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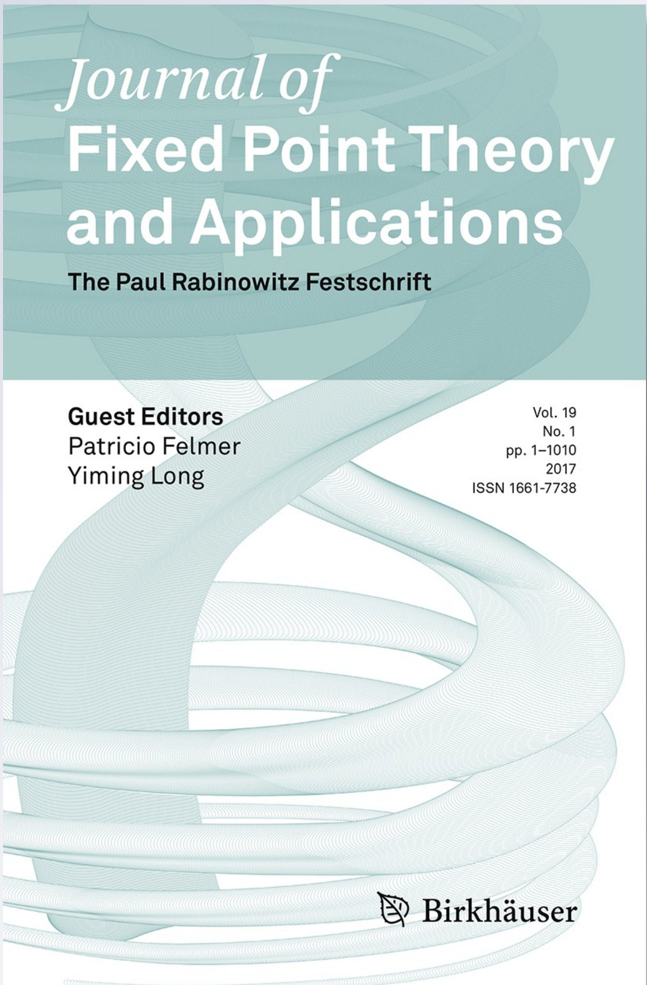
Journal of Fixed Point Theory and Applications

ISSN 1661-7738
Volume 19
Number 1

J. Fixed Point Theory Appl. (2017) 19:129-156
DOI 10.1007/s11784-016-0375-z



J. Fixed Point Theory Appl. 19 (2017) 129–156
DOI 10.1007/s11784-016-0375-z
Published online November 12, 2016

Journal of Fixed Point Theory and Applications

Quasi-periodic water waves

Massimiliano Berti and Riccardo Montalto

In honour of Professor Paul Rabinowitz.

Abstract. We present the result and the ideas of the recent paper (Berti and Montalto, Quasi-periodic standing wave solutions of gravity-capillary water waves, http://arxiv.org/abs/1602.02411, 2016) concerning the existence of Cantor families of small-amplitude time *quasi-periodic* standing wave solutions (i.e. periodic and even in the space variable x) of a 2-dimensional ocean, with infinite depth, in irrotational regime, under the action of gravity and surface tension at the free boundary. These quasi-periodic solutions are linearly stable.

Mathematics Subject Classification. 76B15, 37K55, 76D45 (37K50, 35S05).

Keywords. KAM for PDEs, water waves, quasi-periodic solutions, standing waves.

1. Introduction

An important research stream of the last decades has been the development of Kolmogorov–Arnold–Moser (KAM) theory for partial differential equations (PDEs). Actually, many PDEs arising in Physics are infinite-dimensional Hamiltonian or reversible dynamical systems

$$u_t = X(u), \quad u \in \mathcal{H}, \tag{1.1}$$

defined on some phase space $\mathcal{H}$ of functions $u := u(x)$. Main examples are the nonlinear wave, Klein–Gordon and Schrödinger equations, in one and more space dimension, the KdV (Korteweg de Vries) equation, the water waves equations for fluids. The Hamiltonian and the reversible structure of such equations eliminates dissipative phenomena, like "friction". When such PDEs are defined on a bounded domain, like a compact interval $x \in [0, \pi]$, or $x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ (periodic boundary conditions), or, more in general, x belongs to a *compact* manifold, their dynamics is expected to have a "recurrent" behaviour in time, and it is natural to expect the existence of many time-*periodic* and *quasi-periodic* solutions.

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We recall that a quasi-periodic solution of the Eq. (1.1) with ν frequencies is a *smooth* solution defined for *all* times, of the form

$$u(t) = U(\omega t) \in \mathcal{H} \qquad \text{where} \qquad \mathbb{T}^\nu \ni \varphi \mapsto U(\varphi) \in \mathcal{H}$$

is 2π-periodic in the angular variables $\varphi := (\varphi_1, \ldots, \varphi_\nu)$ and the frequency vector $\omega \in \mathbb{R}^\nu$ is irrational, namely $\omega \cdot k \neq 0$, $\forall k \in \mathbb{Z}^\nu \setminus \{0\}$. In such a case, the linear flow $\{\omega t\}_{t\in\mathbb{R}}$ is *dense* on $\mathbb{T}^\nu$ and the torus-manifold

$$\mathcal{T} := U(\mathbb{T}^\nu) \subset \mathcal{H}$$

is invariant under the flow Φ^t of (1.1). Denoting by $\Psi^t_\omega : \mathbb{T}^\nu \to \mathbb{T}^\nu$ the linear flow

$$\Psi^t_\omega(\varphi) := \varphi + \omega t, \quad \varphi \in \mathbb{T}^\nu,$$

the search of a quasi-periodic solution amounts to look for U such that

$$\Phi^t \circ U = U \circ \Psi^t_\omega. \tag{1.2}$$

Note that (1.2) *only* requires that the flow Φ^t is defined and smooth on the compact manifold $\mathcal{T} := U(\mathbb{T}^\nu)$. This remark is important because, for PDEs, the flow could be ill-posed in a neighborhood of the torus $\mathcal{T}$. From a functional point of view, (1.2) is equivalent to the equation

$$\omega \cdot \partial_\varphi U(\varphi) - X(U(\varphi)) = 0, \quad \forall \varphi \in \mathbb{T}^\nu.$$

When $\nu = 1$, the solution $u(t)$ is periodic in time, with period $2\pi/\omega$.

In the seventies, existence of periodic solutions for semilinear wave equations has been obtained by Rabinowitz [31] and Brezis–Coron–Nirenberg [12] (followed by many others) via minimax variational methods. These proofs work only to find periodic orbits with a rational frequency, because the other periods give rise to a "small divisors" problem. A fortiori, they cannot be extended for the search of quasi-periodic solutions.

Independently of these global results, local bifurcation results of periodic and quasi-periodic solutions were also proved. We cite the pioneering bifurcation results of periodic solutions by Rabinowitz [32,33] for fully nonlinear forced wave equations with a small dissipation term. Then, we refer to the beginning of KAM theory for PDEs started with works of Kuksin [27] and Wayne [37], concerning also quasi-periodic solutions, and subsequently developed in the nineties by Craig–Wayne [16], Bourgain [11], Pöeschel [30], among others. All the latter results hold for 1-dimensional semilinear wave and Schrödinger equations. In the last years, KAM theory has been considerably further extended, mainly in two important directions:

1. Wave and Schrödinger equations in higher space dimensions,
2. quasi-linear PDEs, namely equations of the form $u_t = L(u) + N(u)$ where L is a linear differential (or pseudo-differential) operator of order $\mathrm{ord}(L)$ and the nonlinearity N depends on the derivatives $\partial_x^\alpha u$ of the same order $|\alpha| = \mathrm{ord}(L)$.

We have no space to report, here, the complete literature concerning KAM theory for PDEs for which we refer to the recent survey paper [6]. Here, we want to present the recent KAM result in [7] for the water waves equations, which are, indeed, a quasi-linear system.

1.1. Capillary–gravity standing wave solutions

Consider a 2-dimensional ocean, with infinite depth, filled by an incompressible fluid, in irrotational regime, under the action of gravity and capillarity at the surface. The fluid satisfies periodic boundary conditions and occupies the free boundary region

$$\mathcal{D}_\eta := \{(x,y) \in \mathbb{T} \times \mathbb{R} \,:\, y < \eta(t,x), \quad \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})\}\,.$$

Since the velocity field is irrotational, it is the gradient of a velocity potential $\Phi(t,x,y)$. The incompressibility condition means that Φ is an harmonic function on $\mathcal{D}_\eta$. In this context, the Euler equation for the motion of the fluid reduces to the Bernoulli equation. The water waves equations are

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = \kappa \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{at } y = \eta(x) \\ \Delta\Phi = 0 & \text{in } \mathcal{D}_\eta \\ \nabla\Phi \to 0 & \text{as } y \to -\infty \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } y = \eta(x) \end{cases} \tag{1.3}$$

where g is the acceleration of gravity, $\kappa \in [\kappa_1, \kappa_2]$, $\kappa_1 > 0$, is the surface tension coefficient and

$$\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)$$

is the mean curvature of the free surface. The unknowns of the problem are the free surface $y = \eta(x)$ and the velocity potential $\Phi : \mathcal{D}_\eta \to \mathbb{R}$. The first equation in (1.3) is the Bernoulli condition (also called dynamics condition) according to which the jump of pressure across the free surface is proportional to the mean curvature. The last equation in (1.3) (also called Kinematic condition) expresses that the velocity of the free surface coincides with the one of the fluid particles, and therefore, the fluid particles on the free surface $y = \eta(x,t)$ remain on it along the fluid evolution. In the sequel, we shall assume (with no loss of generality) that the gravity constant $g = 1$.

Following Zakharov [35] and Craig–Sulem [14], the evolution problem (1.3) may be written as an infinite-dimensional Hamiltonian system. At each time $t \in \mathbb{R}$, the profile $\eta(t,x)$ of the fluid and the value

$$\psi(t,x) = \Phi(t,x,\eta(t,x))$$

of the velocity potential Φ restricted to the free boundary uniquely determine the velocity potential Φ in the whole $\mathcal{D}_\eta$, solving (at each t) the elliptic problem

$$\Delta\Phi = 0 \text{ in } \mathcal{D}_\eta,\ \ \Phi(x+2\pi, y) = \Phi(x,y),\ \ \Phi|_{y=\eta} = \psi,\ \ \nabla\Phi(x,y) \to 0 \text{ as } y \to -\infty.$$

As proved in [35], [14], system (1.3) is then equivalent to the system

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + \eta + \frac{1}{2}\psi_x^2 - \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x^2} = \kappa \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \end{cases} \tag{1.4}$$

where $G(\eta)$ is the so-called Dirichlet–Neumann operator defined by

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\partial_y\Phi)(x,\eta(x)) - \eta_x(x)\,(\partial_x\Phi)(x,\eta(x))$$

(we denote by η_x the space derivative $\partial_x\eta$.) The operator $G(\eta)$ is linear in ψ, self-adjoint with respect to the L^2 scalar product and semi-positive definite, actually its Kernel are only the constants. It depends in an analytic way with respect to the free boundary $\eta(x)$ and its derivative with respect to η is

$$d_\eta G(\eta)[\hat\eta]\psi = -G(\eta)(B\hat\eta) - \partial_x(V\hat\eta) \tag{1.5}$$

where

$$B := B(\eta,\psi) := \frac{\eta_x\psi_x + G(\eta)\psi}{1+\eta_x^2}, \qquad V := V(\eta,\psi) := \psi_x - B\eta_x. \tag{1.6}$$

The vector $(V,B) = \nabla_{x,y}\Phi$ is the velocity field evaluated at the free surface $y=\eta(x)$. It is well known since Calderon that the Dirichlet–Neumann operator $G(\eta)$ is a *pseudo-differential* operator with principal symbol $|D|$; actually, $G(\eta) - |D| \in OPS^{-\infty}$, if η is $\mathcal{C}^\infty$.

