ON A FLOW OF TRANSFORMATIONS OF A WIENER SPACE

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Abstract. In this paper, we define, via Fourier transform, an ergodic flow of transformations of a Wiener space which preserves the law of the Ornstein-Uhlenbeck process and which interpolates the iterations of a transformation previously defined by Jeulin and Yor. Then, we give a more explicit expression for this flow, and we construct from it a continuous gaussian process indexed by $\mathbb{R}^2$, such that all its restriction obtained by fixing the first coordinate are Ornstein-Uhlenbeck processes.

1. Introduction

An abstract Wiener space is a triple $(H, E, \mathcal{W})$ consisting of a separable, real Hilbert space H, a separable real Banach space E in which H is continuously embedded as a dense subspace, and a Borel probability measure $\mathcal{W}$ on E with the property that, for each $x^* \in E^*$, the $\mathcal{W}$-distribution of the map $x \in E \longmapsto \langle x, x^* \rangle \in \mathbb{R}$, from E to $\mathbb{R}$, is a centered gaussian random distribution with variance $\|h_{x^*}\|_H^2$, where h_{x^*} is the element of H determined by $(h, h_{x^*})_H = \langle h, x^* \rangle$ for all $h \in H$. See Chapter 8 of [5] for more information on this topic.

Because $\{h_{x^*} : x^* \in E^*\}$ is dense in H and $\|h_{x^*}\|_H = \|\langle \cdot, x^* \rangle\|_{L^2(\mathcal{W})}$, there is a unique isometry, known as the Paley–Wiener map, $\mathcal{I} : H \longmapsto L^2(\mathcal{W})$ such that $\mathcal{I}(h) = \langle \cdot, x^* \rangle$ if $h = h_{x^*}$. In fact, for each $h \in H$, $\mathcal{I}(h)$ under $\mathcal{W}$ is a centered Gaussian variable with variance $\|h\|_H^2$. Because, when $h = h_{x^*}$, $\mathcal{I}(h)$ provides an extention of $(\cdot, h)_H$ to E, for intuitive purposes one can think of $x \rightsquigarrow [\mathcal{I}(h)](x)$ as a giving meaning to the inner product $x \rightsquigarrow (x, h)_H$, although for general h this will be defined only up to a set of $\mathcal{W}$-measure 0.

An important property of abstract Wiener spaces is that they are invariant under orthogonal transformations on H. To be precise, given an orthogonal transformation $\mathcal{O}$ on H, there is a $\mathcal{W}$-almost surely unique $T_{\mathcal{O}} : E \longrightarrow E$ with the property that, for each $h \in H$, $\mathcal{I}(h) \circ T_{\mathcal{O}} = \mathcal{I}(\mathcal{O}^\top h)$ $\mathcal{W}$-almost surely. Notice that this is the relation which one would predict if one thinks of $[\mathcal{I}(h)](x)$ as the inner product of x with h. In general, $T_{\mathcal{O}}$ can be constructed by choosing $\{x_m^* : m \geq 1\} \subseteq E^*$ so the $\{h_{x_m^*} : m \geq 1\}$ is an orthonormal basis in H and then taking

$$T_{\mathcal{O}} x = \sum_{m=1}^{\infty} \langle x, x_m^* \rangle \mathcal{O} h_{x_m^*},$$

where the series converges in E for $\mathcal{W}$-almost every x as well as in $L^p(\mathcal{W}; E)$ for every $p \in [1, \infty)$. See Theorem 8.3.14 in [5] for details. In the case when $\mathcal{O}$ admits an extension as a continuous map on E into itself, $T_{\mathcal{O}}$ can be the taken equal to that extension. In any case, it is an easy matter to check that the measure $\mathcal{W}$ is preserved by $T_{\mathcal{O}}$. Less obvious is a theorem, originally formulated by I.M. Segal (cf. [6]), which says that $T_{\mathcal{O}}$ is ergodic if and only $\mathcal{O}$ admits no non-trivial, finite dimensional, invariant subspace. Equivalently, $T_{\mathcal{O}}$

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is ergodic if and only if the complexification $\mathcal{O}_c$ has a continuous spectrum as a unitary operator on the complexification H_c of H.

The classical Wiener space provides a rich source of examples to which the preceding applies. Namely, take $H = H_0^1$ to be the space of absolutely continuous $h \in \Theta$ whose derivative $\dot{h}$ is in $L^2([0,\infty))$, and set $\|h\|_{H_0^1} = \|\dot{h}\|_{L^2([0,\infty))}$. Then H_0^1 with norm $\|\cdot\|_{H_0^1}$ is a separable Hilbert space. Next, take $E = \Theta$, where Θ is the space of continuous paths $\theta : [0,\infty) \longrightarrow \mathbb{R}$ such that $\theta(0) = 0$ and

$$\frac{|\theta(t)|}{t^{\frac{1}{2}} \log(e + |\log t|)} \longrightarrow 0 \quad \text{as } t > 0 \text{ tends to } 0 \text{ or } \infty,$$

and set

$$\|\theta\|_\Theta = \sup_{t>0} \frac{|\theta(t)|}{t^{\frac{1}{2}} \log(e + |\log t|)}.$$

Then Θ with norm $\|\cdot\|_\Theta$ is a separable Banach space in which H_0^1 is continuously embedded as a dense subspace. Finally, the renowned theorem of Wiener combined with the Brownian law of the iterated logarithm says that there is a Borel probability measure $\mathcal{W}_{H_0^1}$ on Θ for which $(H_0^1, \Theta, \mathcal{W}_{H_0^1})$ is an abstract Wiener space. Indeed, it is the classical Wiener space on which the abstraction is modeled, and $\mathcal{W}_{H_0^1}$ is the distribution of an $\mathbb{R}$-valued Brownian motion.

