STABLE RATIONALITY OF BRAUER-SEVERI SURFACE BUNDLES

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Abstract. For sufficiently ample linear systems on rational surfaces we show that a very general associated Brauer-Severi surface bundle is not stably rational.

1. Introduction

This paper extends the study of stable rationality of conic bundles over rational surfaces in [13] to the case of Brauer-Severi surface bundles. Our main result is:

Theorem 1. *Let k be an uncountable algebraically closed field of characteristic different from 3, S a rational smooth projective surface over k, and L a very ample line bundle on S such that the complete linear system $|L|$ contains a nodal reducible curve $D = D_1 \cup D_2$, where D_1 and D_2 are smooth of positive genus, and contains a curve with E_6-singularity. In case k has positive characteristic, we suppose further that $H^1(S, L) = 0$ and some curve in $|L|$ with E_6-singularity may be lifted, with S and L, to a curve with E_6-singularity in characteristic zero. Then the Brauer-Severi surface bundle corresponding to a very general element of $|L|$ with nontrivial unramified cyclic degree 3 cover is not stably rational.*

This is applicable, for instance, to the complete linear system of degree d curves in $\mathbb{P}^2$ for $d \geq 6$.

The proof of Theorem 1 relies on the construction of good models of Brauer-Severi surface bundles in [15]. A new ingredient is a variant of the standard elementary transformation of vector bundles. This is needed to apply the specialization method, which was introduced by Voisin [21] and developed further in [8], [19], [14] and which tells us that in a family where one (mildly singular) member has an obstruction to stable rationality, the very general member fails to be stably rational. In our case, the family is a family of Brauer-Severi surface bundles, where one member has nontrivial 3-torsion in its unramified Brauer group.

Date: November 2, 2018.

In Section 2 we recall some facts on Brauer groups, and in Section 3 we describe the variant of the standard elementary transformation that will be used in the proof of Theorem 1, which occupies Section 4.

Acknowledgments: We are grateful to Asher Auel for helpful comments. The first author is partially supported by the Swiss National Science Foundation. The second author is partially supported by NSF grant 1601912.

2. Basic facts

Throughout, all cohomology groups are étale cohomology groups. We recall that the Brauer group of a Noetherian scheme S is defined as the torsion subgroup of $H^2(S,\mathbb{G}_m)$ [10]. The same definition extends to Noetherian Deligne-Mumford stacks.

In this section, we work over an algebraically closed field k of characteristic different from 3. We start with two basic facts:

Proposition 2 ([5])**.** *Let S be a smooth surface over k that is* (i) *projective and rational, or* (ii) *quasiprojective. Then there are residue maps fitting in an exact sequence*

$$0\to \mathrm{Br}(K)[3]\to \bigoplus_{\xi\in S^{(1)}} H^1(k(\xi),\mathbb{Z}/3\mathbb{Z})\to \bigoplus_{\xi\in S^{(2)}}\mathbb{Z}/3\mathbb{Z}\qquad \textit{in case (i)},$$

$$0\to \mathrm{Br}(S)[3]\to \mathrm{Br}(K)[3]\to \bigoplus_{\xi\in S^{(1)}} H^1(k(\xi),\mathbb{Z}/3\mathbb{Z})\qquad \textit{in case (ii)}.$$

Here $K=k(S)$, and $S^{(i)}$ denotes the set of codimension i points of S.

Let S be a smooth variety over k. The *root stack* $\sqrt[3]{(S,D)}$ along an effective Cartier divisor D in S is a Deligne-Mumford stack, locally, for D defined by the vanishing of a regular function f on an affine chart $\mathrm{Spec}(A)$ of S, isomorphic to the stack quotient

$$[\mathrm{Spec}(A[t]/(t^3-f))/\mu_3],$$

where the roots of unity μ_3 act by scalar multiplication on t; cf. [6, §2], [1, App. B]. There is a closed substack with morphism to D known as the *gerbe of the root stack* and given Zariski locally as

$$[\mathrm{Spec}(A[t]/(t,f))/\mu_3].$$

This is a gerbe since this μ_3 acts trivially, i.e.,

$$[\mathrm{Spec}(A[t]/(t,f))/\mu_3]\cong \mathrm{Spec}(A/(f))\times B\mu_3,$$

where $B\mu_3$ denotes the classifying stack of μ_3 (stack quotient of a point by the trivial action of μ_3). The complement of the gerbe of the root stack maps isomorphically to $S\smallsetminus D$.

A stack with the étale local structure of a product with $B\mu_3$ is a gerbe. Our gerbes are, furthermore, *banded by* μ_3, meaning that compatible étale local identifications with products with $B\mu_3$ are given. Isomorphism classes of gerbes, banded by μ_3, are classified by the second cohomology group with values in μ_3; cf. [13, §2.2].

