = O\left(\frac{1}{T^2}\frac{\log T}{(\log\log T)^2}\right) \tag{1.3}$$

*and*

$$I(T) = \Omega\left(\frac{1}{T^2}\frac{\sqrt{\log T}}{(\log\log T)^{3/2}}\right) \tag{1.4}$$

*as $T\to\infty$.*

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Additionally, by estimating the tails of the integral in (1.1) we obtain an unconditional formula for $I(T)$ in terms of the nontrivial zeros of the Riemann zeta-function.

**Theorem 1.2.** *For $T \geq 3$, we have*

$$I(T) = 2\pi \sum_{\substack{-T\leq\gamma\leq T\\ \beta>1/2}} \log\left|\frac{\rho}{1-\rho}\right| + O\bigg(\frac{1}{T^2}\log T\bigg). \tag{1.5}$$

Through a straightforward modification of our argument it can be shown that the Lindelöf Hypothesis implies that the error term in (1.5) is $o(T^{-2}\log T)$ as $T\to\infty$. We remark that the proof of Theorem 1.2 does not give a new proof of (1.1) since we merely truncate the integral. However, we will show how to adapt the method used to prove Theorem 1.2 to give a simple, new proof of (1.1) that relies only on standard techniques in complex analysis.

In the final section, we give a new proof of a classical Omega theorem of Montgomery for the function $S(t)$.

## 2. Various lemmas

Our first two lemmas concern integrals of the logarithm of the Riemann zeta-function (one unconditional and the other conditional upon RH).

**Lemma 2.1.** *Uniformly for $1\leq c\leq 2$ and $t\geq 3$ we have*

$$\int_{1/2}^{c} \big|\log\zeta(\sigma+it)\big|\, d\sigma \ll \log t.$$

*Proof.* See Lemma $\beta$ of Titchmarsh [11]. □

**Lemma 2.2.** *Assume RH. Then for $t\geq T\geq 3$ we have*

$$\int_T^t \log\big|\zeta(\tfrac12+iu)\big|\, du \ll \frac{\log t}{(\log\log t)^2}.$$

*Proof.* Under the assumption of RH, Cauchy's theorem implies that

$$\int_T^t \log\big|\zeta(\tfrac12+iu)\big|\, du = -\int_{1/2}^{3/2} \arg\zeta(\sigma+it)\, d\sigma + \int_{1/2}^{3/2} \arg\zeta(\sigma+iT)\, d\sigma + O(1).$$

We will bound the first integral on the right-hand side of this equation. The second integral can be handled similarly.

Let $\sigma_t = 1/2 + (\log\log t)^{-1}$ and write

$$\int_{1/2}^{3/2} \arg\zeta(\sigma+it)\, d\sigma = I_1 + I_2 + I_3, \tag{2.1}$$

where $I_1$ is the portion of the integral over $[1/2,\sigma_t)$, $I_2$ is the portion over $[\sigma_t, 3/4)$, and $I_3$ is the portion over $[3/4,3/2]$. By Theorem 13.21 of [9], we have $\arg\zeta(\sigma+it)\ll \log t/\log\log t$ for $\sigma\geq 1/2$. Thus,

$$I_1 \ll \frac{\log t}{(\log\log t)^2}.$$

For $\sigma_t\leq\sigma<3/4$ it follows from Corollary 13.16 of [9] that $\arg\zeta(\sigma+it)\ll(\log t)^{(2-2\sigma)}/\log\log t$. Hence

$$I_2 \ll \frac{\log t}{(\log\log t)^2}.$$

Finally, Corollary 13.16 of [9] also implies that $\arg\zeta(\sigma+it)\ll(\log t)^{1/2}$ uniformly for $3/4\leq\sigma\leq 3/2$, and we have

$$I_3 \ll (\log t)^{1/2}.$$

The lemma now follows by inserting the estimates for $I_1, I_2$ and $I_3$ into (2.1). □

Next we prove two key lemmas which are used to prove the estimate (1.4) in Theorem 1.1.