One of the simplest examples of an orthogonal transformation on H_0^1 for which the associated transformation on Θ is ergodic is the Brownian scaling map S_α given by $S_\alpha\theta(t) = \alpha^{-\frac{1}{2}}\theta(\alpha t)$ for $\alpha > 0$. It is an easy matter to check that the restriction $\mathcal{O}_\alpha$ of S_α to H_0^1 is orthogonal, and so, since S_α is continuous on Θ, we can take $T_{\mathcal{O}_\alpha} = S_\alpha$. Furthermore, as long as $\alpha \neq 1$, an elementary computation shows that $\lim_{n\to\infty}\big(g, \mathcal{O}_\alpha^n h\big)_H = 0$, first for smooth g, $h \in H_0^1$ with compact support in $(0,\infty)$ and thence for all g, $h \in H_0^1$. Hence, when $\alpha \neq 1$, $\mathcal{O}_\alpha$ admits no non-trivial, finite dimensional subspace, and therefore S_α is ergodic; and so, by the Birkoff's Individual Ergodic Theorem, for $p \in [1,\infty)$ and $f \in L^p(\mathcal{W}_{H_0^1})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f \circ S_\alpha^n = \int f \, d\mathcal{W}_{H_0^1}$$

both $\mathcal{W}_{H_0^1}$-almost surely and in $L^p(\mathcal{W}_{H_0^1})$. Moreover, since $\{S_\alpha : \alpha \in (0,\infty)\}$ is a multiplicative semigroup in the sense that $S_{\alpha\beta} = S_\alpha \circ S_\beta$, one has the continuous parameter version

$$\lim_{a\to\infty} \frac{1}{\log a} \int_1^a (f \circ S_\alpha) \frac{d\alpha}{\alpha} = \int f \, d\mathcal{W}_{H_0^1}$$

of the preceding result.

A more challenging ergodic transformation of the classical Wiener space was studied by Jeulin and Yor (see [1], [2] and [4]), and, in the framework of this article, it is obtained by considering the transformation $\mathcal{O}$ on H_0^1, defined by

$$[\mathcal{O}h](t) = h(t) - \int_0^t \frac{h(s)}{s} \, ds. \tag{1.1}$$

An elementary calculation shows that $\mathcal{O}$ is orthogonal. Moreover, $\mathcal{O}$ admits a continuous extension to Θ given by replacing $h \in H_0^1$ in (1.1) by $\theta \in \Theta$. That is

$$[T_{\mathcal{O}}\theta] = \theta(t) - \int_0^t \frac{\theta(s)}{s}\, ds \quad \text{for } \theta \in \Theta \text{ and } t \geq 0. \tag{1.2}$$

In addition, one can check that $\lim_{n\to\infty}\big(g, \mathcal{O}^n h\big)_{H_0^1} = 0$ for all g, $h \in H_0^1$, which proves that $T_{\mathcal{O}}$ is ergodic for $\mathcal{W}_{H_0^1}$.

In order to study the transformation $T_{\mathcal{O}}$ in greater detail, it will be convenient to reformulate it in terms of the Ornstein–Uhlenbeck process. That is, take H^U to be the space of absolutely continuous functions $h : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\|h\|_{H^U} \equiv \sqrt{\int_{\mathbb{R}} \left(\tfrac{1}{4}h(t)^2 + \dot{h}(t)^2\right) dt} < \infty.$$

Then H^U becomes a separable Hilbert space with norm $\|\cdot\|_{H^U}$. Moreover, the map $F : H_0^1 \longrightarrow H^U$ given by

$$[F(g)](t) = e^{-\frac{t}{2}} g(e^t), \quad \text{for } g \in H_0^1 \text{ and } t \in \mathbb{R}, \tag{1.3}$$

is an isometric surjection which extends as an isometry from Θ onto Banach space $\mathcal{U}$ of continuous $\omega :\longrightarrow \mathbb{R}$ satisfying $\lim_{|t|\to\infty} \frac{|\omega(t)|}{\log|t|} = 0$ with norm $\|\omega\|_{\mathcal{U}} = \sup_{t\in\mathbb{R}} \big(\log(e+|t|)\big)^{-1}|\omega(t)|$. Thus, $(H^U, \mathcal{U}, \mathcal{W}_{H^U})$ is an abstract Wiener space, where $\mathcal{W}_{H^U} = F_* \mathcal{W}_{H_0^1}$ is the image of $\mathcal{W}_{H_0^1}$ under the map F. In fact, $\mathcal{W}_{H^U}$ is the distribution of a standard, reverisible Ornstein–Uhlenbeck process.

