

Large KAM tori for perturbations of the dNLS equation

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Abstract. We prove that small, semi-linear Hamiltonian perturbations of the defocusing nonlinear Schrödinger (dNLS) equation on the circle have an abundance of invariant tori of any size and (finite) dimension which support quasi-periodic solutions. When compared with previous results the novelty consists in considering perturbations which do not satisfy any symmetry condition (they may depend on x in an arbitrary way) and need not be analytic. The main difficulty is posed by pairs of almost resonant dNLS frequencies. The proof is based on the integrability of the dNLS equation, in particular the fact that the nonlinear part of the Birkhoff coordinates is one smoothing. We implement a Newton-Nash-Moser iteration scheme to construct the invariant tori. The key point is the reduction of linearized operators, coming up in the iteration scheme, to 2×2 block diagonal ones with constant coefficients together with sharp asymptotic estimates of their eigenvalues.

Keywords: defocusing NLS equation, KAM for PDE, Nash-Moser theory, invariant tori

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1 Introduction

Consider the defocusing nonlinear Schrödinger (dNLS) equation in one space dimension

$$i\partial_t u = -\partial_x^2 u + 2|u|^2 u \quad (1.1)$$

on the standard Sobolev space $H^\sigma \equiv H^\sigma(\mathbb{T}_1, \mathbb{C})$ of complex valued functions on $\mathbb{T}_1 := \mathbb{R}/\mathbb{Z}$. It is well known that for $\sigma \geq 0$, (1.1) is wellposed and for $\sigma \geq 1$, it is a Hamiltonian PDE with Poisson bracket and Hamiltonian given by

$$\{\mathcal{F}, \mathcal{G}\}(u_1, u_2) = -i \int_0^1 (\nabla_u \mathcal{F} \nabla_{\bar{u}} \mathcal{G} - \nabla_{\bar{u}} \mathcal{F} \nabla_u \mathcal{G}) dx, \quad \mathcal{H}^{nls}(u_1, u_2) = \int_0^1 (\partial_x u \partial_x \bar{u} + u^2 \bar{u}^2) dx. \quad (1.2)$$

Here u_1, u_2 are the real valued functions, defined in terms of u by $u_1 = \sqrt{2}\operatorname{Re}(u)$, $u_2 = -\sqrt{2}\operatorname{Im}(u)$, the L^2 -gradients $\nabla_u, \nabla_{\bar{u}}$ are given by $\nabla_u := (\nabla_{u_1} + i\nabla_{u_2})/\sqrt{2}$, $\nabla_{\bar{u}} := (\nabla_{u_1} - i\nabla_{u_2})/\sqrt{2}$, and \mathcal{F}, \mathcal{G} , viewed as functions of u_1 and u_2 , are \mathcal{C}^1 -smooth, real valued functionals on H^σ with sufficiently regular L^2 -gradients. The Hamiltonian vector field corresponding to \mathcal{H}^{nls} can then be computed to be $-i\nabla_{\bar{u}} \mathcal{H}^{nls}$ and when written in Hamiltonian form, equation (1.1) becomes $\partial_t u = -i\nabla_{\bar{u}} \mathcal{H}^{nls}$. According to [19], (1.1) is an integrable PDE in the strongest possible sense, meaning that it admits global Birkhoff coordinates on H^σ , $\sigma \in \mathbb{Z}_{\geq 0}$ – see Subsection 3.1 for more details. In these coordinates, equation (1.1) can be solved by quadrature and the phase space H^σ is the union of compact, connected tori, invariant under the flow of (1.1). All the solutions are periodic, quasi-periodic or almost periodic in time. These invariant tori are parametrized by the action variables $I = (I_k)_{k \in \mathbb{Z}}$, the latter being defined in terms of the Birkhoff coordinates and filling out the whole positive quadrant $\ell_+^{1,2\sigma}$ of the weighted sequence space $\ell^{1,2\sigma} \equiv \ell^{1,2\sigma}(\mathbb{Z}, \mathbb{R})$. The dimension of such a torus, denoted by \mathcal{T}_I , coincides with the cardinality of the index set $S \equiv S_I \subseteq \mathbb{Z}$, given by $S = \{k \in \mathbb{Z} \mid I_k > 0\}$. In case $|S| < \infty$, it can be shown that elements in \mathcal{T}_I are \mathcal{C}^∞ -smooth and that solutions of (1.1) with initial data in \mathcal{T}_I wrap around \mathcal{T}_I with speed, defined in terms of the frequencies $\omega_k^{nls}(I)$, $k \in S$. They are called S -gap solutions.

Our aim is to prove that for Hamiltonian perturbations

$$i\partial_t u = -\partial_x^2 u + 2|u|^2 u + \varepsilon f(x, u) \quad (1.3)$$

of equation (1.1), many of these finite dimensional tori persist, provided that ε is sufficiently small. The perturbation f is assumed to be given by $f(x, u) = \nabla_{\bar{u}} \mathcal{P}$ where \mathcal{P} is a real valued Hamiltonian of the form

$$\mathcal{P}(u) = \int_0^1 p(x, u_1(x), u_2(x)) dx \quad (1.4)$$

and p a real valued function

$$p : \mathbb{T}_1 \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, \zeta_1, \zeta_2) \mapsto p(x, \zeta_1, \zeta_2)$$

which is then related to $f : \mathbb{T}_1 \times \mathbb{C} \rightarrow \mathbb{C}$ by the identity, valid for any $\zeta = (\zeta_1 - i\zeta_2)/\sqrt{2}$ with $\zeta_1, \zeta_2 \in \mathbb{R}$,

$$f(x, \zeta) = \partial_{\bar{\zeta}} p(x, \zeta_1, \zeta_2), \quad \partial_{\bar{\zeta}} := (\partial_{\zeta_1} - i\partial_{\zeta_2})/\sqrt{2}. \quad (1.5)$$

We assume that f is C^{σ, s_*} -smooth, meaning that

$$\partial_x^\alpha \partial_{\zeta_1}^{\beta_1} \partial_{\zeta_2}^{\beta_2} f \in \mathcal{C}(\mathbb{T}_1 \times \mathbb{C}, \mathbb{C}), \quad \forall 0 \leq \alpha \leq \sigma, \quad \forall 0 \leq \beta_1, \beta_2 \leq s_*. \quad (1.6)$$

Note that $f(x, \zeta)$ need not be complex differentiable in ζ . To state our result in detail, introduce for any given $S \subseteq \mathbb{Z}$ with cardinality $|S| < \infty$, the parameter space

$$\Pi_S := \{(\xi_k)_{k \in \mathbb{Z}} \subset \mathbb{R} \mid \xi_k = 0 \forall k \in \mathbb{Z} \setminus S; \xi_k > 0 \forall k \in S\},$$

which we identify with $\mathbb{R}_{>0}^S$. The elements of S are referred to as tangential sites. By the non-degeneracy property (3.9) of Proposition 3.1, the action-to-frequency map

$$\omega^S : \Pi_S \rightarrow \mathbb{R}^S, \quad I \mapsto (\omega_k^{nls}(I))_{k \in S} \quad (1.7)$$

is a local diffeomorphism on an open, dense subset of Π_S . Finally, let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. The main result of this paper is the following one.

Theorem 1.1. *Let $\sigma \in \mathbb{Z}_{\geq 4}$ and $S \subset \mathbb{Z}$ with $|S| < \infty$, $0 \in S$, and $-S = S$ be given and assume that $\Pi \subseteq \Pi_S$ is a compact subset of positive Lebesgue measure, $\text{meas}(\Pi) > 0$, with the property that the action-to-frequency map $\omega^{nls} : \Pi \rightarrow \mathbb{R}^S$, $I \mapsto (\omega_k^{nls}(I))_{k \in S}$, is a bi-Lipschitz homeomorphism onto its image Ω . Then there is an integer $s_* > \max(\sigma, |S|/2)$ so that for any Hamiltonian \mathcal{P} of the form (1.4) with $f = \nabla_{\bar{u}} \mathcal{P}$ of class C^{σ, s_*} , there exist $\varepsilon_0 > 0$ and $|S|/2 < s < s_*$ so that for any $0 < \varepsilon \leq \varepsilon_0$ the following holds: there exist a closed subset $\Omega_\varepsilon \subseteq \Omega$, satisfying*

$$\lim_{\varepsilon \rightarrow 0} \frac{\text{meas}(\Omega_\varepsilon)}{\text{meas}(\Omega)} = 1, \quad (1.8)$$

and a Lipschitz family of maps $\iota_\omega : \mathbb{T}^S \rightarrow H^\sigma$, $\omega \in \Omega_\varepsilon$, so that ι_ω are H^s -smooth embeddings with the property that for any initial data $\varphi \in \mathbb{T}^S$, the curves

$$t \mapsto \iota_\omega(\varphi + t\omega)$$

are quasi-periodic solutions of (1.3). The torus described by the map ι_ω is invariant under the flow of the perturbed Hamiltonian $\mathcal{H}^{nls} + \varepsilon \mathcal{P}$.

In Theorem 4.1 we will show in addition that, for $\omega \in \Omega_\varepsilon$, the distance of the invariant torus $\iota_\omega(\mathbb{T}^S)$ to the unperturbed torus $\mathcal{T}_{\xi(\omega)}$ is of the order $O(\varepsilon\gamma^{-2})$ where $0 < \gamma < 1$ is the constant appearing in the diophantine condition of ω introduced in (1.22). Here $\xi(\omega)$ denotes the element in Π , corresponding to ω by the action-to-frequency map defined in (1.7). Expressing equation (1.3) in suitable coordinates, one sees that actually the distance of the invariant torus to the unperturbed one is $O(\varepsilon\gamma^{-1})$, see Corollary 8.2. Note that the frequency vector ω of the quasi-periodic solution $\iota_\omega(\varphi + t\omega)$ of (1.3) is the same as the one of the quasi-periodic solutions on the invariant torus $\mathcal{T}_{\xi(\omega)}$ of (1.1).

Comments:

1. Using a covering argument one can show that Theorem 1.1 actually holds for any compact subset $\Pi \subseteq \Pi_S$ with $\text{meas}(\Pi) > 0$. See the comment after Theorem 4.1.
2. In Theorem 9.1 we prove that for some $\nu > 0$, $\text{meas}(\Omega \setminus \Omega_\varepsilon) = O(\varepsilon^\nu)$ as $\varepsilon \rightarrow 0$.
3. The assumption $0 \in S$ and $S = -S$ are introduced just for simplicity, so that all elements in the complement $\mathbb{Z} \setminus S$ of S come in pairs, so that in the reduction procedure in section 7 we only have to deal with 2×2 blocks.

4. By (1.6) the perturbation f is assumed to be C^{σ, s^*} -smooth where a lower bound for s_* is given in Theorem 8.1 (Nash-Moser). Note that the regularity with respect to the space variable is just $\sigma \in \mathbb{Z}_{\geq 4}$. No special effort has been made to get optimal lower bounds for s_* and σ .

Outline of the proof of Theorem 1.1 : The starting point of our proof is to write the perturbed dNLS equation (1.3) in complex Birkhoff coordinates $(w_k)_{k \in \mathbb{Z}}$, the latter being briefly reviewed in Subsection 3.1. The dNLS-Hamiltonian \mathcal{H}^{nls} , expressed in these coordinates, is a real analytic function H^{nls} of the actions $I_k = w_k \bar{w}_k$, $k \in \mathbb{Z}$, and the dNLS frequencies ω_k^{nls} are given by

$$\omega_k^{nls} = \partial_{I_k} H^{nls}, \quad k \in \mathbb{Z}.$$

Denoting by P the Hamiltonian \mathcal{P} , expressed in these coordinates, equation (1.3) then becomes the following infinite dimensional Hamiltonian system

$$i\dot{w}_k = \omega_k^{nls} w_k + \varepsilon \partial_{\bar{w}_k} P, \quad k \in \mathbb{Z}, \quad (1.9)$$

on the phase space $h^\sigma \equiv h^\sigma(\mathbb{Z}, \mathbb{C})$, $\sigma \in \mathbb{Z}_{\geq 4}$, where

$$h^\sigma := \left\{ w = (w_k)_{k \in \mathbb{Z}} \subset \mathbb{C} \mid \|w\|_\sigma < \infty \right\}, \quad \|w\|_\sigma := \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2\sigma} |w_k|^2 \right)^{1/2}, \quad \langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}. \quad (1.10)$$

The sequence space h^σ is endowed with the symplectic form $i \sum_{k \in \mathbb{Z}} dw_k \wedge d\bar{w}_k$. Given a finite subset $S \subset \mathbb{Z}$, introduce the space of S -gap potentials,

$$M_S := \{ w = (w_k)_{k \in \mathbb{Z}} \subset \mathbb{C} \mid w_k = 0 \text{ iff } k \in S^\perp \} \subset h^\sigma, \quad S^\perp := \mathbb{Z} \setminus S,$$

which is symplectic. Note that this space is invariant under the flow of (1.9) with $\varepsilon = 0$. On M_S , we introduce the angle-action variables $(\theta, I) := (\theta_k, I_k)_{k \in S} \in \mathbb{T}^S \times \mathbb{R}_{>0}^S$, defined by

$$I_k := w_k \bar{w}_k, \quad w_k = \sqrt{I_k} e^{-i\theta_k}, \quad k \in S$$

and consider the symplectic space

$$\mathbb{T}^S \times \mathbb{R}_{>0}^S \times h_\perp^\sigma, \quad h_\perp^\sigma := \{ z := (z_k)_{k \in S^\perp} \in h^\sigma(S^\perp, \mathbb{C}) \},$$

referring to the coordinates $z_k := w_k$, $k \in S^\perp$, as normal coordinates. On $\mathbb{T}^S \times \mathbb{R}_{>0}^S \times h_\perp^\sigma$, the symplectic form $i \sum_{k \in \mathbb{Z}} dw_k \wedge d\bar{w}_k$ then becomes

$$\Lambda := \sum_{k \in S} d\theta_k \wedge dI_k + i \sum_{k \in S^\perp} dz_k \wedge d\bar{z}_k \quad (1.11)$$

and the Hamiltonian system (1.9) reads

$$\dot{\theta} = \omega^{nls} + \varepsilon \nabla_I P, \quad \dot{I} = -\varepsilon \nabla_\theta P, \quad i\dot{z}_k = \omega_k^{nls} z_k + \varepsilon \partial_{\bar{z}_k} P, \quad \forall k \in S^\perp, \quad (1.12)$$

where $\omega^{nls} = (\omega_k^{nls})_{k \in S}$ and $\omega_k^{nls} = \omega_k^{nls}(I, z\bar{z})$, $k \in \mathbb{Z}$, with $z\bar{z} \equiv (z_k \bar{z}_k)_{k \in S^\perp}$. Here, the Hamiltonian P is viewed as a function of the new coordinates θ, I, z and by a slight abuse of terminology, also made in the sequel in other contexts, $(I, z\bar{z})$ denotes the conveniently regrouped sequence of actions $(w_k \bar{w}_k)_{k \in \mathbb{Z}}$. Note that for any $\xi := (\xi_k)_{k \in S} \in \mathbb{R}_{>0}^S$, the torus

$$\mathcal{T}_\xi := \mathbb{T}^S \times \{I = \xi\} \times \{z = 0\}, \quad \xi \in \mathbb{R}_{>0}^S, \quad (1.13)$$

is invariant under the flow of the unperturbed system. In fact, the solutions of (1.9) with $\varepsilon = 0$ are of the form

$$t \mapsto (\theta + \omega^{nls}(\xi, 0)t, \xi, 0). \quad (1.14)$$

Here $\theta \in \mathbb{T}^S$ parametrizes the initial data and $\omega_k^{nls}(\xi, 0)$, $k \in S$, are referred to as the unperturbed *tangential* frequencies of \mathcal{T}_ξ . Our aim is to prove that for $\varepsilon > 0$ sufficiently small, most of the tori \mathcal{T}_ξ persist. This is a

small divisors problem. To be able to apply KAM type techniques requires that for $\varepsilon = 0$, the Hamiltonian system (1.12), linearized at the quasi-periodic solution (1.14) of the unperturbed system, has constant coefficients. Indeed this is the case since this linearized system is given by

$$\hat{\theta} = (\partial_I \omega^{nls}(\xi, 0)) \hat{I}, \quad \hat{I} = 0, \quad i \hat{z}_k = \omega_k^{nls}(\xi, 0) \hat{z}_k, \quad k \in S^\perp. \quad (1.15)$$

Since the linearization of (1.3) at a S -gap solution is *not* a linear PDE with constant coefficients, this is one of the main reasons to express equation (1.3) in Birkhoff coordinates. System (1.15) shows that each torus \mathcal{T}_ξ is *elliptic*. Furthermore it can be proved (cf Subsection 3.1 ; [25]) that the dNLS frequencies have the asymptotics

$$\omega_k^{nls}(\xi, 0) = 4\pi^2 k^2 + 4 \sum_{j \in S} \xi_j + O\left(\frac{1}{k}\right), \quad |k| \rightarrow \infty, \quad (1.16)$$

implying that $\omega_k^{nls}(\xi, 0) - \omega_{-k}^{nls}(\xi, 0)$ cannot be bounded away from 0 uniformly in k . However bounds of such type are part of a set of non resonance conditions, referred to as second order Melnikov conditions which are one of the main assumptions in the KAM perturbation theory for elliptic tori as developed in [26], [27], [30]. Hence the latter does not apply.

It turns out to be convenient to study (1.12) in the canonical coordinates (θ, y, z) where y is in a neighborhood $U_0 \subset \mathbb{R}^S$ of 0 chosen such that $\Pi + U_0 \Subset \mathbb{R}_{>0}^S$, where $\Pi \subset \mathbb{R}_{>0}^S$ is the compact set of actions in Theorem 1.1. The Hamiltonian system (1.12) then reads

$$\dot{\theta} = \nabla_y H_\varepsilon, \quad \dot{y} = -\nabla_\theta H_\varepsilon, \quad i \dot{z} = \nabla_{\bar{z}} H_\varepsilon \quad (1.17)$$

where the Hamiltonian H_ε is given by

$$H_\varepsilon(\theta, y, z) \equiv H_\varepsilon(\theta, y, z; \xi) = H^{nls}(\xi + y, z\bar{z}) + \varepsilon P(\theta, y, z) \quad (1.18)$$

and, by a slight abuse of notation, P is now viewed as a function of θ, y, z , given by $P(\theta, \xi + y, z)$. We want to find invariant tori of (1.17) close to the tori \mathcal{T}_ξ of (1.13), admitting quasi-periodic solutions with frequency vector ω . It amounts to solve the equation

$$F_\omega(\iota) = 0, \quad F_\omega(\iota) := (\omega \cdot \partial_\varphi \theta - \nabla_y H_\varepsilon \circ \check{\iota}, \omega \cdot \partial_\varphi y + \nabla_\theta H_\varepsilon \circ \check{\iota}, \omega \cdot \partial_\varphi z + i \nabla_{\bar{z}} H_\varepsilon \circ \check{\iota}) \quad (1.19)$$

where the unknown is the torus embedding $\check{\iota}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with ι being the map

$$\iota: \mathbb{T}^S \rightarrow M^\sigma, \quad \varphi \mapsto (\theta(\varphi) - \varphi, y(\varphi), z(\varphi)),$$

and the phase space

$$M^\sigma \equiv M_S^\sigma := \mathbb{T}^S \times U_0 \times h_\perp^\sigma, \quad \sigma \geq 4. \quad (1.20)$$

In this paper we fix the space regularity σ . In the sequel we will always choose the vector ξ in (1.18) (1.19) to be the function of the parameter $\omega \in \Omega$ given by

$$\xi = (\omega^{nls})^{-1}(\omega). \quad (1.21)$$

Note that other KAM theorems, such as in [26], [30], are formulated for perturbations of parameter dependent families of isochronous systems, with ξ being the independent parameter.

Due to the small divisors problem coming up in the course of the proof, we will look for quasi-periodic solutions whose frequencies are diophantine, namely $\omega \in \Omega_{\gamma, \tau}$ where

$$\Omega_{\gamma, \tau} := \left\{ \omega \in \Omega : |\omega \cdot \ell| \geq \frac{\gamma}{|\ell|^\tau} \quad \forall \ell \in \mathbb{Z}^S \setminus \{0\} \right\} \subset \Omega \quad \text{with} \quad 0 < \gamma < 1, \quad \tau \geq |S| + 1. \quad (1.22)$$

In addition, in order to control the resonant interactions between the tangential and the normal frequencies of such solutions, we will impose on ω also first and second order Melnikov non resonance conditions. At

the starting point of the iteration, we choose finite-gap solutions of the unperturbed system which satisfy first and second order Melnikov conditions of the type

$$\begin{aligned}
|\omega \cdot \ell + \omega_k^{nls}(\xi(\omega), 0)| &\geq \frac{\gamma k^2}{\langle \ell \rangle^\tau}, \quad \forall (\ell, k) \in \mathbb{Z}^S \times S^\perp, \\
|\omega \cdot \ell + \omega_k^{nls}(\xi(\omega), 0) - \omega_j^{nls}(\xi(\omega), 0)| &\geq \frac{\gamma(k^2 - j^2)}{\langle \ell \rangle^\tau}, \quad \forall (\ell, k, j) \in \mathbb{Z}^S \times S^\perp \times S^\perp, \quad (\ell, k, j) \neq (0, k, \pm k), \\
|\omega \cdot \ell + \omega_k^{nls}(\xi(\omega), 0) + \omega_j^{nls}(\xi(\omega), 0)| &\geq \frac{\gamma(k^2 + j^2)}{\langle \ell \rangle^\tau}, \quad \forall (\ell, k, j) \in \mathbb{Z}^S \times S^\perp \times S^\perp.
\end{aligned}$$

Using the asymptotics (3.8) of the dNLS frequencies in Theorem 3.2 and the non-degeneracy conditions (3.10) in Proposition 3.1, the above conditions are fulfilled for most values of the parameter ω . We will then need to impose conditions of this type at each step of the iteration. In the setup chosen in this paper they take the form (7.75) and (7.58) - (7.59).

Let us now explain the main parts of the proof of Theorem 1.1. In view of our non analytic setup, we use a Newton-Nash-Moser iteration scheme for solving $F_\omega(u) = 0$. At each step of the scheme, the subsequent approximation is constructed with the help of an approximate right inverse of the differential dF_ω using a smoothing procedure to counterbalance the loss of regularity of the latter. The construction of an approximate right inverse of dF_ω at an embedding $\check{\iota}$ near $\check{\iota}_0(\varphi) = (\varphi, 0, 0)$ and the proof of tame estimates for it are at the core of the implementation of such a scheme. Following the strategy developed in [5], [2], [3] the task of getting such right inverses can be reduced to construct an approximate right inverse of the part of dF_ω , acting (as an unbounded operator) on h_\perp^σ (cf Section 5). It amounts to solve a φ -dependent linear system of the form

$$\begin{aligned}
\omega \cdot \partial_\varphi h_k(\varphi) + i\omega_k^{nls} h_k(\varphi) + i \sum_{j \in S^\perp} \partial_{I_j} \omega_k^{nls} z_k(\varphi) \left(\bar{z}_j(\varphi) h_j(\varphi) + z_j(\varphi) \bar{h}_j(\varphi) \right) \\
+ i\varepsilon \sum_{j \in S^\perp} \left(\partial_{z_j} \partial_{\bar{z}_k} P(\check{\iota}(\varphi)) h_j(\varphi) + \partial_{\bar{z}_j} \partial_{z_k} P(\check{\iota}(\varphi)) \bar{h}_j(\varphi) \right) = 0, \quad k \in S^\perp \quad (1.23)
\end{aligned}$$

where ω_k^{nls} and $\partial_{I_j} \omega_k^{nls}$ are evaluated at $(\xi + y(\varphi), z(\varphi) \bar{z}(\varphi))$. We analyze such systems in detail in Section 6 and Section 7. In view of the small divisors problems, we would like to apply a KAM scheme to reduce it to a linear system in diagonal form with φ -independent coefficients. However, since according to (1.16), the dNLS frequencies do not satisfy the second order Melnikov conditions with $(\ell, k, j) = (0, k, \pm k)$, this is not possible. Instead we reduce the corresponding linear operator to a self-adjoint, 2×2 block diagonal operator with φ -independent coefficients, by grouping together the variables z_{-k} and z_k . For small amplitude solutions of nonlinear wave (NLW) equations with an external potential, such a scheme has been successfully implemented by Chierchia-You [11], using that the NLW equation can be written as a symmetric first order Hamiltonian system, for which the nonlinear part of the Hamiltonian vector field is one smoothing. It implies that the non constant part of the asymptotic expansion of the normal frequencies is of the size $O(\varepsilon/|k|)$ as $|k| \rightarrow +\infty$, where ε is related to the amplitude of the (small) solution. In contrast, for the dNLS equation, according to (1.16), the non-constant part of the asymptotic expansion of the frequencies $\omega_k^{nls}(\xi, 0)$ is of size $O(1)$ and the nonlinear part of the perturbative Hamiltonian vector field is not regularizing so that the 'perturbed normal frequencies', denoted by ω_k , $k \in S^\perp$, will behave asymptotically as $4\pi^2 k^2 + O(1)$. This information alone does not allow to verify that along the KAM iteration scheme, for any $\ell \neq 0$ and most values of ξ , one has $|\omega \cdot \ell + \omega_k - \omega_{-k}| \geq \gamma \langle \ell \rangle^{-\tau}$. However such non resonance conditions are needed to eliminate along the KAM scheme the φ -dependent monomials $e^{i\ell \cdot \varphi} z_k \bar{z}_{-k}$ and $e^{i\ell \cdot \varphi} z_{-k} \bar{z}_k$ in the perturbed Hamiltonian. One of the main tasks in our proof of Theorem 1.1 is to derive for the perturbed normal frequencies an asymptotic expansion of the form (cf (9.30))

$$\omega_k^{nls}(\xi, 0) + c + O(\varepsilon \gamma^{-2} |k|^{-1}), \quad |k| \rightarrow \infty, \quad (1.24)$$

where $c \in \mathbb{R}$ satisfies $c = O(\varepsilon \gamma^{-2})$, see Lemma 9.3. It allows to show that the required second order Melnikov non resonance conditions hold true for a large set of ω 's – see the arguments of section 9. It turns out that

in (1.24) the constant c is independent of the sign of k , but this fact is irrelevant for the applicability of this approach.

The asymptotic expansion (1.24) is achieved by adapting the strategy of [1] - [2], developed for quasi-linear perturbations of the KdV equation. The main idea is to perform a symplectic transformation which reduces the linearized operator to a diagonal operator with φ -independent coefficients up to a one smoothing remainder. This is achieved in three steps in Subsections 6.2 - 6.4. One of the key ingredients is that, by [24], the Birkhoff map is a perturbation of the Fourier transform by a 1-smoothing nonlinear map. Thus the highest order term of the linearized equation, expressed in the Birkhoff coordinates, is the same as the one in the original coordinates. In contrast to the KdV equation, treated in [1], [2], [3], the NLS equation is a vector valued system, requiring to analyze commutators of matrix valued pseudodifferential operators. Actually, strictly speaking, the operators involved are not pseudodifferential since their symbols are not C^∞ . The regularity assumption (1.6) on the perturbation allows to perform the Nash-Moser iteration in Sobolev spaces of fixed regularity with respect to the space variable. As a consequence we have to choose the transformations in Sections 6.2 - 6.3 with care. After these preliminary changes of coordinates have been performed, we apply a KAM type scheme, described in detail in Section 7, to reduce, for ω 's satisfying the second order Melnikov non-resonance conditions, the above linear operator to a 2×2 block diagonal infinite dimensional matrix with φ -independent coefficients. We express the set of ω 's satisfying the second order Melnikov non-resonance conditions at each step of the induction in terms of the reduced operator only, see (7.57) as well as Lemma 7.6. The measure estimates for these sets are performed in section 9.

Related results: The first KAM theorem for analytic perturbations of the dNLS equation was established by Kuksin and Pöschel [27] for finite dimensional tori near zero. To avoid the difficulties caused by the near resonances of ω_k^{nls} and ω_{-k}^{nls} for $|k| \rightarrow \infty$, they considered the dNLS equation on the dNLS invariant subspace of H^σ of odd functions, requiring the perturbation to be odd. Further results of this kind can be found for instance in [28]. Using the integrability of the dNLS equation this result was shown in Grébert and Kappeler [20] to hold for finite dimensional tori of arbitrary size contained in one of the subspaces defined by the fixed point sets of the maps $R_\alpha : u(x) \mapsto e^{i\alpha}u(1-x)$, $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. Again, these subspaces are invariant under the dNLS flow and the KAM result holds for perturbations which preserve this symmetry. For $\alpha = 0$, or $\alpha = \pi$, it is the subspace of even, respectively odd, functions in H^σ . In another approach, Geng and You [15] proved a KAM result for the dNLS equation for tori near zero in case the perturbation $f(u)$ in (1.3) is analytic and does not explicitly depend on x , see also [18]. In this case, the momentum is an additional integral for the perturbed PDE, allowing to deal with the difficulties caused by the near resonances of ω_k^{nls} and ω_{-k}^{nls} . It can be shown that this result actually holds for perturbations of finite gap solutions of arbitrary size, see Liang and Kappeler [22].

The difficulty posed by resonant frequencies has been also solved for analytic perturbations of the dNLS equation in 1-space dimension by Craig and Wayne [12] for small periodic solutions, and by Bourgain [8] for small quasi-periodic solutions by an approach which does not require second order Melnikov conditions. These results do not prove the linear stability of the quasi-periodic solutions. In higher space dimensions this approach has been extended in [9], [10], [4], [33]. A KAM theorem with second order Melnikov non-resonance conditions for the Schrödinger equation with convolution potential and analytic perturbations has been developed by Eliasson and Kuksin in [13] where they introduced the notion of Töplitz-Lipschitz matrices. Further KAM results have been proved by [16], [17], [31] using the conservation of momentum.

Our approach is completely different from the one of the KAM result of Eliasson and Kuksin. As mentioned above, the key point is the expansion (1.24) for the frequencies of the perturbed equations, which is obtained by conjugating the linearized equation (1.23) to a system of equations decoupled up to order $|k|^{-1}$, with leading coefficients given by (1.24) – see Section 6. This allows to verify the second order Melnikov conditions for perturbations of the 1-dimensional dNLS equation with periodic boundary conditions. Our approach does not require the perturbation to be analytic. We also mention the recent related work [14] where small quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations are constructed.

Organization: The paper is organized as follows: In Section 2 and Section 3 we introduce additional notation and discuss auxiliary results used throughout the paper. In Section 4 we restate Theorem 1.1 in our functional setup, and outline the organisation of its proof. In Section 5 we analyze the differential of F_ω and prove the results on the approximate right inverse needed in the proof of the Nash-Moser iteration scheme, assuming

results on the approximate right inverse of the part of the differential, acting in normal directions. The latter results are proved in Section 6 (preliminary transformations) and Section 7 (reduction to a constant 2×2 block diagonal operator by a KAM interaction scheme). In Section 8 we construct solutions of $F_\omega(\iota) = 0$ by the aforementioned Nash-Moser iteration scheme for ω 's, satisfying appropriate non-resonance conditions. Finally, in Section 9 we obtain the claimed measure estimates of Theorem 1.1 of the subset Ω_ε .

For the convenience of the reader all the above arguments are proved in a self-contained way.