**Lemma 2.3.** *Assume RH. For any sequence of complex numbers $\{r(n)\}$ let*

$$R(t) = \sum_{n\le N} \frac{r(n)}{n^{it}}.$$

*Then uniformly for $1/2 \le \alpha \le 2$ , $h \in \mathbb{R}$, $N > 1$, $T \ge 3$, and $\varepsilon > 0$ we have*

$$\int_T^{2T} \log\zeta(\alpha+it+ih)\big|R(t)\big|^2 dt = T \sum_{mn\le N} \frac{\Lambda(n) r(m)\overline{r(mn)}}{n^{\alpha+ih}\log n} + O\Big(N(\log TN)^{3/2+\varepsilon}\sum_{n\le N}|r(n)|^2\Big).$$

*Proof.* Let $c = 1 + (\log N)^{-1}$, $\mathcal{R}(s) = \sum_{n\le N} r(n)\, n^{-s}$, and $\overline{\mathcal{R}}(s) = \sum_{n\le N} \overline{r(n)}\, n^{-s}$. We shall consider the case $1/2 \le \alpha \le c$. The remaining case $c \le \alpha \le 2$ is treated similarly to $\mathcal{I}_3$ below.

By the elementary inequality $2|ab| \le |a|^2 + |b|^2$ it follows that

$$|r(m)r(n)|\Big(\frac{m}{n}\Big)^{\sigma-\alpha} \le \frac{1}{2}\left(\frac{|r(m)|^2\Delta}{n^{2(\sigma-\alpha)}} + \frac{|r(n)|^2 m^{2(\sigma-\alpha)}}{\Delta}\right)$$

for any $\Delta > 0$. Thus,

$$\begin{aligned}
\big|\mathcal{R}(s-\alpha-ih)\overline{\mathcal{R}}(\alpha+ih-s)\big| &\le \sum_{m,n\le N} |r(m)r(n)|\Big(\frac{m}{n}\Big)^{\sigma-\alpha} \\
&\ll \left(\Delta \sum_{m\le N}\frac{1}{m^{2(\sigma-\alpha)}} + \frac{1}{\Delta}\sum_{m\le N} m^{2(\sigma-\alpha)}\right)\sum_{n\le N}|r(n)|^2 \\
&\ll \left(\Delta N^{1-2(\sigma-\alpha)}\log N + \frac{N^{1+2(\sigma-\alpha)}}{\Delta}\right)\sum_{n\le N}|r(n)|^2
\end{aligned}$$

uniformly for $\alpha \le \sigma \le c$. Choosing $\Delta = N^{2(\sigma-\alpha)}(\log N)^{-1/2}$, we conclude that

$$\big|\mathcal{R}(s-\alpha-ih)\overline{\mathcal{R}}(\alpha+ih-s)\big| \ll N(\log N)^{1/2}\sum_{n\le N}|r(n)|^2 \tag{2.2}$$

uniformly for $\alpha \le \sigma \le c$.

Let $\mathscr{C}$ be the positively oriented rectangle with vertices at $\alpha + i(T+h)$, $c + i(T+h)$, $c + i(2T+h)$, and $\alpha + i(2T+h)$. We write

$$i\int_{\mathscr{C}} \log\zeta(s)\,\mathcal{R}(s-\alpha-ih)\,\overline{\mathcal{R}}(\alpha+ih-s)\,ds = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,$$

where $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ are the parts of the integral over the left, bottom, right, and top edges of $\mathscr{C}$, respectively. Cauchy's theorem implies that

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 = 0.$$

Thus, after an obvious variable change, we have

$$\int_T^{2T} \log\zeta(\alpha+it+ih)\,\big|R(t)\big|^2\,dt = -\mathcal{I}_3 + O\big(|\mathcal{I}_2|+|\mathcal{I}_4|\big). \tag{2.3}$$

By (2.2) and Lemma 2.1 we have

$$|\mathcal{I}_2|+|\mathcal{I}_4| \ll N(\log NT)^{3/2}\sum_{n\le N}|r(n)|^2. \tag{2.4}$$

It remains to estimate $\mathcal{I}_3$.

In $\mathcal{I}_3$, we express $\log\zeta(s)$ as an absolutely convergent Dirichlet, interchange summation and integration, and then integrate term-by-term to obtain

$$-I_3 = T\sum_{mn\le N}\frac{\Lambda(n)r(m)\overline{r(mn)}}{n^{\alpha+ih}\log n} + O\Bigg(\sum_{k=2}^{\infty}\sum_{\substack{m,n\le N\\ n\ne km}}\frac{\Lambda(k)}{k^c\log k}\frac{|r(m)r(n)|}{|\log\frac{n}{km}|}\Big(\frac{n}{m}\Big)^{c-\alpha}\Bigg). \tag{2.5}$$

