DESCENT OF VECTOR BUNDLES UNDER WILDLY RAMIFIED EXTENSIONS

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Abstract. Given an irreducible normal Noetherian scheme and a finite Galois extension of the field of rational functions, we discuss the comparison of the categories of vector bundles on the scheme and equivariant vector bundles on the integral closure in the extension. This is well understood in the tame case (geometric stabilizer groups of order invertible in the local rings), so we focus on the wild (non-tame) case, which may be reduced to the case of cyclic extensions of prime order. In this case, under an additional flatness hypothesis, we give a characterization of the equivariant vector bundles that arise by base change from vector bundles on the scheme.

1. Introduction

Let X be an irreducible normal Noetherian scheme with field of rational functions K, let L/K be a finite Galois field extension with Galois group G, and let Y denote the integral closure of X in L. In case $X = \mathrm{Spec}(A)$ is affine, then, Y will be $\mathrm{Spec}(B)$ where B is the integral closure of A in L, and in general, Y will be locally of this form [7, 6.3.4]. Then G acts on Y, and morphism $Y \to X$ is G-invariant. The morphism $Y \to X$ is finite [22, Prop. I.8], and X is the scheme quotient of Y by G (i.e., in the affine case, the ring of invariants B^G is equal to A). We are interested in comparing the categories of vector bundles on X and G-equivariant vector bundles on Y.

In the language of Deligne-Mumford stacks [5], we have a morphism $[Y/G] \to X$, identifying X with the coarse moduli space in the sense of [12] of the quotient stack $[Y/G]$, and we wish to compare the categories of vector bundles on X and on $[Y/G]$. In fact, we may start with any Noetherian base scheme (which could be X, or any Noetherian scheme over which X is of finite type) and consider an irreducible normal finite-type Deligne-Mumford stack $\mathfrak{X}$ with finite, generically trivial stabilizer and coarse moduli space $\pi\colon \mathfrak{X} \to X$ and ask to compare the categories of vector bundles on X and on $\mathfrak{X}$. They are related by the pullback functor π^*, which is fully faithful, with essential image that can be characterized as follows: a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ gives rise to a coherent sheaf $\pi_*\mathcal{E}$ on X (not necessarily a vector bundle) with adjunction morphism

$$\pi^*\pi_*\mathcal{E} \to \mathcal{E}, \tag{1}$$

and $\mathcal{E} \cong \pi^*\mathcal{F}$ for some vector bundle $\mathcal{F}$ on X if and only if (1) is an isomorphism, in which case we may take $\mathcal{F} = \pi_*\mathcal{E}$. (That π^* is fully faithful follows from the property $\pi_*\mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X$ of the coarse moduli space. It is clear that if $\mathcal{E}$ is in the essential image of π^*, then $\pi_*\mathcal{E}$ is locally free and (1) is an isomorphism. Now the characterization of the essential image follows from the fact that on a reduced Noetherian scheme, a

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coherent sheaf has upper semi-continuous fiber rank, and this is locally constant if and only if the coherent sheaf is locally free.)

With the notation of the previous paragraph we may reduce the question, for a vector bundle $\mathcal{E}$ on $\mathfrak{X}$, whether the morphism (1) is an isomorphism, to the analogous question in the setting of $[Y/G] \to X$ from the beginning of the Introduction, by the following general fact, essentially contained in [12, §4] (see also [3, Lem. 3.4]): For any $x \in X$, with geometric stabilizer group G at the unique point of $\mathfrak{X}$ over x, there exists an affine étale neighborhood $X' \to X$ of x such that $X' \times_X \mathfrak{X}$ is isomorphic to a quotient stack $[Y'/G]$ for an action of G on an affine scheme Y' (and since formation of the coarse moduli space commutes with flat base change, $[Y'/G] \to X'$ is again a coarse moduli space). Then, for $x' \in X'$ mapping to x there is necessarily a unique point of Y' over x', and this is a fixed point for the G-action.

Suppose, now, that $\pi\colon \mathfrak{X} \to X$ has the property that for all $x \in X$ the residue characteristic at x does not divide the order of the geometric stabilizer group at x; such $\mathfrak{X}$ is called *tame* [1, §2.3]. Then $\pi\colon \mathfrak{X} \to X$ is a *good moduli space* [2]. Theorem 10.3 of op. cit. gives the following characterization of the essential image of the functor π^* on vector bundles: for a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ we have $\mathcal{E} \cong \pi^*\mathcal{F}$ for some vector bundle $\mathcal{F}$ on X if and only if for every point, equivalently, for every closed point $x \in X$, the action on the fiber of $\mathcal{E}$ of the geometric stabilizer group at x is trivial.

