# THE STACK OF ADMISSIBLE COVERS IS ALGEBRAIC

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Abstract. We show that the moduli stack of admissible $G$-covers of prestable curves is an algebraic stack, loosely following [ACV03, App. B]. As preparation, we discuss finite group actions on algebraic spaces.

Contents



## 1. Introduction

One of the most-studied objects in algebraic geometry is the moduli space $\mathscr{M}_g$ parametrizing smooth curves of genus $g$. From early on, there has been considerable interest in describing the finite étale covers of $\mathscr{M}_g$. Their understanding is not only of intrinsic interest, but also offers technical advantages. For example, while $\mathscr{M}_g$ is only smooth when considered as an algebraic stack, many of its finite étale covers are given by smooth *schemes*; in some cases, statements for $\mathscr{M}_g$ can then be reduced to scheme-theoretic statements for these covers.

Another important tool in the study of $\mathscr{M}_g$ is its compactification by the moduli stack $\overline{\mathscr{M}}_g$ of stable curves of genus $g$. Admissible $G$-covers were introduced in [ACV03] to give modular interpretations for compactifications of many finite étale covers of $\mathscr{M}_g$. The authors showed that the stacks $\mathscr{A}dm_g(G)$ of admissible $G$-covers of curves of genus $g$ are still smooth and the maps $\mathscr{A}dm_g(G) \to \overline{\mathscr{M}}_g$ still proper and quasi-finite. Moreover, they constructed groups $G$ for which $\mathscr{A}dm_g(G)$ is a smooth projective scheme which is finite over $\overline{\mathscr{M}}_g$. In this chapter, we give a detailed

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proof of one the fundamental theorems of [ACV03] in the more general setting of prestable curves, relying solely on results from the Stacks Project.

**Theorem A.** *$\mathscr{A}dm(G) = \bigsqcup \mathscr{A}dm_g(G)$ is an algebraic stack over* $\mathrm{Spec}(\mathbf{Z}[1/|G|])$.

As explained in [ACV03, Sec 4.2], the stack of admissible $G$-covers is also the normalization of (a generalization of) the stack of admissible covers studied initially in [HM82] and also in [Moc95], [Wew98]. A general reference for these covers and their associated moduli stacks is [BR11]. In the next subsections, we define $\mathscr{A}dm(G)$ in full detail and then outline our proof of Theorem A.

**Definition of admissible $G$-covers.** An admissible $G$-cover is a finite cover $X \to Y$ of prestable curves that is generically a $G$-torsor and has prescribed behavior on the nodes and markings. We note that we allow a family of prestable curves to have disconnected geometric fibers (Definition 9).

**Definition 1.** Let $G$ be a finite group and let $S$ be a scheme over $\mathrm{Spec}(\mathbf{Z}[1/|G|])$. Let $(Y \to S, \{q_i\}_{i=1}^n)$ be a marked connected prestable curve over $S$. An *admissible $G$-cover* is a finite morphism $f\colon X \to Y$ from a prestable curve $X$ together with a $G$-action on $X$ which leaves $f$ invariant, such that

(i) Away from the nodes and markings of $Y$, $f$ is a $G$-torsor.
(ii) Nodes of $X$ map to nodes of $Y$.
(iii) For any geometric point $\bar{s}\colon \mathrm{Spec}\, k \to S$ and any node $x \in X_{\bar{s}}$, the map $f^{\mathrm{sh}}\colon \mathscr{O}^{\mathrm{sh}}_{Y_{\bar{s}}, f(x)} \to \mathscr{O}^{\mathrm{sh}}_{X_{\bar{s}},x}$ can be identified with the strict henselization of the map
$$\big(k[u', v']/(u'v')\big)_{(u',v')} \to \big(k[u, v]/(uv)\big)_{(u,v)}, \quad u' \mapsto u^e,\ v' \mapsto v^e$$
for some $e \in \mathbf{Z}_{\geq 1}$. Under this identification, the action of the stabilizer $G_x$ on $\mathscr{O}^{\mathrm{sh}}_{X_{\bar{s}},x}$ is *balanced*, i.e., given by $u \mapsto \xi(g)u$, $v \mapsto \xi^{-1}(g)v$ for some character $\xi\colon G_x \to k^\times$.
(iv) For any geometric point $\bar{s}\colon \mathrm{Spec}\, k \to S$ and any $x \in X_{\bar{s}}$ lying over a marked point in $Y_{\bar{s}}$, the map $f^{\mathrm{sh}}\colon \mathscr{O}^{\mathrm{sh}}_{Y_{\bar{s}},\phi(x)} \to \mathscr{O}^{\mathrm{sh}}_{X_{\bar{s}},x}$ can be identified with the strict henselization of
$$k[v]_{(v)} \to k[u]_{(u)}, \quad v \mapsto u^e$$
for some $e \in \mathbf{Z}_{\geq 1}$. Under this identification, the action of the stabilizer $G_x$ on $\mathscr{O}^{\mathrm{sh}}_{X_{\bar{s}},x}$ is given by $u \mapsto \xi(g)u$ for some character $\xi\colon G_x \to k^\times$.

Admissible $G$-covers form a category fibered in groupoids $\mathscr{A}dm(G)$; for a rigorous description, see Definition 67.

**Remark 2.** Our definition of an admissible $G$-cover differs slightly from that of [ACV03, Def 4.3.1] in that we only require the descriptions (iii) and (iv) in geometric fibers.

**Remark 3.** We follow [ACV03, Def 4.3.1] in allowing a prestable curve $X \to S$ to have disconnected geometric fibers. If one modifies Definition 1 to require $X \to S$ to be a prestable curve with geometrically connected fibers, the discussion in this article shows (with very minor changes) that the resulting moduli stack is algebraic.

**Remark 4.** In part (iii) of Definition 1 we have required $G_x$ to act on $u$ and $v$ via inverse characters; that is, we require the action to be *balanced*. To motivate this requirement, recall from see [SP21, Tag 0CBX] that if $f\colon X \to S$ is a morphism of schemes and a nodal curve and if $x \in X$ is a split node, then there is an isomorphism
$$\mathscr{O}^\wedge_{X,x} = \mathscr{O}^\wedge_{S,f(x)}[\![u, v]\!]/(uv - h)$$

for some $h \in \mathfrak{m}_{f(x)}$. If moreover $G$ acts on $X$ leaving $X \to S$ invariant, then since $h$ comes from $S$ it is $G$-invariant, so if $u$ and $v$ are weight vectors for the $G_x$-action they must have inverse weights. In other words, a smoothable split node must be balanced.

**Remark 5.** The analytically minded reader may replace conditions (iii) and (iv) on strict henselizations in Definition 1 by the corresponding conditions on completions. This will follow from Lemma 57 below.

**Roadmap of the proof.** One standard approach to proving a theorem like Theorem A is to use Artin's axioms [SP21, Tag 07SZ]. We provide a proof that is logically independent of Artin's axioms by constructing $\mathscr{A}dm(G)$ as a locally closed substack of a certain Hom-stack $\mathfrak{H}_0$.

To construct $\mathfrak{H}_0$, we begin with the stacks $\mathfrak{M}$ and $\mathfrak{N}$ of connected prestable curves and (possibly disconnected) prestable curves, respectively. Algebraicity of these stacks is proved in [Hal13] by finding explicit presentations in terms of Hilbert schemes of projective spaces. From these we construct universal curves $\mathfrak{C}_{\mathfrak{M}}$ and $\mathfrak{C}_{\mathfrak{N}}$, as well as the stack of marked connected prestable curves $\mathfrak{M}_\star$ (Definition 17), and we explain why they are algebraic (Lemma 16 and Lemma 18).

To finish the construction of $\mathfrak{H}_0$, we construct the stack of prestable curves with balanced $G$-action $\mathfrak{N}^{\mathrm{bal}}$. We first define the stack $\mathfrak{N}(G)$ of prestable curves with $G$-action and exhibit it as a locally closed substack of the $|G|$-fold product $\prod_G \mathscr{I}$ of the inertia stack $\mathscr{I}$ of $\mathfrak{N}$ (Lemma 59). An analysis of the local picture at the nodes reveals that the balancedness of the $G$-action is an open condition on the base, so $\mathfrak{N}^{\mathrm{bal}}$ is an open substack of $\mathfrak{N}(G)$ and hence also algebraic (Lemma 62). Finally we set $\mathfrak{H}_0 := \underline{\mathrm{Hom}}_{\mathfrak{N}^{\mathrm{bal}} \times \mathfrak{M}_\star}(\mathfrak{C}_{\mathfrak{N}^{\mathrm{bal}}}, \mathfrak{C}_{\mathfrak{M}_\star})$. It is algebraic by a general argument about Hom-stacks (Lemma 19.

After constructing $\mathfrak{H}_0$, we realize $\mathscr{A}dm(G)$ as a locally closed substack in the following way. A general element of $\mathfrak{H}_0$ over a scheme $S$ defines a morphism of prestable curves $f\colon X \to Y$ over $S$ such that $G$ acts on $X$ and $Y$ has markings. We prove that the family $f\colon X \to Y$ is in $\mathscr{A}dm(G)$ if and only if $Y$ is the (coarse) quotient of $X$ by $G$ and $f$ maps smooth $G$-fixed points to marked points (Theorem 77). Since these conditions are locally closed and closed (Lemma 68, Lemma 69, and Lemma 70), respectively, Theorem A is proven.

**Outline of the contents.** Section 2 provides some of the prerequisites that are necessary in the proof of Theorem A and are interesting in their own right. Namely, in 2.1, we review and extend results of [SP21] on prestable curves. Then, we recall some general algebraicity criteria for stacks in 2.2. Finally, we explain many basic facts about finite group actions on algebraic spaces (2.3), including coarse quotients by those actions (2.4).

Section 3 contains a discussion of the objects of Theorem A and its proof. We begin in 3.1 by introducing balanced group actions on prestable curves and their complete local pictures. In 3.2, we study the deformation theory of these objects, which we use in 3.3 to show that balanced prestable $G$-curves form the algebraic stack $\mathfrak{N}^{\mathrm{bal}}$. We conclude our analysis of the substack $\mathscr{A}dm(G) \subset \mathfrak{H}_0$ as outlined above in 3.4.

**Notation and conventions.** We adopt the following notation and conventions, listed in order of appearance:

- For $x \in X$ a point of a scheme, $\mathscr{O}_{X,x}$ is the local ring of $X$ at $x$, $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$ and $\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$ is the residue field of $X$ at $x$.
- $\bar{k}$ is the algebraic closure of a field $k$.

- A geometric point of a scheme $X$ is a map $\mathrm{Spec}(k) \to X$ from an algebraically closed field $k = \bar{k}$ ([SP21, Tag 0486]).
- $|X|$ is the topological space of an algebraic space ([SP21, Tag 03BY])
- $\mathrm{Hom}_T(X, Y)$ is the set of morphisms of algebraic spaces over $T$.
- $f^{-1}$ will refer to the scheme theoretic inverse image ([SP21, Tag 01JV]).
- $\Delta$ is the diagonal morphism.
- 1 is the identity element of a group. ([SP21, Tag 03BU]).
- $[X/G]$ denotes the stack quotient ([SP21, Tag 044Q]), while $X/G$ denotes the coarse quotient (Def. 31).
- $\mathbf{Z}[1/|G|]$ is the smallest subring of $\mathbf{Q}$ containing $\mathbf{Z}$ and $1/|G|$.

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## 2. Setup

2.1. **Prestable curves.** We recall some definitions leading up to that of a family of prestable curves. The following is [SP21, Tags 0C47, 0CBV].

**Definition 6.** Let $X$ be a 1-dimensional scheme locally of finite type over a field $k$.

(i) A closed point $x \in X$ is a *split node* if there exists an isomorphism $\mathcal{O}^{\wedge}_{X,x} \simeq k[\![u, v]\!]/(uv)$.

(ii) A closed point $x \in X$ is a *node* if there is a split node $\bar{x} \in X_{\bar{k}}$ mapping to $x$.

(iii) The scheme $X$ has *at-worst-nodal singularities* if every closed point of $X$ is either contained in the smooth locus of $X \to \mathrm{Spec}(k)$ or is a node of $X$.

Observe that by definition a split node of $X \to \mathrm{Spec}(k)$ has residue field $k$. By [SP21, Tag 0C4D], one may equivalently demand in Definition 6.(ii) that all $\bar{x} \in X_{\bar{k}}$ mapping to $x$ are split nodes.

We will need some properties of (split) nodes beyond what is currently in the stacks project, so we write these out in Lemmas 7 and 8 below. The proofs of these lemmas both use the very general statement of Lemma 14. Since Lemma 14 applies to more schemes than just nodal curves we defer it to the end of this subsection.

**Lemma 7.** *Let $X$ be a* 1*-dimensional scheme locally of finite type over a field $k$, let $x \in X$ be a split node, and let $K/k$ be a field extension. Then there is a unique $y \in X_K$ mapping to $x$ and $y$ is a split node.*

*Proof.* We note that the lemma does not immediately follow from Lemma 14 because our field extension $K/k$ may not be finite. However, since $x \in X$ is $k$-rational, we can at least use Lemma 14 to say that there is a unique point $y \in X_K$ mapping to $x$ and that $y$ has residue field $K$. By [SP21, Tag 0CBW], the fiber of the normalization $X^\nu \to X$ over $x$ has precisely two points. By [SP21, Tag 0C3N] and surjectivity of the map $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ the fiber of $X^\nu_K \to X_K$ over $y$ has at least two points, and by [SP21, Tag 0CBT] and [SP21, Tag 0C56] it has exactly two points. By another application of [SP21, Tag 0CBW], the node $y$ is split. $\square$

General nodes have a complete local description similar to that of split nodes.

**Lemma 8.** *Let $X$ be a 1-dimensional scheme locally of finite type over a field $k$ and let $x \in X$ be a closed point. The following are equivalent:*

(i) *The point $x$ is a node.*
(ii) *The extension $\kappa(x)/k$ is separable, and $\mathscr{O}^\wedge_{X,x} \simeq \kappa(x)[[u,v]]/(q)$ for some nondegenerate quadratic form $q = au^2 + buv + cv^2$ over $\kappa(x)$.*
(iii) *There exists a finite separable field extension $\kappa'$ of $k$ and a split node $y \in X_{\kappa'}$ mapping to $x$.*

*Proof.* The equivalence of (i) and (ii) is in [SP21, Tag 0C4D], and that (iii) implies (i) is immediate from the definitions and Lemma 7.

Now assume (ii). Set $\kappa := \kappa(x)$ and

$$\kappa' := \begin{cases} \kappa & \text{if } q \text{ is a split quadratic form} \\ \kappa[t]/(a + bt + ct^2) & \text{if } q \text{ is not a split quadratic form.} \end{cases}$$

The latter is the splitting field of $q$: when $q$ is not split over $\kappa$, it factors over $\kappa'$ as $q = (v - tu)(cv + (b + ct)u)$. Since $\kappa$ is finite separable over $k$ and $q$ is nondegenerate, $\kappa'/\kappa/k$ must be a tower of finite separable field extensions.

First, we construct a closed point $\tilde{x} \in X_\kappa(\kappa)$ lying over $x$. By the Primitive Element Theorem [SP21, Tag 030N], there is a polynomial $P \in k[t]$ such that $\kappa = k[t]/(P)$. Then $\kappa \otimes_k \kappa \simeq \kappa[t]/(P)$ has a factor $\kappa$ because $P$ has a zero in $\kappa$. The prime ideal given as the kernel of the associated surjection $\kappa \otimes_k \kappa \twoheadrightarrow \kappa$ determines the point $\tilde{x}$ over $x$ with residue field $\kappa$ [SP21, Tag 01JT]. By [SP21, Tag 0C50, Tag 0AGX], we have an isomorphism $\mathscr{O}^\wedge_{X,x} \simeq \mathscr{O}^\wedge_{X_\kappa,\tilde{x}}$. Finally, by Lemma 14, there is a unique $y \in X_{\kappa'}$ with residue field $\kappa'$ and complete local ring

$$\begin{aligned} \mathscr{O}^\wedge_{X_{\kappa'},y} &\simeq \mathscr{O}^\wedge_{X_\kappa,\tilde{x}} \otimes_\kappa \kappa' \simeq \mathscr{O}^\wedge_{X,x} \otimes_\kappa \kappa' \simeq \kappa[[u,v]]/(q) \otimes_\kappa \kappa' \\ &\simeq \kappa'[[u,v]]/(q) \simeq \kappa'[[u,v]]/(uv) \end{aligned}$$

of the desired form. $\square$

**Definition 9.** A morphism $\pi \colon X \to S$ from an algebraic space $X$ to a scheme $S$ is a *nodal curve* if $\pi$ is flat and of finite presentation and every geometric fiber has pure dimension 1 and at-worst-nodal singularities. We say $\pi$ is a *connected prestable curve* if in addition $\pi$ is proper and every geometric fiber is connected. A *prestable curve* is a finite disjoint union $\sqcup_i X_i \to S$ where each $X_i \to S$ is a connected prestable curve.

**Remark 10.** Definition 9 is a compilation of [SP21, Tags 0D4Z, 0DSE, 0C54, 0C56, 0E6S] with some important differences in nomenclature:

- A nodal curve in our sense is a morphism that is *at-worst-nodal of relative dimension* 1 in the sense of the Stacks Project [SP21, Tag 0C5A]; this recovers the notion of nodal curves from [SP21, Tag 0C46] when $S$ is the spectrum of a field.
- A prestable curve in our sense is a *nodal family of curves* in the sense of the Stacks Project [SP21, Tag 0DSX] (noting that families of curves are by definition always proper in the Stacks Project [SP21, Tag 0D4Z])

- A connected prestable curve in our sense is a prestable curve in the sense of the Stacks Project [SP21, Tag 0E6T].

In the next definition, recall from [SP21, Tag 03BU] that $|X|$ is the set of points of an algebraic space, and elements of this set are equivalence classes of morphisms from fields.

**Definition 11.** Let $\pi\colon X \to S$ be a nodal curve. A point $x \in |X|$ is a *node in its fiber* if it is the image of a node in $X_{\pi(x)}$.

**Definition 12.** If $\pi : X \to S$ is a nodal curve, the *smooth locus* $X^{sm} \subset X$ is the maximal open subspace such that $\pi|_{X^{sm}} : X^{sm} \to S$ is smooth (see [SP21, Tag 0DZI]).