To bound the error term, we first note that

$$\begin{aligned}\sum_{\substack{m,n\le N\\ n\ne km}}\frac{|r(m)r(n)|}{|\log\frac{n}{km}|}\Big(\frac{n}{m}\Big)^{c-\alpha} \ll\ & \Delta\sum_{n\le N}|r(n)|^2\sum_{\substack{m\le N\\ n\ne km}}\frac{1}{m^{2(c-\alpha)}|\log\frac{n}{km}|}\\ &+\frac{1}{\Delta}\sum_{m\le N}|r(m)|^2\sum_{\substack{n\le N\\ n\ne km}}\frac{n^{2(c-\alpha)}}{|\log\frac{n}{km}|}\end{aligned}$$

for any $\Delta>0$. Next, using standard techniques, we have

$$\sum_{\substack{m\le N\\ n\ne km}}\frac{1}{m^{2(c-\alpha)}|\log\frac{n}{km}|}\ll N^{1-2(c-\alpha)}(\log N)^2 \quad\text{and}\quad \sum_{\substack{n\le N\\ n\ne km}}\frac{n^{2(c-\alpha)}}{|\log\frac{n}{km}|}\ll N^{1+2(c-\alpha)}\log N$$

uniformly in $k$. Hence

$$\sum_{\substack{m,n\le N\\ n\ne km}}\frac{|r(m)r(n)|}{|\log\frac{n}{km}|}\Big(\frac{n}{m}\Big)^{c-\alpha}\ll\bigg(\Delta N^{1-2(c-\alpha)}(\log N)^2+\frac{N^{1+2(c-\alpha)}\log N}{\Delta}\bigg)\sum_{n\le N}|r(n)|^2.$$

Choosing $\Delta = N^{2(c-\alpha)}(\log N)^{-1/2}$, it follows that the big-$O$ term in (2.5) is

$$\ll N(\log N)^{3/2}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^c\log k}\sum_{n\le N}|r(n)|^2\ll N(\log N)^{3/2}\log\log N\sum_{n\le N}|r(n)|^2.$$

The lemma now follows from this estimate and (2.3)–(2.5). □

**Lemma 2.4.** *Let $\mu$ and $\nu$ be fixed non-negative integers, $N>1$, and $h\in[0,(\log\log N)^{-1}]$. Then there exist two real-valued arithmetic functions $r^{\pm}(n)$ and a positive constant $C$ (depending on $\mu$ and $\nu$) such that*

$$\sum_{mn\le N}\frac{\Lambda(n)\sin^{\mu}(h\log n)r^{+}(m)r^{+}(mn)}{\sqrt{n}(\log n)^{\nu}}\bigg/\sum_{n\le N}|r^{+}(n)|^2\ \ge\ C\,h^{\mu}(\log N)^{1/2}(\log\log N)^{\mu-\nu+1/2}$$

*and*

$$\sum_{mn\le N}\frac{\Lambda(n)\sin^{\mu}(h\log n)r^{-}(m)r^{-}(mn)}{\sqrt{n}(\log n)^{\nu}}\bigg/\sum_{n\le N}|r^{-}(n)|^2\ \le\ -C\,h^{\mu}(\log N)^{1/2}(\log\log N)^{\mu-\nu+1/2}.$$

*Proof.* Our proof of this lemma is based upon the ideas in the proof of Theorem 2.1 of Soundararajan [10]. We shall prove the first inequality. The second inequality can be proved similarly by choosing $r^{-}(n)=\mu(n)r(n)$, where $\mu(n)$ is the Möbius function and $r(n)$ is defined below. Throughout the proof, the letter $p$ denotes a prime number.

We choose $r^{+}(n)$ to be the multiplicative function $r(n)$ supported on square-free integers and defined on primes $p$ by

$$r(p)=\begin{cases}\frac{L(\log p)^{\nu}}{\sqrt{p}}, & \text{if } A<p<B,\\ 0, & \text{otherwise.}\end{cases}$$

Here the parameters $A$, $B$ and $L$ are chosen so that

$$A = L^2(\log L)^{2\nu+1}, \quad B = L^3, \quad \text{and} \quad L^2(\log B)^{2\nu+1} = (2\nu+1)\log N.$$

We note that with our choice we have $r(p) \ll 1$, $L \asymp (\log N)^{1/2}(\log\log N)^{-\nu-1/2}$, and $\log B < (3/2)\log\log N$, so that $\sin(h\log p) > (h\log p)/2$ for $h \in [0, (\log\log N)^{-1}]$ and $p < B$.

