CW COMPLEXES FOR COMPLEX ALGEBRAIC SURFACES

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Abstract. We describe CW complexes for complex projective algebraic surfaces in the context of practical computation of topological invariants.

1. Introduction

Let X be a nonsingular projective algebraic surface over the complex numbers given by explicit defining equations. The practical computation of topological invariants of the underlying complex manifold $X(\mathbb{C})$ is a desirable goal in many settings in algebraic geometry. That topological invariants are effectively computable is a well-established fact, since $X(\mathbb{C})$ may be embedded in a Euclidean space, and a cell complex on $X(\mathbb{C})$ may be obtained from a suitable subdivision of the Euclidean space. This paper explores the practicality of the computation.

For the computation, one first uses the defining equations of X to obtain discrete data, and then computes the desired invariants from those data. The discrete data could be a structure of CW complex on $X(\mathbb{C})$, presented in such a way that the boundary of a cell can be effectively represented in terms of cells of the next lower dimension. Then the computation of, say, homology or cohomology with integer coefficients reduces to computation of Smith normal form of an integer matrix. The focus in this paper is on the first step; practicality is assessed by comparing the numbers of cells of various dimensions obtained in several examples against the practical capability of Smith normal form computation as presently known.

The link between the algebraic geometry of X and the algebraic topology of $X(\mathbb{C})$ underlies some of the anticipated applications. For instance, the verification that a collection of algebraic divisor classes on X (given by explicit equations) generates a saturated sublattice of $H_2(X(\mathbb{C}),\mathbb{Z})/\mathrm{tors}$ could be approached directly, given the ability to compute the topological intersection number of an algebraic divisor class and a topological homology class. (The need for this arises in arithmetic geometry, such as in [19], where it is addressed by a purely algebraic technique of Stoll and Testa.)

2. Background

The computational topology of manifolds has been treated extensively. From Whitney we have not only his famous embedding theorem [20], but also an effective algorithm for the computation of topological invariants of an (explicitly given) compact manifold in Euclidean space, by restricting a subdivision of the ambient Euclidean space [21]. A general algorithm with complexity analysis is given in [18].

Date: August 23, 2010.

2000 *Mathematics Subject Classification.* 32Q55 (primary); 14E20, 57R19 (secondary).

2.1. **Cell decompositions of manifolds.** The practicality of the computation of a cell decomposition of a manifold is the subject of intensive study in dimensions one and two, with surfaces treated, for instance, in [2] and [3]. See also [1], and especially the introduction, with its thorough overview and references. As mentioned there, an analysis of cylindrical algebraic decomposition [5], an algorithm that has been both well studied and widely implemented, yields a bound of $O(d^{2^n+3^n})$ for a degree-d real algebraic hypersurface of dimension n. A recent preprint [13] focuses on the case of hypersurfaces and delivers triangulations with $O(d^{3\cdot 2^{n-1}-1})$ cells.

2.2. **Smith normal form computation.** Focusing on the case of (co)homology with integer coefficients, once the discrete data have been obtained as in Section 2.1, the computation is reduced to finding the Smith normal form of an integer matrix. This is known to be possible in polynomial time [12], and algorithms that do not exploit any particular sparseness structure can in practice handle matrices with up to 1000 rows and columns [11].

Exploiting sparseness, which is a characteristic of the matrices representing the boundary maps that arise, probabilistic analysis yields an algorithm quadratic in the number of cells [7]. This raises the practicality limit to the tens of thousands, with improvements for simplicial or cubical cell complexes permitting computations with hundreds of thousands of cells [16, 17]. Regular CW complexes have been treated more recently [6], but as the CW complexes that we produce below are not regular, we accept tens of thousands of cells as a practical limit.

2.3. **Case of a complex surface.** Given a nonsingular projective surface X, a first approach to the computation of topological invariants of $X(\mathbb{C})$ would be to view it as a real manifold, embed it in a Euclidean space, and apply existing techniques. If we seek a triangulation, then we face the bounds attached to known algorithms described in Section 2.1. These are theoretical bounds (we have no other guide due to the lack of treatment of practicality issues in dimensions greater than two in the literature), applicable to the case of a hypersurface (which $X(\mathbb{C})$ will not generally be; for example, in the simplest case $X = \mathbb{P}^2$ it is known that a 7-dimensional Euclidean space is required [20]), but they suggest that for the present technology it is infeasible to determine a cell decomposition of $X(\mathbb{C})$ this way.

