ON SOME UNIVERSAL σ-FINITE MEASURES AND SOME EXTENSIONS OF DOOB'S OPTIONAL STOPPING THEOREM

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Abstract. In this paper, we associate, to any submartingale of class (Σ), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, which satisfies some technical conditions, a σ-finite measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$, such that for all $t \geq 0$, and for all events $\Lambda_t \in \mathcal{F}_t$:

$$\mathcal{Q}[\Lambda_t, g \leq t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t]$$

where g is the last hitting time of zero of the process X. This measure $\mathcal{Q}$ has already been defined in several particular cases, some of them are involved in the study of Brownian penalisation, and others are related with problems in mathematical finance. More precisely, the existence of $\mathcal{Q}$ in the general case solves a problem stated by D. Madan, B. Roynette and M. Yor, in a paper studying the link between Black-Scholes formula and last passage times of certain submartingales. Moreover, the equality defining $\mathcal{Q}$ remains true if one replaces the fixed time t by any bounded stopping time. This generalization can be viewed as an extension of Doob's optional stopping theorem.

1. Introduction

This work finds its origin in a recent paper by Madan, Roynette and Yor [7] and a set of lectures by Yor [4] where the authors are able to represent the price of a European put option in terms of the probability distribution of some last passage time. More precisely, they prove that if $(M_t)_{t\geq 0}$ is a continuous nonnegative local martingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions, and such that $\lim_{t\to\infty} M_t = 0$, then

$$(K - M_t)^+ = K\mathbb{P}(g_K \leq t|\mathcal{F}_t) \tag{1.1}$$

where $K \geq 0$ is a constant and $g_K = \sup\{t \geq 0 : M_t = K\}$. Formula (1.1) tells that it is enough to know the terminal value of the submartingale $(K - M_t)^+$ and its last zero g_K to reconstruct it. Yet a nicer interpretation of (1.1) is suggested in [4] and [7]: there exists a measure $\mathcal{Q}$, a random time g, such that the submartingale $X_t = (K - M_t)^+$ satisfies

$$\mathcal{Q}\left[F_t\, \mathbb{1}_{g\leq t}\right] = \mathbb{E}\left[F_t X_t\right], \tag{1.2}$$

for any $t \geq 0$ and for any bounded $\mathcal{F}_t$-measurable random variable F_t. Indeed it easily follows from (1.1) that in this case $\mathcal{Q} = K.\mathbb{P}$ and $g = g_K$. It is also clear that if a stochastic process X satisfies (1.2), then it is a submartingale. The problem of finding the class of submartingales which satisfy (1.2) is posed in [4] and [7] and is the main motivation of this paper:

Problem 1 ([4] and [7]): for which submartingales X can we find a σ-finite measure $\mathcal{Q}$ and the end of an optional set g such that

$$\mathcal{Q}\left[F_t\, \mathbb{1}_{g\leq t}\right] = \mathbb{E}\left[F_t X_t\right]? \tag{1.3}$$

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Identity (1.3) is reminiscent of the stopping theorem for uniformly integrable martingales. Indeed, if M is a càdlàg, uniformly integrable martingale, then it can be represented as $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$, and hence the terminal value of M, i.e. M_∞, is enough to obtain the martingale M. But we also note that if we write $g = \sup\{t \geq 0 : \; M_t = 0\}$, then

$$M_t = \mathbb{E}[M_\infty \mathbb{1}_{g\leq t}|\mathcal{F}_t],$$

since $\mathbb{E}[M_\infty \mathbb{1}_{g>t}|\mathcal{F}_t] = 0$. Thus (1.3) holds for M, where the measure $\mathcal{Q}$ is the signed measure $\mathcal{Q} = M_\infty.\mathbb{P}$. Consequently, the stopping theorem can also be interpreted as the existence of a (signed) measure and the end of an optional set from which one can recover the uniformly integrable martingale M. But (1.3) does not admit a straightforward generalization to martingales which are not uniformly integrable: indeed, such a measure $\mathcal{Q}$ would be real valued and infinite. We hence propose the following problem:
Problem 2: given a continuous martingale M, can we find two σ-finite measures $\mathcal{Q}^{(+)}$ and $\mathcal{Q}^{(-)}$, such that for all $t \geq 0$ and for all bounded $\mathcal{F}_t$-measurable variables F_t:

$$\mathcal{Q}^{(+)}\left[F_t\,\mathbb{1}_{g\leq t}\right] - \mathcal{Q}^{(-)}\left[F_t\,\mathbb{1}_{g\leq t}\right] = \mathbb{E}\left[F_t M_t\right], \tag{1.4}$$

with $g = \sup\{t \geq 0 : \; M_t = 0\}$?
Identities (1.3) and (1.4) can hence be interpreted as an extension of Doob's optional stopping theorem for fixed times t.
It is also noticed in [7] that other instances of formula (1.2) have already been discovered: for example, in [3], Azéma and Yor proved that for any continuous and uniformly martingale M, (1.3) holds for $X_t = |M_t|$, $\mathcal{Q} = |M_\infty|.\mathbb{P}$ and $g = \sup\{t \geq 0 : M_t = 0\}$, or equivalently

$$|M_t| = \mathbb{E}[|M_\infty|\mathbb{1}_{g\leq t}|\mathcal{F}_t].$$

Here again the measure $\mathcal{Q}$ is finite. Recently, other particular cases where the measure $\mathcal{Q}$ is not finite were obtained by Najnudel, Roynette and Yor in their study of Brownian penalisation (see [9]). For example, they prove the existence of the measure $\mathcal{Q}$ when $X_t = |W_t|$ is the absolute value of the standard Brownian Motion. In this case, the measure $\mathcal{Q}$ is not finite but only σ-finite and is singular with respect to the Wiener measure: it satisfies $\mathcal{Q}(g = \infty) = 0$, where $g = \sup\{t \geq 0 : \; W_t = 0\}$.

In the special case where the submartingale X is of class (D), Problem 1 was recently solved[1] in [6] in relation with the study of the draw-down process. In this case, the measure $\mathcal{Q}$ is finite. The relevant family of submartingales is the class (Σ):

Definition 1.1 ([10, 12])**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A nonnegative (local) submartingale $(X_t)_{t\geq 0}$ is of class (Σ), if it can be decomposed as $X_t = N_t + A_t$ where $(N_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted processes satisfying the following assumptions:

- $(N_t)_{t\geq 0}$ is a càdlàg (local) martingale;
- $(A_t)_{t\geq 0}$ is a continuous increasing process, with $A_0 = 0$;
- The measure (dA_t) is carried by the set $\{t \geq 0, X_t = 0\}$.