Note that the scaling transformations for the classical Wiener space become translations in the Ornstein–Uhlenbeck setting. Namely, for each $\alpha > 0$, $F \circ S_\alpha = \tau_{\log\alpha} \circ F$, where τ_s denotes the time-translation map given by $[\tau_s \omega](t) = \omega(s+t)$. Thus, for $s \neq 0$, the results proved about the scaling maps say that τ_s is an ergodic transformation for $\mathcal{W}_{H^U}$. In particular, for $p \in [1,\infty)$ and $f \in L^p(\mathcal{W}_{H^U})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f \circ \tau_{ns} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f \circ \tau_s \, ds = \int f \, d\mathcal{W}_{H^U}$$

both $\mathcal{W}_{H^U}$-almost surely and in $L^p(\mathcal{W}_{H^U})$.

The main goal of this article is to show that the reformulation of transformation $T_{\mathcal{O}}$ coming from the Jeulin–Yor transformation in terms of the Ornstein–Uhlenbeck process allows us to embed $T_{\mathcal{O}}$ in a continuous-time flow of transformations on the space $\mathcal{U}$, each of which is $\mathcal{W}_{H_0^1}$-measure preserving and all but one of which is ergodic. In Section 2, this flow is described via Fourier transforms. In Section 3, a direct and more explicit expression, involving hypergeometric functions and principal values, is computed. In Section 4, we study the two-parameter gaussian process which is induced by the flow introduced in Section 2. In particular, we compute its covariance and prove that it admits a version which is jointly continuous in its parameters.

2. Preliminary description of the flow

Let $\mathcal{O}$ and $T_{\mathcal{O}}$ be the transformations on H_0^1 and Θ given by (1.1) and (1.2), and recall the unitary map $F : H_0^1 \longrightarrow H^U$ in (1.3) and its continuous extension as an isometry from

Θ onto $\mathcal{U}$. Clearly, the inverse of F is given by

$$F^{-1}(\omega)(t) = \sqrt{t}\,\omega(\log t) \quad \text{ for } t > 0.$$

Because F is unitary and $\mathcal{O}$ is orthogonal on H^1_0, $-F \circ \mathcal{O} \circ F^{-1}$ is an orthognal transformation on H^U, and because

$$S := -F \circ T_{\mathcal{O}} \circ F^{-1}$$

is continuous extension of $-F \circ \mathcal{O} \circ F^{-1}$ to $\mathcal{U}$, we can identify S as $T_{-F\circ\mathcal{O}\circ F^{-1}}$.

Another expression for action of S is

$$[S(\omega)](t) = -\omega(t) + \int_0^\infty e^{-\frac{s}{2}}\omega(t-s)\,ds \quad \text{for } t \in \mathbb{R}.$$

Equivalently,

$$S(\omega) = \omega * \mu,$$

where μ is the finite, signed measure μ given by

$$\mu := -\delta_0 + e^{-\frac{t}{2}}\mathbb{1}_{t\geq 0}dt.$$

To confirm that $\omega * \mu$ is well-defined as a Lebesgue integral and that it maps $\mathcal{U}$ continuously into itself, note that, for any $\omega \in \mathcal{U}$ and $t \in \mathbb{R}$,

$$\begin{aligned}\int_0^\infty e^{-\frac{s}{2}}|\omega(t-s)|\,ds &\leq \|\omega\|_{\mathcal{U}}\int_0^\infty e^{-\frac{s}{2}}\log\bigl(e+|t|+s\bigr)\,ds \\ &\leq \|\omega\|_{\mathcal{U}}\log(e+|t|)\int_0^\infty e^{-\frac{s}{2}}(1+s)\,ds \leq 9\|\omega\|_{\mathcal{U}}\log(e+|t|)\end{aligned}$$

The Fourier transform $\widehat{\mu}$ of μ is given by

$$\widehat{\mu}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x}d\mu(x) = -1 + \int_0^\infty e^{-x(1/2+i\lambda)}dx = -1 + \frac{1}{1/2+i\lambda} = \frac{1-2i\lambda}{1+2i\lambda} = e^{-2i\,\mathrm{Arctg}(2\lambda)}.$$

Hence, for all $h \in H^U$ and $\lambda \in \mathbb{R}$,

$$\widehat{h*\mu}(\lambda) = e^{-2i\,\mathrm{Arctg}(2\lambda)}\widehat{h}(\lambda), \tag{2.1}$$

which, since

$$\|h\|^2_{H^U} = \frac{1}{8\pi}\int_{\mathbb{R}}|\widehat{h}(\lambda)|^2\bigl(1+4\lambda^2\bigr)\,d\lambda,$$

provides another proof that $S \restriction H^U$ is isometric.

