BRAUER GROUPS OF INVOLUTION SURFACE BUNDLES

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To David Mumford, with admiration.

1. Introduction

A fundamental breakthrough in the study of rationality properties of complex algebraic varieties was the construction, by Artin and Mumford, of examples of projective unirational threefolds with nontrivial Brauer group. Along with the examples by Iskovskikh–Manin and Clemens–Griffiths, these provided the first instances of nonrational unirational complex threefolds, settling the long-standing Lüroth problem. Even more important was the introduction of new tools and concepts:

- Brauer groups [1],
- Birational rigidity [22], and
- Intermediate Jacobians [9].

All of these have triggered major developments in algebraic geometry; see, e.g., [27], [4], [31], [8], and the references therein.

The closely related Zariski problem concerns stable rationality, i.e., rationality of the product of the variety in question with some projective space. The Artin–Mumford examples are not stably rational, while there exist threefolds with a nontrivial intermediate Jacobian obstruction to rationality and which are nevertheless stably rational [6]. It is currently unknown whether or not birational rigidity obstructs stable rationality.

Recent years have seen a tremendous revival of interest in the Artin–Mumford construction in connection with the Specialization method, introduced by Voisin [37], and developed by Colliot-Thélène–Pirutka [14], Nicaise–Shinder [28], and Kontsevich–Tschinkel [23]. These new techniques relate the failure of (stable) rationality of a very general member of a family to the presence of a Brauer group obstruction in a single member of the family. Often the general members of the family possess no evident obstructions to rationality, while the (mildly singular) special member is of Artin–Mumford type. This led to tremendous advances in the study of stable rationality, see, e.g., [5], [13], [36], [35] and the surveys [38], [29]. In particular, this allowed to:

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- essentially settle the long-standing open problem of stable rationality for non-rational rationally connected threefolds, excluding the case of cubic threefolds: every such threefold is birational to a Fano threefold, a del Pezzo fibration over $\mathbb{P}^1$, or a conic bundle over a rational surface, and for the families of such smooth projective threefolds, the very general members are not stably rational [16], [21], [26];
- understand the behavior of rationality under deformations, giving rise to smooth families of complex three- and fourfolds with varying (stable) rationality properties [17], [19], as well as interesting families with constant (stable) rationality, e.g., some special cubic fourfolds [7], [32].

These developments focused the attention on varieties with nontrivial Brauer group and mild singularities arising in interesting families of rationally connected varieties, e.g., conic and higher-dimensional quadric bundles over projective spaces [2], [34]. The computation of the Brauer group on such varieties is an interesting problem by itself, studied, e.g., by Colliot-Thélène–Ojanguren in [12] and by Colliot-Thélène, in the case of conic bundles over rational surfaces [30, Thm. 3.13]. More recently, Pirutka gave an explicit combinatorial algorithm for the computation of the Brauer group of quadric surface bundles over rational surfaces [30]. It became a crucial ingredient in proofs of failure of stable rationality in [19], [18], [20], [33].

In these investigations it was important to construct good, i.e., mildly singular, birational models of varieties fibered over rational surfaces. Already the case of conic and Brauer–Severi surface bundles is quite involved [24]. In [25], we studied quadric surface bundles and, more generally, involution surfaces bundles, with special attention to producing and deforming such models. In this paper, we use these models to give a combinatorial algorithm for the computation of the Brauer group (Theorem 6), generalizing Pirutka's algorithm. The inspiration comes from the work of Artin and Mumford in the conic bundle case.

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2. Geometry of involution surface bundles

Let K be a field of characteristic different from 2. A surface over K that is geometrically isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ is called an *involution surface*. Such a surface is classified by the pair $(L/K, \beta)$, where

- L/K is the *discriminant extension*, a degree 2 étale K-algebra, and
- β is a 2-torsion element of the Brauer group of L that is the class of a quaternion algebra.

We remark that for the unique nontrivial K-automorphism τ of L, the Brauer group elements β and $\tau^*(\beta)$ determine isomorphic involution surfaces. This ambiguity is eliminated by fixing a compatible collection, for any K-algebra Λ, of identifications of the set of rulings (maps onto a conic) of X_Λ with $\mathrm{Hom}_K(\Lambda, L)$. We assume that such identifications are fixed, without explicit mention, whenever a degree 2 étale K-algebra and a Brauer group element are mentioned in connection with an involution surface.

We work over an algebraically closed ground field k of characteristic different from 2 and let S be a nonsingular algebraic variety over k. An *involution surface bundle* over S is a flat generically smooth projective morphism

$$\pi\colon X \to S,$$

such that if $U \subset S$ denotes the locus over which π is smooth, then the fiber over every point of U is an involution surface. Involution surface bundles were studied in [25], where we identified four geometric types of degenerations of involution surfaces, Types I, II, III, and IV. Involution surface bundles with only these geometric types of degenerations, and satisfying further conditions restricting the singularities of the total space X, were called *mildly degenerating simple* involution surface bundles.

