CANONICAL RATIONAL EQUIVALENCE OF INTERSECTIONS OF DIVISORS

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1. INTRODUCTION

One way to define an operation in intersection theory is to define a map on the group of algebraic cycles together with a map on the group of rational equivalences which commutes with the boundary operation. Assuming the maps commute with smooth pullback, the extension of the operation to the setting of algebraic stacks is automatic. The goal of section 2 of this paper is to present the operation of intersecting with a principal Cartier divisor in this light.

We then show how this operation lets us obtain a rational equivalence which is fundamental to intersection theory. A one-dimensional family of cycles on an algebraic variety always admits a unique limiting cycle, but a family of cycles over the punctured affine plane may yield different limiting cycles if one approaches the origin from different directions. An important step in the historical development of intersection theory was realizing how to prove that any two such limiting cycles are rationally equivalent. The results of section 2 yield, as a corollary, a new, explicit formula for this rational equivalence.

Another important rational equivalence in intersection theory is the one that is used to demonstrate commutativity of Gysin maps associated to regularly embedded subschemes. In section 3, we exhibit a two-dimensional family of cycles such that the cycles we obtain from specializing in two different ways are precisely the ones we need to show to be rationally equivalent to obtain the commutativity result. Our explicit rational equivalence respects smooth pullback, and hence the generalization to stacks is automatic. This simplifies intersection theory on Deligne-Mumford stacks as in [8], where construction of such a rational equivalence fills the most difficult section of that important paper.

Since our rational equivalence arises by considering families of cycles on a larger total space, we are able to deduce (section 4) that the rational equivalence is invariant under a certain naturally arising group action. The key observation is that we can manipulate the situation so that the group action extends to the total space. This equivariance result is used, but appears with mistaken proof, in [2], where an important new tool of modern intersection theory – the theory of virtual fundamental classes – is developed.

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2. Intersection with divisors

In this section we work exclusively on schemes of finite type over a fixed base field. The term variety denotes integral scheme, and by a subvariety we mean an integral closed subscheme. We denote by Z_*X, W_*X, and A_*X, respectively, the group of algebraic cycles, group of rational equivalences, and Chow group of a scheme X. The boundary map $W_*X \to Z_*X$ is denoted ∂. We refer to [3] for basic definitions and properties from intersection theory. Given a Cartier divisor D we denote by $[D]$ the associated Weil divisor (it is important to note that the notion of Weil divisor makes sense on arbitrary varieties, [3], §1.2). If X is a variety then we denote by X^1 the set of subvarieties of codimension 1.

Definition 1. *Let X be a variety and let D be a Cartier divisor. Let $\pi\colon \widehat{X} \to X$ be the normalization map. The support of D, denoted $|D|$, is defined to be $\pi(\bigcup_{W \in \widehat{X}^1,\, \mathrm{ord}_W\, \pi^*D \neq 0} W)$.*

Remark 1. This agrees with the naïve notion of support (the union of all subvarieties appearing with nonzero coefficient in $[D]$) when X is normal or when D is effective.

Remark 2. There is yet another notion of support which appears in [3]. There, the support of a divisor is a piece of data that must be specified along with the divisor. Given a Cartier divisor D on a variety X, let Z be any closed subscheme such that away from Z the canonical section of $\mathcal{O}(D)$ is well-defined and nonvanishing. Then, [3] defines an intersection operation $A_k(X) \to A_{k-1}(Z)$. Unfortunately, the support $|D|$ which we have defined is not generally a support in this sense. Hence in the definition below we require that our divisors be specified by defining functions which are regular away from their supports.

We shall denote by $|D|^0$ the set of irreducible components of $|D|$.

Definition 2. *Let X be a variety. A P-divisor on X is a triple (U, U', x) such that*
  (i) *U and U' are nonempty open subschemes of X such that $U \cup U' = X$;*
  (ii) *$x \in k(U)^*$;*
  (iii) *$x|_{U \cap U'} \in \mathcal{O}^*(U \cap U')$; and*
  (iv) *the data $(x \in k(U)^*, 1 \in k(U')^*)$ specifies a Cartier divisor D such that $|D| = X \setminus U'$.*

By abuse of terminology, we call D a P-divisor if D is the Cartier divisor associated to a P-divisor as in (iv). Given a P-divisor as above, we call x the *local defining function*. Given a P-divisor D on a variety X and a morphism of varieties $f\colon Y \to X$, the pullback f^*D makes sense as a P-divisor when f is smooth or when Y is normal and $f(Y) \not\subset |D|$.