Another method, also for real manifolds, is Morse theory. This has been implemented for surfaces, for example in [8], but in ways that rely on the structure of the nonsingular fibers of a Morse function (unions of circles). In higher dimensions we would face practicality issues such as the determination of boundary relations between the critical points (corresponding to gradient flow lines). The literature offers computational approaches such as [10], but these require a cell decomposition of the manifold as part of the input data. Beyond the practicality issues, a further issue would be to maintain the link between algebraic geometry and topology mentioned in the introduction.

A third approach, using that X is a complex projective variety, is to consider a Lefschetz pencil on X (an algebraic analogue of a Morse function that yields topological information about X). While computations abound (see, e.g., [9]), the general use of Lefschetz pencils brings up the same kinds of issues as with Morse theory.

3. Cell complexes on complex curves and surfaces

Following [15], we start with X presented as a branched cover of $\mathbb{P}^2$ with some branch curve $B \subset \mathbb{P}^2$, and by projecting, B as a branched cover of $\mathbb{P}^1$. We will construct a cell complex on $\mathbb{P}^2$ which extends one on B; this then lifts to X.

3.1. **Cell complexes on curves.** Given a nonsingular complex projective algebraic curve, a general projection to $\mathbb{P}^1$ of suitable degree will represent the curve as a branched cover of $\mathbb{P}^1$ such that over each branch point there is just one ramification point, which is a double point. The curve may have nodes and cusps as singularities, in which case for a general projection the branch points will include as well the images of the nodes and cusps. A cell complex on $\mathbb{P}^1$ with the branch points as 0-cells will lift to a cell complex on the algebraic curve.

We define a finite cell complex on $\mathbb{P}^1$ to be *standard* when the union of 1-cells is homeomorphic to a circle. The complement of the union of 1-cells then has two connected components, which must be the 2-cells.

3.2. **Cell complexes on surfaces.** Now let X be a nonsingular complex projective algebraic surface. A morphism $f : X \to \mathbb{P}^2$ will be called a *generic covering* if f is a finite covering whose only singularities are double points (analytically equivalent to the projection to the (x, y)-plane from the surface $x = z^2$) and cuspidal-type singular points (analytically equivalent to $y = z^3 + xz$), the branch curve $B \subset \mathbb{P}^2$ has only nodes and cusps as singularities, and the restriction of f to the ramification locus of f is a birational isomorphism onto B. Given an embedding of X in a projective space, it is known that a general linear projection $X \to \mathbb{P}^2$ is a generic covering [4].

We suppose that a generic covering $f : X \to \mathbb{P}^2$ has been fixed, and we choose coordinates $(x : y : z)$ on $\mathbb{P}^2$ so that (i) the point $(0 : 0 : 1)$ does not lie on the branch curve B of f, and the projection $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $g(x : y : z) = (x : y)$ has the property that (ii) the restriction of g to B is unramified over $\infty = (0 : 1)$, and the preimage in B of every branch point contains just one ramification point, which is a double point (possibly a singular point of B).

We let d denote the degree of the curve B. Our construction is the following.

Algorithm 3.1. *Step 1.* Choose an ordering $p_1, \ldots, p_N$ of the points in the branch locus of $B \to \mathbb{P}^1$ and a standard cell complex

$$\begin{array}{ll} \{p_1, \ldots, p_N\} & \text{(0-cells)}, \\ \{L_1, \ldots, L_N\} & \text{(1-cells)}, \\ \{H_+, H_-\} & \text{(2-cells)}, \end{array}$$

on $\mathbb{P}^1$ such that the 1-cell L_1 joining p_N with p_1 passes through ∞.
Step 2. Choose a continuously varying family of standard cell complexes on $g^{-1}(t) \cong \mathbb{P}^1$ for $t \in L_1 \cup \cdots \cup L_N$:

$$\begin{array}{ll} \{(0:0:1)\} \cup (g^{-1}(t) \cap B) & \text{(0-cells)} \\ \{M_1(t), \ldots, M_e(t)\} & \text{(1-cells)} \end{array}$$