**Remark 13.** If $X \to S$ is a nodal curve, the smooth locus $X^{sm} \to S$ is precisely the complement of the points in $X$ that are nodes in their fibers. In particular, $X^{sm}$ commutes with arbitrary base change by [SP21, Tag 0C56].

Define $\mathfrak{M}$ and $\mathfrak{N}$ to be the moduli stacks over $Sch_{fppf}$ (see [SP21, Tag 021R]) whose groupoid fibers over a scheme $S$ are given by

$$\mathfrak{M}(S) = \{\text{connected prestable curves } X \to S\}$$

$$\mathfrak{N}(S) = \{\text{prestable curves } X \to S\}.$$

By [SP21, Tag 0D5A] and [SP21, Tag 0DSQ] there is a larger moduli stack *Curves* over $Sch_{fppf}$ with a diagonal that is separated and of finite presentation. By [SP21, Tag 0E6U] and [SP21, Tag 0DSY], respectively, the stacks $\mathfrak{M}$ and $\mathfrak{N}$ are open substacks of *Curves*, and hence algebraic stacks with diagonals that are separated and of finite presentation. The proof of algebraicity in [SP21, Tag 0D5A] uses Artin's axioms; for a proof using only Hilbert schemes of projective spaces, see [Hal13].

We used the following general lemma to prove Lemmas 7 and 8 in this section.

**Lemma 14.** *Let $X$ be a $k$-scheme of locally finite type. Let $K/k$ be a field extension and let $x \in X$ be a rational point. Then there is a unique point $y \in X_K$ lying over $x$ with residue field $\kappa(y) = K$. Moreover, if $K/k$ is finite, then $\mathscr{O}_{X,x} \otimes_k K \simeq \mathscr{O}_{X_K,y}$ and $\mathscr{O}^\wedge_{X,x} \otimes_k K \simeq \mathscr{O}^\wedge_{X_K,y}$.*

*Proof.* By [SP21, Tag 01JT], $y$ corresponds to the unique prime ideal in $k \otimes_k K \simeq K$ which has residue field equal to $K$.

Now assume $K/k$ is a finite extension. By [SP21, Tag 0C4Y], we have $\mathscr{O}_{X_K,y} = (\mathscr{O}_{X,x} \otimes_k K)_{\mathfrak{p}}$, where $\mathfrak{p}$ is the kernel of $\mathscr{O}_{X,x} \otimes_k K \to \kappa(x) \otimes_k K \simeq K$. Since $K$ is flat over $k$, there is an exact sequence

$$0 \to \mathfrak{m} \otimes_k K \to \mathscr{O}_{X,x} \otimes_k K \to K \to 0.$$

This shows that $\mathfrak{p} = \mathfrak{m} \otimes_k K$ and that this ideal is maximal.

In fact, we claim that it is the unique maximal ideal, so $(\mathscr{O}_{X,x} \otimes_k K)_{\mathfrak{p}} \simeq \mathscr{O}_{X,x} \otimes_k K$ as desired. To see this, let $\mathfrak{n} \subset \mathscr{O}_{X,x} \otimes_k K$ be a maximal ideal. Since $K/k$ is finite, $\mathscr{O}_{X,x} \otimes_k k \subset \mathscr{O}_{X,x} \otimes_k K$ is integral, so $(\mathscr{O}_{X,x} \otimes_k k)/(\mathfrak{n} \cap \mathscr{O}_{X,x} \otimes_k k) \to (\mathscr{O}_{X,x} \otimes_k K)/\mathfrak{n}$ is also integral. However, $(\mathscr{O}_{X,x} \otimes_k K)/\mathfrak{n}$ is a field, so [SP21, Tag 00GR] implies that $(\mathscr{O}_{X,x} \otimes_k k)/(\mathfrak{n} \cap \mathscr{O}_{X,x} \otimes_k k)$ is a field as well. Hence $\mathfrak{n} \cap \mathscr{O}_{X,x} \otimes_k k$ is maximal and thus equals $\mathfrak{m} \otimes_k k$. In particular, $\mathfrak{n}$ contains $\mathfrak{m} \otimes_k k$, so $\mathfrak{n}$ contains $\mathfrak{m} \otimes_k K$, so $\mathfrak{n}$ must equal $\mathfrak{m} \otimes_k K$ because we already knew the latter was maximal.

Finally, the natural map $\mathscr{O}^\wedge_{X,x} \otimes_k K \to (\mathscr{O}_{X,x} \otimes_k K)^\wedge$ is an isomorphism by [SP21, Tag 00MA(3)]. $\square$

2.2. **Some algebraic stacks.** We will use the following lemma to construct algebraic stacks. Let $S$ be a scheme in $Sch_{fppf}$ (see [SP21, Tag 021R]).

**Lemma 15.** *Let $\mathscr{Y} \to (Sch/S)_{fppf}$ and $\mathscr{X} \to \mathscr{Y}$ be categories fibered in groupoids, and assume $\mathscr{X} \to \mathscr{Y}$ is representable by algebraic spaces. If $\mathscr{Y}$ is an algebraic stack then so is $\mathscr{X}$.*

*Proof.* This is [SP21, Tag 09WW] and [SP21, Tag 05UN]. □

Define the *universal curve* $\mathfrak{C} \to \mathfrak{M}$ to be the category whose groupoid fiber over an object $S \in Sch_{fppf}$ is

$$\mathfrak{C}(S) := \{(X \to S, q) \mid (X \to S) \in \mathfrak{M}(S) \text{ and } q \in \mathrm{Hom}_S(S, X)\}$$

together with the apparent forgetful morphism to $\mathfrak{M}$. The universal curve $\mathfrak{C} \to \mathfrak{N}$ is defined similarly.

**Lemma 16.** *The morphisms $\mathfrak{C} \to \mathfrak{M}$ and $\mathfrak{C} \to \mathfrak{N}$ are representable morphisms of algebraic stacks.*

*Proof.* We prove the statement for $\mathfrak{C} \to \mathfrak{M}$; the argument for $\mathfrak{C} \to \mathfrak{N}$ is identical. Let $T \to \mathfrak{M}$ be a map from a scheme $T$ corresponding to a connected prestable curve $X \to T$. By Lemma 15 it suffices to show that $T \times_{\mathfrak{M}} \mathfrak{C}$ is representable. Using the construction of [SP21, Tag 0040], objects of $T \times_{\mathfrak{M}} \mathfrak{C}$ over a scheme $S$ are given by quadruples

$$(S, S \to T,\ (Y \to S, f \in \mathrm{Hom}_S(S, Y)),\ \phi \colon Y \simeq S \times_T X)$$

where $Y \to S$ is a connected prestable curve and arrows are unique when they exist. The data of such a triple is equivalent to an element of $\mathrm{Hom}_T(S, X)$, so $X \to T$ represents $T \times_{\mathfrak{M}} \mathfrak{C}$. □

If $\pi \colon X \to S$ is a connected prestable curve, let $X^{sm} \subset X$ be the open subspace where $\pi$ is smooth. We will use this notation even when $S$ is an algebraic stack (see [SP21, Tag 0DZR]). Define $\mathfrak{M}_n \to Sch_{fppf}$ to be the category fibered in groupoids whose objects over $S$ are

$$\mathfrak{M}_n(S) = \left\{ (X \to S, q_1, \dots, q_n) \;\middle|\; \begin{array}{l} (X \to S) \in \mathfrak{M},\ q_i \in \mathrm{Hom}_S(S, X^{sm}) \\ \text{are disjoint sections} \end{array} \right\}$$

**Definition 17.** A *marked connected prestable curve* is an object of $\mathfrak{M}_n$ for some $n \geq 0$. We set $\mathfrak{M}_\star = \bigsqcup_{n \in \mathbf{Z}_{\geq 0}} \mathfrak{M}_n$.

**Lemma 18.** *The category $\mathfrak{M}_\star$ is an algebraic stack.*

*Proof.* We prove that $\mathfrak{M}_n$ is an algebraic stack. Let $\mathfrak{C}_n$ denote the $n$-fold fiber product of $\mathfrak{C}$ over $\mathfrak{M}$. The category $\mathfrak{M}_n$ naturally a full subcategory of $\mathfrak{C}_n$. We show that the inclusion $\mathfrak{M}_n \to \mathfrak{C}_n$ is representable and apply Lemma 15.

Let $T \to \mathfrak{C}_n$ be an arbitrary map from a scheme, and let $(X \to T, q_i)$ be the corresponding prestable curve and sections. The fiber product $\mathfrak{M}_n \times_{\mathfrak{C}_n} T$ is the full subcategory of $T$ whose objects over $S$ are maps $f \colon S \to T$ such that the pullback $(X_S \to S, q_i : S \to X_S)$ of $(X \to T, q_i)$ is an object of $\mathfrak{M}_n$; that is, the $q_i$ are pairwise disjoint and land in $(X_S)^{sm}$. Let $U_1 \subset T$ be the intersection of the finitely many open sets $q_i^{-1}(X^{sm})$. Since $X \to T$ is separated, the equalizer of any two of the $q_i$ is a closed subspace (see e.g. [SP21, Tag 01KM]; let $U_2$ be the open subspace of $T$ equal to the complement of the union of the equalizers. Now $U_1 \cap U_2$ is an open subscheme of $T$ representing $\mathfrak{M}_n \times_{\mathfrak{C}_n} T$. □

Let $\mathscr{X} \to \mathscr{Z}$ and $\mathscr{Y} \to \mathscr{Z}$ be representable morphisms of algebraic stacks. The Hom stack $\underline{\mathrm{Hom}}_{\mathscr{Z}}(\mathscr{X}, \mathscr{Y})$ is the category fibered in groupoids over $\mathscr{Z}$ whose objects over a scheme $T \to \mathscr{Z}$ are

$$\underline{\mathrm{Hom}}_{\mathscr{Z}}(\mathscr{X}, \mathscr{Y})(T) = \mathrm{Hom}_T(\mathscr{X}_T, \mathscr{Y}_T).$$

**Lemma 19.** *If $\mathscr{X} \to \mathscr{Z}$ is representable, of finite presentation, flat, and proper, and $\mathscr{Y} \to \mathscr{Z}$ is representable, of finite presentation, and separated, then $\underline{\mathrm{Hom}}_{\mathscr{Z}}(\mathscr{X}, \mathscr{Y})$ is an algebraic stack.*

*Proof.* Let $T \to \mathscr{Z}$ be a scheme. By [SP21, Tag 0D1C], the fiber product $T \times_{\mathscr{Z}} \underline{\mathrm{Hom}}_{\mathscr{Z}}(\mathscr{X}, \mathscr{Y})$ is representable by an algebraic space, so the lemma follows from Lemma 15. $\square$

**Remark 20.** The proof of representability in [SP21, Tag 0D1C] uses Artin's axioms. If $\mathscr{X} \to \mathscr{Y}$ and $\mathscr{Y} \to \mathscr{Z}$ are families of curves (as they will be in our applications), one may alternatively use an étale base change of $T$ to reduce to the case where the families are H-projective ([SP21, Tag 0E6F]), and then use the more classical [AK80, Thm. 2.6].

2.3. **Finite group actions on algebraic spaces.** Let $G$ be a finite group and let $S$ be a scheme. A *$G$-space (resp. scheme) over $S$* is an algebraic space (resp. scheme) $X \to S$ equipped with a group homomorphism $G \to \mathrm{Aut}_S(X)$. If $X \to S$ is a morphism of affine schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$, then it is equivalent to give a homomorphism from $G$ to the automorphisms of $A$ as a $B$-algebra.

**Definition 21.** For $g \in G$ the *fixed locus* $X^g$ is the fiber product of the diagram

$$\begin{tikzcd} X^g \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "(g,id)"] & X \times_S X \end{tikzcd} \tag{1}$$

If $X$ is separated, then $X^g$ is a closed subspace of $X$. We define the fixed locus of $G$ to be $X^G = \cap_{g \in G} X^g$.

For any scheme $T$, $G$ acts on $X(T)$ and we have $X^g(T) = (X(T))^g$, where the right hand side is the subset fixed by $g$.

**Remark 22.** If $X \to S$ and $Y \to S$ are $G$-spaces over $S$, then the fiber product $X \times_S Y$ has a unique structure of a $G$-space over $S$ such that the projection maps to $X$ and $Y$ are $G$-equivariant. In terms of functors of points, this action is given by the diagonal action of $G$ on $(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T)$.

**Lemma 23.** *Let $X$ be a $G$-space over $S$. If $S' \to S$ is an $S$-scheme with trivial $G$-action then $X^g \times_S S' \simeq (X \times_S S')^g$ for all $g \in G$.*

*Proof.* We argue using functors of points. If $T \to S'$ is a scheme, then an element of $(X \times_S Y)^g$ is given by a pair $(\phi_x, \phi_y) \in X(T) \times_{S(T)} S'(T)$ such that $g \circ \phi_x = \phi_x$ and $g \circ \phi_y = \phi_y$. An element of $X^g \times_S S'$ is a pair $(\phi_x, \phi_y) \in X(T) \times_{S(T)} S'(T)$ such that $g \circ \phi_x = \phi_x$. Since $G$ acts trivially on $S'$ these are the same set. $\square$

In the next lemma, recall that $|X|$ is the set of points of an algebraic space [SP21, Tag 03BU], and elements of this set are equivalence classes of morphisms from fields.

**Lemma 24.** *Let $X$ be a $G$-space over $S$. Let $x \in |X|$ and $g \in G$. The following are equivalent:*

(i) *For every field $K$ and morphism $\alpha\colon \mathrm{Spec}(K) \to X$ representing $x$, we have $g \circ \alpha = \alpha$.*
(ii) *For some field $K$ and morphism $\alpha\colon \mathrm{Spec}(K) \to X$ representing $x$, we have $g \circ \alpha = \alpha$.*

(iii) *We have* $x \in |X^g|$.

*The subset of elements* $g \in G$ *satisfying conditions* (i)–(iii) *is a subgroup of* $G$.

*Proof.* For the equivalence of (i) and (ii), observe that if $\Omega/L$ is a field extension then the induced map $\beta : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(L)$ is an epimorphism. So if $\gamma : \mathrm{Spec}(L) \to X$ represents $x$, we have $g \circ \gamma = \gamma$ if and only if $g \circ \gamma \circ \beta = \gamma \circ \beta$. This means that the representative $\gamma$ is fixed by $g$ if and only if the representative $\gamma \circ \beta$ is fixed by $g$. For the equivalence of (ii) and (iii), use (1). For the last assertion, it is clear that the set of elements satisfying (ii) is a subgroup. □

**Definition 25.** If $X$ is a $G$-space over $S$, the *stabilizer* of a point $x \in |X|$ is the subgroup $G_x \subset G$ of elements satisfying the equivalent conditions of Lemma 24.

The next definition is a generalization of the fact that, for any open subspace $U$ of a $G$-space $X$, there is a natural $G$-action on $\cap_{g \in G}(g \cdot U)$.

**Definition 26.** Let $X$ be a $G$-space over $S$ and let $\mu\colon U \to X$ be a morphism of algebraic spaces. Define $\mu_g$ to be the composition $g \circ \mu\colon U \to X$. The *equivariantization* of $U$, denoted $\mu^{\mathsf{e}}\colon U^{\mathsf{e}} \to X$, is the fiber product $U \times_X \cdots \times_X U \to X$ of all the $\mu_g$ (see [SP21, Tag 02XC]).

**Lemma 27.** *Let $X$ be a $G$-space over $S$ and let $\mu\colon U \to X$ be a morphism of algebraic spaces. Then $U^{\mathsf{e}}$ has a natural $G$-action making $\mu^e$ equivariant. Furthermore, if $\mu$ is an open immersion or étale, then so is $\mu^{\mathsf{e}}$.*

Note that the equivariantization $U^{\mathsf{e}}$ is nonempty if the image of $\mu$ has a fixed point of $X$.

*Proof.* Let $T \to X$ be an $X$-scheme. An element $u = (u_g)_{g \in G} \in U^{\mathsf{e}}(T) \subseteq \prod_{g \in G} U(T)$ is given by morphisms of $X$-schemes $u_g : T \to U$ such that $g \circ \mu \circ u_g = \mu \circ u_1$ for all $g \in G$, where 1 denotes the identity of $G$. Given $h \in G$, set $h \cdot (u_g)_{g \in G} := (u_{h^{-1}g})_{g \in G}$. One checks that this defines the desired action. The last assertion of the lemma follows from the stability of open immersions and étale morphisms under pullback. □

**Example 28.** Recall from [SP21, Tag 02LE] that if $X$ is a scheme, an elementary étale neighborhood $\mu\colon (U, u) \to (X, x)$ is an étale morphism $\mu\colon U \to X$ of schemes such that $\mu(u) = x$ and the induced morphism $\kappa(x) \to \kappa(u)$ on residue fields is an isomorphism. If $X$ is moreover a $G$-space over $S$ and $x \in X^G$ with an elementary étale neighborhood $\mu\colon (U, u) \to (X, x)$, then define

$$u^{\mathsf{e}} \in U^{\mathsf{e}} \quad \text{by} \quad (u^{\mathsf{e}})_g := u. \tag{2}$$

Note that $u^{\mathsf{e}} \in (U^{\mathsf{e}})^G$ and that $\mu^{\mathsf{e}}\colon (U^{\mathsf{e}}, u^{\mathsf{e}}) \to (X, x)$ is an elementary étale neighborhood by Lemma 27 and [SP21, Tag 01JT].

