

ON PENALISATION RESULTS RELATED WITH A REMARKABLE CLASS OF SUBMARTINGALES

JOSEPH NAJNUDEL AND ASHKAN NIKEGHBALI

ABSTRACT. In this paper, we study penalisations of diffusions satisfying some technical conditions, generalizing a result obtained by Najnudel, Roynette and Yor in [12]. If one of these diffusions has probability distribution \mathbb{P} , then our result can be described as follows: for a large class of families of probability measures $(\mathbb{Q}_t)_{t \geq 0}$, each of them being absolutely continuous with respect to \mathbb{P} , there exists a probability \mathbb{Q}_∞ such that for all events Λ depending only on the canonical trajectory up to a fixed time, $\mathbb{Q}_t(\Lambda)$ tends to $\mathbb{Q}_\infty(\Lambda)$ when t goes to infinity. In the cases we study here, the limit measure \mathbb{Q}_∞ is absolutely continuous with respect to a sigma-finite measure \mathcal{Q} , which does not depend on the choice of the family of probabilities $(\mathbb{Q}_t)_{t \geq 0}$, but only on \mathbb{P} . The relation between \mathbb{P} and \mathcal{Q} is obtained in a very general framework by the authors of this paper in [11].

1. INTRODUCTION

In a series of articles by Roynette, Vallois and Yor, summarized in [15], the authors study many examples of probability measures on functional spaces, which are obtained as weak limits of measures which are absolutely continuous with respect to a given probability. The setting generally used is the following: one considers \mathbb{W} , the Wiener measure on the space of continuous functions from \mathbb{R}_+ to \mathbb{R} , denoted $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and endowed with its canonical filtration $(\mathcal{F}_s)_{s \geq 0}$ (not completed). One defines the σ -algebra \mathcal{F} by

$$\mathcal{F} := \bigvee_{s \geq 0} \mathcal{F}_s.$$

One then considers $(\Gamma_t)_{t \geq 0}$, a family of nonnegative random variables on the same space such that

$$0 < \mathbb{W}[\Gamma_t] < \infty,$$

and for $t \geq 0$, one defines the probability measure

$$\mathbb{Q}_t := \frac{\Gamma_t}{\mathbb{W}[\Gamma_t]} \cdot \mathbb{W}.$$

(In this paper, if \mathbb{P} is a probability measure and Y a random variable, we denote by $\mathbb{P}[Y]$ the expectation of Y with respect to \mathbb{P}). Under these assumptions, Roynette, Vallois and Yor have proved that for many examples of families of functionals $(\Gamma_t)_{t \geq 0}$, there exists a probability measure \mathbb{Q}_∞ which can be considered as the weak limit of $(\mathbb{Q}_t)_{t \geq 0}$ when t goes to infinity, in the following sense: for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$,

$$\mathbb{Q}_t[\Lambda_s] \xrightarrow{t \rightarrow \infty} \mathbb{Q}_\infty[\Lambda_s].$$

For example, the measure \mathbb{Q}_∞ exists for the following families of functionals $(\Gamma_t)_{t \geq 0}$:

Date: December 24, 2009.

- $\Gamma_t = \phi(L_t)$, where $(L_t)_{t \geq 0}$ is the local time at zero of the canonical process X , and ϕ is a nonnegative, integrable function from \mathbb{R}_+ to \mathbb{R}_+ .
- $\Gamma_t = \phi(S_t)$, where S_t is the supremum of X on the interval $[0, t]$, and ϕ is, again, a nonnegative, integrable function from \mathbb{R}_+ to \mathbb{R}_+ .
- $\Gamma_t = e^{-\int_0^t q(X_s) ds}$, where q is a measurable function from \mathbb{R} to \mathbb{R}_+ , such that:

$$0 < \int_{\mathbb{R}} (1 + |x|)q(x) dx < \infty.$$

- $\Gamma_t = e^{\lambda L_t + \mu |X_t|}$, where $(L_t)_{t \geq 0}$ is, again, the local time at zero of X .

There are still other interesting particular cases which can be studied. For instance in [8] Najnudel has proved that the limit measure exists for

$$\Gamma_t = \exp \left(- \int_{\mathbb{R}} (L_t^y)^2 dy \right)$$

where $(L_t^y)_{t \geq 0, y \in \mathbb{R}}$ is the regular family of local times of X . This example can be interpreted as the construction of a one-dimensional self-avoiding Brownian motion, and hence, a one-dimensional version of Edwards' model, for polymers of infinite length (this model was studied with several points of view: see for example [6], [19], [3], [18]). Another family of penalisation of probability measures on general functional spaces has been introduced by the authors of this paper in [10], where the functional (Γ_t) is of the form $\Gamma_t = F_t X_t$, for a large class of functionals (F_t) . Note that in all these examples, the proof of the existence of \mathbb{Q}_∞ given by the authors cited above contains two steps which need to be clarified here. The first point is that the functional Γ_t is, in general, not defined everywhere but only almost everywhere. For example, since $(\mathcal{F}_s)_{s \geq 0}$ is not completed, there does not exist a càdlàg and adapted version of the local time which is defined everywhere (a more detailed discussion of this problem is given, for example, in our article [9]). However, the almost sure existence of Γ_t is sufficient to define the measure \mathbb{Q}_t , and to study its weak convergence towards \mathbb{Q}_∞ . The second point is that the existence of \mathbb{Q}_∞ depends on the possibility one has to extend compatible families of probability measures. More precisely, the penalisation results cited above are proved as follows: by studying the asymptotics (for fixed $s \geq 0$ and for t going to infinity) of a suitable version of the conditional expectation of Γ_t given \mathcal{F}_s , one proves that for all $s \geq 0$, there exists an \mathcal{F}_s -measurable, nonnegative random variable M_s such that for all events $\Lambda_s \in \mathcal{F}_s$:

$$\mathbb{Q}_t(\Lambda_s) \xrightarrow[t \rightarrow \infty]{} \mathbb{W}(M_s \mathbf{1}_{\Lambda_s}). \quad (1.1)$$

One immediately deduces that $(M_s)_{s \geq 0}$ is a martingale, and that there exists a compatible family $(\mathbb{Q}_\infty^{(s)})_{s \geq 0}$ of probability measures, $\mathbb{Q}_\infty^{(s)}$ defined on \mathcal{F}_s , such that for $s \geq 0$, $\Lambda_s \in \mathcal{F}_s$:

$$\mathbb{Q}_t(\Lambda_s) \xrightarrow[t \rightarrow \infty]{} \mathbb{Q}_\infty^{(s)}(\Lambda_s).$$

Hence, if one can construct a probability measure on the space $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F})$, such that its restriction to \mathcal{F}_s is $\mathbb{Q}_\infty^{(s)}$, the existence of \mathbb{Q}_∞ is proved. This possibility of extension of measures is not obvious at all: for example, as explained in [9], \mathbb{Q}_∞ does not exist in general if one replaces the filtered probability space $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{W})$ by its usual augmentation. However, in the setting described above, \mathbb{Q}_∞ exists, because the filtered measurable space $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (\mathcal{F}_s)_{s \geq 0})$ satisfies the property (P), described as follows:

Definition 1.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0})$ be a filtered measurable space, such that \mathcal{F} is the σ -algebra generated by \mathcal{F}_s , $s \geq 0$: $\mathcal{F} = \bigvee_{s \geq 0} \mathcal{F}_s$. We say that the property (P) holds if and only if $(\mathcal{F}_s)_{s \geq 0}$ enjoys the following conditions:

- For all $s \geq 0$, \mathcal{F}_s is generated by a countable number of sets.
- For all $s \geq 0$, there exists a Polish space Ω_s , and a surjective map π_s from Ω to Ω_s , such that \mathcal{F}_s is the σ -algebra of the inverse images, by π_s , of Borel sets in Ω_s , and such that for all $B \in \mathcal{F}_s$, $\omega \in \Omega$, $\pi_s(\omega) \in \pi_s(B)$ implies $\omega \in B$.
- If $(\omega_n)_{n \geq 0}$ is a sequence of elements of Ω , such that for all $N \geq 0$,

$$\bigcap_{n=0}^N A_n(\omega_n) \neq \emptyset,$$

where $A_n(\omega_n)$ is the intersection of the sets in \mathcal{F}_n containing ω_n , then:

$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

This definition is given in [9], but the corresponding conditions are not new: they are already stated by Parthasarathy in [14], p. 141. A fundamental example of filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0})$ satisfying the property (P) is the following: for some integer $d \geq 1$, Ω is the space of continuous functions from \mathbb{R}_+ to \mathbb{R}^d , or the space of càdlàg functions from \mathbb{R}_+ to \mathbb{R}^d , for all $s \geq 0$, \mathcal{F}_s is the σ -algebra generated by the canonical process up to time s , and \mathcal{F} is the σ -algebra generated by $(\mathcal{F}_s)_{s \geq 0}$. This example proves that in the examples studied by Najnudel, Roynette, Vallois and Yor, the measure \mathbb{Q}_∞ can be constructed. Indeed, one has the following result, proved in [9], with methods coming from Stroock and Varadhan (see [17]):

Proposition 1.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0})$ be a filtered measurable space satisfying the property (P), and let, for $s \geq 0$, \mathbb{Q}_s be a probability measure on (Ω, \mathcal{F}_s) , such that for all $t \geq s \geq 0$, \mathbb{Q}_s is the restriction of \mathbb{Q}_t to \mathcal{F}_s . Then, there exists a unique measure \mathbb{Q}_∞ on (Ω, \mathcal{F}) such that for all $s \geq 0$, its restriction to \mathcal{F}_s is equal to \mathbb{Q}_s .*

All the examples of penalisation results described above involve the Wiener space. In this paper, we need to deal with a more general setting, for which it is important to have, at the same time, the classical results of stochastic calculus generally proved under usual conditions, and the possibility of extending compatible families of probability measures. Since this extension is in general impossible under usual conditions, we need to complete more carefully the probability spaces. The good way to do this completion, intermediate between the right-continuous version and the usual augmentation, involves the so-called natural conditions or N-usual conditions. They were first introduced by Bichteler in [2], and then rediscovered in [9] where it is shown that most of the properties which are generally proved under usual conditions remain true under natural conditions (for example, existence of càdlàg versions of martingales, Doob-Meyer decomposition, début theorem, etc.). Let us recall here the definition:

Definition 1.3. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ satisfies the natural conditions if and only if the two following assumptions hold:

- The filtration $(\mathcal{F}_s)_{s \geq 0}$ is right-continuous;

- For all $s \geq 0$, and for every \mathbb{P} -negligible set $A \in \mathcal{F}_s$, all the subsets of A are contained in \mathcal{F}_0 .