The preceeding, and especially (2.1), suggests a natural way to embed $S \restriction H^U$ into a continuous group of orthogonal transformations. Namely, for $u \in \mathbb{R}$, let μ^{*u} to be the unique tempered distribution whose Fourier transform is given by

$$\widehat{\mu^{*u}}(\lambda) = e^{-2iu\,\mathrm{Arctg}(2\lambda)}, \tag{2.2}$$

and define $\mathcal{S}^u\varphi = \varphi * \mu^{*u}$ for φ in the Schwartz test function class $\mathscr{S}$ of smooth functions which, together with all their derivatives, are rapidly decreasing. Because

$$\widehat{\mathcal{S}^u\varphi}(\lambda) = e^{-2iu\,\mathrm{Arctg}(2\lambda)}\hat{\varphi}(\lambda),$$

it is obvious that $\mathcal{S}^u$ has a unique extension as an orthogonal transformation on H^U, which we will again denote by $\mathcal{S}^u$. Furthermore, it is clear that $\mathcal{S}^{u+v} = \mathcal{S}^u \circ \mathcal{S}^v$ for all $u, v \in \mathbb{R}$. Finally, for all $g, h \in H^U$, $u \in \mathbb{R}$,

$$\begin{aligned}(g, \mathcal{S}^u h)_{H^U} &= \frac{1}{8\pi} \int_{\mathbb{R}} \overline{\widehat{g}(\lambda)}\, \widehat{h}(\lambda)\, e^{-2iu\,\mathrm{Arctg}(2\lambda)} (1+4\lambda^2)\, d\lambda \\ &= \frac{1}{16\pi} \int_{-\pi/2}^{\pi/2} \overline{\widehat{g}\left(\frac{\tan(\tau)}{2}\right)}\, \widehat{h}\left(\frac{\tan(\tau)}{2}\right) \left(1+\tan^2(\tau)\right)^2 e^{-2iu\tau}\, d\tau,\end{aligned}$$

where

$$\frac{1}{16\pi} \int_{-\pi/2}^{\pi/2} \left|\widehat{g}\left(\frac{\tan(\tau)}{2}\right)\right| \left|\widehat{h}\left(\frac{\tan(\tau)}{2}\right)\right| \left(1+\tan^2(\tau)\right)^2 d\tau = \frac{1}{8\pi} \int_{\mathbb{R}} |\widehat{g}(\lambda)|\, |\widehat{h}(\lambda)|\, (1+4\lambda^2)\, d\lambda$$

$$\leq \frac{1}{8\pi} \left(\int_{\mathbb{R}} |\widehat{g}(\lambda)|^2 (1+4\lambda^2) d\lambda \right)^{1/2} \left(\int_{\mathbb{R}} |\widehat{h}(\lambda)|^2 (1+4\lambda^2) d\lambda \right)^{1/2} = ||g||_{H^U} ||h||_{H^U} < \infty.$$

Hence, by Riemann–Lebesgue lemma, shows that $(g, \mathcal{S}^u h)_{H^U}$ tends to zero when $|u|$ goes to infinity.

Now define the associated transformations $S^u := T_{\mathcal{S}^u}$ on $\mathcal{U}$ for each $u \in \mathbb{R}$. By the general theory summarized in the introduction and the preceding discussion, we know that $\{S^u : u \in \mathbb{R}\}$ is a flow of $\mathcal{W}_{H^U}$-measure preserving transformations and that, for each $u \neq 0$, S^u is ergodic.

3. A more explicit expression

So far we know very little about the transformations S^u for general $u \in \mathbb{R}$. By getting a handle on the tempered distributions μ^{*u}, in this section we will attempt to find out a little more.

We begin with the case when u is an integer $n \in \mathbb{Z}$. Recalling that $\mu = -\delta_0 + e^{-\frac{t}{2}} \mathbb{1}_{t\geq 0}\, dt$, one can use induction to check that, for $n \geq 0$,

$$\mu^{*n} = (-1)^n \big(\delta_0 + e^{-\frac{t}{2}} L_n'(t) \mathbb{1}_{t\geq 0} dt\big),$$

where L_n is the nth Laguerre polynomial. Indeed, the Laguerre polynomials satisfy the following relations: for all $n \geq 0$,

$$L_n(0) = 1$$

and for all $n \geq 0$, $t \in \mathbb{R}$,

$$L_{n+1}'(t) = L_n'(t) - L_n(t).$$

Similarly, starting from $\mu^{*-1} = -\delta_0 + e^{\frac{t}{2}} \mathbb{1}_{t\geq 0}\, dt$, one finds that

$$\mu^{*n} = (-1)^n \big(\delta_0 + e^{\frac{t}{2}} L_n'(-t) \mathbb{1}_{t\leq 0} dt\big)$$

for $n \leq 0$. In particular, μ^{*n} is a finite, signed measure for $n \in \mathbb{Z}$ and $S^n\omega$ can be identified as $\mu^{*n} * \omega$ for all $\omega \in \mathcal{U}$ and $n \in \mathbb{Z}$.

As the next result shows, when $u \notin \mathbb{Z}$, μ^{*u} is more singular tempered distribution than a finite, signed measure.