**Lemma 29.** *Let $X$ be a $G$-space over $S$ and let $\mu\colon U \to X$ be an open morphism of algebraic spaces. Any $x \in |X^G \cap \mu(U)|$ has an open neighborhood $V$ such that for any $h \in G$ the restriction of $\mu^{\mathsf{e}}$ to $\left((\mu^{\mathsf{e}})^{-1}(V)\right)^h \to V^h$ is surjective.*

*Proof.* Let $V := \bigcap_{g \in G}(g \circ \mu)(U)$; this is an open neighborhood of $x$. The equivariant map $\mu^{\mathsf{e}}$ necessarily sends $\left((\mu^{\mathsf{e}})^{-1}(V)\right)^h$ to $V^h$. Let $v \in V^h$, and let $g_1, \ldots, g_r \in G$ be a set of right coset representatives for the subgroup $\langle h \rangle$ of $G$ and write $[g_i]$ for the coset of $g_i$. Choose $u_{[g_i]} \in U$ such that $\mu(u_{[g_i]}) = g_i^{-1} v$ and define $(u_g)_{g \in G}$ by $u_g := u_{[g]}$. Since $v \in V^h$, one checks that

$g\mu(u_{[g]}) = \mu(u_{[1]}) = v$ and hence $(u_g)_{g\in G}$ defines a point of $U^{\mathrm{e}}$ mapping to $v$. Likewise we compute

$$(h \cdot (u_g)_{g\in G})_g = u_{h^{-1}g} = u_{[h^{-1}g]} = u_{[g]} = u_g. \qquad \square$$

**Lemma 30.** *Let $\mu\colon U \to X$ be an equivariant morphism of $G$-algebraic spaces over $S$. If $\mu$ is a monomorphism in the category of algebraic spaces over $S$, then the equivariantization $U^{\mathrm{e}}$ of $U$ is canonically isomorphic to $U$.*

*Proof.* For any $g \in G$, let $\mathrm{pr}_g\colon U^{\mathrm{e}} \to U$ be the projection to the copy of $U$ whose given map to $X$ is $\mu_g$; note that $\mu_g \circ \mathrm{pr}_g = \mu \circ \mathrm{pr}_1$ where $1 \in G$ is the identity. Let $\widetilde{\Delta}\colon U \to U^{\mathrm{e}}$ be the unique map such that $\mathrm{pr}_g \circ \widetilde{\Delta} = g^{-1}$ for all $g \in G$. Clearly, $\mathrm{pr}_1 \circ \widetilde{\Delta}$ is the identity on $U$. Conversely, $\widetilde{\Delta} \circ \mathrm{pr}_1$ is the identity on $U^{\mathrm{e}}$ if and only if $\mathrm{pr}_g \circ \widetilde{\Delta} \circ \mathrm{pr}_1 = \mathrm{pr}_g$ for each $g \in G$, and this holds if and only if

$$\mu_g \circ \mathrm{pr}_g \circ \widetilde{\Delta} \circ \mathrm{pr}_1 = \mu_g \circ \mathrm{pr}_g \qquad \text{for each } g \in G \tag{3}$$

since $\mu$ (and hence $\mu_g$) is a monomorphism. Using the identities $\mu_g \circ \mathrm{pr}_g = \mu \circ \mathrm{pr}_1$ and $\mathrm{pr}_1 \circ \widetilde{\Delta} = id_U$, we see that (3) holds. $\square$

2.4. **Quotients by finite group actions on algebraic spaces.** Let $G$ be a finite group acting on an algebraic space $X$ over a base scheme $S$.

**Definition 31.** The *coarse quotient* of $X$ by $G$ is a morphism $X \to X/G$ to an algebraic space $X/G$ with the following universal property: every $G$-invariant morphism $X \to Y$ of algebraic spaces factors uniquely through $X \to X/G$.

The universal property of $X/G$ implies that it is canonically an algebraic space over $S$ and that if $X/G$ exists, it is unique up to unique isomorphism.

**Lemma 32.** *If $X$ is a $G$-space over $S$ such that $X \to S$ is separated and locally of finite type, then $X/G$ exists and its formation commutes with flat base change on $S$.*

The construction of $X/G$ and the proof of Lemma 32 use the quotient stack $[X/G]$. This stack is defined in [SP21, Tag 044Q(2)] to be the quotient stack associated to a certain groupoid [SP21, Tag 0444] in algebraic spaces, which in particular has $U = X$ and relations $R = G \times X$. Note that the definition of $[X/G]$ is independent of the base $S$ (notated $B$ in [SP21, Tag 044Q(2)]). One may also describe $[X/G]$ as the moduli stack parametrizing $G$-torsors together with maps to $X$ (see [SP21, Tag 04UV]).

**Remark 33.** Formation of $[X/G]$ commutes with arbitrary base change on $S$. More precisely, let $S' \to S$ be a map of schemes and $X' := X \times_S S'$; we claim that the natural 1-morphism $[X'/G] \to [X/G] \times_S S'$ is an equivalence. Indeed, $X' = X \times_S S' \to [X/G] \times_S S'$ is a smooth cover, and the associated relations $R$ (in the sense of [SP21, Tag 04T4]) are precisely the pullback of the relations associated with the smooth cover $X \to [X/G]$, which in turn are given by $G \times X$ with the maps defining the group action (using [SP21, Tag 04M9]). So a computation shows that $R$ is $G \times X'$ with the maps defining the $G$-action on $X'$. Now use [SP21, Tag 04T4, Tag 04T5] to conclude that $[X'/G]$ is $F$.

*Proof of Lemma 32.* We consider the stack quotient $\mathscr{X} := [X/G]$ as in [SP21, Tag 044Q(2)], noting that this operation commutes with arbitrary base change (as follows from Remark 33) and then observe that our definition of $X/G$ agrees with the definition of categorical moduli space for

$\mathscr{X}$ in the sense of [SP21, Tag 0DUG] (this need not exist, in general). So it suffices to show that we can apply the Keel-Mori theorem [SP21, Tag 0DUT]; i.e., we need to show that $\mathscr{X}$ has finite inertia.[1] From the description of the inertia stack as the fiber product of diagonal morphisms [SP21, Tag 034H], it suffices to show that the diagonal $\Delta\colon \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is finite.

By [SP21, Tag 02XE] and [SP21, Tag 04M9] there is a fiber square

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd}$$

where the top horizontal morphism is given by the action map $(g, x) \mapsto (g.x, x)$. By [SP21, Tag 04X0] the projection $X \to \mathscr{X}$ is smooth, hence $X \times_S X \to \mathscr{X} \times_S \mathscr{X}$ is also smooth since it factors as a composition of smooth maps $X \times_S X \to X \times_S \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$. By [SP21, Tag 04XD], we have reduced to showing that $G \times X \to X \times_S X$ is finite, or equivalently that it is proper and locally quasi-finite ([SP21, Tag 0A4X]). This follows if the composition

$$G \times X \to X \times_S X \xrightarrow{\mathrm{pr}_2} X$$

is proper and locally quasi-finite (this uses [SP21, Tags 04NX, 03XN] and the fact that $X \to S$ is separated). These properties hold since the composition is just $\mathrm{pr}_2\colon G \times X \to X$ and $G$ is finite. □

**Lemma 34.** *If $X$ is a $G$-space over $S$ such that $X \to S$ is separated and locally of finite type, then $X \to X/G$ is finite and surjective.*

*Proof.* We recall from the proof of Lemma 32 the factorization

$$X \to [X/G] \to X/G \tag{4}$$

of the map $X \to X/G$. We will show that each of the maps in (4) is proper, locally quasi-finite, and surjective, hence the composition is as well, and by [SP21, Tag 0A4X] it is finite and surjective.

By [SP21, Tag 04M9] there is a fiber square

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d] & X \arrow[d] \\ X \arrow[r] & {[X/G]}, \end{tikzcd}$$

where the left vertical arrow is given by the action of $G$ on $X$. By [SP21, Tag 04X0], the (vertical) map $X \to [X/G]$ is smooth, so by [SP21, Tag 04XD] the (horizontal) map $X \to [X/G]$ is proper, locally quasi-finite, and surjective since $\mathrm{pr}_2\colon G \times X \to X$ has those properties.

Since $X \to S$ is locally of finite type, the morphism $[X/G] \to S$ is also locally of finite type, and hence $[X/G] \to X/G$ is as well (using [SP21, Tag 06FM] and [SP21, Tag 06U9] respectively). By the Keel-Mori theorem [SP21, Tag 0DUT], the morphism $[X/G] \to X/G$ is also separated, quasi-compact, and a universal homeomorphism, hence proper and surjective. Since $[X/G] \to X/G$ is a universal homeomorphism it has discrete fibers, so by [SP21, Tag 06RW] it is locally quasi-finite. □

**Lemma 35.** *Let $X$ be a $G$-space over $S$ such that $X \to S$ is separated and locally of finite type. Let* $\operatorname{Spec} C \to X/G$ *be an étale morphism from an affine scheme. Then* $X \times_{X/G} \operatorname{Spec} C \simeq \operatorname{Spec} B$ *is an*

[1] Note that the final assertion of Lemma 32, namely that $X/G$ commutes with flat base change, also follows from the Keel-Mori theorem [SP21, Tag 0DUT]. This is because the cited tag asserts that $X/G$ exists and is a *uniform* categorical moduli space, meaning that its formation commutes with flat base change.

*affine $G$-scheme étale over $X$, and the projection* $\operatorname{Spec} B \to \operatorname{Spec} C$ *is induced by the inclusion of the invariant ring $C \simeq B^G \hookrightarrow B$.*

*Proof.* By Lemma 34 the map $X \to X/G$ is finite, hence affine, so the fiber product $X \times_{X/G} \operatorname{Spec} C$ is isomorphic to an affine scheme $\operatorname{Spec} B$. By Remark 22, the ring $B$ has a $G$-action and $C \to B$ factors through the invariant subring $B^G$. In fact, by Remark 33 with $S = X/G$ and $S' = \operatorname{Spec} C$, we have

$$[X/G] \times_{X/G} \operatorname{Spec} C = [(\operatorname{Spec} B)/G]. \tag{5}$$

Since formation of the coarse space commutes with flat base change ([SP21, Tag 0DUT]), the map $[(\operatorname{Spec} B)/G] \to \operatorname{Spec} C$ is a coarse moduli space. In the parlance of [SP21, Tag 0DUK], the stack $[(\operatorname{Spec} B)/G]$ is well-nigh affine, so its coarse moduli space is constructed in [SP21, Tag 0DUP]; as explained there, $C \hookrightarrow B^G$ is an isomorphism. □

The following is a special case of [AOV08, Cor. 3.3].

**Lemma 36.** *Let $X$ be a $G$-space over $S$ such that $X \to S$ is separated and locally of finite type, where $S$ is a scheme over* $\operatorname{Spec}(\mathbf{Z}[1/|G|])$.

(i) *The formation of $X/G$ commutes with arbitrary base change on $S$. That is, for any morphism of schemes $T \to S$ the natural map $(X \times_S T)/G \to (X/G) \times_S T$ is an isomorphism.*
(ii) *If $X \to S$ is flat, then $X/G \to S$ is flat.*

*Proof.* If $A$ is a $G$-module on which $|G| \in \mathbf{Z}$ acts as a unit (so $A$ is also a $\mathbf{Z}[1/|G|]$-module), then we may define the Reynolds operator $R_A := (1/|G|)\sum_{g \in G} g \colon A \to A^G$, a projection onto the submodule $A^G \subset A$. The existence of this operator has two important applications: (a) taking $G$-invariants is an exact functor from $\mathbf{Z}[1/|G|]$-modules with $G$-action to $\mathbf{Z}[1/|G|]$-modules (use functoriality of $R_A$), and (b) if $R$ is an algebra with a homomorphism $R \to A^G$, then for any $R$-module $N$ the inclusion $A^G \otimes_R N \subset (A \otimes_R N)^G$ is an equality, where $G$ acts on $A \otimes_R N$ by the rule $g \cdot (a \otimes n) = (g \cdot a) \otimes n$ (show that $R_A \otimes_R \mathrm{id}_N = R_{A \otimes_R N}$).

To prove (i) we may take $S = X/G$. Since the property of being an isomorphism is étale local on the target [SP21, Tag 041Y] and formation of $X/G$ commutes with flat base change, we reduce to the case when $S = \operatorname{Spec} B$, $T = \operatorname{Spec} B'$, and hence by Lemma 35 we have $X = \operatorname{Spec} A$ with $B = A^G$. Now $A^G \otimes_B B' \to (A \otimes_B B')^G$ is an isomorphism by (b) above.

For (ii) we may check flatness locally, so assume $S = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine. By (b) above, the functor $A^G \otimes_B (-)$ is isomorphic to $(A \otimes_R (-))^G$, but $A \otimes_B (-)$ and taking $G$-invariants are both exact functors, by assumption and (a) above. □

The next lemma is similar to [SP21, Tag 0DUZ] but we remove the hypothesis that the base is locally Noetherian.

**Lemma 37.** *Let $X$ be a $G$-space over a $\mathbf{Z}[1/|G|]$-scheme $S$. If $f \colon X \to S$ is proper and of finite presentation, then so is $X/G \to S$.*

*Proof.* Since the properties of being finitely presented and proper are local on the base ([SP21, Tag 041V] and [SP21, Tag 0422]), we may assume $S = \operatorname{Spec} A$ is affine. Write $A = \operatorname{colim}_{i \in I} A_i$ as the filtered colimit of its finitely generated (hence Noetherian) $\mathbf{Z}[1/|G|]$-subalgebras and set $S_i = \operatorname{Spec} A_i$. By [SP21, Tag 07SK], we can find $i \in I$ and a $G$-space $X_i$ over $S_i$ with $f_i \colon X_i \to S_i$

of finite presentation, such that $f$ is the base change of $f_i$. By [SP21, Tag 08K1] we can increase $i$ and assume $f_i$ is still proper. By Lemma 32 the coarse quotient $X_i/G \to S_i$ exists. Since $S_i$ is Noetherian, $X_i/G \to S_i$ is proper (using [SP21, Tag 0DUZ]) and of finite presentation (using [SP21, Tag 06G4(2)]). Lastly, $X/G \simeq (X_i/G) \times_{S_i} S$ by Lemma 36, so $X/G \to S$ is proper and of finite presentation. ◻

## 3. Admissible $G$-covers

### 3.1. **Balanced group actions on nodal curves.** Let $G$ be a finite group.

**Definition 38.** A *nodal $G$-curve* is a $G$-space $X$ over a scheme $S$ where

- (i) $S$ is a $\mathrm{Spec}(\mathbf{Z}[1/|G|])$-scheme
- (ii) $X \to S$ is a nodal curve and
- (iii) for every geometric fiber $X_{\bar{k}}$ and $g \in G$ with $g \neq 1$, the locus $X_{\bar{k}}^g$ does not contain any irreducible component of $X_{\bar{k}}$

A *prestable $G$-curve* is a nodal $G$-curve where we replace (ii) with the condition that $X \to S$ is a prestable curve.

We observe that the notation $X_{\bar{k}}^g$ is well-defined by Lemma 23.

**Remark 39.** Unlike our definition of a $G$-space $X$ over $S$, our definitions of nodal and prestable $G$-curves $X \to S$ require $S$ to be a $\mathrm{Spec}(\mathbf{Z}[1/|G|])$-scheme.

**Remark 40.** The pullback of a nodal (resp. prestable) $G$-curve is a nodal (resp. prestable) $G$-curve.

**Lemma 41.** *Let $X$ be a nodal $G$-curve over a field $k$ and $x \in X$ be a smooth point with residue field $k$ and stabilizer $G_x$. Then $G_x$ is cyclic, and there is a faithful character $\xi : G_x \to k^\times$ such that we have a $G_x$-equivariant isomorphism*

$$\mathscr{O}_{X,x}^\wedge \simeq k[\![u]\!], \qquad g \cdot u = \xi(g)u \text{ for } g \in G_x$$

*sending $u$ to an element of $\mathfrak{m}_x \subset \mathscr{O}_{X,x}^\wedge$.*

*Proof.* Let $\mathfrak{m} \subset \mathscr{O}_{X,x}^\wedge$ be the maximal ideal and let $r := |G_x|$. Note that $\mathrm{char}(k) \nmid r$. Since $x$ is a smooth point of $X$, we have $\mathscr{O}_{X,x}^\wedge \simeq k[\![u]\!]$ for any $u \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ by the proof of [SP21, Tag 0C0S].

Since $x$ is a $G_x$-fixed point, the action of $G_x$ on $X$ induces an action on $\mathscr{O}_{X,x}$ and thus a character $\xi\colon G_x \to \mathrm{Aut}_k(\mathfrak{m}/\mathfrak{m}^2) \simeq k^\times$. We show that $\xi$ is faithful. Let $g \in G_x$ such that $\xi(g) = 1$. Then $g$ also acts trivially on $\mathscr{O}_{X,x}^\wedge \simeq k[\![u]\!]$. Indeed, otherwise we can choose $\ell \geq 2$ minimal and $a \in k^\times$ such that $g \cdot u \equiv u + au^\ell \mod \mathfrak{m}^{\ell+1}$. An inductive argument then shows that

$$u = g^r \cdot u \equiv v + rau^\ell \not\equiv v \mod \mathfrak{m}^{\ell+1}$$

because $r \neq 0$ in $k$, a contradiction. Further, if $Y$ denotes the irreducible component of $X$ containing $x$, then $g$ must act trivially on $k(Y) \subset \mathrm{Frac}\, \mathscr{O}_{X,x}^\wedge$ and thus on $Y$ by [SP21, Tag 0BY1]. Hence $g = 1$ as desired.