The definition of the class (Σ) goes back to Yor ([12]) when X is continuous and some of its main properties which we shall use frequently in this paper were studied in [10]. It

[1]In fact, as mentioned in [6], the solution is essentially contained and somehow hidden in [1].

is shown in [1] and [6] that if X is of class (Σ) and of class (D), then it satisfies (1.2) with $g = \sup\{t \geq 0 : \ X_t = 0\}$ and $\mathcal{Q} = X_\infty.\mathbb{P}$, or equivalently

$$X_t = \mathbb{E}[X_\infty \mathbb{1}_{g \leq t} | \mathcal{F}_t].$$

Now, what happens if X is of class (Σ) but satisfies $A_\infty = \infty$ almost surely? If we work on the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous functions endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the coordinate process $(Y_t)_{t\geq 0}$ and with the Wiener measure $\mathbb{W}$, and if $X_t = |Y_t|$, then, as it was already mentioned, the existence of the measure $\mathcal{Q}$, which is singular with respect to $\mathbb{W}$, was established in [9]. Note that in this case, the submartingale $(X_t)_{t\geq 0}$ is not rigorously of class (Σ), because the local time of the canonical process can only be defined almost everywhere. More precisely, it is impossible to construct a continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(L_t)_{t\geq 0}$, defined everywhere, and such that for all $t \geq 0$, L_t is almost surely the local time at zero of $(X_s)_{s\leq t}$ (this fact is discussed in a detailed way in [8]). In order to avoid this technical problem, and to be able to define the local time everywhere (or, more generally, the process $(A_t)_{t\geq 0}$ for more general submartingales), one needs to complete, in a sense which has to be made precise, the filtration $(\mathcal{F}_t)_{t\geq 0}$. However, one cannot prove the exitence of $\mathcal{Q}$ in the most general case if one considers the standard completion. Indeed, let X be a submartingale of class (Σ), let us assume that in the filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets (i.e. the filtration is complete), and let us suppose that under $\mathbb{P}$, $A_\infty = \infty$ almost surely, and then $g = \infty$ a.s. (this property is satisfied, for example, if X is the absolute value of a Brownian motion). For all $t \geq 0$, the event $\{g > t\}$ has probability one (under $\mathbb{P}$) and then, is in $\mathcal{F}_0$ and, a fortiori, in $\mathcal{F}_t$. If one assumes that $\mathcal{Q}$ exists, one has

$$\mathcal{Q}[g > t, g \leq t] = \mathbb{E}_\mathbb{P}[\mathbb{1}_{g>t} X_t],$$

and then:

$$\mathbb{E}_\mathbb{P}[A_t] \leq \mathbb{E}_\mathbb{P}[X_t] = 0$$

which is absurd.
Because of these technical issues, one needs to introduce some special conditions on the filtrations which are considered. These conditions were first introduced by Bichteler in [5] and rediscovered independently by the authors of the present paper in [8]: let us recall them shortly.

Definition 1.2. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$, satisfies the natural conditions if and only if the two following assumptions hold:

- The filtration $(\mathcal{F}_s)_{s\geq 0}$ is right-continuous;
- For all $s \geq 0$, and for every $\mathbb{P}$-negligible set $A \in \mathcal{F}_s$, all the subsets of A are contained in $\mathcal{F}_0$.

This definition is slightly different from the definitions given in [5] and [8] but one can easily check that it is equivalent. The natural enlargement of a filtered probability space can be defined by using the following proposition:

Proposition 1.3. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$ be a filtered probability space. There exists a unique filtered probability space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_s)_{s\geq 0}, \widetilde{\mathbb{P}})$ (with the same set Ω), such that:*

- *For all $s \geq 0$, $\widetilde{\mathcal{F}}_s$ contains $\mathcal{F}_s$, $\widetilde{\mathcal{F}}$ contains $\mathcal{F}$ and $\widetilde{\mathbb{P}}$ is an extension of $\mathbb{P}$;*
- *The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_s)_{s\geq 0}, \widetilde{\mathbb{P}})$ satisfies the natural conditions;*

- *For any filtered probability space $(\Omega, \mathcal{F}', (\mathcal{F}'_s)_{s\geq 0}, \mathbb{P}')$ satisfying the two items above, $\mathcal{F}'_s$ contains $\widetilde{\mathcal{F}}_s$ for all $s \geq 0$, $\mathcal{F}'$ contains $\widetilde{\mathcal{F}}$ and $\mathbb{P}'$ is an extension of $\widetilde{\mathbb{P}}$.*

The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_s)_{s\geq 0}, \widetilde{\mathbb{P}})$ is called the natural enlargement of $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$.

Intuitively, the natural enlargement of a filtered probability space is its smallest extension which satisfies the natural conditions. Let us now define a class of filtered measurable space on which it is always possible to extend compatible families of probability measures.

Definition 1.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered measurable space, such that $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_t$, $t \geq 0$: $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We say that the property[2] (P) holds if and only if $(\mathcal{F}_t)_{t\geq 0}$ enjoys the following properties:

- for all $t \geq 0$, $\mathcal{F}_t$ is generated by a countable number of sets;
- for all $t \geq 0$, there exists a Polish space Ω_t, and a surjective map π_t from Ω to Ω_t, such that $\mathcal{F}_t$ is the σ-algebra of the inverse images, by π_t, of Borel sets in Ω_t, and such that for all $B \in \mathcal{F}_t$, $\omega \in \Omega$, $\pi_t(\omega) \in \pi_t(B)$ implies $\omega \in B$;
- if $(\omega_n)_{n\geq 0}$ is a sequence of elements of Ω, such that for all $N \geq 0$,
$$\bigcap_{n=0}^{N} A_n(\omega_n) \neq \emptyset,$$
where $A_n(\omega_n)$ is the intersection of the sets in $\mathcal{F}_n$ containing ω_n, then:
$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

A fundamental example of a filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ satisfying the property (P) can be constructed as follows: we take Ω to be equal to $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, or $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$, the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ (for some $d \geq 1$), and for $t \geq 0$, we define $(\mathcal{F}_t)_{t\geq 0}$ as the natural filtration of the canonical process, and we set
$$\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t.$$
Now, if we combine the natural enlargement with the property (P), we obtain the following definition:

Definition 1.5. Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$ be a filtered probability space. We say that it satisfies the property (NP) if and only if it is the natural enlargement of a filtered probability space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_s)_{s\geq 0}, \mathbb{P}^0)$ such that the filtered measurable space $(\Omega, \mathcal{F}^0, (\mathcal{F}^0_s)_{s\geq 0})$ enjoys property (P).

The interest of spaces satisfying property (NP) is that they both satisfy natural conditions and the following proposition, which concerns the extension of compatible families of probability measures:

Proposition 1.6. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, satisfying property (NP). Then, the σ-algebra $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_t)_{t\geq 0}$, and for all coherent families of probability measures $(\mathbb{Q}_t)_{t\geq 0}$ such that $\mathbb{Q}_t$ is defined on $\mathcal{F}_t$, and is absolutely*

[2](P) stands for Parthasarathy since such conditions where introduced by him in [11].

continuous with respect to the restriction of $\mathbb{P}$ *to* $\mathcal{F}_t$*, there exists a unique probability measure* $\mathbb{Q}$ *on* $\mathcal{F}$ *which coincides with* $\mathbb{Q}_t$ *on* $\mathcal{F}_t$ *for all* $t \geq 0$*.*

The main goal of this paper is to show that Problem 1 can be solved for all submartingales of the class (Σ) defined on a space satisfying property (NP). The extension of families of probabilities is involved in a crucial way. The measure $\mathcal{Q}$ is constructed explicitly. Since for continuous martingales, M^+ and M^- are of class (Σ), we shall be able to solve Problem 2 and hence interpret our results as an extension of Doob's optional stopping theorem. Our approach is based on martingale techniques only and we are hence able to obtain the measure $\mathcal{Q}$ for a wide range of processes which can possibly jump, thus including the generalized Azéma submartingales in the filtration of the zeros of Bessel processes of dimension in $(0, 2)$ and the draw-down process $X_t = S_t - M_t$ where M is a martingale with no positive jumps and $S_t = \sup_{u \leq t} M_u$. In particular, the existence of $\mathcal{Q}$ does not require any scaling or Markov property for X. More precisely, the paper is organized as follows:

- in Section 2, we state and prove our main theorem about the existence and the uniqueness of the measure $\mathcal{Q}$ for submartingales of the class (Σ). We then deduce the solution to Problem 2, hence interpreting (1.3) and (1.4) together as an extension of Doob's optional stopping theorem. We also give the image of the measure $\mathcal{Q}$ by the functional A_∞;
- in Section 3, we give several examples of such a measure $\mathcal{Q}$ in classical and less classical settings.