Proposition 3.1. *For each $u \notin \mathbb{Z}$, the distribution μ^{*u} is given by the following formula:*

$$\mu^{*u} = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi} pv(1/x) + \Phi_u(x), \tag{3.1}$$

where pv denotes the principal value, and $\Phi_u \in L^2(\mathbb{R})$ is the function for which $\Phi_u(x)$ equals

$$e^{-|x|/2}\left(-\frac{u\,\sin(\pi u)}{\pi}\sum_{k=0}^{\infty}\frac{(1-u\,\mathrm{sgn}(x))_k|x|^k}{k!(k+1)!}\left[\frac{\Gamma'}{\Gamma}(1+k-u\,\mathrm{sgn}(x))-\frac{\Gamma'}{\Gamma}(1+k)\right.\right.$$
$$\left.\left.-\frac{\Gamma'}{\Gamma}(2+k)+\log(|x|)\right]+\frac{\sin(\pi u)}{\pi x}\right)-\frac{\sin\pi u}{\pi x},$$

Γ'/Γ being the logarithmic derivative of the Euler gamma function and $(\quad)_k$ being the Pochhammer symbol.

Proof. Define the functions ψ_u and θ_u from $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ to $\mathbb{R}$ so that $\theta_u(x) = e^{-\frac{x}{2}}\psi_u(x)$ and $\psi_u(x)$ equals

$$-\frac{u\,\sin(\pi u)}{\pi}\sum_{k=0}^{\infty}\frac{(1-u\,\mathrm{sgn}(x))_k|x|^k}{k!(k+1)!}\left[\frac{\Gamma'}{\Gamma}(1+k-u\,\mathrm{sgn}(x))-\frac{\Gamma'}{\Gamma}(1+k)\right.$$
$$\left.-\frac{\Gamma'}{\Gamma}(2+k)+\log(|x|)\right]+\frac{\sin(\pi u)}{\pi x}.$$

From Lebedev [3], p. 264, equation (9.10.6), with the parameters $\alpha = 1-u$ or $\alpha = 1+u$, $n = 1$, $z = x$ or $z = -x$, the function ψ_u satisfies, for all $x \in \mathbb{R}^*$, the differential equation:

$$x\psi_u''(x) + (2-|x|)\psi_u'(x) + (u - \mathrm{sgn}(x))\psi_u(x) = 0,$$

and grows at most polynomially at infinity. One then deduces that θ_u decreases as least exponentially at infinity, and satisfies (for $x \neq 0$) the following equation:

$$x\theta_u''(x) + 2\theta_u'(x) + \left(u - \frac{x}{4}\right)\theta_u(x) = 0. \tag{3.2}$$

At the same time, by writing

$$e^{-|x|/2} = (e^{-|x|/2} - 1) + 1$$

and expanding $\theta_u(x)$ accordingly, we obtain:

$$\theta_u(x) = \frac{\sin(\pi u)}{\pi x} - \frac{u\sin(\pi u)}{\pi}\left[\frac{\Gamma'}{\Gamma}(1-u\,\mathrm{sgn}(x)) - \frac{\Gamma'}{\Gamma}(1) - \frac{\Gamma'}{\Gamma}(2) + \log(|x|)\right]$$
$$-\frac{\sin(\pi u)}{2\pi}\mathrm{sgn}(x) + \eta_u(x),$$

for

$$\eta_u(x) = x\eta_u^{(1)}(x) + |x|\eta_u^{(2)}(x) + x\log(|x|)\eta_u^{(3)}(x) + |x|\log(|x|)\eta_u^{(4)}(x),$$

where $\eta_u^{(1)}$, $\eta_u^{(2)}$, $\eta_u^{(3)}$, $\eta_u^{(4)}$ are all smooth functions. The derivatives of the functions x, $|x|$, $x\log|x|$, $|x|\log|x|$ in the sense of the distributions are obtained by interpreting their ordinary derivatives as distributions. Similarly, the product by x of their second distributional derivatives are obtained by multiplying their ordinary second derivatives by x. Hence, both $\eta_u'(x)$ and $x\eta_u''(x)$ as distributions can be obtained by computing $\eta_u'(x)$ and $x\eta_u''(x)$ as functions on $\mathbb{R}^*$.

Now, let ν_u be the distribution given by the expression:

$$\nu_u(x) = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi}pv(1/x) + \left[\theta_u(x) - \frac{\sin(\pi u)}{\pi x}\right]. \tag{3.3}$$

Note that the term in brackets, in the definition of ν_u, is a locally integrable function, and that ν_u coincides with the function θ_u in the complement of the neighborhood of zero. Let us now prove that ν_u satisfies the analog of the equation (3.2), in the sense of the distributions. One has:

$$\nu_u(x) = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi} pv(1/x) - \frac{u\sin(\pi u)}{\pi}\left[\frac{\Gamma'}{\Gamma}(1 - u\,\mathrm{sgn}(x)) - \frac{\Gamma'}{\Gamma}(1) - \frac{\Gamma'}{\Gamma}(2) + \log(|x|)\right] - \frac{\sin(\pi u)}{2\pi}\,\mathrm{sgn}(x) + \eta_u(x).$$