Notations: Throughout the paper, for $\sigma \in \mathbb{Z}_{\geq 0}$, $H^\sigma \equiv H^\sigma(\mathbb{T}_1, \mathbb{C})$ denotes the Sobolev space

$$H^\sigma = \left\{ f \in L^2(\mathbb{T}_1, \mathbb{C}) : \|f\|_\sigma < \infty \right\}, \quad \|f\|_\sigma \equiv \|f\|_{H^\sigma} := \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2\sigma} |f_n|^2 \right)^{1/2} \quad (1.25)$$

where

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{i2\pi n x}, \quad f_n = \int_0^1 f(x) e^{-i2\pi n x} dx, \quad n \in \mathbb{Z}, \quad (1.26)$$

and $\langle n \rangle := \max\{1, |n|\}$. Since the Fourier transform is an isometry between H^σ and the sequence space $h^\sigma \equiv h^\sigma(\mathbb{Z}, \mathbb{C})$, we will not distinguish between the two spaces and frequently identify a function $f(x) = \sum_{n \in \mathbb{Z}} f_n e^{i2\pi n x}$ with the sequence of its Fourier coefficients $(f_n)_{n \in \mathbb{Z}}$. Similarly, we will identify the subspace

$$H^\sigma_\perp := \left\{ f(x) = \sum_{n \in \mathbb{Z}} f_n e^{i2\pi n x} \in H^\sigma : f_n = 0, \quad \forall n \in S \right\} \quad (1.27)$$

of H^σ with the corresponding subspace $h^\sigma_\perp = h^\sigma(S^\perp, \mathbb{C})$ of h^σ where, throughout the paper, S^\perp denotes the complement $\mathbb{Z} \setminus S$ of a given finite subset $S \subset \mathbb{Z}$. We denote by π_\perp the standard L^2 -orthogonal projection of H^σ onto H^σ_\perp ,

$$\pi_\perp : H^\sigma \rightarrow H^\sigma_\perp. \quad (1.28)$$

Let

$$\langle f, g \rangle := \int_{\mathbb{T}_1} f(x) \bar{g}(x) dx, \quad \langle f, g \rangle_r := \int_{\mathbb{T}_1} f(x) g(x) dx. \quad (1.29)$$

For a linear operator A acting in $L^2(\mathbb{T}_1)$ we denote by A^* its adjoint with respect to the complex inner product $\langle \cdot, \cdot \rangle$ and by A^t the one with respect to the bilinear form $\langle \cdot, \cdot \rangle_r$. We also denote

$$\overline{A}(f) := \overline{A(\bar{f})}$$

and note that $A^* = \overline{A^t}$. We shall use the notation A^* , A^t , \overline{A} also for an operator A acting on the sequence space h^σ . Furthermore, we need to consider maps $f : \mathbb{T}^S \rightarrow X$ with values in a \mathbb{C} -Banach space X . Given any L^2 -map $f : \mathbb{T}^S \rightarrow X$ (in the sense of Bochner), we define its Fourier coefficients

$$\hat{f}(\ell) := \frac{1}{(2\pi)^{|S|}} \int_{\mathbb{T}^S} f(\varphi) e^{-i\ell \cdot \varphi} d\varphi \in X, \quad \ell \in \mathbb{Z}^S, \quad (1.30)$$

and for any $s \in \mathbb{Z}_{\geq 0}$ the norm

$$\|f\|_s := \left(\sum_{\ell \in \mathbb{Z}^S} \|\hat{f}(\ell)\|_X^2 \langle \ell \rangle^{2s} \right)^{1/2}, \quad (1.31)$$

where for $\ell = (\ell_k)_{k \in S} \in \mathbb{Z}^S$,

$$\langle \ell \rangle := \max\{1, |\ell|\}, \quad |\ell| := \sum_{k \in S} |\ell_k|.$$

We denote by $L^2(\mathbb{T}^S, X)$ the space of L^2 -maps $f : \mathbb{T}^S \rightarrow X$ and introduce for any $s \in \mathbb{Z}_{\geq 0}$ the Banach space

$$H^s(\mathbb{T}^S, X) := \left\{ f \in L^2(\mathbb{T}^S, X) : \|f\|_s < \infty \right\}. \quad (1.32)$$

Usually, we write $L^2(\mathbb{T}^S, X)$ instead of $H^0(\mathbb{T}^S, X)$.

For any $s \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}^s(\mathbb{T}^S, X)$ denotes the Banach space of \mathcal{C}^s -smooth maps on \mathbb{T}^S with values in X , equipped with the norm

$$\|f\|_{\mathcal{C}^s} := \sum_{0 \leq |\alpha| \leq s} \|\partial_\varphi^\alpha f\|_X^{\text{sup}}, \quad \|\partial_\varphi^\alpha f\|_X^{\text{sup}} := \sup_{\varphi \in \mathbb{T}^S} \|\partial_\varphi^\alpha f(\varphi)\|_X \quad (1.33)$$

where we have used the customary multi-index notation, i.e., for any $\alpha = (\alpha_k)_{k \in S} \in \mathbb{Z}_{\geq 0}^S$, ∂_φ^α is the differential operator given by $\prod_{k \in S} (\partial_{\varphi_k})^{\alpha_k}$ and $|\alpha| = \sum_{k \in S} \alpha_k$. Frequently, we will identify $f : \mathbb{T}^S \rightarrow X$ with its lift $\mathbb{R}^S \rightarrow X$, which is periodic with respect to the lattice $(2\pi\mathbb{Z})^S$. Furthermore, we define

$$s_0 := \lceil |S|/2 \rceil + 1 \in \mathbb{Z}$$

so that $H^s(\mathbb{T}^S, X) \hookrightarrow \mathcal{C}^0(\mathbb{T}^S, X)$ for any $s \geq s_0$, cf Lemma 2.1.

For a map $f : \Omega \rightarrow X$, $\omega \mapsto f_\omega$ with domain of definition $\Omega \subset \mathbb{R}^S$ and target a \mathbb{C} -Banach space X , we define its sup-norm and its Lipschitz semi-norm by

$$\|f\|_{X,\Omega}^{\text{sup}} := \sup_{\omega \in \Omega} \|f_\omega\|_X, \quad \|f\|_{X,\Omega}^{\text{lip}} := \sup_{\substack{\omega_1, \omega_2 \in \Omega \\ \omega_1 \neq \omega_2}} \frac{\|f_{\omega_1} - f_{\omega_2}\|_X}{|\omega_1 - \omega_2|}, \quad (1.34)$$

and, for $0 < \gamma < 1$ as in (1.22), the Lipschitz norm

$$\|f\|_{X,\Omega}^{\gamma \text{lip}} := \|f\|_{X,\Omega}^{\text{sup}} + \gamma \|f\|_{X,\Omega}^{\text{lip}}. \quad (1.35)$$

If $X = H^s(\mathbb{T}^S, \mathbb{C})$ or $X = H^s(\mathbb{T}^S, H^\sigma)$, we simply write $\|f\|_s^{\gamma \text{lip}}$ for $\|f\|_{H^s}^{\gamma \text{lip}}$. In the sequel we will typically suppress Ω in the above norms, whenever the context permits.

Finally, throughout the paper, the expression $a \leq_s b$ means that there exists a constant $C(s)$ such that $a \leq C(s)b$ where s refers to the index of the Sobolev space $H^s(\mathbb{T}^S, X)$. The constant may depend on data such as $|S|$, τ , Ω , the perturbation P , \dots . The notation $a \leq b$ means that in addition, the constant C is independent of the Sobolev index s . The constants $C(s)$ and C may change from one argument to another. If a constant κ depends only on $|S|$ and τ such as the number s_0 , we often will write \leq for \leq_κ .

2 Functional analytic prerequisites

In this section we introduce additional notation and discuss some auxiliary results from functional analysis, needed in the sequel.

2.1 Sobolev spaces

We discuss elementary properties of the Banach spaces $H^s(\mathbb{T}^S, X)$.

Lemma 2.1. *Let f be an element in $H^{s_0}(\mathbb{T}^S, X)$ with $s_0 := \lceil |S|/2 \rceil + 1$. Then the following holds:*

- (i) *For any $\varphi \in \mathbb{T}^S$, the series $\sum_{\ell \in \mathbb{Z}^S} \hat{f}(\ell) e^{i\ell \cdot \varphi}$ converges absolutely and $f(\varphi) = \sum_{\ell \in \mathbb{Z}^S} \hat{f}(\ell) e^{i\ell \cdot \varphi}$.*
- (ii) *If $\|f\|_{s+1} < +\infty$ for some $s \geq s_0$, then for any $\omega \in \mathbb{R}^S$,*

$$\|(\omega \cdot \partial_\varphi) f\|_s \leq \|f\|_{s+1}$$

where $\omega \cdot \partial_\varphi = \sum_{k \in S} \omega_k \partial_{\varphi_k}$.

- (iii) *For any $s \in \mathbb{Z}_{\geq 0}$,*

$$\|f\|_{\mathcal{C}^s} \leq_s \|f\|_{s+s_0}, \quad \|f\|_s \leq_s \|f\|_{\mathcal{C}^{s+s_0}} \quad (2.1)$$

where the Banach spaces $(\mathcal{C}^s, \|\cdot\|_{\mathcal{C}^s})$ were introduced at the end of Section 1, see (1.33).

If $(X, \langle \cdot, \cdot \rangle)$ is a \mathbb{C} -Hilbert space then Plancherel's theorem holds, i.e. (cf (1.30))

$$\frac{1}{(2\pi)^{|S|}} \int_{\mathbb{T}^S} \langle f(\varphi), g(\varphi) \rangle d\varphi = \sum_{\ell \in \mathbb{Z}^S} \langle \hat{f}(\ell), \hat{g}(\ell) \rangle, \quad \forall f, g \in L^2(\mathbb{T}^S, X),$$

implying that for any $s \geq 0$,

$$\|f\|_s \stackrel{(1.31)}{=} (2\pi)^{-|S|/2} \left\| \sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^s \hat{f}(\ell) e^{i\ell \cdot \varphi} \right\|_{L^2(\mathbb{T}^S, X)} \quad (2.2)$$

and that in this case, the L^2 -Fourier theory for scalar valued functions extends in a straightforward way.

In the iteration schemes considered in this paper, we will frequently encounter equations of the form

$$(\omega \cdot \partial_\varphi) f = g \quad (2.3)$$

where $\omega \in \mathbb{R}^S$ is assumed to satisfy the diophantine conditions (1.22) and $g : \mathbb{T}^S \rightarrow X$ the compatibility assumption $\hat{g}(0) = 0$. The solution $f = (\omega \cdot \partial_\varphi)^{-1} g$ is given by

$$\hat{f}(0) = 0, \quad \hat{f}(\ell) := \frac{\hat{g}(\ell)}{i\omega \cdot \ell}, \quad \forall \ell \in \mathbb{Z}^S \setminus \{0\}, \quad (2.4)$$

and satisfies the following standard estimates.

Lemma 2.2. *Let $s \geq s_0$ and assume that $\omega \in \mathbb{R}^S$ satisfies the diophantine conditions (1.22). Then for any $g \in H^{s+\tau}(\mathbb{T}^S, X)$ with $\hat{g}(0) = 0$, the linear equation (2.3) has a unique solution $f \in H^s(\mathbb{T}^S, X)$ with $\hat{f}(0) = 0$. It satisfies the estimate*

$$\|f\|_s \leq \gamma^{-1} \|g\|_{s+\tau}.$$

If $g = g_\omega \in H^{s+2\tau+1}(\mathbb{T}^S, X)$ is Lipschitz continuous in $\omega \in \Omega \subseteq \mathbb{R}^S$, then the solution $f = f_\omega \in H^s(\mathbb{T}^S, X)$ is Lipschitz continuous in ω and satisfies

$$\|f\|_s^{\gamma_{\text{lip}}} \leq \gamma^{-1} \|g\|_{s+2\tau+1}^{\gamma_{\text{lip}}}. \quad (2.5)$$

For the class of semilinear perturbations considered in (1.5) – (1.6), it is possible to keep the index $\sigma \geq 4$ of the Sobolev space $H^\sigma \equiv H^\sigma(\mathbb{T}_1, \mathbb{C})$ fixed, whereas the index s of the Sobolev spaces $H^s(\mathbb{T}^S, X)$ varies due to a possible loss of regularity in the (time) variable φ along the various iteration schemes. Nonetheless, since the dNLS equation (1.1) contains the differential operator ∂_x^2 , we also will need to consider functions with values in $H^{\sigma'}$ with σ' such as $\sigma - 2$. We recall that we identify $H^{\sigma'}$ with $h^{\sigma'}$ via the Fourier transform. In the sequel, we will frequently consider the Sobolev space $(H^s(\mathbb{T}^S, h^{\sigma'}), \|\cdot\|_{s, \sigma'})$ of maps with values in the Hilbert space $h^{\sigma'}$ where $\sigma' \in \mathbb{Z}_{\geq 0}$ and the norm $\|u\|_{s, \sigma'}$ of u is given by

$$\|u\|_{s, \sigma'} := \left(\sum_{\ell \in \mathbb{Z}^S} \|\hat{u}(\ell)\|_{h^{\sigma'}}^2 \langle \ell \rangle^{2s} \right)^{1/2}. \quad (2.6)$$

In the case where $\sigma' = \sigma$, we simply write $\|u\|_s$ instead of $\|u\|_{s, \sigma}$. For any $\ell \in \mathbb{Z}^S$, the Fourier coefficient $\hat{u}(\ell)$ is a sequence in $h^{\sigma'}$, which we denote by $(\hat{u}_n(\ell))_{n \in \mathbb{Z}}$. Note that $\hat{u}_n(\ell)$, $\ell \in \mathbb{Z}^S$, are the Fourier coefficients of the function $\varphi \mapsto u_n(\varphi)$, which is the n 'th component of $u(\varphi) = (u_j(\varphi))_{j \in \mathbb{Z}}$, i.e., $u_n(\varphi) = \sum_{\ell \in \mathbb{Z}^S} \hat{u}_n(\ell) e^{i\ell \cdot \varphi}$. Furthermore,

$$\|u\|_{s, \sigma'}^2 = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^S} |\hat{u}_n(\ell)|^2 \langle n \rangle^{2\sigma'} \langle \ell \rangle^{2s} = \sum_{n \in \mathbb{Z}} \|u_n\|_s^2 \langle n \rangle^{2\sigma'} \quad (2.7)$$

where $\|u_n\|_s = \|u_n\|_{H^s(\mathbb{T}^S, \mathbb{C})}$. We shall also consider functions $\varphi \mapsto y(\varphi)$ with values in \mathbb{R}^S in the Sobolev space $H^s(\mathbb{T}^S, \mathbb{R}^S)$ whose norm is also denoted by

$$\|y\|_s := \|y\|_{H^s(\mathbb{T}^S, \mathbb{R}^S)}.$$

Another class of Sobolev spaces used in this paper are the spaces of operator valued maps, $H^s(\mathbb{T}^S, \mathcal{L}(h^{\sigma'}))$, where $\mathcal{L}(h^{\sigma'})$ denotes the Banach space of bounded linear operators on $h^{\sigma'}$, endowed with the operator norm. A linear operator A has a natural matrix representation $(A_k^j)_{j,k \in \mathbb{Z}}$ determined by

$$(A(h))_k = \sum_{j \in \mathbb{Z}} A_k^j h_j \in \mathbb{C}, \quad k \in \mathbb{Z}. \quad (2.8)$$

We will also consider such Sobolev spaces with $h^{\sigma'}(\mathbb{Z}, \mathbb{C}) \times h^{\sigma'}(\mathbb{Z}, \mathbb{C})$ or $h_{\perp}^{\sigma'}$ instead of $h^{\sigma'}$. For an element $\varphi \mapsto A(\varphi)$ in $H^s(\mathbb{T}^S, \mathcal{L}(h^{\sigma'}))$, the corresponding norm is conveniently denoted by $|A|_{s, \sigma'}$, i.e.,

$$|A|_{s, \sigma'} := \left(\sum_{\ell \in \mathbb{Z}^S} \|\hat{A}(\ell)\|_{\sigma'}^2 \langle \ell \rangle^{2s} \right)^{1/2}, \quad \|\hat{A}(\ell)\|_{\sigma'} := \|\hat{A}(\ell)\|_{\mathcal{L}(h^{\sigma'})}. \quad (2.9)$$

In case $\sigma' = \sigma$, we simply write $|A|_s$ instead of $|A|_{s, \sigma}$. We remark that $|A|_s$ is a quite strong norm but particularly convenient for estimating solutions of homological equations – see e.g. Lemma 7.3.

According to (2.9), (2.1), (1.31) one has

$$|A|_{s, \sigma'} \leq_s \|A\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathcal{L}(h^{\sigma'}))} \quad \text{and} \quad \|A\|_{\mathcal{C}^s(\mathbb{T}^S, \mathcal{L}(h^{\sigma'}))} \leq_s |A|_{s+s_0, \sigma'}. \quad (2.10)$$

To state our next result, let D be the operator defined for $h = (h_j)_{j \in \mathbb{Z}}$ by setting

$$(Dh)_j := 2\pi j h_j, \quad \forall j \in \mathbb{Z}, \quad (2.11)$$

and let $\langle\langle D \rangle\rangle := (1 + D^2)^{1/2}$, i.e.

$$\langle\langle D \rangle\rangle h_j := \langle\langle j \rangle\rangle h_j, \quad \langle\langle j \rangle\rangle := (1 + (2\pi j)^2)^{1/2} \quad \forall j \in \mathbb{Z}. \quad (2.12)$$

Note that D is the operator corresponding to the Fourier multiplier $\frac{1}{i} \partial_x$.

Lemma 2.3. *Let $s \in \mathbb{Z}_{\geq 0}$ and $\sigma \in \mathbb{Z}_{\geq 2}$ and assume that A is in $H^s(\mathbb{T}^S, \mathcal{L}(h^{\sigma-2}, h^{\sigma-1}))$. Then the following holds:*

- (i) $|A|_{s, \sigma-2} \leq |A \langle\langle D \rangle\rangle|_{s, \sigma-1}$ and $|A|_{s, \sigma-1} \leq |A \langle\langle D \rangle\rangle|_{s, \sigma-1}$.
- (ii) If $A = A_{\omega}$ is Lipschitz continuous in $\omega \in \Omega \subseteq \mathbb{R}^S$ then

$$|A|_{s, \sigma-2}^{\gamma \text{lip}} \leq |A \langle\langle D \rangle\rangle|_{s, \sigma-1}^{\gamma \text{lip}} \quad \text{and} \quad |A|_{s, \sigma-1}^{\gamma \text{lip}} \leq |A \langle\langle D \rangle\rangle|_{s, \sigma-1}^{\gamma \text{lip}}.$$

Proof. Since for any $\ell \in \mathbb{Z}^S$, $\hat{A}(\ell)$ satisfies

$$\|\hat{A}(\ell)\|_{\sigma-2} \leq \|\hat{A}(\ell)\|_{\mathcal{L}(h^{\sigma-2}, h^{\sigma-1})} \leq \|\hat{A}(\ell) \langle\langle D \rangle\rangle\|_{\sigma-1} \|\langle\langle D \rangle\rangle^{-1}\|_{\mathcal{L}(h^{\sigma-2}, h^{\sigma-1})} \leq \|\hat{A}(\ell) \langle\langle D \rangle\rangle\|_{\sigma-1},$$

and similarly,

$$\|\hat{A}(\ell)\|_{\sigma-1} \leq \|\hat{A}(\ell) \langle\langle D \rangle\rangle\|_{\mathcal{L}(h^{\sigma}, h^{\sigma-1})} \|\langle\langle D \rangle\rangle^{-1}\|_{\mathcal{L}(h^{\sigma-1}, h^{\sigma})} \leq \|\hat{A}(\ell) \langle\langle D \rangle\rangle\|_{\sigma-1},$$

item (i) holds. The claimed estimates of item (ii) are an immediate consequence of item (i). \square

Finally, we consider the operator, defined by multiplication with a map. More precisely, assume that q is in $H^s(\mathbb{T}^S, H^{\sigma'})$ with $s \geq s_0$ and $\sigma' \geq 1$. The latter conditions imply that $H^{\sigma'}$ and in turn $H^s(\mathbb{T}^S, H^{\sigma'})$ are algebras and hence the operator Λ_q of multiplication by q , defined on $H^s(\mathbb{T}^S, H^{\sigma'})$ by setting for any $\varphi \in \mathbb{T}^S$,

$$\Lambda_q(\varphi) : H^{\sigma'} \rightarrow H^{\sigma'}, \quad f \mapsto \Lambda_q(\varphi)f(\cdot) := q(\varphi, \cdot)f(\cdot)$$

is well defined. In the following lemma we again identify the Hilbert spaces $H^{\sigma'}$ and $h^{\sigma'}$ by the Fourier transform.

Lemma 2.4. (Multiplication and commutator estimates) *Let $q \in H^s(\mathbb{T}^S, H^\sigma)$ with $s \geq s_0$ and $\sigma \geq 4$. Then the following holds:*

(i) *For any $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2, \sigma - 3\}$, $|\Lambda_q|_{s, \sigma'} \leq \|q\|_{s, \sigma'}$.*

(ii) *For any $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$, the commutator $[\langle\langle D \rangle\rangle, \Lambda_q]$ of $\langle\langle D \rangle\rangle$ with Λ_q satisfies*

$$|[\langle\langle D \rangle\rangle, \Lambda_q]|_{s, \sigma' - 1} \leq \|q\|_{s, \sigma'}.$$

Proof. (i) Since $\sigma \geq 4$, one has $\sigma' \geq 1$ for σ' in $\{\sigma, \sigma - 1, \sigma - 2, \sigma - 3\}$. Furthermore, the Fourier coefficient $\hat{\Lambda}_q(\ell) : H^{\sigma'} \rightarrow H^{\sigma'}$, $\ell \in \mathbb{Z}^S$, is the multiplication operator by the function $\hat{q}(\ell) \in H^{\sigma'}$. Its operator norm is bounded by $C\|\hat{q}(\ell)\|_{H^{\sigma'}}$ with $C \equiv C(\sigma')$ and thus, recalling (2.9),

$$|\Lambda_q|_{s, \sigma'} \leq C \left(\sum_{\ell \in \mathbb{Z}^S} \|\hat{q}(\ell)\|_{H^{\sigma'}}^2 \langle \ell \rangle^{2s} \right)^{1/2} \leq C \|q\|_{s, \sigma'}.$$

(ii) Let $A := [\langle\langle D \rangle\rangle, \Lambda_q]$. Then the operator $\hat{A}(\ell)$ is represented by the matrix

$$\hat{A}(\ell)_j^{j'} = (\langle\langle j \rangle\rangle - \langle\langle j' \rangle\rangle) \hat{q}_{j-j'}(\ell), \quad j, j' \in \mathbb{Z}.$$

Since $\langle j \rangle^{\sigma' - 1} \leq \langle j - j' \rangle^{\sigma' - 1} + \langle j' \rangle^{\sigma' - 1}$ and $|\langle\langle j \rangle\rangle - \langle\langle j' \rangle\rangle| \leq \langle j - j' \rangle$, one gets that, for any $h = (h_j)_{j \in \mathbb{Z}}$ in $h^{\sigma' - 1}$,

$$\begin{aligned} \|\hat{A}(\ell)h\|_{H^{\sigma' - 1}}^2 &= \sum_{j \in \mathbb{Z}} \langle j \rangle^{2(\sigma' - 1)} \left| \sum_{j' \in \mathbb{Z}} \hat{A}(\ell)_j^{j'} h_{j'} \right|^2 \\ &\leq \sum_{j \in \mathbb{Z}} \left(\sum_{j' \in \mathbb{Z}} \langle j - j' \rangle^{\sigma'} |\hat{q}_{j-j'}(\ell)| |h_{j'}| \right)^2 + \sum_{j \in \mathbb{Z}} \left(\sum_{j' \in \mathbb{Z}} \langle j - j' \rangle |\hat{q}_{j-j'}(\ell)| \langle j' \rangle^{\sigma' - 1} |h_{j'}| \right)^2 =: I + II. \end{aligned}$$

Since, by assumption, $\sigma' - 1 \geq 1$, we get, by the Cauchy Schwartz inequality

$$\begin{aligned} I &\leq \sum_{j \in \mathbb{Z}} \left(\sum_{j' \in \mathbb{Z}} \langle j - j' \rangle^{\sigma'} |\hat{q}_{j-j'}(\ell)| \langle j' \rangle^{\sigma' - 1} |h_{j'}| \frac{1}{\langle j' \rangle^{\sigma' - 1}} \right)^2 \\ &\leq \sum_{j \in \mathbb{Z}} \left(\sum_{j' \in \mathbb{Z}} \langle j - j' \rangle^{2\sigma'} |\hat{q}_{j-j'}(\ell)|^2 \langle j' \rangle^{2(\sigma' - 1)} |h_{j'}|^2 \right) \left(\sum_{j' \in \mathbb{Z}} \frac{1}{\langle j' \rangle^{2(\sigma' - 1)}} \right) \\ &\leq \sum_{j \in \mathbb{Z}} \langle j - j' \rangle^{2\sigma'} |\hat{q}_{j-j'}(\ell)|^2 \sum_{j' \in \mathbb{Z}} \langle j' \rangle^{2(\sigma' - 1)} |h_{j'}|^2 \leq \|\hat{q}(\ell)\|_{H^{\sigma'}}^2 \|h\|_{H^{\sigma' - 1}}^2. \end{aligned}$$

The term II is estimated in the same way, yielding altogether

$$\|\hat{A}(\ell)\|_{\mathcal{L}(H^{\sigma' - 1})} \leq \|\hat{q}(\ell)\|_{H^{\sigma'}}. \quad (2.13)$$

Finally

$$|A|_{s, \sigma' - 1} = \left(\sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} \|\hat{A}(\ell)\|_{\mathcal{L}(H^{\sigma' - 1})}^2 \right)^{1/2} \stackrel{(2.13)}{\leq} \left(\sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} \|\hat{q}(\ell)\|_{H^{\sigma'}}^2 \right)^{1/2} \leq \|q\|_{s, \sigma'},$$

which is the claimed estimate of item (ii). \square

2.2 Smoothing operators and interpolation

In this subsection, we review the notion of families of smoothing operators for scales of Banach spaces and discuss specific examples, needed on the sequel. Assume that $(X_k)_{k \in \mathbb{Z}_{\geq 0}}$ is a scale of Banach spaces $\cdots \subseteq X_{k+1} \subseteq X_k \subseteq \cdots \subseteq X_1 \subseteq X_0$, with norms $\|\cdot\|_k := \|\cdot\|_{X_k}$, so that for any $0 \leq n \leq k$, $\|\cdot\|_n \leq \|\cdot\|_k$. Let us define $X_\infty := \bigcap_{k \geq 0} X_k$.

Definition 2.1 (Smoothing operators). *A one parameter family of linear operators $S_t : X_0 \rightarrow X_\infty$, $t \geq 1$ is said to be a family of smoothing operators for the scale $(X_k)_{k \in \mathbb{Z}_{\geq 0}}$ if the following three conditions are satisfied:*

(SM1) For any $f \in X_0$,

$$\lim_{t \rightarrow +\infty} \|S_t f - f\|_0 = 0.$$

(SM2) For any $k, n \in \mathbb{Z}_{\geq 0}$ with $n \leq k$, there exists a constant $C_{k,n} > 0$ such that

$$\|S_t f\|_k \leq C_{k,n} t^n \|f\|_{k-n}, \quad \forall f \in X_{k-n}, \quad \forall t \geq 1.$$

(SM3) For any $k, n \in \mathbb{Z}_{\geq 0}$, there exists a constant $C'_{k,n} > 0$ such that

$$\|S_t f - f\|_k \leq C'_{k,n} t^{-n} \|f\|_{k+n}, \quad \forall f \in X_{k+n}, \quad \forall t \geq 1.$$

Smoothing operators have the following interpolation property.

Proposition 2.1 (Interpolation estimates). *Given any integers $0 \leq k_1 \leq k \leq k_2$ with $k_2 - k_1 \geq 1$, there exists a constant $C_{k,k_1,k_2} > 0$ such that*

$$\|f\|_k \leq C_{k,k_1,k_2} \|f\|_{k_1}^{1-\lambda} \|f\|_{k_2}^\lambda, \quad \forall f \in X_{k_2}$$

where $0 \leq \lambda \leq 1$ is $\lambda := (k - k_1)/(k_2 - k_1)$.

Proof. Write $\|f\|_k \leq \|S_t f\|_k + \|S_t f - f\|_k$ and use (SM2) - (SM3), to see that the claimed estimate follows by choosing t for minimizing the right hand side. For more details see for instance [6], Lemma 1.1. \square

Smoothing operators for scales of Sobolev spaces: Let $H^s(\mathbb{T}^S, X)$, $s \in \mathbb{Z}_{\geq 0}$, be the Banach spaces defined in (1.32). Note that $\mathcal{C}^\infty(\mathbb{T}^S, X) = \bigcap_{s \geq 0} H^s(\mathbb{T}^S, X)$. We define the one parameter family of operators Π_t , $t \geq 1$

$$\Pi_t : L^2(\mathbb{T}^S, X) \rightarrow \mathcal{C}^\infty(\mathbb{T}^S, X), \quad f(\varphi) \mapsto \Pi_t f(\varphi) := \sum_{|\ell| \leq t} \hat{f}(\ell) e^{i\ell \cdot \varphi}, \quad \forall t \geq 1. \quad (2.14)$$

In the sequel, we will also consider Lipschitz maps $f = f_\omega$, $\omega \in \Omega \subset \mathbb{R}^S$, with values in $H^s(\mathbb{T}^S, X)$, equipped with the norm $\|f\|_s^{\text{lip}} = \|f\|_s^{\text{sup}} + \gamma \|f\|_{s,\Omega}^{\text{lip}}$ defined in (1.35) and (1.31). The following lemma can be proved in a straightforward way.

Lemma 2.5 (Smoothing operators for scales of H^s -spaces). *The one parameter family of operators Π_t , $t \geq 1$, defined in (2.14), is a family of smoothing operators for the scale of Banach spaces $(H^s(\mathbb{T}^S, X), \|\cdot\|_s)$, $s \in \mathbb{Z}_{\geq 0}$.*

At the same time, it is also a family of smoothing operators for the scale of Banach spaces of Lipschitz families in $H^s(\mathbb{T}^S, X)$ equipped with the norms $\|\cdot\|_s^{\text{lip}}$, $s \in \mathbb{Z}_{\geq 0}$.