This definition is slightly different from the definitions given in [2] and [9] but one can easily check that it is equivalent. The natural enlargement of a filtered probability space can be defined by using the following proposition:

Proposition 1.4 ([9]). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ be a filtered probability space. There exists a unique filtered probability space $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_s)_{s \geq 0}, \tilde{\mathbb{P}})$ (with the same set Ω), such that:*

- For all $s \geq 0$, $\tilde{\mathcal{F}}_s$ contains \mathcal{F}_s , $\tilde{\mathcal{F}}$ contains \mathcal{F} and $\tilde{\mathbb{P}}$ is an extension of \mathbb{P} ;
- The space $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_s)_{s \geq 0}, \tilde{\mathbb{P}})$ satisfies the natural conditions;
- For any filtered probability space $(\Omega, \mathcal{F}', (\mathcal{F}'_s)_{s \geq 0}, \mathbb{P}')$ satisfying the two items above, \mathcal{F}'_s contains $\tilde{\mathcal{F}}_s$ for all $s \geq 0$, \mathcal{F}' contains $\tilde{\mathcal{F}}$ and \mathbb{P}' is an extension of $\tilde{\mathbb{P}}$.

The space $(\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_s)_{s \geq 0}, \tilde{\mathbb{P}})$ is called the natural enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$.

Intuitively, the natural enlargement of a filtered probability space is its smallest extension which satisfies the natural conditions. Now, if we combine the natural enlargement with the property (P), we obtain the following definition:

Definition 1.5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ be a filtered probability space. We say that it satisfies the property (NP) if and only if it is the natural enlargement of a filtered probability space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_s)_{s \geq 0}, \mathbb{P}^0)$ such that the filtered measurable space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_s)_{s \geq 0})$ enjoys property (P).

The following result about extension of probability measures is proved in [9] (in a slightly more general form):

Proposition 1.6 ([9]). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ be a filtered probability space, satisfying property (NP). Then, the σ -algebra \mathcal{F} is the σ -algebra generated by $(\mathcal{F}_s)_{s \geq 0}$, and for all coherent families of probability measures $(\mathbb{Q}_s)_{s \geq 0}$, such that \mathbb{Q}_s is defined on \mathcal{F}_s , and is absolutely continuous with respect to the restriction of \mathbb{P} to \mathcal{F}_s , there exists a unique probability measure \mathbb{Q} on \mathcal{F} which coincides with \mathbb{Q}_s on \mathcal{F}_s for all $s \geq 0$.*

The possibility of extension of coherent families of probability measures, on spaces satisfying the property (NP) implies that one can obtain penalisation results in this framework. For the functionals we shall study in this article, the limit measure is absolutely continuous with respect to a remarkable σ -finite measure constructed in a very general setting in [11]. This σ -finite measure was already encountered in different special cases (see [1], [5], [7] and [12] for example), and it involves a remarkable class of submartingales, called (Σ) . The submartingales of class (Σ) were first introduced by Yor in [20], and some of their main properties were studied in [13]. Let us recall the definition:

Definition 1.7. Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ be a filtered probability space. A nonnegative submartingale (resp. local submartingale) $(X_s)_{s \geq 0}$ is of class (Σ) , if and only if it can be decomposed as $X_s = N_s + A_s$ where $(N_s)_{s \geq 0}$ and $(A_s)_{s \geq 0}$ are $(\mathcal{F}_s)_{s \geq 0}$ -adapted processes satisfying the following assumptions:

- $(N_s)_{s \geq 0}$ is a càdlàg martingale (resp. local martingale);
- $(A_s)_{s \geq 0}$ is a continuous increasing process, with $A_0 = 0$;
- The measure (dA_s) is carried by the set $\{s \geq 0, X_s = 0\}$.

In [11], it is proved that for any submartingale of class (Σ) defined on a space satisfying property (NP), one can associate with it a σ -finite measure as follows:

Theorem 1.8. *Let $(X_s)_{s \geq 0}$ be a submartingale of the class (Σ) (in particular X_s is integrable for all $s \geq 0$), defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ which satisfies the property (NP). In particular $(\mathcal{F}_s)_{s \geq 0}$ satisfies the natural conditions and \mathcal{F} is the σ -algebra generated by \mathcal{F}_s , $s \geq 0$. Then, there exists a unique σ -finite measure \mathcal{Q} , defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that for $g := \sup\{s \geq 0, X_s = 0\}$:*

- $\mathcal{Q}[g = \infty] = 0$;
- For all $s \geq 0$, and for all \mathcal{F}_s -measurable, bounded random variables F_s ,

$$\mathcal{Q}[F_s \mathbf{1}_{g \leq s}] = \mathbb{P}[F_s X_s].$$

In this article, we prove that there is a link between penalisations and the measure \mathcal{Q} involved in Theorem 1.8. More precisely, we show that if $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ is the natural augmentation of the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ endowed with its canonical filtration and the law of a diffusion satisfying some technical conditions, if $(X_s)_{s \geq 0}$ is the canonical process, and if $(\Gamma_t)_{t \geq 0}$ is a nonincreasing family of nonnegative random variables on the same space, tending to Γ_∞ when t goes to infinity, and such that

$$0 < \mathbb{P}[\Gamma_t] < \infty$$

and

$$0 < \mathcal{Q}[\Gamma_\infty] < \infty,$$

then under some precise assumptions stated below, the probability measure

$$\mathbb{Q}_t := \frac{\Gamma_t}{\mathbb{P}[\Gamma_t]} \cdot \mathbb{P},$$

converges weakly (in the sense of penalisations) to the limit

$$\mathbb{Q}_\infty := \frac{\Gamma_\infty}{\mathcal{Q}[\Gamma_\infty]} \cdot \mathcal{Q}.$$

In the Brownian case, a similar penalisation result, involving the measure \mathcal{Q} , is obtained in [12].

The present paper is organized as follows: in Section 2, we detail the precise setting under which our penalisation result is available and we state rigorously this result, in Section 3, we give some estimates of hitting times of the canonical diffusion, and in Section 4, we use these estimates in order to complete the proof of our main theorem.

2. THE GENERAL SETTING

The general framework in which one can state our penalisation result was introduced by Salminen, Vallois and Yor in [16], in a slightly different and more general form, and it was also used by Najnudel, Roynette and Yor in [12], Chapter 3. Note that in both cases the filtration is not completed. In this present paper, we use the natural augmentation, which allows us to define the local time everywhere, and to apply Theorem 1.8.

The underlying filtered probability space is constructed as follows. Let Ω be the space of continuous functions from \mathbb{R}_+ to \mathbb{R}_+ , $(\mathcal{F}_s^0)_{s \geq 0}$, the natural filtration of Ω (not completed), and \mathcal{F}^0 , the σ -algebra generated by $(\mathcal{F}_s^0)_{s \geq 0}$. We assume that the following holds:

- the probability \mathbb{P}^0 , defined on (Ω, \mathcal{F}^0) , is such that under \mathbb{P}^0 , the canonical process is a recurrent diffusion in natural scale, starting from a fixed point $x_0 \geq 0$, with zero as an instantaneously reflecting barrier;
- the speed measure $m(x)$ of this diffusion is absolutely continuous with respect to Lebesgue measure on \mathbb{R}_+ , with a continuous density $m : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$;
- we assume that $m(x)$ is equivalent to cx^β when x goes to infinity, for some $c > 0$ and $\beta > -1$, and we suppose that there exists $C > 0$ such that for all $x > 0$, $m(x) \leq Cx^\beta$ if $\beta \leq 0$, and $m(x) \leq C(1 + x^\beta)$ if $\beta > 0$;
- the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ is constructed as the natural augmentation of the space $(\Omega, \mathcal{F}^0, (\mathcal{F}_s^0)_{s \geq 0}, \mathbb{P}^0)$: $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ satisfies the property (NP) and under \mathbb{P} , the law of the canonical process is a diffusion with the same parameters as under \mathbb{P}^0 .