With $r^+(n) = r(n)$, the denominator on the left-hand side of the first inequality is

$$\sum_{n\le N} |r(n)|^2 \le \sum_{n=1}^{\infty} r(n)^2 = \prod_p \big(1 + r(p)^2\big).$$

To estimate the numerator, we use Rankin's trick which asserts that for any sequence of non-negative real numbers $\{a_n\}$, and any $\alpha > 0$ we have

$$\sum_{n>x} a_n \le x^{-\alpha}\sum_{n>x} a_n n^{\alpha} \le x^{-\alpha}\sum_{n=1}^{\infty} a_n n^{\alpha}.$$

Therefore,

$$\begin{aligned}
&\sum_{mn\le N} \frac{\Lambda(n)\sin^{\mu}(h\log n) r(m) r(mn)}{\sqrt{n}\,(\log n)^{\nu}} \\
&\qquad = \sum_{n\le N} \frac{\Lambda(n)\sin^{\mu}(h\log n) r(n)}{\sqrt{n}\,(\log n)^{\nu}} \sum_{\substack{m\le N/n\\ (m,n)=1}} r(m)^2 \\
&\qquad = \sum_{n\le N} \frac{\Lambda(n)\sin^{\mu}(h\log n) r(n)}{\sqrt{n}\,(\log n)^{\nu}} \prod_{p\nmid n}\big(1 + r(p)^2\big) \qquad (2.6)\\
&\qquad\qquad + O\left(h^{\mu}\sum_{n\le N} \frac{\Lambda(n) r(n)}{\sqrt{n}\,(\log n)^{\nu-\mu}}\Big(\frac{n}{N}\Big)^{\alpha}\prod_{p\nmid n}\big(1 + p^{\alpha} r(p)^2\big)\right).
\end{aligned}$$

Here we have used the inequality $|\sin x| \le x$ for $x \ge 0$ in the big-$O$ term. Note that $r(n)$ is supported on square-free integers, and the inequalities $\sin(h\log p) \gg h\log p$ and $r(p) \ll 1$ hold for all $p < B$. Using these observations we see that the ratio of the main term in (2.6) to $\sum_{n\le N}|r(n)|^2$ is

$$\begin{aligned}
&\gg \sum_{p\le N} \frac{\sin^{\mu}(h\log p)\, r(p)}{\sqrt{p}\,(\log p)^{\nu-1}(1+r(p)^2)} = L\sum_{A<p<B} \frac{\sin^{\mu}(h\log p)\log p}{p\,(1+r(p)^2)} \\
&\gg L\sum_{A<p<B} \frac{h^{\mu}(\log p)^{\mu+1}}{p} = Lh^{\mu}\left(\frac{(\log B)^{\mu+1}}{\mu+1} - \frac{(\log A)^{\mu+1}}{\mu+1} + O\big((\log B)^{\mu}\big)\right) \\
&\gg Lh^{\mu}(\log\log N)^{\mu+1} \gg h^{\mu}(\log N)^{1/2}(\log\log N)^{\mu-\nu+1/2}.
\end{aligned}$$

On the other hand, the error term in (2.6) is

$$\begin{aligned}
&\ll h^{\mu} L N^{-\alpha}\left(\sum_{A<p<B} \frac{(\log p)^{\mu+1}}{p^{1-\alpha}(1+p^{\alpha} r(p)^2)}\right)\prod_p \big(1 + p^{\alpha} r(p)^2\big) \\
&\ll \big(1+\alpha\log B\big)\, h^{\mu} L N^{-\alpha}\left(\sum_{A<p<B} \frac{(\log p)^{\mu+1}}{p}\right)\prod_p \big(1 + p^{\alpha} r(p)^2\big).
\end{aligned} \qquad (2.7)$$

Note that $B = L^3$ and $L \ll (\log N)^{1/2}$. So by Rankin's trick (with exponent taken to be 1/2) we have

$$\sum_{n\le N}|r(n)|^2 = \sum_{n=1}^{\infty}|r(n)|^2 + O\bigg(\frac{L^2}{N^{1/2}}\sum_{A<p<B}\frac{(\log p)^{2\nu}}{\sqrt{p}}\bigg)$$
$$= \prod_p(1+r(p)^2) + O\bigg(\frac{L^2}{N^{1/2}}B^{3/2}(\log B)^{2\nu}\bigg) \gg \prod_p(1+r(p)^2).$$