For later reference, we briefly mention the smoothing operators for the special scales of the spaces $H^s(\mathbb{T}^S, \mathcal{L}(h^\sigma))$. For any $t \geq 1$ and $A = \sum_{\ell \in \mathbb{Z}^S} \hat{A}(\ell) e^{i\ell \cdot \varphi} \in H^s(\mathbb{T}^S, \mathcal{L}(h^\sigma))$, $\Pi_t A$ is an operator valued map with Fourier coefficients given by

$$\widehat{\Pi_t A}(\ell) := \begin{cases} \hat{A}(\ell) & \text{if } |\ell| \leq t \\ 0 & \text{otherwise.} \end{cases} \quad (2.15)$$

The operator $\Pi_t^\perp := \text{Id} - \Pi_t$ satisfies for any $n \in \mathbb{Z}_{\geq 0}$

$$\|\Pi_t^\perp A\|_s \leq t^{-n} \|A\|_{s+n}, \quad \|\Pi_t^\perp A\|_s^{\text{lip}} \leq t^{-n} \|A\|_{s+n}^{\text{lip}}. \quad (2.16)$$

Smoothing operators for scales of \mathcal{C}^s spaces: Let us consider the scale of Banach spaces $\mathcal{C}^s(\mathbb{T}^S, X)$, $s \in \mathbb{Z}_{\geq 0}$, equipped with the norm $\|\cdot\|_{\mathcal{C}^s}$ defined in (1.33). A one parameter family of smoothing operators can be constructed as follows (cf e.g. Lemma 6.2.2, Lemma 6.2.4 in [29]): let χ be a \mathcal{C}^∞ -smooth, real valued function on \mathbb{R}^S , which is even and satisfies

$$\chi(\xi) = 1, \quad \forall |\xi| \leq 1, \quad \text{and} \quad \chi(\xi) = 0, \quad \forall |\xi| \geq 2,$$

and denote by ρ its Fourier transform,

$$\rho(\varphi) := \frac{1}{(2\pi)^{|S|}} \int_{\mathbb{R}^S} \chi(\xi) e^{-i\varphi \cdot \xi} d\xi.$$

Then ρ is of Schwartz class and, since by assumption χ is even, real-valued. Furthermore,

$$\chi(\xi) = \int_{\mathbb{R}^S} \rho(\varphi) e^{i\varphi \cdot \xi} d\varphi$$

implies that $\int_{\mathbb{R}^S} \rho(\varphi) d\varphi = \chi(0) = 1$, and for any multi-index $\alpha \in \mathbb{Z}_{\geq 0}^S$, $\int_{\mathbb{R}^S} (i\varphi)^\alpha \rho(\varphi) d\varphi = \partial_\xi^\alpha \chi(\xi)|_{\xi=0} = 0$ where $(i\varphi)^\alpha = \prod_{k \in S} (i\varphi_k)^{\alpha_k}$. For any $t \geq 1$, we define the function $\rho_t(\varphi) := t^{|S|} \rho(t\varphi)$, which satisfies the identities

$$\int_{\mathbb{R}^S} \rho_t(\varphi) d\varphi = 1, \quad \int_{\mathbb{R}^S} (i\varphi)^\alpha \rho_t(\varphi) d\varphi = 0, \quad \forall \alpha \in \mathbb{Z}_{\geq 0}^S.$$

The ρ_t 's now yield the following one parameter family of operators,

$$S_t f(\varphi) := (\rho_t \star f)(\varphi) = \int_{\mathbb{R}^S} \rho_t(\varphi - \psi) f(\psi) d\psi, \quad \forall f \in \mathcal{C}^0(\mathbb{T}^S, X). \quad (2.17)$$

The maps $S_t f$ are \mathcal{C}^∞ -smooth and $(2\pi\mathbb{Z})^S$ -periodic, i.e.,

$$S_t : \mathcal{C}^0(\mathbb{T}^S, X) \rightarrow \mathcal{C}^\infty(\mathbb{T}^S, X) = \bigcap_{s \geq 0} \mathcal{C}^s(\mathbb{T}^S, X).$$

The following lemma can be proved in a straightforward way.

Lemma 2.6 (Smoothing operators for scales of \mathcal{C}^s -spaces). *The one parameter family of operators S_t , $t \geq 1$, defined in (2.17), is a family of smoothing operators for the scale of Banach spaces $(\mathcal{C}^s(\mathbb{T}^S, X), \|\cdot\|_{\mathcal{C}^s})$, $s \in \mathbb{Z}_{\geq 0}$.*

2.3 Tame estimates

The aim of this subsection is to discuss various tame estimates with respect to the φ -variable. Since the class of semilinear perturbations (1.5) – (1.6) considered in this paper, do not lose regularity with respect to the x -variable, tame estimates with respect to the space variable are not needed. We begin with establishing tame estimates for the product of maps u, v in $H^s(\mathbb{T}^S, H^\sigma)$. Recall that for $s \geq s_0$ and $\sigma \geq 1$, $H^s(\mathbb{T}^S, H^\sigma)$ is an algebra. Establishing tame estimates for the product uv means to bound the norm $\|uv\|_s$ by an expression which is linear in the high norms $\|u\|_s$ and $\|v\|_s$. More precisely, we have the following result.

Lemma 2.7 (Tame estimates for products of maps). *Let $s \in \mathbb{Z}_{\geq s_0}$ and $\sigma \geq 1$. Then there are constants $C_{prod}(s) \geq C_{prod}(s_0) \geq 1$ (which also might depend on σ), so that the following holds:*

(i) *for any $u, v \in H^s(\mathbb{T}^S, H^\sigma)$,*

$$\|uv\|_s \leq C_{prod}(s_0) \|u\|_{s_0} \|v\|_s + C_{prod}(s) \|u\|_s \|v\|_{s_0}; \quad (2.18)$$

(ii) *for any $u \equiv u_\omega, v \equiv v_\omega$ in $H^s(\mathbb{T}^S, H^\sigma)$, which are Lipschitz continuous in the parameter $\omega \in \Omega \subseteq \mathbb{R}^S$,*

$$\|uv\|_s^{\gamma_{lip}} \leq C_{prod}(s_0) \|u\|_{s_0}^{\gamma_{lip}} \|v\|_s^{\gamma_{lip}} + C_{prod}(s) \|u\|_s^{\gamma_{lip}} \|v\|_{s_0}^{\gamma_{lip}}. \quad (2.19)$$

In the case where $u, v \in H^s(\mathbb{T}^S, \mathbb{C})$, the same tame estimates hold with $\|\cdot\|_s$ replaced by $\|\cdot\|_{H^s(\mathbb{T}^S, \mathbb{C})}$.

Proof. The proof follows the classical argument, see e.g. [6]. We have to estimate the $\|\cdot\|_s$ -norm of the map

$$\varphi \mapsto u(\varphi)v(\varphi) = \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{k \in \mathbb{Z}^S} \hat{u}(k) \hat{v}(\ell - k) \right) e^{i\ell \cdot \varphi}.$$

Using that H^σ is an algebra and that for any two elements f, g in H^σ , $\|fg\|_\sigma \leq C\|f\|_\sigma\|g\|_\sigma$ with $C \equiv C(\sigma)$, one gets

$$\|uv\|_s^2 = \sum_{\ell \in \mathbb{Z}^S} \left\| \sum_{k \in \mathbb{Z}^S} \hat{u}(k) \hat{v}(\ell - k) \right\|_\sigma^2 \langle \ell \rangle^{2s} \leq C^2 \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{k \in \mathbb{Z}^S} \|\hat{u}(k)\|_\sigma \|\hat{v}(\ell - k)\|_\sigma \right)^2 \langle \ell \rangle^{2s} \leq 2C^2 T_1 + 2C^2 T_2 \quad (2.20)$$

where with $c(s) := 2^{1/s} - 1$,

$$T_1 := \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{\langle k \rangle > \langle \ell \rangle / (1+c(s))} \|\hat{u}(k)\|_\sigma \|\hat{v}(\ell - k)\|_\sigma \right)^2 \langle \ell \rangle^{2s},$$

and

$$T_2 := \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{\langle k \rangle \leq \langle \ell \rangle / (1+c(s))} \|\hat{u}(k)\|_\sigma \|\hat{v}(\ell - k)\|_\sigma \right)^2 \langle \ell \rangle^{2s}.$$

Estimate of T_1 . We estimate T_1 using the Cauchy-Schwartz inequality

$$\begin{aligned} T_1 &= \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{\langle k \rangle > \langle \ell \rangle / (1+c(s))} \langle k \rangle^s \|\hat{u}(k)\|_\sigma \langle \ell - k \rangle^{s_0} \|\hat{v}(\ell - k)\|_\sigma \frac{\langle \ell \rangle^s}{\langle k \rangle^s \langle \ell - k \rangle^{s_0}} \right)^2 \\ &\leq \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{\langle k \rangle > \langle \ell \rangle / (1+c(s))} \langle k \rangle^s \|\hat{u}(k)\|_\sigma \langle \ell - k \rangle^{s_0} \|\hat{v}(\ell - k)\|_\sigma \frac{2}{\langle \ell - k \rangle^{s_0}} \right)^2 \\ &\leq 4 \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{k \in \mathbb{Z}^S} \langle k \rangle^{2s} \|\hat{u}(k)\|_\sigma^2 \langle \ell - k \rangle^{2s_0} \|\hat{v}(\ell - k)\|_\sigma^2 \right) \sum_{k \in \mathbb{Z}^S} \langle k \rangle^{-2s_0}. \end{aligned}$$

Exchanging the order of the sums leads to the bound

$$T_1 \leq \tilde{C}(s_0) \sum_{k, \ell \in \mathbb{Z}^S} \langle k \rangle^{2s} \|\hat{u}(k)\|_\sigma^2 \langle \ell \rangle^{2s_0} \|\hat{v}(\ell)\|_\sigma^2 \leq \tilde{C}(s_0) \|u\|_s^2 \|v\|_{s_0}^2$$

where we emphasize that the constant $\tilde{C}(s_0)$ is independent of s .

Estimate of T_2 . In the sum T_2 we have $\langle \ell - k \rangle \geq \langle \ell \rangle - \langle k \rangle \geq \langle \ell \rangle - \frac{\langle \ell \rangle}{1+c(s)}$ and so $\frac{\langle \ell \rangle}{\langle \ell - k \rangle} \leq \frac{1+c(s)}{c(s)}$. Thus, arguing as above,

$$T_2 \leq \left(\frac{1+c(s)}{c(s)} \right)^2 \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{k \in \mathbb{Z}^S} \langle k \rangle^{2s_0} \|\hat{u}(k)\|_\sigma^2 \langle \ell - k \rangle^{2s} \|\hat{v}(\ell - k)\|_\sigma^2 \right) \sum_{k \in \mathbb{Z}^S} \langle k \rangle^{-2s_0} \leq \tilde{C}(s) \|v\|_s^2 \|u\|_{s_0}^2.$$

The claimed estimate (2.18) now follows from (2.20) with the above bounds for T_1 and T_2 . The bound (2.19) follows by applying (2.18) to the difference quotient

$$\frac{(uv)_{\omega_1} - (uv)_{\omega_2}}{\omega_1 - \omega_2} = \frac{u_{\omega_1} - u_{\omega_2}}{\omega_1 - \omega_2} v_{\omega_1} + u_{\omega_2} \frac{u_{\omega_1} - u_{\omega_2}}{\omega_1 - \omega_2}$$

for any $\omega_1, \omega_2 \in \Omega$. □

Since for any σ , the space of operators $\mathcal{L}(H^\sigma)$ is an algebra with multiplication given by the composition of operators and for any two operators A, B in $\mathcal{L}(H^\sigma)$, the operator norm $\|AB\|_\sigma$ of AB is bounded by $\|A\|_\sigma \|B\|_\sigma$, the proof of Lemma 2.7 also shows that the composition of operator valued maps satisfies tame estimates with respect to the norm $|\cdot|_s = |\cdot|_{s, \sigma}$ introduced in (2.9).

Lemma 2.8. (Tame estimates for the composition of operator valued maps) *Let $s \in \mathbb{Z}_{\geq s_0}$ and $\sigma \geq 0$. Then there are constants $C_{op}(s) \geq C_{op}(s_0) \geq 1$ (which also might depend on σ), so that the following holds:*

(i) *for any operator valued maps A, B in $H^s(\mathbb{T}^S, \mathcal{L}(H^\sigma))$,*

$$|BA|_s, |AB|_s \leq C_{op}(s) |A|_{s_0} |B|_s + C_{op}(s_0) |A|_s |B|_{s_0}; \quad (2.21)$$

(ii) for any operator valued maps $A \equiv A_\omega$ and $B \equiv B_\omega$ in $H^s(\mathbb{T}^S, \mathcal{L}(H^\sigma))$, which are Lipschitz continuous in the parameter $\omega \in \Omega \subset \mathbb{R}^S$,

$$|AB|_s^{\gamma^{\text{lip}}}, |BA|_s^{\gamma^{\text{lip}}} \leq C_{op}(s)|A|_{s_0}^{\gamma^{\text{lip}}}|B|_s^{\gamma^{\text{lip}}} + C_{op}(s_0)|A|_s^{\gamma^{\text{lip}}}|B|_{s_0}^{\gamma^{\text{lip}}}. \quad (2.22)$$

As a consequence, for any $n \geq 1$,

$$|A^n|_{s_0} \leq (2C_{op}(s_0))^{n-1}|A|_{s_0}^n \quad \text{and} \quad |A^n|_s \leq n \cdot (2C_{op}(s_0)|A|_{s_0})^{n-1} \cdot C_{op}(s)|A|_s, \quad (2.23)$$

and similar estimates hold for the Lipschitz norm $|\cdot|_s^{\gamma^{\text{lip}}}$.

(iii) The same estimates as in items (i)-(ii) hold for operator valued maps in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^\sigma \times h_\perp^\sigma))$ where the space $h_\perp^\sigma = h^\sigma(S^\perp, \mathbb{C})$ is introduced in Notations at the end of Section 1.

Remark 2.1. Occasionally we need a straightforward generalization of the estimates (2.21), (2.22). More precisely: for $A \in H^s(\mathbb{T}^S, \mathcal{L}(H^{\sigma_1}, H^{\sigma_2}))$ and $B \in H^s(\mathbb{T}^S, \mathcal{L}(H^{\sigma_2}, H^{\sigma_3}))$, $BA \in H^s(\mathbb{T}^S, \mathcal{L}(H^{\sigma_1}, H^{\sigma_3}))$ satisfies the tame estimate

$$\begin{aligned} \|BA\|_{H^s(\mathbb{T}^S, \mathcal{L}(H^{\sigma_1}, H^{\sigma_3}))} &\leq C_{op}(s)\|B\|_{H^s(\mathbb{T}^S, \mathcal{L}(H^{\sigma_2}, H^{\sigma_3}))}\|A\|_{H^{s_0}(\mathbb{T}^S, \mathcal{L}(H^{\sigma_1}, H^{\sigma_2}))} \\ &\quad + C_{op}(s_0)\|B\|_{H^{s_0}(\mathbb{T}^S, \mathcal{L}(H^{\sigma_2}, H^{\sigma_3}))}\|A\|_{H^s(\mathbb{T}^S, \mathcal{L}(H^{\sigma_1}, H^{\sigma_2}))}. \end{aligned}$$

Moreover if $A = A_\omega$, $B = B_\omega$ are Lipschitz continuous in Ω , then the above estimate holds for the corresponding Lipschitz norms.

We also need to derive tame estimates for maps of the form $\varphi \mapsto A(\varphi)u(\varphi)$ where $\varphi \mapsto u(\varphi)$ is in the Sobolev space $H^s(\mathbb{T}^S, h^\sigma)$ and $\varphi \mapsto A(\varphi)$ is an operator valued map in $H^s(\mathbb{T}^S, \mathcal{L}(H^\sigma))$. Writing A and u as Fourier series, $A(\varphi) = \sum_{\ell \in \mathbb{Z}^S} \hat{A}(\ell) e^{i\ell \cdot \varphi}$ respectively $u(\varphi) = \sum_{\ell \in \mathbb{Z}^S} \hat{u}(\ell) e^{i\ell \cdot \varphi}$, one gets

$$A(\varphi)u(\varphi) = \sum_{\ell \in \mathbb{Z}^S} \left(\sum_{k \in \mathbb{Z}^S} \hat{A}(\ell - k) \hat{u}(k) \right) e^{i\ell \cdot \varphi}.$$

Note that $\hat{A}(\ell - k) \hat{u}(k)$ is in H^σ and that its norm can be estimated as $\|\hat{A}(\ell - k) \hat{u}(k)\|_\sigma \leq \|\hat{A}(\ell - k)\|_\sigma \|\hat{u}(k)\|_\sigma$ where $\|\hat{A}(\ell - k)\|_\sigma$ denotes the operator norm of $\hat{A}(\ell - k)$ in $\mathcal{L}(H^\sigma)$. Hence the proof of Lemma 2.7 also shows that the action of operators on functions satisfies tame estimates in the following sense:

Lemma 2.9 (Tame estimates for the action of operators on maps). *Let $s \in \mathbb{Z}_{\geq s_0}$ and $\sigma \geq 0$. Then there are constants $C_{act}(s) \geq C_{act}(s_0) \geq 1$ (which also might depend on σ), so that the following holds:*

(i) for any operator valued map A in $H^s(\mathbb{T}^S, \mathcal{L}(H^\sigma))$ and any map $u \in H^s(\mathbb{T}^S, h^\sigma)$ one has

$$\|Au\|_s \leq C_{act}(s)|A|_{s_0}\|u\|_s + C_{act}(s_0)|A|_s\|u\|_{s_0}; \quad (2.24)$$

(ii) for any operator valued map $A \equiv A_\omega$ and any map $u \equiv u_\omega$, which are both Lipschitz continuous in the parameter $\omega \in \Omega \subseteq \mathbb{R}^S$,

$$\|Au\|_s^{\gamma^{\text{lip}}} \leq C_{act}(s)|A|_{s_0}^{\gamma^{\text{lip}}}\|u\|_s^{\gamma^{\text{lip}}} + C_{act}(s_0)|A|_s^{\gamma^{\text{lip}}}\|u\|_{s_0}^{\gamma^{\text{lip}}}. \quad (2.25)$$

Lemma 2.8 can be used to derive tame estimates for the exponential of an operator valued map. We state them in the specific form needed in Section 6 where we consider operator valued maps in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^\sigma \times h_\perp^\sigma))$ with $h_\perp^\sigma = h^\sigma(S^\perp, \mathbb{C})$. We introduce the vector valued Fourier multiplier

$$\mathfrak{D} := \text{diag}(\langle\langle D \rangle\rangle, \langle\langle D \rangle\rangle) : h_\perp^\sigma \times h_\perp^\sigma \rightarrow h_\perp^\sigma \times h_\perp^\sigma \quad (2.26)$$

where we recall that $\langle\langle D \rangle\rangle$ is defined in (2.12). Let \mathbb{I}_2 be the identity operator on $h_\perp^\sigma \times h_\perp^\sigma$.

Lemma 2.10. (Tame estimates for the exponential of operators) *Assume that $s \in \mathbb{Z}_{\geq s_0}$, $\sigma \geq \mathbb{Z}_{\geq 4}$ and $C_{op}(s_0) \geq 1$ is the constant in Lemma 2.8-(iii). Then for any Lipschitz continuous map $A \equiv A_\omega$, $\omega \in \Omega \subset \mathbb{R}^S$, with values in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^\sigma \times h_\perp^\sigma))$, the following holds:*

(i) if A satisfies the smallness condition $2C_{op}(s_0)|A|_{s_0}^{\gamma_{\text{lip}}} \leq 1$, then $\Phi := \exp(A)$ and its inverse $\Phi^{-1} = \exp(-A)$ satisfy

$$|\Phi^{\pm 1} - \mathbb{I}_2|_s \leq_s |A|_s \quad \text{and} \quad |\Phi^{\pm 1} - \mathbb{I}_2|_s^{\gamma_{\text{lip}}} \leq_s |A|_s^{\gamma_{\text{lip}}}; \quad (2.27)$$

(ii) if A satisfies $2C_{op}(s_0)|A\mathfrak{D}|_{s_0}^{\gamma_{\text{lip}}} \leq 1$ and in addition $A(\varphi) \in \mathcal{L}(h_{\perp}^{\sigma-1} \times h_{\perp}^{\sigma-1}, h_{\perp}^{\sigma} \times h_{\perp}^{\sigma})$ for any $\varphi \in \mathbb{T}^S$, then

$$|(\Phi^{\pm 1} - \mathbb{I}_2)\mathfrak{D}|_s \leq_s |A\mathfrak{D}|_s \quad \text{and} \quad |(\Phi^{\pm 1} - \mathbb{I}_2)\mathfrak{D}|_s^{\gamma_{\text{lip}}} \leq_s |A\mathfrak{D}|_s^{\gamma_{\text{lip}}}; \quad (2.28)$$

(iii) if A satisfies $2C_{op}(s_0)|A|_{s_0, \sigma} \leq 1$ and in addition for any $\sigma' \in \{\sigma, \sigma-1, \sigma-2, \sigma-3\}$, $A \in H^s(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma'} \times h_{\perp}^{\sigma'}))$ with $|A|_{s, \sigma'} < |A|_{s, \sigma}$ and $|A|_{s_0, \sigma'} < |A|_{s_0, \sigma}$, then

$$\left| \sum_{n \geq 2} \frac{1}{n!} \mathfrak{D}^2 (\mathfrak{D}^{-1} A \mathfrak{D}^{-1})^n \mathfrak{D} \right|_{s, \sigma-1}, \quad \left| \sum_{n \geq 2} \frac{1}{n!} (\mathfrak{D}^{-1} A \mathfrak{D}^{-1})^n \mathfrak{D}^3 \right|_{s, \sigma-1} \leq_s |A|_{s, \sigma} |A|_{s_0, \sigma};$$

(iv) if A satisfies $2C_{op}(s_0)|A|_{s_0, \sigma} \leq 1$ and in addition for any $\sigma' \in \{\sigma+1, \sigma, \sigma-1, \sigma-2, \sigma-3, \sigma-4\}$, $A \in H^s(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma'} \times h_{\perp}^{\sigma'}))$ with $|A|_{s, \sigma'} < |A|_{s, \sigma+1}$ and $|A|_{s_0, \sigma'} < |A|_{s_0, \sigma}$, then

$$\begin{aligned} & \left| \sum_{n \geq 3} \frac{1}{n!} \mathfrak{D}^2 (\mathfrak{D}^{-1} A)^n \mathfrak{D} \right|_{s, \sigma-1}, \quad \left| \sum_{n \geq 3} \frac{1}{n!} \mathfrak{D}^2 (A \mathfrak{D}^{-1})^n \mathfrak{D} \right|_{s, \sigma-1} \leq_s |A|_{s, \sigma+1} |A|_{s_0, \sigma+1}^2, \\ & \left| \sum_{n \geq 3} \frac{1}{n!} (\mathfrak{D}^{-1} A)^n \mathfrak{D}^3 \right|_{s, \sigma-1}, \quad \left| \sum_{n \geq 3} \frac{1}{n!} (A \mathfrak{D}^{-1})^n \mathfrak{D}^3 \right|_{s, \sigma-1} \leq_s |A|_{s, \sigma+1} |A|_{s_0, \sigma+1}^2; \end{aligned}$$

(v) assume that $\Phi_i = \exp(A_i)$, $i = 1, 2$, with $A_i \in H^s(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma} \times h_{\perp}^{\sigma}))$ such that

$$2C_{op}(s_0)|A_i|_{s_0} \leq 1. \quad (2.29)$$

Then the difference $\Phi_2^{-1} - \Phi_1^{-1}$ satisfies the estimate

$$|\Phi_2^{-1} - \Phi_1^{-1}|_s \leq_s |A_2 - A_1|_s + (|A_1|_s + |A_2|_s)|A_2 - A_1|_{s_0}. \quad (2.30)$$

Similarly, if $A_i(\varphi) \in \mathcal{L}(h_{\perp}^{\sigma-1} \times h_{\perp}^{\sigma-1}, h_{\perp}^{\sigma} \times h_{\perp}^{\sigma})$, $\varphi \in \mathbb{T}^S$, and $2C_{op}(s_0)|A_i\mathfrak{D}|_{s_0} \leq 1$, then

$$|(\Phi_2^{-1} - \Phi_1^{-1})\mathfrak{D}|_s \leq_s |(A_2 - A_1)\mathfrak{D}|_s + (|A_1\mathfrak{D}|_s + |A_2\mathfrak{D}|_s)|(A_2 - A_1)\mathfrak{D}|_{s_0}. \quad (2.31)$$

Proof. (i) Let us prove the estimate (2.27) for $|\cdot|_s$. The estimate with the norm $|\cdot|_s^{\gamma_{\text{lip}}}$ can be proven similarly. We have, with $C_{op}(s)$, $C_{op}(s_0)$ given as in Lemma 2.8-(iii),

$$|\Phi^{\pm 1} - \mathbb{I}_2|_s \leq \sum_{n \geq 1} \frac{|A^n|_s}{n!} \stackrel{(2.23)}{\leq} C_{op}(s)|A|_s \sum_{n \geq 1} \frac{(2C_{op}(s_0)|A|_{s_0})^{n-1}}{(n-1)!} = C_{op}(s)|A|_s \exp(2C_{op}(s_0)|A|_{s_0}) \leq_s |A|_s.$$

(ii) Now let us prove the inequality (2.28) for $|\cdot|_s$. The corresponding estimate with the norm $|\cdot|_s^{\gamma_{\text{lip}}}$ is shown in a similar way. For any $n \geq 2$,

$$\begin{aligned} |A^n \mathfrak{D}|_s & \leq C_{op}(s)|A^{n-1}|_{s_0}|A\mathfrak{D}|_s + C_{op}(s_0)|A^{n-1}|_s|A\mathfrak{D}|_{s_0} \\ & \stackrel{(2.23)}{\leq_s} C_{op}(s)C_{op}(s_0)(n(2C_{op}(s_0)|A|_{s_0})^{n-2}|A|_s|A\mathfrak{D}|_{s_0} + (2C_{op}(s_0))^{n-2}|A|_{s_0}^{n-1}|A\mathfrak{D}|_s) \\ & \leq_s (C_{op}(s))^2 n(|A|_s + |A\mathfrak{D}|_s) \leq_s 2(C_{op}(s))^2 n|A\mathfrak{D}|_s. \end{aligned}$$

Hence

$$|(\Phi^{\pm 1} - \mathbb{I}_2)\mathfrak{D}|_s \leq_s |A\mathfrak{D}|_s \sum_{n \geq 1} \frac{1}{(n-1)!} \leq_s |A\mathfrak{D}|_s.$$

(iii) For any $n \geq 2$, one has

$$\mathfrak{D}^2 (\mathfrak{D}^{-1} A \mathfrak{D}^{-1})^n \mathfrak{D} = \mathfrak{D} A \mathfrak{D}^{-1} B^{n-2} \mathfrak{D}^{-1} A, \quad B := \mathfrak{D}^{-1} A \mathfrak{D}^{-1}.$$

Let us estimate separately the norms of $\mathfrak{D}A\mathfrak{D}^{-1}$, B^{n-2} , and $\mathfrak{D}^{-1}A$. We have

$$|\mathfrak{D}A\mathfrak{D}^{-1}|_{s,\sigma-1} \leq \|\mathfrak{D}\|_{\mathcal{L}(h^\sigma, h^{\sigma-1})} |A|_{s,\sigma} \|\mathfrak{D}^{-1}\|_{\mathcal{L}(h^{\sigma-1}, h^\sigma)} \ll |A|_{s,\sigma}, \quad |\mathfrak{D}A\mathfrak{D}^{-1}|_{s_0,\sigma-1} \ll |A|_{s_0,\sigma}.$$

Since for $n \geq 3$

$$|B^{n-2}|_{s_0,\sigma} \stackrel{(2.23)}{\leq} (2C_{op}(s_0))^{n-3} |B|_{s_0,\sigma}^{n-2}, \quad |B^{n-2}|_{s,\sigma} \stackrel{(2.23)}{\leq} nC_{op}(s)(2C_{op}(s_0))^{n-3} |B|_{s_0,\sigma}^{n-3} |B|_{s,\sigma},$$

it then follows from

$$|B|_{s_0,\sigma} = |\mathfrak{D}^{-1}A\mathfrak{D}^{-1}|_{s_0,\sigma} \leq |A|_{s_0,\sigma}, \quad |B|_{s,\sigma} = |\mathfrak{D}^{-1}A\mathfrak{D}^{-1}|_{s,\sigma} \leq |A|_{s,\sigma},$$

and $2C_{op}(s_0)|A|_{s_0,\sigma} \leq 1$ that for $n \geq 3$,

$$|B^{n-2}|_{s_0,\sigma} \leq 1, \quad |B^{n-2}|_{s,\sigma} \leq nC_{op}(s)|A|_{s,\sigma}.$$

Using that

$$|\mathfrak{D}^{-1}A|_{s,\sigma-1} \leq |A|_{s,\sigma-1} \ll |A|_{s,\sigma} \quad \text{and} \quad |\mathfrak{D}^{-1}A|_{s_0,\sigma-1} \leq |A|_{s_0,\sigma-1} \ll |A|_{s_0,\sigma}$$

one then concludes from (2.21) that for any $n \geq 3$,

$$|\mathfrak{D}A\mathfrak{D}^{-1}B^{n-2}\mathfrak{D}^{-1}A|_{s,\sigma-1} \leq n|A|_{s,\sigma}|A|_{s_0,\sigma}$$

and in turn

$$\left| \sum_{n \geq 2} \frac{1}{n!} \mathfrak{D}^2 (\mathfrak{D}^{-1}A\mathfrak{D}^{-1})^n \mathfrak{D} \right|_{s,\sigma-1} \leq |A|_{s,\sigma} |A|_{s_0,\sigma} \sum_{n \geq 2} \frac{n}{n!} \leq |A|_{s,\sigma} |A|_{s_0,\sigma}.$$

The estimate for $|\sum_{n \geq 2} \frac{1}{n!} (\mathfrak{D}^{-1}A\mathfrak{D}^{-1})^n \mathfrak{D}^3|_{s,\sigma-1}$ follows by similar arguments.

(iv) The four series are estimated in the same way. Let us just comment how to prove the estimate for $\sum_{n \geq 3} \frac{1}{n!} (\mathfrak{D}^{-1}A)^n \mathfrak{D}^3$ which we write as the composition $B_1 B_2$ where

$$B_1 := \sum_{n \geq 3} \frac{1}{n!} (\mathfrak{D}^{-1}A)^{n-3}, \quad B_2 := (\mathfrak{D}^{-1}A)^3 \mathfrak{D}^3.$$

The norm $|B_2|_{s,\sigma-1}$ is treated separately using Remark 2.1, whereas the series B_1 is estimated in the same way as the ones of item (iii). To obtain the claimed estimate we then apply Lemma 2.8 to the composition $B_1 B_2$.

(v) Since $\Phi_i^{-1} = \exp(-A_i)$ the estimate (2.30) for $\Phi_2^{-1} - \Phi_1^{-1}$ is obtained from the one for $\Phi_2 - \Phi_1$ by replacing A_i by $-A_i$. Observe that

$$\Phi_2 - \Phi_1 = \sum_{n \geq 1} \frac{A_2^n - A_1^n}{n!} = \sum_{n \geq 1} \frac{1}{n!} \left(\widehat{A} A_2^{n-1} + A_1 \widehat{A} A_2^{n-2} + \dots + A_1^{n-2} \widehat{A} A_2 + A_1^{n-1} \widehat{A} \right),$$

where $\widehat{A} := A_2 - A_1$. The terms $A_1^k \widehat{A} A_2^{n-k-1}$, $1 \leq k \leq n-2$, of the above sum can be estimated as follows

$$\begin{aligned} |A_1^k \widehat{A} A_2^{n-k-1}|_s &\stackrel{(2.21)}{\leq} C_{op}(s) C_{op}(s_0) (|A_1^k|_s 2|\widehat{A}|_{s_0} |A_2^{n-k-1}|_{s_0} + |A_1^k|_{s_0} |\widehat{A}|_s |A_2^{n-k-1}|_{s_0} + |A_1^k|_{s_0} |\widehat{A}|_{s_0} |A_2^{n-k-1}|_s) \\ &\stackrel{(2.23),(2.29)}{\leq} nC_{op}(s)^2 (|A_1|_s + |A_2|_s) |\widehat{A}|_{s_0} + |\widehat{A}|_s. \end{aligned}$$

The terms $|\widehat{A} A_2^{n-1}|_s$ and $|A_1^{n-1} \widehat{A}|_s$ can be estimated in the same way and admit similar bounds. Hence

$$|\Phi_2 - \Phi_1|_s \leq s \left(\sum_{n \geq 1} \frac{n^2}{n!} \right) (|A_1|_s + |A_2|_s) |\widehat{A}|_{s_0} + |\widehat{A}|_s$$

implying (2.30). The proof of the estimate (2.31) is similar. \square

Finally we want to derive tame estimates for the composed map $f \circ \check{i}$ where \check{i} denotes a map $\check{i} : \mathbb{T}^S \rightarrow M^\sigma$ and $f : M^\sigma \rightarrow Y$ takes values in the Banach space Y .