(where $e = d$ if $t \in \{p_1, \ldots, p_N\}$, and $e = d + 1$ otherwise),

$$\{H_+(t), H_-(t)\} \quad \text{(2-cells)}.$$

Step 3. Describe a continuous extension of the cell complexes of Step 2 to cell complexes on $g^{-1}(t)$ for $t \in H_+$, as well as to $g^{-1}(t)$ for $t \in H_-$, such that the cells of $g^{-1}(p_j)$ for $j = 1, \ldots, N$, the unions over $t \in L_j$ of the respective cells of $g^{-1}(t)$ for

$j = 1, \ldots, N$, and the unions over $t \in H_\pm$ of the respective cells of $g^{-1}(t)$ together form a cell complex on $\mathbb{P}^2$.
Step 4. Lift the cell complex on $\mathbb{P}^2$ to obtain one on X.

In Step 2, a natural definition of the $M_k(t)$ for t in the interior of some L_j suggests itself, provided that the points of $g^{-1}(t)$ that approach each other as t approaches an endpoint of L_j are adjacent for the ordering in the standard cell complexes. In the first example (Section 4.1), this holds. Generally, we carry out Step 2 using a scheme described in Section 4.2 based on adding extra 0-cells to $\{p_1, \ldots, p_N\}$. We will also need to add extra 0-cells in order to accomplish Step 3, as explained in Section 4.1. With these modifications, the standard cell complexes of Step 2 will be supplemented with 1- and 2-cells; for instance, for some t, we may have $M_{1+}(t)$ and $M_{1-}(t)$ instead of just $M_1(t)$. The construction is then determined by fixing the standard cell complexes used: in Section 4.1 they are given explicitly, while in Sections 4.2 and 4.3 the 0-cells are connected in the order of their real parts, using line segments in $\mathbb{C}$ as 1-cells. The latter choice is one that could be applied in any example, as long the points of $g^{-1}(p_j)$ have distinct real parts for each j (which after a complex rescaling of coordinates is always the case).

4. Examples

4.1. **Quadric.** For the first example we take X to be the double cover of

$$B : x^2 + y^2 + z^2 = 0$$

in $\mathbb{P}^2$, i.e., X is a quadric surface. The projection $B \to \mathbb{P}^1$ is branched over the points $p_1 = i$ and $p_2 = -i$, where $i = \sqrt{-1}$ and we identify $t \in \mathbb{C}$ with $(1 : t) \in \mathbb{P}^1 \smallsetminus \{\infty\}$.

In Step 1, we take the 1-cells to be

$$L_1 = \{i + t \,|\, t \in \mathbb{R}_{\le 0}\} \cup \{\infty\} \cup \{-i + t \,|\, t \in \mathbb{R}_{\ge 0}\}, \quad L_2 = \{ti \,|\, t \in \mathbb{R},\ |t| \le 1\}.$$

For Step 2, when $t = \infty$ we have $g^{-1}(\infty) \cong \mathbb{P}^1$ by $(0 : y : z) \mapsto (y : z)$, and there is the standard cell complex with 1-cells

$$M_1(\infty) = i + \mathbb{R}_{\le 0} \cup \{\infty\}, \quad M_2(\infty) = i[-1, 1], \quad M_3(\infty) = -i + \mathbb{R}_{\ge 0}.$$