2. Construction of the σ-finite measure

2.1. **The main theorem.** We can now state the main result of the paper:

Theorem 2.1. *Let* $(X_t)_{t\geq 0}$ *be a (true) submartingale of the class* (Σ) *(in particular* X_t *is integrable for all* $t \geq 0$*), defined on a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ *which satisfies the property (NP). In particular,* $(\mathcal{F}_t)_{t\geq 0}$ *is right-continuous and* $\mathcal{F}$ *is the* σ*-algebra generated by* $\mathcal{F}_t$*,* $t \geq 0$*. Then, there exists a unique* σ*-finite measure* $\mathcal{Q}$*, defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*, such that for* $g := \sup\{t \geq 0, X_t = 0\}$*:*

- $\mathcal{Q}[g = \infty] = 0$*;*
- *For all* $t \geq 0$*, and for all* $\mathcal{F}_t$*-measurable, bounded random variables* F_t*,*

$$\mathcal{Q}\left[F_t \, \mathbb{1}_{g\leq t}\right] = \mathbb{E}_{\mathbb{P}}\left[F_t X_t\right].$$

Remark 2.2. If $g < \infty$, then $A_\infty = A_g < \infty$. Hence, the first condition satisfied by $\mathcal{Q}$ implies that:

$$\mathcal{Q}[A_\infty = \infty] = 0.$$

In other words, A_∞ is finite, $\mathcal{Q}$-almost everywhere.

Proof. Let f be a Borel function from $\mathbb{R}_+$ to $\mathbb{R}_+$, bounded and integrable, and let, for $x \geq 0$:

$$G(x) := \int_x^\infty f(y)\, dy.$$

By [10] (Theorem 2.1), one immediately checks that the process

$$(M_t^f := G(A_t) + f(A_t)X_t)_{t\geq 0},$$

where A is the increasing process of X, is a nonnegative local martingale. Moreover, for all $t \geq 0$, if N is the martingale part of X and $\mathcal{T}_t$ is the set containing all the stopping times bounded by t, then the family $(N_T)_{T \in \mathcal{T}_t}$ is uniformly integrable (it is included in the set of conditional expectations of N_t, by stopping theorem), and $(A_T)_{T \in \mathcal{T}_t}$ is bounded by A_t (A is increasing), which is integrable (it has the same expectation as $X_t - X_0$). Hence, $(X_T)_{T \in \mathcal{T}_t}$ is uniformly integrable, which implies, since f and G are uniformly bounded, that $(M^f_T)_{T \in \mathcal{T}_t}$ is also uniformly integrable. Hence, M^f is a true martingale. Therefore, by Proposition 1.6, it is possible to construct a finite measure $\mathcal{P}^f$ on $(\Omega, \mathcal{F}, \mathbb{P})$, uniquely determined by:

$$\mathcal{P}^f[\Lambda_s] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_s} M^f_s]$$

for all $s \geq 0$ and for all events $\Lambda_s \in \mathcal{F}_s$. Let us now prove that:

$$\mathcal{P}^f[A_\infty = \infty] = 0.$$

Indeed, for $u \geq 0$, let us consider, as in [10], the right-continuous inverse of A:

$$\tau_u := \inf\{t \geq 0, A_t > u\}.$$

It is easy to check that for $t, u \geq 0$, the event $\{\tau_u \leq t\}$ is equivalent to $\{\forall t' > t, A_{t'} > u\}$, which implies that τ_u is a stopping time (recall that $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous). Moreover, if $\tau_u < \infty$, then $A_{\tau_u} = u$ and $X_{\tau_u} = 0$. Indeed, for all $t > \tau_u$, $A_t > u$, and for all $t < \tau_u$, $A_t \leq u$, which implies the first equality by continuity of A, for $0 < \tau_u < \infty$ (if $\tau_u = 0$ then $u = 0$ and the equality is also true). Moreover, if $X_{\tau_u} > 0$, by right-continuity of X, there exists a.s. $\epsilon > 0$ such that $X > 0$ on the interval $[\tau_u, \tau_u + \epsilon]$, which implies that A is constant on this interval, and then $A_{\tau_u} = A_{\tau_u + \epsilon} > u$, which is a contradiction. Now, for all $t, u \geq 0$,

$$\begin{aligned}
\mathcal{P}^f[A_t > u] &= \mathbb{E}_{\mathbb{P}}\left[(G(A_t) + f(A_t)X_t)\ \mathbb{1}_{A_t > u}\right] \\
&\leq \mathbb{E}_{\mathbb{P}}\left[(G(A_t) + f(A_t)X_t)\ \mathbb{1}_{\tau_u \leq t}\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[(G(A_{\tau_u \wedge t}) + f(A_{\tau_u \wedge t})X_{\tau_u \wedge t})\ \mathbb{1}_{\tau_u \leq t}\right]
\end{aligned}$$

by applying stopping theorem to the stopping time $\tau_u \wedge t$. Therefore:

$$\begin{aligned}
\mathcal{P}^f[A_t > u] &\leq \mathbb{E}_{\mathbb{P}}\left[(G(A_{\tau_u}) + f(A_{\tau_u})X_{\tau_u})\ \mathbb{1}_{\tau_u \leq t}\right] \\
&= G(u)\, \mathbb{P}\left[\tau_u \leq t\right].
\end{aligned}$$

By taking the increasing limit for t going to infinity, one deduces:

$$\mathcal{P}^f\left[\exists t \geq 0, A_t > u\right] \leq G(u)\mathbb{P}\left[\tau_u < \infty\right].$$

This implies:

$$\mathcal{P}^f\left[A_\infty > u\right] \leq G(u),$$

and by taking $u \to \infty$,

$$\mathcal{P}^f\left[A_\infty = \infty\right] = 0.$$

Let us now suppose that $f(x) > 0$ for all $x \geq 0$, and that G/f is uniformly bounded on $\mathbb{R}_+$ (for example, one can take $f(x) = e^{-x}$). Since $\mathcal{P}^f[A_\infty = \infty] = 0$ and $f(A_\infty) > 0$, one can define a measure $\mathcal{Q}^f$ by the following equality:

$$\mathcal{Q}^f[\Lambda] = \mathcal{P}^f\left[\frac{\mathbb{1}_\Lambda}{f(A_\infty)}\right]$$

for all events $\Lambda \in \mathcal{F}$. This measure is σ-finite, since for all $\epsilon > 0$:

$$\mathcal{Q}^f[f(A_\infty) \geq \epsilon] \leq \frac{1}{\epsilon}\,\mathcal{P}^f(1) < \infty.$$