Next, the finiteness of $G_x$ guarantees that $\xi$ identifies $G_x$ with the $r$-th roots of unity $\mu_r(k)$. Thus, $G_x$ is cyclic ([SP21, Tag 09HX]). To finish the proof, it suffices to exhibit a uniformizer $t \in \mathfrak{m}$ with $g \cdot t = \xi(g)t$ because the natural map

$$k[\![u]\!] \to \mathscr{O}_{X,x}^\wedge, \quad u \mapsto t$$

will then be a $G_x$-equivariant isomorphism with the requested properties. Pick $v \in \mathfrak{m} \setminus \mathfrak{m}^2$ arbitrary and set $t := \sum_{g \in G} \xi(g)^{-1}(g \cdot v)$. Since $g \cdot v \equiv \xi(g)v \mod \mathfrak{m}^2$, we have $t \equiv rv \neq 0 \mod \mathfrak{m}^2$ (again using that $r \in k^\times$) and thus $t$ is not in $\mathfrak{m}^2$; i.e., it is a uniformizer of $\mathfrak{m}$. On the other hand, $g \cdot t = \xi(g)t$, so $t$ does the job. □

**Lemma 42.** *Let $X$ be a nodal $G$-curve over a field $k$ and $x \in X$ be a split node with stabilizer $G_x$. If $X^\nu \to X$ is the normalization let $x_1, x_2$ be the two preimages of $x$ in $X^\nu$ (see [SP21, Tag 0CBW]) and assume that $G_{x_1} = G_x$. Then $G_x$ is cyclic and there exist faithful characters $\xi_1, \xi_2 \colon G_x \to k^\times$ such that we have a $G_x$-equivariant isomorphism*

$$\mathscr{O}^\wedge_{X,x} = k[\![u, v]\!]/(uv), \quad g \cdot u = \xi_1(g)u, \quad g \cdot v = \xi_2(g)v \ \textit{for } g \in G_x \tag{6}$$

*identifying $u$ and $v$ with elements of $\mathfrak{m}_x \subset \mathscr{O}^\wedge_{X,x}$.*

*Proof.* Let $A = \mathscr{O}^\wedge_{X,x}$ and let $A'$ be the integral closure of $A$ in its total ring of fractions. By [SP21, Tag 0C3V], we have $\operatorname{Spec}(A') = X^\nu \times_X \operatorname{Spec}(A)$. Since the node is split, the $\delta$-invariant of $A$ (as defined in [SP21, Tag 0C3T]) is 1 and there are two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ in $A'$ lying over $\mathfrak{m}_x \subset A$: indeed, these properties hold by definition and [SP21, Tag 0C4A] when $A$ is replaced with $\mathscr{O}_{X,x}$, and they pass to the completion by [SP21, Tag 0C3W] and the fact that $\operatorname{Spec}(A') = X^\nu \times_X \operatorname{Spec}(A)$.

We observe that $(A')_{\mathfrak{m}_i}$ is by definition the local ring of $X^\nu \times_X \operatorname{Spec}(A)$ at $x_i$, and this is in turn equal to $\mathscr{O}^\wedge_{X^\nu,x_i}$ by [SP21, Tag 07N9]. Our assumptions guarantee that $G_x$ acts on the branch of $X^\nu$ containing $x_i$ and hence $\mathscr{O}^\wedge_{X^\nu,x_i}$ has a description as in Lemma 41 for some character $\xi_i$. (In particular, $G_x = G_{x_1}$ is cyclic.)

We are in the situation of [SP21, Tag 0C4A(2)], in fact of Case I of the proof. There it is shown that $\kappa(\mathfrak{m}_1) = \kappa(\mathfrak{m}_2) = \kappa(x)$ and $A$ is equal to the subring of $A'$ given by the *wedge* ([SP21, Tag 0C41]) of $(A')_{\mathfrak{m}_1}$ and $(A')_{\mathfrak{m}_2}$; i.e. (using the explicit description of $(A')_{\mathfrak{m}_i} = \mathscr{O}^\wedge_{X^\nu,x_i} \simeq k[\![u]\!]$ in the previous paragraph),

$$\mathscr{O}^\wedge_{X,x} \simeq \{(f_1(u), f_2(v)) \in k[\![u]\!] \times k[\![v]\!] \mid f_1(0) = f_2(0)\}. \tag{7}$$

We observe that $G_x$ acts on $A$ and on $A'$ and that the identification (7) as given in [SP21, Tag 0C4A(2)] is completely $G_x$-equivariant. Thus, we get a $G_x$-equivariant isomorphism $\mathscr{O}^\wedge_{X,x} = k[\![u, v]\!]/(uv)$ with the desired properties. □

**Definition 43.** Let $X$ be a nodal $G$-curve over a field $k$.

(i) If $k$ is algebraically closed, a split node $x \in X$ with stabilizer $G_x$ is *balanced* if $G_x$ is cyclic and for some faithful character $\xi$ of $G_x$, there is a $G_x$-equivariant isomorphism as in (6) with $\xi_2 = \xi_1^{-1}$.

(ii) For general $k$, a node $x \in X$ is *balanced* if there is a balanced split node in $X_{\bar{k}}$ that maps to $x$.

We say $X$ is *balanced* if every node is balanced, and a nodal $G$-curve $X \to S$ is balanced if every geometric fiber is balanced.

**Remark 44.** Let $x$ be a split node of a nodal $G$-curve $X$ over a field $k$ with cyclic stabilizer $G_x = \langle g \rangle$ and $G_x$-equivariant trivialization $\mathscr{O}^\wedge_{X,x} = k[\![u, v]\!]/(uv)$ as in (6). The induced action of $g$ on the tangent space $T_x := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ of $x$ is identified with an endomorphism of $k \cdot u \oplus k \cdot v$. Then $\xi_2 = \xi_1^{-1}$ if and only if $\det g = 1$. In particular, $\xi_2 = \xi_1^{-1}$ is independent of the choice of trivialization in (6) because the condition $\det g = 1$ is independent of the choice of basis of the vector space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.

**Lemma 45.** *Let $X$ be a nodal $G$-curve over a field $k$ and $x \in X$ a node.*

(i) *If $x$ is a split node, then $x$ is balanced if and only if $\mathcal{O}^{\wedge}_{X,x}$ may be described as in* (6) *with $\xi_2 = \xi_1^{-1}$.*

(ii) *If $K/k$ is a field extension and $y \in X_K$ has image $x$, then $x$ is a balanced node if and only if $y$ is one.*

(iii) *If $Y \to X$ is an $G$-equivariant étale morphism of nodal $G$-curves over $k$ and $y \in Y$ has image $x$ with $G_x = G_y$, then $x$ is a balanced node if and only if $y$ is one.*

*Proof.* Let $x$ be a split node and $K/k$ be a field extension. If $y \in X_K$ maps to $x$, then $y$ is a split node by Lemma 7. If $x_1$ (resp. $y_1$) is a preimage of $x$ (resp. $y$) in the normalization of $X$ (resp. $X_K$), then $G_x = G_y$ and $G_{x_1} = G_{y_1}$ by Lemma 23. Hence $G_x = G_{x_1}$ if and only if $G_y = G_{y_1}$, and if this is the case, then by Lemma 42 the rings $\mathcal{O}^{\wedge}_{X,x}$ and $\mathcal{O}^{\wedge}_{X_K,y}$ admit trivializations as in (6) with characters $\xi_1$, $\xi_2$ and $\tilde{\xi}_1$, $\tilde{\xi}_2$, respectively. Moreover, the $G_x$-equivariant map $\mathcal{O}^{\wedge}_{X,x} \to \mathcal{O}^{\wedge}_{X_K,y}$ induces a $G_x$-equivariant morphism of tangent spaces $T_y \xrightarrow{\sim} T_x \otimes_k K$. Now Remark 44 shows that $\xi_2 = \xi_1^{-1}$ if and only if $\tilde{\xi}_2 = \tilde{\xi}_1^{-1}$. In the special case $K = \bar{k}$, this implies (i).

Next, we prove (ii). If $y$ is balanced, then by definition we have a balanced split node $y' \in X_{\bar{K}}$ mapping to $y$. The inclusion $k \hookrightarrow \bar{K}$ factors through $k \hookrightarrow \bar{k}$, so we get a map $X_{\bar{K}} \to X_{\bar{k}}$; let $x' \in X_{\bar{k}}$ be the image of $y'$ under this map. It is a (necessarily split) node by [SP21, Tag 0C56]. Moreover it is balanced by the first paragraph of this proof. Since $x'$ maps to $x$ we see that $x$ is balanced.

Conversely, assume $x$ is balanced. By [SP21, Tag 0C56], $y$ is a node. Let $\bar{k}^{sep} \subset \bar{k}$ denote the separable algebraic closure of $k$. To prove that $y$ is balanced, we may assume $K$ is algebraically closed. Since such $K$ contains $\bar{k} \supset \bar{k}^{sep}$, by the first paragraph it suffices to prove that if $y \in X_{\bar{k}^{sep}}$ has image $x$, then $y$ is a balanced split node. Let $y \in X_{\bar{k}^{sep}}$ have image $x \in X$. By Lemma 8.(iii) and Lemma 7 we know there is a split node $y' \in X_{\bar{k}^{sep}}$ mapping to $x$. By [SP21, Tag 04KP] there is an automorphism in $\mathrm{Gal}(\bar{k}^{sep}/k) \subset \mathrm{Aut}_k(X_{\bar{k}^{sep}})$ that maps $y$ to $y'$; hence $y$ is split (as is every node in the preimage of $x$). Since $x$ is balanced the discussion in the first paragraph shows that there is some balanced split node $y'' \in X_{\bar{k}^{sep}}$ mapping to $x$. There is an element of $\mathrm{Gal}(\bar{k}^{sep}/k) \subset \mathrm{Aut}_k(X_{\bar{k}^{sep}})$ sending $y$ to $y''$, and moreover this automorphism is are $G$-equivariant (consider the action in terms of the functor of points). From the characterization of balanced split nodes in Lemma 42 it follows that $y$ is balanced.

To prove (iii), let $y' \in Y_{\bar{k}}$ be a node mapping to $y$ and let $x' \in X_{\bar{k}}$ be its image. By (ii), $x$ (resp. $y$) is balanced if and only if $x'$ (resp. $y'$) is balanced. Both $x'$ and $y'$ are split nodes; let $x_1', x_2'$ and $y_1', y_2'$ be the two preimages in the respective normalizations indexed so that the $G$-equivariant map of normalizations sends $x_i$ to $y_i$. Note that $G_{y_1'} \subset G_{x_1'} \subset G_{x'} = G_{y'}$ (the equality is our hypothesis). So $G_{y_1'} = G_{y'}$ implies $G_{x_1'} = G_{x'}$, and conversely if $g \in G_{y'} \smallsetminus G_{y_1'}$ then $gy_1' = y_2'$ implies $gx_1' = x_2'$, so in fact $G_{y_1'} = G_{y'}$ if and only if $G_{x_1'} = G_{x'}$. In the situation $G_{x_1'} = G_{y_1'} = G_{x'} = G_{y'}$, the local rings of $x'$ and $y'$ can be trivialized as in (6) and the nodes are balanced if and only if the characters $\xi_1$ and $\xi_2$ are inverse. On the other hand since $\mathcal{O}^{\wedge}_{X_{\bar{k}},x'} \to \mathcal{O}^{\wedge}_{Y_{\bar{k}},y'}$ is $G$-equivariant and $k$-linear, the characters for $\mathcal{O}^{\wedge}_{X_{\bar{k}},x'}$ and $\mathcal{O}^{\wedge}_{Y_{\bar{k}},y'}$ must be the same. □

3.2. **Families of balanced nodal curves.** In this section we study the deformation theory of balanced nodal $G$-curves. This problem is simplest when we work over a base ring that contains the $|G|$-th roots of unity because the group homomorphisms $G_x \to k(s)^{\times}$ of Lemmas 41 and 42 can be lifted to morphisms of group schemes (see Remark 49).

**Remark 46.** If $X$ is a balanced nodal $G$-curve over $k$ and $x \in X$ has residue field $k$, it follows from Lemmas 41 and 42 that $k$ contains all the $|G_x|$-th roots of unity.

**Definition 47.** Let $r \geq 1$ be an integer, and define the *ring of cyclotomic integers* is $\mathbf{Z}[\zeta_r] := \mathbf{Z}[x]/(\Phi_r(x))$, where $\Phi_r(x)$ denotes the $r$-th cyclotomic polynomial. This is the integral closure of $\mathbf{Z}$ in the $r$-th cyclotomic field $\mathbf{Q}(\zeta_r)$, and it is also equal to the smallest subring of $\mathbf{C}$ containing $\mathbf{Z}$ and $\zeta_r$.

**Remark 48.** If $S = \mathrm{Spec}(\mathbf{Z}[1/r, \zeta_r])$, then there is an isomorphism of group schemes $(\mathbf{Z}/r\mathbf{Z})_S \to \mu_{r,S}$. In terms of functors of points, for any $S$-scheme $T$ this morphism is given by group homomorphisms $(\mathbf{Z}/r\mathbf{Z})(T) \to \mu_r(T)$ sending the generator $1 \in \mathbf{Z}/r\mathbf{Z}$ to $\zeta_r \in \Gamma(T, \mathscr{O}_T)$. One checks it is an isomorphism by noting that as a morphism of affine schemes it is given by the ring map

$$\mathbf{Z}[1/r, \zeta_r, x]/(x^r - 1) \to \prod_{i=0}^{r-1} \mathbf{Z}[1/r, \zeta_r] \tag{8}$$

sending $x$ to $(1, \zeta_r, \ldots, \zeta_r^{r-1})$. The homomorphism (8) is an isomorphism by the Chinese Remainder Theorem ([SP21, Tag 00DT]) if each pair of ideals $(x - \zeta_r^i)$ and $(x - \zeta_r^j)$ are coprime (for $i \neq j$). To show that these ideals are coprime it is enough to show that $\zeta_r^i - \zeta_r^j$ is invertible in $\mathbf{Z}[1/r, \zeta_r]$, or equivalently that $1 - \zeta_r^k$ is invertible in $\mathbf{Z}[1/r, \zeta_r]$ for each integer $k$. To see this, observe that

$$\prod_{k=1}^{r-1}(x - \zeta_r^k) = 1 + x + \ldots + x^{r-1}$$

and then set $x = 1$.

**Remark 49.** Let $S$ be a scheme over $\mathrm{Spec}(\mathbf{Z}[1/r, \zeta_r])$, let $s \in S$, and let $G$ be a group of order $r$ with $\xi : G \to k(s)^x$ a group homomorphism. Then there is a morphism of group schemes $G_S \to \mathbf{G}_{m,S}$ such that the map on sections over $\mathrm{Spec}(k(s))$ is given by $\xi$: one may see this as follows. Since $G$ has order $r$, the homomorphism $\xi$ factors through $\mu_r(k(s)) \subset k(s)^\times$. By Remark 48 we have a group homomorphism $G \to \mathbf{Z}/r\mathbf{Z}$. This defines a morphism $G_S \to (\mathbf{Z}/r\mathbf{Z})_S$ of constant group schemes, and by Remark 48 again this yields the desired $G_S \to \mu_{r,S}$. (Note that $G_S \to \mathbf{G}_{m,S}$ is uniquely determined by $\xi$ on the connected component of $s$.)

The following lemma is the analog of [SP21, Tag 0CBX].

**Lemma 50.** *Let $X$ and $S$ be schemes over $\mathbf{Z}[1/|G|, \zeta_{|G|}]$ with $X \to S$ a nodal $G$-curve, and choose $x \in X$ with $G_x = G$. Let $s \in S$ be the image of $x$ and assume $\kappa(x) = \kappa(s)$ and that $\mathscr{O}_{S,s}$ is Noetherian. Let $\xi \colon G \to \kappa(s)^\times$ be defined as in Lemma 41 or 42. Then the morphism of group schemes in Remark 49 defines a group homomorphism $\xi : G \to (\mathscr{O}_{S,s}^\wedge)^\times$ such that*

(i) *If $x$ is a smooth point of $X_s$, then there is a $G$-equivariant isomorphism of $\mathscr{O}_{S,s}^\wedge$-algebras*

$$\mathscr{O}_{X,x}^\wedge \simeq \mathscr{O}_{S,s}^\wedge[\![u]\!], \qquad g \cdot u = \xi(g)u \ \text{ for } g \in G. \tag{9}$$

(ii) *If $x$ is a balanced split node of $X_s$, then there exists $h \in \mathfrak{m}_s \subset \mathscr{O}_{S,s}^\wedge$ and a $G$-equivariant isomorphism of $\mathscr{O}_{S,s}^\wedge$-algebras*

$$\mathscr{O}_{X,x}^\wedge \simeq \mathscr{O}_{S,s}^\wedge[\![u, v]\!]/(uv - h), \quad g \cdot u = \xi(g)u, \quad g \cdot v = \xi^{-1}(g)v \ \text{ for } g \in G. \tag{10}$$

*Proof.* We replace $S$ by $\mathrm{Spec}(\mathscr{O}_{S,s})$ and $X$ by $\mathrm{Spec}(\mathscr{O}_{S,s}) \times_S X$. Let $A = \mathscr{O}_{S,s}^\wedge$ and $B = \mathscr{O}_{X,x}^\wedge$ and let $\mathfrak{m} := \mathfrak{m}_A$ be the maximal ideal of $A$. Then $\mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$ is a flat homomorphism of Noetherian rings and by [SP21, Tag 0C4G] the map $A \to B$ is also flat (we will use this at the end of the proof).

Note that $\mathcal{O}^{\wedge}_{X_s,x} = B/\mathfrak{m}B$. By Remark 49 $\xi$ defines a faithful character $G \to \mathbf{G}_{m,S}$ that identifies $G_S$ with $\mu_{|G|,S}$. Hence, any $\mathcal{O}_{S,s}$-module $R$ with a $G$-action may be written as a direct sum

$$R = \bigoplus_{i=1}^{|G|} R_{\xi^i} \tag{11}$$

of $\mathcal{O}_{S,s}$-modules such that for $g \in G$ and $r \in R_{\xi^i}$ we have $g \cdot r = \xi(g)^i r$. We say that $r \in R_{\xi^i}$ has weight $\xi^i$. To see (11), set $r_i := \frac{1}{|G_x|}\sum_{g\in G}\xi^{-i}(g)g\cdot r$ and check that $\sum_i r_i = r$ and that $r_i$ has weight $\xi^i$.

To prove (i), observe that by Lemma 41 there is an isomorphism

$$(A/\mathfrak{m})[\![u]\!] \to B/\mathfrak{m}B.$$

where $u$ maps to some element of $\mathfrak{m}_B/\mathfrak{m}B$ of weight $\xi$. Since the map $B \to B/\mathfrak{m}$ respects the decompositions (11), we can choose a lift $\tilde{t} \in \mathfrak{m}_B$ of weight $\xi$. This defines a morphism $f\colon A[\![u]\!] \to B$ of Noetherian rings that is an isomorphism modulo $\mathfrak{m}$. We conclude by [SP21, Tags 0315, 00ME] using flatness of $A \to B$ that $f$ is an isomorphism.

To prove (ii) observe that by Lemma 45.(i) there is an isomorphism

$$(A/\mathfrak{m})[\![u,v]\!]/(uv) \to B/\mathfrak{m}B \qquad u \mapsto \bar{x},\ \ v \mapsto \bar{y}$$

where $\bar{x}$ (resp. $\bar{y}$) has weight $\xi$ (resp. $\xi^{-1}$) and $A/\mathfrak{m}$ maps to $(B/\mathfrak{m}B)_{\xi^0}$. Let $x_1$ and $y_1$ be lifts of $\bar{x}$ and $\bar{y}$ of weights $\xi$ and $\xi^{-1}$, respectively, and observe that $x_1y_1 \in \mathfrak{m}B$ has weight $\xi^0$.