From now on we suppose that S is a smooth projective surface over k. Good models will be mildly degenerating simple involution surface bundles over the complement of a codimension 2 set $Z \subset S$ (finitely many points), with additional degeneration types permitted at points of Z. Specifically, an involution surface bundle over S is defined to be *simple* if the complement of U is a simple normal crossing divisor

$$D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

where we take $Z = D^{\mathrm{sing}}$. The geometric fibers should have Type I over D_1, Type II over D_2, Type III over D_3, and Type IV over D_4. Additionally, D_1 and D_3 are required to be smooth, disjoint from each other, and disjoint from D_4, and π is required to have one of 6 explicit isomorphism types étale locally at every point of Z.

By [25, Thm. 10], any fibration $\pi\colon X \to S$ whose generic fiber is an involution surface admits a model $\tilde{\pi}\colon \widetilde{X} \to \widetilde{S}$ over some smooth surface $\widetilde{S}$ with proper birational morphism to S, which is a simple involution surface bundle. The proof translates into the following recipe. Let K denote the function field of S, and L, the discriminant extension of the generic fiber. If L is a quadratic field extension, then from a model we obtain, by birational modification, a finite degree 2 morphism of smooth surfaces $\widetilde{T} \to \widetilde{S}$, while in case $L \cong K \times K$ we take $\widetilde{S} = S$ and $\widetilde{T} = S \sqcup S$. Let β denote the 2-torsion element of L, corresponding to the generic fiber of π, represented geometrically by a conic bundle over $\widetilde{T}$. This is put into a standard form after further blowing up $\widetilde{T}$, which may be done compatibly with blow-ups of $\widetilde{S}$. The standard conic bundle determines, by [25, Thm. 13], a simple involution surface bundle, with following data:

- The branch locus of $\widetilde{T} \to \widetilde{S}$ is $D_1 \cup D_3$, where over D_1 the element β is unramified and the conic bundle has smooth fibers, and over D_3 the Brauer group element is ramified and the conic bundle has, generically, reduced singlar fibers.
- The additional divisors where β is ramified lie over D_2 and D_4.

Every generically smooth quadric surface bundle determines a simple involution surface bundle with $D_3 = D_4 = \emptyset$. However, there exist simple involution surface bundles with $D_3 = D_4 = \emptyset$ which are not models of quadric surface bundles. A necessary and sufficient condition to be a model of a quadric surface bundle is that β lies in the kernel of the corestriction homomorphism $\mathrm{Br}(L)[2] \to \mathrm{Br}(K)[2]$, or equivalently lies in the image of the restriction homomorphism $\mathrm{Br}(K)[2] \to \mathrm{Br}(L)[2]$.

3. Resolution

The hypersurface singularity defined by $uv = xyz$ has singular locus consisting of the union of three curves and may be resolved by first blowing up one of the curves, then the proper transforms of the other two. These assertions are straightforward to verify (over an arbitrary field k), either by recognizing $uv = xyz$ as defining the affine toric variety given by the cone

$$\mathbb{R}_{\ge 0}\langle (1,0,0,0), (0,1,0,0), (1,0,1,0), (0,1,1,0), (1,0,0,1), (0,1,0,1)\rangle$$

in $N \otimes_{\mathbb{Z}} \mathbb{R}$, where $N = \mathbb{Z}^4$, or by computing the blow-ups directly in local coordinates. For the first blow-up, the fiber over the origin is the union of two copies of $\mathbb{P}^2$ along a line, and smooth quadric surfaces over all other points of the center of the blow-up. All the fibers of the second blow-up (over the center of blow-up) are smooth quadric surfaces.

We recall that a proper morphism of finite-type schemes over a field k is said to be universally CH_0-trivial if the induced push-forward morphism on the groups CH_0 of zero-cycles up to rational equivalence is an isomorphism, not only over the given field but also after base-change to an arbitrary extension field; a proper scheme is said to be universally CH_0-trivial if the structure morphism to $\mathrm{Spec}(k)$ is. A sufficient condition for a proper morphism to be universally CH_0-trivial is that its fibers over all points (closed or not) are universally CH_0-trivial [14, Prop. 1.8].

Theorem 1. *Let k be an algebraically closed field of characteristic different from 2, S a smooth projective surface over k, and $\pi\colon X \to S$ a simple involution surface bundle, with singular fibers over*

$$D = D_1 \cup D_2 \cup D_3 \cup D_4.$$

Then by

- *blowing up X along the copy of the normalization of D_2, which is the closure in X of the singular locus of $\pi^{-1}(S \smallsetminus D^{\mathrm{sing}})$:*
$$\varphi\colon X' \to X,$$
- *blowing up X' along its singular locus, which consists of two disjoint curves in the fiber over every point of D_2^{sing}:*
$$\varphi'\colon X'' \to X',$$

we obtain $\varphi \circ \varphi'$, a universally CH_0-trivial desingularization $X'' \to X$.

Proof. As indicated in [25, Defn. 5], the only singularities of X over the complement of D^{sing} are double point singularities along the section over $D_2 \smallsetminus D^{\mathrm{sing}}$. Over D^{sing} the singularities of X are described in [25, §3], and by these descriptions, the closure in X of the indicated section is isomorphic to the normalization of D_2. We recall the description.

- I meets II: X has only ordinary double point singularities along the copy of D_2.
- II meets II: over a Zariski neighborhood of a point $z \in D_2^{\mathrm{sing}}$,
  - the copies in X of the two components of D_2 containing z intersect $\pi^{-1}(z)$ at distinct points z' and z'';
  - X has ordinary double point singularities generically along the two components and two additional curves in $\pi^{-1}(z)$, both containing z' and z'';
  - the étale local isomorphism type of the singularities of X at z' and z'' is that of the hypersurface singularity $uv = xyz$.
- III meets II and IV meets II: X has singularity of type D_∞ along the copy of D_2.