The root stack is smooth when D is smooth, and singular when D is singular. For $D = D_1 \cup D_2$ as in Theorem 1, however, we may consider the *iterated root stack* [6, Def. 2.2.4]

$$\sqrt[3]{(S,\{D_1,D_2\})} := \sqrt[3]{(S,D_1)} \times_S \sqrt[3]{(S,D_2)}, \tag{1}$$

which is smooth with stabilizer group μ_3 over the smooth locus of D and $\mu_3 \times \mu_3$ over $D_1 \cap D_2$. Base change by the inclusion of the gerbe of the root stack $\sqrt[3]{(S,D_i)}$ leads to a closed substack of $\sqrt[3]{(S,\{D_1,D_2\})}$ with morphism to the pre-image of D_i in $\sqrt[3]{(S,D_{3-i})}$ which we call the *gerbe over the ith component*, for $i = 1, 2$:

$$\mathfrak{D}_i \to D_i \times_S \sqrt[3]{(S,D_{3-i})}.$$

Proposition 3 ([17])**.** *Let S be a smooth quasiprojective surface over k, D a curve on S that is either* (i) *smooth or* (ii) *nodal, consisting of two intersecting smooth components, and $U := S \smallsetminus D$. Then the restriction map induces an isomorphism*

$$\begin{aligned} &\mathrm{Br}\left(\sqrt[3]{(S,D)}\right)[3] \to \mathrm{Br}(U)[3] && \textit{in case (i)},\\ &\mathrm{Br}\left(\sqrt[3]{(S,\{D_1,D_2\})}\right)[3] \to \mathrm{Br}(U)[3] && \textit{in case (ii)}. \end{aligned}$$

In each case, nonzero elements of the indicated Brauer groups are represented by sheaves of Azumaya algebras of degree 3*.*

In case (ii) of Proposition 3, we have a morphism

$$\rho\colon \sqrt[3]{(S,\{D_1,D_2\})} \to \sqrt[3]{(S,D)}. \tag{2}$$

Let $\alpha \in \mathrm{Br}\left(\sqrt[3]{(S,\{D_1,D_2\})}\right)$ be the class of a sheaf of Azumaya algebras $\mathcal{A}$ of degree 3.

Assumption 4. The restriction of α to $\mathrm{Br}(U)$ does not extend across the generic point of D_1 or of D_2 in S.

To a sheaf of Azumaya algebras of degree 3 there is an associated PGL_3-torsor (see [10, §I.2]) and hence, attached to $x \in D$, a projective representation of μ_3, respectively $\mu_3 \times \mu_3$, when x is a smooth, respectively singular point of D.

Lemma 5. *With notation as above, let $x \in D_1 \cap D_2$ and let*

$$\mu_3 \times \mu_3 \to PGL_3 \tag{3}$$

be the projective representation associated with the restriction of $\mathcal{A}$ to the copy of the classifying stack $B(\mu_3\times\mu_3)$ in $\sqrt[3]{(S,\{D_1,D_2\})}$ over x, where the factors μ_3 correspond to the stabilizer along D_1 and along D_2. Then the restriction of (3) *to each factor μ_3 is balanced, i.e., is isomorphic to the projectivization of the sum of the three distinct one-dimensional linear representations of μ_3.*

Proof. It suffices to treat just the first factor μ_3. With the fiber product description (1) of the iterated root stack we have the projection morphism

$$p_2\colon \sqrt[3]{(S,\{D_1,D_2\})}\to\sqrt[3]{(S,D_2)}.$$

There is a criterion due to Alper [3, Thm. 10.3] for a vector bundle (e.g., the sheaf of Azumaya algebras $\mathcal{A}$) to descend via a morphism such as p_2. Specifically, Alper considers so-called good moduli spaces, e.g., the coarse moduli space of a finite-type separated Deligne-Mumford stack over k whose stabilizer groups have order not divisible by the characteristic of k. However, by reasoning étale locally, his criterion applies as well to a relative moduli space as in [2, §3]. Applied to p_2, this reveals that there exists a sheaf of Azumaya algebras $\mathcal{A}'$ on $\sqrt[3]{(S,D_2)}$ and an isomorphism $p_2^*\mathcal{A}'\cong\mathcal{A}$ if and only if the relative stabilizer of p_2 acts trivially on fibers of $\mathcal{A}$.

Now, and several times further below, we use the Kummer sequence

$$0\to\mu_3\to\mathbb{G}_m\to\mathbb{G}_m\to 0$$

and the corresponding long exact sequence of cohomology groups. We take

$$\alpha_0\in H^2(\sqrt[3]{(S,\{D_1,D_2\})},\mu_3)$$

to be a lift of the class

$$\alpha\in\mathrm{Br}\big(\sqrt[3]{(S,\{D_1,D_2\})}\big)[3].$$

To α_0 there is a corresponding gerbe

$$\mathfrak{G}\xrightarrow{\tau}\sqrt[3]{(S,\{D_1,D_2\})}$$

banded by μ_3. We have $\tau^*\alpha=0$, hence

$$\tau^*\mathcal{A}\cong\mathrm{End}(\mathcal{E})$$

for some rank 3 vector bundle $\mathcal{E}$ on $\mathfrak{G}$. The stabilizer group of $\mathfrak{G}$ is a central μ_3-extension G of $\mu_3\times\mu_3$:

$$1\to\mu_3\to G\to\mu_3\times\mu_3\to 1, \tag{4}$$

and by convention we take $\mathcal{E}$ so that the action of the central μ_3 is by scalar multplication.