The Eq. (1.4) are the Hamiltonian system (see [14,35])

$$\partial_t\eta = \nabla_\psi H(\eta,\psi), \quad \partial_t\psi = -\nabla_\eta H(\eta,\psi)$$

$$\partial_t u = J\nabla_u H(u), \quad u := \begin{pmatrix}\eta\\ \psi\end{pmatrix}, \quad J := \begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}, \tag{1.7}$$

where ∇ denotes the L^2-gradient, and the Hamiltonian

$$H(\eta,\psi) := \frac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} + \int_{\mathbb{T}}\frac{\eta^2}{2}\,\mathrm{d}x + \kappa\int_{\mathbb{T}}\sqrt{1+\eta_x^2}\,\mathrm{d}x \tag{1.8}$$

is the sum of the kinetic energy

$$K := \frac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} = \frac{1}{2}\int_{\mathcal{D}_\eta}|\nabla\Phi|^2(x,y)\mathrm{d}x\mathrm{d}y,$$

the potential energy and the energy of the capillarity forces (area surface integral) expressed in terms of the variables (η,ψ). In light of (1.7), the variables (η,ψ) are symplectic "Darboux coordinates". The symplectic structure induced by (1.7) is the standard Darboux 2-form

$$\mathcal{W}(u_1,u_2) := (u_1, Ju_2)_{L^2(\mathbb{T}_x)} = (\eta_1,\psi_2)_{L^2(\mathbb{T}_x)} - (\psi_1,\eta_2)_{L^2(\mathbb{T}_x)} \tag{1.9}$$

for all $u_1 = (\eta_1,\psi_1)$, $u_2 = (\eta_2,\psi_2)$.

The water-waves system (1.4)–(1.7) exhibits several symmetries. First of all, the *mass*

$$\int_{\mathbb{T}}\eta\,\mathrm{d}x$$

is a prime integral of (1.4). Moreover,

$$\partial_t\int_{\mathbb{T}}\psi\,\mathrm{d}x = -\int_{\mathbb{T}}\eta\,\mathrm{d}x - \int_{\mathbb{T}}\nabla_\eta K\,\mathrm{d}x = -\int_{\mathbb{T}}\eta\,\mathrm{d}x$$

because $\int_{\mathbb{T}} \nabla_\eta K \, \mathrm{d}x = 0$. This follows because $\mathbb{R} \ni c \mapsto K(c+\eta, \psi)$ is constant (the bottom of the ocean is at $-\infty$) and so, $0 = d_\eta K(\eta, \psi)[1] = (\nabla_\eta K, 1)_{L^2(\mathbb{T})}$. As a consequence, the subspace

$$\int_{\mathbb{T}} \eta \, \mathrm{d}x = \int_{\mathbb{T}} \psi \, \mathrm{d}x = 0 \tag{1.10}$$

is invariant under the evolution of (1.4) and we shall restrict to solutions satisfying (1.10).

Also, the subspace of functions which are *even* in x,

$$\eta(x) = \eta(-x), \quad \psi(x) = \psi(-x), \tag{1.11}$$

is invariant under (1.4). Thus, we restrict (η, ψ) to the phase space of 2π-periodic even functions with zero mean, i.e., which admit the Fourier expansion

$$\eta(x) = \sum_{j \geq 1} \eta_j \cos(jx), \quad \psi(x) = \sum_{j \geq 1} \psi_j \cos(jx). \tag{1.12}$$

In this case also, the velocity potential $\Phi(x, y)$ is even and 2π-periodic in x, and so the x-component of the velocity field $v = (\Phi_x, \Phi_y)$ vanishes at $x = k\pi$, $\forall k \in \mathbb{Z}$. Hence, there is no flux of fluid through the lines $x = k\pi$, $k \in \mathbb{Z}$, and a solution of (1.4) satisfying (1.11) physically describes the motion of a liquid confined between two walls.

Another important symmetry of the capillary water waves system is *reversibility*, namely the Eqs. (1.4)–(1.7) are reversible with respect to the involution $\rho : (\eta, \psi) \mapsto (\eta, -\psi)$, or, equivalently, the Hamiltonian is even in ψ:

$$H \circ \rho = H, \quad H(\eta, \psi) = H(\eta, -\psi), \quad \rho : (\eta, \psi) \mapsto (\eta, -\psi). \tag{1.13}$$

As a consequence, it is natural to look for solutions of (1.4) satisfying

$$u(-t) = \rho u(t), \quad \text{i.e.} \quad \eta(-t, x) = \eta(t, x), \;\; \psi(-t, x) = -\psi(t, x), \;\; \forall t, x \in \mathbb{R}, \tag{1.14}$$

namely η is even in time and ψ is odd in time. Solutions of the water-waves equations (1.4) satisfying (1.12) and (1.14) are called capillary–gravity *standing water waves*.

Existence of small-amplitude time-periodic pure gravity (without surface tension) standing wave solutions has been proved by Iooss, Plotnikov, Toland in [23], see also [19,20], and in [29] in finite depth. Existence of time-periodic capillary–gravity standing wave solutions has been recently proved by Alazard–Baldi [1]. The above results are proved via a Lyapunov–Schmidt decomposition combined with a Nash–Moser iterative scheme.

In [7], we have extended this result proving also the existence of time *quasi-periodic* capillary–gravity standing wave solutions of (1.4) as well as their linear stability. This is the result of Theorem 1.1. The reducibility of the linearized equations at the quasi-periodic solutions is not only an interesting dynamical information but it is also the key for the existence proof.

We also mention that existence of small-amplitude 2-d travelling gravity water wave solutions dates back to Levi-Civita [24] (standing waves are not traveling because they are even in space, see (1.11)). Existence of small-amplitude 3-d traveling gravity–capillary water wave solutions with space-periodic boundary conditions has been proved by Craig–Nicholls [13] (it is not a small divisor problem) and by Iooss–Plotnikov [21,22] in the case of zero surface tension (in such a case, it is a small divisor problem).

The first existence results of quasi-periodic solutions of PDEs with unbounded perturbations (i.e., the nonlinearity contains derivatives) have been obtained by Kuksin [28] for KdV, see also Kappeler–Pöschel [26], by Liu–Yuan [25], Zhang–Gao–Yuan [36] for derivative NLS, by Berti–Biasco–Procesi [8]-[9] for derivative NLW. All these previous results still refer to semilinear perturbations, i.e., the order of the derivatives in the nonlinearity is strictly lower than the order of the constant coefficient (integrable) linear differential operator.

For quasi-linear, also fully nonlinear, perturbations, the first KAM results have been recently proved by Baldi–Berti–Montalto in [2–4] for Hamiltonian perturbations of Airy, KdV and mKdV equations. These techniques have been applied by Feola–Procesi [18] also to quasi-linear perturbations of 1-d Schrödinger equations.

The gravity–capillary water-waves system (1.4) is, indeed, a quasi-linear PDE. In suitable complex coordinates (having introduced the good unknown of Alinhac), it can be written in the symmetric form

$$\mathtt{u}_t = \mathrm{i}T(D)\mathtt{u} + N(\mathtt{u}, \bar{\mathtt{u}}), \quad \mathtt{u} \in \mathbb{C},$$

where

$$T(D) := |D|^{1/2}(1 - \kappa\partial_{xx})^{1/2}$$

is the Fourier multiplier which describes the linear dispersion relation of the water-waves equations linearized at $(\eta, \psi) = 0$ (see (1.15)–(1.18)), and the nonlinearity $N(\mathtt{u}, \bar{\mathtt{u}})$ depends on the highest order term $|D|^{3/2}\mathtt{u}$ as well.

1.2. Main result

We look for small-amplitude quasi-periodic solutions of (1.4), and therefore, it is of main importance the dynamics of the linearized system at the equilibrium $(\eta, \psi) = (0, 0)$ (flat ocean and fluid at rest), namely

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + \eta = \kappa \eta_{xx} \end{cases} \tag{1.15}$$

where $G(0) = |D_x|$ is the Dirichlet–Neumann operator at the flat surface $\eta = 0$, namely

$$|D_x| \cos(jx) = |j| \cos(jx), \quad |D_x| \sin(jx) = |j| \sin(jx), \ \forall j \in \mathbb{Z}.$$

In compact Hamiltonian form, the system (1.15) reads

$$\partial_t u = J\Omega u, \quad \Omega := \begin{pmatrix} 1 - \kappa\partial_{xx} & 0 \\ 0 & G(0) \end{pmatrix}, \tag{1.16}$$

which is the Hamiltonian system generated by the quadratic Hamiltonian (see (1.8))

$$H_L := \frac{1}{2}(u, \Omega u)_{L^2(\mathbb{T}_x)} = \frac{1}{2}(\psi, G(0)\psi)_{L^2(\mathbb{T}_x)} + \frac{1}{2}\int_{\mathbb{T}} \left(\eta^2 + \kappa \eta_x^2\right) \mathrm{d}x. \quad (1.17)$$

The standing wave solutions of the linear system (1.15) are

$$\eta(t,x) = \sum_{j\geq 1} a_j \cos(\omega_j t)\cos(jx), \quad \psi(t,x) = -\sum_{j\geq 1} a_j j^{-1}\omega_j \sin(\omega_j t)\cos(jx),$$

where $a_j \in \mathbb{R}$, and the linear frequencies of oscillations are

$$\omega_j := \omega_j(\kappa) := \sqrt{j(1+\kappa j^2)}, \quad j \geq 1. \quad (1.18)$$

Fix an arbitrary finite subset $\mathbb{S}^+ \subset \mathbb{N}^+ := \{1, 2, \dots\}$ (tangential sites) and consider the linear standing-wave solutions

$$\begin{aligned} \eta(t,x) &= \sum_{j\in\mathbb{S}^+} \sqrt{\xi_j}\cos(\omega_j t)\cos(jx), \\ \psi(t,x) &= -\sum_{j\in\mathbb{S}^+} \sqrt{\xi_j} j^{-1}\omega_j \sin(\omega_j t)\cos(jx), \ \ \xi_j > 0, \end{aligned} \quad (1.19)$$

which are Fourier supported in $\mathbb{S}^+$. The main result of [7] proves that such linear standing-wave solutions can be continued to solutions of the nonlinear water-waves Hamiltonian system (1.4) for most values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$. Theorem 1.1 below states the existence of quasi-periodic solutions

$$u(\tilde{\omega}t, x) = (\eta, \psi)(\tilde{\omega}t, x)$$

of (1.4), with frequency $\tilde{\omega} := (\tilde{\omega}_j)_{j\in\mathbb{S}^+}$ (to be determined), close to the solutions (1.19) of (1.15), for most values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$.

Let $\nu := |\mathbb{S}^+|$ denote the cardinality of $\mathbb{S}^+$. The function $u(\varphi, x) = (\eta, \psi)(\varphi, x)$, $\varphi \in \mathbb{T}^\nu$, belongs to the Sobolev spaces of $(2\pi)^{\nu+1}$-periodic real functions

$$H^s(\mathbb{T}^{\nu+1}, \mathbb{R}^2) := \{u = (\eta, \psi) : \eta, \psi \in H^s\}$$

$$H^s := H^s(\mathbb{T}^{\nu+1}, \mathbb{R}) = \Big\{ f = \sum_{(\ell,j)\in\mathbb{Z}^{\nu+1}} \widehat{f}_{\ell j}\, e^{\mathrm{i}(\ell\cdot\varphi + jx)} : \quad \|f\|_s^2 := \sum_{(\ell,j)\in\mathbb{Z}^{\nu+1}} |\widehat{f}_{\ell j}|^2 \langle \ell, j\rangle^{2s} < +\infty \Big\} \quad (1.20)$$

where $\langle \ell, j\rangle := \max\{1, |\ell|, |j|\}$ with $|\ell| := \max_{i=1,\dots,\nu} |\ell_i|$. For

$$s \geq s_0 := \left[\frac{\nu+1}{2}\right] + 1 \in \mathbb{N},$$

the Sobolev spaces $H^s \subset L^\infty(\mathbb{T}^{\nu+1})$ are an algebra with respect to the product of functions.

Theorem 1.1 (KAM for capillary–gravity periodic standing water waves [7]). *For every choice of finitely many tangential sites* $\mathbb{S}^+ \subset \mathbb{N}^+$*, there exists* $\bar{s} > s_0$*,* $\varepsilon_0 \in (0,1)$ *such that for every* $|\xi| \le \varepsilon_0^2$*,* $\xi := (\xi_j)_{j\in\mathbb{S}^+}$*, there exists a Cantor-like set* $\mathcal{G} \subset [\kappa_1, \kappa_2]$ *with asymptotically full measure as* $\xi \to 0$*, i.e.*

$$\lim_{\xi\to 0} |\mathcal{G}| = \kappa_2 - \kappa_1,$$

such that, for any surface tension coefficient $\kappa \in \mathcal{G}$*, the capillarity-gravity system* (1.4) *has a time quasi-periodic standing wave solution* $u(\tilde{\omega}t, x) = (\eta(\tilde{\omega}t, x), \psi(\tilde{\omega}t, x))$*, with Sobolev regularity* $(\eta, \psi)(\varphi, x) \in H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$*, of the form*

$$\begin{aligned} \eta(\tilde{\omega}t, x) &= \sum_{j\in\mathbb{S}^+} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|}), \\ \psi(\tilde{\omega}, x) &= -\sum_{j\in\mathbb{S}^+} \sqrt{\xi_j} j^{-1} \omega_j \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|}) \end{aligned} \tag{1.21}$$

with a Diophantine frequency vector $\tilde{\omega} := \tilde{\omega}(\kappa, \xi) \in \mathbb{R}^\nu$ *satisfying* $\tilde{\omega}_j - \omega_j(\kappa) \to 0$*,* $j \in \mathbb{S}^+$*, as* $\xi \to 0$*. The terms* $o(\sqrt{|\xi|})$ *are small in* $H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$*. Also, these quasi-periodic solutions are linearly stable.*

Let us make some comments.