This diffusion is in natural scale and, as in [12], one can deduce that the canonical process $(X_s)_{s \geq 0}$ is a submartingale of class (Σ) . In particular, X_s is integrable for all $s \geq 0$, moreover, the local time $(L_s)_{s \geq 0}$ of $(X_s)_{s \geq 0}$ at level zero, is its increasing process. One deduces that Theorem 1.8 applies, and one can construct the corresponding measure \mathcal{Q} . Let us check that we are in the situation where $L_\infty = \infty$, \mathbb{P} -almost surely. Indeed, let $T^{(0)}$ be the first hitting time of zero by $(X_s)_{s \geq 0}$, and for $n \geq 1$, let $T^{(n)}$ be the first hitting time of zero after $T^{(n-1)} + 1$. The variables $(T^{(n)})_{n \geq 0}$ are stopping times, \mathbb{P} -almost surely finite since $(X_s)_{s \geq 0}$ is recurrent. By the strong Markov property the variables $(L_{T^{(n)}} - L_{T^{(n-1)}})_{n \geq 1}$ are i.i.d., which implies that $L_\infty = 0$ almost surely, or $L_\infty = \infty$ almost surely. Let us suppose that the first case holds. One deduces that $(X_s)_{s \geq 0}$ is a martingale, and by the optional sampling theorem, for all $s, u \geq 0$,

$$\mathbb{P}[X_{(T^{(0)} \wedge u) + s} | \mathcal{F}_{T^{(0)} \wedge u}] = X_{T^{(0)} \wedge u},$$

which implies:

$$\mathbb{P}[X_{T^{(0)} + s} \mathbf{1}_{T^{(0)} \leq u}] = \mathbb{P}[X_{T^{(0)}} \mathbf{1}_{T^{(0)} \leq u}] = 0,$$

and then $X_{T^{(0)} + s} = 0$ a.s. on the event $\{T^{(0)} \leq u\}$. Since u can be arbitrarily chosen and $T^{(0)} < \infty$ a.s., $X_{T^{(0)} + s} = 0$ a.s., which contradicts the fact that zero is an instantaneously reflecting barrier. The fact that $L_\infty = \infty$ implies that $g^{[a]} < \infty$, \mathcal{Q} -almost everywhere, for

$$g^{[a]} := \sup\{s \geq 0, X_s \leq a\},$$

as proved in [10]. This property is an important element in the proof of our penalisation result, which can now be rigorously stated. The relevant class of functionals $(\Gamma_t)_{t \geq 0}$ is defined as follows:

Definition 2.1. Let us suppose that the assumptions of Theorem 1.8 are satisfied. We say that a process $(\Gamma_t)_{t \geq 0}$ belongs to the class (C) if it is nonnegative, uniformly bounded, nonincreasing, càdlàg and adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$, if there exists $a > 0$ such that for all $t \geq 0$, $\Gamma_t = \Gamma_{g^{[a]}}$ on the set $\{t \geq g^{[a]}\}$, and if the decreasing limit of Γ_t at infinity, denoted Γ_∞ , is \mathcal{Q} -integrable.

This definition is stated in [10], and in a slightly different way in [12]. The main result of this article is the following:

Theorem 2.2. *We suppose that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, \mathbb{P})$ and the diffusion process X are constructed as above. Let \mathcal{Q} be the σ -finite measure associated with X*

from Theorem 1.8 and let $(\Gamma_t)_{t \geq 0}$ be a process in the class (C), such that $\mathcal{Q}[\Gamma_\infty] > 0$. For all $t \geq 0$:

$$0 < \mathbb{P}[\Gamma_t] < \infty,$$

and one can define a probability measure \mathbb{Q}_t on (Ω, \mathcal{F}) by

$$\mathbb{Q}_t := \frac{\Gamma_t}{\mathbb{P}[\Gamma_t]} \cdot \mathbb{P}.$$

Then the probability measure:

$$\mathbb{Q}_\infty := \frac{\Gamma_\infty}{\mathcal{Q}[\Gamma_\infty]} \cdot \mathcal{Q}$$

is the weak limit of \mathbb{Q}_t in the sense of penalisations, i.e. for all $s \geq 0$, and for all events $\Lambda_s \in \mathcal{F}_s$,

$$\mathbb{Q}_t[\Lambda_s] \xrightarrow{t \rightarrow \infty} \mathbb{Q}_\infty[\Lambda_s].$$

The proof of Theorem 2.2 is given in Section 3 and Section 4.

Remark 2.3. This penalisation result applies in particular for the power $2r$ of a Bessel process of dimension $2(1-r)$, if $0 < r < 1$. For $r = 1/2$, the corresponding diffusion is a reflected Brownian motion, and our result is very similar to the result obtained for Brownian motion in [12]. The proof of Theorem 2.2 is given in Sections 3 and 4.

3. ESTIMATES OF HITTING TIMES

In order to prove Theorem 2.2, we need to estimate the distribution of the hitting times of the canonical process $(X_s)_{s \geq 0}$ under \mathbb{P} . These estimates involve Sturm-Liouville equation in a crucial way, that is why we first prove the following lemma, giving some information about the solutions of this equation:

Lemma 3.1. *Let A be a continuous function from \mathbb{R}_+^* to \mathbb{R}_+^* , integrable in the neighborhood of zero, but not in the neighborhood of infinity. Then there exists a unique function Φ_A from \mathbb{R}_+ to \mathbb{R} , continuous on \mathbb{R}_+ , bounded, twice differentiable on \mathbb{R}_+^* , such that $\Phi_A(0) = 1$ and satisfying Sturm-Liouville equation:*

$$\Phi_A''(t) = A(t)\Phi_A(t) \tag{3.1}$$

for all $t > 0$. This function is strictly positive, decreasing to zero at infinity, and continuously differentiable everywhere in \mathbb{R}_+ . Moreover, if A_1 and A_2 are two functions satisfying the same assumptions as A , and if $A_1(t) \geq A_2(t)$ for all $t > 0$, then $|\Phi_{A_1}'(0)| \geq |\Phi_{A_2}'(0)|$.

Proof. By Cauchy-Lipschitz theorem, for all $t_0 > 0$, $a, b \in \mathbb{R}$, there exists a unique maximal solution $\Theta_{t_0, a, b}$ of (3.1) on an interval of the form $[t_0, v)$ for some $v \in (t_0, \infty]$, which satisfies $\Theta_{t_0, a, b}(t_0) = a$ and $\Theta_{t_0, a, b}'(t_0) = b$. Since A is locally bounded on \mathbb{R}_+^* , $\Theta_{t_0, a, b}$ is in fact well-defined on $[t_0, \infty)$. By linearity, one has:

$$\Theta_{t_0, a, b} = a\Theta_{t_0, 1, 0} + b\Theta_{t_0, 0, 1}.$$

For all $t > t_0$, $\Theta_{t_0, 1, 0}(t)$, $\Theta_{t_0, 1, 0}'(t)$, $\Theta_{t_0, 0, 1}(t)$ and $\Theta_{t_0, 0, 1}'(t)$ are strictly positive, moreover, $\Theta_{t_0, 1, 0}$ and $\Theta_{t_0, 0, 1}$ tend to infinity at infinity. On the other hand, let us suppose that $\Theta_{t_0, a, b}$ is not bounded from above (resp. from below). Then there exists $t > t_0$ such that $\Theta_{t_0, a, b}(t) > 0$ and $\Theta_{t_0, a, b}'(t) \geq 0$ (resp. $\Theta_{t_0, a, b}(t) < 0$ and $\Theta_{t_0, a, b}'(t) \leq 0$): one deduces that $\Theta_{t_0, a, b}$ tends to

infinity (resp. minus infinity) at infinity. Hence, for $b_1 > b_2$, at least one of the two following holds:

- Θ_{t_0,a,b_1} tends to infinity at infinity;
- Θ_{t_0,a,b_2} tends to minus infinity at infinity.