The projective representation of the first factor μ_3 is induced by the linear representation of the subgroup of G, pre-image in (4) of $\mu_3 \times \{1\}$ in $\mu_3 \times \mu_3$. We suppose that this representation is not balanced. If this representation is trivial then the criterion mentioned above is applicable, and $\mathcal{A} \cong p_2^*\mathcal{A}'$ for some sheaf of Azumaya algebras $\mathcal{A}'$ on $\sqrt[3]{(S, D_2)}$. But then the restriction of α to $\mathrm{Br}(U)$ extends across the generic point of D_1, in contradiction to Assumption 4. A nontrivial unbalanced representation is the projectivization of a linear representation which is a sum of two copies of one and one copy of another one-dimensional linear representation of μ_3. Then the restriction of $\mathcal{E}$ to

$$\mathfrak{G} \times_{\sqrt[3]{(S,\{D_1,D_2\})}} \mathfrak{D}_1$$

splits canonically according to multiplicity as $\mathcal{E}_1 \oplus \mathcal{E}_2$. Let us denote by h the inclusion in $\mathfrak{G}$ of the above fiber product. Then we may form an exact sequence

$$0 \to \widetilde{\mathcal{E}}^{(j)} \to \mathcal{E} \to h_*\mathcal{E}_j \to 0 \tag{5}$$

for $j = 1$, 2, and consider the respective corresponding sheaf of Azumaya algebras $\widetilde{\mathcal{A}}^{(j)}$ on $\sqrt[3]{(S, \{D_1, D_2\})}$, which is characterized by $\tau^*\widetilde{\mathcal{A}}^{(j)} \cong \mathrm{End}(\widetilde{\mathcal{E}}^{(j)})$ and which exists since the generic stabilizer μ_3 of $\mathfrak{G}$ acts trivially on $\mathrm{End}(\widetilde{\mathcal{E}}^{(j)})$. Reasoning étale locally, we see that for appropriate j the sheaf of Azumaya algebras $\widetilde{\mathcal{A}}^{(j)}$ descends to $\sqrt[3]{(S, D_2)}$, and we have again reached a contradiction to Assumption 4. $\square$

Remark 6. The condition on the action of the central μ_3 in (4) makes the locally free coherent sheaf $\mathcal{E}$ on the gerbe $\mathfrak{G}$ in the proof of Lemma 5 a *twisted sheaf*; cf. [16]. Using the language of twisted sheaves the conclusion of Lemma 5 may be expressed in the terminology of Lieblich [18, Defn. 7.2]: the twisted sheaf $\mathcal{E}$ is *regular*.

Assumption 7. The restriction of α to $\mathrm{Br}(K)$ (where $K = k(S)$) is an element whose residue (image under the map to $H^1(k(\xi), \mathbb{Z}/3\mathbb{Z})$ in Proposition 2) at the generic point of D_i is the class of an *unramified* cyclic degree 3 cover $\widetilde{D}_i \to D_i$ for $i = 1$, 2.

We are interested in knowing whether $\mathcal{A}$ descends to $\sqrt[3]{(S, D)}$, i.e., is isomorphic to $\rho^*\mathcal{A}'$ for some sheaf of Azumaya algebras $\mathcal{A}'$ on $\sqrt[3]{(S, D)}$.

Lemma 8. *With notation and assumptions as above, let $x \in D_1 \cap D_2$. Then there exists an étale neighborhood $S' \to S$ of x such that α lies in the kernel of*

$$\mathrm{Br}(U) \to \mathrm{Br}(S' \times_S U).$$

Proof. We take $S' \to S$ trivializing the cyclic covers $\widetilde{D}_i \to D_i$ for $i = 1, 2$; to see that this exists we apply the fact, for $i = 1, 2$, that the unramified morphism $\widetilde{D}_i \to S$ factors locally as a closed immersion followed by an étale morphism [9, 18.4.8]. Application of Proposition 2 to S' shows that the pullback of α to $\mathrm{Br}(S' \times_S U)$ is the restriction of an element of $\mathrm{Br}(S')$. This is trivialized upon passage to a suitable further étale neighborhood of x. $\square$

Proposition 9. *With notation and assumptions as above, let $x \in D_1 \cap D_2$. Then the kernel of the projective representation* (3) *is a subgroup, isomorphic to μ_3, embedded either as the diagonal or the antidiagonal in $\mu_3 \times \mu_3$.*

Proof. By Lemma 8, with its notation, the pullback of α to

$$S' \times_S \sqrt[3]{(S, \{D_1, D_2\})} \tag{6}$$

vanishes, and hence the projective representation lifts to a linear representation, which is well-defined up to twist by a character of $\mu_3 \times \mu_3$ and hence may be written as trivial $\oplus\, \chi \oplus \chi'$, for some characters χ and χ' of $\mu_3 \times \mu_3$. By Lemma 5, the restriction of χ and χ' to the first factor μ_3 are nontrivial and opposite, and the same holds for the restrictions to the second factor μ_3.