Example 1. (i) Let X be a normal variety. Let $x \in k(X)^*$ specify a principal Cartier divisor D. Then $(X, X \setminus |D|, x)$ is a P-divisor.

  (ii) Let X be a variety. Every effective principal Cartier divisor is a P-divisor.

  (iii) Let X be a variety, and let $\pi\colon X \to \mathbb{P}^1$ be a dominant morphism. Then the fiber of π over $\{0\}$ is a P-divisor.

The operation of intersecting with a Cartier divisor is generally well-defined only on the level of rational equivalence classes of cycles. When $V \subset |D|$, we have $D \cdot [V] = c_1(\mathcal{O}(D)|_V) \cap [V]$, and there is generally no way to pick canonically a cycle representing this first Chern class. The exception is when $\mathcal{O}(D)|_{|D|}$ is trivial, or in our terminology, D is a P-divisor. Then, we may define a cycle-level intersection operation (see [3], Remark 2.3).

Definition 3. *Let X be a variety, and let D be a P-divisor on X. The cycle-level intersection operation*

$$D \cdot - : Z_k(X) \to Z_{k-1}(|D|)$$

is given by

$$D \cdot [V] = \begin{cases} [D|_V] & \text{if } V \not\subset |D| \ ; \\ 0 & \text{if } V \subset |D| \ . \end{cases}$$

The claim that this map passes to rational equivalence and hence gives an intersection operation $D \cdot : A_k(X) \to A_{k-1}(|D|)$ is proved in [3], but not in a way that makes it easy to see how $D \cdot \alpha$ is to be rationally equivalent to zero if α is a cycle that is rationally equivalent to zero. Following the program set out in the introduction, we would like to demonstrate this fact by giving an explicit map on rational equivalences which commutes with the boundary operation.

Definition 4. *Let X be a variety, and let D be a P-divisor on X with local defining function x. Suppose V is a subvariety of X with normalization $\pi \colon \widehat{V} \to V$, and suppose $y \in k(V)^*$. We define the intersection operation on the level of rational equivalences*

$$D \cdot - : W_k(X) \to W_{k-1}(|D|)$$

by

$$D \cdot y = \begin{cases} \pi_* \left(\displaystyle\sum_{W \in |\pi^* D|^0} (y^{\operatorname{ord}_W x} / x^{\operatorname{ord}_W y})|_W \right) & \text{if } V \not\subset |D| \ ; \\ 0 & \text{if } V \subset |D| \ . \end{cases} \tag{1}$$

Here, $\pi_* \colon W_* \widehat{V} \to W_* V$ is pushforward of rational equivalence.

Remark 3. This definition explains why we require the definition of a P-divisor to include more data than just that of the underlying Cartier divisor. The map (1) actually depends on the choice of defining function.

Remark 4. Definitions 3 and 4 can be viewed as the first two pieces of the specialization map defined on the entire Gersten complex, cf. [5].

Remark 5. The function $y^{\operatorname{ord}_W x} / x^{\operatorname{ord}_W y}$ of (1) and similar formulas often appears in the literature with sign $(-1)^{(\operatorname{ord}_W x)(\operatorname{ord}_W y)}$. For this paper the sign convention is irrelevant; for the sake of brevity we omit signs in Definition 4.

Proposition 1. *Let X be a variety and let D be a P-divisor on X. Then the diagram*

$$\begin{array}{ccc} W_k(X) & \xrightarrow{\ D\cdot\ } & W_{k-1}(|D|) \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ Z_k(X) & \xrightarrow{\ D\cdot\ } & Z_{k-1}(|D|) \end{array}$$

commutes.

This follows easily from

Proposition 2. *Let X be a normal variety and let x and y be rational functions on X with associated P-divisors D and E. For $V \in X^1$ set $a_V = \operatorname{ord}_V x$ and $b_V = \operatorname{ord}_V y$.*

Then

$$\sum_{V \in X^1} \partial(y^{a_V}/x^{b_V}|_V) = 0 \; ; \tag{2}$$

$$\partial(D \cdot y) = D \cdot (\partial y) \; ; \tag{3}$$

$$D \cdot [E] - E \cdot [D] = \sum_{V \in |D|^0 \cap |E|^0} \partial(y^{a_V}/x^{b_V}|_V) \; . \tag{4}$$