Ordinary double point singularities along a curve in X are resolved by blowing up the curve. As indicated in [39], the same holds for singularities of type D_∞. Over a neighborhood of a singular point of D_2, blowing up the copy of the normalization of D_2 amounts to the first step of the indicated resolution of the hypersurface singularity $uv = xyz$, and the remaining singularities, ordinary double points along curves, are resolved by blowing up those curves. With each blow-up, the fiber over any point is a union of two copies of $\mathbb{P}^2$ along $\mathbb{P}^1$ over k, a nodal quadric surface over k, or a nonsingular quadric surface over k or over the function field of a curve over k. Each of these is universally CH_0-trivial. □

4. Brauer group computation

Let Y be an algebraic variety over k. For an extension field F/k we let $Y(F)$ denote the set of F-rational points and Y_F the base-change to F of Y.

Let $L = k(Y)$ be the function field of Y. We let $\mathcal{V}_L$ denote the set of (geometric) divisorial valuations of L; in particular any $v \in \mathcal{V}_L$ is discrete of rank one and is trivial on k. If Y is normal, we let $\mathcal{V}_Y \subset \mathcal{V}_L$ denote the subset of divisorial valuations whose centers on Y are irreducible divisors. For $v \in \mathcal{V}_L$ we write $\mathfrak{o}_v$ for the corresponding local ring and κ_v for the residue field. We denote the henselization by $\mathfrak{o}_v^h$ and its field of fractions by K_v^h.

For a positive integer ℓ invertible in k we fix an isomorphism $\mu_\ell \simeq \mathbb{Z}/\ell\mathbb{Z}$. We write

$$H^i(Y) := H^i_{et}(Y, \mathbb{Z}/\ell\mathbb{Z}), \qquad H^i(L) := H^i(\mathrm{Spec}(L)),$$

when the coefficients are clear from the context. For every $v \in \mathcal{V}_L$ we have residue homomorphisms

$$H^i(L) \xrightarrow{\partial_v} H^{i-1}(\kappa_v).$$

The unramified cohomology of Y is an invariant of its function field L as an extension of k, defined by

$$H^i_{nr}(L/k) := \bigcap_{v \in \mathcal{V}_L} \ker(\partial_v),$$

see [12]. When the base field is clear from context, we will write $H^i_{nr}(L)$. When Y is smooth and projective, we also have

$$H^i_{nr}(L) = \bigcap_{v \in \mathcal{V}_Y} \ker(\partial_v),$$

with isomorphisms

$$H^1(Y) \cong H^1_{nr}(L) \qquad \text{and} \qquad \mathrm{Br}(Y)[\ell] \cong H^2_{nr}(L).$$

Let K be a field and

$$G_K = \mathrm{Gal}(\overline{K}/K)$$

the Galois group of a separable closure $\overline{K}$ of K.

Proposition 2. *Let K be a field of characteristic different from 2 and W an involution surface over K with discriminant extension L/K and Brauer group element $\beta \in \mathrm{Br}(L)$. The restriction map*

$$\mathrm{Br}(K) \to \mathrm{Br}(W),$$

is surjective, with kernel

$$\begin{cases} \langle \mathrm{cores}_{L/K}(\beta) \rangle, & \text{if } L \text{ is a field}, \\ \langle \beta_1, \beta_2 \rangle, & \text{if } L \cong K \times K,\ \beta = (\beta_1, \beta_2) \in \mathrm{Br}(K) \times \mathrm{Br}(K). \end{cases}$$

Proof. Since $\mathrm{Br}(W_{\overline{K}}) = 0$, the Hochschild-Serre spectral sequence

$$H^p(G_K, H^q(W_{\overline{K}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(W, \mathbb{G}_m)$$

gives rise to the exact sequence

$$0 \to \mathrm{Pic}(W) \to \mathrm{Pic}(W_{\overline{K}})^{G_K} \to \mathrm{Br}(K) \to \mathrm{Br}(W) \to H^1(G_K, \mathrm{Pic}(W_{\overline{K}})),$$

The Galois group G_K acts on $\mathrm{Pic}(W_{\overline{K}}) \cong \mathbb{Z}^2$ via the permutation action on rulings when L is a field, and trivially when $L \cong K \times K$. In either case, the Galois cohomology H^1 vanishes, and the surjectivity of the restriction map follows. The description of the kernel is given in [11, Prop. 5.3]. □