1. No global-in-time existence results concerning the initial value problem of the water waves equations (1.4) under *periodic* boundary conditions are known so far. The present Nash–Moser-KAM iterative procedure selects many values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$ which give rise to the quasi-periodic solutions (1.21), which are defined for all times. Clearly, by a Fubini-type argument, it also results that, for most values of $\kappa \in [\kappa_1, \kappa_2]$, there exist quasi-periodic solutions of (1.4) for most values of the amplitudes $|\xi| \le \varepsilon_0^2$. The fact that we find quasi-periodic solutions restricting to a proper subset of parameters is not a technical issue. The capillarity–gravity water-waves equations (1.4) are not expected to be integrable (albeit a rigorous proof is still lacking): yet, the third-order Birkhoff normal form possesses multiple resonant triads (Wilton ripples), see Craig–Sulem [15].
2. In the proof of Theorem 1.1, all the estimates depend on the surface tension coefficient $\kappa > 0$ and the result does not hold at the limit of zero surface tension $\kappa \to 0$. Because of capillarity, the linear frequencies (1.18) grow asymptotically $\sim \sqrt{\kappa} j^{3/2}$ as $j \to +\infty$. Without surface tension, the linear frequencies grow asymptotically as $\sim j^{1/2}$ and a different proof is required.
3. The quasi-periodic solutions (1.21) are mainly supported in Fourier space on the tangential sites $\mathbb{S}^+$. The dynamics of the water waves equations (1.4) restricted to the symplectic subspaces

$$
\begin{aligned}
H_{\mathbb{S}^+} &:= \left\{ v = \sum_{j\in\mathbb{S}^+} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \right\}, \\
H_{\mathbb{S}^+}^{\perp} &:= \left\{ z = \sum_{j\in\mathbb{N}\setminus\mathbb{S}^+} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \in H_0^1(\mathbb{T}_x) \right\}.
\end{aligned}
\tag{1.22}
$$

is quite different. We call $v \in H_{\mathbb{S}^+}$ the *tangential* variable and $z \in H_{\mathbb{S}^+}^{\perp}$ the *normal* one. On the finite-dimensional subspace $H_{\mathbb{S}^+}$, we shall describe the dynamics by introducing the action–angle variables $(\theta, I) \in \mathbb{T}^\nu \times \mathbb{R}^\nu$ as in (2.2). The quasi-periodic solutions (1.21) of (1.4) are, therefore, close to $\mathbb{T}^\nu \times \{\xi\} \times \{z = 0\}$, $\xi \in \mathbb{R}^\nu_+$. On the infinite-dimensional subspace $H_{\mathbb{S}^+}^{\perp}$, the solution stays, forever, close to the elliptic equilibrium $z = 0$, in some Sobolev norm.

A first key observation is that, for most values of the surface tension parameter $\kappa \in [\kappa_1, \kappa_2]$, the unperturbed linear frequencies (1.18), regrouped on the tangential and normal components

$$
\vec{\omega}(\kappa) := (\omega_j(\kappa))_{j\in\mathbb{S}^+}, \quad \vec{\Omega}(\kappa) := (\Omega_j(\kappa))_{j\in\mathbb{N}^+\setminus\mathbb{S}^+} := (\omega_j(\kappa))_{j\in\mathbb{N}^+\setminus\mathbb{S}^+},
\tag{1.23}
$$

are Diophantine, namely

$$
|\vec{\omega}(\kappa) \cdot \ell| \geq \frac{\gamma}{|\ell|^\tau}, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\},
$$

and also satisfy stronger non-resonance properties, the so-called first- and second-order Melnikov non-resonance conditions, see (2.14), which are non-resonance conditions between the tangential and the normal frequencies. We shall prove this fact by degenerate KAM theory, see Sect. 2.2. For such values of $\kappa \in [\kappa_1, \kappa_2]$, the solutions (1.19) of the linear equation (1.15) are already sufficiently good approximate quasi-periodic solutions of the nonlinear water-waves system (1.4). Since the parameter space $[\kappa_1, \kappa_2]$ is fixed, the small divisor constant γ can be taken $\gamma = o(\varepsilon^a)$ with $a > 0$ small as needed, see (2.15). As a consequence, for proving the continuation of (1.19) to solutions of the nonlinear water-waves system (1.4), all the terms which are at least quadratic in (1.4) are already perturbative (i.e. in (2.1), it is sufficient to regard the vector field $\varepsilon X_{P_\varepsilon}$ as a perturbation of the linear vector field $J\Omega$).

Linear stability. The quasi-periodic solutions $u(\tilde{\omega}t) = (\eta(\tilde{\omega}t), \psi(\tilde{\omega}t))$ found in Theorem 1.1 are linearly stable. This is not only a dynamically relevant information but also an essential ingredient of the existence proof (it is not necessary for time-periodic solutions as in [1,19,20,23]). Let us state precisely the result. Around each invariant torus, there exist symplectic coordinates

$$
(\phi, y, w) = (\phi, y, \eta, \psi) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H_{\mathbb{S}^+}^{\perp}
$$

(see [10]) in which the water-waves Hamiltonian reads

$$\omega \cdot y + \frac{1}{2} K_{20}(\phi) y \cdot y + (K_{11}(\phi) y, w)_{L^2(\mathbb{T}_x)} + \frac{1}{2} (K_{02}(\phi) w, w)_{L^2(\mathbb{T}_x)} + K_{\geq 3}(\phi, y, w)$$

where $K_{\geq 3}$ collects the terms at least cubic in the variables (y, w). In these coordinates, the quasi-periodic solution reads $t \mapsto (\omega t, 0, 0)$ (for simplicity, we denote the frequency $\tilde{\omega}$ of the quasi-periodic solution by ω) and the corresponding linearized water-waves equations are

$$\begin{cases} \dot{\phi} = K_{20}(\omega t)[y] + K_{11}^T(\omega t)[w] \\ \dot{y} = 0 \\ \dot{w} = J K_{02}(\omega t)[w] + J K_{11}(\omega t)[y] \, . \end{cases}$$

Thus, the actions $y(t) = y(0)$ do not evolve in time, and the third equation reduces to the PDE

$$\dot{w} = J K_{02}(\omega t)[w] + J K_{11}(\omega t)[y(0)]. \tag{1.24}$$

The self-adjoint operator $K_{02}(\omega t)$ is, up to a finite-dimensional remainder, the restriction to $H_{\mathbb{S}+}^{\perp}$ of the linearized water-waves vector field $\partial_u \nabla H(u(\omega t))$, which is explicitly computed in (2.19).

Denote $H_{\perp}^s := H_{\perp}^s(\mathbb{T}_x) := H^s(\mathbb{T}_x) \cap H_{\mathbb{S}}^{\perp}$ (real or complex valued). We prove the existence of bounded and invertible "symmetrizer" maps $\mathbf{W}_{m,\infty}(\varphi)$, $m = 1, 2$ such that $\forall \varphi \in \mathbb{T}^\nu$, $s \geq s_0$,

$$\mathbf{W}_{m,\infty}(\varphi) : (H^s(\mathbb{T}_x, \mathbb{C}) \times H^s(\mathbb{T}_x, \mathbb{C})) \cap H_{\mathbb{S}_+}^{\perp} \to \big(H^s(\mathbb{T}_x, \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{T}_x, \mathbb{R})\big) \cap H_{\mathbb{S}_+}^{\perp}, \tag{1.25}$$

$$\mathbf{W}_{m,\infty}^{-1}(\varphi) : \big(H^s(\mathbb{T}_x, \mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{T}_x, \mathbb{R})\big) \cap H_{\mathbb{S}_+}^{\perp} \to \big(H^s(\mathbb{T}_x, \mathbb{C}) \times H^s(\mathbb{T}_x, \mathbb{C})\big) \cap H_{\mathbb{S}_+}^{\perp}, \tag{1.26}$$

and such that, under the quasi-periodic-in-time change of variables

$$w = (\eta, \psi) = \mathbf{W}_{1,\infty}(\omega t) w_\infty, \quad w_\infty = (\mathtt{w}_\infty, \overline{\mathtt{w}}_\infty),$$

the Eq. (1.24) transforms into the diagonal system

$$\begin{aligned} \partial_t w_\infty &= -\mathrm{i} \mathbf{D}_\infty w_\infty + f_\infty(\omega t), \\ f_\infty(\omega t) &:= \mathbf{W}_{2,\infty}(\varphi)(\omega t)^{-1} J K_{11}(\omega t)[y(0)] = \begin{pmatrix} \mathtt{f}_\infty(\omega t) \\ \overline{\mathtt{f}}_\infty(\omega t) \end{pmatrix} \end{aligned} \tag{1.27}$$

where, denoting $\mathbb{S}_0 := \mathbb{S}_+ \cup (-\mathbb{S}_+) \cup \{0\} \subseteq \mathbb{Z}$,

$$\mathbf{D}_\infty := \begin{pmatrix} D_\infty & 0 \\ 0 & -D_\infty \end{pmatrix}, \quad D_\infty := \mathrm{diag}_{j \in \mathbb{S}_0^c} \{\mu_j^\infty\}, \quad \mu_j^\infty \in \mathbb{R}, \tag{1.28}$$

is a Fourier multiplier operator of the form

$$\mu_j^\infty := \mathtt{m}_3^\infty \sqrt{|j|(1 + \kappa j^2)} + \mathtt{m}_1^\infty |j|^{\frac{1}{2}} + r_j^\infty, \ j \in \mathbb{S}_0^c, \quad r_j^\infty = r_{-j}^\infty, \tag{1.29}$$

where, for some $\mathtt{a} > 0$,

$$\mathtt{m}_3^\infty = 1 + O(\varepsilon^{\mathtt{a}}),\ \mathtt{m}_1^\infty = O(\varepsilon^{\mathtt{a}}),\ \sup_{j\in\mathbb{S}_0^c} |r_j^\infty| = O(\varepsilon^{\mathtt{a}}),\ \forall |k| \le k_0,$$

see (2.12)–(2.13), (2.15) and $k_0 \in \mathbb{N}$ is a constant which depends only on the linear frequencies $\omega_j(\kappa)$ defined in (1.18). The $\mathrm{i}\mu_j^\infty$ are the *Floquet exponents* of the quasi-periodic solution. The fact that they are purely imaginary is a consequence of the reversible structure of the water-waves equations.

The second equation of system (1.27) is actually the complex conjugated of the first one, and (1.27) reduces to the infinitely many decoupled scalar equations

$$\partial_t \mathtt{w}_{\infty,j} = -\mathrm{i}\mu_j^\infty \mathtt{w}_{\infty,j} + \mathtt{f}_{\infty,j}(\omega t), \quad \forall j \in \mathbb{S}_0^c.$$

By variation of constants, the solutions are

$$\mathtt{w}_{\infty,j}(t) = c_j e^{-\mathrm{i}\mu_j^\infty t} + \mathtt{v}_{\infty,j}(t), \quad \mathtt{v}_{\infty,j}(t) := \sum_{\ell\in\mathbb{Z}^\nu} \frac{\mathtt{f}_{\infty,j,\ell}\, e^{\mathrm{i}\omega\cdot\ell t}}{\mathrm{i}(\omega\cdot\ell + \mu_j^\infty)}, \quad \forall j \in \mathbb{S}_0^c. \tag{1.30}$$

Since the Melnikov conditions (2.14) hold at a solution, then $\mathtt{v}_j^\infty(t)$ in (1.30) is well defined. Moreover, (1.25) implies $\|f_\infty(\omega t)\|_{H_x^s\times H_x^s} \le C|y(0)|$. As a consequence, the Sobolev norm of the solution of (1.27) with initial condition $w_\infty(0) \in H^{\mathfrak{s}_0}(\mathbb{T}_x)$, $\mathfrak{s}_0 < s$ (in a suitable range of values), satisfies

$$\|w_\infty(t)\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0}} \le C(s)(|y(0)| + \|w_\infty(0)\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0}}),$$

and, for all $t \in \mathbb{R}$, using (1.25), (1.26), we get

$$\|(\eta,\psi)(t)\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0-\frac12}} \le \|(\eta(0),\psi(0))\|_{H_x^{\mathfrak{s}_0}\times H_x^{\mathfrak{s}_0-\frac12}}$$

which proves the linear stability of the quasi-periodic solution. Note that the profile $\eta \in H^{\mathfrak{s}_0}(\mathbb{T}_x)$ is more regular than the velocity potential $\psi \in H^{\mathfrak{s}_0-\frac12}(\mathbb{T}_x)$, as it is expected in the presence of surface tension.