Now, for $t \geq 0$, and F_t, bounded, $\mathcal{F}_t$-measurable:

$$\begin{aligned}\mathcal{Q}^f[F_t\,\mathbb{1}_{g\leq t}] &= \mathcal{P}^f\left[\frac{F_t}{f(A_t)}\,\mathbb{1}_{g\leq t}\right]\\ &= \mathcal{P}^f\left[\frac{F_t}{f(A_t)}\,\mathbb{1}_{d_t=\infty}\right]\end{aligned}$$

since $A_\infty = A_t$ on the event $\{g \leq t\}$, which is equivalent to $\{d_t = \infty\}$, where $d_t = \inf\{v > t, X_v = 0\}$. By the début theorem, proved in [8] under natural conditions, d_t is a stopping time with respect to the filtration $(\mathcal{F}_s)_{s\geq 0}$. One deduces, by applying stopping theorem to the stopping time $d_t \wedge u$,

$$\begin{aligned}\mathcal{P}^f\left[\frac{F_t}{f(A_t)}\,\mathbb{1}_{d_t\leq u}\right] &= \mathbb{E}_{\mathbb{P}}\left[\frac{F_t}{f(A_t)}\,M_u^f\,\mathbb{1}_{d_t\leq u}\right]\\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{F_t}{f(A_t)}\,M_{d_t}^f\,\mathbb{1}_{d_t\leq u}\right]\\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{F_t G(A_t)}{f(A_t)}\,\mathbb{1}_{d_t\leq u}\right]\end{aligned}$$

By taking u going to infinity, one obtains:

$$\mathcal{P}^f\left[\frac{F_t}{f(A_t)}\,\mathbb{1}_{d_t<\infty}\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{F_t G(A_t)}{f(A_t)}\mathbb{1}_{d_t<\infty}\right]$$

Moreover,

$$\mathcal{P}^f\left[\frac{F_t}{f(A_t)}\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{F_t G(A_t)}{f(A_t)} + F_t X_t\right]$$

Therefore,

$$\begin{aligned}\mathcal{P}^f\left[\frac{F_t}{f(A_t)}\,\mathbb{1}_{d_t=\infty}\right] &= \mathbb{E}_{\mathbb{P}}[F_t X_t] + \mathbb{E}_{\mathbb{P}}\left[\frac{F_t G(A_t)}{f(A_t)}\mathbb{1}_{d_t=\infty}\right]\\ &= \mathbb{E}_{\mathbb{P}}[F_t X_t] + \mathbb{E}_{\mathbb{P}}\left[\frac{F_t G(A_\infty)}{f(A_\infty)}\mathbb{1}_{d_t=\infty}\right]\end{aligned}$$

and then:

$$\mathcal{Q}^f[F_t\,\mathbb{1}_{g\leq t}] = \mathbb{E}_{\mathbb{P}}[F_t X_t] + \mathbb{E}_{\mathbb{P}}\left[\frac{F_t G(A_\infty)}{f(A_\infty)}\mathbb{1}_{g\leq t}\right]$$

Now, let us define the measure:

$$\mathcal{P}_1^f := G(A_\infty)\,.\mathbb{P}.$$

and the unique measure $\mathcal{P}_2^f$ such that for all $t \geq 0$, its restriction to $\mathcal{F}_t$ has density:

$$N_t^f := G(A_t) - \mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_t] + f(A_t)X_t$$

with respect to $\mathbb{P}$ (note that $N_t^f \geq 0$, $\mathbb{P}$-a.s.). It is easy to check that the measures $\mathcal{P}^f$ and $\mathcal{P}_1^f + \mathcal{P}_2^f$ have the same restriction to $\mathcal{F}_t$, and by monotone class theorem, they are equal.

Under $\mathcal{P}_1^f$ and $\mathcal{P}_2^f$, the measure of the event $\{A_\infty = \infty\}$ is zero, since these two measures are dominated by $\mathcal{P}^f$. Then, one can define the σ-finite measures:

$$\mathcal{Q}_1^f := \frac{1}{f(A_\infty)} . \mathcal{P}_1^f$$

and

$$\mathcal{Q}_2^f := \frac{1}{f(A_\infty)} . \mathcal{P}_2^f .$$

The measure $\mathcal{Q}^f$ is the sum of $\mathcal{Q}_1^f$ and $\mathcal{Q}_2^f$. Now, we have:

$$\mathcal{Q}_1^f [F_t \, \mathbb{1}_{g \leq t}] = \mathbb{E}_{\mathbb{P}} \left[\frac{F_t G(A_\infty)}{f(A_\infty)} \mathbb{1}_{g \leq t} \right],$$

by using directly the definition of $\mathcal{Q}_1^f$. Moreover, let us recall that:

$$\mathcal{Q}^f [F_t \, \mathbb{1}_{g \leq t}] = \mathbb{E}_{\mathbb{P}}[F_t X_t] + \mathbb{E}_{\mathbb{P}} \left[\frac{F_t G(A_\infty)}{f(A_\infty)} \mathbb{1}_{g \leq t} \right].$$

In particular, since G/f is assumed to be uniformly bounded:

$$\mathcal{Q}^f [F_t \, \mathbb{1}_{g \leq t}] < \infty,$$

This implies that the following equalities are meaningful, and then satisfied, since $\mathcal{Q}^f = \mathcal{Q}_1^f + \mathcal{Q}_2^f$:

$$\begin{aligned}
\mathcal{Q}_2^f [F_t \, \mathbb{1}_{g \leq t}] &= \mathcal{Q}^f [F_t \, \mathbb{1}_{g \leq t}] - \mathcal{Q}_1^f [F_t \, \mathbb{1}_{g \leq t}] \\
&= \left(\mathbb{E}_{\mathbb{P}}[F_t X_t] + \mathbb{E}_{\mathbb{P}} \left[\frac{F_t G(A_\infty)}{f(A_\infty)} \mathbb{1}_{g \leq t} \right] \right) \\
&- \mathbb{E}_{\mathbb{P}} \left[\frac{F_t G(A_\infty)}{f(A_\infty)} \mathbb{1}_{g \leq t} \right] \\
&= \mathbb{E}_{\mathbb{P}}[F_t X_t]
\end{aligned}$$

Hence, the measure $\mathcal{Q}_2^f$ satisfies the second property given in Theorem 2.1. By applying this property to $F_t = f(A_t)$ (which is bounded, since f is supposed to be bounded) and by using the fact that $A_t = A_\infty$ on $\{g \leq t\}$, one deduces:

$$\mathcal{P}_2^f [g \leq t] = \mathbb{E}_{\mathbb{P}}[f(A_t) X_t]$$

and then (by using the fact that for all $t \geq 0$, N_t^f has an expectation equal to the total mass of $\mathcal{P}_2^f$):

$$\mathcal{P}_2^f [g > t] = \mathbb{E}_{\mathbb{P}}[G(A_t) - G(A_\infty)].$$

Since $G(A_t) - G(A_\infty) \leq G(0)$ tends $\mathbb{P}$-a.s. to zero when t goes to infinity, one obtains:

$$\mathcal{P}_2^f [g = \infty] = 0,$$

and

$$\mathcal{Q}_2^f [g = \infty] = 0$$

since $\mathcal{Q}_2^f$ is absolutely continuous with respect to $\mathcal{P}_2^f$. Therefore, the measure $\mathcal{Q}$ exists: let us now prove its uniqueness (which implies, in particular, that $\mathcal{Q}_2^f$ is in fact independent of the choice of f). If $\mathcal{Q}'$ and $\mathcal{Q}''$ satisfy the conditions defining $\mathcal{Q}$, one has, for all $t \geq 0$ and all events $\Lambda_t \in \mathcal{F}_t$:

$$\mathcal{Q}'[\Lambda_t, g \leq t] = \mathcal{Q}''[\Lambda_t, g \leq t]$$

Now let $u > t \geq 0$, and let Λ_u be in $\mathcal{F}_u$. One can check that:

$$\mathcal{Q}'[\Lambda_u, g \leq t] = \mathcal{Q}'[\Lambda_u, d_t > u, g \leq u]$$