Proposition 3. *Let k be an algebraically closed field of characteristic different from 2, S a nonsingular algebraic variety over k, and*

$$\pi \colon X \to S$$

a mildly degenerating simple involution surface bundle, smooth over $U \subset S$, with degenerate fibers over

$$D = D_1 \cup D_2 \cup D_3 \cup D_4.$$

With $K = k(S)$ and $W = X_K$ we adopt the further notation of Proposition 2. For $v \in \mathcal{V}_S$ and coefficients $\mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a positive integer, invertible in k, the restriction map

$$\rho_v \colon H^1(\kappa_v) \to \bigoplus_{\substack{w \in \mathcal{V}_X \\ w|_K = v}} H^1(\kappa_w) \tag{4.1}$$

is injective when ℓ is odd and has kernel

$$\begin{cases} 0, & \text{if } v \in U \text{ or } v \in D_1, \\ \langle \partial_v(\mathrm{cores}_{L/K}(\beta))\rangle, & \text{if } v \in D_2 \text{ and } v \text{ is inert in } L, \\ \langle \partial_{v_1}(\beta), \partial_{v_2}(\beta)\rangle, & \text{if } v \in D_2, \text{ extending to distinct } v_1,\, v_2 \in \mathcal{V}_L, \\ \langle \partial_{v'}(\beta)\rangle, & \text{if } v \in D_3 \text{ with unique extension } v' \in \mathcal{V}_L, \\ \langle \partial_\varepsilon(\beta)\rangle, & \text{if } v \in D_4, \text{ marked by } \varepsilon \text{ over } v, \end{cases}$$

when ℓ is even.

By abuse of notation, in case $L \cong K \times K$ we consider v as split in L, with $\partial_{v_i}(\beta) = \partial_v(\beta_i)$ for $i = 1, 2$. Every component of a Type IV divisor has a *marking* $\varepsilon \in \mathcal{V}_L$ extending v, with the property that β extends to an element of the Brauer group of $\mathrm{Spec}(\mathfrak{o}'_v) \smallsetminus \{\varepsilon\}$, where $\mathfrak{o}'_v$ denotes the integral closure of $\mathfrak{o}_v$ in L. We will employ the notation $\mathcal{V}_T$ analogously when $T \cong S \sqcup S$.

Proof. We proceed via a case-by-case analysis:

- $v \in U$ or $v \in D_1$. Then X_{κ_v} is geometrically integral, hence the kernel is trivial.
- $v \in D_2$ and v is inert, extending uniquely to a valuation v' on L. Then, according to the description of mildly degenerating simple involution surface bundles from [25], X_{κ_v} is the restriction of scalars under $\kappa_{v'}/\kappa_v$ of the singular conic in $\mathbb{P}^2_{\kappa_{v'}}$, defined by an equation of the form
$$X^2 - rY^2 = 0,$$
where $r \in \kappa_{v'}^\times$ is a representative of
$$\partial_{v'}(\beta) \in H^1(\kappa_{v'}, \mathbb{Z}/2\mathbb{Z}) \cong \kappa_{v'}^\times/\kappa_{v'}^{\times 2}.$$
The conic contains a dense open subscheme isomorphic to $\mathbb{A}^1_{\kappa_{v'}(\sqrt{r})}$, hence a dense open subscheme of X_{κ_v} is isomorphic to $\mathbb{A}^2_\Lambda$, where Λ is the coordinate ring of the restriction of scalars under $\kappa_{v'}/\kappa_v$ of $\mathrm{Spec}(\kappa_{v'}(\sqrt{r}))$.
  - If $r \in \kappa_v$, then
  $$\Lambda \cong \kappa_v(\sqrt{r}) \times \kappa_v(\sqrt{rs}),$$
  where $s \in \kappa_v$ is such that $\kappa_{v'} \cong \kappa_v(\sqrt{s})$.
  - If $r \notin \kappa_v$ but $N_{\kappa_{v'}/\kappa_v}(r) = c^2 \in \kappa_v^{\times 2}$, then
  $$\Lambda \cong \kappa\big(\sqrt{\mathrm{tr}_{\kappa_{v'}/\kappa_v}(r) + 2c}\big) \times \kappa\big(\sqrt{\mathrm{tr}_{\kappa_{v'}/\kappa_v}(r) - 2c}\big),$$
  where we observe that
  $$(\mathrm{tr}_{\kappa_{v'}/\kappa_v}(r) + 2c)(\mathrm{tr}_{\kappa_{v'}/\kappa_v}(r) - 2c)$$

is equal to a square times s.

So, in these two cases, the kernel is trivial. (The kernel is also trivial when $\partial_{v'}(\beta)=0$, $\Lambda\cong\kappa_v\times\kappa_v\times\kappa_{v'}$.)

– If $r\notin\kappa_v$ and $N_{\kappa_{v'}/\kappa_v}(r)\notin\kappa_v^{\times 2}$, then Λ is a quadratic extension of $\kappa_v(\sqrt{N_{\kappa_{v'}/\kappa_v}(r)})$ which as extension of κ_v is either cyclic or non-Galois. So the kernel (when ℓ is even) is

$$\langle N_{\kappa_{v'}/\kappa_v}(r)\rangle=\langle\partial_v(\mathrm{cores}_{L/K}(\beta))\rangle.$$

- $v\in D_2$ and v is split. Then X_{κ_v} is a product of singular conics. We leave details of this case to the reader.
- $v\in D_3$. Then the construction of mildly degenerating involution surface bundles given in [25, Thm. 6] (out of a conic bundle corresponding to the Brauer class β) leads to a description of a dense open subscheme of X_{κ_v} as $\mathbb{A}^2_{\kappa_v(\sqrt{r})}$ where $r\in\kappa_v^\times=\kappa_{v'}^\times$ is a representative of $\partial_{v'}(\beta)$ (or two copies of $\mathbb{A}^2_{\kappa_v}$ when $\partial_{v'}(\beta)=0$).
- $v\in D_4$. Then $X_{\kappa(v)}$ is a product of a singular conic and a nonsingular conic, and the kernel is as claimed. □