Recall that $M^\sigma = \mathbb{T}^S \times U_0 \times h_\perp^\sigma$ denotes the phase space introduced in (1.20). We assume that \check{i} has a lift of the form $(\varphi, 0, 0) + \iota(\varphi)$ where $\iota : \mathbb{R}^S \rightarrow \mathbb{R}^S \times U_0 \times h_\perp^\sigma$ is $(2\pi\mathbb{Z})^S$ -periodic. Whenever the context permits, we will identify \check{i} with its lift and denote both by the same letter. Similarly, we will identify maps $\mathbb{T}^S \rightarrow Y$ with their lifts $\mathbb{R}^S \rightarrow Y$, which are $(2\pi\mathbb{Z})^S$ -periodic.

Lemma 2.11. (Tame estimates for the composition of maps in C^s -spaces) *Assume that f is a map in $C^s(\mathbb{T}^S \times V, Y)$ where V is an open neighborhood in $\mathbb{R}^S \times h_\perp^\sigma$ and $s \in \mathbb{Z}_{\geq 0}$. Then for any map $\check{i}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \in C^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$ and $\check{i}(\mathbb{T}^S) \subset \mathbb{T}^S \times V$, the following holds:*

(i) *The composition $f \circ \check{i} \in C^s(\mathbb{T}^S, Y)$ satisfies the tame estimate*

$$\|f \circ \check{i}\|_{C^s} \leq C(s, \|f\|_{C^s}, \|\iota\|_{C^0}) \cdot (1 + \|\iota\|_{C^s}). \quad (2.32)$$

(ii) *If $f \in C^{s+1}(\mathbb{T}^S \times V, Y)$, then for any $\hat{\iota}$ in $C^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$,*

$$\|df(\check{i})[\hat{\iota}]\|_{C^s} \leq C(s, \|f\|_{C^{s+1}}, \|\iota\|_{C^0}) \cdot (\|\hat{\iota}\|_{C^s} + \|\iota\|_{C^s} \|\hat{\iota}\|_{C^0}). \quad (2.33)$$

(iii) *If $f \in C^{s+1}(\mathbb{T}^S \times V, Y)$ and V is in addition convex, then for any two maps, $\check{i}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$ with $\iota^{(a)} \in C^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$ and $\check{i}^{(a)}(\mathbb{T}^S) \subset \mathbb{T}^S \times V$, $a = 1, 2$, the difference $\Delta_{12}f = f \circ \check{i}^{(1)} - f \circ \check{i}^{(2)}$ satisfies the estimate*

$$\|\Delta_{12}f\|_{C^s} \leq C(s, \|f\|_{C^{s+1}}, \|\iota^{(1)}\|_{C^0}, \|\iota^{(2)}\|_{C^0}) \cdot (\|\Delta_{12}\iota\|_{C^s} + (\|\iota^{(1)}\|_{C^s} + \|\iota^{(2)}\|_{C^s}) \|\Delta_{12}\iota\|_{C^0})$$

where $\Delta_{12}\iota := \iota^{(1)} - \iota^{(2)}$.

(iv) *If $f \in C^{s+1}(\mathbb{T}^S \times V, Y)$ and in addition V is convex and $\iota \equiv \iota_\omega$ Lipschitz continuous in the parameter $\omega \in \Omega \subset \mathbb{R}^S$, the composition $f \circ \check{i} \in C^s(\mathbb{T}^S, Y)$ is also Lipschitz continuous in ω and satisfies the estimate*

$$\|f \circ \check{i}\|_{C^s}^{\text{lip}} \leq C(s, \|f\|_{C^{s+1}}, \|\iota\|_{C^0}^{\text{sup}}) \cdot (\|\iota\|_{C^s}^{\text{lip}} + \|\iota\|_{C^s}^{\text{sup}} \|\iota\|_{C^0}^{\text{lip}}). \quad (2.34)$$

Proof. (i) For any multi-index $\alpha \in \mathbb{Z}_{\geq 0}^S$ with $1 \leq |\alpha| \leq s$, one computes

$$\partial_\varphi^\alpha (f \circ \check{i})(\varphi) = \sum_{\substack{1 \leq m \leq |\alpha| \\ \alpha = \alpha_1 + \dots + \alpha_m}} c_{\alpha_1, \dots, \alpha_m} (d^m f)(\check{i}(\varphi)) [\partial_\varphi^{\alpha_1} \check{i}(\varphi), \dots, \partial_\varphi^{\alpha_m} \check{i}(\varphi)]$$

where $c_{\alpha_1, \dots, \alpha_m}$ are combinatorial constants and $\alpha_1, \dots, \alpha_m$ are nonzero integer vectors in $\mathbb{Z}_{\geq 0}^S$. Hence

$$\begin{aligned} \|\partial_\varphi^\alpha (f \circ \check{i})\|_{C^0} &\leq C(s, \|f\|_{C^s}) \sum_{\substack{1 \leq m \leq |\alpha| \\ \alpha = \alpha_1 + \dots + \alpha_m}} \|\partial_\varphi^{\alpha_1} \check{i}\|_{C^0} \cdots \|\partial_\varphi^{\alpha_m} \check{i}\|_{C^0} \\ &\leq C(s, \|f\|_{C^s}) \sum_{\substack{1 \leq m \leq |\alpha| \\ \alpha = \alpha_1 + \dots + \alpha_m}} (1 + \|\iota\|_{C^{|\alpha_1|}}) \cdots (1 + \|\iota\|_{C^{|\alpha_m|}}). \end{aligned} \quad (2.35)$$

We claim that for any $0 \leq k \leq |\alpha|$, there exists a constant $C_{|\alpha|, k} > 0$ such that

$$1 + \|\iota\|_{C^k} \leq C_{|\alpha|, k} (1 + \|\iota\|_{C^0})^{1 - \frac{k}{|\alpha|}} (1 + \|\iota\|_{C^{|\alpha|}})^{\frac{k}{|\alpha|}}. \quad (2.36)$$

Indeed, by the interpolation estimates for C^s -spaces (Proposition 2.1, Lemma 2.6) one has $\|\iota\|_{C^k} \leq \|\iota\|_{C^0}^{1 - \frac{k}{|\alpha|}} \|\iota\|_{C^{|\alpha|}}^{\frac{k}{|\alpha|}}$ yielding

$$1 + \|\iota\|_{C^k} \leq C'_{|\alpha|, k} (1 + \|\iota\|_{C^0}^{1 - \frac{k}{|\alpha|}}) (1 + \|\iota\|_{C^{|\alpha|}}^{\frac{k}{|\alpha|}}). \quad (2.37)$$

Since for any $0 \leq \lambda \leq 1$, $f_\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}$, $t \mapsto t^\lambda$ is concave, one has

$$\frac{1}{2}(1 + t^\lambda) = \frac{1}{2}f_\lambda(1) + \frac{1}{2}f_\lambda(t) \leq f_\lambda\left(\frac{1+t}{2}\right) = 2^{-\lambda}(1+t)^\lambda$$

implying that $(1 + t^\lambda) \leq 2^{1-\lambda}(1 + t)^\lambda$ for any $t \geq 0$. Thus we conclude that

$$1 + \|\iota\|_{\mathcal{C}^0}^{1-\frac{k}{|\alpha|}} \leq 2^{\frac{k}{|\alpha|}}(1 + \|\iota\|_{\mathcal{C}^0})^{1-\frac{k}{|\alpha|}}, \quad 1 + \|\iota\|_{\mathcal{C}^{|\alpha|}}^{\frac{k}{|\alpha|}} \leq 2^{1-\frac{k}{|\alpha|}}(1 + \|\iota\|_{\mathcal{C}^{|\alpha|}})^{\frac{k}{|\alpha|}}.$$

Combining this with (2.37) yields (2.36). Applying the estimate (2.36) to the products in (2.35), one gets

$$(1 + \|\iota\|_{\mathcal{C}^{|\alpha_1|}}) \cdots (1 + \|\iota\|_{\mathcal{C}^{|\alpha_m|}}) \leq C_s \prod_{j=1}^m (1 + \|\iota\|_{\mathcal{C}^0})^{1-\frac{|\alpha_j|}{|\alpha|}} (1 + \|\iota\|_{\mathcal{C}^{|\alpha_j|}})^{\frac{|\alpha_j|}{|\alpha|}} \leq C_s (1 + \|\iota\|_{\mathcal{C}^0})^{m-1} (1 + \|\iota\|_{\mathcal{C}^{|\alpha|}})$$

which proves the estimate (2.32).

(ii) By the Leibnitz rule, for any multi-index $\beta \in \mathbb{Z}_{\geq 0}^S$ with $0 \leq |\beta| \leq s$, and any $\hat{\iota} \in \mathcal{C}^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma_1})$, one has

$$\partial_{\varphi}^{\beta} df(\check{\iota}(\varphi))[\hat{\iota}(\varphi)] = \sum_{\beta_1 + \beta_2 = \beta} c_{\beta_1, \beta_2} \partial_{\varphi}^{\beta_1} (df(\check{\iota}(\varphi))) [\partial_{\varphi}^{\beta_2} \hat{\iota}(\varphi)]$$

where c_{β_1, β_2} are combinatorial constants. Each term in the latter sum is estimated individually. For the term with $\beta_1 = 0, \beta_2 = \beta$ one gets

$$\|df(\check{\iota})[\partial_{\varphi}^{\beta} \hat{\iota}]\|_{\mathcal{C}^0} \leq \|f\|_{\mathcal{C}^1} \|\hat{\iota}\|_{\mathcal{C}^{|\beta|}} \leq \|f\|_{\mathcal{C}^1} \|\hat{\iota}\|_{\mathcal{C}^s}$$

whereas in the case $1 \leq |\beta_1| \leq s$, one has

$$\partial_{\varphi}^{\beta_1} (df(\check{\iota}(\varphi))) [\partial_{\varphi}^{\beta_2} \hat{\iota}(\varphi)] = \sum_{\substack{1 \leq m \leq |\beta_1| \\ \alpha_1 + \cdots + \alpha_m = \beta_1}} c_{\alpha_1, \dots, \alpha_m} d^{m+1} f(\check{\iota}(\varphi)) [\partial_{\varphi}^{\alpha_1} \check{\iota}(\varphi), \dots, \partial_{\varphi}^{\alpha_m} \check{\iota}(\varphi), \partial_{\varphi}^{\beta_2} \hat{\iota}(\varphi)]$$

yielding

$$\|\partial_{\varphi}^{\beta_1} (df(\check{\iota})) [\partial_{\varphi}^{\beta_2} \hat{\iota}]\|_{\mathcal{C}^0} \leq C(s, \|f\|_{\mathcal{C}^{s+1}}) \sum_{\substack{1 \leq m \leq |\beta_1| \\ \alpha_1 + \cdots + \alpha_m = \beta_1}} (1 + \|\iota\|_{\mathcal{C}^{|\alpha_1|}}) \cdots (1 + \|\iota\|_{\mathcal{C}^{|\alpha_m|}}) \|\hat{\iota}\|_{\mathcal{C}^{|\beta_2|}}.$$

Since $|\alpha_1| + \cdots + |\alpha_m| + |\beta_2| = |\beta_1| + |\beta_2| = |\beta|$, the interpolation estimates for \mathcal{C}^s -spaces (Proposition 2.1, Lemma 2.6) and the estimate (2.36), then lead to

$$(1 + \|\iota\|_{\mathcal{C}^{|\alpha_1|}}) \cdots (1 + \|\iota\|_{\mathcal{C}^{|\alpha_m|}}) \|\hat{\iota}\|_{\mathcal{C}^{|\beta_2|}} \leq C_s \|\hat{\iota}\|_{\mathcal{C}^0}^{1-\frac{|\beta_2|}{|\beta|}} \|\hat{\iota}\|_{\mathcal{C}^{|\beta_1|}}^{\frac{|\beta_2|}{|\beta|}} \prod_{j=1}^m (1 + \|\iota\|_{\mathcal{C}^0})^{1-\frac{|\alpha_j|}{|\beta|}} (1 + \|\iota\|_{\mathcal{C}^{|\alpha_j|}})^{\frac{|\alpha_j|}{|\beta|}}.$$

Using that $\frac{\sum_{j=1}^m |\alpha_j|}{|\beta|} = \frac{|\beta_1|}{|\beta|} = 1 - \frac{|\beta_2|}{|\beta|}$ it then follows that

$$(1 + \|\iota\|_{\mathcal{C}^{|\alpha_1|}}) \cdots (1 + \|\iota\|_{\mathcal{C}^{|\alpha_m|}}) \|\hat{\iota}\|_{\mathcal{C}^{|\beta_2|}} \leq C(s, \|\iota\|_{\mathcal{C}^0}) \cdot \|\hat{\iota}\|_{\mathcal{C}^0}^{\frac{|\beta_1|}{|\beta|}} (1 + \|\iota\|_{\mathcal{C}^{|\beta_1|}})^{\frac{|\beta_1|}{|\beta|}} \cdot \|\hat{\iota}\|_{\mathcal{C}^{|\beta_1|}}^{\frac{|\beta_2|}{|\beta|}}$$

and by Young's inequality with exponents $|\beta|/|\beta_1|, |\beta|/|\beta_2|$ we conclude that

$$(1 + \|\iota\|_{\mathcal{C}^{|\alpha_1|}}) \cdots (1 + \|\iota\|_{\mathcal{C}^{|\alpha_m|}}) \|\hat{\iota}\|_{\mathcal{C}^{|\beta_2|}} \leq C(s, \|\iota\|_{\mathcal{C}^0}) (\|\hat{\iota}\|_{\mathcal{C}^{|\beta_1|}} + \|\iota\|_{\mathcal{C}^{|\beta_1|}} \|\hat{\iota}\|_{\mathcal{C}^0}).$$

Combining the estimates obtained so far, the estimate (2.33) follows.

(iii) Since by assumption, V is convex, the claimed estimates for $\Delta_{12} f$ can be derived from the estimates of item (ii) by the mean value theorem.

(iv) The estimate (2.34) directly follows from the estimates of item (iii). \square

When combined with the inequalities (2.1), Lemma 2.11 leads to tame estimates in the case where $\check{\iota}$ are maps in Sobolev spaces. We state them in the form needed in the sequel.

Lemma 2.12. (Tame estimates for the composition of maps in H^s -spaces) Assume that f is in $\mathcal{C}^{s+s_0}(\mathbb{T}^S \times V, Y)$, where V is an open subset contained in $\mathbb{R}^S \times h_{\perp}^{\sigma}$ and $s \in \mathbb{Z}_{\geq 0}$. Then the following holds:

(i) There exists a constant $C(s) > 0$ (depending on $\|f\|_{\mathcal{C}^{s+s_0}}$) so that for any map $\check{i}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \in H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma})$, $\|\iota\|_{s_0} \leq 1$, and $\check{i}(\mathbb{T}^S) \subset \mathbb{T}^S \times V$, the composition $f \circ \check{i}$ is in $H^s(\mathbb{T}^S, Y)$ and satisfies the tame estimate

$$\|f \circ \check{i}\|_{s,Y} \leq C(s)(1 + \|\iota\|_{s+2s_0}). \quad (2.38)$$

(ii) Assume in addition that $f \in \mathcal{C}^{s+s_0+1}(\mathbb{T}^S \times V, Y)$ and V is convex. Then there exists a constant $C(s) > 0$ (depending on $\|f\|_{\mathcal{C}^{s+s_0+1}}$) so that for any two maps, $\check{i}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(j)}(\varphi)$ with $\iota^{(a)} \in H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma})$, $\|\iota^{(a)}\|_{s_0} \leq 1$, and $\check{i}^{(a)}(\mathbb{T}^S) \subset \mathbb{T}^S \times V$, $a = 1, 2$, the difference $\Delta_{12}f = f \circ \check{i}^{(1)} - f \circ \check{i}^{(2)}$ satisfies the tame estimate

$$\|\Delta_{12}f\|_{s,Y} \leq C(s) \cdot (\|\Delta_{12}\iota\|_{s+2s_0} + (\|\iota^{(1)}\|_{s+2s_0} + \|\iota^{(2)}\|_{s+2s_0})\|\Delta_{12}\iota\|_{s_0})$$

where $\Delta_{12}\iota := \iota^{(1)} - \iota^{(2)}$.

(iii) Assume in addition that $f \in \mathcal{C}^{s+s_0+1}(\mathbb{T}^S \times V, Y)$ and V is convex. Then there exists a constant $C(s) > 0$ (depending on $\|f\|_{\mathcal{C}^{s+s_0+1}}$) so that for any map $\check{i}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\check{i}(\mathbb{T}^S) \subset \mathbb{T}^S \times V$ and $\iota \equiv \iota_{\omega} \in H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma})$ having the property that it is Lipschitz continuous in the parameter $\omega \in \Omega \subset \mathbb{R}^S$ and satisfies $\|\iota\|_{s_0}^{\text{sup}} \leq 1$, the composition $f \circ \check{i}$ is in $H^s(\mathbb{T}^S, Y)$, is Lipschitz continuous in ω , and admits the tame estimate

$$\|f \circ \check{i}\|_s^{\text{lip}} \leq C(s) \cdot (\|\iota\|_{s+2s_0}^{\text{lip}} + \|\iota\|_{s+2s_0}^{\text{sup}} \|\iota\|_{s_0}^{\text{lip}}).$$

3 Setup and preliminary estimates

In this section we review properties of the Birkhoff coordinates, constructed in [19], discuss asymptotic estimates of the dNLS frequencies, and describe the Hamiltonian setup for the perturbation of the dNLS equation. Furthermore we provide (tame) estimates of the composition and its derivatives of torus embeddings with the dNLS Hamiltonian H^{nls} and with the perturbation P , needed in the sequel.

3.1 Normal form of the dNLS equation

Introduce the \mathbb{R} -subspaces H_r^{σ} of $H^{\sigma} \times H^{\sigma}$ and h_r^{σ} of $h^{\sigma} \times h^{\sigma}$, defined by

$$H_r^{\sigma} := \{(u, \bar{u}) : u \in H^{\sigma}\}, \quad h_r^{\sigma} := \left\{ ((w_k)_{k \in \mathbb{Z}}, (\bar{w}_k)_{k \in \mathbb{Z}}) : (w_k)_{k \in \mathbb{Z}} \in h^{\sigma} \right\}$$

with H^{σ} and h^{σ} defined in (1.25) and (1.10). Denote by F_{nls} the following version of the Fourier transform in the space variable introduced in [19]

$$F_{nls} : H^0 \times H^0 \rightarrow h^0 \times h^0, \quad (u^{(1)}, u^{(2)}) \rightarrow \left((-u_{-k}^{(1)})_{k \in \mathbb{Z}}, (-u_k^{(2)})_{k \in \mathbb{Z}} \right) \quad (3.1)$$

where the Fourier coefficients $u_k^{(1)}, u_k^{(2)}$ are defined as in (1.26). Note that for $(u^{(1)}, u^{(2)}) \in H_r^0$, one has $u^{(2)} = \bar{u}^{(1)}$, implying that for any $k \in \mathbb{Z}$, $u_k^{(2)} = \bar{u}_{-k}^{(1)}$. Hence F_{nls} maps H_r^0 into h_r^0 . In fact, for any $\sigma \geq 0$, $F_{nls} : H_r^{\sigma} \rightarrow h_r^{\sigma}$ is a linear isomorphism. The definition of F_{nls} in (3.1) is related to the specific choices made in the construction of the Birkhoff coordinates in [19] – see Theorem 3.1 below.

In addition we introduce the bilinear bounded map

$$I : h^{\sigma} \times h^{\sigma} \rightarrow \ell^{1,2\sigma}, \quad ((z_k)_{k \in \mathbb{Z}}, (w_k)_{k \in \mathbb{Z}}) \rightarrow (z_k w_k)_{k \in \mathbb{Z}},$$

where $\ell^{1,2\sigma} \equiv \ell^{1,2\sigma}(\mathbb{Z}, \mathbb{C})$ denotes the weighted ℓ^1 sequence space

$$\ell^{1,2\sigma} := \left\{ (y_k)_{k \in \mathbb{Z}} \subseteq \mathbb{C} : \sum_{k \in \mathbb{Z}} \langle k \rangle^{2\sigma} |y_k| < +\infty \right\}. \quad (3.2)$$

Clearly, for $\sigma' \leq \sigma$ we have the continuous embedding $\ell^{1,2\sigma} \hookrightarrow \ell^{1,2\sigma'}$. Note that for $(w_k)_{k \in \mathbb{Z}}$ in h_r^σ , $(I_k)_{k \in \mathbb{Z}} = (w_k \bar{w}_k)_{k \in \mathbb{Z}}$ is in the positive quadrant

$$\ell_+^{1,2\sigma} = \{(y_k)_{k \in \mathbb{Z}} \in \ell^{1,2\sigma} : y_k \geq 0, \forall k \in \mathbb{Z}\}.$$

The following theorem summarizes the pertinent properties of the Birkhoff coordinates for the dNLS equation, used in the sequel.

Theorem 3.1 ([19], [24]). **(Birkhoff coordinates)** (i) *There exists a neighbourhood \mathcal{W} in $H^0 \times H^0$ and an analytic map $\Phi^{nls} : \mathcal{W} \rightarrow h^0 \times h^0$ with the following properties:*

(BC1) *For any $\sigma \in \mathbb{Z}_{\geq 0}$, $\Phi^{nls}(H_r^\sigma) \subseteq h_r^\sigma$ and $\Phi^{nls} : H_r^\sigma \rightarrow h_r^\sigma$ is a real analytic diffeomorphism.*

(BC2) *The map Φ^{nls} is canonical on H_r^0 with respect to the Poisson bracket (1.2), i.e., $\{w_k, \bar{w}_k\} = -i$ for any $k \in \mathbb{Z}$, whereas all other Poisson brackets between coordinate functions vanish.*

(BC3) *The Hamiltonian \mathcal{H}^{nls} of dNLS, when expressed in Birkhoff coordinates on h_r^1 , is a function of the actions $I = (I_k)_{k \in \mathbb{Z}} \in \ell_+^{1,2}$ only and $H^{nls} = \mathcal{H}^{nls} \circ (\Phi^{nls})^{-1} : \ell_+^{1,2} \rightarrow \mathbb{R}$ is real analytic.*

(BC4) *The differential $d_0 \Phi^{nls}$ of Φ^{nls} at 0 is the Fourier transform F_{nls} .*

(ii) *The nonlinear parts $A^{nls} := \Phi^{nls} - F_{nls}$ of Φ^{nls} and $B^{nls} := (\Phi^{nls})^{-1} - F_{nls}^{-1}$ of $(\Phi^{nls})^{-1}$ are one smoothing in the sense that for any $\sigma \in \mathbb{Z}_{\geq 1}$*

$$A^{nls} : H_r^\sigma \rightarrow h_r^{\sigma+1} \quad \text{and} \quad B^{nls} : h_r^\sigma \rightarrow H_r^{\sigma+1}$$

are real analytic and bounded, meaning that the image of any bounded subset is bounded.

The map Φ^{nls} is referred to as Birkhoff map and the coordinates $(w_k)_{k \in \mathbb{Z}}$ are called (complex) Birkhoff coordinates for the dNLS equation.

PROOF. Item (i) of Theorem 3.1 is the reformulation of the corresponding theorem of [19] for the dNLS equation in complex coordinates

$$w_k = (x_k - iy_k)/\sqrt{2}, \quad \forall k \in \mathbb{Z}, \quad (3.3)$$

where x_k, y_k are the real coordinates of Theorem in [19], page 5. For item (ii), we refer to [24]. ■

According to Theorem 3.1 (i), the Hamiltonian equations of motion, when expressed in Birkhoff coordinates on h_r^1 , take the form

$$\dot{w}_k = \{w_k, H^{nls}\} = -i \partial_{\bar{w}_k} H^{nls} = -i \partial_{I_k} H^{nls} \cdot \partial_{\bar{w}_k} I_k.$$

Since $I_k = w_k \bar{w}_k$, one then gets

$$\dot{w}_k = -i \omega_k^{nls} w_k, \quad \omega_k^{nls} = \partial_{I_k} H^{nls}, \quad \forall k \in \mathbb{Z}.$$

Note that by Theorem 3.1 (i), $H^{nls} : \ell_+^{1,2} \rightarrow \mathbb{R}$ is real analytic and hence so are the frequencies $\omega_k^{nls} = \partial_{I_k} H^{nls}$, $k \in \mathbb{Z}$. In [20], asymptotic estimates for ω_k^{nls} as $|k| \rightarrow \infty$ were obtained

$$\omega_k^{nls} = 4\pi^2 k^2 + O(1).$$

Actually, they can be refined on the space of actions $\ell_+^{1,4}$, corresponding to potentials in H_r^2 ([25]),

$$\omega_k^{nls} = 4\pi^2 k^2 + 4 \sum_{j \in \mathbb{Z}} I_j + O(1/k).$$

To state these results more precisely, let $\ell^\infty \equiv \ell^\infty(\mathbb{Z}, \mathbb{C})$ denote the Banach space of complex valued, bounded sequences, endowed with the sup-norm $\|\cdot\|_{\ell^\infty}$.

Theorem 3.2. (dNLS frequencies) *There exists an open complex neighbourhood V of $\ell_+^{1,2}$ in $\ell^{1,2}$ so that the following holds:*

(i) *The map*

$$V \rightarrow \ell^\infty, \quad (I_k)_{k \in \mathbb{Z}} \mapsto (\omega_n^{nls}(I) - 4\pi^2 n^2)_{n \in \mathbb{Z}} \quad (3.4)$$

is real analytic and bounded. Furthermore for any $I^{(0)} \in \ell_+^{1,2}$ there exist a complex neighbourhood $V(I^{(0)}) \subseteq V$ and a constant $C > 0$ so that on $V(I^{(0)})$

$$\sup_{n \in \mathbb{Z}} \left\| \left(\frac{1}{\langle k \rangle^2} \partial_{I_k} \omega_n^{nls} \right)_{k \in \mathbb{Z}} \right\|_{\ell^\infty} \leq C. \quad (3.5)$$

As a consequence, for any $n \in \mathbb{Z}$, the map

$$\ell_+^{1,2} \rightarrow \ell^\infty, \quad I \mapsto \left(\frac{1}{\langle k \rangle^2} \partial_{I_k} \omega_n^{nls} \right)_{k \in \mathbb{Z}} \quad (3.6)$$

is real analytic and locally bounded uniformly in n . More generally, for any $N \in \mathbb{Z}_{\geq 1}$ and $I^{(0)} \in \ell_+^{1,2}$, there exist a complex neighbourhood $V_N(I^{(0)}) \subseteq V(I^{(0)})$ and a constant $C_N > 0$ so that on $V_N(I^{(0)})$

$$\sup_{|\alpha|=N} \sup_{n \in \mathbb{Z}} \left| \left(\prod_{k \in \mathbb{Z}} \langle k \rangle^{-2\alpha_k} \right) \partial_I^\alpha \omega_n^{nls}(I) \right| \leq C_N \quad (3.7)$$

where the supremum is taken over all multi-indices $\alpha = (\alpha_k)_{k \in \mathbb{Z}}$ with $\alpha_k \in \mathbb{Z}_{\geq 0}$ and $|\alpha| := \sum_{k \in \mathbb{Z}} \alpha_k = N$.

(ii) *The map*

$$V \cap \ell^{1,4} \rightarrow \ell^\infty, \quad I = (I_k)_{k \in \mathbb{Z}} \mapsto (r_n)_{n \in \mathbb{Z}}, \quad r_n := n \left(\omega_n^{nls} - 4\pi^2 n^2 - 4 \sum_{k \in \mathbb{Z}} I_k \right) \quad (3.8)$$

is real analytic and bounded.

Proof. (i) The analyticity and boundedness of the map $(I_k)_{k \in \mathbb{Z}} \mapsto (\omega_n^{nls} - 4\pi^2 n^2)_{n \in \mathbb{Z}}$ (cf (3.4)) is proved in [25], Corollary 2.1. Let $I^{(0)} \in \ell_+^{1,2}$. Then there exist a closed complex ball $B_r(I^{(0)}) \subseteq \ell^{1,2}$ of radius $r > 0$, centered at $I^{(0)}$, and $C > 0$ so that for any $n \in \mathbb{Z}$, the real analytic map $\omega_n^{nls} - 4\pi^2 n^2 : B_r(I^{(0)}) \rightarrow \mathbb{C}$ satisfies

$$\sup_{I \in B_r(I^{(0)})} |\omega_n^{nls}(I) - 4\pi^2 n^2| \leq C/2.$$

By Cauchy's estimate, the differential $d\omega_n^{nls} : \ell^{1,2} \rightarrow \mathbb{C}$ satisfies the estimate

$$\sup_{I \in B_{r/2}(I^{(0)})} \|d\omega_n^{nls}\|_{(\ell^{1,2})^*} \leq C/r$$

where $(\ell^{1,2})^*$ is the dual of $\ell^{1,2}$ and given by $\ell^{\infty, -2}$. Hence $(\frac{1}{\langle k \rangle^2} \partial_{I_k} \omega_n^{nls}(I))_{k \in \mathbb{Z}} \in \ell^\infty$ and

$$\sup_{I \in B_{r/2}(I^{(0)})} \left\| \left(\frac{1}{\langle k \rangle^2} \partial_{I_k} \omega_n^{nls}(I) \right)_{k \in \mathbb{Z}} \right\|_{\ell^\infty} \leq C/r, \quad \forall n \in \mathbb{Z},$$

proving (3.5) with $V(I^{(0)}) := B_{r/2}(I^{(0)})$. The analyticity of the map (3.6) then follows from the characterization of analytic maps with values in ℓ^∞ , see e.g. [23, Theorem A.3]. The estimates (3.7) of the higher derivatives of the dNLS frequencies ω_n^{nls} are proved in a similar way. Since we need to apply again Cauchy's estimate we might have to choose the neighborhood $V_N(I^{(0)})$ smaller than $V(I^{(0)})$.

(ii) The claimed statement is proved in [25], Theorem 2.3. \square

Finally we recall from [20] that the dNLS frequencies satisfy Kolmogorov and Melnikov conditions. In [20] (cf also [27]), the Birkhoff normal form of the Hamiltonian \mathcal{H}^{nls} of (1.2) has been computed near $u = 0$ up to order four, yielding

$$\omega_n^{nls}(I) = 4\pi^2 n^2 + 4 \sum_{k \in \mathbb{Z}} I_k - 2I_n + O(I^2).$$

In particular, it follows that for any $S \subseteq \mathbb{Z}$ with $|S| < \infty$,

$$\det((\partial_{I_k} \omega_n^{nls})_{k,n \in S})|_{I=0} = -(-2)^{|S|} (2|S| - 1) \neq 0.$$

Hence by the analyticity of ω_n^{nls} we have the following result.