Indeed, if none of these holds, then Θ_{t_0,a,b_1} is bounded from above, and Θ_{t_0,a,b_2} is bounded from below, which implies that $\Theta_{t_0,0,1}$ is bounded from above: a contradiction. Now, for b large enough, $\Theta_{t_0,a,b}(t_0 + 1)$ and $\Theta'_{t_0,a,b}(t_0 + 1)$ are strictly positive, which implies that $\Theta_{t_0,a,b}$ tends to infinity at infinity, similarly, for b small enough, $\Theta_{t_0,a,b}$ tends to minus infinity at infinity. One deduces that there exists $b(t_0, a)$ such that for $b > b(t_0, a)$, $\Theta_{t_0,a,b}$ tends to infinity and for $b < b(t_0, a)$, $\Theta_{t_0,a,b}$ tends to minus infinity. Let us suppose that for some $t \geq t_0$, $\Theta_{t_0,a,b(t_0,a)}(t)$ and $\Theta'_{t_0,a,b(t_0,a)}(t)$ have the same sign, and that at least one of them is different from zero. Since one can increase t by a small quantity, one can suppose that $\Theta_{t_0,a,b(t_0,a)}(t)$ and $\Theta'_{t_0,a,b(t_0,a)}(t)$ have strictly the same sign (recall that $\Theta_{t_0,a,b(t_0,a)}(t) > 0$ implies that $\Theta''_{t_0,a,b(t_0,a)}(t) > 0$). One deduces that for b sufficiently close to $b(t_0, a)$, $\Theta_{t_0,a,b}(t)$ and $\Theta'_{t_0,a,b}(t)$ have the same sign, independent of b . Therefore, $\Theta_{t_0,a,b}$ and $\Theta'_{t_0,a,b}$ tend to a limit equal to infinity or minus infinity, independently of b : this is a contradiction. At this step, we know that for all $t \geq t_0$, $\Theta_{t_0,a,b(t_0,a)}(t) = \Theta'_{t_0,a,b(t_0,a)}(t) = 0$ or $\Theta_{t_0,a,b(t_0,a)}(t)\Theta'_{t_0,a,b(t_0,a)}(t) < 0$. If the first case holds for some $t \geq t_0$, by using Cauchy-Lipschitz theorem in both directions of the time, we deduce that $a = b(t_0, a) = 0$ and $\Theta_{t_0,a,b(t_0,a)}$ is identically zero. Otherwise, $\Theta_{t_0,a,b(t_0,a)}^2$ is strictly decreasing, which implies that it remains strictly positive, hence, $\Theta_{t_0,a,b(t_0,a)}$ is strictly positive and strictly decreasing for $a > 0$, strictly negative and strictly increasing for $a < 0$. In particular, it converges almost surely to a limit l . If $l > 0$, then for all $t \geq t_0$, $\Theta_{t_0,a,b(t_0,a)}(t) \geq l$ and $\Theta''_{t_0,a,b(t_0,a)}(t) \geq lA(t)$, which implies that $\Theta'_{t_0,a,b(t_0,a)}$ tends to infinity at infinity, since A is not integrable at infinity: this is impossible. Since $l < 0$ gives also a contradiction, one has $l = 0$. To summarize, we have proved that for all $t_0 > 0$ and all $a \in \mathbb{R}$, there exists a unique solution $\Theta_{t_0,a}$ of (3.1) defined on $[t_0, \infty)$, equal to a at t_0 , and which does not tend to infinity or minus infinity at infinity. This solution is identically zero if $a = 0$, it is strictly monotone and tends to zero at infinity, if $a \neq 0$. In fact, by linearity, one has $\Theta_{t_0,a} = a\Theta_{t_0,1}$ for all $a \in \mathbb{R}$. The uniqueness of $\Theta_{t_0,a}$ implies that for $t_0 > t_1 > 0$, $a \in \mathbb{R}$, the restriction of $\Theta_{t_1,a}$ to $[t_0, \infty)$ is equal to $\Theta_{t_0,b}$ where $b = \Theta_{t_1,a}(t_0)$. This compatibility implies that if we define Θ from \mathbb{R}_+^* to \mathbb{R}_+^* by:

$$\Theta(t) = \Theta_{1,1}(t)$$

for $t \geq 1$, and

$$\Theta(t) = \frac{1}{\Theta_{t,1}(1)}$$

for $t < 1$, then Θ is a solution of (3.1), strictly decreasing and tending to zero at infinity. Now, since $|\Theta'|$ is decreasing, one has, for all $t \in (0, 1)$,

$$\Theta(t) \leq \Theta(1) + \int_t^1 |\Theta'(s)| ds \leq 1 + |\Theta'(t)| \leq L|\Theta'(t)|$$

for $L := 1 + 1/|\Theta'(1)|$. One deduces that

$$\Theta''(t) \leq LA(t)|\Theta'(t)|,$$

and then

$$\left| \frac{d}{dt} \log |\Theta'(t)| \right| \leq LA(t).$$

Since A is integrable in the neighborhood of zero, $|\Theta'(t)|$, and then $\Theta(t)$, is bounded for $t \in (0, 1)$. Since Θ is decreasing, it can be defined at zero by continuity. Moreover, $\Theta'(t)$ converges when t goes to zero, and one can check that its limit is equal to the derivative at zero of the extension of Θ . One then deduces the existence and the properties of regularity of Φ_A by setting, for all $t \geq 0$:

$$\Phi_A(t) = \frac{\Theta(t)}{\Theta(0)}.$$

To prove uniqueness, let us suppose that Φ_1 and Φ_2 satisfy the conditions given in Lemma 3.1. The functions Φ_1 and Φ_2 cannot vanish on \mathbb{R}_+^* : otherwise, by the properties of the solutions of (3.1) on the intervals of the form $[t_0, \infty)$ for $t_0 > 0$, they would be identically zero. Then Φ_1, Φ_2 are strictly positive, and they are proportional to each other on every interval of the form $[t_0, \infty)$ for $t_0 > 0$. In other words, Φ_1/Φ_2 is constant on $[t_0, \infty)$ for all $t_0 > 0$, and then on \mathbb{R}^* , finally, on \mathbb{R}_+ by continuity at zero. Since $\Phi_1(0) = \Phi_2(0) = 1$, $\Phi_1 = \Phi_2$ everywhere. Let us now suppose that $A_1 \geq A_2$ satisfy the assumptions of Lemma 3.1, and that $|\Phi'_{A_1}(0)| < |\Phi'_{A_2}(0)|$. One has $\Phi'_{A_1}(0) > \Phi'_{A_2}(0)$. Let $T > 0$ be the infimum of the times t such that $\Phi'_{A_1}(t) \leq \Phi'_{A_2}(t)$. On the interval $(0, T)$, one has $\Phi'_{A_1} > \Phi'_{A_2}$, and then $\Phi_{A_1} \geq \Phi_{A_2}$, since $\Phi_{A_1}(0) = \Phi_{A_2}(0)$. Now, since $A_1 \geq A_2$, one deduces that

$$\Phi''_{A_1} = A_1 \Phi_{A_1} \geq A_2 \Phi_{A_2} = \Phi''_{A_2},$$

which implies:

$$\Phi'_{A_1} \geq \Phi'_{A_2} + \Phi'_{A_1}(0) - \Phi'_{A_2}(0). \quad (3.2)$$

If T is supposed to be finite, one deduces a contradiction at time T . Hence T is infinite and 3.2 holds everywhere. One deduces that

$$\Phi_{A_1}(t) - \Phi_{A_2}(t) \geq t (\Phi'_{A_1}(0) - \Phi'_{A_2}(0))$$

for all $t \geq 0$, and then $\Phi_{A_1} - \Phi_{A_2}$ tends to infinity at infinity, which is absurd. \square

Once Lemma 3.1 is proved, one can state and show the results on hitting times which are involved in the proof of Theorem 2.2. In the sequel of the paper, for any $x \geq 0$, we denote by \mathbb{P}_x the distribution of a diffusion which starts at x , with the same parameters as the canonical process under \mathbb{P} . In particular, $\mathbb{P}_{x_0} = \mathbb{P}$. With this notation, one has the following lemma:

Lemma 3.2. *For all $\lambda > 0$, there exists a unique function Φ_λ such that $\Phi_\lambda(0) = 1$ and for all $x \geq y \geq 0$:*

$$\mathbb{P}_x[e^{-\lambda T_y}] = \frac{\Phi_\lambda(x\lambda^\alpha)}{\Phi_\lambda(y\lambda^\alpha)},$$

where T_y is the first hitting time of y by the canonical process, and $\alpha := 1/(\beta + 2)$ lies in the interval $(0, 1)$. Moreover, there exists $\lambda_0 > 0$ such that the following properties hold:

- $\Phi_\lambda(t)$ converges to one when (λ, t) goes to zero.
- $\Phi_\lambda(t)$ converges to zero when t goes to infinity, uniformly in $\lambda \in (0, \lambda_0)$.
- For all $\lambda > 0$, Φ_λ is continuously differentiable everywhere in \mathbb{R}_+ .
- The map: $(\lambda, t) \longrightarrow \Phi'_\lambda(t)$ is uniformly bounded on $(0, \lambda_0) \times \mathbb{R}_+$.
- $|\Phi'_\lambda(t)|$ converges to a constant $K > 0$ when (λ, t) goes to zero.

Proof. Let us define, for all $u \geq 0$:

$$\Psi_\lambda(u) := \mathbb{P}_u[e^{-\lambda T_0}].$$

By the strong Markov property one has

$$\mathbb{P}_x[e^{-\lambda T_0}] = \mathbb{P}_x[e^{-\lambda T_y}] \mathbb{P}_y[e^{-\lambda T_0}],$$

and then

$$\mathbb{P}_x[e^{-\lambda T_y}] = \frac{\Phi_\lambda(x\lambda^\alpha)}{\Phi_\lambda(y\lambda^\alpha)},$$

where for $t \geq 0$,

$$\Phi_\lambda(t) := \Psi_\lambda(t\lambda^{-\alpha}).$$

One deduces the existence of Φ_λ , its uniqueness is clear. Now, by classical properties of diffusions (see [4] for example), the functions Ψ_λ , and then Φ_λ , are continuous on \mathbb{R}_+ , twice differentiable on \mathbb{R}_+^* and Ψ_λ satisfies:

$$\mathcal{G}\Psi_\lambda = \lambda \Psi_\lambda$$

where \mathcal{G} is the infinitesimal generator of the diffusion X . One deduces:

$$\Phi_\lambda''(t) = t^\beta p(t\lambda^{-\alpha}) \Phi_\lambda(t),$$

for

$$p(u) := u^{-\beta} m(u).$$

By assumption, $p(u)$ is strictly positive, continuous with respect to $u > 0$, it tends to c when u goes to infinity, and for all $u > 0$,

$$p(u) \leq C (1 + u^{-\beta} \mathbf{1}_{\beta > 0}).$$

One deduces that

$$A_\lambda : t \longrightarrow t^\beta p(t\lambda^{-\alpha})$$

satisfies the assumptions of Lemma 3.1, moreover, with the notation of this Lemma:

$$\Phi_\lambda = \Phi_{A_\lambda},$$

in particular, this function is continuously differentiable everywhere in \mathbb{R}_+ (third item of Lemma 3.2). Now, let us define the functions:

$$A_0 : t \longrightarrow c t^\beta,$$

and for $\epsilon > 0$, $\lambda \geq 0$:

$$A_{\lambda, \epsilon} : t \longrightarrow A_\lambda(t + \epsilon).$$

Since p tends to c at infinity, for all $\epsilon > 0$, there exists $\lambda(\epsilon) \in (0, 1)$ such that for all $\lambda \in [0, \lambda(\epsilon)]$:

$$(1 - \epsilon)A_{0, \epsilon} \leq A_{\lambda, \epsilon} \leq (1 + \epsilon)A_{0, \epsilon}.$$