Suppose, now, S is a nonsingular projective surface over k, and X is a simple involution surface bundle over S. We are interested in knowing when $\alpha\in\mathrm{Br}(K)[\ell]$ (with ℓ invertible in k) restricts under π to an element of $\mathrm{Br}(X_K)[\ell]$ that is unramified, i.e., has trivial residue for all valuations in $\mathcal{V}_{k(X)}$, or equivalently, for all valuations in $\mathcal{V}_{\widetilde{X}}$, where $\widetilde{X}$ is a desingularization of X (see [10, Thm. 4.1.1]). By Proposition 3, a *necessary* condition for this is that $\partial_v(\alpha)$ should belong to the kernel of the map ρ_v in (4.1), for all $v\in\mathcal{V}_S$. Indeed, there is a commutative diagram

$$\begin{array}{ccc}\mathrm{Br}(k(X))[\ell] & \xrightarrow{\partial_w} & H^1(\kappa_w)\\ {\scriptstyle\rho}\uparrow & & \uparrow\\ \mathrm{Br}(K)[\ell] & \xrightarrow{\partial_v} & H^1(\kappa_v)\end{array}$$

as in [12, §1], where the vertical maps are restriction maps and the factor coming from the valuation under w of a uniformizer of v is always 1, since π is smooth outside of a locus of codimension at least 2. The next result shows that this condition is also sufficient.

Proposition 4. *Let k be an algebraically closed field of characteristic different from 2, S a nonsingular surface over k with function field K, and $\pi\colon X\to S$ a simple involution surface bundle. Let $\alpha\in\mathrm{Br}(K)[\ell]$, with ℓ invertible in k, be an element such that*

$$\partial_v(\alpha)\in\ker(\rho_v),\qquad \textit{for all } v\in\mathcal{V}_S.$$

Then, for every $v \in \mathcal{V}_S$ with $\partial_v(\alpha) \neq 0$, we have

$$\alpha \in \ker(\mathrm{Br}(K)[\ell] \to \mathrm{Br}(X_{K_v^h})[\ell]),$$

and, for every $z \in D^{\mathrm{sing}}$, we have

$$\alpha \in \ker(\mathrm{Br}(K)[\ell] \to \mathrm{Br}(X_{K_z^h})[\ell]),$$

where K_z^h denotes the fraction field of the henselization $\mathfrak{o}_z^h$ of the local ring at z.

Proof. Suppose first that v is in D_2 and is inert in L. Then ℓ is even, and $\partial_v(\alpha) = \partial_v(\mathrm{cores}_{L/K}(\beta))$. Since

$$\mathrm{cores}_{L/K}(\beta) \in \ker(\mathrm{Br}(K) \to \mathrm{Br}(X_K)),$$

it suffices to show that

$$\alpha - \mathrm{cores}_{L/K}(\beta) \in \ker(\mathrm{Br}(K)[\ell] \to \mathrm{Br}(K_v^h)[\ell]).$$

By the equality of residues, $\alpha - \mathrm{cores}_{L/K}(\beta)$ is the restriction of an element of $\mathrm{Br}(\mathfrak{o}_v^h)$. But $\mathrm{Br}(\mathfrak{o}_v^h) = 0$, since κ_v is a C_1-field (Tsen's theorem). The same argument takes care of the cases $v \in D_3$ and $v \in D_4$.

It remains to treat the case that v in D_2 is split in L, and the case of $z \in D^{\mathrm{sing}}$. In this case, we have $L \otimes_K K_v^h \cong K_v^h \times K_v^h$. We are then reduced to the case $L \cong K \times K$, and we may argue as above, using

$$\partial_v(\alpha) \in \{\partial_v(\beta_1), \partial_v(\beta_2), \partial_v(\beta_1 + \beta_2)\}.$$

For $z \in D^{\mathrm{sing}}$, either α vanishes on an étale neighborhood of z, in which case the assertion is trivial, or else after passing to a suitable étale neighborhood we have $\alpha = (x, y)$ where x and y are local defining equations of the components of D containing z. We divide into subcases according to the étale local isomorphism type of X, using the notation from [25] and noting that cases $\widehat{X}_{I,II}$ and $\widehat{X}''_{IV,IV}$ are trivial for the above reason. In all of the remaining cases, except $\widehat{X}_{III,II}$, we may assume $L \cong K \times K$ and obtain, by Proposition 3 (applied after base change to a suitable étale neighborhood) kernel generated by (x, y). In case $\widehat{X}_{III,II}$ we adopt the notation of [25, §3.3]: x is a local defining equation of D_3, y of D_2, and $L = K(s)$ where $s^2 = x$. Now [15, 18.8.10] is applicable to the branched degree 2 covering of S and tells us that the étale local form (s, y) of $\beta \in \mathrm{Br}(L)$ (which we have by the same argument as above) is achieved after passing to a suitable étale neighborhood of z in S. This corestricts to (x, y), and we conclude as before. $\square$

Corollary 5. *Let k be an algebraically closed field of characteristic different from 2, S a nonsingular projective surface over k with function field K, and $\pi \colon X \to S$ be a simple involution surface bundle. Let $\alpha \in \mathrm{Br}(K)[\ell]$, with ℓ invertible in k. Then the following are equivalent:*

(i) $\partial_v(\alpha) \in \ker(\rho_v)$, *for every* $v \in \mathcal{V}_S$*;*
(ii) $\rho(\alpha) \in H^2_{nr}(k(X)/k)$, *i.e., if* $\widetilde{X}$ *denotes any desingularization of* X *then* $\rho(\alpha)$ *is the restriction of an element of* $\mathrm{Br}(\widetilde{X})[\ell]$.