Proposition 3.1 ([20]). **(Non-degeneracy of dNLS frequencies)** *For any $S \subset \mathbb{Z}$ with $|S| < \infty$, $\Pi_S \rightarrow \mathbb{R}$, $I \mapsto \det((\partial_{I_k} \omega_n^{nls})_{k,n \in S})$ is a real analytic map satisfying*

$$\det((\partial_{I_k} \omega_n^{nls})_{k,n \in S}) \neq 0 \quad \text{a.e. on} \quad \Pi_S = \{(I_k)_{k \in \mathbb{Z}} : I_k > 0 \forall k \in S; I_k = 0 \forall k \in S^\perp\}. \quad (3.9)$$

In addition, for any $\ell \in \mathbb{Z}^S$, $a, b \in S^\perp$, with $a \neq b$, the following functions are real analytic and satisfy a.e. on Π_S

$$\sum_{n \in S} \ell_n \omega_n^{nls} \pm \omega_a^{nls} \neq 0, \quad \sum_{n \in S} \ell_n \omega_n^{nls} \pm (\omega_a^{nls} + \omega_b^{nls}) \neq 0, \quad \sum_{n \in S} \ell_n \omega_n^{nls} + \omega_a^{nls} - \omega_b^{nls} \neq 0. \quad (3.10)$$

3.2 Hamiltonian setup

Recall that in (1.20) we introduced as phase space

$$M^\sigma := \mathbb{T}^S \times U_0 \times h_\perp^\sigma, \quad h_\perp^\sigma = h^\sigma(S^\perp, \mathbb{C}),$$

with coordinates denoted by (θ, y, z) . Note that the tangent space of M^σ is independent of the base point (θ, y, z) of M^σ . It is denoted by TM^σ and given by

$$TM^\sigma = \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma.$$

Denote by Id_\perp the identity operator on h_\perp^σ and by Id_S the one on \mathbb{R}^S . The Poisson bracket between functionals $F, G : M^\sigma \rightarrow \mathbb{R}$ with sufficiently regular gradient is given by

$$\{F, G\} := \begin{pmatrix} \nabla_\theta F \\ \nabla_y F \end{pmatrix} \cdot \begin{pmatrix} 0 & \text{Id}_S \\ -\text{Id}_S & 0 \end{pmatrix} \begin{pmatrix} \nabla_\theta G \\ \nabla_y G \end{pmatrix} + \begin{pmatrix} \nabla_z F \\ \nabla_{\bar{z}} F \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \text{Id}_\perp \\ i \text{Id}_\perp & 0 \end{pmatrix} \begin{pmatrix} \nabla_z G \\ \nabla_{\bar{z}} G \end{pmatrix}, \quad (3.11)$$

where in the latter expression, the dot denotes the bilinear form on $(h_\perp^\sigma)^2 \times (h_\perp^\sigma)^2$ given by

$$((w, \tilde{w}), (z, \tilde{z})) \mapsto \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} \cdot \begin{pmatrix} z \\ \tilde{z} \end{pmatrix} := w \cdot z + \tilde{w} \cdot \tilde{z}, \quad w \cdot z = \sum_{k \in S^\perp} w_k z_k \in \mathbb{C} \quad (3.12)$$

and $\nabla_z F = (\partial_{z_k} F)_{k \in S^\perp}$, $\nabla_{\bar{z}} F = (\partial_{\bar{z}_k} F)_{k \in S^\perp}$ with

$$\partial_{z_k} F := \frac{1}{\sqrt{2}}(\partial_{x_k} F + i\partial_{y_k} F), \quad \partial_{\bar{z}_k} F := \frac{1}{\sqrt{2}}(\partial_{x_k} F - i\partial_{y_k} F)$$

and $x_k = \sqrt{2}\text{Re}z_k$, $y_k = -\sqrt{2}\text{Im}z_k$ defined as in (3.3). For such a functional F , the corresponding Hamiltonian vector field is written as

$$X_F := (\nabla_y F, -\nabla_\theta F, -i\nabla_{\bar{z}} F). \quad (3.13)$$

The Hamiltonian vector field X_F may be in TM^σ or lose regularity as the dNLS Hamiltonian vector field which takes values in $TM^{\sigma-2}$. In complex notations, the differential dX_F of the vector field X_F is given by

$$\begin{pmatrix} \hat{\theta} \\ \hat{y} \\ \hat{z} \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta \nabla_y F[\hat{\theta}] + \partial_y \nabla_y F[\hat{y}] + \partial_z \nabla_y F[\hat{z}] + \partial_{\bar{z}} \nabla_y F[\hat{\bar{z}}] \\ -\partial_\theta \nabla_\theta F[\hat{\theta}] - \partial_y \nabla_\theta F[\hat{y}] - \partial_z \nabla_\theta F[\hat{z}] - \partial_{\bar{z}} \nabla_\theta F[\hat{\bar{z}}] \\ -i\partial_\theta \nabla_{\bar{z}} F[\hat{\theta}] - i\partial_y \nabla_{\bar{z}} F[\hat{y}] - i\partial_z \nabla_{\bar{z}} F[\hat{z}] - i\partial_{\bar{z}} \nabla_{\bar{z}} F[\hat{\bar{z}}] \end{pmatrix}$$

where ∂_θ , ∂_y , ∂_z , and $\partial_{\bar{z}}$ are defined in the standard way, i.e., for instance

$$\partial_z \nabla_y F[\hat{z}] = \sum_{k \in S^\perp} \hat{z}_k \partial_{z_k} \nabla_y F.$$

It turns out to be convenient to add to the domain of dX_F as fourth component the complex conjugate of the third one and to extend the resulting map to the following linear operator defined on $\mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma$, still denoted by dX_F ,

$$dX_F : \begin{pmatrix} \widehat{\theta} \\ \widehat{y} \\ \widehat{z}_1 \\ \widehat{z}_2 \end{pmatrix} \mapsto \begin{pmatrix} \partial_\theta \nabla_y F[\widehat{\theta}] + \partial_y \nabla_y F[\widehat{y}] + \partial_z \nabla_y F[\widehat{z}_1] + \partial_{\bar{z}} \nabla_y F[\widehat{z}_2] \\ -\partial_\theta \nabla_\theta F[\widehat{\theta}] - \partial_y \nabla_\theta F[\widehat{y}] - \partial_z \nabla_\theta F[\widehat{z}_1] - \partial_{\bar{z}} \nabla_\theta F[\widehat{z}_2] \\ -i\partial_\theta \nabla_{\bar{z}} F[\widehat{\theta}] - i\partial_y \nabla_{\bar{z}} F[\widehat{y}] - i\partial_z \nabla_{\bar{z}} F[\widehat{z}_1] - i\partial_{\bar{z}} \nabla_{\bar{z}} F[\widehat{z}_2] \\ i\partial_\theta \nabla_z F[\widehat{\theta}] + i\partial_y \nabla_z F[\widehat{y}] + i\partial_z \nabla_z F[\widehat{z}_1] + i\partial_{\bar{z}} \nabla_z F[\widehat{z}_2] \end{pmatrix}. \quad (3.14)$$

Here we use that by assumption F is real valued and hence $\overline{\nabla_z F} = \nabla_{\bar{z}} F$.

The symplectic form corresponding to the Poisson bracket (3.11) is the restriction to the real subspace $\{(\theta, y, z, \bar{z}) : (\theta, y, z) \in TM^\sigma\}$ of $\mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma$ of the skew symmetric \mathbb{C} -bilinear form

$$(\mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma) \times (\mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma) \rightarrow \mathbb{C},$$

associating to two elements $(\widehat{\theta}^{(i)}, \widehat{y}^{(i)}, \widehat{z}_1^{(i)}, \widehat{z}_2^{(i)})$, $i = 1, 2$, the complex number

$$\begin{pmatrix} 0 & \text{Id}_S \\ -\text{Id}_S & 0 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\theta}^{(1)} \\ \widehat{y}^{(1)} \end{pmatrix} \cdot \begin{pmatrix} \widehat{\theta}^{(2)} \\ \widehat{y}^{(2)} \end{pmatrix} + \begin{pmatrix} 0 & -i \text{Id}_\perp \\ i \text{Id}_\perp & 0 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{z}_1^{(1)} \\ \widehat{z}_2^{(1)} \end{pmatrix} \cdot \begin{pmatrix} \widehat{z}_1^{(2)} \\ \widehat{z}_2^{(2)} \end{pmatrix}. \quad (3.15)$$

This symplectic form Λ can be expressed as in (1.11).

It immediately follows from the above definition that for any $Y \in TM^\sigma$ and any \mathcal{C}^1 functional $F : M^\sigma \rightarrow \mathbb{C}$ with sufficiently regular gradient, one has $dF(Y) = \Lambda(X_F, Y)$. We also introduce the Liouville 1-form $\lambda : TM^\sigma \rightarrow \mathbb{C}$ defined by

$$\lambda = - \sum_{k \in S} y_k d\theta_k + i \sum_{k \in S^\perp} z_k d\bar{z}_k. \quad (3.16)$$

At any given point (θ, y, z) , λ is the bounded \mathbb{R} -linear functional

$$TM^\sigma \rightarrow \mathbb{C}, \quad (\widehat{\theta}, \widehat{y}, \widehat{z}) \rightarrow - \sum_{k \in S} y_k \widehat{\theta}_k + i \sum_{k \in S^\perp} z_k \widehat{\bar{z}}_k.$$

A diffeomorphism $\Gamma : \mathcal{U} \rightarrow M^\sigma$, defined on an open subset \mathcal{U} of M^σ , is said to be symplectic if $\Gamma^* \Lambda = \Lambda$ at any point $(\theta, y, z) \in \mathcal{U}$. Note that h_\perp^σ is a symplectic subspace of h^σ . Indeed the pull back Λ_\perp of the symplectic form Λ by the inclusion $h_\perp^\sigma \hookrightarrow M^\sigma$, is given by

$$\Lambda_\perp = i \sum_{k \in S^\perp} dz_k \wedge d\bar{z}_k,$$

which is clearly a non-degenerate bilinear form on h_\perp^σ . Now we consider φ -dependent canonical transformations on h_\perp^σ .

Definition 3.1. (Symplectic operator) An operator valued map $\mathbb{T}^S \rightarrow \mathcal{L}(h_\perp^\sigma)$ of the form $h \mapsto \Phi_1(\varphi)h + \Phi_2(\varphi)\bar{h}$ is said to be symplectic if $\Phi(\varphi)^* \Lambda_\perp = \Lambda_\perp$ for any $\varphi \in \mathbb{T}^S$. The map $\Phi(\varphi)$, when extended as a \mathbb{C} -linear map to $h_\perp^\sigma \times h_\perp^\sigma$,

$$h_\perp^\sigma \times h_\perp^\sigma \rightarrow h_\perp^\sigma \times h_\perp^\sigma, \quad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \mapsto \begin{pmatrix} \Phi_1(\varphi) & \Phi_2(\varphi) \\ \overline{\Phi_2(\varphi)} & \overline{\Phi_1(\varphi)} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \quad (3.17)$$

is also denoted by $\Phi(\varphi)$. We denote by $\overline{\Phi_i}$ the operators given by $\overline{\Phi_i}(h) := \overline{\Phi_i(\bar{h})}$ where $\bar{h} := (\bar{h}_k)_{k \in S^\perp}$.

In view of (3.15), the property of $\Phi(\varphi)$ being symplectic can be expressed in terms of the map (3.17) as follows

$$\Phi(\varphi)^t \mathbb{J}_2 \Phi(\varphi) = \mathbb{J}_2, \quad (3.18)$$

where

$$\Phi(\varphi)^t = \begin{pmatrix} \Phi_1(\varphi)^t & \overline{\Phi_2(\varphi)^t} \\ \Phi_2(\varphi)^t & \overline{\Phi_1(\varphi)^t} \end{pmatrix}, \quad \mathbb{J}_2 := i \begin{pmatrix} 0 & \text{Id}_\perp \\ -\text{Id}_\perp & 0 \end{pmatrix} \quad (3.19)$$

where $[\Phi_i(\varphi)]^t$ denotes the transpose with respect to the bilinear form defined in (3.12).

Next, let us consider a family of quadratic Hamiltonians $F(\varphi, \cdot) : h_\perp^\sigma \rightarrow \mathbb{R}$, $\varphi \in \mathbb{T}^S$, of the form

$$F(\varphi, z) = \bar{z} \cdot A_1(\varphi)z + \frac{1}{2}\bar{z} \cdot A_2(\varphi)\bar{z} + \frac{1}{2}z \cdot A_3(\varphi)z, \quad z \in h_\perp^\sigma, \quad (3.20)$$

where $A_i(\varphi)$, $1 \leq i \leq 3$, $\varphi \in \mathbb{T}^S$, are (possibly unbounded) linear operators on h_\perp^σ . Without loss of generality we may require that for $i = 2, 3$, one has $A_i^t = A_i$. The assumption that F is real valued implies that

$$A_1^* = A_1, \quad \bar{A}_2 = A_3,$$

where for any $\varphi \in \mathbb{T}^S$, $A_1^*(\varphi)$ is the adjoint operator of $A_1(\varphi)$ with respect to the standard complex scalar product on h_\perp^0 ,

$$(z, w) := \sum_{n \in S^\perp} z_n \bar{w}_n, \quad \forall z, w \in h_\perp^0. \quad (3.21)$$

Note that $A_1 = \partial_z \nabla_{\bar{z}} F$, $A_2 = \partial_{\bar{z}} \nabla_{\bar{z}} F$ and $A_3 = \partial_z \nabla_z F$. The φ -dependent Hamiltonian vector field X_F , associated to the Hamiltonian F , is the map $\varphi \mapsto X_F(\varphi)$ with $X_F(\varphi)$ given for any $\varphi \in \mathbb{T}^S$ by

$$h_\perp^\sigma \rightarrow h_\perp^\sigma, \quad h \mapsto -i(A_1(\varphi)h + A_2(\varphi)\bar{h}).$$

In the case at hand, the formula analogous to (3.14) is then given by

$$- \begin{pmatrix} i\text{Id}_\perp & 0 \\ 0 & -i\text{Id}_\perp \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ \bar{A}_2 & A_1 \end{pmatrix}, \quad A_1^* = A_1, \quad A_2^t = A_2.$$

Definition 3.2. (Hamiltonian operator) *The operator $JA(\varphi)$ where*

$$J := \begin{pmatrix} i\text{Id}_\perp & 0 \\ 0 & -i\text{Id}_\perp \end{pmatrix}, \quad A(\varphi) := \begin{pmatrix} \frac{A_1(\varphi)}{A_2(\varphi)} & \frac{A_2(\varphi)}{A_1(\varphi)} \end{pmatrix}, \quad A_1^* = A_1, \quad A_2^t = A_2, \quad (3.22)$$

as well as the operator $\mathfrak{L}(\varphi)$ defined, for $\varphi \in \mathbb{T}^S$, by

$$\mathfrak{L}(\varphi) = \omega \cdot \partial_\varphi \mathbb{I}_2 + JA(\varphi), \quad \mathbb{I}_2 = \begin{pmatrix} \text{Id}_\perp & 0 \\ 0 & \text{Id}_\perp \end{pmatrix} \quad (3.23)$$

are referred to as linear Hamiltonian operators associated to the Hamiltonian F in (3.20).

Equivalently the Hamiltonian operator $JA(\varphi)$ can be written in the form

$$JA(\varphi) = \mathbb{J}_2 \mathbb{A}(\varphi), \quad \mathbb{A}(\varphi) := \begin{pmatrix} \overline{A_2(\varphi)} & \overline{A_1(\varphi)} \\ A_1(\varphi) & A_2(\varphi) \end{pmatrix} \quad \mathbb{A}^t(\varphi) = \mathbb{A}(\varphi) \quad (3.24)$$

where \mathbb{J}_2 is defined in (3.19) and $\mathbb{A}^t(\varphi) = \mathbb{A}(\varphi)$, since $A_1^t = \bar{A}_1$ and $A_2^t = A_2$.

Lemma 3.1. *Assume that $\Phi \in \mathcal{C}^1(\mathbb{T}^S, \mathcal{L}(h_\perp^\sigma \times h_\perp^\sigma))$ is a map with $\Phi(\varphi)$ a linear symplectic transformation for any $\varphi \in \mathbb{T}^S$ (cf Definition 3.1) and $\mathfrak{L}(\varphi)$ a Hamiltonian operator (cf Definition 3.2). Then the transformed operator $\mathfrak{L}_+(\varphi) := \Phi^{-1}(\varphi)\mathfrak{L}(\varphi)\Phi(\varphi)$ is Hamiltonian and of the form $\mathfrak{L}_+(\varphi) = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbb{J}_2 \mathbb{A}_+(\varphi)$, where*

$$\mathbb{A}_+(\varphi) := \Phi^t(\varphi)\mathbb{A}(\varphi)\Phi(\varphi) + \Phi^t(\varphi)\mathbb{J}_2(\omega \cdot \partial_\varphi)(\Phi(\varphi)), \quad (3.25)$$

and satisfies $\mathbb{A}_+(\varphi) = \mathbb{A}_+^t(\varphi)$. Here we denoted by $\Phi^{-1}(\varphi)$ the operator $\Phi^{-1}(\varphi) := (\Phi(\varphi))^{-1}$ for any $\varphi \in \mathbb{T}^S$.

Proof. Using the representation (3.24) for the Hamiltonian operator $\mathfrak{L}(\varphi) = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbb{J}_2 \mathbb{A}(\varphi)$ we have

$$\mathfrak{L}_+(\varphi) = \Phi^{-1}(\varphi) \mathfrak{L}(\varphi) \Phi(\varphi) = \omega \cdot \partial_\varphi \mathbb{I}_2 + \Phi^{-1}(\varphi) \mathbb{J}_2 \mathbb{A}(\varphi) \Phi(\varphi) + \Phi^{-1}(\varphi) (\omega \cdot \partial_\varphi) (\Phi(\varphi)). \quad (3.26)$$

By the condition (3.18) and using that $\mathbb{J}_2^{-1} = \mathbb{J}_2$, one has $\Phi^{-1}(\varphi) \mathbb{J}_2 = \mathbb{J}_2 \Phi^t(\varphi)$, yielding

$$\Phi^{-1}(\varphi) \mathbb{J}_2 \mathbb{A}(\varphi) \Phi(\varphi) = \mathbb{J}_2 \Phi^t(\varphi) \mathbb{A}(\varphi) \Phi(\varphi). \quad (3.27)$$

Since $\mathbb{J}_2^2 = \mathbb{I}_2$, and using that by (3.18) $\mathbb{J}_2 \Phi^{-1}(\varphi) = \Phi^t(\varphi) \mathbb{J}_2$, we have

$$\Phi^{-1}(\varphi) (\omega \cdot \partial_\varphi) (\Phi(\varphi)) = \mathbb{J}_2 (\mathbb{J}_2 \Phi^{-1}(\varphi) (\omega \cdot \partial_\varphi) (\Phi(\varphi))) = \mathbb{J}_2 (\Phi^t(\varphi) \mathbb{J}_2 (\omega \cdot \partial_\varphi) (\Phi(\varphi))). \quad (3.28)$$

Combining (3.26), (3.27), (3.28) we get the claimed formula $\mathfrak{L}_+(\varphi) = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbb{J}_2 \mathbb{A}_+(\varphi)$ with $\mathbb{A}_+(\varphi)$ given in (3.25).

It remains to verify that $\mathbb{A}_+(\varphi) = \mathbb{A}_+^t(\varphi)$. To see that $\Phi^t(\varphi) \mathbb{J}_2 (\omega \cdot \partial_\varphi) (\Phi(\varphi))$ is symmetric, note that by (3.18), for any $\varphi \in \mathbb{T}^S$,

$$0 = (\omega \cdot \partial_\varphi) (\Phi^t(\varphi) \mathbb{J}_2 \Phi(\varphi)) = (\omega \cdot \partial_\varphi) (\Phi^t(\varphi)) \mathbb{J}_2 \Phi(\varphi) + \Phi^t(\varphi) \mathbb{J}_2 (\omega \cdot \partial_\varphi) (\Phi(\varphi)),$$

implying that

$$\Phi^t(\varphi) \mathbb{J}_2 (\omega \cdot \partial_\varphi) (\Phi(\varphi)) = -(\omega \cdot \partial_\varphi) (\Phi^t(\varphi)) \mathbb{J}_2 \Phi(\varphi) = (\omega \cdot \partial_\varphi) (\Phi^t(\varphi)) \mathbb{J}_2^t \Phi(\varphi) = (\Phi^t(\varphi) \mathbb{J}_2 (\omega \cdot \partial_\varphi) (\Phi(\varphi)))^t.$$

Since by assumption $\mathbb{A}(\varphi)$ is symmetric, so is $\Phi^t(\varphi) \mathbb{A}(\varphi) \Phi(\varphi)$. In view of the formula for $\mathbb{A}_+(\varphi)$, it then follows that $\mathbb{A}_+(\varphi)$ is symmetric. \square

In the sequel we use the shorthand notations F_{nls}^\perp and $(F_{nls}^{-1})_{\hookrightarrow}$, the latter being identified by a slight abuse of terminology with F_{nls}^{-1} , i.e.,

$$F_{nls}^\perp := \mathbb{I}_\perp F_{nls} \quad \text{and} \quad F_{nls}^{-1} \equiv (F_{nls}^{-1})_{\hookrightarrow} := F_{nls}^{-1} \mathbb{I}_{\hookrightarrow} \quad (3.29)$$

where, recalling that π_\perp denotes the L^2 projector (1.28) onto H_\perp^σ ,

$$\mathbb{I}_\perp := \begin{pmatrix} \pi_\perp & 0 \\ 0 & \pi_\perp \end{pmatrix} \quad \text{and} \quad \mathbb{I}_{\hookrightarrow} : h_\perp^\sigma \times h_\perp^\sigma \rightarrow h^\sigma \times h^\sigma \quad (3.30)$$

denotes the inclusion map. Note that

$$F_{nls}^{-1} F_{nls}^\perp = \mathbb{I}_\perp. \quad (3.31)$$

According to (3.1)

$$F_{nls}^\perp = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad F_{nls}^{-1} = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \quad (3.32)$$

where for any $u \in H^\sigma$

$$F_1(u) = -(u_{-n})_{n \in S^\perp}, \quad F_2(u) = -(u_n)_{n \in S^\perp}$$

and for any $z = (z_n)_{n \in S^\perp} \in h_\perp^\sigma$

$$G_1(z) = - \sum_{n \in S^\perp} z_{-n} e^{2\pi i n x}, \quad G_2(z) = - \sum_{n \in S^\perp} z_n e^{2\pi i n x}.$$

In view of the definitions (1.29), (3.12), (3.21) one verifies that

$$F_2 = \overline{F_1}, \quad G_2 = \overline{G_1}, \quad (3.33)$$

$$z \cdot F_1(u) = \langle G_2(z), u \rangle_r, \quad z \cdot F_2(u) = \langle G_1(z), u \rangle_r, \quad (3.34)$$

$$(z, F_1(u)) = \langle G_1(z), u \rangle, \quad (z, F_2(u)) = \langle G_2(z), u \rangle. \quad (3.35)$$

Lemma 3.2. *Assume that A is a linear operator acting on $H^\sigma \times H^\sigma$ of the form*

$$A = \begin{pmatrix} B & C \\ \overline{C} & \overline{B} \end{pmatrix}, \quad B^* = B, \quad C^t = C \quad (3.36)$$

where B^* is the adjoint of B with respect to the complex $L^2(\mathbb{T}_1)$ scalar product $\langle \cdot, \cdot \rangle$ and C^t is the transposed with respect to the real bilinear form $\langle \cdot, \cdot \rangle_r$, where $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_r$ are defined in (1.29). Then the operator $JF_{nls}^\perp AF_{nls}^{-1}$ is Hamiltonian.

Proof. By (3.32) one has

$$F_{nls}^\perp AF_{nls}^{-1} = \begin{pmatrix} F_1 B G_1 & F_1 C G_2 \\ F_2 \overline{C} G_1 & F_2 \overline{B} G_2 \end{pmatrix}.$$

Using the identities (3.33)-(3.35) one verifies that all the conditions listed in the Definition 3.2 of a Hamiltonian operator are satisfied. \square

3.3 Tame estimates for the Hamiltonian vector fields $X_{H^{nls}} \circ \check{\iota}$ and $X_P \circ \check{\iota}$

In this subsection we derive tame estimates for the compositions of torus embeddings $\check{\iota} : \mathbb{T}^S \rightarrow M^\sigma$ with the dNLS Hamiltonian H^{nls} and with the perturbation P where M^σ is the phase space introduced in (1.20).

Recall that the dNLS Hamiltonian H^{nls} is a function of the actions $I_n, n \in \mathbb{Z}$, alone and that $I_n = \xi_n + y_n, n \in S$, and $I_n = z_n \bar{z}_n, n \in S^\perp$. To simplify notation, given a map $\check{\iota} : \mathbb{T}^S \rightarrow M^\sigma$, we will frequently suppress the variable φ in $\check{\iota}(\varphi) = (\theta(\varphi), y(\varphi), z(\varphi))$. The main results are the following ones.

Proposition 3.2. *Given an integer $s \geq s_0$, there exists $0 < \rho_1 \leq 1$ so that for any map $\check{\iota}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \in H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$ and $\|\iota\|_{3s_0} \leq \rho_1$, one has $\check{\iota}(\mathbb{T}^S) \subset M^\sigma$ and the following holds:*

(i) *The dNLS frequencies ω_n^{nls} satisfy the tame estimate*

$$\sup_{n \in \mathbb{Z}} \|\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)\|_s \leq_s \|\iota\|_{s+2s_0}. \quad (3.37)$$

Moreover, for any $N \in \mathbb{Z}_{\geq 1}$, there exists $0 < \rho_N \leq \rho_1$ so that in case $\|\iota\|_{3s_0} \leq \rho_N$,

$$\sup_{1 \leq |\alpha| \leq N} \sup_{n \in \mathbb{Z}} \left\| \left(\prod_{j \in \mathbb{Z}} \langle j \rangle^{-2\alpha_j} \right) \partial_I^\alpha \omega_n^{nls}(\xi + y, z\bar{z}) \right\|_s \leq_s 1 + \|\iota\|_{s+2s_0} \quad (3.38)$$

where the supremum is taken over all multi-indices $\alpha = (\alpha_j)_{j \in \mathbb{Z}}$ with $\alpha_j \in \mathbb{Z}_{\geq 0}$ and $1 \leq |\alpha| = \sum_{j \in \mathbb{Z}} \alpha_j \leq N$.

(ii) *The derivatives of $\nabla_y H^{nls}(\xi + y, z\bar{z})$ and $\nabla_z H^{nls}(\xi + y, z\bar{z})$ with respect to y satisfy the tame estimates*

$$\|\partial_y \nabla_y H^{nls}(\xi + y, z\bar{z}) - \partial_y \nabla_y H^{nls}(\xi, 0)\|_s \leq_s \|\iota\|_{s+2s_0}, \quad \|\partial_y \nabla_z H^{nls}(\xi + y, z\bar{z})\|_s \leq_s \|\iota\|_{s+2s_0}.$$

Since $\nabla_{\bar{z}} H^{nls} = \overline{\nabla_z H^{nls}}$, the derivative $\partial_y \nabla_{\bar{z}} H^{nls}(\xi + y, z\bar{z})$ satisfies the same tame estimate.

(iii) *For any map \hat{z} in $H^s(\mathbb{T}^S, h_\perp^\sigma)$, the derivatives of $\nabla_y H^{nls}$, $\nabla_z H^{nls}$, and $\nabla_{\bar{z}} H^{nls}$ with respect to z in direction \hat{z} satisfy the tame estimates*

$$\begin{aligned} \|\partial_z \nabla_y H^{nls}(\xi + y, z\bar{z})[\hat{z}]\|_s &\leq_s \|\iota\|_{3s_0} \|\hat{z}\|_s + \|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0}, \\ \|\partial_z \nabla_z H^{nls}(\xi + y, z\bar{z})[\hat{z}]\|_s &\leq_s \|\iota\|_{3s_0} \|\hat{z}\|_s + \|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0}, \end{aligned}$$

and

$$\|(\partial_z \nabla_{\bar{z}} H^{nls}(\xi + y, z\bar{z}) - \partial_z \nabla_{\bar{z}} H^{nls}(\xi, 0))[\hat{z}]\|_s \leq_s \|\iota\|_{3s_0} \|\hat{z}\|_s + \|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0}. \quad (3.39)$$

Since $\partial_{\bar{z}} = \overline{\partial_z}$, the derivatives of $\nabla_y H^{nls}(\xi + y, z\bar{z})$, $\nabla_z H^{nls}(\xi + y, z\bar{z})$, and $\nabla_{\bar{z}} H^{nls}(\xi + y, z\bar{z})$ with respect to \bar{z} in direction $\bar{\hat{z}}$ satisfy corresponding tame estimates.

(iv) *If in addition $\iota \equiv \iota_\omega$ is Lipschitz continuous in $\omega \in \Omega$ and satisfies $\|\iota\|_{3s_0}^{\text{lip}} \leq \rho_1$ it follows that for any map $\hat{z} \equiv \hat{z}_\omega$ in $H^s(\mathbb{T}^S, h_\perp^\sigma)$, which is also Lipschitz continuous in $\omega \in \Omega$, all the previous estimates hold with $\|\cdot\|_s$ replaced by $\|\cdot\|_s^{\text{lip}}$.*

Remark 3.1. The estimate (3.38) is only used in this paper for $N \leq 3$. See for instance Lemma 3.3 and Lemmata 6.1, 6.2.

Proof. (i) To obtain the claimed tame estimates, we want to apply Lemma 2.12 (ii). First we need to make some preliminary considerations. By (3.2), for any $(z_n)_{n \in S^\perp} \in h_\perp^\sigma$, $(z_n \bar{z}_n)_{n \in S^\perp}$ is in $\ell_{+, \perp}^{1, 2\sigma} := \ell_+^{1, 2\sigma}(S^\perp, \mathbb{R})$ and

$$h_\perp^\sigma \rightarrow \ell_{+, \perp}^{1, 2\sigma}, (z_n)_{n \in S^\perp} \mapsto (z_n \bar{z}_n)_{n \in S^\perp}, \quad \|(z_n \bar{z}_n)_{n \in S^\perp}\|_{\ell^{1, 2\sigma}} = \|(z_n)_{n \in S^\perp}\|_\sigma^2,$$

is a bounded quadratic map. In particular, this map is in $\mathcal{C}^\infty(h_\perp^\sigma, \ell_{+, \perp}^{1, 2\sigma})$. By Theorem 3.2, for any $\xi \in \mathbb{R}_{>0}^S$, there exists an open neighborhood V' of $(\xi, 0)$ in $\ell_+^{1, 2\sigma}$ so that the map

$$(\omega_n^{nls} - 4n^2\pi^2)_{n \in \mathbb{Z}} : V' \rightarrow \ell^\infty$$

is in $\mathcal{C}^\infty(V', \ell^\infty)$. Altogether it then follows that there is an open convex neighborhood V of $(0, 0)$ in $U_0 \times h_\perp^\sigma$ so that the composition $f : V \rightarrow \ell^\infty$, defined by $f(y, z) := (\omega_n^{nls}(\xi + y, z\bar{z}) - 4n^2\pi^2)_{n \in \mathbb{Z}}$, is in $\mathcal{C}^{s+s_0}(V', \ell^\infty)$. Choose $0 < \rho_1 \leq 1$ so that the closed ball in $U_0 \times h_\perp^\sigma$ of radius ρ_1 , centered at $(0, 0)$, is contained in V . By Lemma 2.1(iii) (Sobolev embedding), it then follows that for any map $\check{\iota}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\|\iota\|_{s_0} \leq \rho_1$, one has $(y(\varphi), z(\varphi)) \in V$ and hence by Lemma 2.12(ii) with $\check{\iota}^{(1)} := \check{\iota}$, $\check{\iota}^{(2)}$ given by $\check{\iota}^{(2)}(\varphi) = (\varphi, 0, 0)$, and $\check{\iota}^{(1)} - \check{\iota}^{(2)} = \iota$

$$\sup_{n \in \mathbb{Z}} \|\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)\|_s \leq_s \|\iota\|_{s+2s_0}.$$

The tame estimates (3.38) can be derived in a similar way, using this time item (i) of Lemma 2.12 as well as Theorem 3.2.