Set $h_1 = 0$ and $\delta_1 = x_1y_1$. For $n > 1$ we recursively construct elements $x_n \in B_\xi$, $y_n \in B_{\xi^{-1}}$, $h_n \in A$, and $\delta_n \in (\mathfrak{m}^nB)_{\xi^0}$ such that $x_n - x_{n+1}, y_n - y_{n+1} \in \mathfrak{m}^nB$, $h_n - h_{n+1} \in \mathfrak{m}^n$, and $x_ny_n = h_n + \delta_n$. To define the $(n+1)^{st}$ collection of elements, write $\delta_n = \sum f_ib_i$ with $f_i \in \mathfrak{m}^n$ and $b_i \in B$. Since $\delta_n$ and $f_i$ have weight $\xi^0$, we can replace each summand of $\sum f_ib_i$ with its weight-$\xi^0$ part and thus assume $b_i$ has weight $\xi^0$. Since $A/\mathfrak{m} \simeq B/\mathfrak{m}_B$ and $\mathfrak{m}_B = x_1B + y_1B + \mathfrak{m}B = x_nB + y_nB + \mathfrak{m}B$, we can write $b_i = a_i + x_nb_{i,1} + y_nb_{i,2} + \delta_{i,n}$ where $a_i \in A$; $b_{i,1}, b_{i,2} \in B$; and $\delta_{i,n} \in \mathfrak{m}B$. The element $b_i$ already has weight $\xi^0$, so by replacing every summand of the right side with its weight-$\xi^0$ part we can choose $\delta_{i,n}$ of weight $\xi^0$ and $b_{i,1}, b_{i,2}$ of weight $\xi^{-1}$ and $\xi$, respectively.

Define $x_{n+1} := x_n - \sum b_{i,2}f_i$ and $y_{n+1} := y_n - \sum b_{i,1}f_i$; note that these have the desired weights. One checks that

$$x_{n+1}y_{n+1} = h_n + \sum f_ia_i + \sum f_i\delta_{i,n} + \sum c_{ij}f_if_j$$

for some $c_{ij} \in B_{\xi^0}$. Thus, $h_{n+1} := h_n + \sum f_ia_i$ and $\delta_{n+1} := \sum f_i\delta_{i,n} + \sum c_{ij}f_if_j$ satisfy the recursive assumptions. Since $A$ and $B$ are complete, we can define $x_\infty := \lim x_n$, $y_\infty := \lim y_n$, and $h_\infty := \lim h_n$, noting that $x_\infty y_\infty = h_\infty$ and that $x_\infty$ and $y_\infty$ have weights $\xi$ and $\xi^{-1}$, respectively. We define $f\colon A[\![u,v]\!]/(uv - h_\infty) \to B$ to send $u$ to $x_\infty$ and $v$ to $y_\infty$. Since $f$ is an isomorphism modulo $\mathfrak{m}$, it is an isomorphism as in the proof of (i). □

**Lemma 51.** *Let $X \to S$ be a nodal $G$-curve and a morphism of schemes, and choose $x \in X$ with image $s$ such that $\kappa(x) = \kappa(s)$ and $\mathcal{O}_{S,s}$ is Noetherian. Assume $x$ is either a smooth point or a balanced split node of its fiber. After passing to an elementary étale neighborhood of $s \in S$ we can arrange that the hypotheses of Lemma 50 are satisfied.*

*Proof.* By restricting the $G$-action to $G_x$, we obtain the hypothesis that $G = G_x$. We may assume $S = \mathrm{Spec}(A)$. Let $r := |G|$ and set $S' := \mathrm{Spec}(A[t]/(\Phi_r(t)))$. Since $r$ is invertible in $A$ by Definition 38, the map $S' \to S$ is étale and $S'$ is defined over $\mathrm{Spec}(\mathbf{Z}[1/r, \zeta_r])$. Let $x' \in X_{S'}$ be any point mapping to $x$ and let $s'$ be its image in $S'$. As $s'$ lies in the fiber $\kappa(s) \times_S S' \simeq \mathrm{Spec}(\kappa(s)[t]/\Phi_r(t))$ and $r$ is invertible in $\kappa(s)$, we have $\kappa(s) = \kappa(s')$, so $(S', s') \to (S, s)$ is even an elementary étale neighborhood. Moreover, if $x$ is a smooth rational point (resp. balanced split node) of its fiber, then so is $x'$ by Lemma 7 and Lemma 45.(ii). By Lemma 23 and Lemma 24, we have $G_{x'} = G$. □

The remainder of this section is dedicated to "spreading out" the description of $\mathscr{O}^\wedge_{X,x}$ in Lemma 50.(ii) to an étale neighborhood of $x$. That is, we prove the following equivariant analog of [SP21, Tag 0CBY] (recall from [SP21, Tag 03BU] that $|X|$ is the topological space of an algebraic space $X$).

**Proposition 52.** *Let $X \to S$ be a nodal $G$-curve over $S$. Let $x \in |X|$ be a point which is a balanced node of the fiber $X_s$. Then the stabilizer $G_x$ of $x \in X_s$ is a cyclic group of order $r$, and there exists a commutative diagram of $G_x$-spaces and equivariant morphisms*

$$\begin{tikzcd}
X \arrow[d] & U \arrow[l, "\mu"'] \arrow[rr] \arrow[dr] & & W \arrow[r] \arrow[dl] & \mathrm{Spec}(\mathbf{Z}[u,v,a,1/r,\zeta_r]/(uv-a) \arrow[d] \\
S & & V \arrow[ll] \arrow[rr] & & \mathrm{Spec}(\mathbf{Z}[a,1/r,\zeta_r])
\end{tikzcd}$$

*such that the arrows $X \stackrel{\mu}{\leftarrow} U$, $S \leftarrow V$, and $U \to W$ are étale morphisms, the right hand square cartesian, the $G_x$-action on the top right is given by $g \cdot u = \zeta_r^i u$, $g \cdot v = \zeta_r^{-i} v$ for some integer $i$, the $G_x$-action on $V$ is trivial, and there exists a point $u \in U$ mapping to $x \in X$ with $G_u = G_x$. Moreover $U^g$ surjects onto $\mu(U) \cap X^g$ for all $g \in G_x$.*

**Remark 53.** Since the description of complete local rings in Lemma 50 may be read as a kind of flat-local description of nodal $G$-curves, one may wonder why the étale-local description on Proposition 52 is necessary. The reason is that flat maps do not preserve smooth and nodal points of curves. A naive observation is that flat maps need not preserve dimension, so the notion "$x \in X$ is a node in its fiber" does not make sense after replacing $X$ with an arbitrary flat cover. However, even flat maps of relative dimension zero need not preserve nodes and smooth points: for example, the ring map

$$k[\![u,v]\!]/(u-v^2) \to k[\![u^{1/2},v]\!]/(u-v^2) \simeq k[\![u^{1/2},v]\!]/((u^{1/2}-v)(u^{1/2}+v))$$

induces a flat morphism of schemes that sends a node to a smooth point. On the other hand, by [SP21, Tag 0C57] (resp. Lemma 45.(iii)), étale covers *do* preserve nodes and smooth points (resp. balanced split nodes). Looking at the proof of [SP21, Tag 0C57], one sees that the additional condition that $f$ is unramified (hence étale) is precisely what is needed to make the tag work.

Our proof of Proposition 52 uses Artin approximation and closely follows the proof of [SP21, Tag 0CBY]. Before proving Proposition 52, we state and prove a version of Artin approximation for limit preserving functors from [Art69] as well as equivariant analogs of many of the tags used to prove [SP21, Tag 0CBY].

Following [SP21, Tag 049J], a contravariant functor $F\colon (Sch/X)^{opp} \to Sets$ is limit preserving if for every directed inverse system of affine schemes $T_i$ in $(Sch/X)$ with affine inverse limit $T$, the natural map $\left(\mathrm{colim}_i\, F(T_i)\right) \to F(T)$ is a bijection. The following is [Art69, Cor. 2.2].

**Proposition 54.** *Let $X$ be a locally Noetherian scheme and $x \in X$ such that $\mathscr{O}_{X,x}$ is a* G*-ring in the sense of [SP21, Tag 07GH]. Let $F\colon (Sch/X)^{opp} \to Sets$ be a limit preserving contravariant functor and $\varphi \in F(\mathrm{Spec}(\mathscr{O}^\wedge_{X,x}))$. Then for any natural number $N$, there exists an elementary étale* *neighborhood* $(U,u) \to (X,x)$ *and an element* $\Phi \in F(U)$ *such that* $\varphi$ *and* $\Phi$ *map to the same element in* $F\big(\mathrm{Spec}(\mathscr{O}^{\wedge}_{U,u}/\mathfrak{m}_u^N)\big)$.

*Proof.* We may assume that $N \geq 2$. By [SP21, Tag 02Y2], $F$ corresponds to a category fibered in sets $p\colon \mathscr{S}_F \to (Sch/X)$. Under this correspondence, the fact that $F$ is limit preserving translates to the fact that $p$ is limit preserving on objects in the sense of [SP21, Tag 06CT]. Now [SP21, Tag 07XB] (applied to $\mathscr{X} = \mathscr{S}_F$ and $R = \mathscr{O}^{\wedge}_{X,x}$) gives a ring $A$, a morphism $U := \mathrm{Spec}(A) \to X$ of finite type, a closed point $u \in U$ lying over $x$ and an object $\Phi \in \mathscr{S}_F(U)$ such that the map $U \to X$ induces an isomorphism $\mathscr{O}^{\wedge}_{X,x}/\mathfrak{m}_x^N \to \mathscr{O}_{U,u}/\mathfrak{m}_u^N$. Over this isomorphism, it guarantees the existence of an isomorphism $\varphi|_{\mathrm{Spec}(\mathscr{O}^{\wedge}_{X,x}/\mathfrak{m}_x^N)} \simeq \Phi|_{\mathrm{Spec}(\mathscr{O}_{U,u}/\mathfrak{m}_u^N)}$. Lastly, there exists an isomorphism of graded $\mathscr{O}_{X,x}/\mathfrak{m}_x$-algebras $\mathrm{Gr}_{\mathfrak{m}_x}(\mathscr{O}^{\wedge}_{X,x}) \to \mathrm{Gr}_{\mathfrak{m}_u}(\mathscr{O}_{U,u})$, where $\mathrm{Gr}_{\mathfrak{m}}(B) := \bigoplus_{\ell \geq 0} \mathfrak{m}^{\ell}/\mathfrak{m}^{\ell+1}$ denotes the graded ring attached to a local ring $B$ with maximal ideal $\mathfrak{m}$; note that this graded algebra is generated in degree 1.

The natural map $\mathscr{O}_{X,x} \to \mathscr{O}_{U,u}$ induces isomorphisms $\mathrm{Gr}^{\ell}_{\mathfrak{m}_x}(\mathscr{O}^{\wedge}_{X,x}) \to \mathrm{Gr}^{\ell}_{\mathfrak{m}_u}(\mathscr{O}_{U,u})$ for $\ell = 0, 1$ because $N \geq 2$. Since the graded algebras are generated in degree 1 and their graded pieces have the same dimension (by the existence of the abstract isomorphism), $\mathscr{O}_{X,x} \to \mathscr{O}_{U,u}$ must in fact induce isomorphisms $\mathrm{Gr}^{\ell}_{\mathfrak{m}_x}(\mathscr{O}^{\wedge}_{X,x}) \to \mathrm{Gr}^{\ell}_{\mathfrak{m}_u}(\mathscr{O}_{U,u})$ for all $\ell$. For $\ell = 1$, we see that $\kappa(x) \to \kappa(u)$ is an isomorphism. Via induction and repeated application of the five lemma on the diagram of natural morphisms

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Gr}^{\ell}_{\mathfrak{m}}(\mathscr{O}_{X,x}) & \longrightarrow & \mathscr{O}_{X,x}/\mathfrak{m}_x^{\ell+1} & \longrightarrow & \mathscr{O}_{X,x}/\mathfrak{m}_x^{\ell} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Gr}^{\ell}_{\mathfrak{m}}(\mathscr{O}_{U,u}) & \longrightarrow & \mathscr{O}_{U,u}/\mathfrak{m}_u^{\ell+1} & \longrightarrow & \mathscr{O}_{U,u}/\mathfrak{m}_u^{\ell} & \longrightarrow & 0,
\end{array}
$$

the middle map $\mathscr{O}_{X,x}/\mathfrak{m}_x^{\ell} \to \mathscr{O}_{U,u}/\mathfrak{m}_u^{\ell}$ is an isomorphism for all $\ell$. Hence, $\mathscr{O}^{\wedge}_{X,x} \to \mathscr{O}^{\wedge}_{U,u}$ is an isomorphism, and $U \to X$ is étale at $u$ by [SP21, Tag 039N, Tag 039M]. After shrinking $U$ if necessary, $(U,u) \to (X,x)$ is an elementary étale neighborhood. Finally, under the correspondence of $\mathscr{S}_F$ with $F$, the isomorphism $\varphi|_{\mathrm{Spec}(\mathscr{O}^{\wedge}_{U,u}/\mathfrak{m}_u^N)} \simeq \Phi|_{\mathrm{Spec}(\mathscr{O}^{\wedge}_{U,u}/\mathfrak{m}_u^N)}$ translates to the statement that $\varphi$ and $\Phi$ map to the same element in $F\big(\mathrm{Spec}(\mathscr{O}^{\wedge}_{U,u}/\mathfrak{m}_u^N)\big)$, so we win. $\square$

Now we add group actions to many of the tags used in the proof of [SP21, Tag 0CBY]. The following is an equivariant analog of [SP21, Tag 0CAU] and [SP21, Tag 0CAV].

**Lemma 55.** *Let $G$ be a finite group and let $S$ be a locally Noetherian scheme. Let $X$ and $Y$ be $G$-schemes with $G$-invariant maps $X \to S$, $Y \to S$ locally of finite type. Choose $x \in X^G$ and $y \in Y^G$ lying over the same point $s \in S$ with $\mathscr{O}_{S,s}$ a* G*-ring in the sense of [SP21, Tag 07GH]. Suppose we are given a $G$-equivariant local $\mathscr{O}_{S,s}$-algebra map*

$$\varphi\colon \mathscr{O}_{Y,y} \to \mathscr{O}^{\wedge}_{X,x}.$$

*Then for each $N \geq 1$, there is a $G$-equivariant elementary étale neighborhood $\mu\colon (U,u) \to (X,x)$, with $u \in U^G$, such that $U^g$ surjects onto $\mu(U) \cap X^g$ for all $g \in G$, and a $G$-equivariant $S$-morphism $f\colon U \to Y$ mapping $u$ to $y$ such that the following diagram commutes modulo $\mathfrak{m}_u^N$:*

$$
\begin{array}{ccc}
\mathscr{O}^{\wedge}_{X,x} & \longrightarrow & \mathscr{O}^{\wedge}_{U,u} \\
{\scriptstyle\varphi}\uparrow & & \uparrow \\
\mathscr{O}_{Y,y} & \xrightarrow{f_u^{\#}} & \mathscr{O}_{U,u}
\end{array}
\qquad (12)
$$

*Moreover, if $\varphi$ induces an isomorphism $\mathscr{O}^\wedge_{Y,y} \simeq \mathscr{O}^\wedge_{X,x}$, then we can choose $f$ so that $(U,u) \to (Y,y)$ is an elementary étale neighborhood.*

*Proof.* Define a contravariant functor $F$ on $X$-schemes as follows. For $U \to X$ let

$$F(U) = \mathrm{Hom}^G_S(U^{\mathrm{e}}, Y)$$

be the set of $G$-equivariant morphisms of $S$-schemes from the equivariantization $U^{\mathrm{e}}$ (see Definition 26) to $Y$. If $U \to V$ is a morphism of $X$-schemes, the induced map of equivariantizations $U^{\mathrm{e}} \to V^{\mathrm{e}}$ defines a restriction map $F(V) \to F(U)$ by precomposition. We check that $F$ is limit preserving; that is, if $\{U_i\}$ is a directed inverse system of affine $X$-schemes, we show that the canonical map

$$\varinjlim \mathrm{Hom}^G_S(U_i^{\mathrm{e}}, Y) \longrightarrow \mathrm{Hom}^G_S((\varprojlim U_i)^{\mathrm{e}}, Y) \tag{13}$$

is a bijection. First observe that equivariantization is a type of limit and hence commutes with limits. Next observe that (13) is the $G$-invariant part of the canonical morphism

$$\varinjlim \mathrm{Hom}_S(U_i^{\mathrm{e}}, Y) \longrightarrow \mathrm{Hom}_S(\varprojlim U_i^{\mathrm{e}}, Y), \tag{14}$$

where to commute the $G$-invariants and the colimit on the left side we use that finite limits commute with filtered colimits ([SP21, Tag 002W]). But (14) is a bijection since the functor $U \mapsto \mathrm{Hom}_S(U, Y)$ is locally finitely presented by [SP21, Tag 01TX] and [SP21, Tag 01ZC].

The given morphism $\varphi$ defines an element of $\mathrm{Hom}^G_S(\mathrm{Spec}(\mathscr{O}^\wedge_{X,x}), Y)$. We compute

$$\begin{aligned}\mathrm{Spec}(\mathscr{O}^\wedge_{X,x})^{\mathrm{e}} &= \left[\varinjlim \mathrm{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}_x^n)\right]^{\mathrm{e}} \\ &= \varinjlim \mathrm{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}_x^n)^{\mathrm{e}} \\ &= \varinjlim \mathrm{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}_x^n) \\ &= \mathrm{Spec}(\mathscr{O}^\wedge_{X,x})\end{aligned} \tag{15}$$

where the first and last equalities are the definition of completion,[2] the second again uses [SP21, Tag 002W], and the third is Lemma 30 together with [SP21, Tag 01L6]. Hence $\varphi$ produces an element of $F(\mathrm{Spec}(\mathscr{O}^\wedge_{X,x}))$. Now Proposition 54 produces for any $N \geq 1$ an elementary étale neighborhood $\mu\colon (U,u) \to (X,x)$ and an element $\Phi \in F(U)$ such that $\varphi$ and $\Phi$ map to the same element of $F(\mathrm{Spec}(\mathscr{O}^\wedge_{U,u}/\mathfrak{m}_u^N))$.