Since

$$\frac{\Gamma'}{\Gamma}(1+u) - \frac{\Gamma'}{\Gamma}(1-u) = \frac{\frac{d}{du}\left(\Gamma(1+u)\Gamma(1-u)\right)}{\Gamma(1+u)\Gamma(1-u)} = \frac{\frac{d}{du}(\pi u/\sin(\pi u))}{\pi u/\sin(\pi u)} = \frac{1}{u} - \pi\cot(\pi u),$$

one obtains, after straightforward computation,

$$\nu_u(x) = \cos(\pi)\delta_0(x) + \frac{\sin(\pi u)}{\pi} pv(1/x) - \frac{u\cos(\pi u)}{2}\,\mathrm{sgn}(x) - \frac{u\sin(\pi u)}{\pi}\log(|x|) + c(u) + \eta_u(x),$$

where $c(u)$ does not depend on x. One deduces that

$$\nu_u(x) = \cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi} pv(1/x) + \chi_{u,1}(x),$$

where $\chi_{u,1}$ denotes a locally integrable function. Moreover,

$$\nu_u'(x) = \cos(\pi u)\delta_0'(x) - \frac{\sin(\pi u)}{\pi} fp(1/x^2) - u\cos(\pi u)\delta_0(x) - \frac{u\sin(\pi u)}{\pi} pv(1/x) + \eta_u'(x),$$

where $fp(1/x^2)$ denotes the finite part of $1/x^2$, and then

$$x\nu_u'(x) = -\cos(\pi u)\delta_0(x) - \frac{\sin(\pi u)}{\pi} pv(1/x) - \frac{u\sin(\pi u)}{\pi} + x\eta_u'(x).$$

By differentiating again, one obtains:

$$\nu_u'(x) + x\nu_u''(x) = -\cos(\pi u)\delta_0'(x) + \frac{\sin(\pi u)}{\pi} fp(1/x^2) + \eta_u'(x) + x\eta_u''(x).$$

Therefore,

$$\begin{aligned} x\nu_u''(x) + 2\nu_u'(x) + \left(u - \frac{x}{4}\right)\nu_u(x) &= \chi_{u,2}(x) + \left(-\cos(\pi u)\delta_0'(x) + \frac{\sin(\pi u)}{\pi} fp(1/x^2)\right) \\ &+ \left(\cos(\pi u)\delta_0'(x) - \frac{\sin(\pi u)}{\pi} fp(1/x^2) - u\cos(\pi u)\delta_0(x) - \frac{u\sin(\pi u)}{\pi} pv(1/x)\right) \\ &+ u\left(\cos(\pi u)\delta_0(x) + \frac{\sin(\pi u)}{\pi} pv(1/x)\right) = \chi_{u,2}(x), \end{aligned}$$

where $\chi_{u,2}$ is a locally integrable function. Since θ_u satisfies (3.2), $\chi_{u,2}$ is identically zero. Hence, ν_u is a tempered distribution solving the differential equation:

$$x\nu_u''(x) + 2\nu_u'(x) + \left(u - \frac{x}{4}\right)\nu_u(x) = 0,$$

or equivalently,

$$\frac{x}{4}\nu_u(x) - \frac{d^2}{d^2x}(x\nu_u(x)) - u\nu_u(x) = 0.$$

Multiplying by $-4i$ and taking the Fourier transform (in the sense of the distributions), one deduces:

$$\widehat{\nu_u}'(\lambda)(1+4\lambda^2) = -4iu\widehat{\nu_u}(\lambda).$$

This linear equation admits a unique solution, up to a multiplicative factor c:

$$\widehat{\nu_u}(\lambda) = c\exp\left(\int_0^\lambda \frac{-4iu}{1+4t^2}dt\right) = c\exp(-2iu\operatorname{Arctg}(2\lambda)).$$

Hence, ν_u is proportional to μ^{*u}. In order to determine the constant c, let us observe that the distribution $\nu_{u,0}$ given by

$$\nu_{u,0}(x) = \nu_u(x) - c\cos(\pi u)\delta_0(x) - \frac{c\sin(\pi u)}{\pi}pv(1/x)$$

admits the Fourier transform:

$$\widehat{\nu_{u,0}}(\lambda) = c\,e^{-2iu\operatorname{Arctg}(2\lambda)} - c\,e^{-\pi iu\operatorname{sgn}(\lambda)}.$$

One deduces that $\widehat{\nu_{u,0}}$ is a function in L^2, which implies that $\nu_{u,0}$ is also a function in L^2, and then locally integrable. Since the last term in (3.3) is also a locally integrable function, one deduces that $c = 1$, and then

$$\mu^{*u} = \nu_u,$$

which proves Proposition 3.1. □

The reasonably explicit expression for μ^{*u} found in Proposition 3.1 yields a reaonably explicit expression for the action of $\mathcal{S}^u$. Indeed, only the term $pv(1/x)$ is a source of concern. However, convolution with respect of $pv(1/x)$ is, apart from a multiplicative constant, just the Hilbert transform, whose properties are well-known. In particular, it is a translation invariant, bounded map on $L^2(\mathbb{R})$, and as such it is also a bounded map on H^U. Thus, we can unambiguously write $\mathcal{S}^u(h) = h * \mu^{*u}$ for all $h \in H^U$. On the other hand, the interpretation of $\omega * \mu^{*u}$ for $\omega \in \mathcal{U}$ needs some thought. No doubt, $\omega * \mu^{*u}$ is well-defined as an element of $\mathscr{S}'$, the space tempered distributions, but it is not immediately obvious that it is can be represented by an element of $\mathcal{U}$ or, if it can, that the element of $\mathcal{U}$ which represents it can be identified as $S^u\omega$. In fact, the best that we should expect is that such statements will be true of $\mathcal{W}_{H^U}$-almost every $\omega \in \mathcal{U}$. The following result justifies that expectation.