The opposite to the tame case is the *wild* case: the order of the geometric stabilizer group G at some $x \in X$ is divisible by the residue characteristic p at x. We let G' be a p-Sylow subgroup of G. As mentioned, by passing to an affine étale neighborhood we may suppose that $\mathfrak{X} = [Y/G]$, which we may suppose to be of the form $Y = \mathrm{Spec}(B)$ with $[G : G']$ invertible in B. Now we consider $\mathfrak{X}' = [Y/G']$, with coarse moduli space $\pi'\colon \mathfrak{X}' \to X'$. Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$, with pullback $\mathcal{E}'$ to $\mathfrak{X}'$. Then $\mathcal{E} \cong \pi^*\mathcal{F}$ for a vector bundle $\mathcal{F}$ on X if and only if $\mathcal{E}' \cong \pi'^*\mathcal{F}'$ for a vector bundle $\mathcal{F}'$ on X' and $\mathcal{E}$ satisfies the above-mentioned criterion of [2]. Indeed, the forwards implication is clear, and for the reverse implication we let $(s_i)_{1\le i\le r}$ be a basis of the G-equivariant projective B-module corresponding to $\mathcal{E}$ consisting of G'-invariant elements (which exists after replacing B by $B[1/f]$ for suitable $f \in B^G$ not vanishing at x) and obtain, with $\big([G : G']^{-1} \sum_{gG' \in G/G'} g \cdot s_i\big)_{1\le i\le r}$, a basis of G-invariant elements.

By the argument of the previous paragraph, the wild case may be reduced to the case of quotients by finite p-groups. Since finite p-groups are solvable, we may reduce further to the case of quotients by $\mathbb{Z}/p\mathbb{Z}$ (by expressing $Y \to X$ as a succession of quotients by $\mathbb{Z}/p\mathbb{Z}$-actions). We come to the main result of this paper.

Theorem 1.1. *Let X be an irreducible normal Noetherian scheme with function field K, and let L/K be a cyclic field extension of prime degree. We let Y denote the integral closure of X in L and $G = \mathrm{Gal}(L/K)$. We suppose that the morphism $Y \to X$ is flat. Then, base change identifies the category of vector bundles on X with the category of G-equivariant vector bundles on Y such that the G-action on the restriction to $\mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n_y})$ is trivial, for every $y \in Y$ with $\dim(\mathcal{O}_{Y,y}) = 1$, fixed by G with trivial G-action on the residue field $\mathcal{O}_{Y,y}/\mathfrak{m}_y$, where n_y denotes the largest integer such that G acts trivially on $\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n_y}$.*

An integer n_y as in the statement of Theorem 1.1 exists, since G acts nontrivially on $\mathcal{O}_{Y,y}$.

Thanks to flatness, the standard theory of faithfully flat descent is available [9]. But its direct applicability is limited by a measure of complexity developed in [21], with complexity ≤ 1 required for optimal applicability. But this is not generally attained even in the tame case, as we see from Lemma 3.3.3 of op. cit. We remark that the flatness hypothesis is always satisfied when X is an (irreducible Noetherian) Dedekind scheme (i.e., regular of dimension ≤ 1). Another situation where the flatness hypothesis is satisfied is the case that X is regular of dimension 2. As well, standard flatification results [20] allow to perform a birational modification of X, so that the flatness hypothesis will be satisfied.

In Section 2 we give some examples, illustrating the hypotheses in Theorem 1.1. In Section 3 we give a proof of Theorem 1.1. A geometric application is presented in Section 4.

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2. Examples

In this section we give some examples, to illustrate the hypotheses of Theorem 1.1.

Example 2.1. The flatness hypothesis of Theorem 1.1 is necessary. Indeed, let us take $X = \mathrm{Spec}(\mathbb{C}[r,s,t]/(rt-s^2))$. Then $K = \mathbb{C}(r,t,\sqrt{rt})$, which has the quadratic extension $L = K(\sqrt{r})$. The integral closure of X in L is $Y = \mathrm{Spec}(\mathbb{C}[u,v])$, where $u = \sqrt{r}$ and $v = \sqrt{t}$, and Y is not flat over X. But there is a unique point $y_0 \in Y$ which is fixed by $G = \mathbb{Z}/2\mathbb{Z}$ and has trivial residue field action, and y_0 satisfies $\dim(\mathcal{O}_{Y,y_0}) = 2$. However, there is a G-equivariant line bundle on Y for which the action of G on the fiber at y_0 is nontrivial.