Since $\mathscr{O}_{U,u}$ is Noetherian, by [SP21, Tag 031C] we have $\mathscr{O}^\wedge_{U,u}/\mathfrak{m}_u^N \simeq \mathscr{O}_{U,u}/\mathfrak{m}_u^N$. Unwinding the definition of $F$, we see that we have an equivariant map $U^{\mathrm{e}} \to Y$ fitting into the black part of the following $G$-equivariant commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(\mathscr{O}^\wedge_{X,x}) \arrow[d, no head, "(15)", "\sim"'] & \mathrm{Spec}(\mathscr{O}^\wedge_{U^{\mathrm{e}},u^{\mathrm{e}}}/\mathfrak{m}^N_{u^{\mathrm{e}}}) \arrow[l] \arrow[r, "\sim"] & \mathrm{Spec}(\mathscr{O}_{U^{\mathrm{e}},u^{\mathrm{e}}}/\mathfrak{m}^N_{u^{\mathrm{e}}}) \arrow[d, no head, "\sim"'] \\ \mathrm{Spec}(\mathscr{O}^\wedge_{X,x})^{\mathrm{e}} \arrow[d] & \mathrm{Spec}(\mathscr{O}^\wedge_{U,u}/\mathfrak{m}^N)^{\mathrm{e}} \arrow[l] \arrow[r, "\sim"] & \mathrm{Spec}(\mathscr{O}_{U,u}/\mathfrak{m}^N)^{\mathrm{e}} \arrow[d] \\ Y & & U^{\mathrm{e}} \arrow[ll] \end{tikzcd} \tag{16}$$

[2] These direct limits are taken in the category of schemes (not locally ringed spaces). In fact, they can be computed in the category of affine schemes since a map from $\mathrm{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}_x^n)$ an arbitrary scheme $Y$ factors through every affine open neighborhood of the image of $\mathfrak{m}$. See e.g. [Eme10].

We have $u^{\mathsf{e}} \in U^{\mathsf{e}}$ defined as in (2) and $\mu^{\mathsf{e}} \colon (U^{\mathsf{e}}, u^{\mathsf{e}}) \to (X, x)$ is an elementary étale neighborhood. Note that the black part of (16) implies that $u^{\mathsf{e}}$ maps to $y$. For the elementary étale neighborhood of the lemma, we take $((\mu^{\mathsf{e}})^{-1}(V), u^{\mathsf{e}})$ as constructed in Lemma 29.

To show that (12) commutes, we claim that we can construct the entire diagram in (16) to be commutative, such that the composition of the vertical arrows on the right is the canonical map $\operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N) \to U^{\mathsf{e}}$. Granting this, (12) is obtained from (16) by taking the induced maps on local rings.

We derive (16). First, we have a commutative diagram

$$\begin{tikzcd}
 & \operatorname{Spec}(\mathscr{O}^{\wedge}_{U,u}/\mathfrak{m}_u^N) \arrow[r, "\sim"] \arrow[dl] & \operatorname{Spec}(\mathscr{O}_{U,u}/\mathfrak{m}_u^N) \\
\operatorname{Spec}(\mathscr{O}^{\wedge}_{X,x}) & \operatorname{Spec}(\mathscr{O}^{\wedge}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N) \arrow[l] \arrow[r, "\sim"] \arrow[u, "\sim"] & \operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N) \arrow[u, "\sim"]
\end{tikzcd} \tag{17}$$

induced by the map $\mathrm{pr}_1 \colon (U^{\mathsf{e}}, u^{\mathsf{e}}) \to (U, u)$ of elementary étale neighborhoods over $(X, x)$, where $\mathrm{pr}_1$ was defined in Lemma 30. In this diagram, the left vertical map is an isomorphism by [SP21, Tag 0AGX] and the right vertical map is an isomorphism because the other three are. Next, the equivariantization of (17) maps to (17) under $\mathrm{pr}_1$, and we call the resulting three-dimensional commuting diagram $D$.

The map $\mathrm{pr}_1 \colon \operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N)^{\mathsf{e}} \to \operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N)$ in the diagram $D$ is an isomorphism by Lemma 30 (using the factorization $\operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N) \to \operatorname{Spec}(\mathscr{O}_{X,x}/\mathfrak{m}_x^N) \to X$, where the first map is an isomorphism by the same reasoning we used for the right vertical map of (17), and the second is a monomorphism by [SP21, Tag 01L6]).

From $D$, one may extract the gray part of (16). Specifically, the bottom line of (17) is the top (gray) line of (16), and the equivariantization of the top line of (17) is the middle line of (16). By contemplating $D$, one may also check that the composition of the vertical arrows on the right side of (16) is equal to the composition

$$\operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N) \xrightarrow{\widetilde{\Delta}} \left[\operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N \to U^{\mathsf{e}} \xrightarrow{\mathrm{pr}_1} U\right]^{\mathsf{e}}$$

where the notation $[-]^{\mathsf{e}}$ means we have applied the equivariantization functor to the canonical maps in the square brackets, and $\widetilde{\Delta}$ was defined in Lemma 30. After replacing $X$ and $U$ with affine schemes one can write down the corresponding ring map and see that it induces the canonical map $\operatorname{Spec}(\mathscr{O}_{U^{\mathsf{e}},u^{\mathsf{e}}}/\mathfrak{m}_{u^{\mathsf{e}}}^N) \to U^{\mathsf{e}}$.

For the "moreover," we first note that by taking $N \geq 2$ we can choose $f$ to induce an isomorphism on residue fields. Next, the proof of [SP21, Tag 0CAV] shows that when $\varphi$ induces an isomorphism of completed local rings, the morphism $f$ is étale at $u$. Hence there is an open subset $U'$ of $U$ containing $u$ where $f$ is étale, and we replace $U$ with $(U')^{\mathsf{e}}$. □

The next lemma is the analog of [SP21, Tag 0GDX].

**Lemma 56.** *Let $G$ be a finite group acting on schemes $X$ and $Y$. Moreover, let $S, T, x, y, s, t, \sigma, y_\sigma$, and $\varphi$ be given as follows: we have $G$-invariant morphisms of schemes*

$$\begin{array}{ccccc} X & Y & & x & y \\ \downarrow & \downarrow & \text{with } G\text{-fixed points} & \downarrow & \downarrow \\ S & T & & s & t \end{array}$$

*such that $S$ is locally Noetherian, $T$ is of finite type over $\mathbf{Z}$, the morphisms $X \to S$ and $Y \to T$ are locally of finite type, and $\mathscr{O}_{S,s}$ is a* G*-ring in the sense of [SP21, Tag 07GH].*

*The map*

$$\sigma\colon \mathscr{O}_{T,t} \to \mathscr{O}^{\wedge}_{S,s}$$

*is a local homomorphism. Set* $Y_\sigma = Y \times_{T,\sigma} \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s})$*. The point* $y_\sigma$ *is a point of* $Y_\sigma$ *mapping to* $y$ *and the closed point of* $\mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s})$*. Finally,*

$$\varphi\colon \mathscr{O}^{\wedge}_{X,x} \xrightarrow{\sim} \mathscr{O}^{\wedge}_{Y_\sigma,y_\sigma}$$

*is an isomorphism of* $\mathscr{O}^{\wedge}_{S,s}$*-algebras. In this situation there exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[d] & W \arrow[l, "\mu"'] \arrow[rr] \arrow[dr] & & Y \times_{T,\tau} V \arrow[r] \arrow[dl] & Y \arrow[d] \\
S & & V \arrow[ll] \arrow[rr, "\tau"] & & T
\end{tikzcd}$$

*with* $W$ *a* $G$*-scheme,* $V$ *a scheme, and all morphisms* $G$*-equivariant, together with points* $w \in W^G$ *and* $v \in V$ *such that*

(i) $(V, v) \to (S, s)$ *is an elementary étale neighborhood*

(ii) $\mu\colon (W, w) \to (X, x)$ *is an elementary étale neighborhood and* $W^g$ *surjects onto* $\mu(W) \cap X^g$ *for all* $g \in G$

(iii) $\tau(v) = t$

*Let* $y_\tau \in Y \times_{T,\tau} V$ *correspond to* $y_\sigma$ *via the identification* $(Y_\sigma)_s = (Y \times_T V)_v$*. Then*

(iv) $(W, w) \to (Y \times_{T,\tau} V, y_\tau)$ *is an elementary étale neighborhood.*

*Proof.* The proof of [SP21, Tag 0GDX] works equivariantly. We will outline that proof here (omitting critical details that are checked in the original!) and explain what changes in the equivariant setting.

One initially constructs the black part of the diagram

$$\begin{tikzcd}
X \arrow[d] & X \times_S \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s}) \arrow[l] \arrow[dr] & U \arrow[l] \arrow[r] \arrow[d] & Y \times_T \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s}) \arrow[r] \arrow[dl] & Y \arrow[d] \\
S & & \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s}) \arrow[ll] \arrow[rr] \arrow[d] & & T \\
& & S_i \arrow[ull, "\text{smooth}"] \arrow[urr] & &
\end{tikzcd} \tag{18}$$

with $U \to X \times_S \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s})$ and $U \to U \times_T \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s})$ étale, using Lemma 55 for the existence of $U$—in particular, the top row of (18) consists of $G$-schemes and equivariant homomorphisms, $u \in U^G$ maps to $x \in X$ and $y \in Y$, and $U^g$ surjects onto the image of $U$ in $(X \times_S \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s}))^g$. Next one uses our hypothesis that $\mathscr{O}_{S,s}$ is a G-ring to write $\mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s})$ as the limit of smooth affine $S$-schemes $S_i$; after increasing $i$, by [SP21, Tag 01ZC], we may assume we have the gray part of the diagram (18). By [SP21, Tags 01ZM, 07RP] we can, after increasing $i$, find a diagram of $G$-schemes over $S_i$

$$\begin{tikzcd}
X \times_S S_i \arrow[dr] & U_i \arrow[l] \arrow[r] \arrow[d] & Y \times_T S_i \arrow[dl] \\
& S_i &
\end{tikzcd} \tag{19}$$

with $U_i \to X \times_S S_i$, $U_i \to Y \times_T S_i$ étale, such that this diagram pulls back to the relevant part of (18) and the map $U \to U_i$ is $G$-equivariant. (To see that we can choose $U_i$ to have a $G$-action, observe that by [SP21, Tag 01ZM], for each $g \in G$, we can increase $i$ to find an automorphism $g_i$ of $U_i$

that restricts to $g$. After further increasing $i$ we can assume that all necessary diagrams relating to group actions and equivariance of maps commute.) After increasing $i$ again, using Lemma 23 and [SP21, Tag 07RR] we can ensure that $U_i^g$ surjects onto the image of $U$ in $(X \times_S S_i)^g$. Set $u_i \in U_i$ to be the image of $u \in U$; since $u$ was $G$-fixed so is $u_i$. Finally we use [SP21, Tag 057G] to find a closed subscheme $V \subset S_i$ containing the image of $u_i$ such that $V \to S$ is étale. We replace (19) with its pullback to $V$ to get the desired diagram of the lemma. In particular, we set $W := V \times_{S_i} U_i$ and $w := u_i \in U_i^G$ is contained in this closed subscheme. The restriction to $V$ preserves surjections and $g$-fixed loci by Lemma 23, so for all $g \in G$, $W^g$ surjects onto $\mu(W) \cap X^g$ in the final diagram. □

*Proof of Proposition 52.* We follow the proof of [SP21, Tag 0CBY], so we will be brief here. Replace $S$ with an affine open neighborhood $\mathrm{Spec}(R)$ of $s$ and replace $X$ with $X'$, where $X'$ is the $G_x$-equivariantization of an affine étale neighborhood of the pullback, so $X$ is a $G_x$-scheme. By Lemma 29 we can after shrinking $X'$ assume that $(X')^g$ surjects onto the image of $X'$ in $X^g$. Now use absolute Noetherian approximation: write $R$ as a filtered colimit of finite type $\mathbf{Z}$-algebras $R_i$. By [SP21, Tags 01ZM, 0C5F] we can find $i$ and a nodal $G$-curve $X_i \to \mathrm{Spec}(R_i)$ whose pullback to $S$ is $X \to S$. By [SP21, Tag 0C3I] the image of $x$ in $X_i$ is a node and by Lemma 45 it is balanced. Hence (using Lemma 23 for the properties of $X^g$) it suffices to prove the lemma for $X_i \to \mathrm{Spec}(R_i)$.

We have reduced to the case where $S$ is affine and of finite type over $\mathbf{Z}$. By Lemma 8 and [SP21, Tag 02LF] there is an affine étale neighborhood $(V, v) \to (S, s)$ such that, if $X_v$ denotes the base change of $X$ to $v$, there is a split node $x_v \in X_v$ with image $x$. By Lemma 45, $x_v$ is balanced since $x$ was.

We have reduced to the situation where $x$ is a balanced split node of its fiber and $G = G_x$.

By Lemma 51 we can after another étale base change assume that $X \to S$ satisfies all the assumptions of Lemma 50. We conclude the argument as in the proof of [SP21, Tag 0CBY], using Lemma 50 in place of [SP21, Tag 0CBX] and Lemma 56 in place of [SP21, Tag 0GDX]. □

We give a first application of Lemma 56 and Proposition 52 to the étale local structure of nodal $G$-curves.

**Lemma 57.** *Let $X$ be a nodal $G$-curve over an algebraically closed field $k$ and let $x \in X(k)$.*

(i) *The point $x$ is smooth if and only if there is a $G_x$-equivariant isomorphism $\mathcal{O}_{X,x}^{\mathrm{sh}} \simeq k[u]_{(u)}^{\mathrm{sh}}$ with the $G_x$-action on the target given by $g \cdot u = \xi(g)u$ for some faithful $\xi\colon G_x \to k^\times$.*

(ii) *The point $x$ is a balanced node if and only if there is a $G_x$-equivariant isomorphism $\mathcal{O}_{X,x}^{\mathrm{sh}} \simeq \big(k[u,v]/(uv)\big)_{(u,v)}^{\mathrm{sh}}$, with the $G_x$-action on the target given by $g \cdot u = \xi(g)u$ and $g \cdot v = \xi^{-1}(g)v$ for some faithful $\xi : G_x \to k^\times$.*

*Proof.* (i). If $x$ is a smooth point, Lemma 41 gives a faithful character $\xi : G_x \to k^\times$ and a $G_x$-equivariant isomorphism $\mathcal{O}_{X,x}^\wedge \simeq k[\![u]\!]$. Let $Y = \mathrm{Spec}\, k[u]$ with $k$-linear $G_x$-action given by $\xi$ on $u$. Lemma 56 applied to $X$ and $Y$ over $k$ (together with the fact that $\mathrm{Spec}\, k$ has no nontrivial connected étale covers) then guarantees the existence of a pointed $G_x$-scheme $(W, w)$ together with $G_x$-equivariant étale morphisms $(W, w) \to (X, x)$ and $(W, w) \to (Y, (u))$. Since étale morphisms induce isomorphisms on strict henselizations, we get the desired description of $\mathcal{O}_{X,x}^{\mathrm{sh}}$. Conversely, assume $\mathcal{O}_{X,x}^{\mathrm{sh}}$ has the form given in (i) above. Since $\mathbf{A}_k^1$ is smooth and $\mathcal{O}_{X,x}^{\mathrm{sh}} \simeq k[u]_{(u)}^{\mathrm{sh}} \simeq \mathcal{O}_{\mathbf{A}^1,0}^{\mathrm{sh}}$, [SP21, Tag 00TV] and [SP21, Tag 06LN] show that $x$ is a smooth point of $X$.

(ii). If $x$ is a node, then an argument similar to that used in (i) above shows that $\mathcal{O}_{X,x}^{\mathrm{sh}}$ has the desired form (use Proposition 52 in place of Lemmas 41 and 56). Conversely, assume $\mathcal{O}_{X,x}^{\mathrm{sh}}$

has the form given in (ii) above. Completing the isomorphism $\mathscr{O}^{\mathrm{sh}}_{X,x} \simeq \left(k[u,v]/(uv)\right)^{\mathrm{sh}}_{(u,v)}$ yields a $G_x$-equivariant isomorphism $\mathscr{O}^{\wedge}_{X,x} \simeq k[\![u,v]\!]/(uv)$, by [SP21, Tag 06LJ(b)] and the fact that henselizations and strict henselizations coincide for local rings whose residue fields are algebraically closed (follows from [SP21, Tag 0BSL]). This shows that $x$ is a balanced node in the sense of Definition 43. □

3.3. **The stack of prestable balanced curves.** Let $G$ be a finite group. Recall that if $X \to S$ is a prestable $G$-curve then $S$ is a $\mathrm{Spec}(\mathbf{Z}[1/|G|])$-scheme. In this section we construct a locally closed substack of $\mathfrak{N}_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}$ parametrizing balanced prestable $G$-curves. We will repeatedly use Proposition 52 to work étale-locally (see Remark 53 for a discussion of why a flat local picture is not enough).

**Remark 58.** As a warmup for the next lemma, recall from [SP21, Tag 034H] the definition of the inertia stack $\mathscr{I}$ of an algebraic stack $\mathscr{X}$. An element of $\mathscr{I}(S)$ is a pair $(x,\alpha)$ with $x \in \mathscr{X}(S)$ and $\alpha \in \mathrm{Aut}_S(x)$ (by definition, $\alpha$ is an arrow $x \to x$ lying over $id : S \to S$). A morphism $(x_1,\alpha_1) \to (x_2,\alpha_2)$ between objects of $\mathscr{I}(S)$ is an arrow $f : x_1 \to x_2$ such that $f \circ \alpha_1 = \alpha_2 \circ f$.