Clearly, a crucial point is the diagonalization of (1.24) into (1.28). With respect to [1], this requires to analyze more in detail the pseudo-differential nature of the operators obtained after each conjugation and to implement a KAM scheme with second-order Melnikov non-resonance conditions, as we shall explain in detail below. We now present the main ideas of the proof.

2. Ideas of the proof

We prove Theorem 1.1 by a Nash–Moser iterative scheme in Sobolev spaces formulated as a "Theorem of hypothetic conjugation" á la Herman (Sect. 2.1) plus a degenerate KAM theory argument to perform the measure estimates in κ (Sect. 2.2). The core of the Nash–Moser scheme is to prove that the linearized operators obtained at any approximate solution are invertible, with an inverse that satisfies tame estimates in Sobolev spaces. We explain how to prove this property in Sect. 2.3.

First of all, instead of working in a shrinking neighborhood of the origin, it is a convenient device to rescale the variable $u \mapsto \varepsilon u$ with $u = O(1)$, writing (1.4), (1.7) as

$$\partial_t u = J\Omega u + \varepsilon X_{P_\varepsilon}(u) \tag{2.1}$$

where $J\Omega$ is defined in (1.16) and $X_{P_\varepsilon}(u)$ is the Hamiltonian vector field generated by the Hamiltonian

$$\mathcal{H}_\varepsilon(u) := \varepsilon^{-2} H(\varepsilon u) = H_L(u) + \varepsilon P_\varepsilon(u)$$

where H is the water-waves Hamiltonian (1.8) and H_L is defined in (1.17).

We decompose the phase space as in (1.22),

$$H^1_{0,\text{even}} = H_{\mathbb{S}^+} \oplus H^\perp_{\mathbb{S}^+},$$

and we introduce action–angle variables on the tangential sites by setting

$$\eta_j := \sqrt{\frac{2}{\pi}}\, \Lambda_j^{1/2} \sqrt{\xi_j + I_j} \cos(\theta_j), \quad \psi_j := -\sqrt{\frac{2}{\pi}}\, \Lambda_j^{-1/2} \sqrt{\xi_j + I_j}\, \sin(\theta_j)$$

$$\Lambda_j := \sqrt{j(1 + \kappa j^2)^{-1}}, \quad j \in \mathbb{S}^+, \tag{2.2}$$

where $\xi_j > 0$, $j = 1, \ldots, \nu$, are positive constants, and $|I_j| \le \xi_j$. The symplectic 2-form in (1.9) then reads

$$\mathcal{W} := \left(\sum_{j \in \mathbb{S}^+} d\theta_j \wedge dI_j \right) \oplus \mathcal{W}_{|H^\perp_{\mathbb{S}^+}}$$

and the Hamiltonian system (2.1) transforms into the new Hamiltonian system

$$\dot\theta = \partial_I H_\varepsilon(\theta, I, z), \quad \dot I = -\partial_\theta H_\varepsilon(\theta, I, z), \quad z_t = J\nabla_z H_\varepsilon(\theta, I, z) \tag{2.3}$$

generated by the Hamiltonian

$$H_\varepsilon := \mathcal{H}_\varepsilon \circ A = \varepsilon^{-2} H \circ \varepsilon A \tag{2.4}$$

where

$$A(\theta, I, z) := v(\theta, I) + z := \sum_{j \in \mathbb{S}^+} \sqrt{\frac{2}{\pi}} \begin{pmatrix} \Lambda_j^{1/2} \sqrt{\xi_j + I_j}\, \cos(\theta_j) \\ -\Lambda_j^{-1/2} \sqrt{\xi_j + I_j}\, \sin(\theta_j) \end{pmatrix} \cos(jx) + z. \tag{2.5}$$

We denote by

$$X_{H_\varepsilon} := (\partial_I H_\varepsilon, -\partial_\theta H_\varepsilon, J\nabla_z H_\varepsilon)$$

the Hamiltonian vector field in the variables $(\theta, I, z) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}$. The involution ρ in (1.13) becomes

$$\tilde\rho : (\theta, I, z) \mapsto (-\theta, I, \rho z). \tag{2.6}$$

By (1.8) and (2.4), the Hamiltonian H_ε reads (up to a constant)

$$H_\varepsilon = \mathcal{N} + \varepsilon P, \quad \mathcal{N} := H_L \circ A = \vec\omega(\kappa) \cdot I + \frac{1}{2}(z, \Omega z)_{L^2_x}, \quad P := P_\varepsilon \circ A, \tag{2.7}$$

where $\vec\omega(\kappa)$ is defined in (1.23) and Ω in (1.16). We look for an embedded invariant torus

$$i : \mathbb{T}^\nu \to \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}, \quad \varphi \mapsto i(\varphi) := (\theta(\varphi), I(\varphi), z(\varphi))$$

of the Hamiltonian vector field X_{H_ε} filled by quasi-periodic solutions with Diophantine frequency $\omega \in \mathbb{R}^\nu$.

2.1. Nash–Moser theorem of hypothetic conjugation

The Hamiltonian H_ε in (2.7) is a perturbation of the isochronous system with Hamiltonian $\mathcal{N}$. The expected quasi-periodic solutions of the corresponding Hamiltonian system (2.3) will have a shifted frequency—to be found—close to the linear frequencies $\omega_j(\kappa)$ in (1.18), which depend on the nonlinear term P and the amplitudes ξ_j.

In view of that, we introduce the family of Hamiltonians

$$H_\alpha := \mathcal{N}_\alpha + \varepsilon P, \quad \mathcal{N}_\alpha := \alpha \cdot I + \frac{1}{2}(z, \Omega z)_{L^2_x}, \quad \alpha \in \mathbb{R}^\nu, \tag{2.8}$$

which depend on a constant vector $\alpha \in \mathbb{R}^\nu$. For the value $\alpha = \vec{\omega}(\kappa)$, we have $H_\alpha = H_\varepsilon$. Then, we look for a zero (i, α) of the nonlinear operator

$$\begin{aligned} \mathcal{F}(i,\alpha) &:= \mathcal{F}(i,\alpha,\omega,\kappa,\varepsilon) := \omega\cdot\partial_\varphi i(\varphi) - X_{H_\alpha}(i(\varphi)) \\ &= \omega\cdot\partial_\varphi i(\varphi) - (X_{\mathcal{N}_\alpha} + \varepsilon X_P)(i(\varphi)) \\ &:= \begin{pmatrix} \omega\cdot\partial_\varphi\theta(\varphi) - \alpha - \varepsilon\partial_I P(i(\varphi)) \\ \omega\cdot\partial_\varphi I(\varphi) + \varepsilon\partial_\theta P(i(\varphi)) \\ \omega\cdot\partial_\varphi z(\varphi) - J(\Omega z + \varepsilon\nabla_z P(i(\varphi))) \end{pmatrix} \end{aligned} \tag{2.9}$$

for some Diophantine vector $\omega \in \mathbb{R}^\nu$. If $\mathcal{F}(i,\alpha) = 0$, then $\varphi \mapsto i(\varphi)$ is an embedded torus, invariant for the Hamiltonian vector field X_{H_α}, filled by quasi-periodic solutions with frequency ω.

Since each Hamiltonian H_α in (2.8) is reversible, we look for reversible solutions of $\mathcal{F}(i,\alpha) = 0$, namely satisfying $\tilde{\varrho} i(\varphi) = i(-\varphi)$ (see (2.6)), i.e.

$$\theta(-\varphi) = -\theta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\varrho z)(\varphi). \tag{2.10}$$

The Sobolev norm of the periodic component of the embedded torus

$$\mathfrak{I}(\varphi) := i(\varphi) - (\varphi, 0, 0) := (\Theta(\varphi), I(\varphi), z(\varphi)), \quad \Theta(\varphi) := \theta(\varphi) - \varphi,$$

is

$$\|\mathfrak{I}\|_s := \|\Theta\|_{H^s_\varphi} + \|I\|_{H^s_\varphi} + \|z\|_s$$

where $\|z\|_s := \|z\|_{H^s_{\varphi,x}} = \|\eta\|_s + \|\psi\|_s$, see (1.20).

For the next theorem, we recall that k_0 is the index of non-degeneracy provided by Proposition 2.3 and it depends only on the linear unperturbed frequencies $\omega_j(\kappa)$. Therefore, it is considered as an absolute constant and we will often omit to write explicitly the dependence of the constants with respect to k_0. We look for quasi-periodic solutions with frequency ω belonging to a δ-neighborhood (independent of ε)

$$\Omega := \{\omega \in \mathbb{R}^\nu \,:\, \mathrm{dist}\,(\omega, \vec{\omega}[\kappa_1, \kappa_2]) < \delta, \ \delta > 0\}$$

of the unperturbed linear frequencies $\vec{\omega}[\kappa_1, \kappa_2]$ defined in (1.23).

Theorem 2.1 (Nash–Moser theorem of hypothetic conjugation). *Fix finitely many tangential sites $\mathbb{S}^+ \subset \mathbb{N}^+$ and let $\nu := |\mathbb{S}^+|$. Let $\tau \geq 1$. There exist constants $\varepsilon_0 > 0$, $a_0 := a_0(\nu, \tau, k_0) > 0$ and $k_1 := k_1(\nu, k_0, \tau) > 0$ such that,* *for all* $\gamma = \varepsilon^a$, $0 < a < a_0$, $\varepsilon \in (0, \varepsilon_0)$, *there exists a* k_0*-times differentiable function*

$$\alpha_\infty : \Omega \times [\kappa_1, \kappa_2] \mapsto \mathbb{R}^\nu, \quad \alpha_\infty(\omega, \kappa) = \omega + r_\varepsilon(\omega, \kappa), \quad |r_\varepsilon|^{k_0,\gamma} \le C\varepsilon\gamma^{-(1+k_1)}, \tag{2.11}$$

a family of embedded tori i_∞ *defined for all* $\omega \in \Omega$ *and* $\kappa \in [\kappa_1, \kappa_2]$ *satisfying the reversibility property* (2.10) *and*

$$\|i_\infty(\varphi) - (\varphi, 0, 0)\|_{s_0}^{k_0,\gamma} \le C\varepsilon\gamma^{-(1+k_1)},$$

a sequence of k_0*-times differentiable functions* $\mu_j^\infty : \Omega \times [\kappa_1, \kappa_2] \to \mathbb{R}$, $j \in \mathbb{N}^+ \setminus \mathbb{S}^+$, *of the form*

$$\mu_j^\infty(\omega, \kappa) = \mathtt{m}_3^\infty(\omega, \kappa) j^{\frac{1}{2}}(1 + \kappa j^2)^{\frac{1}{2}} + \mathtt{m}_1^\infty(\omega, \kappa) j^{\frac{1}{2}} + r_j^\infty(\omega, \kappa) \tag{2.12}$$

satisfying

$$|\mathtt{m}_3^\infty - 1|^{k_0,\gamma} + |\mathtt{m}_1^\infty|^{k_0,\gamma} \le C\varepsilon, \qquad \sup_{j \in \mathbb{S}^c} |r_j^\infty|^{k_0,\gamma} \le C\varepsilon\gamma^{-k_1}, \tag{2.13}$$

such that for all (ω, κ) *in the Cantor-like set*

$$\begin{aligned} \mathcal{C}_\infty^\gamma := \Big\{ & (\omega, \kappa) \in \Omega \times [\kappa_1, \kappa_2] \,:\, |\omega \cdot \ell| \ge \gamma \langle \ell \rangle^{-\tau}, \, \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}, \\ & |\omega \cdot \ell + \mu_j^\infty(\omega, \kappa)| \ge 4\gamma j^{\frac{3}{2}} \langle \ell \rangle^{-\tau}, \, \forall \ell \in \mathbb{Z}^\nu, \, j \in \mathbb{N}^+ \setminus \mathbb{S}^+ \\ & \quad (1\text{-}\mathit{Melnikov\ conditions}) \\ & |\omega \cdot \ell + \mu_j^\infty(\omega, \kappa) - \varsigma \mu_{j'}^\infty(\omega, \kappa)| \ge \frac{4\gamma |j^{\frac{3}{2}} - \varsigma j'^{\frac{3}{2}}|}{\langle \ell \rangle^\tau}, \\ & \quad \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+, \varsigma = \pm 1, (2\text{-}\mathit{Melnikov}) \Big\} \end{aligned} \tag{2.14}$$

the function $i_\infty(\varphi) := i_\infty(\omega, \kappa, \varepsilon)(\varphi)$ *is a solution of* $\mathcal{F}(i_\infty, \alpha_\infty(\omega, \kappa), \omega, \kappa, \varepsilon) = 0$. *As a consequence, the embedded torus* $\varphi \mapsto i_\infty(\varphi)$ *is invariant for the Hamiltonian vector field* $X_{H_{\alpha_\infty(\omega,\kappa)}}$ *and it is filled by quasi-periodic solutions with frequency* ω.