Let χ_i for $i \in \{0, 1, 2\}$ denote the ith character of μ_3. Swapping χ and χ' if necessary, we may suppose that

$$\chi|_{\mu_3 \times \{1\}} = \chi_1 \quad \text{and} \quad \chi'|_{\mu_3 \times \{1\}} = \chi_2.$$

Now there are two possibilities. If

$$\chi|_{\{1\} \times \mu_3} = \chi_1 \quad \text{and} \quad \chi'|_{\{1\} \times \mu_3} = \chi_2,$$

then the kernel is the antidiagonal copy of μ_3. If

$$\chi|_{\{1\} \times \mu_3} = \chi_2 \quad \text{and} \quad \chi'|_{\{1\} \times \mu_3} = \chi_1,$$

then the kernel is the diagonal copy of μ_3. $\square$

Definition 10. In the two cases in the proof of Proposition 9, leading to antidiagonal μ_3 and diagonal μ_3, we say that the sheaf of Azumaya algebras $\mathcal{A}$ at x is *good*, respectively *bad*.

Proposition 11. *With notation and assumptions as above, the sheaf of Azumaya algebras $\mathcal{A}$ descends to $\sqrt[3]{(S, D)}$ if and only if $\mathcal{A}$ is good at every point of $D_1 \cap D_2$.*

Proof. The morphism ρ in (2) is a relative coarse moduli space. Indeed, if near $x \in D_1 \cap D_2$ in S we denote a defining equation of D_i by f_i for

$i = 1, 2$, then ρ has the local form

$$[\mathrm{Spec}(A[t_1,t_2]/(t_1^3 - f_1, t_2^3 - f_2))/\mu_3 \times \mu_3] \to [\mathrm{Spec}(A[t]/(t^3 - f_1 f_2))/\mu_3]$$

where $t = t_1 t_2$ and $\mu_3 \times \mu_3$ maps to μ_3 by multiplication. Letting $\tilde{\mu}_3$ denote the antidiagonal copy of μ_3 in $\mu_3 \times \mu_3$, we obtain

$$[\mathrm{Spec}(A[t_1,t_2]/(t_1^3 - f_1, t_2^3 - f_2))/\tilde{\mu}_3] \to \mathrm{Spec}(A[t]/(t^3 - f_1 f_2))$$

upon base change to an étale chart of $\sqrt[3]{(S,D)}$. Triviality of the action of $\tilde{\mu}_3$ is thus necessary and sufficient for the descent of $\mathcal{A}$ to $\sqrt[3]{(S,D)}$. $\square$

3. Elementary transformation

Already the proof of Lemma 5 exhibits the use of an elementary transformation (5) to alter the representation type of fibers of a vector bundle. In this section we use a variant of this to change the type of a sheaf of Azumaya algebras at a point from bad to good (Definition 10).

As in the previous section, S is a smooth quasiprojective surface over an algebraically closed field k of characteristic different from 3, and $D = D_1 \cup D_2$ is a nodal divisor with intersecting irreducible smooth components D_1 and D_2. We are given nontrivial unramified cyclic degree 3 covers

$$\widetilde{D}_i \to D_i, \quad \text{for} \quad i = 1, 2,$$

and an element

$$\alpha \in \mathrm{Br}\left(\sqrt[3]{(S, \{D_1, D_2\})}\right)[3],$$

whose residue along D_i is the class of $\widetilde{D}_i \to D_i$, for $i = 1, 2$. Let $\mathcal{A}$ be a sheaf of Azumaya algebras of degree 3 on $\sqrt[3]{(S, \{D_1, D_2\})}$ representing α. At a point $x \in D_1 \cap D_2$, the sheaf of Azumaya algebras $\mathcal{A}$ has a type, good or bad, according to the type of the associated projective representation at the point of $\sqrt[3]{(S, \{D_1, D_2\})}$ with stabilizer $\mu_3 \times \mu_3$ over x.

Let C_0 be a general nonsingular curve in S through x. Specifically, we suppose that C_0 meets D_i transversely, for $i = 1, 2$, and does not pass through any point of $D_1 \cap D_2$ besides x. The pre-image C of C_0 in $\sqrt[3]{(S, \{D_1, D_2\})}$ has a D_4-singularity over x (meaning that at a point over x of an étale chart $U \to \sqrt[3]{(S, \{D_1, D_2\})}$ the pre-image of C has a D_4-singularity).

Lemma 12. *With the above notation, α restricts to zero in* $\mathrm{Br}(C)$*.*

Remark 13. Even though C has a dense open substack isomorphic to a scheme with trivial Brauer group (a curve over k), examples such as $[(\mathrm{Proj}\, k[x,y,z]/(x^3 - y^3))/\mu_3 \times \mu_3]$, with factors μ_3 acting by scalar multiplication on distinct coordinates, illustrate the possible nonvanishing of $\mathrm{Br}(C)$ and the need to use some property of α in the proof of Lemma

12. Notice, it is essential in such an example to have a singularity, since by [4, Prop. 2.5(iv)], restriction to a dense open substack of a regular Noetherian Deligne-Mumford stack induces an injective map on Brauer groups. Since $B(\mu_3\times\mu_3)$ already has nontrivial Brauer group (see [17, Prop. 4.3.2]), we obtain a nontrivial Brauer group for any stack quotient of an action of $\mu_3\times\mu_3$ that has a fixed point.