In the situation of Theorem 1.1 we let x denote the image in X of a point $y \in Y$ satisfying the indicated conditions. Then $\mathcal{O}_{Y,y}$ is the integral closure of $\mathcal{O}_{X,x}$ in L, and these are discrete valuation rings. The classical formula

$$[L : K] = ef$$

with $\mathfrak{m}_x\mathcal{O}_{Y,y} = \mathfrak{m}_y^e$ and $f = [\lambda : \kappa]$, where $\kappa = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and $\lambda = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ are the residue fields, leads to the following possibilities; we write $p = [L : K]$:

- $e = p$, $f = 1$, the ramification can be tame or wild:
  - (tame) $\mathrm{char}(\kappa) \neq p$, then $n_y = 1$, i.e., G must act nontrivially on $\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2$. Indeed, a uniformizing element of $\mathcal{O}_{Y,y}$ that is invariant modulo $\mathfrak{m}_{Y,y}^2$ under G would lift to an invariant uniformizing element, a contradiction.
  - (wild) $\mathrm{char}(\kappa) = p$, so there is only the trivial representation of G on $\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2$, and $n_y \geq 2$.
- $e = 1$, $f = p$, the ramification is wild:
  - λ/κ is an inseparable extension of fields of characteristic p, and a uniformizing element for $\mathcal{O}_{X,x}$ is also a uniformizing element for $\mathcal{O}_{Y,y}$. (Separable λ/κ would be Galois with $G = \mathrm{Gal}(\lambda/\kappa)$ acting nontrivially on the residue field.)

The tamely ramified case, $e = p$ with $\mathrm{char}(\kappa) \neq p$, is not very interesting, since the criterion from Theorem 1.1 in that case just reproduces that of [2]. The two interesting cases, therefore, are those with wild ramification:

- *Case* e: $e = p$, with $\mathrm{char}(\kappa) = p$ and $n_y \geq 2$.
- *Case* f: $f = p$, with $\mathrm{char}(\kappa) = p$, λ/κ inseparable of degree p, and $n_y \geq 1$.

Example 2.2. In the literature, the case of inseparable λ/κ is often excluded (e.g., in [22, Chap. IV]), then wild ramification puts us in Case e. Examples come from arithmetic and geometry and include *weakly ramified* extensions ($n_y = 2$) such as $\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2$ and $(\mathbb{F}_3((t))[u]/(u^3 - tu^2 + t))/\mathbb{F}_3((t))$, as well as extensions with larger n_y, such as $\mathbb{Q}_2(\sqrt{-2})/\mathbb{Q}_2$ and $(\mathbb{F}_3((t))[u]/(u^3 - t^2u + t))/\mathbb{F}_3((t))$ with $n_y = 3$. The most widely studied are the weakly ramified extensions, e.g., [6], [19], [23], [13].

Example 2.3. Case f includes examples such as $G = \mathbb{Z}/p\mathbb{Z}$ acting on $\mathbb{A}^2_k$ (k any field of characteristic p), where a generator acts by $(u, v) \mapsto (u, u + v)$, i.e., $X = \mathrm{Spec}(k[t, u])$, $L = K[v]/(v^p - u^{p-1}v - t)$. We have $n_{(u)} = 1$.

3. Proof of the main result

In this section we give a proof of Theorem 1.1. The only nontrivial task is to show that a G-equivariant vector bundle on Y satisfying the indicated conditions comes from a vector bundle on X. We use the formulation in terms of stacks, with vector bundle $\mathcal{E}$ on $\mathfrak{X} = [Y/G]$ and $\pi\colon \mathfrak{X} \to X$, and the characterization mentioned in the Introduction, i.e., we set ourselves the task of showing that (1) is an isomorphism. We use the notion of *reflexive sheaves*, i.e., sheaves isomorphic to their double duals [10]: $\pi_*\mathcal{E}$ is reflexive (by Proposition 1.6 of op. cit.), and since π is flat, so is $\pi^*\pi_*\mathcal{E}$ (by Proposition 1.8 of op. cit.). Now (1) is an isomorphism if and only if its restriction to the complement of any closed substack of codimension ≥ 2 is an isomorphism. Thus, we may suppose that $\pi_*\mathcal{E}$ is locally free (since, if not, then the locus where $\pi_*\mathcal{E}$ fails to be locally free has codimension ≥ 2). Since a morphism of locally free sheaves on $\mathfrak{X}$ that is generically an isomorphism can only drop rank on a divisor, we are reduced to carrying out the task for $[\mathrm{Spec}(\mathcal{O}_{Y,y})/G] \to \mathrm{Spec}(\mathcal{O}_{X,x})$.