For $t \in \mathbb{P}^1 \smallsetminus \{\infty\}$ we identify $g^{-1}(t)$ with $\mathbb{P}^1$ by $(x : tx : z) \mapsto (x : z)$. We have the following cell complexes on $g^{-1}(t)$, $t \in L_1 \smallsetminus \{\infty, i, -i\}$, compatible with the one at ∞:

$$M_{2-\mathrm{sgn}(\mathrm{Re}(t))}(t) = i\sqrt{1 + t^2} + \mathbb{R}_{\le 0}, \quad M_2(t) = i\sqrt{1 + t^2}[-1, 1],$$
$$M_{2+\mathrm{sgn}(\mathrm{Re}(t))}(t) = -i\sqrt{1 + t^2} + \mathbb{R}_{\ge 0},$$

where the analytic function $\sqrt{1 + z^2}$ is extended to $\mathbb{C} \smallsetminus \{ti \,|\, t \in \mathbb{R},\ |t| \ge 1\}$, by convention taking the value 1 at $z = 0$. The cell complex at $t = \pm i$ is taken to have 1-cells $\mathbb{R}_{\le 0}$ and $\mathbb{R}_{\ge 0}$. For $t \in L_2 \smallsetminus \{i, -i\}$ we take

$$M_1(t) = -i\sqrt{1 + t^2} + \mathbb{R}_{\ge 0}, \quad M_2(t) = i\sqrt{1 + t^2}[-1, 1], \quad M_3(t) = i\sqrt{1 + t^2} + \mathbb{R}_{\le 0}.$$

With only these definitions it is impossible to complete Step 3. This is because when the 1-cells $M_1(\infty)$ and $M_3(\infty)$ vary continuously to t in a neighborhood of ∞, there must be 1-cells with paths to ∞ along line segments of varying slope, not only line segments parallel to $\mathbb{R} \subset \mathbb{C}$. Here $g^{-1}(t)$ for $t \ne \infty$ is identified with $\mathbb{P}^1$ and $\mathbb{C}$ with $\mathbb{P}^1 \smallsetminus \{\infty\}$ as indicated above.

$j = 3$:	$M_1(t) = -i\sqrt{1+t^2} + \mathbb{R}_{\geq 0}, M_2(t) = i\sqrt{1+t^2}[-1,1], M_3(t) = i\sqrt{1+t^2} + \mathbb{R}_{\leq 0},$
$j = 4$:	$M_{1+} = -i\sqrt{1+t^2} + \mathbb{R}_{\geq 0}, M_{1-} = -i\sqrt{1+t^2} + e^{-\pi i \frac{t-t_3}{t_4-t_3}} \mathbb{R}_{\geq 0},$ $M_2 = i\sqrt{1+t^2}[-1,1], M_{3+} = i\sqrt{1+t^2} + \mathbb{R}_{\leq 0}, M_{3-} = i\sqrt{1+t^2} + e^{-\pi i \frac{t-t_3}{t_4-t_3}} \mathbb{R}_{\leq 0},$
$j = 5$:	$M_{1\pm} = -i\sqrt{1+t^2} \pm \mathbb{R}_{\geq 0}, M_{2+} \cup M_{2-} = i\sqrt{1+t^2}[-1,1],$ $M_{3+} = i\frac{t_4+t_5-2t}{t_5-t_4}\sqrt{1+t^2} + \mathbb{R}_{\leq 0}, M_{3-} = i\sqrt{1+t^2} + \mathbb{R}_{\geq 0},$
$j = 6$:	$M_1 = i\frac{t_5+t_6-2t}{t_5-t_6}\sqrt{1+t^2} + \mathbb{R}_{\geq 0}, M_{2+} \cup M_{2-} = i\sqrt{1+t^2}[-1,1],$ $M_{3+} = -i\sqrt{1+t^2} + \mathbb{R}_{\leq 0}, M_{3-} = i\sqrt{1+t^2} + \mathbb{R}_{\geq 0},$
$j = 7$:	$M_{1+} = i\sqrt{1+t^2} + \mathbb{R}_{\geq 0}, M_{1-} = i\sqrt{1+t^2} + e^{\pi i \frac{t-t_6}{t_7-t_6}} \mathbb{R}_{\geq 0}, M_2 = i\sqrt{1+t^2}[-1,1],$ $M_{3+} = -i\sqrt{1+t^2} + \mathbb{R}_{\leq 0}, M_{3-} = -i\sqrt{1+t^2} + e^{\pi i \frac{t-t_6}{t_7-t_6}} \mathbb{R}_{\leq 0},$
$j = 8$:	$M_{1\pm} = i\sqrt{1+t^2} \pm \mathbb{R}_{\geq 0}, M_{2+} \cup M_{2-} = i\sqrt{1+t^2}[-1,1],$ $M_{3+} = i\frac{t_7+t_8-2t}{t_7-t_8}\sqrt{1+t^2} + \mathbb{R}_{\leq 0}, M_{3-} = -i\sqrt{1+t^2} + \mathbb{R}_{\geq 0},$
$j = 9$:	$M_1 = i\frac{t_8+t_9-2t}{t_9-t_8}\sqrt{1+t^2} + \mathbb{R}_{\geq 0}, M_{2+} \cup M_{2-} = i\sqrt{1+t^2}[-1,1],$ $M_{3+} = i\sqrt{1+t^2} + \mathbb{R}_{\leq 0}, M_{3-} = -i\sqrt{1+t^2} + \mathbb{R}_{\geq 0}.$