One then deduces, by setting $\Lambda'_u := \Lambda_u \cap \{d_t > u\}$ (this event is in $\mathcal{F}_u$):

$$\begin{aligned}\mathcal{Q}'[\Lambda_u, g \leq t] &= \mathcal{Q}'[\Lambda'_u, g \leq u] \\ &= \mathcal{Q}''[\Lambda'_u, g \leq u] \\ &= \mathcal{Q}''[\Lambda_u, g \leq t].\end{aligned}$$

By monotone class theorem, applied to the restrictions of $\mathcal{Q}'$ and $\mathcal{Q}''$ to the set $\{g \leq t\}$, one has for all $\Lambda \in \mathcal{F}$:

$$\mathcal{Q}'[\Lambda, g \leq t] = \mathcal{Q}''[\Lambda, g \leq t].$$

By taking the increasing limit for t going to infinity,

$$\mathcal{Q}'[\Lambda, g < \infty] = \mathcal{Q}''[\Lambda, g < \infty].$$

Now, by assumption:

$$\mathcal{Q}'[g = \infty] = \mathcal{Q}''[g = \infty] = 0,$$

which implies:

$$\mathcal{Q}'[\Lambda] = \mathcal{Q}''[\Lambda].$$

This completes the proof of Theorem 2.1. □

A careful look at the proof of Theorem 2.1 shows that the result is valid if t is replaced by a bounded stopping time T. Moreover, for submartingales of the class (Σ) which are also of class (D), we can take a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual assumptions. More precisely, the following result holds:

Corollary 2.3. *Let $(X_t)_{t\geq 0}$ be a submartingale of the class (Σ), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$.*

(1) *If $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the property (NP), then there exists a unique σ-finite measure $\mathcal{Q}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that for $g := \sup\{t \geq 0, X_t = 0\}$:*
- *$\mathcal{Q}[g = \infty] = 0$;*
- *For any bounded stopping time T, and for all $\mathcal{F}_T$-measurable, bounded random variables F_T,*

$$\mathcal{Q}\left[F_T \mathbb{1}_{g\leq T}\right] = \mathbb{E}_{\mathbb{P}}\left[F_T X_T\right].$$

(2) *If X is of class (D) and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the usual assumptions or the property (NP), then for any stopping time T*

$$X_T = \mathbb{E}[X_\infty \mathbb{1}_{g\leq T} | \mathcal{F}_T],$$

where as usual $g := \sup\{t \geq 0, X_t = 0\}$.

Remark 2.4. Part (2) of Corollary 2.3, under the usual assumptions, is proved in [6].

Let us note that, in the proof of Theorem 2.1, if f does not vanish, is bounded and if G/f is also bounded then the finite measure $\mathcal{P}_2^f$ has density $f(A_\infty)$ with respect to $\mathcal{Q}$. Now, one can prove that, in fact, these conditions on f are not needed. More precisely, one has the following:

Proposition 2.5. *Let us suppose that the assumptions of Theorem 2.1 are satisfied, and let f be an integrable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then, there exists a unique finite (positive) measure $\mathcal{M}^f$ such that:*

$$\mathcal{M}^f[F_t] = \mathbb{E}_{\mathbb{P}}[F_t N_t^f]$$

for all $t \geq 0$, and for all bounded, $\mathcal{F}_t$-measurable functionals F_t, where the process $(N_t^f)_{t\geq 0}$ is given by:

$$N_t^f := G(A_t) - \mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_t] + f(A_t)X_t$$

for

$$G(x) := \int_x^\infty f(y)dy.$$

In particular, $(N_t^f)_{t\geq 0}$ is a nonnegative martingale. Moreover, the measure $\mathcal{M}^f$ is absolutely continuous with respect to $\mathcal{Q}$, with density $f(A_\infty)$.

Proof. In the proof of Theorem 2.1, we have shown this result if f is strictly positive, bounded, and if G/f is also bounded (recall that $G(x)$ is the integral of f between x and infinity). One can now prove Proposition 2.5 for any measurable, bounded, nonnegative functions f with compact support. Indeed, if f is such a function, one can find f_1 and f_2, bounded, strictly positive, integrable, such that, with obvious notation, G_1/f_1 and G_2/f_2 are bounded, and $f = f_1 - f_2$ (for example, one can take $f_1(x) := f(x) + e^{-x}$ and $f_2(x) := e^{-x}$). One has, for all $t \geq 0$, and for all bounded, $\mathcal{F}_t$-measurable random variables F_t:

$$\mathcal{M}^{f_1}[F_t] = \mathbb{E}_{\mathbb{P}}[F_t N_t^{f_1}],$$

and

$$\mathcal{M}^{f_2}[F_t] = \mathbb{E}_{\mathbb{P}}[F_t N_t^{f_2}].$$

Now N^f is the difference of N^{f_1} and N^{f_2}, and then, it is a (nonnegative) martingale. By Proposition 1.6, there exists a unique finite measure $\mathcal{M}$ such that:

$$\mathcal{M}[F_t] = \mathbb{E}_{\mathbb{P}}[F_t N_t^f].$$

Therefore $\mathcal{M}^f$ exists, is unique, and since $\mathcal{M}^{f_1}$ and $\mathcal{M}^{f_2} + \mathcal{M}^f$ coincide on $\mathcal{F}_t$ for all $t \geq 0$:

$$\mathcal{M}^f[F_t] = \mathcal{M}^{f_1}[F_t] - \mathcal{M}^{f_2}[F_t]$$

(this equality is meaningful because all the measures involved here are finite). Since Proposition 2.5 is satisfied for f_1 and f_2:

$$\mathcal{M}^f[F_t] = \mathcal{Q}[F_t f_1(A_\infty)] - \mathcal{Q}[F_t f_2(A_\infty)],$$

which implies

$$\mathcal{M}^f[F_t] = \mathcal{Q}[F_t f(A_\infty)].$$

By monotone class theorem, f satisfies Proposition 2.5. Now, let us only suppose that f is nonnegative and integrable. There exists nonnegative, measurable, bounded functions $(f_k)_{k\geq 1}$ with compact support, such that:

$$f = \sum_{k\geq 1} f_k.$$

With obvious notation, one has:

$$G = \sum_{k\geq 1} G_k,$$

and then, for all $t \geq 0$:

$$G(A_t) = \sum_{k\geq 1} G_k(A_t)$$

and

$$\mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_t] = \sum_{k\geq 1} \mathbb{E}_{\mathbb{P}}[G_k(A_\infty)|\mathcal{F}_t],$$