All these functions satisfy the conditions of Lemma 3.1, and one deduces:

$$\left| \Phi'_{(1-\epsilon)A_{0, \epsilon}}(0) \right| \leq \left| \Phi'_{A_{\lambda, \epsilon}}(0) \right| \leq \left| \Phi'_{(1+\epsilon)A_{0, \epsilon}}(0) \right|$$

Now, by checking Sturm-Liouville equation, one sees that for all $t \geq 0$:

$$\Phi_{A_{\lambda, \epsilon}}(t) = \frac{\Phi_{A_\lambda}(t + \epsilon)}{\Phi_{A_\lambda}(\epsilon)},$$

$$\Phi_{(1-\epsilon)A_0,\epsilon}(t) = \frac{\Phi_{A_0} [(1-\epsilon)^\alpha(t+\epsilon)]}{\Phi_{A_0} [(1-\epsilon)^\alpha(\epsilon)]}$$

and

$$\Phi_{(1+\epsilon)A_0,\epsilon}(t) = \frac{\Phi_{A_0} [(1+\epsilon)^\alpha(t+\epsilon)]}{\Phi_{A_0} [(1+\epsilon)^\alpha(\epsilon)]}.$$

Therefore:

$$K_1(\epsilon) \leq \frac{|\Phi'_{A_\lambda}(\epsilon)|}{\Phi_{A_\lambda}(\epsilon)} \leq K_2(\epsilon),$$

where

$$K_1(\epsilon) := (1-\epsilon)^\alpha \frac{|\Phi'_{A_0} [(1-\epsilon)^\alpha(\epsilon)]|}{\Phi_{A_0} [(1-\epsilon)^\alpha(\epsilon)]}$$

and

$$K_2(\epsilon) := (1+\epsilon)^\alpha \frac{|\Phi'_{A_0} [(1+\epsilon)^\alpha(\epsilon)]|}{\Phi_{A_0} [(1+\epsilon)^\alpha(\epsilon)]}$$

do not depend on $\lambda \leq \lambda(\epsilon)$ and tend to $K := |\Phi'_{A_0}(0)|$ when ϵ goes to zero. One deduces that

$$|\Phi'_{A_\lambda}(\epsilon)| \leq K_2(\epsilon)$$

and

$$|\Phi'_{A_\lambda}(0)| \leq K_2(\epsilon) + \int_0^\epsilon A_\lambda(u) \Phi_{A_\lambda}(u) du \leq K_2(\epsilon) + C \int_0^\epsilon u^\beta [1 + (u\lambda^{-\alpha})^{-\beta} \mathbf{1}_{\beta>0}] du.$$

Since one supposes $\lambda \leq \lambda(\epsilon) \leq 1$, one deduces that for all $t \geq 0$:

$$|\Phi'_{A_\lambda}(t)| \leq |\Phi'_{A_\lambda}(0)| \leq K_2(\epsilon) + I(\epsilon),$$

where

$$I(\epsilon) := C \int_0^\epsilon (1 + u^\beta) du < \infty$$

decreases to zero when ϵ goes to zero. In other words:

$$|\Phi'_{A_\lambda}(t)| \leq K_3(\epsilon),$$

where $K_3(\epsilon) < \infty$ decreases to K when ϵ goes to zero. In particular, for $\lambda \leq \lambda(1)$:

$$|\Phi'_{A_\lambda}(t)| \leq K_3(1),$$

which proves the fourth item of Lemma 3.2. Moreover, for all $\epsilon > 0$:

$$\limsup_{(\lambda,t) \rightarrow 0} |\Phi'_{A_\lambda}(t)| \leq \sup_{\lambda \in [0, \lambda(\epsilon)], t \in \mathbb{R}_+} |\Phi'_{A_\lambda}(t)| \leq K_3(\epsilon),$$

and then, by taking ϵ going to zero,

$$\limsup_{(\lambda,t) \rightarrow 0} |\Phi'_{A_\lambda}(t)| \leq K. \tag{3.3}$$

On the other hand, again for $\lambda \in [0, \lambda(\epsilon)]$, one has

$$\Phi_{A_\lambda}(\epsilon) \geq 1 - \int_0^\epsilon |\Phi'_{A_\lambda}(u)| du \geq 1 - \epsilon K_3(\epsilon),$$

and then

$$|\Phi'_{A_\lambda}(\epsilon)| \geq [1 - \epsilon K_3(\epsilon)]_+ \frac{|\Phi'_{A_\lambda}(\epsilon)|}{\Phi_{A_\lambda}(\epsilon)} \geq [1 - \epsilon K_3(\epsilon)]_+ K_1(\epsilon) =: K_4(\epsilon),$$

where $K_4(\epsilon)$ tends to K when ϵ tends to zero. We deduce that

$$|\Phi'_{A_\lambda}(t)| \geq K_4(\epsilon)$$

for all $t \leq \epsilon$, and then

$$\liminf_{(\lambda,t) \rightarrow 0} |\Phi'_{A_\lambda}(t)| \geq \inf_{\lambda \in [0, \lambda(\epsilon)], t \in [0, \epsilon]} |\Phi'_{A_\lambda}(t)| \geq K_4(\epsilon).$$

This implies, by letting ϵ go to zero, that

$$\liminf_{(\lambda,t) \rightarrow 0} |\Phi'_{A_\lambda}(t)| \geq K. \quad (3.4)$$

The inequalities (3.3) and (3.4) imply the fifth item of Lemma 3.2. Moreover, for all $\lambda \geq 0$ and $t \geq 0$:

$$1 - t|\Phi'_{A_\lambda}(0)| \leq \Phi_{A_\lambda}(t) \leq 1.$$

Hence, for $\lambda \leq \lambda(1)$:

$$1 - tK_3(1) \leq \Phi_{A_\lambda}(t) \leq 1.$$

Since $tK_3(1)$ tends to zero when $(\lambda, t) \in [0, \lambda(1)] \times \mathbb{R}_+$ tends to zero, one deduces the first item of Lemma 3.2. It now remains to prove the second item. Let us suppose that $\lambda \leq 1$ and $t \geq 1$. In this case:

$$A_{\lambda,t} \geq \delta A_{0,t},$$

where

$$\delta := \frac{1}{c} \inf_{v \in [1, \infty)} p(v) > 0.$$

One deduces that

$$\frac{|\Phi'_{A_\lambda}(t)|}{\Phi_{A_\lambda}(t)} \geq \delta^\alpha \frac{|\Phi'_{A_0}(\delta^\alpha t)|}{\Phi_{A_0}(\delta^\alpha t)},$$

or

$$\frac{d}{dt} [\log(\Phi_{A_\lambda}(t))] \leq \frac{d}{dt} [\log(\Phi_{A_0}(\delta^\alpha t))].$$

By integrating and taking the exponential, one obtains

$$\Phi_{A_\lambda}(t) \leq \Phi_{A_\lambda}(1) \frac{\Phi_{A_0}(\delta^\alpha t)}{\Phi_{A_0}(\delta^\alpha)},$$

which implies, for all $\lambda \leq 1$ and $t \geq 1$:

$$\Phi_{A_\lambda}(t) \leq \theta(t),$$

where

$$\theta(t) = \frac{\Phi_{A_0}(\delta^\alpha t)}{\Phi_{A_0}(\delta^\alpha)}.$$

We can remark that this upper bound trivially holds for $t < 1$, since $\theta(t)$ is greater than or equal to one in this case. Now, $\theta(t)$ tends to zero when t goes to infinity (since Φ_{A_0} tends to zero), independently of $\lambda \leq 1$, which completes the proof of Lemma 3.2. \square

The following lemma gives some information about the marginal distributions of the canonical process under \mathbb{P}_x :

Lemma 3.3. *One can define a function χ from \mathbb{R}_+^* to \mathbb{R}_+ , tending to zero at zero, and satisfying the following property: for all $u > 0$, there exists $t_0(u) > 0$ such that for all $x \geq 0$, $t \geq t_0(u)$:*

$$\mathbb{P}_x[X_t \leq t^\alpha u] \leq \chi(u).$$

Proof. Let us fix $u > 0$. By coupling, one obtains immediately, for all $x \geq 0$, $t > 0$,

$$\mathbb{P}_x[X_t \leq t^\alpha u] \leq \mathbb{P}_0[X_t \leq t^\alpha u].$$

Now one has (see, for example, [12], p. 89):

$$\mathbb{P}_0[X_t \leq t^\alpha u] = \int_0^{t^\alpha u} p(t, 0, y) m(y) dy = \int_0^{t^\alpha u} p(t, y, 0) m(y) dy,$$

where p is the density of the semi-group of $(X_t)_{t \geq 0}$. Again by coupling, one sees that $p(t, y, 0) \leq p(t, 0, 0)$, which implies that

$$\mathbb{P}_x[X_t \leq t^\alpha u] \leq p(t, 0, 0) \int_0^{t^\alpha u} C (y^\beta + \mathbb{1}_{\beta > 0}) dy \leq C p(t, 0, 0) \left[(t^\alpha u)^{\beta+1} / (\beta + 1) + t^\alpha u \mathbb{1}_{\beta > 0} \right].$$