Proof of Lemma 12. We argue as in [11, Thm. 1.3]. Let $\widehat{C}$ denote the normalization of C, and C' the seminormalization:

$$\widehat{C}\stackrel{\sigma}{\to}C'\stackrel{\nu}{\to}C.$$

Then we have an exact sequence

$$0\to\mathbb{G}_{m,C}\to\nu_*\mathbb{G}_{m,C'}\to i_*\mathcal{L}\to 0,$$

where $\mathcal{L}$ is an invertible sheaf on $B(\mu_3\times\mu_3)$, identified with the singular substack of C with inclusion map i. So ν induces an isomorphism $\mathrm{Br}(C)[3]\to\mathrm{Br}(C')[3]$, and we are reduced to showing that α restricts to zero in $\mathrm{Br}(C')$.

Identifying as well the singular substack of C' with $B(\mu_3\times\mu_3)$, with inclusion i', there is an exact sequence

$$0\to\mathbb{G}_{m,C'}\to\sigma_*\mathbb{G}_{m,\widehat{C}}\to i'_*\mathcal{H}\to 0,$$

for a two-dimensional torus $\mathcal{H}$ over $B(\mu_3\times\mu_3)$, that appears also in another exact sequence

$$0\to\mathbb{G}_{m,B(\mu_3\times\mu_3)}\to j_*\mathbb{G}_{m,B\tilde{\mu}_3}\to\mathcal{H}\to 0$$

that is related to the first by obvious restriction maps. Here we employ the notation $\tilde{\mu}_3$ as in the proof of Proposition 11 and denote by j the morphism $B\tilde{\mu}_3\to B(\mu_3\times\mu_3)$. We obtain a commutative diagram of cohomology groups

$$\begin{tikzcd}
\mathrm{Pic}(\widehat{C}) \arrow[r] \arrow[d] & H^1(B(\mu_3\times\mu_3),\mathcal{H}) \arrow[r] \arrow[d, equal] & \mathrm{Br}(C') \arrow[r] \arrow[d] & 0 \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & H^1(B(\mu_3\times\mu_3),\mathcal{H}) \arrow[r] & \mathrm{Br}(B(\mu_3\times\mu_3)) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since the map on the left is surjective, we have an isomorphism of Brauer groups on the right. So we are further reduced to verifying the triviality of the restriction of α to $B(\mu_3\times\mu_3)$. Recalling that $B(\mu_3\times\mu_3)$ is identified with the singular substack of C and observing that for $S'\to S$ as in Lemma 8, taken to have a unique point $x'\in S'$ over x, we have $B(\mu_3\times\mu_3)$ identified as well with the singular substack of the pre-image of C in the stack (6), we conclude by the vanishing mentioned at the beginning of the proof of Proposition 9. $\square$

With the notation of the proof of Proposition 11 we have

$$R_0 := A[t_1, t_2]/(t_1^3 - f_1, t_2^3 - f_2),$$

with $\mu_3 \times \mu_3$-action, as well as twists by characters $\chi_{i,j}$ of $\mu_3 \times \mu_3$ defined by

$$\chi_{i,j}(\lambda, \lambda') := \lambda^i \lambda'^j.$$

We introduce the following notation:

$$\begin{aligned} R_1 &:= R_0 \otimes \chi_{1,1}, & R_2 &:= R_0 \otimes \chi_{2,2}, \\ R' &:= R_0 \otimes \chi_{1,2}, & R'' &:= R_0 \otimes \chi_{2,1}. \end{aligned}$$

We let I_0 denote the ideal sheaf of $B(\mu_3 \times \mu_3)$ in C, with twists $I_i := I_0 \otimes \chi_{i,i}$. A Zariski neighborhood of the point of $\sqrt[3]{(S, \{D_1, D_2\})}$ over x has the form $[\mathrm{Spec}(R_0)/\mu_3 \times \mu_3]$, so coherent sheaves are given by finitely generated equivariant R_0-modules, and an exact sequence of coherent sheaves is given algebraically by

$$0 \to R_1 \oplus R_2 \oplus R_0 \xrightarrow{\begin{pmatrix} -t_2^2 & t_1 & 0 \\ -t_1^2 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}} R' \oplus R'' \oplus R_0 \xrightarrow{\begin{pmatrix} -t_2 & t_1 & 0 \end{pmatrix}} I_1 \to 0;$$

the maps are equivariant since the matrix entries are eigenfunctions, compatible with each source and target factor (e.g., action of (λ, λ') sending $1 \in R_1$ to $\lambda\lambda'$ and $-t_2^2 \in R'$ to $-\lambda\lambda' t_2^2$). We view this as an analytic local model of an elementary transformation.