Proposition 3.2. *For $\mathcal{W}_{H^U}$-almost every $\omega \in \mathcal{U}$, the tempered distribution $\omega * \mu^{*u}$ is represented by an element of $\mathcal{U}$ which can be can be identified as $S^u\omega$.*

Proof. Recall that, for $\varphi \in \mathscr{S}$, $\varphi * \mu^{*-u}$ is the element of $\mathscr{S}$ whose Fourier transform is given by

$$\widehat{\varphi * \mu^{*-u}}(\lambda) = \widehat{\varphi}(\lambda)e^{2iu\operatorname{Arctg}(2\lambda)} \quad \text{for all } \lambda \in \mathbb{R}.$$

Also, if $T \in \mathscr{S}'$, then $T * \mu^{*u}$ is the tempered distribution whose action on $\varphi \in \mathscr{S}$ is given by

$${}_{\mathscr{S}}\langle \varphi, T * \mu^{*u}\rangle_{\mathscr{S}'} = {}_{\mathscr{S}}\langle \varphi * \mu^{*-u}, T\rangle_{\mathscr{S}'}.$$

Now choose an orthonormal basis $\{h_n : n \geq 1\}$ for H^U all of whose members are elements of $\mathscr{S}$, and, for each $n \geq 1$, set $g_n = \frac{1}{4}h_n + h_n''$. Next, think of g_n as the element of $\mathcal{U}^*$ whose action on $\omega \in \mathcal{U}$ is given by

$${}_{\mathcal{U}}\langle \omega, g_n\rangle_{\mathcal{U}^*} = {}_{\mathscr{S}}\langle g_n, \omega\rangle_{\mathscr{S}'}.$$

It is then an easy matter to check that, in the notation of the introduction, $h_n = h_{g_n}$. Hence, if B is the subset of $\omega \in \mathcal{U}$ for which

$$\omega = \lim_{n\to\infty} \sum_{m=1}^{n} {}_{\mathscr{S}}\langle g_n, \omega\rangle_{\mathscr{S}'} h_n \quad \text{and} \quad S^u\omega = \lim_{n\to\infty} \sum_{m=1}^{n} {}_{\mathscr{S}}\langle g_n, \omega\rangle_{\mathscr{S}'} h_n * \mu^{*u},$$

where the convergence is in $\mathcal{U}$, then $\mathcal{W}_{H^U}(B) = 1$.

Now let $\omega \in B$. Then, for each $\varphi \in \mathscr{S}$,

$$\begin{aligned} {}_{\mathscr{S}}\langle \varphi, \omega * \mu^{*u}\rangle_{\mathscr{S}'} &= {}_{\mathscr{S}}\langle \varphi * \mu^{*-u}, \omega\rangle_{\mathscr{S}'} = \lim_{n\to\infty} \sum_{m=1}^{n} {}_{\mathscr{S}}\langle g_n, \omega\rangle_{\mathscr{S}'} {}_{\mathscr{S}}\langle \varphi, h_n * \mu^{*u}\rangle_{\mathscr{S}'} \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} {}_{\mathscr{S}}\langle g_n, \omega\rangle_{\mathscr{S}'} {}_{\mathscr{S}}\langle \varphi, \mathcal{S}^u h_n\rangle_{\mathscr{S}'} = {}_{\mathscr{S}}\langle \varphi, S^u\omega\rangle_{\mathscr{S}'}. \end{aligned}$$

Thus, for $\omega \in B$, $\omega * \mu^{*u} \in \mathscr{S}'$ is represented by $S^u\omega \in \mathcal{U}$. □

4. A two parameter gaussian process

By construction, $\{S^u\omega(t) : (u,t) \in \mathbb{R}^2\}$ is a gaussian family in $L^2(\mathcal{W}_{H^U})$. In this concluding section, we will show that this family admits a modification which is jointly continuous in (u,t).

Let $\varphi, \psi \in \mathscr{S}$ and $u, v \in \mathbb{R}^2$ be given. Then, by Proposition 3.2, for $\mathcal{W}_{H^U}$-almost every $\omega \in \mathcal{U}$,

$$\iint\limits_{\mathbb{R}^2} \varphi(s)\psi(t)(S^u(\omega))(s)(S^v(\omega))(t)\, dsdt = {}_{\mathscr{S}}\langle \varphi, \omega * \mu^{*u}\rangle_{\mathscr{S}'} {}_{\mathscr{S}}\langle \psi, \omega * \mu^{*v}\rangle_{\mathscr{S}'},$$

where the integral in the left-hand side is absolutely convergent. Because $\mathbb{E}_{\mathcal{W}_{H^U}}\left[S^u\omega(t)^2\right]$ is finite and independent of $(u,t) \in \mathbb{R}^2$, by taking the expectation with respect to $\mathcal{W}_{H^U}$ and using (2.2), one can pass from this to