Now, if $t \geq u^{-1/\alpha}$, one has necessarily $t^\alpha u \leq (t^\alpha u)^{\beta+1}$ for $\beta > 0$, which implies:

$$\mathbb{P}_x[X_t \leq t^\alpha u] \leq \frac{C(\beta + 2)}{\beta + 1} p(t, 0, 0) (t^\alpha u)^{\beta+1} = \chi_0(u) t^{1-\alpha} p(t, 0, 0)$$

where

$$\chi_0(u) := \frac{C(\beta + 2)}{\beta + 1} u^{\beta+1}.$$

Hence, we are done, provided that there exists $L_1 > 0$ such that $p(t, 0, 0) \leq L_1 t^{\alpha-1}$ for t large enough. If $t > t' > 0$, one has

$$p(t, 0, 0) = \mathbb{P}_0[p(t', X_{t-t'}, 0)] \leq p(t', 0, 0),$$

which implies that $p(t, 0, 0)$ is decreasing with respect to t . One deduces:

$$p(t, 0, 0) \leq \frac{2e}{t} \int_{t/2}^t e^{-s/t} p(s, 0, 0) ds \leq \frac{2e}{t} R(1/t),$$

where for $\lambda > 0$,

$$R(\lambda) := \int_0^\infty e^{-s\lambda} p(s, 0, 0) ds.$$

Now from a result by Salminen, Vallois and Yor in [16] (p. 5, equation just after (3)), one obtains

$$\int_0^\infty m(y) \mathbb{P}_y[e^{-\lambda T_0}] dy = \frac{1}{\lambda R(\lambda)},$$

which implies, with the notation of Lemma 3.2, that

$$p(t, 0, 0) \leq 2e \left(\int_0^\infty m(y) \Phi_{1/t}(yt^{-\alpha}) dy \right)^{-1}.$$

Therefore we only need to prove that for some $L_2 > 0$, and for t large enough

$$I := \int_0^\infty m(y) \Phi_{1/t}(yt^{-\alpha}) dy \geq L_2 t^{1-\alpha}.$$

Now, from Lemma 3.2 (first item), we know that there exist $t_0, v_0 > 0$ such that for all $t \geq t_0$ and all $v \leq v_0$, $\Phi_{1/t}(v) \geq 1/2$. One deduces that for all $t \geq t_0 \vee v_0^{-1/\alpha}$, $v_0 t^\alpha \geq 1$:

$$I \geq \frac{1}{2} \int_{1/2}^{v_0 t^\alpha} m(y) dy \geq \delta \frac{(v_0 t^\alpha)^{\beta+1} - (1/2)^{\beta+1}}{2(\beta+1)} \geq \frac{\delta v_0^{\beta+1} (1 - (1/2)^{\beta+1})}{2(\beta+1)} t^{1-\alpha}.$$

where δ is the infimum of $y^{-\beta} m(y)$ on $[1/2, \infty)$, strictly positive since $m(y)$ is strictly positive, continuous and equivalent to cy^β at infinity. \square

From Lemma 3.3, one deduces the following result, which majorizes the probability, for the canonical process, to hit $[0, a]$ during a small interval of time:

Lemma 3.4. *One can define a function ρ from $(0, 1)$ to \mathbb{R}_+^* , tending to zero at zero, and satisfying the following property: for all $a \geq 0$, $\gamma \in (0, 1)$, there exists $t_0(a, \gamma)$ such that for every $x \geq 0$, $t \geq t_0(a, \gamma)$:*

$$\mathbb{P}_x [\exists s \in [(1-\gamma)t, t], X_s \leq a] \leq \rho(\gamma).$$

Proof. Let us fix a and γ . One has immediately:

$$\mathbb{P}_x [\exists s \in [(1-\gamma)t, t], X_s \leq a] = \mathbb{P}_x [\Delta(X_{(1-\gamma)t})],$$

where, for $y \leq a$:

$$\Delta(y) = 1$$

and for $y > a$:

$$\Delta(y) = \mathbb{P}_y [T_a \leq \gamma t].$$

In any case, one has:

$$\Delta(y) \leq e \mathbb{P}_{y \vee a} [e^{-T_a/(\gamma t)}] \leq e \frac{\Phi_{1/(\gamma t)} [y(\gamma t)^{-\alpha}]}{\Phi_{1/(\gamma t)} [a(\gamma t)^{-\alpha}]}.$$

Now, there exist $\lambda_1, v_1 > 0$ such that $\Phi_\lambda(v) \geq e/3$ for all $\lambda \leq \lambda_1$ and $v \leq v_1$ (first item of Lemma 3.2). Hence, there exists $t_1 > 0$ (depending on γ and a) such that for $t \geq t_1$:

$$\Phi_{1/(\gamma t)} [a(\gamma t)^{-\alpha}] \geq e/3$$

and then for all $y \geq 0$:

$$\Delta(y) \leq 3 \Phi_{1/(\gamma t)} [y(\gamma t)^{-\alpha}]$$

By the second item of Lemma 3.2, there exists a function q , bounded by one, decreasing to zero at infinity, such that for all $\lambda \leq \lambda_0$, $v > 0$:

$$\Phi_\lambda(v) \leq q(v).$$

Hence, if $t \geq t_1 \vee 1/(\lambda_0 \gamma)$, for all $y \geq 0$,

$$\Delta(y) \leq 3q [y(\gamma t)^{-\alpha}],$$

which implies:

$$\mathbb{P}_x [\exists s \in [(1-\gamma)t, t], X_s \leq a] \leq 3 \mathbb{P}_x [q (X_{(1-\gamma)t} (\gamma t)^{-\alpha})],$$

and then, for any $v \geq 0$:

$$\mathbb{P}_x [\exists s \in [(1-\gamma)t, t], X_s \leq a] \leq 3 (q(v) + \mathbb{P}_x [X_{(1-\gamma)t} \leq (\gamma t)^\alpha v]).$$

We now fix

$$v := \left(\frac{\gamma}{1-\gamma} \right)^{-\alpha/2}.$$

One deduces, by Lemma 3.3, that there exists $t_2 \geq 0$ (depending on a and γ), such that for $t \geq t_2$:

$$\mathbb{P}_x [\exists s \in [(1-\gamma)t, t], X_s \leq a] \leq 3 \left(q \left[\left(\frac{\gamma}{1-\gamma} \right)^{-\alpha/2} \right] + \chi \left[\left(\frac{\gamma}{1-\gamma} \right)^{\alpha/2} \right] \right) =: \rho(\gamma),$$

which tends to zero at zero. \square

We have now all the estimates of hitting times needed in the proof of Theorem 2.2, which is finished in Section 4.

4. PROOF OF THE MAIN THEOREM

In order to prove Theorem 2.2, we essentially need to estimate the behaviour of the expectation of Γ_t under \mathbb{P} , when t goes to infinity. This expectation will be implicitly splitted as follows:

$$\mathbb{P}[\Gamma_t] = \mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} \leq s}] + \mathbb{P}[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t}] + \mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} > (1-\gamma)t}],$$

where $s \geq 0$, $\gamma \in (0, 1/2)$ and

$$g_t^{[a]} := \sup\{u \in [0, t], X_u \leq a\}$$

(recall that $g^{[a]}$ denotes the supremum of $u \in \mathbb{R}_+$ such that $X_u \leq a$). Moreover, we shall use a Tauberian theorem: this is the reason why we assume that the speed measure $m(x)$ of the diffusion $(X_s)_{s \geq 0}$ behaves like a power of x at infinity. The proof of Theorem 2.2 is divided into several steps, each of them corresponding to a lemma or a proposition. The first step is the following:

Lemma 4.1. *For all $a > 0$, $s > 0$, $u \geq 0$, and for all bounded, \mathcal{F}_s -measurable, nonnegative functionals Γ_s :*

$$\lambda^{1-\alpha} \int_u^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_s \mathbf{1}_{g_t^{[a]} \leq s} \right] dt \xrightarrow{\lambda \rightarrow 0} K \mathcal{Q} \left[\Gamma_s \mathbf{1}_{g^{[a]} \leq s} \right] < \infty,$$

where K is the constant introduced in the last item of Lemma 3.2.