In Theorem 2.1, we are not concerned about the measure of $\mathcal{C}_\infty^\gamma$, in particular in investigating if it is not empty. Note that the Cantor-like set $\mathcal{C}_\infty^\gamma$ in (2.14) for which a solution exists is defined only in terms of the "final" solution i_∞ and the "final" normal perturbed frequencies μ_j^∞, $j \in \mathbb{N}^+ \setminus \mathbb{S}^+$, which are defined for *all* the values of $(\omega, \kappa) \in \Omega \times [\kappa_1, \kappa_2]$ by a Whitney-type extension argument.

Theorem 2.1 is called of "hypothetic conjugation" because it does not prove the existence of a quasi-periodic solution for the original Hamiltonian H_ε but just for a nearby Hamiltonian $H_{\alpha_\infty(\omega,\kappa)}$. The aim is, now, to deduce Theorem 1.1 from Theorem 2.1: we have to prove the existence of quasi-periodic solutions of the water-waves equations (1.4), and not only of the system with modified Hamiltonian H_α with $\alpha := \alpha_\infty(\omega, \kappa)$. Therefore, we have to prove that the curve of the unperturbed linear frequencies

$$[\kappa_1, \kappa_2] \ni \kappa \mapsto \vec{\omega}(\kappa) := (\sqrt{j(1 + \kappa j^2)})_{j\in\mathbb{S}^+} \in \mathbb{R}^\nu$$

intersects the image $\alpha_\infty(\mathcal{C}^\gamma_\infty)$, under the map α_∞ of the Cantor set $\mathcal{C}^\gamma_\infty$, for "most" values of $\kappa \in [\kappa_1, \kappa_2]$.

The above functional setting perspective is in the spirit of the Théoréme de conjugaison hypothétique of Herman proved by Fejoz [17] for finite-dimensional Hamiltonian systems. A relevant difference is that in [17], in addition to α, also the normal frequencies are introduced as independent parameters, unlike in Theorem 2.1. Actually, for PDEs, it seems more convenient the present formulation: it is, indeed, a major point of the work to know the asymptotic expansion (1.29) of the Floquet exponents.

2.2. Measure estimates and degenerate KAM theory

For any $\beta \in \alpha_\infty(\mathcal{C}^\gamma_\infty)$, Theorem 2.1 proves the existence of an embedded invariant torus filled by quasi-periodic solutions with Diophantine frequency $\omega = \alpha^{-1}_\infty(\beta, \kappa)$ for the Hamiltonian

$$H_\beta = \beta \cdot I + \frac{1}{2}(z, \Omega z)_{L^2_x} + \varepsilon P.$$

Theorem 2.2 now proves that for "most" values of $\kappa \in [\kappa_1, \kappa_2]$, the vector $\beta = \vec{\omega}(\kappa) \in \alpha_\infty(\mathcal{C}^\gamma_\infty)$. Hence, for such values of κ, we have found an embedded invariant torus for the Hamiltonian H_ε in (2.7), filled by quasi-periodic solutions with Diophantine frequency $\omega = \alpha^{-1}_\infty(\vec{\omega}(\kappa), \kappa)$.

Theorem 2.2 (Measure estimates). *Let*

$$\gamma = \varepsilon^a, \quad 0 < a < \min\{a_0, 1/(1 + k_0 + k_1)\}, \qquad \tau > k_0(\nu + 4). \tag{2.15}$$

Then, the measure of the set

$$\mathcal{G}_\varepsilon := \big\{\kappa \in [\kappa_1, \kappa_2] \, : \, \big(\alpha^{-1}_\infty(\vec{\omega}(\kappa), \kappa), \kappa\big) \in \mathcal{C}^\gamma_\infty\big\} \tag{2.16}$$

satisfies $|\mathcal{G}_\varepsilon| \geq (\kappa_2 - \kappa_1) - C\varepsilon^{a/k_0}$ *as* $\varepsilon \to 0$.

Clearly, Theorems 2.1 and 2.2 imply Theorem 1.1 with the Cantor-like set $\mathcal{G} := \mathcal{G}_\varepsilon$ and frequency vector $\tilde{\omega} = \alpha^{-1}_\infty(\vec{\omega}(\kappa), \kappa)$.

Let us sketch the proof of Theorem 2.2. By (2.11), for ε small, for any $\kappa \in [\kappa_1, \kappa_2]$, the function $\alpha_\infty(\cdot, \kappa)$ is invertible:

$$\begin{aligned} &\beta = \alpha_\infty(\omega, \kappa) = \omega + r_\varepsilon(\omega, \kappa) \quad \Longleftrightarrow \omega = \alpha^{-1}_\infty(\beta, \kappa) = \beta + \tilde{r}_\varepsilon(\beta, \kappa) \\ &\quad \text{with} \quad |\tilde{r}_\varepsilon|^{k_0,\gamma} \leq C\varepsilon\gamma^{-(1+k_1)}. \end{aligned}$$

We then consider the perturbed tangential frequency vector

$$\omega_\varepsilon(\kappa) := \alpha^{-1}_\infty(\vec{\omega}(\kappa), \kappa) = \vec{\omega}(\kappa) + \mathtt{r}_\varepsilon(\kappa), \quad \mathtt{r}_\varepsilon(\kappa) := \tilde{r}_\varepsilon(\vec{\omega}(\kappa), \kappa), \tag{2.17}$$

which satisfies

$$|\partial^k_\kappa \mathtt{r}_\varepsilon(\kappa)| \leq C\varepsilon\gamma^{-(1+k_1+k)}, \quad \forall 0 \leq k \leq k_0.$$

We also denote, with a small abuse of notations, the perturbed normal Floquet exponents

$$\mu_j^\infty(\kappa) := \mu_j^\infty(\omega_\varepsilon(\kappa), \kappa) := \mathtt{m}_3^\infty(\kappa) j^{\frac{1}{2}}(1+\kappa j^2)^{\frac{1}{2}} + \mathtt{m}_1^\infty(\kappa) j^{\frac{1}{2}} + r_j^\infty(\kappa), \quad \forall j \in \mathbb{N}^+ \setminus \mathbb{S}^+, \tag{2.18}$$

where

$$\mathtt{m}_3^\infty(\kappa) := \mathtt{m}_3^\infty(\omega_\varepsilon(\kappa), \kappa), \quad \mathtt{m}_1^\infty(\kappa) := \mathtt{m}_1^\infty(\omega_\varepsilon(\kappa), \kappa), \quad r_j^\infty(\kappa) := r_j^\infty(\omega_\varepsilon(\kappa), \kappa).$$

For proving the measure estimate of Theorem 2.2, the key point is to prove the following transversality property.

Proposition 2.3 (Transversality). *There exist $k_0 \in \mathbb{N}$, $\rho_0 > 0$ such that, for any $\kappa \in [\kappa_1, \kappa_2]$,*

$$\max_{k \le k_0} |\partial_\kappa^k \{\vec{\omega}(\kappa) \cdot \ell\}| \ge \rho_0 \langle \ell \rangle, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\},$$
$$\max_{k \le k_0} |\partial_\kappa^k \{\vec{\omega}(\kappa) \cdot \ell + \Omega_j(\kappa)\}| \ge \rho_0 \langle \ell \rangle, \quad \forall \ell \in \mathbb{Z}^\nu,\ j \in \mathbb{N}^+ \setminus \mathbb{S}^+,$$
$$\max_{k \le k_0} |\partial_\kappa^k \{\vec{\omega}(\kappa) \cdot \ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa)\}| \ge \rho_0 \langle \ell \rangle, \quad \forall (\ell, j, j') \ne (0, j, j),\ j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+,$$
$$\max_{k \le k_0} |\partial_\kappa^k \{\vec{\omega}(\kappa) \cdot \ell + \Omega_j(\kappa) + \Omega_{j'}(\kappa)\}| \ge \rho_0 \langle \ell \rangle, \quad \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+.$$

We call (following [34]) ρ_0 the "amount of non-degeneracy" and k_0 the "index of nondegeneracy".

The above conditions are stable under perturbations which are small in $\mathcal{C}^{k_0}$-norm, and, therefore, hold for the perturbed tangential frequency $\omega_\varepsilon(\kappa)$ defined in (2.17) and the perturbed Floquet exponents $\mu_j(\kappa)$ introduced in (2.18). It follows that

Lemma 2.4. *For ε small enough, for all $\kappa \in [\kappa_1, \kappa_2]$,*

$$\max_{k \le k_0} |\partial_\kappa^k \{\omega_\varepsilon(\kappa) \cdot \ell\}| \ge \rho_0 \langle \ell \rangle / 2, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\},$$
$$\max_{k \le k_0} |\partial_\kappa^k \{\omega_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\kappa)\}| \ge \rho_0 \langle \ell \rangle / 2, \quad \forall \ell \in \mathbb{Z}^\nu,\ j \in \mathbb{N}^+ \setminus \mathbb{S}^+,$$
$$\max_{k \le k_0} |\partial_\kappa^k \{\omega_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\kappa) - \mu_{j'}^\infty(\kappa)\}| \ge \rho_0 \langle \ell \rangle / 2, \quad \forall (\ell, j, j') \ne (0, j, j),\ j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+,$$
$$\max_{k \le k_0} |\partial_\kappa^k \{\omega_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\kappa) + \mu_{j'}^\infty(\kappa)\}| \ge \rho_0 \langle \ell \rangle / 2, \quad \forall \ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+.$$

As a consequence, the classical Rüssmann lemma (Theorem 17.1 in [34]) implies that the measure of the $\kappa \in [\kappa_1, \kappa_2]$ where

$$|\omega_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\omega_\varepsilon(\kappa), \kappa) - \varsigma \mu_{j'}^\infty(\omega_\varepsilon(\kappa), \kappa)| < \frac{4\gamma |j^{\frac{3}{2}} - \varsigma j'^{\frac{3}{2}}|}{\langle \ell \rangle^\tau},$$
$$\ell \in \mathbb{Z}^\nu, j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+, \varsigma = \pm 1,$$

has measure $O(\gamma^{1/k_0})$, and similarly for the other Melnikov conditions in (2.14). Together with the asymptotic of the eigenvalues, this implies that the Cantor-like set of non-resonant parameters $\mathcal{G}_\varepsilon$ defined in (2.16) has asymptotically full measure.

The transversality Proposition 2.3 is proved by arguments of degenerate KAM theory as developed in [5]. It uses in an essential way that the unperturbed frequencies $\kappa \mapsto \omega_j(\kappa)$ are *analytic*, are simple (on the subspace of the even functions), grow asymptotically as $j^{3/2}$ and are *non-degenerate* in the following sense:

Definition 2.1. A function $f := (f_1, \dots, f_N) : [\kappa_1, \kappa_2] \to \mathbb{R}^N$ is non-degenerate if, for any $c := (c_1, \dots, c_N) \in \mathbb{R}^N \setminus \{0\}$, the function $f \cdot c = f_1 c_1 + \dots + f_N c_N$ is not identically zero on the whole interval $[\kappa_1, \kappa_2]$.