Proof. If we split the sum in (2) into a sum over $V \in |D|^0$ and a sum over $V \notin |D|^0$ we obtain (3). Similarly if we split away the terms with $V \in |D|^0 \cap |E|^0$ we obtain (4) from (2). So, for a fixed variety X and fixed divisors D and E, the three assertions are equivalent. Now, we get (2) as a consequence of the tame symbol in K-theory, cf. [7], §7, or by the following elementary geometric argument. We quickly reduce to the case where D and E are effective. Then, when D and E meet properly, (4) follows from [3], Theorem 2.4, Case 1. An induction on *excess of intersection*

$$\varepsilon(D, E) = \max_{V \in X^1} a_V \cdot b_V$$

completes the proof: if we denote the normalized blow-up along the ideal (x, y) by $\sigma \colon X' \to X$ and denote the exceptional divisor by Z then we may write $\sigma^* D = Z + D'$ and $\sigma^* E = Z + E'$, and now $|D'| \cap |E'| = \emptyset$ and $\max(\varepsilon(D', Z), \varepsilon(E', Z)) < \varepsilon(D, E)$ (assuming D and E do not meet properly), cf. [3], Lemma 2.4. The result pushes forward.

Corollary 1. *Let D and E be P-divisors on a variety X, with respective local defining functions x and y. Let $\pi \colon \widehat{X} \to X$ be the normalization map. Then*

$$D \cdot [E] - E \cdot [D] = \partial \omega$$

where $\omega \in W_*(|D| \cap |E|)$ is given by

$$\omega = \sum_{V \in |\pi^* D|^0 \cap |\pi^* E|^0} \pi_*(y^{\mathrm{ord}_V\, x}/x^{\mathrm{ord}_V\, y}|_V).$$

3. Application to intersection theory on stacks

All stacks (and schemes) in this section are algebraic stacks of Artin type, [1], [6], which are locally of finite type over the base field. The notion of P-divsor on a stack makes sense (it is as in Definition 2 with "open subscheme" replaced by "open substack," where by "Cartier divisor" in part (iv) of the definition we mean a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$ for the Zariski topology, and where normalization, order along a substack of codimension 1, and support of a Cartier divisor are well defined on stacks because they all respect smooth pullback and hence can be defined locally). Since an Artin stack possesses a smooth cover by a scheme, the operation of intersecting with a P-divisor on a stack comes for free once we know that this operation on schemes commutes with smooth pullback. Also for free we get Corollary 1 in the setting of stacks: the formation of ω from X, D, and E commutes with smooth pullback.

Proposition 3. *Let X be a variety, let Y be a scheme, and let $f \colon Y \to X$ be a smooth morphism. Let D be a P-divisor on X. Then $f^* \circ D \cdot = (f^* D) \cdot \circ f^*$, both as maps on cycles and as maps on rational equivalences.*

We note that the definitions and propositions of section 2 are valid for a general reduced scheme (such as the scheme Y appearing in the statement of Proposition 3) and now the proof of Proposition 3 is routine.

We now turn to an application of Corollary 1 to intersection theory on Deligne-Mumford stacks (where a reasonable intersection theory exists, cf. [4], [8]). Central to intersection theory on schemes is the Gysin map corresponding to a regularly embedded subscheme, since the diagonal of a smooth scheme is a regular embedding and this way we obtain an intersection product on smooth schemes. The diagonal morphism for a smooth Deligne-Mumford stack is not generally an embedding, but it is representable and unramified.

Lemma 1. *Let $f\colon F \to G$ be a representable morphism of Artin stacks. Then f is unramified if and only if there exists a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\;g\;} & V \\ \downarrow & & \downarrow \\ F & \xrightarrow{\;f\;} & G \end{array}$$

such that the vertical maps are smooth surjective, g is a closed immersion of schemes, and the induced morphism $U \to F \times_G V$ is étale.

Proof. This is [8], Lemma 1.19. Because this is such a basic fact about properties of morphisms in algebraic geometry, we present an elementary proof in the Appendix.

To describe a representable morphism, we use the terminology *local immersion* as a synonym for *unramified* and call f above a *regular local immersion* if moreover g is a regular embedding of schemes. Since formation of normal cone is of a local nature, an obvious patching construction produces the normal cone $C_X Y$ to a local immersion $X \to Y$; the cone is a bundle in case $X \to Y$ is a regular local immersion.

To get Fulton-MacPherson-style intersection theory on Deligne-Mumford stacks we clearly need to have Gysin maps for regular local immersions. In [8], the author supplies this needed Gysin map by giving a (long, difficult) proof of the stack analogue of [3], Theorem 6.4, namely

Proposition 4. *Let $X \to Y$ and $Y' \to Y$ be local immersions of Artin stacks. Then $[C_{X \times_Y C_{Y'} Y} C_{Y'} Y] = [C_{C_X Y \times_{X \times_Y Y'}} C_X Y]$ in $A_*(C_X Y \times_Y C_{Y'} Y)$.*

Remark 6. Though our focus is on applications to intersection theory on Deligne-Mumford stacks, we continue to make use of constructions which behave well locally with respect to smooth pullback, and hence our results are valid in the more general setting of Artin stacks.