TABLE 1. Definition of the $M_k(t)$ on the segments where $t+i$ is real and strictly between t_{j-1} and t_j, for $j = 3, \ldots, 9$.

A remedy is to choose

$$0 = t_2 < t_3 < t_4 < t_5 < t_6 < t_7 < t_8 < t_9 \in \mathbb{R},$$

set $p_j = -i + t_j$ for $3 \leq j \leq 9$, redefine

$$L_1 = \{i + t \,|\, t \in \mathbb{R}_{\leq 0}\} \cup \{\infty\} \cup \{-i + t_9 + t \,|\, t \in \mathbb{R}_{\geq 0}\}$$

(this is a subset of the original L_1), and introduce

$$L_j = \{-i + t \,|\, t_{j-1} \leq t \leq t_j\}$$

for $3 \leq j \leq 9$ with the $M_k(t)$ appearing in Table 1. (The specification of $M_k(t)$ for t in the interiors of the L_j, given in the table, determines $M_k(p_j)$ for $3 \leq j \leq 9$.)

Then there exist families of cell complexes over $t \in H_+$ and $t \in H_-$, built out of line segments in $\mathbb{C}$, extending the $M_k(t)$ for $t \in L_1 \cup \cdots \cup L_9$. For instance, the line segments that approach ∞ may be taken parallel to $\mathbb{R} \subset \mathbb{C}$ when $|\operatorname{Re}(t)| \leq t_9 + 1$, $|\operatorname{Im}(t)| \leq 2$, and

$$\begin{cases} \operatorname{Re}(t) \leq t_3 \text{ or } \operatorname{Re}(t) \geq t_4, & \text{when } t \in H_+, \\ \operatorname{Re}(t) \leq t_6 \text{ or } \operatorname{Re}(t) \geq t_7, & \text{when } t \in H_-. \end{cases}$$

Suitable cell complexes on $g^{-1}(t)$ for $t \in H_-$ close to L_k for $k = 5$, 6, and 7 are depicted in Figure 1.

The cell complex of Step 3 has 17 0-cells, 53 1-cells, 69 2-cells, 34 3-cells, and 4 4-cells. If we orient the cells so that

- L_j is a path from t_{j-1} (or t_9, when $j = 1$) to t_j,
- $M_1(t)$ (or $M_{1\pm}(t)$) is a path from $(0:0:1)$ to a point of $g^{-1}(t) \cap B$, $M_e(t)$ (or $M_{e\pm}(t)$) is a path from a point of $g^{-1}(t) \cap B$ to $(0:0:1)$, and $M_2(t)$ for $t \neq \pm i$ with nonnegative real part is a path from the point with negative imaginary part to the point with positive imaginary part,
- open subsets of $\mathbb{C}$ are given the canonical orientation,

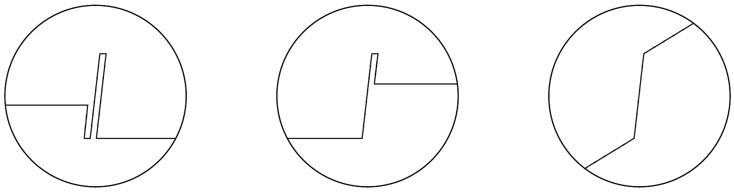

FIGURE 1. Neighborhood of $0 \in \mathbb{C}$ of cell complex on $g^{-1}(t)$ for $t \in H_-$ close to L_5 (left), to L_6 (middle), and to L_7 (right).

then we have, in a straightforward manner, a homological chain complex for $\mathbb{P}^2$. For instance, we may compute $H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z}) = \ker(\partial_2)/\operatorname{im}(\partial_3) \cong \mathbb{Z}$ where $\partial_2 : \mathbb{Z}^{69} \to \mathbb{Z}^{53}$ and $\operatorname{im}(\partial_3)$ are given in Table 2.