From a geometric point of view, f non-degenerate means that the image of the curve $f([\kappa_1, \kappa_2]) \subset \mathbb{R}^N$ is not contained in any hyperplane of $\mathbb{R}^N$. For such reason, a curve f which satisfies the non-degeneracy property of Definition 2.1 is also referred as an *essentially non-planar* curve, or a curve with *full torsion*. Proposition 2.3 is deduced by the following non-degeneracy properties.

Lemma 2.5. *The frequency vectors* $\vec{\omega}(\kappa) \in \mathbb{R}^\nu$, $(\sqrt{\kappa}, \vec{\omega}(\kappa)) \in \mathbb{R}^{\nu+1}$ *and*

$$(\vec{\omega}(\kappa), \Omega_j(\kappa)) \in \mathbb{R}^{\nu+1}, j \in \mathbb{N}^+ \backslash \mathbb{S}^+, \quad (\vec{\omega}(\kappa), \Omega_j(\kappa), \Omega_{j'}(\kappa)) \in \mathbb{R}^{\nu+2},$$
$$\forall j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+, \ j \neq j',$$

are non-degenerate.

This lemma is proved by a direct verification. Setting $\lambda_0(\kappa) := \sqrt{\kappa}$ and $\lambda_j(\kappa) := \sqrt{j(1 + \kappa j^2)}$, $j \geq 1$, it is sufficient to show that, for any N, for any $0 \leq j_1 < \dots < j_N$, the matrix

$$\mathcal{A}(\kappa) := \begin{pmatrix} \lambda_{j_1}(\kappa) & \dots & \lambda_{j_N}(\kappa) \\ \partial_\kappa \lambda_{j_1}(\kappa) & \dots & \partial_\kappa \lambda_{j_N}(\kappa) \\ \vdots & \ddots & \vdots \\ \partial_\kappa^{N-1} \lambda_{j_1}(\kappa) & \dots & \partial_\kappa^{N-1} \lambda_{j_N}(\kappa) \end{pmatrix}$$

is non-singular at some value of $\kappa \in [\kappa_1, \kappa_2]$. Actually, $\mathcal{A}(\kappa)$ turns out to be non-singular for all $\kappa \in [\kappa_1, \kappa_2]$. A direct computation shows that $\det \mathcal{A}(\kappa)$ is proportional to a Van der Monde determinant.

2.3. Analysis of the linearized operators

In addition to the previous bifurcation analysis, the other key step for the Nash–Moser iterative scheme is to prove that the operator $d_{\alpha,i}\mathcal{F}$ obtained linearizing (2.9) at any approximate solution is, for most values of the parameters (ω, κ), invertible, and that its inverse satisfies *tame* estimates in Sobolev spaces.

The linearized operator $d_{\alpha,i}\mathcal{F}$ is quite complicated because all the components (θ, I, z) components in the system (2.9) are coupled among them. Therefore, we first implement the procedure developed in Berti–Bolle [10], and used in [3], which consists in introducing a convenient set of symplectic variables near the approximate invariant torus such that the linearized equations become (approximately) decoupled in the action–angle components and the normal one. As a consequence, the problem reduces to "almost-approximately" invert a quasi-periodic linear operator restricted to the normal directions. Actually, since this symplectic change of variables modifies, up to a translation, only the finite-dimensional action component, this operator turns out to be just the linearized water-waves system in the original coordinates, restricted to the normal directions. More precisely,

$$\Pi_{\mathbb{S}^+}^{\perp}\mathcal{L}_{|H_{\mathbb{S}^+}^{\perp}} \qquad \text{where} \qquad \mathcal{L} := \omega\cdot\partial_\varphi \mathbb{I}_2 - J\partial_u\nabla_u H(U(\varphi))$$

is obtained linearizing (1.4), (1.7) at an approximate solution $U(\varphi) = (\eta, \psi)(\varphi, x)$, changing $\partial_t \rightsquigarrow \omega\cdot\partial_\varphi$, and denoting the 2×2-identity matrix by

$$\mathbb{I}_2 := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Using formula (1.5), the linearized operator $\mathcal{L}$ is computed to be

$$\mathcal{L} = \omega\cdot\partial_\varphi \mathbb{I}_2 + \begin{pmatrix} \partial_x V + G(\eta)B & -G(\eta) \\ (1+BV_x) + BG(\eta)B - \kappa\partial_x c\partial_x & V\partial_x - BG(\eta) \end{pmatrix} \tag{2.19}$$

where the functions $B := B(\varphi, x)$, $V := V(\varphi, x)$ are defined by (1.6) and $c := c(\varphi, x) := (1+\eta_x^2)^{-3/2}$. The operator $\mathcal{L}$ is real, even and reversible.

Notation. In (2.19) and hereafter, any function a is identified with the corresponding multiplication operators $h \mapsto ah$, and, where there is no parenthesis, composition of operators is understood. For example, $\partial_x c\partial_x$ means: $h \mapsto \partial_x(c\partial_x h)$.

The key part of the analysis consists, now, in diagonalizing (actually it is sufficient to "almost" diagonalize) the quasi-periodic linear operator $\mathcal{L}$, via linear changes of variables close to the identity, which map Sobolev spaces into itself and satisfy tame estimates. These changes of variables have two well-different tasks:

1. Transform $\mathcal{L}$ to an operator of the form (2.20) which has constant coefficients up to pseudo-differential remainders of order zero (actually more regularizing on the off-diagonal terms). These steps are exposed in Sects. 2.3.1–2.3.5.
2. Reduce quadratically the size of the perturbative terms $\mathcal{R}$, $\mathcal{Q}$, see Sect. 2.3.6.

For the search of periodic solutions, i.e. [1,19,20,23,29], there is no need to perform the task 2, because it is possible to invert the linearized operator in (2.20) simply by a Neumann-argument. Indeed, for periodic solutions, a sufficiently regularizing operator in the space variable is also regularizing in the time variable, on the characteristic Fourier indices which correspond to the small divisors. This is clearly not true for quasi-periodic solutions. That is why we will completely diagonalize the linear operator in (2.20) by a KAM scheme. For that, we need to analyze more in detail the pseudo-differential nature of the remainders after each conjugation step.

The approximate solution $U(\varphi, x)$ at which we linearize is assumed to be bounded in a low Sobolev norm (as it is satisfied by any approximate solutions along the Nash–Moser iteration). Moreover, $U(\varphi, x)$ is supposed to be $\mathcal{C}^\infty(\mathbb{T}^\nu_\varphi \times \mathbb{T}_x)$ because, along the Nash–Moser iteration, each approximate solution is actually a trigonometric polynomial in (φ, x) (with clearly more and more harmonics). As a consequence, all the coefficients of the linearized operator $\mathcal{L}$ in (2.19) are $\mathcal{C}^\infty$. This allows to work in the usual framework of $\mathcal{C}^\infty$ pseudo-differential symbols. For the Nash–Moser convergence, we shall then perform quantitative estimates in Sobolev spaces.

2.3.1. Reduction of $\mathcal{L}$ to constant coefficients in decreasing symbols. The goal of the first steps is to reduce $\mathcal{L}$ to a quasi-periodic linear operator of the form

$$(h,\bar h) \mapsto \Big(\omega\cdot\partial_\varphi + \mathrm{i}\mathtt{m}_3 T(D) + \mathrm{i}\mathtt{m}_1 |D|^{\frac12}\Big)\, h + \mathcal{R}h + \mathcal{Q}\bar h, \quad h\in\mathbb{C}, \tag{2.20}$$

where $\mathtt{m}_3, \mathtt{m}_1 \in \mathbb{R}$ are constants coefficients, satisfying $\mathtt{m}_3 \approx 1$, $\mathtt{m}_1 \approx 0$, the principal symbol operator

$$T(D) = |D|^{1/2}(1-\kappa\partial_{xx})^{1/2},$$

and the remainders $\mathcal{R} := \mathcal{R}(\varphi)$, $\mathcal{Q} := \mathcal{Q}(\varphi)$ are small bounded operators acting in the Sobolev spaces H^s, which satisfy tame estimates. More precisely, in view of a KAM reducibility scheme that will completely diagonalize the operator (2.20) (Sect. 2.3.6), we need that all the derivatives

$$\partial_\varphi^\beta \partial_{\omega,\kappa}^k \mathcal{R}, \quad \partial_\varphi^\beta \partial_{\omega,\kappa}^k \mathcal{Q}, \quad |\beta|\le\beta_0, \ |k|\le k_0, \tag{2.21}$$

for β_0 large enough (depending on the Diophantine exponent τ), satisfy tame estimates.

2.3.2. Symmetrization and space–time reduction of $\mathcal{L}$ at the highest order. The first part of the analysis is similar to Alazard–Baldi [1]. We first conjugate the linear operator $\mathcal{L}$ in (2.19) by the change of variable

$$\mathcal{Z} := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad \mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

obtaining

$$\mathcal{L}_0 := \mathcal{Z}^{-1}\mathcal{L}\mathcal{Z} = \omega\cdot\partial_\varphi \mathbb{I}_2 + \begin{pmatrix} \partial_x V & -G(\eta) \\ a-\kappa\partial_x c\partial_x & V\partial_x \end{pmatrix} \tag{2.22}$$

for some function $a(\varphi,x)$. This step amounts to introduce (a linearized version of) the “good unknown of Alinhac”.

As a second step, we conjugate $\mathcal{L}_0$ with an operator of the form $\mathcal{S}\mathcal{Q}\mathcal{B}$, where $\mathcal{B}$ is a change of variable

$$(\mathcal{B}h)(\varphi,x) := h(\varphi, x+\beta(\varphi,x)) \tag{2.23}$$

induced by a φ-dependent family of diffeomorphisms of the torus

$$y = x+\beta(\varphi,x) \qquad \Leftrightarrow \qquad x = y+\tilde\beta(\varphi,y), \tag{2.24}$$

$\mathcal{Q}$ is a matrix-valued multiplication operator

$$\mathcal{Q} := \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \qquad \mathcal{Q}^{-1} := \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix},$$

for a function $q(\varphi,x)$ close to 1, and $\mathcal{S}$ is the vector-valued Fourier multiplier

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}, \quad G := |D|^{-\frac12}(1+\kappa D^2)^{\frac12} \in OPS^{1/2}.$$

Choosing properly the small periodic functions $\beta(\varphi,x)$ and $q(\varphi,x)-1$, one gets

$$\mathcal{L}_1 = \mathcal{S}^{-1}\mathcal{Q}^{-1}\mathcal{B}^{-1}\mathcal{L}_0\mathcal{B}\mathcal{Q}\mathcal{S}$$
$$= \omega\cdot\partial_\varphi \mathbb{I}_2 + \begin{pmatrix} a_1\partial_x + a_2 & -m_3(\varphi)T(D)+\sqrt{\kappa}\, a_3\mathcal{H}|D|^{\frac12}+\mathcal{R}_{1,B} \\ m_3(\varphi)T(D)-\frac{a_4}{\sqrt{\kappa}}|D|^{\frac12}\mathcal{H}+m_3(\varphi)\pi_0+\mathcal{R}_{1,C} & a_1\partial_x+\mathcal{R}_{1,D}\end{pmatrix} \tag{2.25}$$

for suitable functions a_1, a_2, a_3, a_4 and pseudo-differential operators remainders $\mathcal{R}_{1,B}$, $\mathcal{R}_{1,C}$, $\mathcal{R}_{1,D} \in OPS^0$ which are $O(\varepsilon)$ small, in low Sobolev norm. All the coefficients and the operators depend in a tame way, i.e. at most linearly, in the high Sobolev norm of the approximate solution $\|u\|_{s+\sigma}$ with a possible fixed loss of derivatives σ. Note that the coefficient $m_3(\varphi)$ of the highest order operator $\mathcal{L}_1$ in (2.25) is independent of the space variable. The operator π_0 is the L^2 projector of the constants, that, for simplicity of exposition, we neglect in the sequel.