Remark 7. Given a stack X which is only locally of finite type over a base field, we must take $Z_* X$ to be the group of *locally finite* formal linear combinations of integral closed substacks. More intrinsically, $Z_* X$ is the group of global sections of the sheaf for the Zariski topology $\mathcal{Z}_*$ which associates to a stack of finite type the free abelian group on integral closed substacks. Similarly, $W_* X$ is the group of global sections of sheaf $\mathcal{W}_*$. As always, $A_* X$ is defined to be $Z_* X / \partial W_* X$.

The methods of the last section allow us to supply a new, simpler proof of this proposition.

Proof. Recall that given a closed immersion $X \to Y$ there are associated spaces

$$M_X Y = \mathrm{B}\ell_{X \times \{0\}} \, Y \times \mathbb{P}^1 \ ,$$
$$M_X^\circ Y = M_X Y \setminus \mathrm{B}\ell_{X \times \{0\}} \, Y \times \{0\} \ ,$$

cf. [3], §5.1. Given a locally closed immersion, say with U an open subscheme of Y and X a closed subscheme of U, then $M_X^\circ Y := M_X^\circ U \amalg_{U \times \mathbb{A}^1} Y \times \mathbb{A}^1$ makes sense and is independent of the choice of U.

This lets us define $M_F^\circ G$ when $F \to G$ is a local immersion of stacks, as follows. Assume we have a diagram as in the statement of Lemma 1, and set $R = U \times_F U$ and $S = V \times_G V$. There are projections $q_1, q_2 \colon S \to V$. Define $s_i \colon M_R^\circ S \to M_U^\circ V$ $(i = 1, 2)$ to be the composite $M_R^\circ S \to M_{U \times_G V}^\circ S \to M_U^\circ V$, where the first map is induced by the open immersion $R \to U \times_G V$ and the second, by pullback via q_i. Then $[M_R^\circ S \rightrightarrows M_U^\circ V]$ is the smooth groupoid presentation of a stack which we denote $M_F^\circ G$. We have, by descent, a morphism $M_F^\circ G \to \mathbb{P}^1$, which is flat and has as general fiber a copy of G and as special fiber the normal cone $C_F G$.

In the situation at hand, this construction gives

$$(s \times t) \colon M_X^\circ Y \times_Y M_{Y'}^\circ Y \to \mathbb{P}^1 \times \mathbb{P}^1 \ ,$$

and hence a pair of P-divisors, D (corresponding to s) and E (corresponding to t). We note that $(s \times t)^{-1}(\{0\} \times \{0\}) = C_X Y \times_Y C_{Y'} Y$. Since the restriction of $s \times t$ to $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \{0\} \times \{0\}$ is flat, we have

$$[D] = [C_X Y \times_Y M_{Y'}^\circ Y] \mod Z_*(C_X Y \times_Y C_{Y'} Y),$$
$$[E] = [M_X^\circ Y \times_Y C_{Y'} Y] \mod Z_*(C_X Y \times_Y C_{Y'} Y) \ .$$

We examine the fiber of $s \times t$ over $\mathbb{P}^1 \times \{0\}$ more closely. The fiber square

$$
\begin{array}{ccc}
i^* C_{Y'} Y & \longrightarrow & C_{Y'} Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ i\ } & Y
\end{array}
$$

gives rise to a closed immersion f making

$$M_{i^* C_{Y'} Y}^\circ C_{Y'} Y \xrightarrow{\ f\ } M_X^\circ Y \times_Y C_{Y'} Y$$

with maps h and g to $\mathbb{P}^1$

commute (where g is first projection followed by s). Since f is an isomorphism away from the fiber over 0, we see in fact that

$$[E] = [M_{i^* C_{Y'} Y}^\circ C_{Y'} Y] \mod Z_*(C_X Y \times_Y C_{Y'} Y) \ ,$$

and since h is flat we find

$$D \cdot [E] = [C_{i^* C_{Y'} Y} C_{Y'} Y] \ .$$

Similarly, if j denotes the map $Y' \to Y$ then

$$E \cdot [D] = [C_{j^* C_X Y} C_X Y] \ ,$$

and so the rational equivalence $\omega \in W_*(C_X Y \times_Y C_{Y'} Y)$ of Corollary 1 satisfies

$$\partial \omega = [C_{X \times_Y C_{Y'} Y} C_{Y'} Y] - [C_{C_X Y \times_Y Y'} C_X Y] \ .$$