In the table, 1-cells $L_j^{\pm}$ denote the point of $g^{-1}(t) \cap B$ ($t \in L_j$) with indicated sign of imaginary part, L_j^0 the point with varying sign of imaginary part, $p_{jk(\pm)}$ (respectively $L_{jk(\pm)}$) the 1- (respectively 2-) cells in $g^{-1}(t)$ (respectively over L_j), and $p_j^>$, $p_j^<$, p_j^+, p_j^- the 2-cells in $g^{-1}(t)$ (with $>$ and $<$ indicating the respective signs of the real parts, and $+$ and $-$ the signs of the imaginary parts).

Step 4 produces a cell complex with 18 0-cells, 88 1-cells, 134 2-cells, 68 3-cells, and 8 4-cells.

4.2. **Cubic.** It is known classically that the branch curve of a general cubic surface is a sextic curve with six cusps lying on a conic [22]. For this example we take

$$X : x^2z + x^2t + y^3 + z^2t + t^3 = 0.$$

This is a triple cover of $\mathbb{P}^2$ branched over

$$B : 4x^6 + 39x^4z^2 + 54x^2y^3z + 12x^2z^4 + 27y^6 + 4z^6 = 0.$$

Under projection $B \to \mathbb{P}^1$ we find 12 ordinary branch points and 6 images of cusps, for a total of 18 branch points.

As mentioned in Section 3.2, for the the $M_k(t)$ we join the points of $g^{-1}(t) \cap B$ in the order of their real parts (under the usual identification of $\mathbb{C}$ with the complement of ∞ in $g^{-1}(t) \cong \mathbb{P}^1$) using line segments in $\mathbb{C}$. This means that on some L_j the order in which the points are joined will change. This may be accomplished by suitably enlarging the set $\{p_1, \ldots, p_N\}$ just as in the first example, e.g., with L_5 and L_6. To accomplish m adjacent point swaps along L_j we replace L_j by $2(m+1)$ 1-cells (by adding $2m+1$ additional points on L_j to $\{p_1, \ldots, p_N\}$).

Choosing $\{L_1, \ldots, L_N\}$ (Step 1), making modifications as in the previous paragraph for Step 2, we find (for Step 3 adding additional cells in analogy with the specifications of Table 1) the following total numbers of cells in Step 3: 2143 0-cells, 5012 1-cells, 3636 2-cells, 768 3-cells, and 4 4-cells. Taking account of the cells that make up B (2142 0-cells, 2160 1-cells, and 12 2-cells, with cusps accounting for 6 0-cells) we obtain 4281 0-cells, 12876 1-cells, 10896 2-cells, 2304 3-cells, and 12 4-cells in Step 4.

4.3. ***K*3 surface.** A $K3$ surface of degree 2 is a double cover of the plane branched along a nonsingular sextic curve [14]. The construction may be carried out just as in Section 4.2; we report the total numbers of cells (Step 4) obtained in a particular example ($B : x^6 + y^5z = z^6$): 1676 0-cells, 6120 1-cells, 5692 2-cells, 1232 3-cells, and 8 4-cells.