Proof. One has by the Markov property that

$$\begin{aligned} \lambda^{1-\alpha} \int_s^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_s \mathbf{1}_{g_t^{[a]} \leq s} \right] dt &= \lambda^{1-\alpha} \int_s^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_s \mathbf{1}_{X_s \geq a} \mathbb{P}_{X_s} [T_a > t - s] \right] \\ &= \lambda^{1-\alpha} e^{-\lambda s} \mathbb{P} \left[\Gamma_s \mathbf{1}_{X_s \geq a} \mathbb{P}_{X_s} \left[\int_0^\infty e^{-\lambda u} \mathbf{1}_{T_a > u} du \right] \right] \\ &= \lambda^{1-\alpha} e^{-\lambda s} \mathbb{P} \left[\Gamma_s \mathbf{1}_{X_s \geq a} \mathbb{P}_{X_s} \left[\frac{1 - e^{-\lambda T_a}}{\lambda} \right] \right] \\ &= \lambda^{-\alpha} e^{-\lambda s} \mathbb{P} \left[\Gamma_s \mathbf{1}_{X_s \geq a} \frac{\Phi_\lambda(a\lambda^\alpha) - \Phi_\lambda(X_s\lambda^\alpha)}{\Phi_\lambda(a\lambda^\alpha)} \right] \\ &= \frac{e^{-\lambda s}}{\Phi_\lambda(a\lambda^\alpha)} \mathbb{P} \left[\Gamma_s (X_s - a)_+ \int_0^1 dv |\Phi'_\lambda((a + v(X_s - a)_+) \lambda^\alpha)| \right]. \end{aligned}$$

Now, by last item of Lemma 3.2, for v, a, X_s fixed:

$$|\Phi'_\lambda((a + v(X_s - a)_+) \lambda^\alpha)| \xrightarrow{\lambda \rightarrow 0} K.$$

Moreover, by the fourth item, for $\lambda \leq \lambda_0$:

$$|\Phi'_\lambda((a + v(X_s - a)_+) \lambda^\alpha)| \leq K',$$

where $K' > 0$ does not depend on λ, v, a or X_s . Since $\mathbb{P}[\Gamma_s(X_s - a)_+]$ is dominated by $\mathbb{P}[X_s] < \infty$, one can apply dominated convergence, which yields

$$\lambda^{1-\alpha} \int_s^\infty e^{-\lambda t} \mathbb{P}[\Gamma_s \mathbf{1}_{g_t^{[a]} \leq s}] \xrightarrow{\lambda \rightarrow 0} K \mathbb{P}[\Gamma_s(X_s - a)_+],$$

and Lemma 4.1 for $u = s$. Now, if $M > 0$ majorizes uniformly Γ_s , we have

$$\lambda^{1-\alpha} \left| \int_s^u e^{-\lambda t} \mathbb{P}[\Gamma_s \mathbf{1}_{g_t^{[a]} \leq s}] \right| \leq M|u - s| \lambda^{1-\alpha},$$

which tends to zero with λ . □

The next step is the following:

Lemma 4.2. *Let $a > 0$ be fixed. Then, there exist $u_0 > 0, L > 0$ such that for all $r > 0, u \geq u_0$ and for all \mathcal{F}_r -measurable, bounded, nonnegative functionals Γ_r :*

$$\mathbb{P}[\Gamma_r \mathbf{1}_{g_{r+u}^{[a]} \leq r}] \leq Lu^{-\alpha} \mathcal{Q}[\Gamma_r \mathbf{1}_{g^{[a]} \leq r}].$$

Proof. One has:

$$\begin{aligned} \mathbb{P}[\Gamma_r \mathbf{1}_{g_{r+u}^{[a]} \leq r}] &= \mathbb{P}[\Gamma_r \mathbf{1}_{X_r \geq a} \mathbb{P}_{X_r}[T_a > u]] \\ &\leq \frac{1}{1 - (1/e)} \mathbb{P}[\Gamma_r \mathbf{1}_{X_r \geq a} \mathbb{P}_{X_r}[1 - e^{-T_a/u}]] \\ &\leq 2 \mathbb{P}\left[\Gamma_r \mathbf{1}_{X_r \geq a} \frac{\Phi_{1/u}(au^{-\alpha}) - \Phi_{1/u}(X_r u^{-\alpha})}{\Phi_{1/u}(au^{-\alpha})}\right] \\ &\leq \frac{2}{\Phi_{1/u}(au^{-\alpha})} u^{-\alpha} \mathbb{P}\left[\Gamma_r (X_r - a)_+ \int_0^1 dv |\Phi'_{1/u}((a + v(X_r - a)_+) u^{-\alpha})|\right]. \end{aligned}$$

From the fourth item of Lemma 3.2, one deduces that $|\Phi'_{1/u}|$ is uniformly bounded by a constant L' for $u \geq 1/\lambda_0$. Under these assumptions:

$$\mathbb{P}[\Gamma_r \mathbf{1}_{g_{r+u}^{[a]} \leq r}] \leq \frac{2L'}{\Phi_{1/u}(au^{-\alpha})} u^{-\alpha} \mathbb{P}[\Gamma_r (X_r - a)_+] \leq 3L' u^{-\alpha} \mathcal{Q}[\Gamma_r \mathbf{1}_{g^{[a]} \leq r}],$$

for u sufficiently large in order to make sure that $\Phi_{1/u}(au^{-\alpha}) \geq 2/3$. □

Now, let us prove the following:

Lemma 4.3. *Let $a > 0$ and let $(\Gamma_t)_{t \geq 0}$ be a càdlàg, adapted, nonnegative, uniformly bounded and nonincreasing process, such that for all $t \geq 0, X_t = X_{g^{[a]}}$ on the set $\{t \geq g^{[a]}\}$. We define Γ_∞ as the limit of Γ_t for t going to infinity (in particular, $\Gamma_\infty = \Gamma_{g^{[a]}}$ for $g^{[a]} < \infty$),*

and we suppose that Γ_∞ is integrable with respect to \mathcal{Q} . Then, for all $\gamma \in (0, 1/2)$, there exists $R > 0$ such that for all $s \geq 0$:

$$\limsup_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t} \right] dt \leq R \mathcal{Q}[\Gamma_\infty \mathbf{1}_{g^{[a]} > s}].$$

Proof. For all $t > 0$:

$$\mathbb{P} \left[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t} \right] \leq \mathbb{P} \left[\Gamma_{(1-\gamma)t} \mathbf{1}_{g_{(1-\gamma)t}^{[a]} > s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t} \right].$$

Remark that the quantities involved here are equal to zero for $t \leq s/(1-\gamma)$. By Lemma 4.2, one deduces that for $t \geq u_0/\gamma$:

$$\mathbb{P} \left[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t} \right] \leq L(\gamma t)^{-\alpha} \mathcal{Q} \left[\Gamma_{(1-\gamma)t} \mathbf{1}_{g_{(1-\gamma)t}^{[a]} > s} \mathbf{1}_{g^{[a]} \leq (1-\gamma)t} \right].$$

Now, if $g^{[a]} \leq (1-\gamma)t$, $\Gamma_{(1-\gamma)t} = \Gamma_\infty$, which implies:

$$\mathbb{P} \left[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t} \right] \leq L(\gamma t)^{-\alpha} \mathcal{Q}[\Gamma_\infty \mathbf{1}_{g^{[a]} > s}].$$

One now deduces:

$$\lambda^{1-\alpha} \int_{u_0/\gamma}^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t} \right] dt \leq L \Gamma(1-\alpha) \gamma^{-\alpha} \mathcal{Q}[\Gamma_\infty \mathbf{1}_{g^{[a]} > s}].$$

Since

$$\lambda^{1-\alpha} \int_0^{u_0/\gamma} e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{s < g_t^{[a]} \leq (1-\gamma)t} \right] dt \xrightarrow{\lambda \rightarrow 0} 0,$$

we are done. □

Lemma 4.4. *Let $a > 0$ and let $(\Gamma_t)_{t \geq 0}$ be a càdlàg, adapted, nonnegative, uniformly bounded and nonincreasing process, such that for all $t \geq 0$, $X_t = X_{g^{[a]}}$ on the set $\{t \geq g^{[a]}\}$. We define Γ_∞ as the limit of Γ_t for t going to infinity, and we suppose that Γ_∞ is integrable with respect to \mathcal{Q} . Then, for all $\gamma \in (0, 1/2)$, for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$:*

$$\lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t} \right] dt \xrightarrow{\lambda \rightarrow 0} K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}].$$

Proof. Let $v > s$. For $t > v$:

$$\begin{aligned}
\mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v}] &= \mathbb{P} \left[\frac{\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \mathbf{1}_{X_t > a}}{X_t - a} (X_t - a)_+ \right] \\
&= \mathcal{Q} \left[\frac{\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \mathbf{1}_{X_t > a}}{X_t - a} \mathbf{1}_{g^{[a]} \leq t} \right] \\
&= \mathcal{Q} \left[\frac{\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{X_t > a}}{X_t - a} \mathbf{1}_{g^{[a]} \leq v} \right] \\
&= \mathcal{Q} \left[\frac{\Gamma_v \mathbf{1}_{\Lambda_s} \mathbf{1}_{X_t > a}}{X_t - a} \mathbf{1}_{g^{[a]} \leq v} \right] \\
&= \mathcal{Q} \left[\frac{\Gamma_v \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \mathbf{1}_{X_t > a}}{X_t - a} \mathbf{1}_{g^{[a]} \leq t} \right] \\
&= \mathbb{P} \left[\frac{\Gamma_v \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \mathbf{1}_{X_t > a}}{X_t - a} (X_t - a)_+ \right] \\
&= \mathbb{P}[\Gamma_v \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v}]
\end{aligned}$$

By Lemma 4.1, one has

$$\lambda^{1-\alpha} \int_v^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \right] dt \xrightarrow{\lambda \rightarrow 0} K \mathcal{Q}[\Gamma_v \mathbf{1}_{\Lambda_s} \mathbf{1}_{g^{[a]} \leq v}],$$

which implies

$$\lambda^{1-\alpha} \int_w^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \right] dt \xrightarrow{\lambda \rightarrow 0} K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s} \mathbf{1}_{g^{[a]} \leq v}]$$

for all $w \geq 0$. Now, one also has:

$$\int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t} \right] dt \geq \int_{v/(1-\gamma)}^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \right] dt,$$

which implies:

$$\liminf_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t} \right] dt \geq K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s} \mathbf{1}_{g^{[a]} \leq v}]. \quad (4.1)$$

On the other hand:

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t} \right] dt &\leq \int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq v} \right] dt \\
&\quad + \int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{v < g_t^{[a]} \leq (1-\gamma)t} \right] dt,
\end{aligned}$$

which implies by Lemma 4.3 that

$$\limsup_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P} \left[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t} \right] dt \leq K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s} \mathbf{1}_{g^{[a]} \leq v}] + R \mathcal{Q}[\Gamma_\infty \mathbf{1}_{g^{[a]} > v}]. \quad (4.2)$$

By comparing (4.1) and (4.2) and by taking $v \rightarrow \infty$, we are done, since Γ_∞ is \mathcal{Q} -integrable and $g^{[a]} < \infty$, \mathcal{Q} -almost everywhere. \square

The following result gives, for all $s \geq 0$ and all events $\Lambda_s \in \mathcal{F}_s$, an equivalent of the expectation of $\Gamma_t \mathbf{1}_{\Lambda_s}$ under \mathbb{P} , for t going to infinity, if $(\Gamma_t)_{t \geq 0}$ belongs to the class (C).