We then write $\mathcal{L}_1$ as an operator acting on the complex variables

$$h := \eta + \mathrm{i}\psi, \quad \bar h := \eta - \mathrm{i}\psi,$$

obtaining

$$\begin{aligned}\mathcal{L}_1 &= \omega\cdot\partial_\varphi\mathbb{I}_2 + \mathrm{i}m_3(\varphi)\mathbf{T}(D) + \mathbf{A}_1(\varphi,x)\partial_x + \mathrm{i}(\mathbf{A}_0^{(I)}(\varphi,x) \\ &\quad + \mathbf{A}_0^{(II)}(\varphi,x))\mathcal{H}|D|^{\frac12} + \mathbf{R}_1^{(I)} + \mathbf{R}_1^{(II)}\end{aligned} \tag{2.26}$$

where

$$\mathbf{T}(D) := \begin{pmatrix} T(D) & 0 \\ 0 & -T(D)\end{pmatrix}, \quad \mathbf{A}_1(\varphi,x) := \begin{pmatrix} a_1(\varphi,x) & 0 \\ 0 & a_1(\varphi,x)\end{pmatrix},$$

$$\mathbf{A}_0^{(I)}(\varphi,x) := \begin{pmatrix} a_5(\varphi,x) & 0 \\ 0 & -a_5(\varphi,x)\end{pmatrix},$$
$$\mathbf{A}_0^{(II)}(\varphi,x) := \begin{pmatrix} 0 & a_6(\varphi,x) \\ -a_6(\varphi,x) & 0\end{pmatrix},$$

and

$$\mathbf{R}_1^{(I)} := \begin{pmatrix} r_1^{(I)}(x,D) & 0 \\ 0 & \overline{r_1^{(I)}(x,D)}\end{pmatrix},$$
$$\mathbf{R}_1^{(II)} := \begin{pmatrix} 0 & r_1^{(II)}(x,D) \\ \overline{r_1^{(II)}(x,D)} & 0\end{pmatrix} \in OPS^0 \tag{2.27}$$

are $O(\varepsilon)$-pseudo-differential operators. Note that $\mathcal{L}_1$ in (2.26) is block-diagonal (in $(u,\bar u)$) up to order $|D|^{1/2}$.

The next step is to remove the dependence on φ from the highest order term $\mathrm{i}m_3(\varphi)\mathbf{T}(D)$, by applying a quasi-periodic time reparametrization

$$\mathcal{P}\,\mathbb{I}_2 = \begin{pmatrix}\mathcal{P} & 0 \\ 0 & \mathcal{P}\end{pmatrix}, \quad (\mathcal{P}h)(\varphi,x) := h(\varphi+\omega p(\varphi),x),$$

induced by the diffeomorphism

$$\vartheta := \varphi + \omega p(\varphi) \quad \Leftrightarrow \quad \varphi = \vartheta + \omega\tilde p(\vartheta)$$

where $p(\varphi)$ is a small periodic function. Choosing properly p and assuming ω to be Diophantine, we get

$$\begin{aligned}\mathcal{L}_2 := \omega\cdot\partial_\varphi \mathbb{I}_2 + \mathtt{im}_3 \mathbf{T}(D) + \mathbf{B}_1(\varphi,x)\partial_x \\ +\mathrm{i}\left(\mathbf{B}_0^{(I)}(\varphi,x) + \mathbf{B}_0^{(II)}(\varphi,x)\right)\mathcal{H}|D|^{\frac12} + \mathbf{R}_2^{(I)} + \mathbf{R}_2^{(II)}\end{aligned} \tag{2.28}$$

where

$$\mathbf{B}_1 = \begin{pmatrix} a_7(\varphi,x) & 0 \\ 0 & a_7(\varphi,x)\end{pmatrix}, \quad \mathbf{B}_0^{(I)} = \begin{pmatrix} a_8(\varphi,x) & 0 \\ 0 & -a_8(\varphi,x)\end{pmatrix},$$
$$\mathbf{B}_0^{(II)} = \begin{pmatrix} 0 & a_9(\varphi,x) \\ a_9(\varphi,x) & 0\end{pmatrix}$$

and $\mathbf{R}_2^{(I)}$ $\mathbf{R}_2^{(II)}$ are $O(\varepsilon)$ pseudo-differential operators $\mathbf{R}_2^{(I)}$ $\mathbf{R}_2^{(II)}$. All the previous transformations are real, even, and reversibility-preserving, so that $\mathcal{L}_2$ is a real, even and reversible operator.

From this point, we have to proceed quite differently with respect to [1].

2.3.3. Block-decoupling. The next step is to conjugate the operator $\mathcal{L}_2$ in (2.28) to an operator of the form

$$\begin{aligned}\mathcal{L}_M := \mathbf{\Phi}_M^{-1}\mathcal{L}_2\mathbf{\Phi}_M = \omega\cdot\partial_\varphi\mathbb{I}_2 \\ +\mathtt{im}_3\mathbf{T}(D) + \mathbf{B}_1(\varphi,x)\partial_x + \mathrm{i}\mathbf{B}_0^{(I)}(\varphi,x)\mathcal{H}|D|^{\frac12} + \mathbf{R}_M^{(I)} + \mathbf{R}_M^{(II)}\end{aligned} \tag{2.29}$$

where the remainders

$$\mathbf{R}_M^{(I)} := \begin{pmatrix} r_M^{(I)}(\varphi,x,D) & 0 \\ 0 & \overline{r_M^{(I)}(\varphi,x,D)}\end{pmatrix} \in OPS^0,$$
$$\mathbf{R}_M^{(II)} := \begin{pmatrix} 0 & \mathcal{R}_M^{(II)} \\ \overline{\mathcal{R}}_M^{(II)} & 0\end{pmatrix} \in OPS^{\frac12 - M}$$

are ε small. This is achieved by applying iteratively M-times a conjugation map which transforms the off-diagonal block operators into 1-smoother ones. Notice that the operator $\mathcal{L}_M$ in (2.29) is block-diagonal up to the smoothing remainder $\mathbf{R}_M^{(II)} \in OPS^{\frac12-M}$. The coefficients of $\mathbf{R}_M^{(II)}$ depend on $O(M)$-derivatives of the approximate solution. In any case, the number of regularizing steps M will be fixed (independently on s, depending just on the Diophantine exponent τ), determined by the KAM reducibility scheme.

2.3.4. Egorov analysis. Space reduction of the order ∂_x. The goal is now to eliminate the first-order vector field $\mathbf{B}_1(\varphi,x)\partial_x$ from $\mathcal{L}_M$. We conjugate $\mathcal{L}_M$ by the flow

$$\mathbf{\Phi}(\varphi,t) := \begin{pmatrix} \Phi(\varphi,t) & 0 \\ 0 & \overline{\Phi}(\varphi,t)\end{pmatrix}$$

generated by the system

$$\partial_t \begin{pmatrix} u \\ \overline{u}\end{pmatrix} = \mathrm{i}\begin{pmatrix} a(\varphi,x) & 0 \\ 0 & -a(\varphi,x)\end{pmatrix}|D|^{\frac12}\begin{pmatrix} u \\ \overline{u}\end{pmatrix} \tag{2.30}$$

where $a(\varphi, x)$ is a small *real* valued function to be determined. Thus, $\Phi(\varphi, t)$ is the flow of the scalar linear pseudo-PDE

$$\partial_t u = \mathrm{i} a(\varphi, x)|D|^{\frac{1}{2}} u. \tag{2.31}$$

Conjugating the operator $\mathcal{L}_M$ in (2.29) by the time one flow operator $\mathbf{\Phi}(\varphi) := \mathbf{\Phi}(\varphi, 1)$, we get

$$\begin{aligned}\mathcal{L}_M^{(1)} &= \mathbf{\Phi}\mathcal{L}_M\mathbf{\Phi}^{-1} = \omega\cdot\partial_\varphi\mathbb{I}_2 + \mathbf{\Phi}(\varphi)\mathbf{P}_0(\varphi, x, D)\mathbf{\Phi}(\varphi)^{-1} \\ &\quad + \mathbf{\Phi}(\varphi)\omega\cdot\partial_\varphi\{\mathbf{\Phi}(\varphi)^{-1}\} + \mathbf{\Phi}\mathbf{R}_M^{(II)}\mathbf{\Phi}^{-1}\end{aligned}$$

where we have denoted

$$\mathbf{P}_0(\varphi, x, D) = \mathrm{i}\mathtt{m}_3\mathbf{T}(D) + \mathbf{B}_1(\varphi, x)\partial_x + \mathrm{i}\mathbf{B}_0^{(I)}(\varphi, x)\mathcal{H}|D|^{\frac{1}{2}} + \mathbf{R}_M^{(I)}$$

the diagonal part of $\mathcal{L}_M$. Note that the terms $\mathbf{\Phi}(\varphi)\mathbf{P}_0(\varphi, x, D)\mathbf{\Phi}(\varphi)^{-1}$ and $\mathbf{\Phi}(\varphi)\,\omega\cdot\partial_\varphi\{\mathbf{\Phi}(\varphi)^{-1}\}$ are block-diagonal. They are classical pseudo-differential operators and can be analyzed by an Egorov-type argument. On the other hand, the off-diagonal term $\mathbf{\Phi}\mathbf{R}_M^{(II)}\mathbf{\Phi}^{-1}$ is very regularizing and satisfies tame estimates. Let us see how evolves the operator

$$\begin{aligned}\mathbf{P}(\varphi, t) &= \mathbf{\Phi}(\varphi, t)\mathbf{P}_0\mathbf{\Phi}(\varphi, t)^{-1} = \begin{pmatrix} P(\varphi, t) & 0 \\ 0 & \overline{P}(\varphi, t)\end{pmatrix}, \\ P(\varphi, t) &:= \Phi(\varphi, t)p_0(\varphi, x, D)\Phi^{-1}(\varphi, t).\end{aligned} \tag{2.32}$$

under the flow of (2.30). The operator $P(\varphi, t)$ solves the usual Heisenberg equation

$$\begin{cases}\partial_t P(\varphi, t) = \mathrm{i}[A(\varphi), P(\varphi, t)] \\ P(\varphi, 0) = P_0 := p_0(\varphi, x, D)\end{cases} \quad \text{where} \quad A(\varphi) = a(\varphi, x)|D|^{\frac{1}{2}}. \tag{2.33}$$

The Eq. (2.33) can be solved in decreasing symbols using the fact that the order of the commutator $[A(\varphi), Q(\varphi)]$ with a classical pseudo-differential operator Q is strictly less than the order of Q. More precisely, (2.33) has an approximate solution $Q(\varphi, t) := q(t, \varphi, x, D)$ expanded in decreasing orders

$$q(t, \varphi, x, \xi) = \sum_{n=0}^{M} q_n(t, \varphi, x, \xi), \quad q_n(t, \varphi, x, \xi) \in S^{\frac{1}{2}(3-n)}, \quad \forall n = 0, \ldots, M \tag{2.34}$$

where $q_0 = p_0$ and the other lower order symbols q_n are recursively computed. This shows that the diagonal term $P(\varphi, t)$ remains pseudo-differential along the conjugation. One can analyze $\mathbf{\Phi}(\varphi)\omega\cdot\partial_\varphi\{\mathbf{\Phi}(\varphi)^{-1}\}$ in the same way.

As an outcome, choosing properly the function $a(\varphi, x)$, and using the fact that the operator $\mathcal{L}_2$ is even, one can eliminate the order ∂_x, getting an operator of the form

$$\mathcal{L}_M^{(1)} = \omega\cdot\partial_\varphi\mathbb{I}_2 + \mathrm{i}\mathtt{m}_3\mathbf{T}(D) + \mathrm{i}\left(\mathbf{C}_1(\varphi, x) + \mathbf{C}_0(\varphi, x)\mathcal{H}\right)|D|^{\frac{1}{2}} + \mathbf{R}_M^{(1)} + \mathbf{Q}_M^{(1)} \tag{2.35}$$

where

$$\mathbf{C}_1(\varphi, x) := \begin{pmatrix} a_{10} & 0 \\ 0 & -a_{10}\end{pmatrix}, \quad \mathbf{C}_0(\varphi, x) := \begin{pmatrix} a_{11} & 0 \\ 0 & -a_{11}\end{pmatrix}.$$

Remark 2.6. Alazard–Baldi [1] uses a semi-Fourier integral operator like $\mathrm{Op}(e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}}) \in OPS^0_{\frac12,\frac12}$. The use of the flow $\Phi(\varphi)$ of (2.31) is simpler because the proof that Φ, as well as its inverse Φ^{-1}, is a bounded operator on Sobolev spaces H^s and satisfies tame estimates, follows by simple energy estimates (the vector field $\mathrm{i}a(\varphi,x)|D|^{1/2}$ is skew-adjoint at the highest order, see Appendix of [7]).