$\mathbb{P}$-a.s., where the two sums are uniformly bounded by $G(0)$. This boundedness implies that one can substract the second sum from the first, and obtain:

$$G(A_t) - \mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_t] = \sum_{k\geq 1} \left(G_k(A_t) - \mathbb{E}_{\mathbb{P}}[G_k(A_\infty)|\mathcal{F}_t]\right)$$

almost surely. Moreover:

$$f(A_t)X_t = \sum_{k\geq 1} f_k(A_t)X_t,$$

and then, $\mathbb{P}$-a.s.:

$$N_t^f = \sum_{k\geq 1} N_t^{f_k}.$$

We know that $\mathcal{M}^{f_k}$ is well-defined for all $k \geq 1$, hence, one can consider the measure:

$$\mathcal{M} := \sum_{k\geq 1} \mathcal{M}^{f_k}.$$

Now, for $t \geq 0$ and F_t, bounded, $\mathcal{F}_t$-measurable:

$$\begin{aligned} \mathcal{M}[F_t] &= \sum_{k\geq 1} \mathcal{M}^{f_k}[F_t] \\ &= \sum_{k\geq 1} \mathbb{E}_{\mathbb{P}}[F_t N_t^{f_k}] \\ &= \mathbb{E}_{\mathbb{P}}[F_t N_t^f]. \end{aligned}$$

Hence, the measure $\mathcal{M}^f$ is well-defined, unique by monotone class theorem, and is equal to $\mathcal{M}$. Now, one has, for all $k \geq 1$:

$$\mathcal{M}^{f_k} = f_k(A_\infty)\,.\mathcal{Q}.$$

Since $\mathcal{M}^f$ is the sum of the measures $\mathcal{M}^{f_k}$,

$$\mathcal{M}^f = \left[\sum_{k\geq 1} f_k(A_\infty)\right].\mathcal{Q} = f(A_\infty).\mathcal{Q}$$

which completes the proof of Proposition 2.5. □

Another question which is quite natural to ask is the following: since $\mathcal{Q}[A_\infty = \infty] = 0$, what is the image of $\mathcal{Q}$ by the functional A_∞ (in other words, what is the "distribution of A_∞ under $\mathcal{Q}$")? This question can be solved in any case:

Proposition 2.6. *Let us suppose the assumptions of Theorem 2.1. Then, if $(A_t)_{t\geq 0}$ is the increasing process of $(X_t)_{t\geq 0}$, the image by the functional A_∞ of the measure $\mathcal{Q}$ is a measure on $\mathbb{R}_+$, equal to the sum of $\mathbb{E}_{\mathbb{P}}[X_0]$ times Dirac measure at zero, and another measure, absolutely continuous with respect to Lebesgue measure, with density $\mathbb{P}[A_\infty > u]$ at any $u \in \mathbb{R}_+$. In particular, if $A_\infty = \infty$, $\mathbb{P}$-almost surely, then this image measure is $\mathbb{E}_{\mathbb{P}}[X_0]\delta_0 + \mathbb{1}_{\mathbb{R}_+}\lambda$, where λ is Lebesgue measure on $\mathbb{R}_+$, and δ_0 is Dirac measure at zero.*

Proof. Let f be an integrable function from $\mathbb{R}_+$ to $\mathbb{R}_+$. By taking the notation of Proposition 2.5, one has:

$$\mathcal{M}^f = f(A_\infty)\,.\mathcal{Q}.$$

Therefore, $\mathcal{Q}[f(A_\infty)]$ is the total mass of $\mathcal{M}^f$, and then, the expectation of

$$N_0^f = G(0) - \mathbb{E}_{\mathbb{P}}[G(A_\infty)|\mathcal{F}_0] + f(0)X_0.$$

By applying this result to $f = \mathbb{1}_{[0,u]}$, one deduces, for any $u \geq 0$:

$$\begin{aligned}\mathcal{Q}[A_\infty \leq u] &= u - \mathbb{E}_{\mathbb{P}}[(u - A_\infty)_+] + f(0)\mathbb{E}_{\mathbb{P}}[X_0]\\ &= \mathbb{E}_{\mathbb{P}}[A_\infty \wedge u] + f(0)\mathbb{E}_{\mathbb{P}}[X_0]\\ &= \int_0^u \mathbb{P}[A_\infty > v]\,dv + f(0)\mathbb{E}_{\mathbb{P}}[X_0].\end{aligned}$$

which implies Proposition 2.6. □

Remark 2.7. When X is also of class (D), $\mathbb{P}[A_\infty > v]$ is computed in [10], Theorem 4.1.

2.2. **An extension of Doob's optional stopping theorem.** We shall now see how Theorem 2.1 and Corollary 2.3 can be interpreted as an extension of Doob's optional theorem to continuous martingales which are not necessarily uniformly integrable on the one hand, and to the larger class of processes of the class (Σ).

Let M be a continuous martingale; then M^+ and M^- are both of class (Σ). If $g = \sup\{t \geq 0 : M_t = 0\}$, then under the assumptions of Theorem 2.1, there exist two σ-finite measures $\mathcal{Q}^{(+)}$ and $\mathcal{Q}^{(-)}$ such that

- $\mathcal{Q}^{(\pm)}[g = \infty] = 0$;
- For all $t \geq 0$, and for all $\mathcal{F}_t$-measurable, bounded random variables F_t,

$$\mathcal{Q}^{(\pm)}\left[F_t\,\mathbb{1}_{g\leq t}\right] = \mathbb{E}_{\mathbb{P}}\left[F_t M_t^{\pm}\right].$$

Now since $M = M^+ - M^-$, we deduce from Theorem 2.1 and Corollary 2.3 the following solution to Problem 2:

Proposition 2.8. *Let M be a continuous martingale defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ which satisfies the property (NP). Then there exist two σ-finite measures $\mathcal{Q}^{(+)}$ and $\mathcal{Q}^{(-)}$, such that for any bounded stopping time T and any bounded $\mathcal{F}_T$-measurable variable F_T,*

$$\mathcal{Q}^{(+)}\left[F_T\,\mathbb{1}_{g\leq T}\right] - \mathcal{Q}^{(-)}\left[F_T\,\mathbb{1}_{g\leq T}\right] = \mathbb{E}\left[F_T M_T\right],$$

with $g = \sup\{t \geq 0 : \ M_t = 0\}$. The measures $\mathcal{Q}^{(+)}$ and $\mathcal{Q}^{(-)}$ are obtained by applying Theorem 2.1 to the submartingales M^+ and M^-.

Remark 2.9. If the martingale M is uniformly integrable, then following Corollary 2.3, one can also work with a filtration satisfying the usual assumptions and take any stopping time T, not necessarily bounded. Consequently, Proposition 2.8 can be viewed as an extension of Doob's optional stopping theorem: the terminal value of the martingale M has to be replaced by $\left(\mathcal{Q}^{(+)} - \mathcal{Q}^{(-)}\right)$ which is a signed measure when restricted to the sets $\mathbb{1}_{g \leq t}$. Theorem 2.1 and Corollary 2.3 can in turn be interpreted as an extension of the stopping theorem to the larger class of submartingales of the class (Σ).

3. Some examples

Now, let us study in more details several consequences of Theorem 2.1.