Images of basis elements of homological chain complex under ∂_2 (with $w^{\pm} = L_3^{\pm} + \cdots + L_9^{\pm}$)

2-cell	boundary	2-cell	boundary	2-cell	boundary
H_+^+	$L_1^+ + L_2^+ + w^+$	L_{53-}	$L_5^+ + p_{53-} - p_{43-}$	p_1^-	$p_{11} + p_{12}$
H_+^-	$L_1^- + L_2^- + w^-$	L_{61}	$p_{61} - L_6^0 - p_{51}$	p_2^+	$-p_{21} - p_{22}$
H_-^+	$-L_1^+ - L_2^- - w^+$	L_{62+}	$L_6^0 - L_6^+ - p_{52}$	p_2^-	$p_{21} + p_{22}$
H_-^-	$-L_1^- - L_2^+ - w^-$	L_{62-}	$L_6^- + p_{62} - L_6^0$	p_3^+	$-p_{31} - p_{32} - p_{33}$
L_{11}	$p_{11} - L_1^+ + p_{93}$	L_{63+}	$L_6^- + p_{63} - p_{53+}$	p_3^-	$p_{31} + p_{32} + p_{33}$
L_{12}	$L_1^- - L_1^+ - p_{92}$	L_{63-}	$L_6^+ - p_{61} - p_{53-}$	p_4^+	$-p_{43+} + p_{43-}$
L_{13}	$L_1^- + p_{12} + p_{91}$	L_{71+}	$p_{71+} - L_7^+ - p_{61}$	$p_4^<$	$p_{41-} + p_{42} + p_{43+}$
L_{21}	$p_{21} - L_2^- - p_{11}$	L_{71-}	$p_{71-} - L_7^+ - p_{61}$	$p_4^>$	$-p_{41+} - p_{42} - p_{43-}$
L_{22}	$L_2^- - L_2^+$	L_{72}	$L_7^- + p_{72} - L_7^+ - p_{62}$	p_4^-	$p_{41+} - p_{41-}$
L_{23}	$L_2^+ + p_{22} - p_{12}$	L_{73+}	$L_7^- + p_{73+} - p_{63}$	p_5^+	$p_{52} - p_{53+} + p_{53-}$
L_{31}	$p_{31} - L_3^- - p_{21}$	L_{73-}	$L_7^- + p_{73-} - p_{63}$	$p_5^>$	$-p_{51} - p_{52} - p_{53-}$
L_{32}	$L_3^- + p_{32} - L_3^+$	L_{81+}	$p_{81} - L_8^+ - p_{71+}$	p_5^-	$p_{51} + p_{53+}$
L_{33}	$L_3^+ + p_{33} - p_{22}$	L_{81-}	$-p_{83+} - L_8^+ - p_{71-}$	p_6^+	$-p_{61} + p_{62} - p_{63}$
L_{41+}	$p_{41+} - L_4^- - p_{31}$	L_{82+}	$L_8^0 - L_8^+ - p_{72}$	p_6^-	$p_{61} - p_{62} + p_{63}$
L_{41-}	$p_{41-} - L_4^- - p_{31}$	L_{82-}	$L_8^- + p_{82} - L_8^0$	p_7^+	$-p_{71+} + p_{71-}$
L_{42}	$L_4^- + p_{42} - L_4^+ - p_{32}$	L_{83+}	$L_8^0 + p_{83+} - p_{73+}$	$p_7^<$	$-p_{71-} + p_{72} - p_{73+}$
L_{43+}	$L_4^+ + p_{43+} - p_{33}$	L_{83-}	$L_8^- + p_{83-} - p_{73-}$	$p_7^>$	$p_{71+} - p_{72} + p_{73-}$
L_{43-}	$L_4^+ + p_{43-} - p_{33}$	L_{91}	$p_{91} - L_9^0 - p_{81}$	p_7^-	$p_{73+} - p_{73-}$
L_{51+}	$p_{51} - L_5^- - p_{41+}$	L_{92+}	$L_9^0 + p_{92} - L_9^+$	p_8^+	$-p_{81} - p_{83+}$
L_{51-}	$-p_{53+} - L_5^- - p_{41-}$	L_{92-}	$L_9^- - L_9^0 - p_{82}$	$p_8^>$	$p_{81} - p_{82} + p_{83-}$
L_{52+}	$L_5^0 + p_{52} - L_5^+$	L_{93+}	$L_9^+ + p_{93} - p_{83+}$	p_8^-	$p_{82} + p_{83+} - p_{83-}$
L_{52-}	$L_5^- - L_5^0 - p_{42}$	L_{93-}	$L_9^- - p_{91} - p_{83-}$	p_9^+	$-p_{91} - p_{92} - p_{93}$
L_{53+}	$L_5^0 + p_{53+} - p_{43+}$	p_1^+	$-p_{11} - p_{12}$	p_9^-	$p_{91} + p_{92} + p_{93}$