**Lemma 59.** *Let $\mathscr{X}$ be an algebraic stack with separated diagonal. Let $\mathscr{I}$ be the inertia stack of $\mathscr{X}$ and let $\prod_G \mathscr{I}$ be the fiber product $\mathscr{I} \times_{\mathscr{X}} \ldots \times_{\mathscr{X}} \mathscr{I}$ of $|G|$ copies of $\mathscr{I}$. Then there is a closed substack $\mathscr{X}(G) \subset \prod_G \mathscr{I}$ given on $S$-valued points by*

$$\mathscr{X}(G)(S) := \{(x,\theta) \mid x \in \mathscr{X}(S),\ \theta \in \mathrm{Hom}^{\mathrm{gp}}_S(G_S, \mathrm{Aut}_S(x))\} \tag{20}$$

*where $\mathrm{Hom}^{\mathrm{gp}}_S(G_S, \mathrm{Aut}_S(x))$ is the set of homomorphisms of group algebraic spaces over $S$.*

**Remark 60.** An object of $\prod_G \mathscr{I}(S)$ may be written as $(x,(\alpha_g)_{g\in G})$ where $x \in \mathscr{X}(S)$ and $\alpha_g \in \mathrm{Aut}_S(x)$ for each $g \in G$. Thus, in Lemma 59, the groupoid fiber $\mathscr{X}(G)(S)$ is viewed as a full subcategory of $\prod_G \mathscr{I}(S)$ by identifying an object $(x,\theta) \in \mathscr{X}(G)(S)$ with the object $\left(x,(\theta(g))_{g\in G}\right) \in \prod_G \mathscr{I}(S)$. In particular, an arrow $(x_1,\theta_1) \to (x_2,\theta_2)$ between objects of $\mathscr{X}(G)(S)$ is given by a "$G$-equivariant arrow" $x_1 \to x_2$; i.e., for each $g \in G$, the arrow $x_1 \to x_2$ commutes with the automorphisms specified by $\theta_1(g)$ and $\theta_2(g)$.

*Proof of Lemma 59.* For an object $x$ of $\mathscr{X}(S)$, a morphism $\theta \in \mathrm{Hom}_S(G_S, \mathrm{Aut}_S(x))$ is equivalent to sections $\theta_g : S \to \mathrm{Aut}_S(x)$ for every $g \in G$. Let $1 \in G$ be the identity and let $1_S : S \to \mathrm{Aut}_S(x)$ be the identity section, $\iota : \mathrm{Aut}_S(x) \to \mathrm{Aut}_S(x)$ the involution, and $c : \mathrm{Aut}_S(x) \times \mathrm{Aut}_S(x) \to \mathrm{Aut}_S(x)$ the composition. That the map $\theta$ is a homomorphism translates to the identities (finite in number)

$$\begin{gathered} \theta_1 = 1_S \qquad\qquad \theta_{g^{-1}} = \iota \circ \theta_g \text{ for each } g \in G \\ \theta_{gh} = c(\theta_g, \theta_h) \text{ for each } (g,h) \in G \times G. \end{gathered} \tag{21}$$

Since the diagonal of $\mathscr{X}$ is separated, the algebraic space $\mathrm{Aut}_T(x) \to T$ is separated, and its diagonal is a closed embedding. So the proof of [SP21, Tag 01KM] shows that the locus where any one of the equalities in (21) holds is represented by a closed subscheme of $S$. Intersecting these, we get a closed subscheme of $S$ representing the property that $\theta$ restricts to a group homomorphism. □

Setting $\mathscr{X} = \mathfrak{N}_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}$ in Lemma 59 we obtain the following corollary.

**Corollary 61.** *There is an algebraic stack $\mathfrak{N}_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}(G)$ whose objects over a scheme $S$ are prestable $G$-curves over $S$, and whose arrows are given by $G$-equivariant morphisms of $G$-curves over $S$.*

Define $\mathfrak{N}^{\mathrm{bal}} \subset \mathfrak{N}_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}(G)$ to be the full subcategory whose objects are balanced prestable $G$-curves.

**Lemma 62.** *The subcategory $\mathfrak{N}^{\mathrm{bal}} \subset \mathfrak{N}_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}(G)$ is an open substack. In particular, $\mathfrak{N}^{\mathrm{bal}}$ is algebraic.*

*Proof.* We show that $\mathfrak{N}^{\mathrm{bal}} \subset \mathfrak{N}_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}(G)$ is an open substack by using the discussion in [SP21, Tag 0E0E]. That is, we show that if $\pi\colon X \to S$ is $G$-space over $S$ and a prestable curve, there is a largest open subscheme $S^{\mathrm{bal}} \subset S$ such that the pullback $X \times_S S^{\mathrm{bal}}$ is a balanced $G$-curve, and formation of $S^{\mathrm{bal}}$ commutes with arbitrary base change. Define $|S^{\mathrm{bal}}| \subset |S|$ to be the locus of points $s \in S$ such that $X_s$ is a balanced prestable $G$-curve. By Remark 40 and Lemma 45 the locus $|S^{\mathrm{bal}}|$ is preserved by arbitrary base change, so it remains to show that $|S^{\mathrm{bal}}|$ is open.

Note that each composition $X^g \to X \to S$ is proper. By [SP21, Tag 0D4Q], there is an open subscheme $S' \subset S$ whose underlying set consists of points $s \in S$ such that $X^g_s$ has dimension $\leq 0$ for each $1 \neq g \in G$; i.e., $X_s$ is a prestable $G$-curve. If $s \in S'$ is contained in $|S^{\mathrm{bal}}|$, Lemma 45.(iii) and Proposition 52 gives for each node $x \in X_s$ an etale map $f_x\colon U_x \to X$ such that $f(U_x) \to \pi \circ f(U_x)$ is a balanced nodal $G$-curve. Let $X'' = X^{sm} \cup \left(\bigcup_x f_x(U_x)\right)$ where the union is over nodes $x$ of $X_s$ and let $Z := X \smallsetminus X''$. Now $S'' := S' \smallsetminus \pi(Z) \subseteq S'$ is open (because $\pi$ is proper) and it is the maximal subscheme over which the restriction of $X$ to $\pi^{-1}(S'')$ is balanced prestable. $\square$

**Lemma 63.** *Let $X \to S$ be a balanced prestable $G$-curve with coarse quotient $f\colon X \to X/G$ and let $x \in |X|$. If $x$ is a smooth point (resp. node) of its fiber, then $f(x)$ is also a smooth point (resp. node) of its fiber.*

*Proof.* Since the question may be checked on geometric fibers of $X \to X/G \to S$ (using Remark 13) and since the map $X \to X/G$ commutes with arbitrary base change (Lemma 36), we reduce to the case where $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field $k = \bar{k}$. Let $V = \mathrm{Spec}(C)$ be an affine étale neighborhood of $f(x) \in X/G$. Let $U = V \times_{X/G} X$ and let $u \in U$ be a closed point mapping to $x$. Let $v \in V$ be the image of $u$.

Note that $U$ is also affine, so $U = \mathrm{Spec}(B)$ for some ring $B$ and $C = B^G$ by Lemma 35. By Lemma 36 we have

$$\mathscr{O}^{\wedge}_{V,v} = (\mathscr{O}^{\wedge}_{V,v} \otimes_C B)^G.$$

Let $u_1, \ldots, u_n$ be the points in the fiber of $v$, numbered so that $u = u_1$, and note that $G$ acts transitively on these points. By [SP21, Tag 07N9] we have

$$\mathscr{O}^{\wedge}_{V,v} \otimes_C B = \mathscr{O}^{\wedge}_{U,u_1} \oplus \ldots \oplus \mathscr{O}^{\wedge}_{U,u_n}.$$

Moreover, the projection

$$(\mathscr{O}^{\wedge}_{U,u_1} \oplus \ldots \oplus \mathscr{O}^{\wedge}_{U,u_n})^G \to (\mathscr{O}^{\wedge}_{U,u_1})^{G_{u_1}}$$

is an isomorphism. Since Lemmas 23 and 24 show that $G_{u_1} \simeq G_x$, we conclude

$$\mathscr{O}^{\wedge}_{V,v} \simeq (\mathscr{O}^{\wedge}_{U,u})^{G_x}. \tag{22}$$

Set $n := |G_x|$. Assume $x$ is a (necessarily balanced) node of its fiber. By Lemma 45.(iii), $u$ is also a balanced node, and by [SP21, Tag 0C57] $f(x)$ is a node if and only if $v$ is. By (22) we have $\mathscr{O}^{\wedge}_{V,v} \xrightarrow{\simeq} (\mathscr{O}^{\wedge}_{U,u})^{G_x} \to \mathscr{O}^{\wedge}_{U,u}$, and hence by Lemma 45 we have $\mathscr{O}^{\wedge}_{V,v} \simeq k[\![t^n, s^n]\!]/(t^n s^n)$. In particular, $\mathscr{O}^{\wedge}_{V,v}$ is of the form in Definition 6.(i) and hence $v$ is a node.

Now assume $x$ is a smooth point of its fiber. Since $U \to X$ is étale, $u$ is a smooth point of its fiber. Using Lemma 41 and arguing as in the nodal case, we get $\mathscr{O}^{\wedge}_{V,v} = (\mathscr{O}^{\wedge}_{U,u})^{G_x} \simeq k[\![t^n]\!]$. This implies that $v$ is smooth ([SP21, Tag 00TV, Tag 07NY]) and so by [SP21, Tag 05AX] $f(x)$ is smooth. □

**Lemma 64.** *If $X \to S$ is a balanced prestable $G$-curve, then the coarse quotient $X/G \to S$ is a prestable curve.*

*Proof.* The map $X/G \to S$ is flat, proper and of finite presentation by Lemma 36.(ii) and Lemma 37. The remaining conditions may be checked on geometric fibers so by Lemma 36.(i) we may assume $S = \mathrm{Spec}(k)$ for some algebraically closed field $k = \bar{k}$. Now $X/G$ is a nodal curve by Lemma 63 and it has finitely many connected components because $X$ does and $X \to X/G$ is surjective (Lemma 34). Finally, $X/G$ has pure dimension 1 because $X$ does and $X \to X/G$ is finite and surjective; cf. [SP21, Tag 0ECG]. □

The following technical lemma will be used to analyze the geometry of $X^g$.

**Lemma 65.** *Let $X \to S$ be an algebraic space over a scheme $S$ with $X_1, X_2 \subset X$ closed subspaces that are finite étale over $S$. Assume $|X_1| \subset |X_2|$. For each $x \in |X_1|$ there exists an open neighborhood $V \subset X$ of $x$ such that $V \cap X_1 = V \cap X_2$.*

*Proof.* Since $X_i \to S$ is finite, the subspace $X_i$ is representable by a scheme (by definition [SP21, Tag 03ZP]). Applying [SP21, Tag 04HN] twice, we find an étale map $S' \to S$ such that the fibers $(X_1)_{S'}$ and $(X_2)_{S'}$ are both isomorphic to disjoint unions of schemes isomorphic to $S'$ by the projection maps. The point $x$ is in one of these. Removing the other ones from $X_{S'}$, we get an open neighborhood $U$ of $x$ in $X_{S'}$ such that $U \cap (X_1)_{S'} = U \cap (X_2)_{S'}$. If two closed subspaces of an algebraic space agree on an étale cover, then (by descent) they agree. Hence the desired neighborhood is the image of $U$ in $X$. □

**Lemma 66.** *Let $X \to S$ be a balanced prestable $G$-curve.*

(i) *For every non-identity $g \in G$, the map $(X^g \cap X^{sm}) \to S$ is finite étale.*
(ii) *For each $x \in |X^{sm}|$ there is an open neighborhood of $x$ where*

$$\bigcup_{g \in (G \smallsetminus \{1\})} (X^g \cap X^{sm}) = X^h \cap X^{sm}$$

*for a generator $h$ of $G_x$. In particular $\bigcup_{g \in (G \smallsetminus \{1\})} (X^g \cap X^{sm}) \to S$ is finite étale.*

*Proof.* (i) We claim there is an open $X' \subset X$ containing the points of $X$ that are nodes in their fibers such that the stabilizer of any smooth point in $X'$ is trivial. Granting this, the complement of $X'$ is a closed set containing $X^g \cap X^{sm}$ and contained in $X^{sm}$. This shows that for $g \neq 1$ the map $(X^g \cap X^{sm}) \to S$ is proper; moreover it is finite because its fibers are closed subsets of $X^{sm}$ that do not contain any irreducible component; cf. [SP21, Tag 0A4X].

We construct $X'$. For a point $x \in |X|$ that is a node in its fiber, we have a diagram as in Proposition 52, where in particular $U^g$ surjects onto $\mu(U) \cap X^g$ for all $g \in G_x$. On the right hand side of the diagram we may directly compute that for any $g \in (G_x \smallsetminus \{1\})$ the fixed locus contains no smooth point, so by Lemma 23 and Remark 13, the locus $(W^{sm})^g$ is empty. This forces $(U^{sm})^g$ to be empty as well (since the smooth locus is preserved by étale covers), and hence the image of $U$ contains no $g$-fixed point of $X^{sm}$.

Intersecting these images for all nonidentity $g \in G_x$ gives a neighborhood of $x$ that contains no smooth $g$-fixed point for each $g \in (G_x \smallsetminus \{1\})$, and furthermore, by removing $X^h$ for $h \notin G_x$, we get an open neighborhood of $x$ whose intersection with $X^g \cap X^{sm}$ is empty for each $g \in (G \smallsetminus \{1\})$. Let $X' \subset X$ be the union of these open sets (one for each node of $X$).

Finally, we show that $(X^g \cap X^{sm}) \to S$ is étale. We may assume $S = \mathrm{Spec}(R)$. As in the proof of Proposition 52 we can find a subring $R_i \subset R$ finitely generated over $\mathbf{Z}$ and a prestable $G$-curve $X_i \to \mathrm{Spec}(R_i)$ whose base change to $S$ is $X \to S$. Since base change preserves $X^g \cap X^{sm}$ by Remark 13 and Lemma 23 and smoothness can be checked locally on the target [SP21, Tag 02VL], we have reduced to the case where $S$ is Noetherian.

By [SP21, Tag 056U], it suffices to prove that the morphism is étale at every closed point $x \in X$ with image $s \in S$ such that the induced field extension $\kappa(s) \subseteq \kappa(x)$ is finite separable. Let $x \in X$ be of that form. By [SP21, Tag 02LF, Tag 01JT], we can find an étale neighborhood $(S', s') \to (S, s)$ and $x' \in X'_S$ closed such that $\kappa(s') = \kappa(x') = \kappa(x)$. We thus reduce to the case where $\kappa(x) = \kappa(s)$. Similarly, by Lemma 51, we can assume that the hypotheses of Lemma 50 are satisfied; taking $G_x$-invariants in (9) and using [SP21, Tag 00MA(2)] this implies that the natural map $\mathscr{O}^\wedge_{S,s} \to \mathscr{O}^\wedge_{X^g,x}$ is an isomorphism.

One may now apply [SP21, Tag 039N, Tag 039M] to see that $X \to S$ is étale at $x$.

(ii) It suffices to consider the case $X = X^{sm}$. Fix $x \in |X|$ and let $(G \smallsetminus G_x) \subset G$ be the subset equal to the complement of $G_x$. Then $U = X \smallsetminus \left(\bigcup_{g \in G \smallsetminus G_x} X^g\right)$ is an open subset containing $x$. Now let $h$ be a generator of $G_x$. For every $n = 1, \ldots, (|G_x| - 1)$ we have $X^h \subset X^{h^n}$ a containment of finite étale spaces over $S$ with $x \in |X^h|$. By Lemma 65 we have an open set $V_n \subset X$ where $X^h \cap V_n = X^{h^n} \cap V_n$. The desired open neighborhood is $U \cap \left(\bigcap_{n=1}^{|G_x|-1} V_n\right)$. □

3.4. **Construction of the moduli stack of admissible covers.** For the rest of this document we fix a finite group $G$ and work over $\mathrm{Spec}(\mathbf{Z}[1/|G|])$. Recall the moduli of marked connected prestable curves $\mathfrak{M}_\star$ from Definition 17. For the rest of this document we will write $\mathfrak{M}_\star$ for $(\mathfrak{M}_\star)_{\mathrm{Spec}(\mathbf{Z}[1/|G|])}$. We observe that $\mathfrak{N}^{\mathrm{bal}}$ is already defined over $\mathrm{Spec}(\mathbf{Z}[1/|G|])$. Set

$$\mathfrak{H}_0 = \underline{\mathrm{Hom}}_{\mathfrak{N}^{\mathrm{bal}} \times \mathfrak{M}_\star}(\mathfrak{C}_{\mathfrak{N}^{\mathrm{bal}}}, \mathfrak{C}_{\mathfrak{M}_\star}),$$

where $\underline{\mathrm{Hom}}$ denotes the Hom stack from Lemma 19. Let $S$ be a scheme over $\mathrm{Spec}(\mathbf{Z}[1/|G|])$. An object in $\mathfrak{N}^{\mathrm{bal}} \times \mathfrak{M}_\star$ over $S$ is a morphism $S \to \mathfrak{N}^{\mathrm{bal}} \times \mathfrak{M}_\star$, or equivalently a pair of morphisms $S \to \mathfrak{N}^{\mathrm{bal}}$ and $S \to \mathfrak{M}_\star$. Set

$$X := (\mathfrak{C}_{\mathfrak{N}^{\mathrm{bal}}})_S := \mathfrak{C}_{\mathfrak{N}^{\mathrm{bal}}} \times_{\mathfrak{N}^{\mathrm{bal}}} S \quad \text{and} \quad Y := (\mathfrak{C}_{\mathfrak{M}_\star})_S := \mathfrak{C}_{\mathfrak{M}_\star} \times_{\mathfrak{M}_\star} S.$$

An object of $\mathfrak{H}_0$ over this pair is a morphism $X \to Y$ of $S$-algebraic spaces. The stack $\mathfrak{H}_0$ is algebraic by Lemmas 18, 62, 16, and 19. Note that if $f : X \to Y$ is an admissible $G$-cover as defined in Definition 1, then $X$ is a balanced nodal $G$-curve by Lemma 57.(ii). Hence we may define the stack of admissible covers as a full subcategory of $\mathfrak{H}_0$.