Choosing $\alpha = (\log L)^{-2}$, we see that the ratio of (2.7) to $\sum_{n\le N}|r(n)|^2 \gg \prod_p(1+r(p)^2)$ is

$$\ll h^{\mu} L N^{-\alpha}(\log B)^{\mu+1}\prod_p\left(\frac{1+p^{\alpha}r(p)^2}{1+r(p)^2}\right)$$
$$\ll h^{\mu}L(\log B)^{\mu+1}\exp\bigg\{-\alpha\log N + \sum_{A<p<B}(p^{\alpha}-1)\frac{L^2(\log p)^{2\nu}}{p}\bigg\}$$
$$\ll h^{\mu}L(\log B)^{\mu+1}\exp\bigg\{-\alpha\log N + \frac{\alpha L^2}{2\nu+1}\Big((\log B)^{2\nu+1}-(\log A)^{2\nu+1}\Big) + O\big(\alpha^2L^2(\log B)^{2\nu+2}\big)\bigg\}$$
$$\ll h^{\mu}L(\log B)^{\mu+1}\exp\bigg\{-\frac{1}{2}\frac{\alpha L^2(\log A)^{2\nu+1}}{2\nu+1}\bigg\} = o\Big(h^{\mu}(\log N)^{1/2}(\log\log N)^{\mu-\nu+1/2}\Big)$$

since $L(\log B)^{\mu+1} \ll (\log N)^{1/2}(\log\log N)^{\mu-\nu+1/2}$ by our choices of $A, B$, and $L$. Combining the estimates, the lemma follows. □

## 3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Our proof of (1.3) follows from Lemma 2.2, while our proof of (1.4) is a consequence of the following Omega theorem.

**Theorem 3.1.** *Assume RH. Then as $t \to \infty$, we have*

$$\int_{t-h}^{t+h}\log|\zeta(\tfrac{1}{2}+iu)|\,du = \Omega_{\pm}\left(h\sqrt{\frac{\log t}{\log\log t}}\right)$$

*uniformly for $h \in [0, (\log\log t)^{-1}]$.*

*Proof.* We prove this theorem using Soundararajan's resonance method.

Let $R(t) = \sum_{n\le N} r(n)n^{-it}$ and observe that

$$\max_{T\le t\le 2T}\int_{t-h}^{t+h}\log\big|\zeta(\tfrac{1}{2}+iu)\big|\,du \;\ge\; \frac{\int_T^{2T}\Big\{\int_{t-h}^{t+h}\log|\zeta(\frac{1}{2}+iu)|\,du\Big\}|R(t)|^2\,dt}{\int_T^{2T}|R(t)|^2\,dt} \tag{3.1}$$

and

$$\min_{T\le t\le 2T}\int_{t-h}^{t+h}\log\big|\zeta(\tfrac{1}{2}+iu)\big|\,du \;\le\; \frac{\int_T^{2T}\Big\{\int_{t-h}^{t+h}\log|\zeta(\frac{1}{2}+iu)|\,du\Big\}|R(t)|^2\,dt}{\int_T^{2T}|R(t)|^2\,dt}. \tag{3.2}$$

Making the substitution $u = t + h_1$, using Lemma 2.3 with $\alpha = 1/2$, and integrating with respect to $h_1$, the double integral in the numerators in (3.1) and (3.2) is

$$= \Re\int_{-h}^{h}\int_T^{2T}\log\zeta(\tfrac{1}{2}+it+ih_1)|R(t)|^2\,dt\,dh_1$$
$$= 2T\sum_{mn\le N}\frac{\Lambda(n)r(m)\overline{r(mn)}\sin(h\log n)}{\sqrt{n}(\log n)^2} + O\bigg(hN(\log TN)^{3/2+\varepsilon}\sum_{n\le N}|r(n)|^2\bigg). \tag{3.3}$$

Furthermore, Montgomery and Vaughan's mean-value theorem for Dirichlet polynomials (Corollary 3 of [8]) implies that

$$\int_T^{2T} |R(t)|^2\,dt = \big(T+O(N)\big)\sum_{n\le N}|r(n)|^2. \tag{3.4}$$

Choosing $N=T(\log T)^{-2}$, Lemma 2.4 and equations (3.1)–(3.4) imply that

$$\max_{T\le t\le 2T}\int_{t-h}^{t+h}\log|\zeta(\tfrac12+iu)|\,du \;\ge\; c_1 h\sqrt{\frac{\log T}{\log\log T}}$$

and

$$\min_{T\le t\le 2T}\int_{t-h}^{t+h}\log|\zeta(\tfrac12+iu)|\,du \;\le\; -c_2 h\sqrt{\frac{\log T}{\log\log T}}$$

uniformly for $h\in[0,(\log\log N)^{-1}]$, where $c_1$ and $c_2$ are (computable) positive constants. The theorem follows. $\square$

We now prove Theorem 1.1.