and ∂_3 (with $u = L_{32} + \cdots + L_{92\pm}$ and $v_{j\pm} = (2-j)H_{\pm}^{2-j} \pm (L_{1j} + L_{2j} + L_{3j})$)

$$
\begin{gathered}
v_{1+} + u + L_{22} + L_{41-} + L_{51-} - L_{63+} - L_{73+} - L_{82-} - L_{83+} - L_{92\pm} - L_{93+}, \\
v_{3+} + u + L_{22} + L_{43-} + L_{53-} + L_{63-} - L_{71+} - L_{81+} - L_{92+} - L_{91}, \\
v_{1-} - L_{32} - L_{42} - L_{41+} - L_{51+} - L_{52\pm} - L_{61} - L_{62+} - L_{71-} - L_{81-} + L_{93+}, \\
v_{3-} - L_{32} - L_{42} - L_{43+} - L_{52-} - L_{53+} - L_{63+} - L_{73-} - L_{83-} - L_{93-}, \\
H_+^+ - H_+^- + u + L_{12} + L_{22},\ H_-^+ - H_-^- - u - L_{12} + L_{22}, \\
p_1^{\pm} - p_9^{\mp} \pm (L_{11} - L_{12} + L_{13}),\ p_j^{\pm} - p_{j-1}^{\pm} \pm (L_{j1} + L_{j2} + L_{j3})\,(j = 2, 3), \\
p_4^> - p_3^+ + L_{41+} + L_{42} + L_{43-},\ p_4^+ + L_{43+} - L_{43-},\ p_4^- - L_{41+} + L_{41-}, \\
p_4^< - p_3^- - L_{41-} - L_{42} - L_{43+},\ p_5^+ - p_4^+ - L_{52+} + L_{53+} - L_{53-}, \\
-p_4^< - L_{51-} - L_{52-} - L_{53+},\ p_5^> - p_4^> + L_{51+} + L_{52\pm} + L_{53-}, \\
p_5^- - p_4^- - L_{51+} + L_{51-},\ p_6^+ - p_5^+ - L_{62\pm} + L_{63+} - L_{63-},\ -p_5^> + L_{61} + L_{62+} + L_{63-}, \\
p_6^- - p_5^- - L_{61} + L_{62-} - L_{63+},\ p_7^< - p_6^+ + L_{71-} - L_{72} + L_{73+},\ p_7^- - L_{73+} + L_{73-}, \\
p_7^+ + L_{71+} - L_{71-},\ p_7^> - p_6^- - L_{71+} + L_{72} - L_{73-},\ p_8^+ - p_7^+ + L_{81+} - L_{81-}, \\
-p_7^< + L_{81-} - L_{82+} + L_{83+},\ p_8^> - p_7^> - L_{81+} + L_{82\pm} - L_{83-}, \\
p_8^- - p_7^- - L_{82-} - L_{83+} + L_{83-},\ p_9^+ - p_8^+ + L_{91} + L_{92+} + L_{93+}, \\
-p_8^> - L_{91} + L_{92-} - L_{93-},\ p_9^- - p_8^- - L_{92\pm} - L_{93+} + L_{93-}
\end{gathered}
$$

Table 2. Data for computation in Section 4.1.

5. Conclusion

We have described a construction of CW complexes for complex algebraic surfaces that in examples yields sizes amenable to computation. The examples encompass simple classes of rational surfaces as well as a first instance of $K3$ surfaces, which are actively studied with the interactions of the algebraic and topological points of view playing an important role due to the presence of transcendental (nonalgebraic) homology classes.

Acknowledgements. The author has benefited from stimulating discussions with Yonatan Harpaz and Tomer Schlank. The author's research has been supported by a grant from the Swiss National Science Foundation.

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