Proof. By Proposition 3, (ii) implies (i). Now suppose (i) is satisfied. We need to show that for any $w \in \mathcal{V}_{k(X)}$ we have $\partial_w(\alpha) = 0$. Since $\rho(\alpha) \in \mathrm{Br}(X_K)[\ell]$, the residue is trivial for all valuations that restrict to the trivial valuation on K. So we consider only valuations w restricting nontrivially to K.

Suppose, first, that w restricts to some $v \in \mathcal{V}_S$. By Proposition 4, there exist an étale morphism $S' \to S$ and $v' \in \mathcal{V}_{S'}$ extending v and inducing an isomorphism on residue fields, such that

$$\alpha \in \ker(\mathrm{Br}(K)[\ell] \to \mathrm{Br}(X_{K'})[\ell]), \tag{4.2}$$

where $K' = k(S')$. Since the residue homomorphism commutes with restriction under an étale morphism, we have $\partial_w(\alpha) = 0$.

It remains to consider the case that the restriction of w to K is centered on some k-point $z \in S$. We will show that there exists a pointed étale neighborhood (S', z') of (S, z) for which (4.2) holds, where $K' = k(S')$. As before, the vanishing of ∂_w follows. If $z \notin D^{\mathrm{sing}}$ then α restricts to 0 in $\mathrm{Br}(K')[\ell]$ for some étale neighborhood. If $z \in D^{\mathrm{sing}}$, then we are done by Proposition 4. $\square$

In case S is a nonsingular projective *rational* surface, $\mathrm{Br}(S) = 0$ and elements of $\mathrm{Br}(K)[\ell]$ are described completely with ramification data, according to the exact sequence from [1, Thm. 1]:

$$0 \to \mathrm{Br}(K)[\ell] \to \bigoplus_{v \in \mathcal{V}_S} H^1(\kappa_v) \to \bigoplus_{z \in S(k)} \mathbb{Z}/\ell\mathbb{Z}. \tag{4.3}$$

In particular, in this setting, condition (i) in Corollary 5 forces α to be 2-torsion in $\mathrm{Br}(K)$. In light of this, we take $\ell = 2$ and work with coefficients $\mathbb{Z}/2\mathbb{Z}$ in the following concrete description of the unramified Brauer group of an involution surface bundle over a projective rational surface.

Theorem 6. *Let* k *be an algebraically closed field of characteristic different from* 2*,* S *a nonsingular projective rational surface over* k *with function field* K*, and* $\pi\colon X \to S$ *a simple involution surface bundle, such that the associated conic bundle under the correspondence of* [25, Thm. 13] *is a standard conic bundle over a degree* 2 *covering* $T \to S$*. Define*

- $L = k(T)$*, when* T *is irreducible, and*
- $L = K \times K$*, when* $T = S \sqcup S$*,*

and let $\beta \in \mathrm{Br}(L)[2]$ be the class of the standard conic bundle over T. Define

$$\mathcal{S} = \begin{cases} 0, & \text{if } L \text{ is a field, } \mathrm{cores}_{L/K}(\beta) = 0, \\ \mathbb{F}_2, & \text{if } L \text{ is a field, } \mathrm{cores}_{L/K}(\beta) \neq 0, \\ \bigoplus_{\substack{i=1 \\ \beta_i \neq 0 \\ \text{or } \beta_1 \neq \beta_2}} \mathbb{F}_2, & \text{if } L = K \sqcup K,\ \beta = (\beta_1, \beta_2) \in \mathrm{Br}(K) \times \mathrm{Br}(K), \end{cases}$$

$$\mathcal{P} = \bigoplus_{v' \in \mathcal{V}_{\mathcal{P}}} \mathbb{F}_2,\ \mathcal{V}_{\mathcal{P}} = \{v' \in \mathcal{V}_T \mid \partial_{v'}(\beta) \neq 0\},$$

$$\mathcal{Q} = \bigoplus_{v \in \mathcal{V}_{\mathcal{Q}}} \mathbb{F}_2,\ \mathcal{V}_{\mathcal{Q}} = \{v \in \mathcal{V}_S \mid \partial_v(\mathrm{cores}_{L/K}(\beta)) = 0,\ \exists\, v' \in \mathcal{V}_{\mathcal{P}}\colon v'|_K = v\},$$

$$\mathcal{R} = \bigoplus_{z \in \mathcal{Z}_{\mathcal{R}}} \mathbb{F}_2,\ \mathcal{Z}_{\mathcal{R}} = \{z \in D^{\mathrm{sing}} \mid \text{type } \widehat{X}_{II,II},\ \widehat{X}_{III,II},\ \widehat{X}_{IV,II}, \text{ or } \widehat{X}'_{IV,IV}\},$$

where D denotes the simple normal crossing divisor over which π has singular fibers. We define homomorphisms