Proposition 14. *With notation as above, we suppose that $\mathcal{A}$ is bad at x. Let $\alpha_0 \in H^2(\sqrt[3]{(S, \{D_1, D_2\})}, \mu_3)$ be a lift of α,*

$$\mathfrak{G} \xrightarrow{\tau} \sqrt[3]{(S, \{D_1, D_2\})}$$

a corresponding gerbe banded by μ_3, and $\mathcal{E}$ a rank 3 vector bundle on $\mathfrak{G}$ such that $\tau^*\mathcal{A} \cong \mathrm{End}(\mathcal{E})$. Then there exist a line bundle $\mathcal{L}$ on $\tau^{-1}(C)$ and an exact sequence

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{E} \to h_*(\mathcal{I} \otimes \mathcal{L}) \to 0,$$

where $\mathcal{I}$ denotes the ideal sheaf in $\tau^{-1}(C)$ of its singular locus, as a reduced substack, and h denotes the inclusion $\tau^{-1}(C) \to \mathfrak{G}$. Furthermore, the sheaf $\widetilde{\mathcal{E}}$ on the left is locally free and determines a sheaf of Azumaya algebras $\widetilde{\mathcal{A}}$ on $\sqrt[3]{(S, \{D_1, D_2\})}$ with $\tau^*\widetilde{\mathcal{A}} \cong \mathrm{End}(\widetilde{\mathcal{E}})$, that is good at x.

We note, by Lemma 12, that for $\mathcal{A}$ to be bad at x means that $\mathcal{E}|_{\tau^{-1}(C)}$, twisted by a suitable line bundle, restricts over x to $\tau^*(\chi_{1,2} \oplus \chi_{2,1} \oplus \chi_{0,0})$.

Analogously, for $\widetilde{\mathcal{A}}$ to be good at x means that the twist of $\widetilde{\mathcal{E}}|_{\tau^{-1}(C)}$, restricted over x, is $\tau^*(\chi_{1,1}\oplus\chi_{2,2}\oplus\chi_{0,0})$. Indeed, by the Kummer sequence,

$$\alpha_0|_C \in \ker\big(H^2(C,\mu_3)\to \mathrm{Br}(C)\big)\cong \mathrm{Pic}(C)/3\,\mathrm{Pic}(C).$$

So, $\tau^{-1}(C)$ is isomorphic to the stack of third roots of a line bundle on C, whose universal line bundle may be used for the twisting; see [1, §B.1].

Proof. As noted above, Lemma 12 tells us that there is a line bundle $\mathcal{T}$ on

$$\mathfrak{G}\times_{\sqrt[3]{(S,\{D_1,D_2\})}} C$$

for which the induced character of the generic stabilizer μ_3 is χ_1. The restriction of $\mathcal{E}$, tensored with $\mathcal{T}^\vee$, has, referring to (4) above, trivial action of the central μ_3, and hence descends to a vector bundle E on C. Since we are free to twist $\mathcal{T}$ by the pullback of any line bundle from C, there is no loss of generality in supposing that the isomorphism type of E over x is $\chi_{1,2}\oplus\chi_{2,1}\oplus\chi_{0,0}$.

Let L be a line bundle on C whose isomorphism type over x is $\chi_{1,1}$. We let I denote the ideal sheaf in C of its singular locus (as a reduced substack); the fiber of I at the point over x is a two-dimensional vector space with representation $\chi_{1,0}\oplus\chi_{0,1}$. So there exists an equivariant surjective linear map from the fiber of E to the fiber of $I\otimes L$. This extends to a module homomorphism, and if we average over translates by the elements of $\mu_3\times\mu_3$ then this becomes an equivariant module homomorphism, which we may view as a surjective morphism of sheaves

$$E|_{V\times_S C}\to (I\otimes L)|_{V\times_S C},$$

for some affine neighborhood $V\subset S$ of x. As explained in [13, §4.3] this extends, after possibly modifying L away from x, to a surjective morphism of sheaves on C. Pulling back to the gerbe and tensoring with $\mathcal{T}$ determines a surjective morphism of sheaves on $\mathfrak{G}$ and hence an exact sequence as in the statement.

The ideal sheaf $\mathcal{I}$ is Cohen-Macaulay of depth 1, so by the Auslander-Buchsbaum formula has projective dimension 1, and $\widetilde{\mathcal{E}}$ is locally free. Since the generic stabilizer μ_3 of $\mathfrak{G}$ acts trivially on $\mathrm{End}(\widetilde{\mathcal{E}})$, we obtain a sheaf of Azumaya algebras $\widetilde{\mathcal{A}}$ with $\tau^*\widetilde{\mathcal{A}}\cong\mathrm{End}(\widetilde{\mathcal{E}})$.

For the analysis of the type of the sheaf of Azumaya algebras $\widetilde{\mathcal{A}}$ at x, which is sensitive only to the projective representation of the $\mu_3\times\mu_3$ stabilizer over x, we may pass to an étale neighborhood of $x\in S$ as in Lemma 8 and thus assume that we have an exact sequence as in the statement of the proposition on $\sqrt[3]{(S,\{D_1,D_2\})}$, rather than on a gerbe. As before, $\mathcal{E}$ is only determined up to twisting by a line bundle. Since the map from the Picard group of $\sqrt[3]{(S,\{D_1,D_2\})}$ to the character group of