3.1. **The case of a the absolute value, or the positive part, of a martingale.** We suppose that $X_t = M_t^+$, $X_t = M_t^-$ or $X_t = |M_t|$, where $(M_t)_{t \geq 0}$ is a continuous martingale, defined on a space satisfying the property (NP). In this case, X is a submartingale of class (Σ), and its increasing process is half of the local time of M at level zero in the two first cases, and the local time of M in the third case. Therefore, one can apply Theorem 2.1. In particular, if $(X_t)_{t \geq 0}$ is a strictly positive martingale, then it is a submartingale of class (Σ), with increasing process identically equal to zero. One deduces that for any nonnegative, integrable function f, $N_t^f = f(0) X_t$, which implies that for all $t \geq 0$, the restriction of $\mathcal{M}^f$ to $\mathcal{F}_t$ has density $f(0) X_t$ with respect to $\mathbb{P}$. Hence, since $f(A_\infty) = f(0)$, the restriction of $\mathcal{Q}$ to $\mathcal{F}_t$ has density X_t with respect to $\mathbb{P}$. In particular, $\mathcal{Q}$ is a finite mesure, and X does not vanish under $\mathcal{Q}$, i.e.

$$\mathcal{Q}[\exists t \geq 0, X_t = 0] = 0.$$

3.2. **The case of the draw-down of a martingale.** Let $(M_t)_{t \geq 0}$ be a càdlàg martingale, starting at zero, without positive jumps, again defined on a space satisfying the property (NP). This assumption implies that its supremum

$$S_t := \sup_{s \leq t} M_s$$

is a.s. continuous with respect to t. The process

$$(X_t := S_t - M_t)_{t \geq 0}$$

is then a submartingale of class (Σ) with martingale part $-M$ and increasing process S. One obtains, for all $t \geq 0$ and F_t bounded, $\mathcal{F}_t$-measurable:

$$\mathcal{Q}\left[F_t \, \mathbb{1}_{g \leq t}\right] = \mathbb{E}_{\mathbb{P}}\left[F_t (S_t - M_t)\right]$$

where, in this case, g is the last time when M reaches its overall supremum.

3.3. **The uniformly integrable case.** Let us suppose that, in Theorem 2.1, the family of variables $(X_t)_{t \geq 0}$ is uniformly integrable. In this case, $(\mathbb{E}_{\mathbb{P}}[X_t])_{t \geq 0}$, and then $(\mathbb{E}_{\mathbb{P}}[A_t])_{t \geq 0}$ are uniformly bounded. By monotone convergence, A_∞ is integrable, and in particular finite a.s. Since $(A_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ are uniformly integrable, $(N_t)_{t \geq 0}$ is a uniformly integrable martingale, which implies that there exists N_∞ such that for all $t \geq 0$, $N_t = \mathbb{E}[N_\infty | \mathcal{F}_t]$ and N_t tends a.s. to N_∞ for t going to infinity. One deduces that X_t tends a.s. to $X_\infty :=$

$N_\infty + A_\infty$. Now, for all nonnegative, bounded, integrable functions f, the martingale N^f is uniformly inegrable. Moreover, if f is continuous, $G(A_t) + X_t f(A_t)$ tends a.s. to $G(A_\infty) + X_\infty f(A_\infty)$ when $t \to \infty$, and any càdlàg version of the uniformly integrable martingale $(\mathbb{E}[G(A_\infty)|\mathcal{F}_t])_{t\geq 0}$ tends a.s. to $G(A_\infty)$ (such a càdlàg version exists since the underlying space satisfies natural conditions, see [8]). Therefore, the terminal value of any càdlàg version of N^f is $X_\infty f(A_\infty)$, which implies that $\mathcal{M}^f$ has density $X_\infty f(A_\infty)$ with respect to $\mathbb{P}$, and finally:

$$\mathcal{Q} = X_\infty \, .\mathbb{P}.$$

This case was essentially obtained by Azéma, Meyer and Yor in [1] and in [6] in relation with problems from mathematical finance. The particular case where $X_t = |M_t|$, where $(M_t)_{t\geq 0}$ is a continuous uniformly integrable martingale, starting at zero, and for which the measure $\mathcal{Q}$ has density $|M_\infty|$ with respect to $\mathbb{P}$, was studied in [2], [3].

3.4. **The case where A_∞ is infinite almost surely.** In this case, for any nonnegative, integrable function f, one has:

$$N_t^f = G(A_t) + f(A_t) X_t.$$

Moreover, if $X_0 = 0$ a.s., then the image of $\mathcal{Q}$ by A_∞ is simply Lebesgue measure. There are several interesting examples of this particular case.

1) It $X_t = M_t^+$, $X_t = M_t^-$ or $X_t = |M_t|$, where M is a continuous martingale, then we are in the case: $A_\infty = \infty$ if and only if and only if the total local time of M is a.s. infinite, or, equivalently, if and only if the overall supremum of $|M|$ is a.s. infinite. This condition is satisfied, in particular, if M is a Brownian motion. More precisely, let us suppose that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is the natural augmentation of the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, equipped with its canonical filtration and the Wiener measure. If $X_t = |Y_t|$, $X_t = Y_t^+$ or $X_t = Y_t^-$, where $(Y_t)_{t\geq 0}$ denotes the canonical process, the σ-finite measure $\mathcal{Q}$ described in Theorem 2.1 was essentially studied in [9], Chapter 1. This measure satisfies a slightly more general result than what is written in Theorem 2.1. Indeed, in their monograph, Najnudel, Roynette and Yor prove (up to the technicalities on the choice of filtration which are discussed above and in [8]) that there exists a unique σ-finite measure $\mathcal{W}$ on Ω such that for all $t \geq 0$, for all bounded, $\mathcal{F}_t$-measurable functionals F_t, and for all $a \in \mathbb{R}$:

$$\mathcal{W}[F_t \mathbb{1}_{g_a \leq t}] = \mathbb{P}[F_t \, |Y_t - a|],$$

$$\mathcal{W}[g_a = \infty] = 0$$

where

$$g_a := \sup\{t \geq 0, Y_t = a\}.$$

Moreover $\mathcal{W}$ can be decomposed (in unique way) as the sum of two σ-finite measures $\mathcal{W}^+$ and $\mathcal{W}^-$, such that:

$$\mathcal{W}^+[F_t \mathbb{1}_{g_a \leq t}] = \mathbb{P}[F_t \, (Y_t - a)^+],$$

$$\mathcal{W}^-[F_t \mathbb{1}_{g_a \leq t}] = \mathbb{P}[F_t \, (Y_t - a)^-],$$

$$\mathcal{W}^+[E_-] = \mathcal{W}^-[E_+] = 0$$

where E_- is the set of trajectories which do not tend to $+\infty$, and E_+ is the set of trajectories which do not tend to $-\infty$. With these definitions, the measure $\mathcal{Q}$ is equal to $\mathcal{W}^+$ if $X_t = Y_t^+$, $\mathcal{W}^-$ if $X_t = Y_t^-$ and $\mathcal{W}$ if $X_t = |Y_t|$.

2) Let $(M_t)_{t\geq 0}$ be a càdlàg martingale, starting at zero, without positive jumps. The process

$$(X_t := S_t - M_t)_{t\geq 0}$$

is a submartingale of class (Σ) with martingale part $-M$ and increasing process S, and one has $A_\infty = \infty$ a.s., if and only if the overall supremum of M is a.s. infinite. A particular case where this condition holds is, again, when M is a Brownian motion. More precisely, if one takes the same filtered probability space as in the previous example, and if $X_t = (\sup_{s\leq t} Y_s) - Y_t$, then the σ-finite measure exists and is in fact equal to $\mathcal{W}^-$. Note that the image of this measure by X is equal to the image of $\mathcal{W}$ by the absolute value.