Thus, from now on, we adopt the notation $X = \mathrm{Spec}(A)$ for a discrete valuation ring A with field of fractions K and residue field $\kappa = A/\mathfrak{m}_A$, cyclic field extension L/K of degree $p = \mathrm{char}(\kappa)$, and integral closure B of A in L, which we suppose to be a discrete valuation ring with trivial action of $G = \mathrm{Gal}(L/K)$ on the residue field $\lambda = B/\mathfrak{m}_B$. We are either in Case e, with residue field $\lambda \cong \kappa$ and $\mathfrak{m}_AB = \mathfrak{m}_B^p$, or in Case f, with $\mathfrak{m}_AB = \mathfrak{m}_B$ and λ/κ inseparable of degree p. We let n denote the largest integer such that G acts trivially on $B/\mathfrak{m}_B^n$ (which exists since G acts nontrivially on B), and we let σ denote a generator of G.

Lemma 3.1. *In Case e we let v be a uniformizing element of B, and in Case f we let $v \in B$ be an element whose residue does not lie in κ.*
(i) *B is free as an A-module with basis $1, v, \ldots, v^{p-1}$.*
(ii) *We have $\sigma(v) = uv$ for some $u \in B$ with residue 1.*
(iii) *The valuation of $v^i - \sigma(v^i)$ is $n+i-1$ in Case e and n in Case f, for $1 \leq i \leq p-1$.*
(iv) *With u as in* (ii), *the valuation of $1 - u$ is $n - 1$ in Case e and n in Case f.*

Proof. The first assertion is clear. The valuation of v is 1 in Case e and 0 in Case f. The second assertion follows from the fact that $n \geq 2$ in Case e and $n \geq 1$ in Case f. We have

$$v^i - \sigma(v^i) = (v - \sigma(v))(v^{i-1} + v^{i-2}\sigma(v) + \cdots + \sigma(v)^{i-1}).$$

By (ii), for $2 \leq i \leq p - 1$ the second factor has the same valuation as v^{i-1}. So the valuations of $v^i - \sigma(v^i)$ for $i = 1, \ldots, p-1$ form an increasing sequence of consecutive integers in Case e and are constant in Case f. The third assertion follows (cf. [22, Lem. IV.1]). Combining (ii) and (iii), we obtain (iv). □

The remainder of the proof consists of computations of the Picard group of $\mathfrak{X} = [Y/G]$ and the cohomology module $H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and discussions of their behavior upon restriction to thickenings of the closed point of $\mathfrak{X}$. To this end, we introduce the following notation:

$$Y_j = \mathrm{Spec}(B/\mathfrak{m}_B^j), \qquad \mathfrak{X}_j = [Y_j/G].$$

Let $\mathcal{A}$ be a sheaf of abelian groups on $\mathfrak{X}$ and m, a positive integer. Suppose that the following condition is satisfied: $H^i(Y,\mathcal{A}) = 0$ for all $1 \le i \le m$. Then by [18, Exa. III.2.6, Prop. III.2.7] (see also [5, §4]), we have the identifications

$$H^m(\mathfrak{X},\mathcal{A}) \cong \check{H}^m(Y/\mathfrak{X},\mathcal{A}) \cong H^m(G,\mathcal{A}(Y)) \tag{2}$$

of sheaf cohomology, Čech cohomology, and group cohomology; the last of these, we recall, may be computed efficiently as the mth cohomology of the complex

$$\mathcal{A}(Y) \xrightarrow{\mathrm{id}-\sigma} \mathcal{A}(Y) \xrightarrow{\sum_{i=0}^{p-1}\sigma^i} \mathcal{A}(Y) \xrightarrow{\mathrm{id}-\sigma} \mathcal{A}(Y) \xrightarrow{\sum_{i=0}^{p-1}\sigma^i} \dots \tag{3}$$

(cf. [17, Thm. IV.7.1]). The condition is always satisfied when $\mathcal{A}$ is a coherent sheaf, and in this case $H^m(\mathfrak{X},\mathcal{A})$ is a finitely generated torsion A-module.