$\mu_3\times\mu_3$ (given by restriction to the copy of $B(\mu_3\times\mu_3)$ over x) is surjective, there is no loss of generality in supposing as before that the isomorphism type of $\mathcal{E}$ over x is $\chi_{1,2}\oplus\chi_{2,1}\oplus\chi_{0,0}$, and of the coherent sheaf on the right is $\chi_{1,2}\oplus\chi_{2,1}$. Restriction to the copy of $B(\mu_3\times\mu_3)$ over x determines a four-term exact sequence with a Tor sheaf on the left

$$0\to \mathrm{Tor}\to \widetilde{\mathcal{E}}|_{B(\mu_3\times\mu_3)}\to \chi_{1,2}\oplus\chi_{2,1}\oplus\chi_{0,0}\to\chi_{1,2}\oplus\chi_{2,1}\to 0.$$

Since the configuration of D_1, D_2, and C in S at x, from which we constructed R_0 with $\mu_3\times\mu_3$-action, has a unique analytic isomorphism type, the model computation just before the statement of the proposition may be used to see that

$$\mathrm{Tor}\cong\chi_{1,1}\oplus\chi_{2,2}.$$

Thus $\widetilde{\mathcal{E}}|_{B(\mu_3\times\mu_3)}$ is (non-canonically) isomorphic to $\chi_{1,1}\oplus\chi_{2,2}\oplus\chi_{0,0}$. It follows that $\widetilde{\mathcal{A}}$ is good at x. □

4. Proof of the main theorem

The argument begins as in the proof of the main theorem of [13]. The hypotheses on the complete linear system $|L|$ guarantee that the monodromy action on nontrivial unramified cyclic degree 3 covers of a nonsingular member of $|L|$ is transitive; cf. the proof of [12, Lem. 3.1], where an additional hypothesis (components meeting in an odd number of points) is required for the analysis of the monodromy representation mod 2 but is not needed for the present analysis. We take the space of reduced nodal curves in $|L|$ with nontrivial degree 3 cyclic étale covering, and the member $D=D_1\cup D_2$ with degree 3 cyclic étale cover, nontrivial over each component, as pointed variety (B,b_0). There is an associated element

$$\alpha\in\mathrm{Br}\left(\sqrt[3]{(S,\{D_1,D_2\})}\right),$$

by Propositions 2 and 3, represented by a sheaf of Azumaya algebras $\mathcal{A}$ of degree 3. By repeated application of Proposition 14, we may suppose that $\mathcal{A}$ is good at all nodes of D. By Proposition 11, $\mathcal{A}$ descends to the (singular) root stack $\sqrt[3]{(S,D)}$; we let

$$\beta\in\mathrm{Br}\left(\sqrt[3]{(S,D)}\right)$$

denote its Brauer class, and

$$\gamma\in H^2\big(\sqrt[3]{(S,D)},\mu_3\big)$$

a choice of lift, with gerbe $\mathfrak{G}_0$ associated with γ and locally free coherent sheaf $\mathcal{E}_0$ of rank 3 associated with the sheaf of Azumaya algebras.

Applying the deformation-theoretic machinery of [13, §4.3], we obtain by (usual) elementary transformation a subsheaf $\widetilde{\mathcal{E}}_0$, also locally free of rank 3, for which the space of obstructions vanishes. Upon replacing B by a suitable étale neighborhood of b_0, we obtain the root stack $\sqrt[3]{(B \times S, \mathcal{D})}$, where $\mathcal{D}$ denotes the corresponding family of divisors in $B \times S$, a class

$$\Gamma \in H^2\big(\sqrt[3]{(B \times S, \mathcal{D})}, \mu_3\big)$$

restricting to γ, a gerbe

$$\mathfrak{G} \to \sqrt[3]{(B \times S, \mathcal{D})}$$

restricting to $\mathfrak{G}_0$, and a locally free sheaf $\widetilde{\mathcal{E}}$ on $\mathfrak{G}$ restricting to $\widetilde{\mathcal{E}}_0$. The locally free sheaf $\widetilde{\mathcal{E}}$ determines a smooth $\mathbb{P}^2$-bundle

$$\mathcal{P} \to \sqrt[3]{(B \times S, \mathcal{D})}.$$

We now apply the final step in the proof of [15, Thm. 1.4] to the $\mathbb{P}^2$-bundle $\mathcal{P}$. The construction of Brauer-Severi surface bundles [15, Prop. 4.4], applied to $\mathcal{P}$, produces a Brauer-Severi surface bundle

$$\mathcal{X} \to B \times S.$$

Over B, this is a flat family of Brauer-Severi surface bundles over S. The fiber X_0 over b_0 has discriminant curve $D = D_1 \cup D_2$ with two smooth components meeting transversely, Brauer class given by nontrivial étale cyclic covers, and nontrivial unramified Brauer group, as we now show. Let

$$\tilde{\pi}\colon \widetilde{X}_0 \to \widetilde{S}$$

be a standard model of $X_0 \to S$ [15, Thm. 1.2]. So $\widetilde{X}_0$ is a smooth projective variety, flat over $\widetilde{S}$ with singular fibers over a divisor isomorphic to $D_1 \sqcup D_2$. We show $\mathrm{Br}(\widetilde{X}_0)[3] \neq 0$, following the two steps of [20, §3.4].