3) Another interesting example is studied in Chapter 3 of [9]. We assume that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ is the natural augmentation of the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, equipped with its canonical filtration and a probability measure under which the canonical process $(Y_t)_{t\geq 0}$ is a recurrent, homogeneous diffusion with values in $\mathbb{R}_+$, starting at zero, and such that zero is an instantaneously reflecting barrier. Moreover, we suppose that the infinitesimal generator $\mathcal{G}$ of Y satisfies (for $x \geq 0$):

$$\mathcal{G}f(x) = \frac{d}{dm}\frac{d}{dS} f(x)$$

where S is a continuous, strictly increasing function such that $S(0) = 0$ and $S(\infty) = \infty$, and m is the speed measure, satisfying $m(\{0\}) = 0$. There exists a jointly continuous family $(L_t^y)_{t,y\geq 0}$ of local times of Y, satisfying:

$$\int_0^t h(Y_s)\, ds = \int_0^\infty h(y) L_t^y\, m(dy)$$

for all borelian functions h from $\mathbb{R}_+$ to $\mathbb{R}_+$. If we define the process $(X_t)_{t\geq 0}$ by:

$$X_t = S(Y_t)$$

then $(X_t - L_t^0)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale. Hence, if $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_t)_{t\geq 0}$, the assumptions of Theorem 2.1 are satisfied, and L_∞^0 is infinite, since the diffusion Y is recurrent. The σ-finite measure $\mathcal{Q}$ is given by the formula:

$$\mathcal{Q} = \int_0^\infty dl\, \mathbb{Q}_l,$$

where $\mathbb{Q}_l$ is the law of a process $(Z_t^l)_{t\geq 0}$, defined in the following way: let τ_l be the inverse local time at l (and level zero) of a diffusion R, which has a law equal to the distribution of Y under $\mathbb{P}$, and let $(\tilde{R}_u)_{u\geq 0}$ be an homogeneous diffusion, independent of R, starting at zero, never hitting zero again, and such that for $0 \leq u < v$, $x, y > 0$:

$$P[\tilde{R}_v \in dy \,|\, \tilde{R}_u = x] = \frac{S(y)}{S(x)} P[R_v \in dy, \forall w \in [u,v], R_w > 0 \,|\, R_u = x]$$

(intuitively, the law of $(\tilde{R}_u)_{u\geq 0}$ is the law of $(R_u)_{u\geq 0}$, conditioned not to vanish), then Z^l satisfies

$$Z_t^l = R_t$$

for $t \leq \tau_l$, and

$$Z_{\tau_l + u}^l = \tilde{R}_u$$

for $u \geq 0$. Theorem 2.1 applies, in particular, if Y is a Bessel process of dimension $d \in (0, 2)$. If $d = 2(1-\alpha)$ (which imples $0 < \alpha < 1$), one obtains:

$$X_t = (Y_t)^{2\alpha} = (Y_t)^{2-d}.$$

In this case, the process $(\tilde{R}_u)_{u \geq 0}$, involved in an essential way in the construction of $\mathcal{Q}$, is a Bessel process of dimension $4 - d = 2(1+\alpha)$. For $d = 1$ ($\alpha = 1/2$), $(X_t = Y_t)_{t \geq 0}$ is the absolute value of a Brownian motion, and $\tilde{R}$ is a Bessel process of dimension 3.

4) Let $(\Omega, \mathcal{H}, \mathbb{P}, (\mathcal{H}_t)_{t \geq 0})$ be the natural augmentation of the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, equipped with its canonical filtration and a probability measure under which the canonical process $(Y_t)_{t \geq 0}$ is a Bessel process of dimension $d := 2(1-\alpha)$ for $0 < \alpha < 1$. For $t \geq 0$, let us take the notation:

$$g(t) := \sup\{u \leq t, Y_u = 0\},$$

and let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration of the zeros of Y, i.e. $\mathcal{F}_t = \mathcal{H}_{g(t)}$. One defines the σ-algebra $\mathcal{F}$ as the σ-algebra generated by $(\mathcal{F}_t)_{t \geq 0}$, i.e. by the zeros of Y. Now, the process

$$(X_t := (t - g(t))^{\alpha})_{t \geq 0}$$

is a $(\mathcal{F}_t)_{t \geq 0}$-submartingale of class (Σ), and its increasing process $(A_t)_{t \geq 0}$ is given by:

$$A_t = \frac{1}{2^{\alpha}\Gamma(1+\alpha)} \, L_t(Y)$$

where $L_t(Y)$ is the local time of Y at zero, defined as the increasing process of the submartingale $(Y_t^{2\alpha})_{t \geq 0}$, which is of class (Σ) with respect to the space $(\Omega, \mathcal{H}, \mathbb{P}, (\mathcal{H}_t)_{t \geq 0})$. (see [10], and the previous example). Since Y is recurrent, $A_{\infty} = \infty$ a.s. Now, let $\mathcal{R}$ be the σ-finite measure on $(\Omega, (\mathcal{H}_t)_{t \geq 0}, \mathcal{H})$ which is equal to the measure $\mathcal{Q}$ of example 3). Because of this example, one has, for all bounded, $\mathcal{H}_t$-measurable functions F_t:

$$\mathcal{R}[F_t \, \mathbb{1}_{g \leq t}] = \mathbb{E}_{\mathbb{P}}[F_t Y_t^{2\alpha}]$$

where g is the last zero of Y, equal to the last zero of X. Now, if F_t is $\mathcal{F}_t$-measurable, then one obtains

$$\mathcal{R}[F_t \, \mathbb{1}_{g \leq t}] = \mathbb{E}_{\mathbb{P}}[F_t \mathbb{E}_{\mathbb{P}}[Y_t^{2\alpha} | \mathcal{F}_t]]$$

which implies:

$$\mathcal{R}[F_t \, \mathbb{1}_{g \leq t}] = 2^{\alpha}\Gamma(1+\alpha) \, \mathbb{E}_{\mathbb{P}}[F_t X_t]$$

Therefore, the measure $\mathcal{Q}$ satisfying the conditions given in Theorem 2.1 is the restriction of the measure

$$\tilde{\mathcal{Q}} := \frac{1}{2^{\alpha}\Gamma(1+\alpha)} \, \mathcal{R}$$

to the σ-algebra $\mathcal{F}$. Moreover, the image of $\mathcal{Q}$ by X is:

$$\mathcal{S} := \frac{1}{2^{\alpha}\Gamma(1+\alpha)} \int_0^{\infty} dl \, \mathbb{S}_l$$

where $\mathbb{S}_l$ is the law of a process $(V_t^l)_{t \geq 0}$, defined in the following way: let τ_l be the inverse local time at l (and level zero) of a diffusion R, with the same law as Y under $\mathbb{P}$, and let $\gamma(t)$ be the last zero of R before time t, for all $t \geq 0$, V^l satisfies

$$V_t^l = (t - g(t))^{\alpha}$$

for $t \leq \tau_l$, and

$$V_{\tau_l + u}^l = u^{\alpha}$$

for $u \geq 0$. Note that in this case, we have not checked that the filtered probability space has property (NP). However, we have proved that the conclusion of Theorem 2.1 holds in this case.

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