(ii) Note that $\nabla_y H^{nls}(\xi + y, z\bar{z}) = (\omega_n(\xi + y, z\bar{z}))_{n \in S}$ and hence

$$\partial_y \nabla_y H^{nls}(\xi + y, z) = (\partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}))_{n, k \in S}.$$

Arguing similarly as in the proof of item (i), the claimed estimates for $\partial_y \nabla_y H^{nls}(\xi + y, z\bar{z}) - \partial_y \nabla_y H^{nls}(\xi, 0)$ follow from Lemma 2.12(ii). Since $\nabla_z H^{nls}(\xi + y, z\bar{z}) = (\omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_n)_{n \in S^\perp}$ vanishes at $z = 0$, one concludes that $\partial_y \nabla_z H^{nls}(\xi, 0) = 0$ and that in turn – again in view of Lemma 2.12(ii) – the tame estimates $\|\partial_y \nabla_z H^{nls}(\xi + y, z\bar{z})\|_s \leq_s \|\iota\|_{s+2s_0}$ hold.

(iii) We only prove estimate (3.39) since the other ones can be derived by similar arguments. Taking the derivative of $\nabla_{\bar{z}} H^{nls}(\xi + y, z\bar{z}) = (\omega_n^{nls}(\xi + y, z\bar{z}) z_n)_{n \in S^\perp}$ with respect to z yields

$$\partial_z \nabla_{\bar{z}} H^{nls}(\xi + y, z\bar{z})[\widehat{z}] = T_1 + T_2,$$

where

$$T_1 := \left(\omega_n^{nls}(\xi + y, z\bar{z}) \widehat{z}_n \right)_{n \in S^\perp} \quad \text{and} \quad T_2 := \left(z_n \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \widehat{z}_k \right)_{n \in S^\perp}.$$

Concerning the term T_1 , note that

$$\partial_z \nabla_{\bar{z}} H^{nls}(\xi, 0)[\widehat{z}] = (\omega_n^{nls}(\xi, 0) \widehat{z}_n)_{n \in S^\perp}.$$

By Lemma 2.7 (tame estimates for products of functions) it follows that for any $n \in S^\perp$, the expression $\|(\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)) \cdot \widehat{z}_n\|_s$ can be \leq_s -bounded by

$$\|\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)\|_{s_0} \|\widehat{z}_n\|_s + \|\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)\|_s \|\widehat{z}_n\|_{s_0}.$$

Together with the estimates (3.37) for $\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)$, this yields

$$\|(\omega_n^{nls}(\xi + y, z\bar{z}) - \omega_n^{nls}(\xi, 0)) \cdot \widehat{z}_n\|_s \leq_s \|\iota\|_{3s_0} \|\widehat{z}_n\|_s + \|\iota\|_{s+2s_0} \|\widehat{z}_n\|_{s_0},$$

implying, by (2.7), that

$$\|T_1 - \partial_z \nabla_{\bar{z}} H^{nls}(\xi, 0)[\widehat{z}]\|_s \leq_s \|\iota\|_{3s_0} \|\widehat{z}\|_s + \|\iota\|_{s+2s_0} \|\widehat{z}\|_{s_0}. \quad (3.40)$$

Towards the term T_2 , note that for any $n, k \in S^\perp$, Lemma 2.7 implies that $\|\partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k\|_s$ is \leq_s -bounded by

$$\|\partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z})\|_s \|z_k\|_{s_0} \|\hat{z}_k\|_{s_0} + \|\partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z})\|_{s_0} (\|z_k\|_s \|\hat{z}_k\|_{s_0} + \|z_k\|_{s_0} \|\hat{z}_k\|_s).$$

By (2.7) we have $\langle k \rangle^\sigma \|z_k\|_s \leq \|z\|_{s, \sigma}$. By assumption, $\langle k \rangle^2 \|z_k\|_{s_0} \leq 1$ (recall that $\sigma \geq 4$) whereas by (3.38),

$$\|\partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z})\|_s \leq_s \langle k \rangle^2 (1 + \|\iota\|_{s+2s_0}).$$

Hence $\sum_{k \in S^\perp} \|\partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k\|_s$ is \leq_s -bounded by

$$(1 + \|\iota\|_{s+2s_0}) \sum_{k \in S^\perp} \|\hat{z}_k\|_{s_0} + (1 + \|\iota\|_{3s_0}) \left(\|\iota\|_s \sum_{k \in S^\perp} \|\hat{z}_k\|_{s_0} + \sum_{k \in S^\perp} \|\hat{z}_k\|_s \right)$$

implying that (recall that $\sigma \geq 4$ and $\|\iota\|_{3s_0} \leq 1$)

$$\left\| \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k \right\|_s \leq_s \|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0} + \|\hat{z}\|_s. \quad (3.41)$$

Using again Lemma 2.7, the term $\|z_n \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k\|_s$ can be \leq_s -bounded by

$$\|z_n\|_s \cdot \left\| \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k \right\|_{s_0} + \|z_n\|_{s_0} \cdot \left\| \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k \right\|_s,$$

yielding, by (3.41), the estimate

$$\left\| z_n \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k \right\|_s \leq_s \|z_n\|_s \cdot \|\hat{z}\|_{s_0} + \|z_n\|_{s_0} \cdot (\|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0} + \|\hat{z}\|_s).$$

Therefore

$$\|T_2\|_s^2 = \sum_{n \in S^\perp} \langle n \rangle^{2\sigma} \|z_n \sum_{k \in S^\perp} \partial_{I_k} \omega_n^{nls}(\xi + y, z\bar{z}) \bar{z}_k \hat{z}_k\|_s^2$$

is \leq_s -bounded by

$$\sum_{n \in S^\perp} \langle n \rangle^{2\sigma} \|z_n\|_s^2 \cdot \|\hat{z}\|_{s_0}^2 + \sum_{n \in S^\perp} \langle n \rangle^{2\sigma} \|z_n\|_{s_0}^2 \cdot (\|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0} + \|\hat{z}\|_s)^2$$

leading to the estimate (recall that $\|\iota\|_{3s_0} \leq 1$)

$$\|T_2\|_s \leq_s \|\iota\|_{s+2s_0} \|\hat{z}\|_{s_0} + \|\iota\|_{s_0} \|\hat{z}\|_s. \quad (3.42)$$

The estimate (3.39) now follows from the bounds (3.40), (3.42) derived for T_1 and T_2 .

(iv) The Lipschitz estimates are obtained by using similar arguments. \square

Proposition 3.2 can be applied to obtain tame estimates for the composition of the differential $dX_{H^{nls}}$ of the Hamiltonian vector field $X_{H^{nls}}$ with a map $\check{\iota} : \mathbb{T}^S \rightarrow M^\sigma$, $\varphi \mapsto (\theta(\varphi), y(\varphi), z(\varphi))$. We denote by dX_F the linear operator in (3.14).

Corollary 3.1. *Given an integer $s \geq s_0$, there exists $0 < \rho \leq 1$ so that for any map $\check{\iota}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \in H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$ and $\|\iota\|_{3s_0} \leq \rho$, one has $\check{\iota}(\mathbb{T}^S) \subset M^\sigma$ and the following holds:*

(i) *For any map $\hat{\iota} = (\hat{\theta}, \hat{y}, \hat{z}_1, \hat{z}_2)$ in $H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$,*

$$\|dX_{H^{nls}}(\xi + y, z\bar{z})[\hat{\iota}] - dX_{H^{nls}}(\xi, 0)[\hat{\iota}]\|_s \leq_s \|\iota\|_{3s_0} \|\hat{\iota}\|_s + \|\iota\|_{s+2s_0} \|\hat{\iota}\|_{s_0}$$

where

$$dX_{H^{nls}}(\xi, 0)[\hat{\iota}] = \left(\partial_y \nabla_y H^{nls}(\xi, 0)[\hat{y}], 0, -i \partial_z \nabla_{\bar{z}} H^{nls}(\xi, 0)[\hat{z}_1], i \partial_{\bar{z}} \nabla_z H^{nls}(\xi, 0)[\hat{z}_2] \right)$$

with $\partial_y \nabla_y H^{nls}(\xi, 0)[\hat{y}] = \left(\sum_{k \in S} \partial_{I_k} \omega_n^{nls}(\xi, 0) \hat{y}_k \right)_{n \in S}$ and $\partial_z \nabla_{\bar{z}} H^{nls}(\xi, 0)[\hat{z}_1] = \left(\omega_n^{nls}(\xi, 0) \hat{z}_{1n} \right)_{n \in S^\perp}$.

(ii) *If in addition $\check{\iota} \equiv \check{\iota}_\omega$ is Lipschitz continuous in $\omega \in \Omega$ and satisfies $\|\iota\|_{3s_0}^{\text{lip}} \leq \rho$, then for any map $\hat{\iota} \equiv \hat{\iota}_\omega$ in $H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$ which are Lipschitz continuous in $\omega \in \Omega$, the estimates of item (i) hold with $\|\cdot\|_s$ replaced by $\|\cdot\|_s^{\text{lip}}$.*

Proof. Since the Hamiltonian vector field $X_{H^{nls}}$ is given by

$$X_{H^{nls}} = (\nabla_y H^{nls}, 0, -i\nabla_{\bar{z}} H^{nls}) = ((\omega_n^{nls})_{n \in S}, 0, -i(\omega_n^{nls} z_n)_{n \in S^\perp}),$$

the first component of $dX_{H^{nls}}[\hat{\imath}]$ is given by

$$\partial_y \nabla_y H^{nls}[\hat{y}] + \partial_z \nabla_y H^{nls}[\hat{z}_1] + \partial_{\bar{z}} \nabla_y H^{nls}[\hat{z}_2],$$

the second component is 0, whereas the third and fourth components are

$$-i(\partial_y \nabla_{\bar{z}} H^{nls}[\hat{y}] + \partial_z \nabla_{\bar{z}} H^{nls}[\hat{z}_1] + \partial_{\bar{z}} \nabla_{\bar{z}} H^{nls}[\hat{z}_2]) \quad \text{and} \quad i(\partial_y \nabla_z H^{nls}[\hat{y}] + \partial_z \nabla_z H^{nls}[\hat{z}_1] + \partial_{\bar{z}} \nabla_z H^{nls}[\hat{z}_2]).$$

In particular, one obtains the claimed formula for $dX_{H^{nls}}(\xi, 0)[\hat{\imath}]$ and items (i) and (ii) follow from items (ii) - (iii), respectively item (iv) of Proposition 3.2. \square

By Proposition 3.2 and the arguments used in its proof, one can also derive the following

Lemma 3.3. *Given an integer $s \geq s_0$, there exists $0 < \rho \leq 1$ so that for any map $\check{\imath}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \equiv \iota_\omega$ in $H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$, which is Lipschitz continuous in $\omega \in \Omega \subset \mathbb{R}^S$ and satisfies $\|\iota\|_{3s_0}^{\text{lip}} \leq \rho$, one has $\check{\imath}(\mathbb{T}^S) \subset M^\sigma$ and for any maps $\hat{\imath}^{(a)} \equiv \hat{\imath}_\omega^{(a)}$ in $H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$, $a = 1, 2$, which are Lipschitz continuous in $\omega \in \Omega$,*

$$\|d^2 X_{H^{nls}}(\xi + y, z\bar{z})[\hat{\imath}^{(1)}, \hat{\imath}^{(2)}]\|_s^{\text{lip}} \leq_s \|\hat{\imath}^{(1)}\|_s^{\text{lip}} \|\hat{\imath}^{(2)}\|_{s_0}^{\text{lip}} + \|\hat{\imath}^{(1)}\|_{s_0}^{\text{lip}} \|\hat{\imath}^{(2)}\|_s^{\text{lip}} + \|\iota\|_{s+2s_0}^{\text{lip}} \|\hat{\imath}^{(1)}\|_{s_0}^{\text{lip}} \|\hat{\imath}^{(2)}\|_{s_0}^{\text{lip}}.$$

We now state tame estimates for the Hamiltonian vector field of the perturbation P . Recall that P is the Hamiltonian \mathcal{P} , expressed in Birkhoff coordinates on M^σ , where $\mathcal{P}(u) = \int_0^1 p(x, u_1(x), u_2(x)) dx$ (cf (1.4)) and $\partial_{\bar{z}} p$ is assumed to be of class $\mathcal{C}^{\sigma, s_*}$ with $s_* > \max(\sigma, s_0)$ sufficiently large. In the following proposition, we restrict the range of s so that Lemma 2.12 applies.

Proposition 3.3. *Given an integer s with $s_0 \leq s \leq s_* - s_0 - 3$, there exists $0 < \rho \leq 1$ so that for any map $\check{\imath}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \equiv \iota_\omega$ in $H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$, which is Lipschitz continuous in $\omega \in \Omega$ and satisfies $\|\iota\|_{3s_0}^{\text{lip}} \leq \rho$, one has $\check{\imath}(\mathbb{T}^S) \subset M^\sigma$ and the following holds:*

(i) $\nabla_\theta P, \nabla_y P$, and $\nabla_z P$ satisfy the tame estimates

$$\|\nabla_\theta P\|_s^{\text{lip}}, \|\nabla_y P\|_s^{\text{lip}}, \|\nabla_z P\|_s^{\text{lip}} \leq_s 1 + \|\iota\|_{s+2s_0}^{\text{lip}}.$$

The derivatives of $\nabla_\theta P, \nabla_y P$, and $\nabla_z P$ with respect to θ and y satisfy the tame estimates

$$\|\partial_\theta \nabla_\theta P \circ \check{\imath}\|_s^{\text{lip}}, \|\partial_y \nabla_\theta P \circ \check{\imath}\|_s^{\text{lip}}, \|\partial_\theta \nabla_y P \circ \check{\imath}\|_s^{\text{lip}}, \|\partial_y \nabla_y P \circ \check{\imath}\|_s^{\text{lip}} \leq_s 1 + \|\iota\|_{s+2s_0}^{\text{lip}}$$

and

$$\|\partial_\theta \nabla_z P \circ \check{\imath}\|_s^{\text{lip}}, \|\partial_y \nabla_z P \circ \check{\imath}\|_s^{\text{lip}} \leq_s 1 + \|\iota\|_{s+2s_0}^{\text{lip}}.$$

Since $\nabla_{\bar{z}} P = \overline{\nabla_z P}$, the derivatives of $\nabla_{\bar{z}} P$ with respect to θ and y also satisfy the same tame estimates.

(ii) For any map $\hat{z}_1 \equiv \hat{z}_{1,\omega}$ in $H^s(\mathbb{T}^S, h_\perp^\sigma)$, which is Lipschitz continuous in $\omega \in \Omega$, the derivatives of $\nabla_\theta P, \nabla_y P, \nabla_z P$, and $\nabla_{\bar{z}} P$ with respect to z in direction \hat{z}_1 satisfy the tame estimates

$$\begin{aligned} & \|\partial_z \nabla_\theta P \circ \check{\imath}[\hat{z}_1]\|_s^{\text{lip}}, \|\partial_z \nabla_y P \circ \check{\imath}[\hat{z}_1]\|_s^{\text{lip}}, \|\partial_z \nabla_z P \circ \check{\imath}[\hat{z}_1]\|_s^{\text{lip}}, \|\partial_z \nabla_{\bar{z}} P \circ \check{\imath}[\hat{z}_1]\|_s^{\text{lip}} \\ & \leq_s \|\hat{z}_1\|_s^{\text{lip}} + \|\iota\|_{s+2s_0}^{\text{lip}} \|\hat{z}_1\|_{s_0}^{\text{lip}}. \end{aligned}$$

Since $\partial_{\bar{z}} = \overline{\partial_z}$, the derivatives of $\nabla_\theta P, \nabla_y P, \nabla_z P$, and $\nabla_{\bar{z}} P$ with respect to \bar{z} in direction $\hat{z}_2 \equiv \hat{z}_{2,\omega}$ admit the same bounds for any \hat{z}_2 in $H^s(\mathbb{T}^S, h_\perp^\sigma)$, which is Lipschitz continuous in $\omega \in \Omega$.

Proof. The stated estimates can be shown in a similar way as the ones for the dNLS Hamiltonian. \square

Finally, one can also derive tame estimates for the second derivative of the Hamiltonian vector field X_P . Again we restrict the range of s so that Lemma 2.12 applies.

Lemma 3.4. *Given an integer s with $s_0 \leq s \leq s_* - s_0 - 4$, there exists $0 < \rho \leq 1$ so that for any map $\check{\iota}(\varphi) = (\varphi, 0, 0) + \iota(\varphi)$ with $\iota \equiv \iota_\omega$ in $H^{s+2s_0}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$, which is Lipschitz continuous in $\omega \in \Omega$ and satisfies $\|\iota\|_{3s_0}^{\gamma_{\text{lip}}} \leq \rho$, one has $\check{\iota}(\mathbb{T}^S) \subset M^\sigma$ and for any maps $\hat{\iota}^{(a)} \equiv \hat{\iota}_\omega^{(a)}$ in $H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$, $a = 1, 2$, which are Lipschitz continuous in $\omega \in \Omega$, one has*

$$\|d^2 X_P \circ \check{\iota}[\hat{\iota}^{(1)}, \hat{\iota}^{(2)}]\|_s^{\gamma_{\text{lip}}} \leq_s \|\hat{\iota}^{(1)}\|_s^{\gamma_{\text{lip}}} \|\hat{\iota}^{(2)}\|_{s_0}^{\gamma_{\text{lip}}} + \|\hat{\iota}^{(1)}\|_{s_0}^{\gamma_{\text{lip}}} \|\hat{\iota}^{(2)}\|_s^{\gamma_{\text{lip}}} + \|\iota\|_{s+2s_0}^{\gamma_{\text{lip}}} \|\hat{\iota}^{(1)}\|_{s_0}^{\gamma_{\text{lip}}} \|\hat{\iota}^{(2)}\|_{s_0}^{\gamma_{\text{lip}}}.$$

Proof. The stated tame estimates correspond to the ones of Lemma 3.3 for the Hamiltonian vector field $X_{H^{nts}}$ and can be derived by the arguments used in the proof of Proposition 3.2. \square

4 Nash-Moser theorem

The purpose of this short section is to reformulate Theorem 1.1 in the functional setup, described in the previous sections, and outline the organisation of its proof.

We consider torus embeddings

$$\check{\iota} : \mathbb{T}^S \rightarrow M^\sigma : \varphi \mapsto (\theta(\varphi), y(\varphi), z(\varphi))$$

whose lifts are assumed to be of the form $(\varphi, 0, 0) + \iota(\varphi)$ where

$$\iota(\varphi) = (\Theta(\varphi), y(\varphi), z(\varphi))$$

with $\Theta : \mathbb{R}^S \rightarrow \mathbb{R}^S$ being 2π -periodic in each component of $\varphi = (\varphi_n)_{n \in S}$. We look for zeros ι of the nonlinear operator F_ω defined in (1.19) by a Nash - Moser theorem.

In the sequel, we will identify such embeddings with their lifts. Furthermore recall that the Sobolev norm $\|\iota\|_{s, \sigma'}$, $\sigma' \leq \sigma$, of the periodic part ι of the map $\check{\iota}$, is given by

$$\|\iota\|_{s, \sigma'} := \|\Theta\|_s + \|y\|_s + \|z\|_{s, \sigma'}$$

where $\|\Theta\|_s := \|\Theta\|_{H^s(\mathbb{T}^S, \mathbb{R}^S)}$, $\|y\|_s := \|y\|_{H^s(\mathbb{T}^S, \mathbb{R}^S)}$, and $\|z\|_{s, \sigma'} := \|z\|_{H^s(\mathbb{T}^S, h_\perp^{\sigma'})}$ (cf (2.7)). In case $\sigma' = \sigma$ we also write $\|\iota\|_s$, $\|z\|_s$, instead of $\|\iota\|_{s, \sigma}$, $\|z\|_{s, \sigma}$.

Theorem 4.1. *Assume the assumptions of Theorem 1.1 hold. Then there is $s_* > \max(\sigma, s_0)$, $s_0 = \lfloor |S|/2 \rfloor + 1$, so that for any $f \in \mathcal{C}^{\sigma, s_*}$ in the perturbed equation (1.3), there exists $0 < \varepsilon_0 < 1$ such that the following holds: for any $0 < \varepsilon \leq \varepsilon_0$, there is a closed subset $\Omega_\varepsilon \subseteq \Omega$ satisfying*

$$\lim_{\varepsilon \rightarrow 0} \frac{\text{meas}(\Omega_\varepsilon)}{\text{meas}(\Omega)} = 1, \quad (4.1)$$

so that for any $\omega \in \Omega_\varepsilon$, there exists a torus embedding $\check{\iota}_\omega : \mathbb{T}^S \rightarrow M^\sigma$, satisfying $\omega \cdot \partial_\varphi \check{\iota}_\omega(\varphi) - X_{H_\varepsilon}(\check{\iota}_\omega(\varphi)) = 0$. This means that the embedded torus $\check{\iota}_\omega(\mathbb{T}^S)$ is invariant for the Hamiltonian vector field $X_{H_\varepsilon(\cdot, \xi)}$ with $\xi = (\omega^{nls})^{-1}(\omega)$, and is filled by quasi-periodic solutions with the frequency ω . The map $\check{\iota}_\omega(\varphi)$ admits a lift of the form $(\varphi, 0, 0) + \iota_\omega(\varphi)$ where ι_ω is in $H^{s_0 + \mu_1}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma)$ for some $\mu_2 > 0$ (depending only on $|S|$) with $s_0 + \mu_2 < s_*$, is Lipschitz continuous in $\omega \in \Omega_\varepsilon$, and satisfies

$$\|\iota_\omega\|_{s_0 + \mu_2}^{\gamma_{\text{lip}}} = O(\varepsilon \gamma^{-2}) \quad \text{with} \quad \gamma \equiv \gamma_\varepsilon := \varepsilon^\alpha (< 1), \quad 0 < \alpha < 1/4.$$

Furthermore the linearized equation at the quasi-periodic solution $\check{\iota}_\omega(\omega t) = \omega t + \iota_\omega(\omega t)$ is stable – see Corollary 8.1 for a precise statement.

Remark 4.1. *In the estimates of the embedded tori we do not distinguish between the different components Θ , y , z of ι . Actually, the estimates for y and z can be sharpened for most ω in Ω_ε . It turns out that an effective way for proving the improved ones is to do so a posteriori, using that $F_\omega(\iota_\omega, 0) = 0$ and that $\|\iota_\omega\|_{s_0 + \mu_2}^{\gamma_{\text{lip}}} = O(\varepsilon \gamma^{-2})$. See Corollary 8.2 and its proof for details.*

Comments:

1. Up to the end of Section 8, $\gamma \in (0, 1)$ is assumed to be a constant independent of ε with $\varepsilon\gamma^{-4}$ small. Only in Section 9 (Theorem 9.1), γ and ε are assumed to be related by requiring that $\gamma_\varepsilon = \varepsilon^a$ for some $0 < a < 1/4$. The set Ω_ε is defined in (8.37).
2. Let $\Pi \subseteq \Pi_S$ be a compact subset with measure $|\Pi| > 0$. By Proposition 3.1, for any $\delta > 0$ there exists an open subset Π_δ of Π_S so that $\text{meas}(\Pi \cap \Pi_\delta) \leq \delta$ and on $\Pi \setminus \Pi_\delta$, $\det((\partial_{I_j} \omega_n^{nls})_{i,j \in S})$ is bounded and uniformly bounded away from 0. Hence on $\Pi \setminus \Pi_\delta$, the action to frequency map $I \mapsto (\omega_n^{nls})_{n \in S}$ is a local diffeomorphism. As $\Pi \setminus \Pi_\delta$ is compact there exists a finite cover $(\Pi^{(i)})_{i \in \mathcal{I}}$ of $\Pi \setminus \Pi_\delta$ with $\Pi^{(i)}$ compact so that $\Pi^{(i)} \rightarrow \mathbb{R}^S, I \mapsto (\omega_n^{nls})_{n \in S}$ is a bi-Lipschitz homeomorphism onto its image. By first choosing $\delta > 0$ and then applying Theorem 4.1 for the finitely many parameter sets $\Pi^{(i)}, i \in \mathcal{I}$, for $0 < \varepsilon \leq \varepsilon_0(\delta)$, one sees that Theorem 4.1 holds for any compact subset $\Pi \subseteq \Pi_S$ with $\text{meas}(\Pi) > 0$ as set of parameters.

Theorem 4.1 – which implies Theorem 1.1 – is shown in Section 5 - 9 by means of a Nash-Moser iteration scheme. Let us give a brief outline of its proof. It is convenient to introduce an auxiliary variable $\zeta \in \mathbb{R}^S$ and consider the modified Hamiltonian vector field $X_{H_{\varepsilon,\zeta}} = X_{H_\varepsilon} + (0, \zeta, 0)$ with Hamiltonian

$$H_{\varepsilon,\zeta}(\theta, y, z) \equiv H_{\varepsilon,\zeta}(\theta, y, z; \omega) := H_\varepsilon(\theta, y, z) + \zeta \cdot \theta, \quad \zeta \in \mathbb{R}^S, \quad (4.2)$$

where H_ε is defined in (1.18) and considered as a function of the parameter $\omega \in \Omega$ by setting $\xi = (\omega^{nls})^{-1}(\omega)$. Lemma 5.1 shows that any invariant torus for $X_{H_{\varepsilon,\zeta}}$ is actually invariant for X_{H_ε} . The variable ζ will allow us to control the average of the y -component of approximations of the linearized Hamiltonian vector fields, adding in this way flexibility for choosing such approximations.

We look for zeros of the map

$$F_\omega(\iota, \zeta) := \omega \cdot \partial_\varphi \check{\iota}(\varphi) - X_{H_{\varepsilon,\zeta}}(\check{\iota}(\varphi)) = \omega \cdot \partial_\varphi \check{\iota}(\varphi) - X_{H_\varepsilon}(\check{\iota}(\varphi)) + (0, \zeta, 0) \quad (4.3)$$

which when written componentwise reads

$$F_\omega(\iota, \zeta) = (\omega \cdot \partial_\varphi \theta - \nabla_y H_\varepsilon, \omega \cdot \partial_\varphi y + \nabla_\theta H_\varepsilon + \zeta, \omega \cdot \partial_\varphi z + i \nabla_{\bar{z}} H_\varepsilon). \quad (4.4)$$

In order to implement a convergent Nash-Moser scheme that leads to a solution of $F_\omega(\iota, \zeta) = 0$, the main task is to construct an *approximate right inverse* of the differential $d_{\iota,\zeta} F_\omega$, satisfying tame estimates – see Theorem 5.2 in the subsequent section. Note that the derivative of $F_\omega(\iota, \zeta)$ in direction $(\widehat{\iota}, \widehat{\zeta})$ is given by

$$d_{\iota,\zeta} F_\omega[\widehat{\iota}, \widehat{\zeta}] = \omega \cdot \partial_\varphi \widehat{\iota} - \partial_\iota X_{H_\varepsilon}(\check{\iota}(\varphi))[\widehat{\iota}] + (0, \widehat{\zeta}, 0), \quad (4.5)$$

which is independent of ζ . According to [32], an approximate right inverse of $d_{\iota,\zeta} F_\omega$ is a map with the property that, when composed with $d_{\iota,\zeta} F_\omega$, it is equal to the identity up to an error of the size of $F_\omega(\iota, \zeta)$. In particular, at a solution (ι, ζ, ω) of $F_\omega(\iota, \zeta) = 0$, an approximate right inverse is an *exact* one. For constructing an approximate right inverse, we implement the strategy developed in [5], [2] which reduces the search of such an operator to the one of an approximate right inverse of the part of $d_{\iota,\zeta} F_\omega$, acting on the normal directions only – see Theorem 5.1, which is proved in Section 6 and Section 7. In these sections we also provide estimates for the variation of the quantities considered with respect to the torus embedding $\check{\iota}$. This information is needed for the proof of the measure estimates of Section 9 (Theorem 9.1). The construction of solutions of $F_\omega(\iota, \zeta) = 0$ via a Nash-Moser iteration scheme and the proof of their linear stability is presented in Section 8 (Theorem 8.1 and Corollary 8.1).

5 Approximate right inverse

The main result of this section is Theorem 5.2. Throughout the remainder of the paper, we always assume that $\check{\iota} \equiv \check{\iota}_\omega : \mathbb{T}^S \rightarrow M^\sigma, \varphi \mapsto \check{\iota}(\varphi)$ is a C^∞ torus embedding of the form $(\varphi, 0, 0) + \iota(\varphi)$ Lipschitz continuous in ω on a closed subset

$$\Omega_o(\iota) \subset \Omega_{\gamma,\tau} \subset \Omega, \quad (5.1)$$

where $\Omega_{\gamma,\tau}$ is the set of diophantine frequencies introduced in (1.22). Furthermore, we assume that ι is small in the sense that

$$\|\iota\|_{s_0+\mu_1}^{\gamma\text{lip}} \ll \varepsilon\gamma^{-2}, \quad \|E\|_{s_0+\mu_1,\sigma-2}^{\gamma\text{lip}} \ll \varepsilon \quad \text{with} \quad \varepsilon\gamma^{-4} \ll 1 \quad \text{and} \quad 0 < \gamma < 1 \quad (5.2)$$

where $E : \mathbb{T}^S \rightarrow \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma-2}$ is the 'error function' of (ι, ζ) ,

$$E(\varphi) := (E_{\theta}(\varphi), E_y(\varphi), E_z(\varphi)) = F_{\omega}(\iota, \zeta)(\varphi). \quad (5.3)$$

It will be verified in Section 8 that the smallness assumptions (5.2) hold along the Nash-Moser iteration scheme. In all of Section 5, if not stated otherwise, the Lipschitz estimates are computed on $\Omega_o(\iota)$. Furthermore, in the estimates in the subsequent subsections, the Sobolev exponent s will be an arbitrary integer satisfying

$$s_0 \leq s \leq s_* - \mu_1, \quad s_0 = [S/2] + 1.$$

Here, $\mu_1 \equiv \mu_1(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ is assumed to be sufficiently large so that it is bigger than various integers $\mu \equiv \mu(|S|, \tau)$, coming up in the lemmas below, and so that the tame estimates of Subsection 2.3 such as the ones of Lemma 2.12 apply in the situations considered.