Remark 8. The map $M_F^\circ G \to G$ associated to a local immersion of stacks is not generally separated, though this should cause the reader no concern, since intersection theory is valid even on non-separated schemes and stacks. In fact, even those operations of [8] which require a so-called finite parametrization may be carried out on arbitrary Deligne-Mumford stacks which are of finite type over a field (no such operations show up in this paper). This is so thanks to the proof, [6], (10.1), that every Deligne-Mumford stack of finite type over a field possesses a finite parametrization, i.e., admits a finite surjective map from a scheme.

Remark 9. The reader who wishes greater generality may see easily that all results in this section are valid in the setting of Artin stacks which are locally of finite type over an excellent Dedekind domain.

4. Equivariance for tangent bundle action

We continue to work with stacks which are locally of finite type over some base field. A special case of Proposition 4 is when $i\colon X \to Y$ is a local immersion of smooth Deligne-Mumford stacks. Suppose $j\colon Y' \to Y$ is a local immersion, with Y' an arbitrary Deligne-Mumford stack. Recall that the local immersion j gives rise to a natural group action of j^*T_Y on $C_{Y'}Y$. In short, the action is given locally (say Y is an affine scheme and Y' is the closed subscheme given by the ideal I) by considering the action of $T_Y|_{Y'}$ on $\operatorname{Spec} \operatorname{Sym}(I/I^2)$ induced by the map $I/I^2 \to \Omega_Y^1$ and proving ([2], Lemma 3.2) that the normal cone $\operatorname{Spec} \bigoplus I^k/I^{k+1}$ is invariant under the group action.

If we let $N_X Y$ be the normal bundle to X in Y and denote simply by N its pullback to $X' := X \times_Y Y'$, then $C_X Y \times_Y C_{Y'}Y$ is identified with $N \times_{X'} i^*C_{Y'}Y$. Viewing T_X as a subbundle of i^*T_Y, we have the natural action of $T_X|_{X'}$ on $i^*C_{Y'}Y$. This plus the trivial action on N gives an action of $T_X|_{X'}$ on $N \times_{X'} i^*C_{Y'}Y$.

Theorem 1. *The rational equivalence between $[C_{i^*C_{Y'}Y}C_{Y'}Y]$ and $[N \times_{X'} C_{X'}X]$ produced in the proof of Proposition 4 is invariant under the action of $T_X|_{X'}$ on $N \times_{X'} i^*C_{Y'}Y$ described above.*

Remark 10. As a consequence, the rational equivalence descends to a rational equivalence on the stack quotient $[N \times_{X'} i^*C_{Y'}Y / T_X|_{X'}]$. This fact is exploited in [2], Lemma 5.9: as justification the authors invoke the stronger claim which appears as their Proposition 3.5 and which asserts that the rational equivalence is invariant under the action of the bigger group $T_Y|_{X'}$. The stronger claim is false (if $X \to Y$ and $Y' \to Y$ meet improperly, then even the cycle $[N \times_{X'} C_{X'}X]$ fails to be invariant under the action of $T_Y|_{X'}$); the stated proof creates the impression that the claim should follow formally from the fact that the rational equivalence of [8], §4 commutes with smooth pullback.

Proof. The question is local, so we may assume Y is an irreducible scheme, smooth and of finite type over the base field, X is a smooth irreducible closed subscheme of Y, and Y' is a closed subscheme of Y. If $X \subset Y'$ then the group action is trivial and there is nothing to prove, so we assume the contrary.

Lemma 2. *Let Y be a smooth irreducible scheme of finite type over a field k, of dimension n, let X be a smooth irreducible closed subscheme of Y of codimension d,*

and let Y' be a closed subscheme of Y such that $X \not\subset Y'$. Let y be a closed point of $Y' \cap X$. Then, after suitable base change by a finite separable extension of the base field, and after shrinking Y to a neighborhood of y in Y, there exists an étale map $f \colon Y \to \mathbb{A}^n$ such that X maps into a linear subspace of $\mathbb{A}^n$ of codimension d and such that $Y' \to f(Y')$ is étale.