**Definition 67.** The *stack of admissible covers* $\mathscr{A}dm(G)$ is the full subcategory of $\mathfrak{H}_0$ whose objects over a scheme $S$ are given by

$$\mathscr{A}dm(G)(S) := \{(f : X \to Y) \in \mathfrak{H}_0(S) \mid f \text{ is an admissible } G\text{-cover}\}.$$

We now define a sequence of full subcategories $\mathfrak{H}_i \subset \mathfrak{H}_0$ and show that each is an algebraic stack, in fact a locally closed substack, of $\mathfrak{H}_0$. Briefly, if $f : X \to Y$ is an object of $\mathfrak{H}_0$, then

- it is in $\mathfrak{H}_1$ if $f$ is $G$-invariant,
- it is in $\mathfrak{H}_2$ if $Y$ is the coarse quotient $X/G$, and
- it is in $\mathfrak{H}_3$ if $f$ sends smooth points of $X$ with nontrivial stabilizer to marked points of $Y$.

In Theorem 77 we show that $\mathfrak{H}_3$ is equal to $\mathscr{A}dm(G)$, thus proving Theorem A that $\mathscr{A}dm(G)$ is algebraic.

Let $\mathfrak{H}_1 \subset \mathfrak{H}_0$ be the full subcategory of $G$-invariant morphisms. The $S$-valued points of $\mathfrak{H}_1$ are given by

$$\mathfrak{H}_1(S) = \{(f\colon X \to Y) \in \mathfrak{H}_0 \mid f \circ g = f \text{ for all } g \in G\}.$$

**Lemma 68.** *The full subcategory $\mathfrak{H}_1$ is an algebraic stack and the inclusion $\mathfrak{H}_1 \to \mathfrak{H}_0$ is a closed immersion.*

*Proof.* Let $S$ be a scheme and let $f\colon X \to Y$ be an object of $\mathfrak{H}_0$. We find a closed subscheme $S' \subset S$ with the following universal property: a morphism $T \to S$ factors through $S'$ if and only if the restriction of $f$ to $X \times_S T$ is $G$-invariant. Granted this, the statement is a consequence of Lemma 15 and [SP21, Tag 04YL].

Define maps $\Gamma, \Gamma'\colon S \to \prod_G \mathrm{Hom}_S(X, Y)$ given by $\Gamma = (f \circ g)_{g\in G}$ and $\Gamma' = (f)_{g\in G}$. The desired locus $S'$ is the equalizer of $\Gamma$ and $\Gamma'$. This is closed as in [SP21, Tag 01KM] since $\mathrm{Hom}_S(X, Y)$ is separated ([SP21, Tag 0DPN]). □

If $f\colon X \to Y$ is an object of $\mathfrak{H}_1(S)$, then $f$ is $G$-invariant and therefore factors through the coarse moduli space $X/G$ of Definition 31. Define $\mathfrak{H}_2 \subset \mathfrak{H}_1$ to be the full subcategory whose objects over a scheme $S$ are given by

$$\mathfrak{H}_2(S) = \left\{ (f\colon X \to Y) \in \mathfrak{H}_1(S) \;\middle|\; \begin{array}{c} \text{the induced map } \bar{f}\colon X/G \to Y \\ \text{is an isomorphism} \end{array} \right\}.$$

**Lemma 69.** *The full subcategory $\mathfrak{H}_2$ is an algebraic stack and the inclusion $\mathfrak{H}_2 \to \mathfrak{H}_1$ is an open immersion.*

*Proof.* Let $S$ be an arbitrary scheme and $f\colon X \to Y$ be an object of $\mathfrak{H}_1(S)$. We need to show that there exists an open subscheme $S' \subset S$ with the following universal property: a morphism $\alpha\colon T \to S$ factors through $S'$ if and only if the morphism $X_T/G \to Y_T$ induced by $\alpha_T : X_T \to Y_T$ is an isomorphism.

This morphism factors through $X_T/G \to X/G \times_S T$, which is always an isomorphism by Lemma 36. Thus, it suffices to find an open $S' \subset S$ such that $\alpha\colon T \to S$ factors through $S'$ if and only if the natural morphism $X/G \times_S T \to Y \times_S T$ is an isomorphism. Since $X/G \to S$ is a family of prestable curves by Lemma 64 and $Y \to S$ is a family of prestable curves by assumption, the existence of $S'$ follows from [SP21, Tag 05XD].

Now the statement is a consequence of Lemma 68, Lemma 15, and [SP21, Tag 04YL]. □

If $Y \to S$ is a marked connected prestable curve, we define $Y_\star \subset Y$ to be the locus of marked points (the union of the images of the sections). Likewise if $X \to S$ is a balanced prestable $G$-curve, let $X_\star := \bigcup_{g\in(G\smallsetminus\{1\})} X^g \cap X^{sm}$. This is a closed subspace of $X$ by the proof of Lemma 66.(i). Define $\mathfrak{H}_3 \subset \mathfrak{H}_2$ to be the full subcategory whose objects over a scheme $S$ are

$$\mathfrak{H}_3(S) = \{(f\colon X \to Y) \in \mathfrak{H}_2(S) \mid f(|X_\star|) \subseteq |Y_\star|\}.$$

**Lemma 70.** *The full subcategory $\mathfrak{H}_3$ is an algebraic stack and the inclusion $\mathfrak{H}_3 \to \mathfrak{H}_2$ is a closed immersion.*

*Proof.* Let $S$ be a scheme and $f\colon X \to Y$ be an object of $\mathfrak{H}_2(S)$. We need to show that there exists a closed subscheme $S' \subseteq S$ with the following universal property: a morphism $T \to S$ factors through $S'$ if and only if $f_T\left(|X_{T,\star}|\right) \subseteq |Y_{T,\star}|$. This is equivalent to the condition that $f_T\left(X_{T,\star}\right) \subseteq Y_{T,\star}$ by Lemma 66.(ii) and Lemma 65 (note that $Y_{T,\star}$ is finite étale over $T$ since it is the union of the images of some disjoint sections of $Y_T \to T$). Since $(X_\star)_T = X_{T,\star}$ by Remark 13 and Lemma 23, this is in turn equivalent to the condition that the closed immersion $j\colon (X_\star)_T \hookrightarrow X$ factors through $f^{-1}(Y_\star)$.

Let $\mathscr{I}$ be the ideal sheaf of $f^{-1}(Y_\star)$ inside $X$. By [SP21, Tag 03MB] the condition is equivalent to the composition $j^*\mathscr{I} \to \mathscr{O}_{(X_\star)_T} = j^*\mathscr{O}_{X_\star}$ being zero. By Lemma 66, the $\mathscr{O}_X$-module $\mathscr{O}_{X_\star}$ is flat and proper over $S$. Therefore, [SP21, Tag 083M] shows that this locus is represented by a closed subscheme $S' \subseteq S$. Now the statement is a consequence of Lemma 69, Lemma 15, and [SP21, Tag 04YL]. □

All that remains is to show that the algebraic stack $\mathfrak{H}_3$ is isomorphic to the category $\mathscr{A}dm(G)$ of admissible $G$-covers. Observe that both $\mathfrak{H}_3$ and $\mathscr{A}dm(G)$ are full subcategories of $\mathfrak{H}_0$, so it suffices to prove that the objects of $\mathfrak{H}_3(S)$ and $\mathscr{A}dm(G)(S)$ give the same subset of $\mathfrak{H}_0(S)$. To that end, we first clarify what is meant by a $G$-torsor in Definition 1.

**Definition 71.** Let $X$ be an algebraic space with an action by $G$ and let $\mathscr{Y}$ a Deligne–Mumford stack. A $G$-invariant morphism $X \to \mathscr{Y}$ is a $G$-torsor if

(i) the map $(\mathrm{act}, \mathrm{pr}_2)\colon G \times_{\mathscr{Y}} X \to X \times_{\mathscr{Y}} X$ is an isomorphism, and
(ii) there is an étale cover $Y \to \mathscr{Y}$ such that the pullback $X \times_{\mathscr{Y}} Y \to Y$ admits a section.

**Remark 72.** A $G$-torsor in our sense defines a $G$-torsor on the small étale site of $\mathscr{Y}$ in the sense of [SP21, Tag 03AH].

**Remark 73.** By [SP21, Tag 03AI], a $G$-torsor $X \to \mathscr{Y}$ is isomorphic to $G \times \mathscr{Y} \to \mathscr{Y}$ with the natural left $G$-action if and only if it admits a section. In particular, Definition 71.(ii) guarantees that a $G$-torsor $X \to \mathscr{Y}$ is an étale morphsim.

**Example 74.** If $X$ is an algebraic space with an action by $G$ then $X \to [X/G]$ is a $G$-torsor. Condition (i) is [SP21, Tag 04M9]. Condition (ii) also follows from [SP21, Tag 04M9] using Remark 73 and the fact that $X \to [X/G]$ is étale. This last fact may be deduced from [SP21, Tag 04M9] using [SP21, Tag 0CIQ(3)] and the fact that $X \to [X/G]$ is smooth by [SP21, Tag 04X0].

The next couple of lemmas show that an admissible cover $X \to Y$ over $S$ belongs to $\mathfrak{H}_3(S)$.

**Lemma 75.** *Let $S$ be a scheme. Let $X \to S$ and $Y \to S$ be two prestable curves and $f\colon X \to Y$ a morphism over $S$. Assume that for every $x \in |X|$, if $x$ is smooth (resp. nodal) in its fiber, then $f(x)$ is smooth (resp. nodal) in its fiber. Finally, suppose there is an open subset $U \subset Y$ such that, for every geometric point $\bar{s}$ of $S$, the fiber $U_{\bar{s}}$ is dense in every irreducible component of the fiber $Y_{\bar{s}}$, and $f^{-1}(U_{\bar{s}})$ is dense in every component of $X_{\bar{s}}$. If $f\colon f^{-1}(U) \to U$ is an isomorphism, then $f$ is an isomorphism.*

*Proof.* By [SP21, Tag 05XD], there is an open subscheme $S' \subseteq S$ such that a map $T \to S$ factors through $S'$ if and only if $f_T : X_T \to Y_T$ is an isomorphism. We want to prove that $S'$ contains all points of $S$. Thus, it suffices to show that $f$ is an isomorphism on every geometric fiber, so we may assume that $S = \operatorname{Spec} k$ for an algebraically closed field $k$.

Let $\nu_X : X^\nu \to X$ and $\nu_Y : Y^\nu \to Y$ denote the normalization maps. Then $f$ induces a map $f^\nu : X^\nu \to Y^\nu$ between disjoint unions of integral, normal curves using [SP21, Tag 035Q(4)] and the fact that every component of $X$ dominates a component of $Y$.

We first show that $f^\nu$ is an isomorphism. Since $f$ gives a bijection between irreducible components of $f^{-1}(U)$ and $U$ (or equivalently $X$ and $Y$ by density), $f^\nu$ induces a bijection between connected components. Moreover, our assumptions imply that $f^\nu$ defines birational maps on the various connected components, hence isomorphisms by [SP21, Tag 0BY1], and we see that $f^\nu$ is an isomorphism as desired. Since $\nu_X$ and $\nu_Y$ become isomorphisms after restriction to $X^{sm}$ and $Y^{sm}$, respectively, we see that $f$ restricts to an isomorphism $X^{sm} \to Y^{sm}$.

Next we show that $f$ induces a bijection on points. Surjectivity follows since $f^\nu$ and $\nu_Y$ are both surjective. For injectivity, since $f$ preserves nodes and smooth points and $f^{sm}$ is an isomorphism, we only need to show that a node $y \in Y$ has a unique node in $X$ mapping to it. By [SP21, Tag 0CBW] applied to $Y$, and the fact that $f^\nu$ is an isomorphism, there are exactly two points in $X^\nu$ mapping to $y$. By [SP21, Tag 0CBW] applied to $X$, there can only be one node in $X$ mapping to $y$. So $f$ is a bijection on points, as desired.

Now let $x_1, \ldots, x_n$ be the points of $X$ that are nodes in their fibers and let $y_1, \ldots, y_n$ be their images in $Y$. Since $f$ is surjective and using [SP21, Tag 00MB], we have that $Y^{sm} \sqcup \bigsqcup_{i=1}^n \operatorname{Spec}(\mathscr{O}^\wedge_{Y,y_i}) \to Y$ is an fpqc covering of $Y$ (see [SP21, Tag 03NW]). By [SP21, Tag 02L4] and [SP21, Tag 07N9], the fact that $f^{sm}$ is an isomorphism, and the fact that $f$ is injective on points, to complete the proof of the lemma we only have to show that the map $\mathscr{O}^\wedge_{Y,y_i} \to \mathscr{O}^\wedge_{X,x_i}$ induced by $f$ is an isomorphism.

For this last step, let $A = \mathscr{O}^\wedge_{X,x_i}$ and $B = \mathscr{O}^\wedge_{Y,y_i}$, and let $A'$ (resp. $B'$) be the integral closure of $A$ (resp. $B$) in its total ring of fractions. By [SP21, Tag 0C3V] and the fact that $f^\nu$ is an isomorphism, we have that $f$ induces an isomorphism $f^\sharp : B' \xrightarrow{\sim} A'$. The proof of Lemma 42 describes $A$ (resp. $B$) as the explicit subring of $A'$ (resp. $B'$) equal to the wedge $(A')_{\mathfrak{m}_1} \wedge (A')_{\mathfrak{m}_2}$ (resp. $(B')_{\mathfrak{n}_1} \wedge (B')_{\mathfrak{n}_2}$) where $\mathfrak{m}_1, \mathfrak{m}_2$ (resp. $\mathfrak{n}_1, \mathfrak{n}_2$) are the two maximal ideals lying over the maximal ideal $\mathfrak{m}_{x_i} \subset A$ (resp. $\mathfrak{n}_{y_i} \subset B$). Since $f^\sharp$ is an isomorphism, it sends the maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ of $A'$ to the maximal ideals $\mathfrak{n}_1, \mathfrak{n}_2$ of $B'$, and hence induces an isomorphism of subrings $B \xrightarrow{\sim} A$. □

**Lemma 76.** *If $f : X \to Y$ is an admissible $G$-cover in the sense of Definition 1, then the induced morphism $q : X/G \to Y$ is an isomorphism.*

*Proof.* Let $Y^{gen} \subset Y$ be the complement of the nodes and markings in $Y$. It is an open subset. Lemma 36 gives the following pullback diagram of open immersions:

$$\begin{tikzcd}
f^{-1}(Y^{gen}) \arrow[r] \arrow[d, hook] & f^{-1}(Y^{gen})/G \arrow[r, "q^{gen}"] \arrow[d, hook] & Y^{gen} \arrow[d, hook] \\
X \arrow[r] & X/G \arrow[r, "q"] & Y
\end{tikzcd}$$

We claim $q^{gen}$ is an isomorphism. Indeed, by assumption $f^{-1}(Y^{gen}) \to Y^{gen}$ is a $G$-torsor. By Definition 71.(i) and Remark 73, after passing to an étale cover of $Y^{gen}$ we may assume that $f^{-1}(Y^{gen}) \simeq G \times Y^{gen}$, so $f^{-1}(Y^{gen})/G \simeq Y^{gen}$ in this situation. Now $q^{gen}$ is an isomorphism by [SP21, Tag 04ZP] and [SP21, Tag 04XD].

The locus $f^{-1}(Y^{gen})$ is dense in every irreducible component of every geometric fiber $X_{\bar{s}}$: if not, some irreducible component of $X_{\bar{s}}$ would map to a node or marking of $Y_{\bar{s}}$, contradicting finiteness of $f$ [SP21, Tag 0397]. Now we apply Lemma 64 and Lemma 75. □

**Theorem 77.** *As full subcategories of $\mathfrak{H}_0$, we have $\mathfrak{H}_3 = \mathscr{A}dm(G)$. In particular, $\mathscr{A}dm(G)$ is an algebraic stack.*

*Proof.* Let $S$ be a test scheme. We show that $\mathfrak{H}_3(S) = \mathscr{A}dm(G)(S)$ as full subcategories of $\mathfrak{H}_0(S)$. Since an admissible $G$-cover $f\colon X \to Y$ is $G$-invariant it belongs to $\mathfrak{H}_1(S)$. By Lemma 76, it is in $\mathfrak{H}_2(S)$. Lastly, let $x \in X_\star$. As $G$ acts freely on a $G$-torsor, $x$ cannot be contained in an open subset of $X$ that is a $G$-torsor over its image. Thus, $f(x)$ is a node or marked point of $Y$ by Definition 1.(i). Lemma 63 rules out that $f(x)$ is a node, hence $f(x) \in Y_\star$. In other words, $f$ is in $\mathfrak{H}_3(S)$, showing "$\supseteq$".

Conversely, assume that $f\colon X \to Y$ is an object of $\mathfrak{H}_3(S)$. Then $f$ is finite by Lemma 34. If $x \in X$ maps to $y \in Y$, then by Lemma 63, $x$ is a node (resp. smooth) if and only if $y$ is a node (resp. smooth). In particular part (ii) of Definition 1 holds. By Lemma 36 the map $X_{\bar{k}} \to Y_{\bar{k}}$ is still a coarse quotient, so by Lemma 35 and [SP21, Tag 05WR] we have

$$\mathscr{O}^{\mathrm{sh}}_{Y_{\bar{k}},y} = (\mathscr{O}^{\mathrm{sh}}_{X_{\bar{k}},x_1} \times \ldots \times \mathscr{O}^{\mathrm{sh}}_{X_{\bar{k}},x_n})^G$$

where $x_1, \ldots x_n$ are the points in the fiber of $y$. Since the right hand side is canonically isomorphic to $(\mathscr{O}^{\mathrm{sh}}_{X_{\bar{k}},x_i})^{G_{x_i}}$ for any $i$ by the proof of Lemma 63, parts (iii) and (iv) follow from Lemma 57.

Finally, let $U := f^{-1}(Y^{sm} \setminus Y_\star)$ be the locus of $X$ away from the markings and nodes of $Y$. By Lemma 36 and the condition defining $\mathfrak{H}_2$, the restriction $f|_U\colon U \to Y^{sm} \setminus Y_\star$ can be identified with $U \to U/G$. Moreover, the condition defining $\mathfrak{H}_3$ guarantees that $G$ acts freely on the locus $U$. Therefore, the map from the inertia stack $\mathscr{I}_{[U/G]} \to [U/G]$ is an isomorphism [SP21, Tag 06PB] so that $[U/G]$ is an algebraic space [SP21, Tag 04SZ] and $[U/G] \simeq U/G$ by the universal property of coarse moduli spaces. In particular, by Example 74 the map $U \to U/G$ is a $G$-torsor, giving part (i) and "$\subseteq$". □

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