1) Application of Leray spectral sequence to deduce: the restriction
$$\varrho\colon\ \mathrm{Br}(k(S)) \to \mathrm{Br}(\mathrm{Spec}(k(S)) \times_{\widetilde{S}} \widetilde{X}_0) \tag{7}$$
is surjective with kernel of order 3.
2) Analysis of residues to deduce: the subgroup $\langle \beta_1, \beta_2 \rangle$ of order 9 of $\mathrm{Br}(\mathrm{Spec}(k(S)))$, where β_i has ramification only along D_i with class in $H^1(k(D_i), \mathbb{Z}/3\mathbb{Z})$ corresponding to the given cyclic covering of D_i (cf. Proposition 2) for $i = 1$, 2, is sent by ϱ into the subgroup $\mathrm{Br}(\widetilde{X}_0)$ of the right-hand side of (7). Since $\widetilde{X}_0$ is smooth, it suffices to show for $i = 1$, 2 that $\varrho(\beta_i)$ has vanishing residue along all divisors of $\widetilde{X}_0$. Any (3-torsion) element of the right-hand side of (7) is unramified along any divisor of $\widetilde{X}_0$ that dominates $\widetilde{S}$. Since $\tilde{\pi}$ is flat, any other divisor $\Xi \subset \widetilde{X}_0$ must dominate a divisor $\Pi \subset \widetilde{S}$. These observations reduce the analysis to application of the compatibility of residues of β_i along Π and $\varrho(\beta_i)$ along Ξ,

stated in [20, Prop. 3.4]. When Π is (the proper transform of) D_i, the kernel of $H^1(k(\Pi),\mathbb{Z}/3\mathbb{Z}) \to H^1(k(\Xi),\mathbb{Z}/3\mathbb{Z})$ is generated by a class corresponding to the given cyclic covering of D_i, hence $\varrho(\beta_i)$ is unramified along Ξ. Otherwise, β_i is unramified along Π, hence $\varrho(\beta_i)$ is unramified along Ξ. We note, since $\ker(\varrho) = \langle \beta_1 + \beta_2 \rangle$, that $\langle \varrho(\beta_1) \rangle = \langle \varrho(\beta_2) \rangle$ in the right-hand side of (7).

Statements 1) and 2) are straightforward extensions of analogous assertions for conic bundles given in [7, Prop. 1.5, proof of Prop. 2.1].

We justify the applicability of the specialization method and thereby conclude that the very general Brauer-Severi surface bundle in the family is not stably rational, by showing that X_0 has a universally CH_0-trivial desingularization. The singular locus of X_0 lies entirely over the nodes of D, and by applying [15, Lem. 2.8] to the action of μ_3 on a cyclic triple cover of the form $r^3 = st$ over the henselian local ring of $\mathrm{Spec}(k[s,t])$ at the origin, we are reduced to verifying the existence of a universally CH_0-trivial desingularization for the variety

$$Y = X \times_{\mathbb{A}^1} \mathbb{A}^2,$$

where $X \to \mathbb{A}^1$ is a regular Brauer-Severi surface bundle with singular fiber over 0 and the morphism $\mathbb{A}^2 \to \mathbb{A}^1$ is given by multiplication of coordinates. We note that a regular Brauer-Severi surface over $\mathbb{A}^1$ with singular fiber over 0 is given explicitly by the forwards construction of [15, Prop. 4.4], applied to

$$\mathbb{P}(L \oplus L' \oplus L'') \to \sqrt[3]{(\mathbb{A}^1, 0)},$$

where L, L', L'' are line bundles representing the three elements of the Picard group of $\sqrt[3]{(\mathbb{A}^1, 0)}$. By the analysis in the last paragraph of [15, §5], this is a variety with singularities along three smooth curves C_1, C_2, C_3 meeting at a single point p where the étale local isomorphism type is that of the hypersurface

$$\widehat{Y} = \mathrm{Spec}(k[u,v,x,y,z]/(uv - xyz))$$

at the origin. We first blow up Y along C_1 to obtain

$$\varphi\colon Y' \to Y$$

with

$$\varphi^{-1}(p) \cong \mathbb{P}^2 \sqcup_{\mathbb{P}^1} \mathbb{P}^2.$$

Then we blow up Y' along the (disjoint) proper transforms C_2', C_3' of the other curves:

$$\varphi'\colon Y'' \to Y'.$$

We claim that Y'' is smooth and the morphisms φ and φ' are universally CH_0-trivial; for the latter assertion we use the sufficient criterion of [8, Prop. 1.8], that fibers over all points (closed or not) are universally CH_0-trivial. For φ, the fiber $\varphi^{-1}(p)$ is universally CH_0-trivial, and the fiber over another closed point, respectively the generic point of C_1, is a smooth quadric surface over k, respectively over $k(C_1)$; these are universally CH_0-trivial. Now by writing the analogous blow-up $\widehat{Y}'$ of $\widehat{Y}$ along the curve $u = v = x = y = 0$ explicitly in coordinate charts, e.g.,

$$\mathrm{Spec}(k[x, z, u', v', y']/(y'z - u'v'))$$

with $u = xu'$, $v = xv'$, $y = xy'$, we see that Y' has ordinary double points along C_1' and C_2', hence Y'' is smooth and φ' is universally CH_0-trivial.

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