Proof. We may assume y is a k-valued point, and moreover that Y sits in $\mathbb{A}^l$ with $X = \mathbb{A}^{l-d} \cap Y$ (for suitable l). Now, we consider $\mathbb{A}^l$ sitting in $\mathbb{P}^l$ with complement the hyperplane at infinity H. Let R be the union of the following loci: the closure $\overline{X}$ of X in $\mathbb{P}^l$, the $(n-d)$-dimensional space tangent to X at y, the join $J(y, (X \cap Y')^-)$ of y and the closure of $X \cap Y'$, and the join $J(y, \overline{X} \cap H)$. Since R has dimension $(n-d)$, we may (after a suitable finite separable base field extension) choose a linear subspace $\Lambda \subset \mathbb{P}^{l-d}$ of dimension $(l-n-1)$ such that $\Lambda \cap R = \emptyset$. Projection from Λ seems a likely candidate for f but is potentially unsuitable due to lack of control over $\overline{Y'} \cap H$. So, we instead build a large linear system which separates X from the boundary of Y' and sends X into a linear space of dimension $(n-d)$. We may then take f to be the composition of the map thus determined with a linear projection to $\mathbb{P}^n$.

We give $\mathbb{P}^l$ homogeneous coordinates $x_0, \ldots, x_l$ so that $H = \{x_0 = 0\}$. Let us suppose that $\overline{X} \subset \mathbb{P}^l$ is cut out by homogeneous polynomials $h_1, \ldots, h_q$ of degree p. Let $b_0, \ldots, b_n$ be linear polynomials which cut out Λ. The map

$$(x_0 : \cdots : x_l) \mapsto (x_0^{p-1} b_0(x) : \cdots : x_0^{p-1} b_n(x) : h_1(x) : \cdots : h_q(x)) \qquad (5)$$

from $\mathbb{P}^l$ to $\mathbb{P}^m$ (with $m = n + q$) is rational but becomes regular if we form the blow-up $\varphi \colon Q \to \mathbb{P}^l$ of the base locus of the indicated linear system. We denote by ψ the (regular) map $Q \to \mathbb{P}^m$ given by (5).

Let us give $\mathbb{P}^m$ homogeneous coordinates $B_0, \ldots, B_n, H_1, \ldots, H_q$. We let $\mathbb{P}^n$ denote the subspace of $\mathbb{P}^m$ given by $H_1 = \cdots = H_q = 0$. Then φ determines an isomorphism $Q \times_{\mathbb{P}^m} \mathbb{P}^n \to \overline{X}$, and the restriction of ψ to $Q \times_{\mathbb{P}^m} \mathbb{P}^n \simeq \overline{X}$ is given by the linear projection $\overline{X} \subset \mathbb{P}^l \setminus \Lambda \to \mathbb{P}^n$ determined by Λ. Indeed, if we define V to be the subscheme of $\mathbb{P}^l \times \mathbb{P}^m$ given by the equations

$$b_j(x) B_i = b_i(x) B_j \qquad \text{for all } i \neq j \;, \qquad (6)$$

$$x_0^{p-1} b_j(x) H_i = h_i(x) B_j \qquad \text{for all } i \text{ and } j \;, \qquad (7)$$

then V contains Q as a closed subscheme. Now the scheme $V \times_{\mathbb{P}^m} \mathbb{P}^n$ is covered by open charts of the form $\{B_j \neq 0\}$, and on any of these charts, (7) implies that all h_i vanish, i.e., that the map $V \times_{\mathbb{P}^m} \mathbb{P}^n \to \mathbb{P}^l$ factors through $\overline{X}$. Since $\Lambda \cap \overline{X} = \emptyset$, we conclude from (6) that $b_j(x) \neq 0$ on the chart where $B_j \neq 0$. Thus $V \times_{\mathbb{P}^m} \mathbb{P}^n \to \overline{X}$ is an isomorphism, and hence $Q \times_{\mathbb{P}^m} \mathbb{P}^n \to \overline{X}$ is an isomorphism. Solving (6), we see that ψ maps $x \in \overline{X}$ to $(b_0(x) : \cdots : b_n(x)) \in \mathbb{P}^n$, i.e., that ψ, restricted to $Q \times_{\mathbb{P}^m} \mathbb{P}^n$, is equal to the projection map determined by Λ.