- *$\mathcal{S} \to \mathcal{P}$ as the diagonal inclusion when L is a field, and the product of diagonal inclusions according to the convention of $\mathcal{V}_T$ stated after Proposition 3, otherwise;*
- *$\mathcal{Q} \to \mathcal{P}$ by the relation of extension of valuations, and*
- *$\mathcal{P} \to \mathcal{R}$ by the relation in Table 1.*

Then

- *$\mathcal{S} \to \mathcal{P}$ and $\mathcal{Q} \to \mathcal{P}$ are injective with trivially intersecting images,*
- *the composite $\mathcal{Q} \to \mathcal{R}$ is zero, and*
- *the recipe of Table 2 identifies $\mathcal{S}$ with $\ker(\mathrm{Br}(K) \to \mathrm{Br}(X_K))$ and $\ker(\mathcal{P}/\mathcal{Q} \to \mathcal{R})$ with the pre-image of the subgroup $H^2_{nr}(k(X)/k)$.*

The elements of $\mathcal{V}_{\mathcal{P}}$ correspond to the components of the pre-image of D_2, the components of D_3, and the marked components over D_4. The last statement of the theorem is summarized by the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, equal] & \ker(\mathcal{P}/\mathcal{Q} \to \mathcal{R}) \arrow[r] \arrow[d, hook] & H^2_{nr}(k(X)/k) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{S} \arrow[r] & \mathrm{Br}(K) \arrow[r] & \mathrm{Br}(X_K) \arrow[r] & 0 \end{tikzcd}$$

Remark 7. We remark that the construction in [25, Thm. 10] of good models of involution surface bundles (i.e., models which are simple involution surface bundles) proceeds via a standard conic bundle over a degree 2 covering of a birational model of the base surface, and hence

	$v'\|_K = v$ Type II	v Type III	v Type IV
z Type $\widehat{X}_{II,II}$	s		
z Type $\widehat{X}_{III,II}$	$\times$	$\times$	
z Type $\widehat{X}_{IV,II}$	m		$\times$
z Type $\widehat{X}'_{IV,IV}$			$\times$

TABLE 1. Relation between $\mathcal{V}_{\mathcal{P}}$ and $\mathcal{Z}_{\mathcal{R}}$. For $v' \in \mathcal{V}_{\mathcal{P}}$, restricting to $v \in \mathcal{V}_S$ corresponding to a divisor containing $z \in \mathcal{Z}_{\mathcal{R}}$ the symbol $\times$ indicates that v' is related to z; s indicates that v' is related to z when v is split in L; m indicates that v' is related to z when the divisor corresponding to v' meets the marked Type IV component at a point above z.

	$v'\|_K = v$ Type II	v Type III	v Type IV
inert	$\partial_v(\mathrm{cores}_{L/K}(\beta))$		
split	$\partial_{v'}(\beta)$		$\partial_{v'}(\beta)$
ramified		$\partial_{v'}(\beta)$	

TABLE 2. Homomorphism $\mathcal{P} \to \bigoplus_{v \in \mathcal{V}_{\mathcal{Q}}} H^1(\kappa_v)$ determining $\ker(\mathcal{P} \to \mathcal{R}) \to \mathrm{Br}(K)[2]$ by the representation of an element of $\mathrm{Br}(K)[2]$ by ramification data in $\bigoplus_{v \in \mathcal{V}_S} H^1(\kappa_v)$.

these particular simple involution surface bundles satisfy the condition stated in Theorem 6.

Proof of Theorem 6. We use the exact sequence (4.3), which identifies $\mathrm{Br}(K)[2]$ with ramification data at divisors satisfying compatibility conditions at points. The assertions about $\mathcal{S} \to \mathcal{P}$, $\mathcal{Q} \to \mathcal{P}$, and $\mathcal{Q} \to \mathcal{R}$ are readily verified. By Corollary 5, the pre-image of $H^2_{nr}(k(X)/k)$ under $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ consists of elements whose ramification data are constrained to lie in the kernels described in Proposition 3. The direct sum of these, we check, is identified with $\mathcal{P}/\mathcal{Q}$ by the homomorphism described in Table 2. We check, as well, that the homomorphism encoded by Table 1 corresponds to the compatibility conditions at points from (4.3). Finally, $\mathcal{S}$ is identified with the kernel of $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ from Proposition 2. $\square$

5. Example

Here we demonstrate Theorem 6 on the example from [19]:

$$X: \quad yzs^2 + xzt^2 + xyu^2 + (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0.$$

This is a hypersurface in $\mathbb{P}^3 \times \mathbb{P}^2$, where the respective factors have homogeneous coordinates s, t, u, v and x, y, z, and is a quadric surface bundle over $S = \mathbb{P}^2$ with discriminant extension given by the degree 2 covering branched over

$$C: \quad x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 0.$$

The fourfold X appears (modulo birational transformations) as the limit of several interesting families of varieties [19], [20], [18], [3], [33]; higher-dimensional variants are also in [35, Sect. 3]. The presence of nontrivial unramified cohomology in X, together with a verification of CH_0-triviality of a resolution of singularities of X, show that very general members of those families fail stable rationality.