5.1 Formula for ζ

For any given torus embedding the vector ζ and the error function E defined in (5.3) are related:

Lemma 5.1. *For any torus embedding $\check{\iota} \equiv \check{\iota}_{\omega}$, we have*

$$\zeta = \frac{1}{(2\pi)^S} \int_{\mathbb{T}^S} \left(-(\partial_{\varphi}\theta(\varphi))^t \cdot E_y + (\partial_{\varphi}y)^t \cdot E_{\theta} - i(\partial_{\varphi}z)^t \cdot \bar{E}_z + i(\partial_{\varphi}\bar{z})^t \cdot E_z \right) d\varphi. \quad (5.4)$$

Hence ζ is Lipschitz continuous in $\omega \in \Omega_o(\iota)$ and satisfies the estimate

$$|\zeta|^{\gamma\text{lip}} \ll \|E\|_{s_0,\sigma-2}^{\gamma\text{lip}}.$$

As a consequence, for any (ι, ζ) with $F_{\omega}(\iota, \zeta) = 0$ one has $\zeta = 0$, and the torus $\check{\iota}(\mathbb{T}^S)$ is invariant for the Hamiltonian vector field $X_{H_{\varepsilon}}$.

Proof. We follow the arguments in [5]. Since H_{ε} is an autonomous Hamiltonian one verifies by a straightforward change of variables that the function

$$G : \mathbb{T}^S \rightarrow \mathbb{C}, \quad \psi \mapsto G(\psi) := \int_{\mathbb{T}^S} \left(-\lambda_{\check{\iota}(\psi)}(\omega \cdot \partial_{\varphi}\check{\iota}^{(\psi)}) - H_{\varepsilon}(\check{\iota}^{(\psi)}) \right) d\varphi$$

is constant, where $\check{\iota}^{(\psi)}(\varphi) := \check{\iota}(\psi + \varphi)$ and $\lambda_{\check{\iota}(\psi+\varphi)}$ is the canonical one form λ defined in (3.16) evaluated at $\check{\iota}(\psi + \varphi)$. Note that $-\lambda_{\check{\iota}}(\omega \cdot \partial_{\varphi}\check{\iota}) - H_{\varepsilon}(\check{\iota})$ is the Lagrangian associated to H_{ε} . Using that $\partial_{\psi}G(0) = 0$, a direct calculation proves (5.4). By Lemma 2.7 (tame estimates for products of maps), the fact that $E \in H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma-2})$ and the smallness assumption (5.2), the claimed estimate follows. \square

5.2 Isotropic torus embeddings

An invariant torus $\check{\iota}(\mathbb{T}^S)$, densely filled by a quasi-periodic solution, is isotropic (cf e.g. Lemma 1 in [5]). It means that the pullback of the symplectic form Λ by $\check{\iota}$ vanishes, $\check{\iota}^*\Lambda = 0$. In our symplectic setup it is useful to work with isotropic torus embeddings. In Lemma 5.3 below we provide a canonical construction for approximating a torus embedding $\check{\iota}$ by an isotropic one. By a straightforward computation one verifies that in our infinite dimensional setup

$$\check{\iota}^*\Lambda = d(\check{\iota}^*\lambda) \quad (5.5)$$

where $\check{\iota}^*\lambda$ is the pullback of the *one-form* λ defined by (3.16). Here d denotes the exterior differential of the one-form $\check{\iota}^*\lambda$ on the torus \mathbb{T}^S . Our task is therefore to provide a canonical construction of approximating $\check{\iota}$

by an embedding $\check{\iota}_{\text{iso}}$ so that $\check{\iota}_{\text{iso}}^* \lambda$ is a closed one form. Any C^2 -smooth one-form $\alpha = \sum_{j \in S} a_j d\varphi_j$ on the torus \mathbb{T}^S admits a Hodge decomposition

$$\alpha = \sum_{j \in S} [[a_j]] d\varphi_j + df + \rho,$$

where the constant one-form $\sum_{j \in S} [[a_j]] d\varphi_j$ is the harmonic part of α with

$$[[a_j]] := \frac{1}{(2\pi)^{|S|}} \int_{\mathbb{T}^S} a_j(\varphi) d\varphi,$$

df is the exact one-form with $f : \mathbb{T}^S \rightarrow \mathbb{C}$ having average 0 and $\rho := \sum_{j \in S} r_j d\varphi_j$ is a co-closed one-form, meaning that $r = (r_j)_{j \in S}$ satisfies $\text{div}(r) = 0$. In the language of differential forms it means that $d^* \rho = 0$, where d^* denotes the adjoint of d with respect to the standard inner product. Using integration by parts, a standard computation yields $d^* \alpha = -\text{div}(a)$ where $a = (a_j)_{j \in S}$. Since $d^* df = d^* \alpha$ it then follows that

$$f = \Delta^{-1}(\text{div}(a)), \quad \Delta = \sum_{j \in S} \partial_{\varphi_j}^2.$$

The expression $\Delta^{-1}(\text{div}(a))$ is well defined as the average of $\text{div}(a)$ vanishes. Similarly, since $d\rho = d\alpha = \sum_{k < j} A_{kj} d\varphi_k \wedge d\varphi_j$ with $A_{kj} := \partial_{\varphi_k} a_j - \partial_{\varphi_j} a_k$, one computes $d^* d\rho = \sum_{k \in S} (\sum_{j \in S} \partial_{\varphi_j} A_{kj}) d\varphi_k$, yielding

$$r_k = -\Delta^{-1} \left(\sum_{j \in S} \partial_{\varphi_j} A_{kj} \right), \quad \forall k \in S. \quad (5.6)$$

In the situation at hand, the one-form $\sum_{j \in S} a_j d\varphi_j$ is given by the pullback $\check{\iota}^* \lambda$ of λ ,

$$a = (a_j)_{j \in S} = -(\partial_{\varphi} \theta)^t y + i(\partial_{\varphi} \bar{z})^t z \quad (5.7)$$

and one has

$$d(\check{\iota}^* \lambda - \rho) = 0, \quad \check{\iota}^* \lambda - \rho = \sum_{k \in S} (a_k - r_k) d\varphi_k \quad (5.8)$$

where $r = (r_k)_{k \in S}$ is of the form (5.6). In view of (5.6), (5.7) define $\check{\iota}_{\text{iso}}(\varphi) := (\varphi, 0, 0) + \iota_{\text{iso}}(\varphi)$ where

$$\iota_{\text{iso}}(\varphi) := (\theta(\varphi) - \varphi, y_{\text{iso}}(\varphi), z(\varphi)), \quad y_{\text{iso}}(\varphi) := y(\varphi) + (\partial_{\varphi} \theta(\varphi))^{-t} r(\varphi). \quad (5.9)$$

We prove in Lemma 5.3 that $\check{\iota}_{\text{iso}}(\mathbb{T}^S) \subseteq M^\sigma$ is an isotropic torus. First we estimate the coefficients A_{kj} , $k, j \in S$, in terms of the error function E . Denoting by $(\underline{e}_j)_{j \in S}$ the standard basis of \mathbb{R}^S , one has

$$A_{kj} \stackrel{(5.5)}{=} \check{\iota}^* \Lambda[\underline{e}_k, \underline{e}_j] = \Lambda[\partial_{\varphi_k} \check{\iota}, \partial_{\varphi_j} \check{\iota}]$$

and hence

$$\omega \cdot \partial_{\varphi} A_{kj} = \Lambda[\partial_{\varphi_k}(\omega \cdot \partial_{\varphi} \check{\iota}), \partial_{\varphi_j} \check{\iota}] + \Lambda[\partial_{\varphi_k} \check{\iota}, \partial_{\varphi_j}(\omega \cdot \partial_{\varphi} \check{\iota})].$$

Recall that $\omega \cdot \partial_{\varphi} \check{\iota} = E + X_{H_\varepsilon} - (0, \zeta, 0)$ and hence $\partial_{\varphi_k} \omega \cdot \partial_{\varphi} \check{\iota} = \partial_{\varphi_k} E + \partial_{\varphi_k} X_{H_\varepsilon}$. In view of the formula (3.15) for Λ and since the Hessian $d^2 H_\varepsilon$ is symmetric one has

$$\Lambda[\partial_{\varphi_k} X_{H_\varepsilon}, \partial_{\varphi_j} \check{\iota}] + \Lambda[\partial_{\varphi_k} \check{\iota}, \partial_{\varphi_j} X_{H_\varepsilon}] = d^2 H_\varepsilon[\partial_{\varphi_k} \check{\iota}, \partial_{\varphi_j} \check{\iota}] - d^2 H_\varepsilon[\partial_{\varphi_j} \check{\iota}, \partial_{\varphi_k} \check{\iota}] = 0$$

implying that

$$\omega \cdot \partial_{\varphi} A_{kj} = \Lambda[\partial_{\varphi_k} E, \partial_{\varphi_j} \check{\iota}] + \Lambda[\partial_{\varphi_k} \check{\iota}, \partial_{\varphi_j} E]. \quad (5.10)$$

This formula allows to prove the following lemma.

Lemma 5.2. *There exists $\mu \equiv \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any integer $s_0 \leq s \leq s_* - \mu$, the following tame estimate holds:*

$$\sup_{k, j \in S} \|A_{kj}\|_s^{\text{lip}} \leq \gamma^{-1} (\|E\|_{s+2\tau+2, \sigma-2}^{\text{lip}} + \|E\|_{s_0+1, \sigma-2}^{\text{lip}} \|\iota\|_{s+2\tau+2}^{\text{lip}}).$$

Proof. In view of the formula (3.15) for Λ , the identity (5.10) for A_{kj} , the estimate of Lemma 2.2 for the solution A_{kj} of (5.10), the tame estimates for products of functions in $H^s(\mathbb{T}^S, \mathbb{C})$ of Lemma 2.7, the assumptions $\sigma \geq 4$, and the smallness condition (5.2), the claimed estimate follows. \square

The main result of this section is the following lemma.

Lemma 5.3. (Isotropic torus) *The torus embedding $\check{\iota}_{\text{iso}}(\varphi) := (\theta(\varphi), y_{\text{iso}}(\varphi), z(\varphi))$, defined by (5.9), is isotropic, $\check{\iota}^* \Lambda = 0$. Expressed in coordinates, it means that*

$$(\partial_\varphi \theta)^t \partial_\varphi y_{\text{iso}} - (\partial_\varphi y_{\text{iso}})^t \partial_\varphi \theta + \text{i}(\partial_\varphi z)^t \partial_\varphi \bar{z} - \text{i}(\partial_\varphi \bar{z})^t \partial_\varphi z = 0. \quad (5.11)$$

Moreover there exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any integer $s_0 \leq s \leq s_* - \mu$

$$\|y_{\text{iso}} - y\|_s^{\text{lip}} \leq_s \gamma^{-1} (\|E\|_{s+\mu, \sigma-2}^{\text{lip}} + \|E\|_{s_0+\mu, \sigma-2}^{\text{lip}} \|\iota\|_{s+\mu}^{\text{lip}}) \quad (5.12)$$

$$\|\iota_{\text{iso}}\|_s^{\text{lip}} \leq_s \|\iota\|_{s+\mu}^{\text{lip}} \quad (5.13)$$

$$\|F_\omega(\iota_{\text{iso}}, \zeta)\|_{s, \sigma-2}^{\text{lip}} \leq_s \gamma^{-1} (\|E\|_{s+\mu, \sigma-2}^{\text{lip}} + \|E\|_{s_0+\mu, \sigma-2}^{\text{lip}} \|\iota\|_{s+\mu}^{\text{lip}}) \quad (5.14)$$

$$\|d_\iota(\iota_{\text{iso}})[\hat{\iota}]\|_s \leq_s \|\hat{\iota}\|_{s+\mu} + \|\iota\|_{s+\mu} \|\hat{\iota}\|_{s_0+\mu}. \quad (5.15)$$

Proof. By (5.3) one sees that $\check{\iota}_{\text{iso}}^* \lambda = \sum_{j \in S} a_j^{\text{iso}}(\varphi) d\varphi_j$ is given by

$$a_{\text{iso}} = (a_j^{\text{iso}})_{j \in S} = -(\partial_\varphi \theta)^t y_{\text{iso}} + \text{i}(\partial_\varphi \bar{z})^t z = -(\partial_\varphi \theta)^t y - r + \text{i}(\partial_\varphi \bar{z})^t z = a - r.$$

Hence $\check{\iota}_{\text{iso}}^* \Lambda \stackrel{(5.5)}{=} d(\check{\iota}_{\text{iso}}^* \lambda) \stackrel{(5.8)}{=} 0$. As a consequence $\Lambda[\partial_{\varphi_k} \check{\iota}_{\text{iso}}, \partial_{\varphi_j} \check{\iota}_{\text{iso}}] = 0$ for any $k, j \in S$. By the formula (3.15) for Λ , the claimed identity (5.11) follows. The estimate (5.12) follows from the definition of y_{iso} (cf (5.9)), the one of r (cf (5.6)), and Lemma 5.2. To obtain (5.13), one expresses r in terms of a (cf formula (5.7)) and uses the tame estimates of products of Lemma 2.7. The estimate (5.14) is obtained by the mean value theorem, using the estimate of $y_{\text{iso}} - y$ of (5.12) and the estimates for $\partial_y X_{H_\varepsilon}$ (cf Proposition 3.2 and Proposition 3.3), and (5.13). The remaining estimate (5.15) is derived in a similar fashion. \square

5.3 Canonical coordinates near an isotropic torus

In order to facilitate the search of an approximate inverse of the differential $d_{\iota, \zeta} F_\omega(\iota_{\text{iso}}, \zeta)$ we introduce suitable coordinates (ψ, v, w) near the isotropic torus $\check{\iota}_{\text{iso}}(\mathbb{T}^S) \subseteq M^\sigma$,

$$\Gamma : \begin{pmatrix} \psi \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \theta(\psi) \\ y_{\text{iso}}(\psi) + Y(\psi, v, w) \\ z(\psi) + w \end{pmatrix} \quad (5.16)$$

where

$$Y(\psi, v, w) := (\partial_\psi \theta)^{-t}(\psi)v + Y_w(\psi)w + Y_{\bar{w}}(\psi)\bar{w} \quad (5.17)$$

and for any $\psi \in \mathbb{T}^S$, $Y_w(\psi)$ is the linear operator

$$Y_w(\psi) : h_\perp^\sigma \rightarrow \mathbb{C}^S, \quad w \mapsto \text{i}(\partial_\psi \theta)^{-t}(\partial_\psi \bar{z})^t w, \quad Y_{\bar{w}} = \overline{Y}_w. \quad (5.18)$$

By the definition (5.16) of the transformation Γ one has

$$\check{\iota}_{\text{iso}} = \Gamma \circ \check{\iota}_0 \quad \text{where} \quad \check{\iota}_0 : \mathbb{T}^S \rightarrow M^\sigma, \quad \varphi \mapsto (\varphi, 0, 0), \quad (5.19)$$

i.e., in the new coordinates, $\check{\iota}_{\text{iso}}$ is given by $\check{\iota}_0$. Furthermore, using (5.11) (since $\check{\iota}_{\text{iso}}(\mathbb{T}^S)$ is an isotropic torus) one verifies that $\Gamma^* \Lambda = \Lambda$, i.e., Γ is canonical, see also [5]. For our purposes, it suffices to consider $d_\iota(\Gamma \circ \check{\iota})$ at $\iota = 0$, which we denote by $d\Gamma \circ \check{\iota}_0$. Following the procedure described in Subsection 3.2, we extend the bilinear map $d_\iota^2(\Gamma \circ \check{\iota})$ to be defined for elements $(\hat{\iota}^{(1)}, \hat{\iota}^{(2)})$ with $\hat{\iota}^{(a)} := (\hat{\psi}^{(a)}, \hat{v}^{(a)}, \hat{w}_1^{(a)}, \hat{w}_2^{(a)})$ in $H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$, $a = 1, 2$, and denote it by $d^2\Gamma \circ \check{\iota}_0$, when evaluated at $\iota = 0$.

Lemma 5.4. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$, so that for any $\widehat{\iota} := (\widehat{\psi}, \widehat{v}, \widehat{w})$ in $H^s(\mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma'})$ with $s_0 \leq s \leq s_* - \mu$ and $\sigma - 2 \leq \sigma' \leq \sigma$,*

$$\|(d\Gamma(\check{\iota}_0(\varphi)) - \text{Id})[\widehat{\iota}]\|_{s, \sigma'} \leq_s \|\iota\|_{s_0 + \mu} \|\widehat{\iota}\|_{s, \sigma'} + \|\iota\|_{s + \mu} \|\widehat{\iota}\|_{s_0, \sigma'}, \quad (5.20)$$

$$\|(d\Gamma(\check{\iota}_0(\varphi)))^{-1}[\widehat{\iota}]\|_{s, \sigma'} \leq_s \|\widehat{\iota}\|_{s, \sigma'} + \|\iota\|_{s + \mu} \|\widehat{\iota}\|_{s_0, \sigma'}. \quad (5.21)$$

Moreover, for any $\widehat{\iota}^{(a)} := (\widehat{\psi}^{(a)}, \widehat{v}^{(a)}, \widehat{w}_1^{(a)}, \widehat{w}_2^{(a)}) \in H^s(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma} \times h_{\perp}^{\sigma})$, $a = 1, 2$,

$$\|d^2\Gamma(\check{\iota}_0(\varphi))[\widehat{\iota}^{(1)}, \widehat{\iota}^{(2)}]\|_s \leq_s \|\widehat{\iota}^{(1)}\|_s \|\widehat{\iota}^{(2)}\|_{s_0} + \|\widehat{\iota}^{(1)}\|_{s_0} \|\widehat{\iota}^{(2)}\|_s + \|\iota\|_{s + \mu} \|\widehat{\iota}^{(1)}\|_{s_0} \|\widehat{\iota}^{(2)}\|_{s_0}.$$

The same estimates hold if the norm $\|\cdot\|_s$ is replaced by $\|\cdot\|_s^{\text{lip}}$.

Proof. The estimate (5.20) is obtained from the formula of the differential of $\Gamma \circ \check{\iota}$ with respect to ι at $\iota = 0$ and the tame estimates for products of maps of Lemma 2.7. As mentioned at the beginning of this section, we choose μ_0 larger than μ . Hence by the smallness condition (5.2), the estimate of $(d\Gamma(\varphi, 0, 0) - \text{Id})[\widehat{\iota}]$ for $s = s_0$ yields

$$\|(d\Gamma(\check{\iota}_0(\varphi)) - \text{Id})[\widehat{\iota}]\|_{s_0} < \varepsilon \gamma^{-2} \|\widehat{\iota}\|_{s_0}.$$

Since $\varepsilon \gamma^{-2}$ is assumed to be sufficiently small, it follows that for any $\varphi \in \mathbb{T}^S$, the operator $d\Gamma(\check{\iota}_0(\varphi))$ on $\mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma}$ is invertible by Neumann series. One then verifies in a straightforward way that $\|(d\Gamma(\check{\iota}_0(\varphi)))^{-1}[\widehat{\iota}]\|_s$ satisfies the bound, stated in (5.21). The claimed bound for $\|d^2\Gamma(\check{\iota}_0(\varphi))[\widehat{\iota}^{(1)}, \widehat{\iota}^{(2)}]\|_s$ is obtained from the formula of the second derivative of $\Gamma \circ \check{\iota}$ and the tame estimates for products of maps, stated in Lemma 2.7. The stated estimates of the γ lip-norms of the expressions considered can be derived by similar arguments. \square

Denote by $K_{\varepsilon, \zeta}$ the Hamiltonian $H_{\varepsilon, \zeta}$, expressed in the new coordinates,

$$K_{\varepsilon, \zeta} := H_{\varepsilon, \zeta} \circ \Gamma = H_{\varepsilon} \circ \Gamma + \zeta \cdot \theta(\psi), \quad K_{\varepsilon} := H_{\varepsilon} \circ \Gamma. \quad (5.22)$$

The corresponding Hamiltonian vector field is then given by

$$X_{K_{\varepsilon, \zeta}} := (\nabla_v K_{\varepsilon}, -\nabla_{\psi} K_{\varepsilon} - (\partial_{\psi} \theta)^t \zeta, -i \nabla_{\bar{w}} K_{\varepsilon}). \quad (5.23)$$

Furthermore, since $\check{\iota}_{\text{iso}}(\varphi) = \Gamma(\check{\iota}_0(\varphi))$, the directional derivative $\omega \cdot \partial_{\varphi} \check{\iota}_{\text{iso}}(\varphi)$ equals $d\Gamma(\check{\iota}_0(\varphi))[(\omega, 0, 0)]$. Using the transformation law of vector fields one concludes that

$$F_{\omega}(\check{\iota}_{\text{iso}}, \zeta)(\varphi) = \omega \cdot \partial_{\varphi} \check{\iota}_{\text{iso}}(\varphi) - X_{H_{\varepsilon, \zeta}}(\check{\iota}_{\text{iso}}(\varphi)) = d\Gamma(\check{\iota}_0(\varphi))[(\omega, 0, 0)] - d\Gamma(\check{\iota}_0(\varphi))X_{K_{\varepsilon, \zeta}}(\check{\iota}_0(\varphi)),$$

or

$$X_{K_{\varepsilon, \zeta}}(\check{\iota}_0(\varphi)) = (\omega, 0, 0) - (d\Gamma(\check{\iota}_0(\varphi)))^{-1}F_{\omega}(\check{\iota}_{\text{iso}}, \zeta)(\varphi). \quad (5.24)$$

Note that if $\check{\iota}_{\text{iso}}$ is a solution, i.e., $F_{\omega}(\check{\iota}_{\text{iso}}, \zeta) = 0$, then by Lemma 5.1, $\zeta = 0$ and hence by the formula above, $X_{K_{\varepsilon, 0}}(\check{\iota}_0(\varphi)) = (\omega, 0, 0)$. Comparing this with this formula (5.23) one gets in this case

$$\nabla_v K_{\varepsilon} \circ \check{\iota}_0(\varphi) = \omega, \quad \nabla_{\psi} K_{\varepsilon} \circ \check{\iota}_0(\varphi) = 0, \quad \nabla_{\bar{w}} K_{\varepsilon} \circ \check{\iota}_0(\varphi) = 0.$$

In the general case one has the following estimates:

Lemma 5.5. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$, so that for any integer $s_0 \leq s \leq s_* - \mu$*

$$\begin{aligned} \|\nabla_{\psi} K_{\varepsilon} \circ \check{\iota}_0\|_s^{\text{lip}}, \|\nabla_v K_{\varepsilon} \circ \check{\iota}_0 - \omega\|_s^{\text{lip}} &\leq_s \gamma^{-1} (\|E\|_{s + \mu, \sigma - 2}^{\text{lip}} + \|E\|_{s_0 + \mu, \sigma - 2}^{\text{lip}} \|\iota\|_{s + \mu}^{\text{lip}}), \\ \|\nabla_{\bar{w}} K_{\varepsilon} \circ \check{\iota}_0\|_{s, \sigma - 2}^{\text{lip}}, \|\nabla_{\bar{w}} K_{\varepsilon} \circ \check{\iota}_0\|_{s, \sigma - 2}^{\text{lip}} &\leq_s \gamma^{-1} (\|E\|_{s + \mu, \sigma - 2}^{\text{lip}} + \|E\|_{s_0 + \mu, \sigma - 2}^{\text{lip}} \|\iota\|_{s + \mu}^{\text{lip}}). \end{aligned}$$

Proof. The claimed estimates follow from the formula (5.24) and the estimates (5.14), (5.21). \square

5.4 Approximate right inverse of the differential of F_ω

By formula (4.3), the differential $d_{\iota, \zeta} F_\omega$ is independent of ζ and hence we write $d_{\iota, \zeta} F_\omega(\iota)$ for its value at ι . To get an approximate right inverse for the differential $d_{\iota, \zeta} F_\omega$ at (ι, ζ) , it suffices to construct an approximate inverse of the differential at $(\check{\iota}_{\text{iso}}, \zeta)$. Indeed

$$G_1[\widehat{\iota}, \widehat{\zeta}] := d_{\iota, \zeta} F_\omega(\iota)[\widehat{\iota}, \widehat{\zeta}] - d_{\iota, \zeta} F_\omega(\iota_{\text{iso}})[\widehat{\iota}, \widehat{\zeta}] \stackrel{(4.5)}{=} -d_\iota X_{H_\varepsilon}(\check{\iota}(\varphi))[\widehat{\iota}] + d_\iota X_{H_\varepsilon}(\check{\iota}_{\text{iso}}(\varphi))[\widehat{\iota}] \quad (5.25)$$

satisfies the following estimates:

Lemma 5.6. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$, so that for any $\widehat{\iota} := (\widehat{\varphi}, \widehat{y}, \widehat{z}_1, \widehat{z}_2)$ in $H^{s+\mu}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$ with $s_0 \leq s \leq s_* - \mu$ and any $\widehat{\zeta} \in \mathbb{R}^S$, which are both Lipschitz continuous in ω*

$$\|G_1[\widehat{\iota}, \widehat{\zeta}]\|_{s, \sigma-2}^{\gamma \text{lip}} \leq_s \gamma^{-1} \left(\|E\|_{s+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\zeta}\|_{s_0+\mu}^{\gamma \text{lip}} + \|E\|_{s_0+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s+\mu}^{\gamma \text{lip}} + \|\iota\|_{s+\mu}^{\gamma \text{lip}} \|E\|_{s_0+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s_0+\mu}^{\gamma \text{lip}} \right).$$

Proof. By the mean value theorem and the definition (5.3) of $\check{\iota}_{\text{iso}}$, one has

$$G_1 = \int_0^1 (y_{\text{iso}} - y) \cdot \partial_y (d_\iota X_{H_\varepsilon}(\check{\iota} + t(\iota_{\text{iso}} - \iota)))[\widehat{\iota}] dt = \int_0^1 d_\iota^2 X_{H_\varepsilon}(\check{\iota} + t(\iota_{\text{iso}} - \iota))[\widehat{\iota}, \widehat{\iota}^{(1)}] dt$$

where $\widehat{\iota}^{(1)} = (0, y_{\text{iso}} - y, 0, 0)$. The claimed estimate then follows from the tame estimate of $y_{\text{iso}} - y$ of (5.12) and the tame estimate for $d^2 X_{H_\varepsilon} \circ \check{\iota}[\widehat{\iota}, \widehat{\iota}^{(1)}]$, obtained from Lemma 3.3 and Lemma 3.4. \square

We consider torus embeddings of the form $\Gamma(\check{\iota})$, where $\check{\iota}(\varphi) := (\psi(\varphi), y(\varphi), z(\varphi))$ and Γ is the coordinate transformation, introduced in (5.16). Since Γ is symplectic

$$X_{H_{\varepsilon, \zeta}} \circ \Gamma = d\Gamma \circ X_{K_{\varepsilon, \zeta}}$$

and one has

$$F_\omega(\Gamma(\check{\iota}) - \check{\iota}_0, \zeta) = d\Gamma(\check{\iota})(\omega \cdot \partial_\varphi \check{\iota} - X_{K_{\varepsilon, \zeta}}(\check{\iota}, \zeta)).$$

Denoting the differential of F_ω with respect to the two arguments temporarily by dF_ω one then gets by the chain and product rule for any $\widehat{\iota}(\varphi) = (\widehat{\psi}(\varphi), \widehat{v}(\varphi), \widehat{w}(\varphi), \widehat{\widetilde{w}}(\varphi))$ and $\widehat{\zeta} \in \mathbb{R}^S$

$$\begin{aligned} dF_\omega(\Gamma(\check{\iota}) - \check{\iota}_0, \zeta)[d\Gamma(\check{\iota})\widehat{\iota}, \widehat{\zeta}] &= d_{\iota, \zeta}(F_\omega(\Gamma(\check{\iota}) - \check{\iota}_0, \zeta))[\widehat{\iota}, \widehat{\zeta}] \\ &= d\Gamma(\check{\iota})(\omega \cdot \partial_\varphi \widehat{\iota} - d_{\iota, \zeta} X_{K_{\varepsilon, \zeta}}(\check{\iota})[\widehat{\iota}, \widehat{\zeta}]) + d^2\Gamma(\check{\iota})[d\Gamma(\check{\iota})^{-1}(F_\omega(\Gamma(\check{\iota}) - \check{\iota}_0, \zeta)), \widehat{\iota}]. \end{aligned}$$

Now we evaluate the above expression at $\check{\iota} = \check{\iota}_0$ and $\widehat{\iota}$ given by $d\Gamma(\check{\iota})^{-1}\widehat{\iota}$. Recalling that $\Gamma(\check{\iota}_0) = \check{\iota}_{\text{iso}}$ we get

$$d_{\iota, \zeta} F_\omega(\iota_{\text{iso}})[\widehat{\iota}, \widehat{\zeta}] = d\Gamma(\check{\iota}_0)(\omega \cdot \partial_\varphi - d_{\iota, \zeta} X_{K_{\varepsilon, \zeta}}(\check{\iota}_0))[d\Gamma(\check{\iota}_0)^{-1}[\widehat{\iota}], \widehat{\zeta}] + G_2[\widehat{\iota}, \widehat{\zeta}], \quad (5.26)$$

where

$$G_2[\widehat{\iota}, \widehat{\zeta}] := d^2\Gamma(\check{\iota}_0)[d\Gamma(\check{\iota}_0)^{-1}[F_\omega(\iota_{\text{iso}})], d\Gamma(\check{\iota}_0)^{-1}[\widehat{\iota}]]. \quad (5.27)$$

Note that $G_2[\widehat{\iota}, \widehat{\zeta}]$ is independent of $\widehat{\zeta}$. It can be estimated as follows:

Lemma 5.7. *There exists $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$, so that for any $\widehat{\iota} := (\widehat{\varphi}, \widehat{y}, \widehat{z}_1, \widehat{z}_2)$ in $H^{s+\mu}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$ with $s_0 \leq s \leq s_* - \mu$ and any $\widehat{\zeta} \in \mathbb{R}^S$, which are both Lipschitz continuous in ω ,*

$$\|G_2[\widehat{\iota}, \widehat{\zeta}]\|_{s, \sigma-2}^{\gamma \text{lip}} \leq_s \gamma^{-1} \left(\|E\|_{s+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\zeta}\|_{s_0+\mu}^{\gamma \text{lip}} + \|E\|_{s_0+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s+\mu}^{\gamma \text{lip}} + \|\iota\|_{s+\mu}^{\gamma \text{lip}} \|E\|_{s_0+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s_0+\mu}^{\gamma \text{lip}} \right).$$

Proof. The claimed estimate follows by the estimates of Lemma 5.4 and (5.14). \square