*Proof of Theorem 1.1.* We first prove (1.3). Assuming RH, (1.2) implies that

$$\int_{-\infty}^{\infty}\frac{\log|\zeta(\frac12+it)|}{\frac14+t^2}\,dt=0.$$

Since the integrand is even, it follows that

$$I(T)=-2\int_T^{\infty}\frac{\log|\zeta(\frac12+it)|}{\frac14+t^2}\,dt.$$

Integrating by parts and applying Lemma 2.2 we have

$$\begin{aligned}I(T)&=-2\int_T^{\infty}\frac{1}{\frac14+t^2}\,d\bigg(\int_T^t\log|\zeta(\tfrac12+iu)|\,du\bigg)\\&=-4\int_T^{\infty}\frac{t}{(\frac14+t^2)^2}\bigg(\int_T^t\log|\zeta(\tfrac12+iu)|\,du\bigg)\,dt\\&\ll\int_T^{\infty}\frac{1}{t^3}\frac{\log t}{(\log\log t)^2}\,dt\ll\frac{1}{T^2}\frac{\log T}{(\log\log T)^2}.\end{aligned}$$

This completes the proof of (1.3).

We now prove (1.4). Let $h\in[0,(\log\log t)^{-1}]$ and suppose, for sake of contradiction, that

$$I(t)=o\bigg(\frac{1}{t^2}\sqrt{\frac{\log t}{(\log\log t)^3}}\bigg).$$

Then for $t-h\le u\le t+h$ we have

$$I(u)-I(t-h)=o\bigg(\frac{1}{t^2}\sqrt{\frac{\log t}{(\log\log t)^3}}\bigg), \tag{3.5}$$

as well. Integrating by parts yields

$$\begin{aligned}\int_{t-h}^{t+h} \log\big|\zeta(\tfrac12+iu)\big|du =& \int_{t-h}^{t+h} \big(\tfrac14+u^2\big)\, d\bigg(\int_{t-h}^{u} \frac{\log|\zeta(\frac12+iv)|}{\frac14+v^2}\,dv\bigg)\\ =&\Big(\tfrac14+(t+h)^2\Big)\int_{t-h}^{t+h}\frac{\log|\zeta(\frac12+iv)|}{\frac14+v^2}\,dv\\ &-\int_{t-h}^{t+h} 2u\left(\int_{t-h}^{u}\frac{\log|\zeta(\frac12+iv)|}{\frac14+v^2}\,dv\right)du.\end{aligned}$$

Using the assumption (3.5) twice, it follows that

$$\int_{t-h}^{t+h} \log|\zeta(\tfrac12+iu)|\,du = o\left(\sqrt{\frac{\log t}{(\log\log t)^3}}\right).$$

If $h=(\log\log t)^{-1}$, this contradicts Theorem 3.1, and thus proves (1.4). □

## 4. Proof of Theorem 1.2

In this section, we use contour integration to prove Theorem 1.2. We also show how this method can be modified to give a new proof of (1.1) that relies solely on standard techniques from complex analysis.

*Proof of Theorem 1.2.* First, suppose that $T$ is not an ordinate of a zero of $\zeta(s)$ and consider

$$\frac{1}{i}\int_{\frac12+iT}^{\frac12+i\infty}\frac{\log\zeta(s)}{s(1-s)}\,ds.$$

Let $\mathcal{S}$ be subset of the region $\sigma>1/2$ and $t>T$, that excludes all the horizontal segments $1/2+i\gamma$ to $\beta+i\gamma$. It follows that $\log\zeta(s)$ is a single-valued analytic function in $\mathcal{S}$. Moreover, along each branch cut from $1/2+i\gamma$ to $\beta+i\gamma$ the values of $\log\zeta(s)$ on the upper and lower cuts differ by $2\pi i$. Therefore, moving the contour in the above integral from $\Re(s)=1/2$ to $\Re(s)=\infty$ yields

$$\frac{1}{i}\int_{\frac12+iT}^{\frac12+i\infty}\frac{\log\zeta(s)}{s(1-s)}\,ds = 2\pi\sum_{\substack{\gamma>T\\ \beta>1/2}}\int_{\frac12+i\gamma}^{\beta+i\gamma}\frac{1}{s(1-s)}\,ds + \frac{1}{i}\int_{\frac12+iT}^{\infty+iT}\frac{\log\zeta(s)}{s(1-s)}\,ds. \tag{4.1}$$