Proposition 4.5. *Let $(\Gamma_t)_{t \geq 0}$ be a process in the class (C). Then, there exists $D > 0$ such that for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$:*

$$t^\alpha \mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s}] \xrightarrow[t \rightarrow \infty]{} D \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}].$$

Proof. Lemma 4.4 implies immediately (by taking any γ) that

$$\liminf_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s}] dt \geq K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}]. \quad (4.3)$$

Now, for $\lambda > 0$, let us define:

$$I(\lambda) := \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t] dt$$

and for $\lambda > 0$, $\gamma < (0, 1/2)$,

$$I(\lambda, \gamma) := \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t}] dt$$

and

$$J(\lambda, \gamma) := \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} > (1-\gamma)t}] dt.$$

Here $a > 0$ is chosen in order to have $\Gamma_t = \Gamma_{g_t^{[a]}}$ on the event $\{t \geq g_t^{[a]}\}$. For all $t \geq 0$, by Markov property,

$$\mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} > (1-\gamma)t}] \leq \mathbb{P}[\Gamma_{t/2}] \sup_{x \in \mathbb{R}_+} \mathbb{P}_x [\exists u \in [(1/2 - \gamma)t, t/2], X_u \leq a].$$

By Lemma 3.4, for $t \geq 2t_0(a, 2\gamma)$:

$$\mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} > (1-\gamma)t}] \leq \rho(2\gamma) \mathbb{P}[\Gamma_{t/2}].$$

If $M > 0$ majorizes uniformly Γ_t for all $t \geq 0$, one deduces:

$$\int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{g_t^{[a]} > (1-\gamma)t}] dt \leq 2Mt_0(a, 2\gamma) + \rho(2\gamma) \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_{t/2}] dt,$$

which implies:

$$J(\lambda, \gamma) \leq 2M\lambda^{1-\alpha} t_0(a, 2\gamma) + 2\rho(2\gamma)I(\lambda).$$

Therefore:

$$I(\lambda) \leq I(\lambda, \gamma) + 2M\lambda^{1-\alpha} t_0(a, 2\gamma) + 2\rho(2\gamma)I(\lambda),$$

which implies

$$I(\lambda) \leq \frac{I(\lambda, \gamma) + 2M\lambda^{1-\alpha} t_0(a, 2\gamma)}{1 - 2\rho(2\gamma)},$$

and finally

$$J(\lambda, \gamma) \leq 2M\lambda^{1-\alpha} t_0(a, 2\gamma) + 2\rho(2\gamma) \frac{I(\lambda, \gamma) + 2M\lambda^{1-\alpha} t_0(a, 2\gamma)}{1 - 2\rho(2\gamma)},$$

for γ such that $\rho(2\gamma) < 1/2$ (this condition is always satisfied if γ is small enough). Since, by Lemma 4.4,

$$I(\lambda, \gamma) \xrightarrow{\lambda \rightarrow 0} K \mathcal{Q}[\Gamma_\infty],$$

one deduces:

$$\limsup_{\lambda \rightarrow 0} J(\lambda, \gamma) \leq \frac{2K\rho(2\gamma)}{1 - 2\rho(2\gamma)} \mathcal{Q}[\Gamma_\infty],$$

and then,

$$\begin{aligned} \limsup_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s}] dt &\leq \limsup_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s} \mathbf{1}_{g_t^{[a]} \leq (1-\gamma)t}] \\ &\quad + \limsup_{\lambda \rightarrow 0} J(\lambda, \gamma) \\ &\leq K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}] + \frac{2K\rho(2\gamma)}{1 - 2\rho(2\gamma)} \mathcal{Q}[\Gamma_\infty]. \end{aligned}$$

By making $\gamma \rightarrow 0$, one deduces:

$$\limsup_{\lambda \rightarrow 0} \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} \mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s}] dt \leq K \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}] \quad (4.4)$$

By taking (4.3) and (4.4) together and by using Tauberian theorem (recall that Γ_t is decreasing with respect to t), we are done. \square

Now, Theorem 2.2 can be deduced from Proposition 4.5 in a very simple way, as follows. Let us suppose that for some $t \geq 0$, $\Gamma_t = 0$ almost surely with respect to \mathbb{P} . Since all the \mathbb{P} -negligible events in \mathcal{F}_t are also \mathcal{Q} -negligible, one has $\mathcal{Q}[\Gamma_t] = 0$, which contradicts the fact that $\mathcal{Q}[\Gamma_\infty] > 0$, since Γ_t decreases with respect to t . Then, $\mathbb{P}[\Gamma_t] > 0$ for all $t \geq 0$, and obviously $\mathbb{P}[\Gamma_t] < \infty$, since Γ_t is uniformly bounded. Hence, \mathbb{Q}_t is well-defined. Moreover, for all $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\mathbb{Q}_t[\Lambda_s] = \frac{\mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s}]}{\mathbb{P}[\Gamma_t]}.$$

Now by Proposition 4.5,

$$\mathbb{P}[\Gamma_t \mathbf{1}_{\Lambda_s}] \underset{t \rightarrow \infty}{\sim} D t^{-\alpha} \mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}]$$

and

$$\mathbb{P}[\Gamma_t] \underset{t \rightarrow \infty}{\sim} D t^{-\alpha} \mathcal{Q}[\Gamma_\infty].$$

Therefore:

$$\mathbb{Q}_t[\Lambda_s] \xrightarrow{t \rightarrow \infty} \frac{\mathcal{Q}[\Gamma_\infty \mathbf{1}_{\Lambda_s}]}{\mathcal{Q}[\Gamma_\infty]} = \mathbb{Q}_\infty[\Lambda_s],$$

which completes the proof of Theorem 2.2.

REFERENCES

1. A. Bentata and M. Yor, *From Black-Scholes and Dupire formulae to last passage times of local martingales. Part A: The infinite time horizon*, 2008.
2. K. Bichteler, *Stochastic integration with jumps*, Cambridge University Press, 2002.
3. E. Bolthausen, *On the construction of the three-dimensional polymer measure*, Probab. Theory Related Fields **97** (1993), no. 1-2, 81–101.
4. A.-N. Borodin and P. Salminen, *Handbook of Brownian Motion. Facts and Formulae*, Birkhäuser, Basel, 1996.

5. P. Cheridito, A. Nikeghbali, and E. Platen, *Processes of the class sigma, last zero and draw-down processes*, 2009.
6. S.-F. Edwards, *The statistical mechanics of polymers with excluded volume*, Proc. Phys. Soc. **85** (1965), 613–624.
7. D. Madan, B. Roynette, and M. Yor, *From Black-Scholes formula, to local times and last passage times for certain submartingales*, Prépublication IECN 2008/14.
8. J. Najnudel, *Construction of an Edwards' probability measure on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$* , <http://arxiv.org/pdf/0801.2751>, 2008.
9. J. Najnudel and A. Nikeghbali, *A new kind of augmentation of filtrations*, 2009.
10. ———, *On some properties of a universal sigma-finite measure associated with a remarkable class of submartingales*, 2009.
11. ———, *On some universal σ -finite measures and some extensions of Doob's optional stopping theorem*, 2009.
12. J. Najnudel, B. Roynette, and M. Yor, *A Global View of Brownian Penalizations*, MSJ Memoirs, vol. 19, Mathematical Society of Japan, Tokyo, 2009.
13. A. Nikeghbali, *A class of remarkable submartingales*, Stochastic Process. Appl. **116** (2006), no. 6, 917–938.
14. K.-R. Parthasarathy, *Probability measures on metric spaces*, New York: Academic Press, 1967.
15. B. Roynette, P. Vallois, and M. Yor, *Some penalizations of the Wiener measure*, Japan Jour. of Math. **1** (2006), no. 1, 263–290.
16. P. Salminen, P. Vallois, and M. Yor, *On the excursion theory for linear diffusions*, Japan Jour. of Math. **2** (2007), no. 1, 97–127.
17. D.-W. Stroock and S.-R.-S. Varadhan, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006, Reprint of the 1997 edition.
18. R. van der Hofstad, F. den Hollander, and W. König, *Central limit theorem for the Edwards model*, Ann. Probab. **25** (1997), no. 2, 573–597.
19. M.-J. Westwater, *On Edwards' model for long polymer chains*, Comm. Math. Phys. **72** (1980), no. 2, 131–174.
20. M. Yor, *Les inégalités de sous-martingales, comme conséquences de la relation de domination*, Stochastics **3** (1979), no. 1, 1–15.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, 8057-ZÜRICH, SWITZERLAND

E-mail address: joseph.najnudel@math.uzh.ch

E-mail address: ashkan.nikeghbali@math.uzh.ch