The map ψ sends X dominantly into $\mathbb{P}^{n-d}$. Since Λ is disjoint from the tangent space to X at y, we have that $\psi \colon X \to \mathbb{P}^{n-d}$ is étale at y. The map ψ, restricted to Y, is defined and is unramified near y; the same is true for Y'. Let $u = \psi(y)$, and let us examine the projective morphism $\psi \colon Q \to \mathbb{P}^m$, restricted to the strict transform of Y'. We now claim that y is the only point of the strict transform of Y' that is mapped via ψ to u. Indeed, let $x \neq y$ be another point of the strict transform of Y' such that $\psi(x) = u$. In particular, we have $\psi(x) \in \mathbb{P}^n \subset \mathbb{P}^m$, and by the identification $Q \times_{\mathbb{P}^m} \mathbb{P}^n \simeq \overline{X}$ we are reduced to showing that any $x \in \overline{X} \cap \overline{Y'}$ with $x \neq y$ satisfies

$J(y,x) \cap \Lambda = \emptyset$. Either $x \in X \cap Y'$, in which case $J(y,x) \subset J(y, (X \cap Y')^-)$ is disjoint from Λ, or else $x \in \overline{X} \cap H$, in which case $J(y,x) \subset J(y, \overline{X} \cap H)$ is disjoint from Λ, and the claim is proved.

To complete the proof of the lemma, we choose a linear subspace Ω of $\mathbb{P}^m$ of dimension $(m - n - 1)$, such that $\Omega \cap [\psi(T_{y,Y}) \cup J(u, \psi(Y')^-)] = \emptyset$ (where $T_{y,Y} =$ tangent space to Y at y), and we let f be the composition of ψ and the projection $\mathbb{P}^m \setminus \Omega \to \mathbb{P}^n$ determined by Ω. We have $f : Y \to \mathbb{P}^n$ defined near y, dominant, and étale at y. Since X was mapped to a linear space of $\mathbb{P}^m$, the image in $\mathbb{P}^n$ is again linear. Let $t = f(y)$. The claim above, plus the condition on Ω, implies that the composite map sending the strict transform of Y' in $\mathbb{P}^l$ to $\mathbb{P}^n$ has fiber over t equal to $\{y\}$. The proof is now done since for any projective morphism $f : S \to T$ of schemes of finite type over a field k, if f is unramified at a k-valued point $s \in S$ and if, with $t = f(s)$, we have $f^{-1}(t) = \{s\}$, then there must exist an open neighborhood of t in T over which f is a closed immersion. So, in fact, $Y' \to f(Y')$ is an isomorphism over a neighborhood of t.

Since the rational equivalence of the proof of Proposition 4 commutes with étale base change, we are reduced by Lemma 2 to the case where $Y = \mathbb{A}^n$ and $X = \mathbb{A}^m$ (as a linear subspace of $\mathbb{A}^n$). Now we need the

Key Observation. *Assume $Y = \mathbb{A}^n$ and Y' is a closed subscheme of Y. Identify T_Y, as a group scheme over Y, with the additive group $\mathbb{A}^n$. Then there is a group action of $\mathbb{A}^n$ on $\widetilde{M}^\circ_{Y'}Y$ (which we define to be the fiber of $M^\circ_{Y'}Y \to \mathbb{P}^1$ over $\mathbb{A}^1$) which restricts to the natural action of T_Y on $C_{Y'}Y$.*

Indeed, we let $\mathbb{A}^n$ act on $Y \times \mathbb{A}^1$ by

$$(a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n, t) = (x_1 + ta_1, \ldots, x_n + ta_n, t) .$$

By the universal property of blowing up, this extends uniquely to an action of $\mathbb{A}^n$ on $\widetilde{M}^\circ_{Y'}Y$. If Y' is given by the ideal $(f_1, \ldots, f_k)$, and if we view $\widetilde{M}^\circ_{Y'}Y$ as the closure of the graph of $(f_1/t, \ldots, f_k/t) : Y \times (\mathbb{A}^1 \setminus \{0\}) \to \mathbb{A}^k = \operatorname{Spec} k[z_1, \ldots, z_k]$, then the action is given algebraically by

$$\mathbf{a} = (a_1, \ldots, a_n) : z_i \mapsto z_i + (f_i(\mathbf{x} + t\mathbf{a}) - f_i(\mathbf{x}))/t ,$$

so at $t = 0$ we recover $z_i \mapsto z_i + D_{\mathbf{a}} f_i(\mathbf{x})$. This is the natural action of T_Y on $C_{Y'}Y$.