Also, we have

$$\int_{\frac12+i\gamma}^{\beta+i\gamma}\frac{1}{s(1-s)}\,ds = \log(\rho)-\log(\tfrac12+i\gamma)-\log(1-\rho)+\log(\tfrac12-i\gamma). \tag{4.2}$$

For $\sigma\ge 2$ we have $\log\zeta(s)\ll 2^{-\sigma}$ uniformly in $t$. From this and Lemma 2.1 it follows that

$$\int_{\frac12+iT}^{\infty+iT}\frac{\log\zeta(s)}{s(1-s)}\,ds \ll \frac{1}{T^2}\bigg(\int_{\frac12}^{2}+\int_{2}^{\infty}\bigg)|\log\zeta(\sigma+iT)|\,d\sigma \ll \frac{1}{T^2}\big(\log T+1\big).$$

Taking the real parts in (4.1), and using the above estimate and (4.2), we deduce that

$$\int_{T}^{\infty}\frac{\log|\zeta(\frac12+it)|}{\frac14+t^2}\,dt = 2\pi\sum_{\substack{\gamma>T\\ \beta>1/2}}\log\left|\frac{\rho}{1-\rho}\right| + O\bigg(\frac{1}{T^2}\log T\bigg).$$

Similarly, it can be shown that

$$\int_{-\infty}^{-T}\frac{\log|\zeta(\frac12+it)|}{\frac14+t^2}dt = 2\pi\sum_{\substack{\gamma<-T\\ \beta>1/2}}\log\left|\frac{\rho}{1-\rho}\right| + O\bigg(\frac{1}{T^2}\log T\bigg).$$

Combining these two estimates and then differencing the resulting formula with (1.1) completes the proof of the theorem in the case when $T \neq \gamma$. If $T = \gamma$, we note that for all sufficiently small $\varepsilon > 0$ the estimate in (1.5) holds for $T = \gamma + \varepsilon$. The theorem now follows in this case by letting $\varepsilon \to 0^+$. $\square$

*Proof of* (1.1). Consider the integral

$$\frac{1}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\log((s-1)\zeta(s))}{s(1-s)}\,ds.$$

Arguing as in the previous proof, we move the contour from $\Re(s) = 1/2$ to $\Re(s) = \infty$ and deduce that

$$\frac{1}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\log((s-1)\zeta(s))}{s(1-s)}\,ds = 2\pi \sum_{\beta>1/2} \int_{\frac{1}{2}+i\gamma}^{\beta+i\gamma} \frac{1}{s(1-s)}\,ds + \frac{1}{i}\int_{\mathcal{C}} \frac{\log((s-1)\zeta(s))}{s(1-s)}\,ds, \tag{4.3}$$

where $\mathcal{C}$ is the positively oriented circle centered at $s = 1$ with radius $1/4$. By the calculus of residues and the fact that $\lim_{s\to 1}((s-1)\zeta(s)) = 1$ the last integral equals zero. Thus, by this and (4.2), taking the real parts in (4.3) gives

$$\int_{-\infty}^{\infty} \frac{\log|(-\frac{1}{2}+it)\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}\,dt = 2\pi \sum_{\beta>1/2} \log\left|\frac{\rho}{1-\rho}\right|.$$

Note that by residue calculus (or otherwise) we have

$$\int_{-\infty}^{\infty} \frac{\log|-\frac{1}{2}+it|}{\frac{1}{4}+t^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(\frac{1}{4}+t^2)}{\frac{1}{4}+t^2}\,dt = 0.$$

This completes the proof. $\square$

## 5. Montgomery's Omega theorem for $S(t)$

Let $N(t)$ denote the number of non-trivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function with $0 < \gamma \leq t$. It is well-known that