$$\begin{aligned} &\iint\limits_{\mathbb{R}^2} \varphi(s)\psi(t)\mathbb{E}_{\mathcal{W}_{H^U}}\left[(S^u(\omega))(s)(S^v(\omega))(t)\right] dsdt = \mathbb{E}_{\mathcal{W}_{H^U}}\left[{}_{\mathscr{S}}\langle \varphi, \omega * \mu^{*u}\rangle_{\mathscr{S}'} {}_{\mathscr{S}}\langle \psi, \omega * \mu^{*v}\rangle_{\mathscr{S}'}\right] \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{2i(u-v)\,\mathrm{Arctg}(2\lambda)}}{1+4\lambda^2}\, \widehat{\varphi}(\lambda)\, \overline{\widehat{\psi}(\lambda)} d\lambda = \frac{2}{\pi}\iiint\limits_{\mathbb{R}^3} \frac{e^{i[(t-s)\lambda + 2(u-v)\,\mathrm{Arctg}(2\lambda)]}}{1+4\lambda^2}\, \varphi(s)\psi(t)\, dsdtd\lambda. \end{aligned}$$

Hence,

$$\mathbb{E}_{\mathcal{W}_{H^U}}[S^u(\omega))(s)(S^v(\omega))(t)] = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{i[(t-s)\lambda + 2(u-v)\,\mathrm{Arctg}(2\lambda)]}}{1+4\lambda^2}\, d\lambda, \tag{4.1}$$

first for almost every and then, by continuity, for all $(s,t) \in \mathbb{R}^2$. In particular, we now know that the $\mathcal{W}_{H^U}$-distribution of $\{S^u(\omega))(t) : (u,t) \in \mathbb{R}^2\}$ is stationary.

To show that there is a continuous version of this process, we will use Kolmogorov's continuity criterion, which, because it is stationary and gaussian, comes down to showing that

$$\left|1 - \mathbb{E}_{\mathcal{W}_{H^U}}[(S^u(\omega))(s)(S^v(\omega))(t)]\right| \le C\left|(u,s) - (v,t)\right|^{\alpha}$$

for some $C < \infty$ and $\alpha > 0$. But

$$\left|1 - \mathbb{E}_{\mathcal{W}_{H^U}}[(S^u(\omega))(s)(S^v(\omega))(t)]\right| \leq \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{1+4\lambda^2}\left|e^{i[(t-s)\lambda + 2(u-v)\operatorname{Arctg}(2\lambda)]} - 1\right|$$

$$\leq \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{1+4\lambda^2}\left|e^{i(t-s)\lambda} - 1\right| + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{1+4\lambda^2}\left|e^{2i(u-v)\operatorname{Arctg}(2\lambda)} - 1\right|$$

$$\leq \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{1+4\lambda^2}(|t-s||\lambda| \wedge 2) + \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{1+4\lambda^2}|(u-v)\operatorname{Arctg}(2\lambda)|,$$

and, after simple estimation, this shows that

$$|1 - \mathbb{E}[(S^u(\omega))(s)(S^v(\omega))(t)]| \leq C\left[|u-v| + |t-s|\left(1 + \log\left(1 + \frac{1}{(t-s)^2}\right)\right)\right],$$

where $C < \infty$. Clearly, the desired conclusion follows.

Remark 4.1. A question about filtrations comes naturally when one considers the group of transformations $(S^u)_{u\in\mathbb{R}}$ on the space $\mathcal{U}$. Indeed, for all $t, u \in \mathbb{R}$, let $\mathcal{F}_t^u$ be the σ-algebra generated by the $\mathcal{W}_{H^U}$-negligible subsets of $\mathcal{U}$ of and the variables $(S^u(\omega))(s)$, for $s \in (-\infty, t]$ (these variables are well-defined up to a negligible set). From the results of Jeulin and Yor, one quite easily deduces the following properties of the filtrations of the form $(\mathcal{F}_t^u)_{t\in\mathbb{R}}$ for $u \in \mathbb{R}$:

- For all $t, u \in \mathbb{R}$, $\mathcal{F}_t^u$ is generated by $\mathcal{F}_t^{u+1}$ and $(S^u(\omega))(t)$.
- For all $t, u \in \mathbb{R}$, $\mathcal{F}_t^{u+1}$ and $(S^u(\omega))(t)$ are independent under $\mathcal{W}_{H^U}$.
- For all $t, u \in \mathbb{R}$, the decreasing intersection of $\mathcal{F}_t^{u+n}$ for $n \in \mathbb{Z}$ is trivial (i.e. it satisfies the zero-one law).
- If $u \in \mathbb{R}$ is fixed, the σ-algebra generated by $\mathcal{F}_t^{u+n}$ for $t \in \mathbb{R}$ does not depend on $n \in \mathbb{Z}$.

All these statements concern the sequence of filtrations $(\mathcal{F}^{u+n})_{n\in\mathbb{Z}}$ for fixed $u \in \mathbb{R}$. A natural question arises: how can these results be extended to the continuous family of filtrations $(\mathcal{F}^u)_{u\in\mathbb{R}}$? Unfortunately, for the moment, we have no answer to this question (in particular the family does not seem to be decreasing with u).

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