Concluding the proof of equivariance, we observe that $\widetilde{M}^\circ_{\mathbb{A}^m}\mathbb{A}^n$ fits into the fiber diagram

$$
\begin{array}{ccc}
\widetilde{M}^\circ_{\mathbb{A}^m}\mathbb{A}^n \times_{\mathbb{A}^n} \widetilde{M}^\circ_{Y'}\mathbb{A}^n & \longrightarrow & \widetilde{M}^\circ_{Y'}\mathbb{A}^n \\
\downarrow & & \downarrow \\
\widetilde{M}^\circ_{\mathbb{A}^m}\mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\
\downarrow & & \downarrow \\
\widetilde{M}^\circ_{\{0\}}\mathbb{A}^{n-m} & \longrightarrow & \mathbb{A}^{n-m}
\end{array}
$$

and now the action from the Key Observation of $\mathbb{A}^m \subset \mathbb{A}^n$ on $\widetilde{M}^\circ_{Y'}\mathbb{A}^n$ and the trivial action of $\mathbb{A}^m$ on $M^\circ_{\{0\}}\mathbb{A}^{n-m}$, combine to give a group action of $\mathbb{A}^m$ on $\widetilde{M}^\circ_X Y \times_Y \widetilde{M}^\circ_{Y'}Y$. The function $\widetilde{M}^\circ_X Y \times_Y \widetilde{M}^\circ_{Y'}Y \to \mathbb{A}^1 \times \mathbb{A}^1$ which is used in Corollary 1 is invariant

for this $\mathbb{A}^m$-action. Since the rational equivalence of the proof of Proposition 4 is compatible with smooth pullback, we get the desired equivariance result.

Appendix A. Appendix: unramified morphisms

We give an elementary algebraic proof of the following fact.

Lemma 3. *Let $S \to T$ be an unramified morphism of affine schemes which are of finite type over a base field k. Then there exists a commutative diagram of affine schemes*

$$
\begin{array}{ccc}
U & \xrightarrow{\ g\ } & V \\
\downarrow & & \downarrow \\
S & \xrightarrow{\ f\ } & T
\end{array}
$$

such that the vertical maps are étale surjective and such that g is a closed immersion.

This fact plus the local nature of the property of being unramified gives us Lemma 1.

Proof. Say $S = \operatorname{Spec} A$, $T = \operatorname{Spec} B$, and f is given algebraically by $f^* \colon B \to A$. Recall that for f to be unramified means that for every maximal ideal $\mathfrak{p}$ of A with $\mathfrak{q} = f(\mathfrak{p})$, we have $f^*(\mathfrak{q}) \cdot A_{\mathfrak{p}} = \mathfrak{p} A_{\mathfrak{p}}$, and the induced field extension $B/\mathfrak{q} \to A/\mathfrak{p}$ is separable.

Case 1: The induced field extension $B/\mathfrak{q} \to A/\mathfrak{p}$ is an isomorphism. Then, if $x_1, \ldots, x_n$ are generators of A as a k-algebra, we may write

$$
x_i = f^*(t_i) + w_i
$$

with $t_i \in B$ and $w_i \in \mathfrak{p}$, for each i. Since f is unramified, we have

$$
w_i = \sum_{j=1}^{m_i} \frac{f^*(y_{ij}) p_{ij}}{q_i}
$$

for some $y_{ij} \in \mathfrak{q}$, $p_{ij} \in A$, and $q_i \in A \setminus \mathfrak{p}$.

Choose representative polynomials P_{ij} and Q_i in $k[X_1, \ldots, X_n]$ such that $P_{ij}(x_1, \ldots, x_n) = p_{ij}$ and $Q_i(x_1, \ldots, x_n) = q_i$. Let

$$
V = \operatorname{Spec} B[X_1, \ldots, X_n] \left/ \left(X_1 Q_1 - t_1 Q_1 - \sum_{j=1}^{m_1} y_{1j} P_{1j}, \ \ldots, \right.\right.
$$

$$
\left.\left. X_n Q_n - t_n Q_n - \sum_{j=1}^{m_n} y_{nj} P_{nj} \right), \right.
$$

and define $g \colon S \to V$ by $B \xrightarrow{f^*} A$ and $X_i \mapsto x_i$, and let $\varphi \colon V \to T$ be given by inclusion of B. Then g is a closed immersion, and by the Jacobian criterion φ is étale in some neighborhood of $g(\mathfrak{p})$.

Case 2: The field extension $B/\mathfrak{q} \to A/\mathfrak{p}$ is separable. We let k' be the maximal subfield of $A/\mathfrak{p}$ which is separable over k and perform the étale base change $\operatorname{Spec} k' \to \operatorname{Spec} k$ to get $f' \colon S' \to T'$. Now S' has an $A/\mathfrak{p}$-valued point which maps to $\mathfrak{p} \in S$, and since k' together with $B/\mathfrak{q}$ generates all of $A/\mathfrak{p}$ we are now in the situation of Case 1.

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