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\Big(\frac{1}{t}\Big)$$

for $t \geq 10$. Here, if $t$ is not equal to an ordinate of a zero of $\zeta(s)$, the function $S(t)$ is defined by

$$S(t) = \frac{1}{\pi}\Im\log\zeta\big(\tfrac{1}{2}+it\big),$$

where the branch of logarithm is obtained by continuous variation along the line segments joining the points $2, 2+it$, and $\frac{1}{2}+it$, starting with $\arg\zeta(2) = 0$. If $t$ corresponds to an ordinate of a zero of $\zeta(s)$ we set

$$S(t) = \frac{1}{2}\lim_{\varepsilon\to 0}\big\{S(t+\varepsilon)+S(t-\varepsilon)\big\}.$$

Assuming RH, it is known that

$$\big|S(t)\big| \leq \Big(\tfrac{1}{4}+o(1)\Big)\frac{\log t}{\log\log t}$$

as $t \to \infty$ [4]. In this section, we illustrate how Lemmas 2.3 and 2.4 in §3 can be used to give a new proof of Montgomery's result [7] that

$$S(t) = \Omega_{\pm}\left(\sqrt{\frac{\log t}{\log\log t}}\right) \tag{5.1}$$

assuming RH. Tsang [13] gave an alternate proof of (5.1). In contrast to the proofs of Montgomery and Tsang, our proof uses the resonance method.

*Proof of* (5.1). Define the auxiliary function

$$S_1(t) = \int_0^t S(u)\, du$$

and note that

$$\max_{t\le u\le t+h} \pm S(u) \ge \frac{1}{h}\int_t^{t+h} \pm S(u)\ du = \frac{\pm\big(S_1(t+h) - S_1(t)\big)}{h}. \tag{5.2}$$

We use a result of Littlewood (see Theorem 3 of [6] or Theorem 9.9 of [12]) that

$$S_1(t) = \frac{1}{\pi}\int_{1/2}^{2} \log|\zeta(\sigma+it)|\, d\sigma + O(1).$$

Now taking the real part of the integral in Lemma 2.3, and integrating with respect to $\alpha$ from 1/2 to 2 yields

$$\int_T^{2T} S_1(t+h)|R(t)|^2\, dt = \frac{T}{\pi}\sum_{mn\le N}\frac{\Lambda(n)r(m)\overline{r(mn)}}{\sqrt{n}\,(\log n)^2}\cos\big(h\log n\big) + O\Big(T\sum_{n\le N}|r(n)|^2\Big)$$
$$+ O\Big(N(\log TN)^{3/2}\sum_{n\le N}|r(n)|^2\Big) + O\Big(\int_T^{2T}|R(t)|^2\, dt\Big).$$

Choosing $N = T(\log T)^{-2}$ and noting that

$$\int_T^{2T}|R(t)|^2\, dt = \Big(T + O(N)\Big)\sum_{n\le N}|r(n)|^2,$$

we obtain

$$\frac{\pm\int_T^{2T}\big(S_1(t+h) - S_1(t)\big)\,|R(t)|^2\, dt}{\int_T^{2T}|R(t)|^2\, dt} = \mp\frac{2}{\pi}\frac{\sum_{mn\le N}\frac{\Lambda(n)r(m)\overline{r(mn)}}{\sqrt{n}\,(\log n)^2}\sin^2\big(\frac{h}{2}\log n\big)}{\sum_{n\le N}|r(n)|^2} + O(1).$$

Using Lemma 2.4 with $\mu=\nu=2$ to estimate the ratio of sums on the right-hand side of the above expression, we deduce that

$$\max_{T\le t\le 2T} \pm\big(S_1(t+h) - S_1(t)\big) \gg h^2\sqrt{\log T\log\log T}$$

uniformly for $h\in[0,(\log\log N)^{-1}]$. Combining this inequality with the observation in (5.2) and choosing $h = (\log\log N)^{-1}$, the estimate (5.1) follows. □

We remark that using the resonance method in a different way, the estimate in (5.1) can be refined. In [3], assuming RH, it is shown that

$$\max_{T\le t\le 2T} S(t) \ge \frac{1}{\pi}\sqrt{\frac{\log t}{\log\log t}} + O\bigg(\frac{\sqrt{\log t}}{\log\log t}\bigg)$$

and

$$\min_{T\le t\le 2T} S(t) \le -\frac{1}{\pi}\sqrt{\frac{\log t}{\log\log t}} + O\bigg(\frac{\sqrt{\log t}}{\log\log t}\bigg).$$

These are conditional analogues of Soundararajan's unconditional Omega theorem for $\big|\zeta(\frac{1}{2}+it)\big|$ in [10]. It does not seem, however, that the method in [3] can be modified to prove Theorem 3.1.

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