Analogous statements hold with Y replaced by Y_j, and $\mathfrak{X}$, by $\mathfrak{X}_j$.

Lemma 3.2. *The Picard group* $\mathrm{Pic}(\mathfrak{X})$ *has order* p *in Case* e *and is trivial in Case* f. *The restriction map*

$$\mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(\mathfrak{X}_n)$$

is injective.

Proof. Since $\mathfrak{X}$ is an irreducible regular Noetherian Deligne-Mumford stack with trivial generic stabilizer, every line bundle has a rational section and thus may be identified with the line bundle associated with a divisor (formal $\mathbb{Z}$-linear combination of integral closed substacks of codimension 1), defined uniquely up to linear equivalence of divisors (see [4, §XIII.2] for a discussion of line bundles and divisors on Deligne-Mumford stacks). The unique integral closed substack of codimension 1 is $\mathfrak{X}_1$. In Case e this is not linear equivalent to zero, but is so when multiplied with p: rational functions on $\mathfrak{X}$ are elements of K, and precisely the multiples of p arise from nonzero elements of K under the valuation of B. In Case f a uniformizing element of A is also uniformizing for B, so $\mathfrak{X}_1$ is linearly equivalent to zero. The claim about injectivity is nontrivial only in Case e and follows, then, from Lemma 3.1 (iv) by recognizing, with $\mathrm{Pic}(\mathfrak{X}) \cong H^1(\mathfrak{X},\mathbb{G}_m)$ and (2)–(3), that the element $1-u$ describes a generator of $\mathrm{Pic}(\mathfrak{X})$. □

The next statement describes $H^1(\mathfrak{X},\mathcal{O}_\mathfrak{X})$ via (2)–(3).

Lemma 3.3. *Letting* $t \in A$ *denote a uniformizing element and* $v \in B$ *be as in Lemma 3.1,* $H^1(\mathfrak{X},\mathcal{O}_\mathfrak{X})$ *is isomorphic as an* A*-module to* $\bigoplus_{i=1}^{p-1} A/t^{\lfloor (n+i-1)/p\rfloor}A$ *with* i*th generator* $t^{-\lfloor (n+i-1)/p\rfloor}(v^i-\sigma(v^i))$ *in Case* e, *to* $(A/t^nA)^{p-1}$ *with* i*th generator* $t^{-n}(v^i-\sigma(v^i))$ *in Case* f. *The restriction map*

$$H^1(\mathfrak{X},\mathcal{O}_\mathfrak{X}) \to H^1(\mathfrak{X}_n,\mathcal{O}_{\mathfrak{X}_n})$$

is injective.

Proof. The kernel of $\mathrm{id}-\sigma$ in (3), with $\mathcal{A} = \mathcal{O}_\mathfrak{X}$ and hence $\mathcal{A}(Y) = B$, is A, so $\mathrm{id}-\sigma$ maps the free A-module B/A with basis $(v^i)_{1\le i\le p-1}$ isomorphically to its image in B. Since $H^1(\mathfrak{X},\mathcal{O}_\mathfrak{X})$ is a torsion A-module, the kernel of $\sum_{i=0}^{p-1}\sigma^i$ consists precisely of elements such that the product with some power of t lies in the image of $\mathrm{id}-\sigma$. The kernel is therefore the free A-module with, as basis, the indicated generators.

The quotient is the indicated torsion A-module. We compute annihilators in $B/\mathfrak{m}_B^n$ of the indicated generators to verify the claim about injectivity. $\square$

Remark 3.4. In Case *e*, in fact, the restriction map $H^1(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})\to H^1(\mathfrak{X}_{n-1},\mathcal{O}_{\mathfrak{X}_{n-1}})$ is injective, and more generally if we replace $\mathcal{O}_{\mathfrak{X}}$ by any line bundle on $\mathfrak{X}$ then the map on H^1 given by restriction to $\mathfrak{X}_{n-1}$ is injective.

With the next statement, the proof of Theorem 1.1 is complete.