The fact that the diagonal terms of the conjugated operator (2.35) are still pseudo-differential is a relevant information. Indeed, the flow $\Phi(\varphi) \sim \mathrm{Op}(e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}})$ maps Sobolev spaces in itself, but each derivative

$$\partial_\varphi \Phi(\varphi) \sim \mathrm{Op}\left(e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}}\,\mathrm{i}\partial_\varphi a(\varphi,x)\sqrt{|\xi|}\right)$$

is an unbounded operator which loses $|D|^{1/2}$ derivatives. Actually, $\partial^k_{\omega,\kappa}\partial^\beta_\varphi \Phi(\varphi)$ loses $|D|^{\frac{|\beta|+|k|}{2}}$ derivatives. Since the conjugated operator

$$P_+(\varphi) := \Phi(\varphi)P_0\Phi(\varphi)^{-1} = \mathrm{Op}(c(\varphi,x,\xi)), \quad c(\varphi,x,\xi) \in S^m, \tag{2.36}$$

is a classical pseudo-differential operator, the differentiated operator

$$\partial_\varphi P_+(\varphi) = \mathrm{Op}(\partial_\varphi c(\varphi,x,\xi)) \in OPS^m$$

is still a pseudo-differential operator of the same order of P_0 with just a symbol $\partial_\varphi c$ less regular in φ. The loss of regularity for $\partial_\varphi c$ may be compensated by the usual Nash–Moser smoothing procedure in φ. This is the reason why we require that the diagonal remainder $\mathcal{R} \in OPS^0$ is just of order zero.

On the other hand, the off-diagonal term $\mathcal{Q}_M \in OPS^{-M}$ evolves, under the flow $\mathbf{\Phi}(\varphi,t)$, according to the "skew-Heisenberg" equation obtained replacing in (2.33) the commutator with the skew-commutator. As a consequence, the symbol of $\mathcal{Q}^+_M := \Phi(\varphi)\mathcal{Q}_M\Phi(\varphi)^{-1}$ assumes the form $e^{\mathrm{i}a(\varphi,x)\sqrt{|\xi|}}q(\varphi,x,\xi)$, where $q(\varphi,x,\xi) \in S^{-M}$ is a classical symbol (actually we do not prove it explicitly because it is not needed). Thus, the action of each ∂_φ on $\mathcal{Q}^+_M$ produces an operator which loses $|D|^{\frac12}$ derivatives in space more than $\mathcal{Q}_M$. This is why we have performed previously a large number M of regularizing steps for the off-diagonal components $\mathcal{Q}$.

2.3.5. Space reduction of the order $|D|^{1/2}$. Finally, we eliminate the x-dependence of the coefficient in front of $|D|^{\frac12}$ in the operator $\mathcal{L}^{(1)}_M$ in (2.35), conjugating $\mathcal{L}^{(1)}_M$ by a matrix-valued multiplication operator of the form

$$\mathbf{V} := \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \overline{\mathcal{V}} \end{pmatrix}, \qquad \mathcal{V} := \mathrm{Op}(v), \qquad v := v(\varphi,x,\xi) \in S^0.$$

Choosing properly the function $v(\varphi,x)$, one finally gets

$$\mathcal{L}^{(2)}_M := \omega\cdot\partial_\varphi \mathbb{I}_2 + \mathrm{i}\mathtt{m}_3\mathbf{T}(D) + \mathrm{i}\mathtt{m}_1\mathbf{\Sigma}|D|^{\frac12} + \mathbf{R}^{(2)}_M + \mathbf{Q}^{(2)}_M, \quad \text{where} \quad \mathbf{\Sigma} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.37}$$

which has the form stated in (2.20). We remark that all the previous transformations are real, even, and reversibility-preserving, so that $\mathcal{L}^{(2)}_M$ is real, even and reversible.

2.3.6. KAM-reducibility scheme. We are now in position to apply an iterative quadratic scheme to reduce the *size* of the terms $\mathbf{R}_M^{(2)}$ and $\mathbf{Q}_M^{(2)}$ (if possible). Let us explain the main idea. Consider a linear real, even, and reversible operator acting on $H_{\mathbb{S}_0}^{\perp}$,

$$\mathbf{L} = \omega\cdot\partial_\varphi \mathbb{I}_2^{\perp} + \mathbf{D} + \varepsilon \mathbf{P}, \tag{2.38}$$

with diagonal part (with respect to the exponential basis)

$$\mathbf{D} = \begin{pmatrix} \mathrm{i}\Lambda & 0 \\ 0 & -\mathrm{i}\Lambda \end{pmatrix}, \quad \Lambda = \mathrm{diag}_j(\mu_j), \quad \mu_j = \mathtt{m}_3\sqrt{j(1+\kappa j^2)} + \mathtt{m}_1 j^{\frac{1}{2}} + r_j,$$
$$\sup_j |r_j| = O(\varepsilon)$$

with $r_j \in \mathbb{R}$ (at the first step, $\mu_j = \mathtt{m}_3\sqrt{j(1+\kappa j^2)} + \mathtt{m}_1 j^{\frac{1}{2}}$ by (2.37)), and a bounded perturbation

$$\mathbf{P} = \begin{pmatrix} P_1 & P_2 \\ \overline{P}_2 & \overline{P}_1 \end{pmatrix}.$$

Transform $\mathbf{L}$ under the flow $\mathbf{\Phi}(\varphi,\tau)$ generated by a linear system

$$\partial_\tau \begin{pmatrix} u \\ \bar u \end{pmatrix} = \varepsilon \mathbf{W}(\varphi)\begin{pmatrix} u \\ \bar u \end{pmatrix} \qquad \text{where} \qquad \mathbf{W}(\varphi) = \begin{pmatrix} W_1 & W_2 \\ \overline{W}_2 & \overline{W}_1 \end{pmatrix}$$

is a bounded map, to be determined. The conjugated operator $\mathbf{L}(\tau) = \mathbf{\Phi}(\varphi,\tau)\, \mathbf{L}\mathbf{\Phi}(\varphi,\tau)^{-1}$ solves as usual the vector Heisenberg equation

$$\begin{cases} \partial_\tau \mathbf{L}(\tau) = [\varepsilon \mathbf{W}(\varphi), \mathbf{L}(\tau)] \\ \mathbf{L}(0) = \mathbf{L} = \omega\cdot\partial_\varphi + \mathbf{D} + \varepsilon\mathbf{P}. \end{cases}$$

This time, we expand $\mathbf{L}(1)$ in power of ε (Lie series)

$$\begin{aligned} \mathbf{L}(1) &= \mathbf{L}(0) + [\varepsilon\mathbf{W}(\varphi), \mathbf{L}(0)] + O(\varepsilon^2) \\ &= \omega\cdot\partial_\varphi + \mathbf{D} + \varepsilon\left\{\mathbf{P} + \omega\cdot\partial_\varphi \mathbf{W}(\varphi) + [\mathbf{W},\mathbf{D}]\right\} + O(\varepsilon^2). \end{aligned}$$

The goal is to eliminate the ε-term $\omega\cdot\partial_\varphi\mathbf{W}(\varphi) + [\mathbf{W},\mathbf{D}] + \mathbf{P}$ (if possible). This amounts to solve

$$\begin{aligned} &\omega\cdot\partial_\varphi W_1(\varphi) + \mathrm{i}(W_1(\varphi)\Lambda - \Lambda W_1(\varphi)) + P_1(\varphi) = [\![P_1(\varphi)]\!] \\ &\omega\cdot\partial_\varphi W_2(\varphi) + \mathrm{i}(W_2(\varphi)\Lambda + \Lambda W_2(\varphi)) + P_2(\varphi) = 0, \end{aligned}$$

where $[\![P_1(\varphi)]\!] := \mathrm{diag}([P_1(\varphi)]_j^j)$. Expanding in Fourier series $W_i(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu} W_i(\ell)e^{\mathrm{i}\ell\cdot\varphi}$ and $P_i(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu} P_i(\ell)e^{\mathrm{i}\ell\cdot\varphi}$, $i=1,2$, we are led to the following equations:

$$\begin{aligned} &\mathrm{i}\omega\cdot\ell W_1(\ell) + \mathrm{i}(W_1(\ell)\Lambda - \Lambda W_1(\ell)) + P_1(\ell) = [\![P_1(\ell)]\!] \\ &\mathrm{i}\omega\cdot\ell W_2(\ell) + \mathrm{i}(W_2(\ell)\Lambda + \Lambda W_1(\ell)) + P_2(\ell) = 0. \end{aligned}$$

Representing $W_i(\ell) = ([W_i(\ell)]_k^j)_{j,k\in\mathbb{Z}}$ as a matrix, we get the infinitely many scalar equations

$$\begin{aligned} &\mathrm{i}\omega\cdot\ell[W_1(\ell)]_k^j + \mathrm{i}[W_1(\ell)]_k^j(\mu_j - \mu_k) + [P_1(\ell)]_k^j = [P_1(\ell)]_j^k \\ &\mathrm{i}\omega\cdot\ell[W_2(\ell)]_k^j + \mathrm{i}[W_2(\ell)]_k^j(\mu_j + \mu_k) + [P_2(\ell)]_k^j = 0. \end{aligned}$$

These equations admit the solutions

$$[W_1(\ell)]_k^j = \frac{[P_1(\ell)]_k^j}{\mathrm{i}(\omega\cdot\ell+\mu_j-\mu_k)}$$

$$\forall(\ell,j,k)\neq(0,j,j),\quad [W_2(\ell)]_k^j = \frac{[P_2(\ell)]_k^j}{\mathrm{i}(\omega\cdot\ell+\mu_j+\mu_k)},\quad \forall(\ell,j,k)$$

if the corresponding denominators do not vanish. Note that since we look for solutions even in x and the eigenvalues μ_j are simple, $\mu_j-\mu_k\neq 0$ for $j\neq k$ (we could expand on the $\cos(jx)$ basis so that $j,k\in\mathbb{N}^+$). We actually require a quantitative lower bound for the denominators, as

$$|\omega\cdot\ell+\mu_j-\mu_k|\geq\frac{|j^{3/2}-k^{3/2}|}{\gamma\langle\ell\rangle^\tau},\quad |\omega\cdot\ell+\mu_j+\mu_k|\geq\frac{|j^{3/2}+k^{3/2}|}{\gamma\langle\ell\rangle^\tau}.$$

These conditions are called the second-order Melnikov non-resonance conditions and appear in the Cantor set (2.14). After this conjugation step, we have obtained a linear operator of the same form (2.38), but with a smaller $O(\varepsilon^2)$ perturbation and a new diagonal part corrected by the matrix $[\![P_1(0)]\!]=\mathrm{diag}[P_1(0)]_j^j$. Since $\mathbf{P}$ is reversible, then $[P_1(0)]_j^j$ are purely imaginary. To apply the above classical KAM reducibility scheme to the operator $\mathcal{L}_M^{(2)}$ in (2.37), a difficulty is that the remainders $\mathbf{R}_M^{(2)}$, $\mathbf{Q}_M^{(2)}$ just satisfy tame estimates. For technical details of the proof, we refer to [7]. Here, we just mention that for the convergence, we need the tame conditions (2.21). In conclusion, the operator $\mathcal{L}_M^{(2)}$ defined in (2.37) may be conjugated to a diagonal operator of the form

$$\omega\cdot\partial_\varphi\mathbb{I}_2^\perp+\mathrm{i}\mathbf{D}_\infty,\quad \mathbf{D}_\infty=\begin{pmatrix}\mathrm{i}\Lambda_\infty & 0\\ 0 & -\mathrm{i}\Lambda_\infty\end{pmatrix},$$

with

$$\Lambda_\infty=\mathrm{diag}_j(\mu_j^\infty),\quad \mu_j^\infty=\mathtt{m}_3\sqrt{j(1+\kappa j^2)}+\mathtt{m}_1 j^{\frac12}+r_j^\infty,\quad \sup_j r_j^\infty=O(\varepsilon).$$

It is then sufficient to require the first-order Melnikov conditions

$$|\omega\cdot\ell+\mu_j^\infty|\geq 2\gamma j^{\frac32}\langle\ell\rangle^{-\tau},\quad \forall\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{N}\setminus\mathbb{S}^+$$

to prove the invertibility of the diagonal operators $\omega\cdot\partial_\varphi\mathbb{I}_2^\perp+\mathrm{i}\mathbf{D}_\infty$. These conditions appear in (2.14).

Since all the transformations that we have performed above are bounded map between Sobolev spaces, we get the required tame estimates for the inverse.

After the above analysis of the linearized operator, a differentiable Nash–Moser iterative scheme constructs a sequence of better and better approximate solutions of (2.9), finally proving Theorem 2.1.

Acknowledgements

This research was supported by PRIN 2012 "Variational and perturbative aspects of nonlinear differential problems" and partially, by the "Swiss National Science Foundation".

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