We write the double cover $T \to S$ as

$$T: \quad w^2 = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz,$$

a nonsingular quadric surface. The Brauer group element is

$$\beta = (xz^{-1}, yz^{-1}) \in \mathrm{Br}(\mathbb{C}(T)).$$

The quadric surface bundle is, however, not a simple involution surface bundle. Indeed, the fibers generically along any coordinate line in $\mathbb{P}^2$ are not of any of the four permitted degeneration types.

According to the construction in [25, Thm. 10] of a simple involution surface bundle, we should blow up S as needed so that the locus

$$D_{xyz} \cup D_{xzy} \cup D_{yxz} \cup D_{yzx} \cup D_{zxy} \cup D_{zyx}, \quad D_{xyz} : x = 0, w = y - z, \text{ etc.}$$

on T where β is ramified is a simple normal crossing divisor, which additionally has normal crossings with the ramification locus

$$w = 0$$

of $T \to S$. The first of these conditions is satisfied, but the additional condition fails since pairs of divisors such as D_{xyz} and D_{xzy} intersect at points with $w = 0$.

Blowing up S at the points $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : 1 : 0)$ yields exceptional divisors D_x, D_y, D_z. When we do this, T transforms to a singular surface, whose resolution requires blowing at 3 more points to obtain $\widetilde{S}$ with 3 more exceptional divisors E_x, E_y, E_z. The degree 2 cover $\widetilde{T}$ is nonsingular, with covering map

$$\tilde{\psi}\colon \widetilde{T} \to \widetilde{S}$$

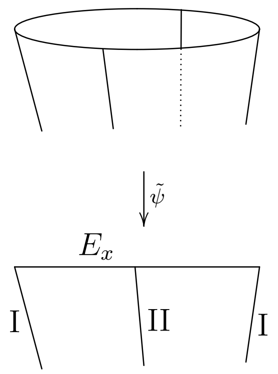

FIGURE 1. Graphical representation of covering $\tilde{\psi}\colon \widetilde{T} \to \widetilde{S}$ near E_x, which meets C' and D'_x (both of Type I) and the proper transform of the coordinate axis $x = 0$ (Type II).

branched over

$$C' \cup D'_x \cup D'_y \cup D'_z,$$

where primes denote proper transforms. The locus on $\widetilde{T}$ where β ramifies is

$$D'_{xyz} \cup D'_{xzy} \cup D'_{yxz} \cup D'_{yzx} \cup D'_{zxy} \cup D'_{zyx}. \tag{5.1}$$

On $\widetilde{S}$, the Type I locus is $C' \cup D'_x \cup D'_y \cup D'_z$, and the Type II locus consists of the proper transforms of the coordinate axes, which all split in $\widetilde{T}$; see Figure 1.

Since every intersection of components in (5.1) is a branch point of the covers which describe the ramification of β, there exist standard conic bundles over $\widetilde{T}$ with Brauer class β and, correspondingly, simple involution surface bundles over $\widetilde{S}$. We may use any such involution surface bundle for the computation of the unramified Brauer group of X via Theorem 6.

We have

$$\mathcal{S} = 0, \qquad \mathcal{P} = \mathbb{F}_2^6, \qquad \mathcal{Q} = \mathbb{F}_2^3, \qquad \mathcal{R} = \mathbb{F}_2^3.$$

Ordering the basis of $\mathcal{P}$ as in (5.1), we have image of $\mathcal{Q}$ spanned by

$$(1,1,0,0,0,0), \qquad (0,0,1,1,0,0), \qquad (0,0,0,0,1,1),$$

and matrix representation

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

of $\mathcal{P} \to \mathcal{R}$, with basis $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ of $\mathcal{R}$.

Now

$$\ker(\mathcal{P}/\mathcal{Q} \to \mathcal{R}) \cong \mathbb{F}_2,$$

generated by, e.g., $(1,0,1,0,1,0)$, which corresponds to a Brauer group element that is ramified along each of the three coordinate axes:

$$(xz^{-1}, yz^{-1}) \in \mathrm{Br}(\mathbb{C}(S)).$$

Key to this example is the presence of components of the Type II locus that split in the double cover. If we start with a quadric surface bundle and hence $\beta \in \mathbb{C}(T)$ with $\mathrm{cores}_{L/K}(\beta) = 0$, then $\mathcal{P}/\mathcal{Q} = 0$ unless some Type II component splits. It is not essential, however, to have singular Type II locus. For instance, there exist nonsingular cubic and quartic curves in $S = \mathbb{P}^2$ that meet with tangency at 6 points lying on a conic. If we let the quartic curve determine T, then the pre-image in T of the cubic curve has two irreducible components. We take $\beta \in \mathrm{Br}(\mathbb{C}(T))$ to be the restriction of the class in $\mathrm{Br}(\mathbb{C}(S))$ determined by a nontrivial unramified degree 2 cover of the cubic curve and $X \to S$ a corresponding quadric surface bundle. In a manner analogous to that described above, $\widetilde{S}$ and $\widetilde{T}$ may be obtained by blowing up 6 points on S and again 6 points. Over $\widetilde{S}$ there is a model of X which is a simple involution surface bundle with disjoint smooth Type I and II loci. Theorem 6 yields $H^2_{nr}(\mathbb{C}(X)/\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

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