Proposition 3.5. *Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$. If the restriction of $\mathcal{E}$ to $\mathfrak{X}_n$ is trivial, then $\mathcal{E}$ is trivial.*

Proof. We prove the result by induction on the rank of $\mathcal{E}$. Lemma 3.2 takes care of rank 1. For rank ≥ 2, the base case applied to the determinant line bundle lets us suppose that $\mathcal{E}$ has trivial determinant. Any nontrivial section at the generic point extends to a nontrivial morphism from some line bundle to $\mathcal{E}$, which by removing the torsion from the cokernel leads to a short exact sequence of vector bundles

$$0\to\mathcal{L}\to\mathcal{E}\to\mathcal{F}\to 0,$$

where $\mathcal{L}$ is a line bundle. The restriction of $\mathcal{F}$ to $\mathfrak{X}_n$ is globally generated, and on $\mathfrak{X}_n$ globally generated vector bundles are trivial, so by the induction hypothesis $\mathcal{F}$ is trivial. By triviality of the determinant, $\mathcal{L}$ is trivial. We have an extension of a trivial vector bundle by a trivial line bundle and may conclude by Lemma 3.3 (using $\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{X}},\mathcal{O}_{\mathfrak{X}})=H^1(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$). $\square$

4. Geometric application

In this section we present a geometric application of Theorem 1.1. One method for constructing fibrations over a scheme X (conic bundles, etc.) is by descent from a Deligne-Mumford stack $\mathfrak{X}$ with coarse moduli space X; see, e.g., [11, Prop. 16], [14, Prop. 3.1, Prop. 4.4]. As a basic tool, following [14, Prop. 2.5, Rem. 2.6] we have:

Proposition 4.1. *Let X be a Noetherian scheme and $\mathfrak{X}$ a tame finite-type Deligne-Mumford stack over X, with finite stabilizer and X as coarse moduli space. Then base change by $\mathfrak{X}\to X$ in one direction and coarse moduli space in the other provide equivalences of categories between schemes, flat and projective over X, and Deligne-Mumford stacks, flat and projective over $\mathfrak{X}$, with trivial actions of geometric stabilizer groups on all fibers, equivalently, on fibers at closed points.*

Remark 4.2. Projective morphisms of Deligne-Mumford stacks are representable; although Deligne-Mumford stacks form a 2-category there is a standard way (recalled, e.g., in [14, §2.5]) to make a category out of stacks with representable morphism to $\mathfrak{X}$.

Remark 4.3. The statement has been adapted to the notational setting of this paper, and an unnecessary hypothesis concerning Gorenstein fibers with ample anticanonical class has been eliminated. (The combination of [15, Lem. 2], realizing a suitable power of a relatively ample line bundle on a stack, projective over $\mathfrak{X}$, as pulled back from the coarse moduli space, existence of a finite cover of $\mathfrak{X}$ by a scheme [16, Thm. 16.6], and the criterion [8, 2.6.2] for ampleness, takes care of the projectivity over X of the coarse moduli space. Now, as in the Introduction, it suffices to treat the case X is affine and $\mathfrak{X}=[Y/G]$. Again by [15, Lem. 2], a suitable power of a G-equivariant ample line bundle will have the property that its restriction to any geometric fiber is equivariantly isomorphic to an ample line bundle with trivial linearization. We conclude as in the proof of [14, Prop. 2.5].)

Theorem 4.4. *Let X be an irreducible normal Noetherian scheme with function field K, and let L/K be a cyclic field extension of prime degree. We let Y denote the integral closure of X in L and $G = \mathrm{Gal}(L/K)$. We suppose that the morphism $Y \to X$ is flat. Then, base change in one direction and quotient by G in the other provide equivalences of categories between schemes, flat and projective over X, and G-schemes, with flat and projective equivariant morphism to Y and trivial G-action on the fiber over $\mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n_y})$, for every $y \in Y$ with $\dim(\mathcal{O}_{Y,y}) = 1$, fixed by G with trivial G-action on the residue field $\mathcal{O}_{Y,y}/\mathfrak{m}_y$, where n_y denotes the largest integer such that G acts trivially on $\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n_y}$.*

Since the tensor product of the translates under G of a relatively ample line bundle on Y is relatively ample and admits a linearization, the G-schemes in the statement of Theorem 4.4 admit G-equivariant relatively ample line bundles.

Proof. We proceed as in the proof of Proposition 4.1, but invoke Theorem 1.1 instead of [2, Thm. 10.3] for the descent of the G-modules of global sections of powers of a given G-equivariant ample line bundle. □

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