In view of the formula (5.26) and Lemma 5.7, the problem of finding an approximate right inverse of $dF_\omega(\check{\iota}_{\text{iso}}, \zeta)$ is reduced to find an approximate right inverse of the operator $\omega \cdot \partial_\varphi - d_{\iota, \zeta} X_{K_\varepsilon, \zeta}(\check{\iota}_0, \zeta)$ where $X_{K_\varepsilon, \zeta}$ is given in (5.23). In order to compute the differential of $X_{K_\varepsilon, \zeta}$ at $\check{\iota}_0(\varphi) = (\varphi, 0, 0)$, we compute the Taylor expansion of K_ε, ζ in v, w, \bar{w} at $(v, w) = (0, 0)$ up to order 2. Denoting $(w, \bar{w}) \in h_\perp^\sigma \times h_\perp^\sigma$ by W , the expansion is given by

$$\theta(\psi) \cdot \zeta + K_{0,0}(\psi) + K_{1,0}(\psi) \cdot v + K_{0,1}(\psi) \cdot W + \frac{1}{2}v \cdot K_{2,0}(\psi)v + v \cdot K_{1,1}(\psi)W + \frac{1}{2}W \cdot K_{0,2}(\psi)W$$

where

$$K_{0,0}(\psi) := K_\varepsilon(\psi, 0, 0), \quad K_{1,0}(\psi) := \nabla_v K_\varepsilon(\psi, 0, 0), \quad K_{2,0}(\psi) := \partial_v \nabla_v K_\varepsilon(\psi, 0, 0), \quad (5.28)$$

$$K_{0,1}(\psi) := \nabla_W K_\varepsilon(\psi, 0, 0) = (\nabla_w K_\varepsilon(\psi, 0, 0), \nabla_{\bar{w}} K_\varepsilon(\psi, 0, 0)), \quad K_{1,1}(\psi) := \partial_W \nabla_v K_\varepsilon(\psi, 0, 0), \quad (5.29)$$

and

$$K_{0,2}(\psi) := \partial_W \nabla_W K_\varepsilon(\psi, 0, 0) = \begin{pmatrix} \partial_w \nabla_w K_\varepsilon(\psi, 0, 0) & \partial_{\bar{w}} \nabla_w K_\varepsilon(\psi, 0, 0) \\ \partial_w \nabla_{\bar{w}} K_\varepsilon(\psi, 0, 0) & \partial_{\bar{w}} \nabla_{\bar{w}} K_\varepsilon(\psi, 0, 0) \end{pmatrix}.$$

With \mathbb{J}_2 given by (3.19), the differential of the map $(\check{\iota}, \zeta) \mapsto \omega \cdot \partial_\varphi \check{\iota} - X_{K_\varepsilon, \zeta}(\check{\iota})$ at $\check{\iota}_0$ in direction $(\widehat{\iota}, \widehat{\zeta})$ reads as

$$\begin{pmatrix} \omega \cdot \partial_\varphi \widehat{\psi} - \partial_\varphi K_{1,0}(\varphi)[\widehat{\psi}] - K_{2,0}(\varphi)[\widehat{v}] - K_{1,1}(\varphi)[\widehat{W}] \\ \omega \cdot \partial_\varphi \widehat{v} + (\partial_\varphi \theta(\varphi))^t \widehat{\zeta} + \partial_\varphi ((\partial_\varphi \theta(\varphi))^t \zeta)[\widehat{\psi}] + \partial_\varphi \nabla_\varphi K_{0,0}(\varphi)[\widehat{\psi}] + \nabla_\varphi (K_{1,0}(\varphi) \cdot \widehat{v} + K_{0,1}(\varphi) \cdot \widehat{W}) \\ \omega \cdot \partial_\varphi \widehat{W} + \mathbb{J}_2(\partial_\varphi K_{0,1}(\varphi)[\widehat{\psi}] + K_{1,1}(\varphi)^t \widehat{v} + K_{0,2}(\varphi)[\widehat{W}]) \end{pmatrix}$$

where $\widehat{\iota}(\varphi) = (\widehat{\psi}(\varphi), \widehat{v}(\varphi), \widehat{W}(\varphi))$ with $\widehat{W}(\varphi) = (\widehat{w}_1(\varphi), \widehat{w}_2(\varphi))$ in $h_\perp^\sigma \times h_\perp^\sigma$. In the above expression, various terms can be estimated in terms of the error function E introduced in (5.3). Indeed, since

$$\nabla_\varphi K_{0,0}(\varphi) = \nabla_\psi K_\varepsilon(\check{\iota}_0(\varphi)), \quad K_{1,0}(\varphi) = \nabla_v K_\varepsilon(\check{\iota}_0(\varphi)), \quad K_{0,1}(\varphi) = (\nabla_w K_\varepsilon(\check{\iota}_0(\varphi)), \nabla_{\bar{w}} K_\varepsilon(\check{\iota}_0(\varphi))), \quad (5.30)$$

it follows from Lemma 5.5 and 5.1 that the operator $\omega \cdot \partial_\varphi - d_{\iota, \zeta} X_{K_\varepsilon, \zeta}(\check{\iota}_0)$ is of the form

$$\omega \cdot \partial_\varphi - d_{\iota, \zeta} X_{K_\varepsilon, \zeta}(\check{\iota}_0) = \mathfrak{T}_\omega + G_3, \quad (5.31)$$

where

$$\mathfrak{T}_\omega[\widehat{\iota}, \widehat{\zeta}] := \begin{pmatrix} \omega \cdot \partial_\varphi \widehat{\psi} - K_{2,0}(\varphi)[\widehat{v}] - K_{1,1}(\varphi)[\widehat{W}] \\ \omega \cdot \partial_\varphi \widehat{v} + (\partial_\varphi \theta(\varphi))^t \widehat{\zeta} \\ \omega \cdot \partial_\varphi \widehat{W} + \mathbb{J}_2(K_{1,1}(\varphi)^t \widehat{v} + K_{0,2}(\varphi)[\widehat{W}]) \end{pmatrix}$$

and

$$G_3[\widehat{\iota}, \widehat{\zeta}] := \begin{pmatrix} -\partial_\varphi K_{1,0}(\varphi)[\widehat{\psi}] \\ \partial_\varphi ((\partial_\varphi \theta(\varphi))^t \zeta)[\widehat{\psi}] + \partial_\varphi \nabla_\varphi K_{0,0}(\varphi)[\widehat{\psi}] + \nabla_\varphi (K_{1,0}(\varphi) \cdot \widehat{v} + K_{0,1}(\varphi) \cdot \widehat{W}) \\ \mathbb{J}_2 \partial_\varphi K_{0,1}(\varphi)[\widehat{\psi}] \end{pmatrix}.$$

Note that $G_3[\widehat{\iota}, \widehat{\zeta}]$ is independent of $\widehat{\zeta}$ and can be estimated as follows.

Lemma 5.8. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$, so that for any $\widehat{\iota} := (\widehat{\psi}, \widehat{v}, \widehat{W})$ in $H^{s+\mu}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma)$ with $s_0 \leq s \leq s_* - \mu$ and any $\widehat{\zeta} \in \mathbb{R}^S$, which are both Lipschitz continuous in ω ,*

$$\|G_3[\widehat{\iota}, \widehat{\zeta}]\|_{s, \sigma-2}^{\gamma \text{lip}} \leq s \gamma^{-1} \left(\|E\|_{s+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s_0+\mu}^{\gamma \text{lip}} + \|E\|_{s_0+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s+\mu}^{\gamma \text{lip}} + \|\iota\|_{s+\mu}^{\gamma \text{lip}} \|E\|_{s_0+\mu, \sigma-2}^{\gamma \text{lip}} \|\widehat{\iota}\|_{s_0+\mu}^{\gamma \text{lip}} \right).$$

Proof. In view of the formula (5.30), the claimed estimates follow from Lemma 5.1 and Lemma 5.5. \square

Our aim is to construct a right inverse of \mathfrak{L}_ω . It means that for given maps $\varphi \mapsto (g_1(\varphi), g_2(\varphi), g_3(\varphi)) \in \mathbb{R}^S \times \mathbb{R}^S \times (h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ of appropriate regularity, we have to solve the inhomogenous linear system

$$\omega \cdot \partial_\varphi \widehat{\psi} - K_{2,0}(\varphi)[\widehat{v}] - K_{1,1}(\varphi)[\widehat{W}] = g_1, \quad (5.32)$$

$$\omega \cdot \partial_\varphi \widehat{v} + (\partial_\varphi \theta(\varphi))^t [\widehat{\zeta}] = g_2, \quad (5.33)$$

$$\mathfrak{L}_\omega \widehat{W} + \mathbb{J}_2 K_{1,1}(\varphi)^t [\widehat{v}] = g_3, \quad (5.34)$$

where for any $\omega \in \Omega_o(\iota)$, the operator $\mathfrak{L}_\omega : H^s(\mathbb{T}^S, h_\perp^\sigma \times h_\perp^\sigma) \rightarrow H^{s-1}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ is defined by

$$\mathfrak{L}_\omega(\varphi) := \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbb{J}_2 K_{0,2}(\varphi), \quad K_{0,2} = \begin{pmatrix} \partial_w \nabla_w K_\varepsilon & \partial_{\bar{w}} \nabla_w K_\varepsilon \\ \partial_w \nabla_{\bar{w}} K_\varepsilon & \partial_{\bar{w}} \nabla_{\bar{w}} K_\varepsilon \end{pmatrix} \circ \iota_0. \quad (5.35)$$

The maps g_1, g_2 are assumed to be in $H^{s+2\tau+1}(\mathbb{T}^S, \mathbb{R}^S)$ and $g_3 \in H^{s+2\tau+1}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ with $s_0 \leq s \leq s_* - \nu$ and $\nu = \nu(|S|, \tau)$ being an integer, which can be explicitly computed.

Note that the above inhomogeneous linear system is in triangular form: We first solve the second equation (5.33). It turns out to be convenient to write $\widehat{v} = \widehat{v}_1 + \widehat{v}_0$ with $[[\widehat{v}_1]] = 0$ and $\widehat{v}_0 = [[\widehat{v}]]$ where we recall that for any given continuous map $f : \mathbb{T}^S \rightarrow X$ with values in a Banach space X , $[[f]]$ denotes its average $(2\pi)^{-|S|} \int_{\mathbb{T}^S} f(\varphi) d\varphi$. The second equation (5.33) is solved for $\widehat{\zeta}$ and \widehat{v}_1 . Next we solve the third equation (5.34) for \widehat{W} and then finally solve the first equation (5.32) for $\widehat{\psi}$ and \widehat{v}_0 . Let us first consider in detail the second equation. Recall that $\theta(\varphi) = \varphi + \Theta(\varphi)$, where $\Theta(\cdot)$ is 2π -periodic in each component. Hence

$$[[(\partial_\varphi \theta)^t]] = \text{Id}_S + [[(\partial_\varphi \Theta)^t]] = \text{Id}_S$$

and the solution of the second equation is given by

$$\widehat{\zeta} := [[g_2]], \quad \widehat{v}_1 := (\omega \cdot \partial_\varphi)^{-1} (g_2 - [[g_2]] - (\partial_\varphi \Theta(\varphi))^t [\widehat{\zeta}]). \quad (5.36)$$

Lemma 5.9. *For any g_2 in $H^{s+2\tau+1}(\mathbb{T}^S, \mathbb{R}^S)$ with $s \geq s_0$, \widehat{v}_1 and $\widehat{\zeta}$ of (5.36) satisfy*

$$\|\widehat{v}_1\|_s^{\text{lip}} \leq \gamma^{-1} (\|g_2\|_{s+2\tau+1}^{\text{lip}} + \|\iota\|_{s+2\tau+2}^{\text{lip}} \|g_2\|_{s_0}^{\text{lip}}), \quad \|\widehat{\zeta}\|_s^{\text{lip}} \leq \|g_2\|_{s_0}^{\text{lip}}. \quad (5.37)$$

Proof. The claimed estimate for $|\widehat{\zeta}|^{\text{lip}}$ is straightforward. To prove the one for $\|\widehat{v}_1\|_s^{\text{lip}}$, we apply Lemma 2.2 to get the bound $\|g_2 - [[g_2]]\|_{s+2\tau+1}^{\text{lip}} + \|(\partial_\varphi \Theta(\varphi))^t [\widehat{\zeta}]\|_{s+2\tau+1}^{\text{lip}}$. Since $\|g_2 - [[g_2]]\|_{s+2\tau+1}^{\text{lip}} \leq \|g_2\|_{s+2\tau+1}^{\text{lip}}$ and $\|(\partial_\varphi \Theta(\varphi))^t [\widehat{\zeta}]\|_{s+2\tau+1}^{\text{lip}} \leq \|\iota\|_{s+2\tau+2}^{\text{lip}} \|\widehat{\zeta}\|_s^{\text{lip}}$ one has $\|\widehat{v}_1\|_s^{\text{lip}} \leq \gamma^{-1} (\|g_2\|_{s+2\tau+1}^{\text{lip}} + \|\iota\|_{s+2\tau+2}^{\text{lip}} \|g_2\|_{s_0}^{\text{lip}})$. \square

We point out that the average \widehat{v}_0 of \widehat{v} will be determined by equation (5.32), but temporarily, we will consider it as a free parameter. Now we have to solve the equation

$$\mathfrak{L}_\omega \widehat{W} = g_3 - \mathbb{J}_2 K_{1,1}(\varphi)^t [\widehat{v}]. \quad (5.38)$$

We summarize our results on the invertibility of \mathfrak{L}_ω with the following theorem.

Theorem 5.1 (Invertibility of \mathfrak{L}_ω). *For any constant $C > 0$, there exist $0 < \delta_0(|S|, \tau, s_*, C) < 1$ and $\mu_0 = \mu_0(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any ι with*

$$\|\iota\|_{s_0+\mu_0}^{\text{lip}} \leq C\varepsilon\gamma^{-2}, \quad \|E\|_{s_0+\mu_0, \sigma-2}^{\text{lip}} \leq C\varepsilon, \quad \varepsilon\gamma^{-4} \leq \delta_0,$$

there exists a subset of $\Omega_o(\iota)$, denoted by $\Omega_{\text{Mel}}^{2\gamma}(\iota) \equiv \Omega_{\text{Mel}}^{2\gamma}(\iota; \Omega_o(\iota))$, with the following properties: for any $g \in H^{s+2\tau+1}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ with $s_0 \leq s \leq s_ - \mu_0$ and any $\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$, the linear equation $\mathfrak{L}_\omega h = g$ has a unique solution $h = \mathfrak{L}_\omega^{-1} g \in H^s(\mathbb{T}^S, h_\perp^\sigma \times h_\perp^\sigma)$. In case g is Lipschitz continuous on $\Omega_{\text{Mel}}^{2\gamma}(\iota)$, the solution h is Lipschitz continuous on $\Omega_{\text{Mel}}^{2\gamma}(\iota)$ and satisfies the estimate*

$$\|\mathfrak{L}_\omega^{-1} g\|_{s, \sigma}^{\text{lip}} \leq_s \gamma^{-1} \left(\|g\|_{s+2\tau+1, \sigma-2}^{\text{lip}} + \|\iota\|_{s+\mu_0}^{\text{lip}} \|g\|_{s_0+2\tau+1, \sigma-2}^{\text{lip}} \right). \quad (5.39)$$

Remark: According to (7.84), a possible choice of μ_0 in Theorem 5.1 is $\mu_0 = 4s_0 + 10\tau + 7$.

Theorem 5.1 is proved in Section 7.6, using the results established in Sections 6 and 7. In the sequel, the integers $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ coming up in lemmas, where Theorem 5.1 is applied, will be chosen larger than the corresponding integer μ_0 , of Theorem 5.1.

In order to apply Theorem 5.1 to solve the equation (5.38) we need the following estimate for the Taylor coefficients $K_{2,0}$ and $K_{1,1}$ defined in (5.28), (5.29):

Lemma 5.10. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any $\widehat{v} \in H^s(\mathbb{T}^S, \mathbb{R}^S)$, $\widehat{W} = (\widehat{w}_1, \widehat{w}_2) \in H^s(\mathbb{T}^S, h_{\perp}^{\sigma} \times h_{\perp}^{\sigma})$ with $s_0 \leq s \leq s_* - \mu$, which are both Lipschitz continuous in ω ,*

$$\begin{aligned} \|K_{2,0} - (\partial_{I_j} \omega_k^{nls}(\xi, 0))_{k,j \in S}\|_s^{\gamma \text{lip}} &\leq \varepsilon + \|\iota\|_{s+\mu}^{\gamma \text{lip}}, \\ \|(K_{1,1})^t[\widehat{v}]\|_s^{\gamma \text{lip}} &\leq \varepsilon \gamma^{-2} \|\widehat{v}\|_s^{\gamma \text{lip}} + \|\iota\|_{s+\mu}^{\gamma \text{lip}} \|\widehat{v}\|_{s_0}^{\gamma \text{lip}}, \\ \|K_{1,1}[\widehat{W}]\|_s^{\gamma \text{lip}} &\leq \varepsilon \gamma^{-2} \|\widehat{W}\|_s^{\gamma \text{lip}} + \|\iota\|_{s+\mu}^{\gamma \text{lip}} \|\widehat{W}\|_{s_0}^{\gamma \text{lip}}. \end{aligned}$$

Proof. By (5.16) - (5.17), $\partial_v K_{\varepsilon} = \partial_y H_{\varepsilon} \circ \Gamma \cdot (\partial_{\psi} \theta(\psi))^{-t}$ or $\nabla_v K_{\varepsilon} = (\partial_{\psi} \theta(\psi))^{-1} \nabla_y H_{\varepsilon} \circ \Gamma$. Hence

$$\begin{aligned} \partial_v \nabla_v K_{\varepsilon}(\check{\iota}(\varphi)) &= (\partial_{\varphi} \theta(\varphi))^{-1} \partial_y \nabla_y H_{\varepsilon}(\check{\iota}_{\text{iso}}(\varphi)) (\partial_{\varphi} \theta(\varphi))^{-t} \\ &\stackrel{(1.18)}{=} (\partial_{\varphi} \theta(\varphi))^{-1} \partial_y \nabla_y H^{nls}(\check{\iota}_{\text{iso}}(\varphi)) (\partial_{\varphi} \theta(\varphi))^{-t} + \varepsilon (\partial_{\varphi} \theta(\varphi))^{-1} \partial_y \nabla_y P(\check{\iota}_{\text{iso}}(\varphi)) (\partial_{\varphi} \theta(\varphi))^{-t}. \end{aligned}$$

We claim that the first term in the latter expression can be bounded by $C(s) \|\iota\|_{s+\mu}^{\gamma \text{lip}}$ and the second one by $\varepsilon C(s) (1 + \|\iota\|_{s+\mu}^{\gamma \text{lip}})$. Indeed, the estimate of the first term is derived from Proposition 3.2 (ii),

$$\|\partial_y \nabla_y H^{nls}(\check{\iota}_{\text{iso}}) - \partial_y \nabla_y H^{nls}(\xi, 0)\|_s^{\gamma \text{lip}} \leq_s \|\iota_{\text{iso}}\|_{s+2s_0}^{\gamma \text{lip}},$$

using that $\partial_{\varphi} \theta(\varphi) = \text{Id}_{\mathbb{R}^S} + \partial_{\varphi} \Theta(\varphi)$ with $\|\partial_{\varphi} \Theta(\varphi)\|_s^{\gamma \text{lip}} < \|\iota\|_{s+1}^{\gamma \text{lip}}$, $\partial_y \nabla_y H^{nls}(\xi, 0) = (\partial_{I_j} \omega_k^{nls}(\xi, 0))_{k,j \in S}$, and $\|\iota_{\text{iso}}\|_s^{\gamma \text{lip}} \leq_s \|\iota\|_{s+\mu}^{\gamma \text{lip}}$ by (5.13). To estimate the second term, one argues in a similar way, using this time that by Proposition 3.3, $\|\partial_y \nabla_y P(\check{\iota}_{\text{iso}})\|_s^{\gamma \text{lip}} \leq_s 1 + \|\iota_{\text{iso}}\|_{s+2s_0}^{\gamma \text{lip}}$. The claimed estimates for $K_{1,1}[\widehat{v}]$ and $(K_{1,1})^t[\widehat{W}]$ can be proved by similar arguments. \square

Combining Theorem 5.1 and Lemma 5.10, we get the following estimate for the solution \widehat{W} of equation (5.38).

Corollary 5.1. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any $g_3 \in H^{s+2\tau+1}(\mathbb{T}^S, h_{\perp}^{\sigma-2} \times h_{\perp}^{\sigma-2})$ and $\widehat{v} \in H^{s+2\tau+1}(\mathbb{T}^S, \mathbb{R}^S)$ with $s_0 \leq s \leq s_* - \mu$, which are both Lipschitz continuous in ω on $\Omega_{\text{Mel}}^{2\gamma}(\iota)$, the solution*

$$\widehat{W} = \mathfrak{L}_{\omega}^{-1}(\varphi)(g_3 - \mathbb{J}_2 K_{1,1}(\varphi)^t[\widehat{v}]) \quad (5.40)$$

of equation (5.38) is Lipschitz continuous on $\Omega_{\text{Mel}}^{2\gamma}(\iota)$ and satisfies the estimate

$$\|\widehat{W}\|_s^{\gamma \text{lip}} \leq_s \gamma^{-1} \left(\|g_3\|_{s+2\tau+1, \sigma-2}^{\gamma \text{lip}} + \varepsilon \gamma^{-2} \|\widehat{v}\|_{s+2\tau+1}^{\gamma \text{lip}} + \|\iota\|_{s+\mu}^{\gamma \text{lip}} \|g_3\|_{s_0+2\tau+1, \sigma-2}^{\gamma \text{lip}} + \varepsilon \gamma^{-2} \|\widehat{v}\|_{s_0+2\tau+1}^{\gamma \text{lip}} \right). \quad (5.41)$$

Finally we solve the first equation (5.32) for $\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$,

$$\omega \cdot \partial_{\varphi} \widehat{\psi} = g_1 + K_{1,1}(\varphi)[\widehat{W}] + K_{2,0}(\varphi)[\widehat{v}] \quad (5.42)$$

where $\widehat{W} \in H^s(\mathbb{T}^S, h_{\perp}^{\sigma} \times h_{\perp}^{\sigma})$ is given by (5.40) and \widehat{v} is of the form $\widehat{v}_1 + \widehat{v}_0$ with $\widehat{v}_1 \in H^s(\mathbb{T}^S, \mathbb{R}^S)$ defined by (5.36). The first task for solving this equation is to prove that we can choose \widehat{v}_0 in such a way that the average of the right hand side of the above equation vanishes. By (5.40), the equation (5.42) can be written as

$$\omega \cdot \partial_{\varphi} \widehat{\psi} = g_1 + K_{1,1}(\varphi) \mathfrak{L}_{\omega}^{-1}(\varphi) g_3 + M_{\omega}(\varphi) \widehat{v} \quad (5.43)$$

where

$$M_{\omega}(\varphi) := K_{2,0}(\varphi) - K_{1,1}(\varphi) \mathfrak{L}_{\omega}^{-1}(\varphi) \mathbb{J}_2 K_{1,1}(\varphi)^t.$$

Taking the average in (5.43) and using that $\widehat{v} = \widehat{v}_1 + \widehat{v}_0$, we get

$$0 = [[g_1]] + [[K_{1,1}\mathbb{J}_2\mathfrak{L}_\omega^{-1}g_3]] + [[M_\omega\widehat{v}_1]] + [[M_\omega]]\widehat{v}_0. \quad (5.44)$$

In order to solve this latter equation for \widehat{v}_0 , we need to show that $[[M_\omega]] : \mathbb{R}^S \rightarrow \mathbb{R}^S$ is invertible. To this end, first note that for any $x \in \mathbb{R}^S$, $\|([M_\omega]] - (\partial_{I_j}\omega_k^{nls}(\xi, 0))_{k,j \in S})x\|$ is bounded by

$$\sup_{\varphi \in \mathbb{T}^S} \|K_{1,1}(\varphi)\mathfrak{L}_\omega^{-1}(\varphi)\mathbb{J}_2K_{1,1}(\varphi)^t x\| + \sup_{\varphi \in \mathbb{T}^S} \|(K_{2,0}(\varphi) - (\partial_{I_j}\omega_k^{nls}(\xi, 0))_{k,j \in S})x\|,$$

yielding

$$\|([M_\omega]] - (\partial_{I_j}\omega_k^{nls}(\xi, 0))_{k,j \in S})x\| \leq \|K_{1,1}\mathfrak{L}_\omega^{-1}\mathbb{J}_2K_{1,1}^t x\|_{s_0} + \|(K_{2,0} - (\partial_{I_j}\omega_k^{nls}(\xi, 0))_{k,j \in S})x\|_{s_0}.$$

It then follows from Lemma 5.10, the tame estimate (5.39) for the inverse \mathfrak{L}_ω^{-1} , and the smallness condition (5.2) that $\|([M_\omega]] - (\partial_{I_j}\omega_k^{nls}(\xi, 0))_{k,j \in S})\| \leq \varepsilon\gamma^{-2}$. En passant we mention that by the same arguments, one sees that

$$\|M_\omega - (\partial_{I_j}\omega_k^{nls}(\xi, 0))_{k,j \in S}\|_{s_0}^{\text{lip}} \leq \varepsilon\gamma^{-2}. \quad (5.45)$$

Since by assumption, the inverse of $(\partial_{I_j}\omega_k^{nls}(\xi(\omega)))_{j,k \in S}$ is bounded uniformly on Ω and $\Omega_{\text{Mel}}^{2\gamma}(\iota) \subset \Omega$, it follows from Lemma 5.10 and the smallness assumption (5.2) that the operator $[[M_\omega]]$ is invertible with the norm of $[[M_\omega]]^{-1}$ uniformly bounded. In fact,

$$\|[[M_\omega]]^{-1}\|^{\text{lip}} \leq 1. \quad (5.46)$$

The operator $[[M_\omega]]$ being invertible implies that for any ω in $\Omega_{\text{Mel}}^{2\gamma}(\iota)$, equation (5.44) can be solved for \widehat{v}_0 ,

$$\widehat{v}_0 = -[[M_\omega]]^{-1} \left([[g_1]] + [[K_{1,1}\mathfrak{L}_\omega^{-1}g_3]] + [[M_\omega\widehat{v}_1]] \right). \quad (5.47)$$

As a consequence, equation (5.42) can be solved for $\widehat{\psi}$,

$$\widehat{\psi} = (\omega \cdot \partial_\varphi)^{-1} \left(g_1 + K_{1,1}(\varphi)\mathfrak{L}_\omega^{-1}(\varphi)g_3 + M_\omega(\varphi)\widehat{v}_1 \right). \quad (5.48)$$

Lemma 5.11. *There exist $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any map $g = (g_1, g_2, g_3)$ in $H^{s+4\tau+2}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ with $s_0 \leq s \leq s_* - \mu$, and any $\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$ with $\Omega_{\text{Mel}}^{2\gamma}(\iota) \equiv \Omega_{\text{Mel}}^{2\gamma}(\iota; \Omega_o(\iota))$ as in Theorem 5.1, \widehat{v}_0 , defined in (5.47), and $\widehat{\psi}$, defined in (5.48), satisfy the estimates*

$$|\widehat{v}_0|^{\text{lip}} \leq \gamma^{-1} \|g\|_{s_0+2\tau+1, \sigma-2}^{\text{lip}} \quad (5.49)$$

$$\|\widehat{\psi}\|_s^{\text{lip}} \leq \gamma^{-2} \|g\|_{s+4\tau+2, \sigma-2}^{\text{lip}} + \gamma^{-3} \|\iota\|_{s+\mu}^{\text{lip}} \|g\|_{s_0+4\tau+2, \sigma-2}^{\text{lip}}. \quad (5.50)$$

Proof. By the formula (5.47) and the estimate (5.46),

$$\begin{aligned} |\widehat{v}_0|^{\text{lip}} &\leq \|[[M_\omega(\varphi)\widehat{v}_1]]\|^{\text{lip}} + \|[[K_{1,1}(\varphi)\mathfrak{L}_\omega^{-1}(\varphi)g_3]]\|^{\text{lip}} + \|[[g_1]]\|^{\text{lip}} \\ &\leq \|M_\omega(\varphi)\widehat{v}_1\|_{s_0}^{\text{lip}} + \|K_{1,1}(\varphi)\mathfrak{L}_\omega^{-1}(\varphi)g_3\|_{s_0}^{\text{lip}} + \|g_1\|_{s_0}^{\text{lip}}. \end{aligned}$$

Since by (5.45)

$$\|M_\omega\|_{s_0}^{\text{lip}} \leq \|(\partial_{I_j}\omega_k^{nls}(\xi(\omega)))_{j,k \in S}\|_{s_0}^{\text{lip}} + \varepsilon\gamma^{-2} \stackrel{\text{Prop 3.1}}{\leq} 1$$

one gets by the estimate (5.37)

$$\|M_\omega(\varphi)\widehat{v}_1\|_{s_0}^{\text{lip}} \leq \gamma^{-1} \|g_2\|_{s_0+2\tau+1}^{\text{lip}}.$$

Furthermore by Lemma 5.10, Theorem 5.1, and the smallness condition (5.2) we get

$$\|K_{1,1}(\varphi)\mathfrak{L}_\omega^{-1}(\varphi)g_3\|_{s_0}^{\text{lip}} \leq \varepsilon\gamma^{-3} \|g_3\|_{s_0+2\tau+1, \sigma-2}^{\text{lip}}.$$

Altogether, this then proves (5.49). The estimate for $\widehat{\psi}$, defined by formula (5.48) is derived from Lemma 2.2, using arguments similar to the ones above. \square

Summarizing our results obtained so far, we have constructed the unique solution $(\widehat{\psi}, \widehat{v}, \widehat{W}, \widehat{\zeta})$ of the linear system (5.32)-(5.34). Combining Lemma 5.9, Corollary 5.1 and Lemma 5.11 we get the following corollary.

Corollary 5.2. *There exists $\mu = \mu(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ so that for any map $g = (g_1, g_2, g_3)$ in $H^{s+\mu}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma-2} \times h_{\perp}^{\sigma-2})$ with $s_0 \leq s \leq s_* - \mu$, and any $\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$ with $\Omega_{\text{Mel}}^{2\gamma}(\iota) \equiv \Omega_{\text{Mel}}^{2\gamma}(\iota; \Omega_0(\iota))$ as in Theorem 5.1, the linear system (5.32)-(5.34) admits a unique solution $\mathfrak{T}_{\omega}^{-1}g = (\widehat{\iota}, \widehat{\zeta})$. It satisfies the tame estimate*

$$\|\mathfrak{T}_{\omega}^{-1}g\|_s^{\gamma\text{lip}} \leq_s \gamma^{-2} (\|g\|_{s+\mu, \sigma-2}^{\gamma\text{lip}} + \|\iota\|_{s+\mu}^{\gamma\text{lip}} \|g\|_{s_0+\mu, \sigma-2}^{\gamma\text{lip}}).$$

Proof. Combining Lemmas 5.9 and 5.11 yields

$$\|\widehat{v}\|_s^{\gamma\text{lip}} \leq_s \|\widehat{v}_1\|_s^{\gamma\text{lip}} + \|\widehat{v}_0\|_s^{\gamma\text{lip}} \leq_s \gamma^{-1} \|g\|_{s+2\tau+1, \sigma-2}^{\gamma\text{lip}} + \gamma^{-1} \|\iota\|_{s+2\tau+2}^{\gamma\text{lip}} \|g\|_{s_0, \sigma-2}^{\gamma\text{lip}}.$$

From this and the estimate (5.41) we conclude the claimed estimate for \widehat{W} . Finally the claimed estimate for $\widehat{\psi}$ is given in (5.50) and the one for $\widehat{\zeta}$ in (5.37). \square

With these preparations we now prove that the operator

$$\mathbf{T}_{\omega} := d\widetilde{\Gamma}(\check{\iota}_0) \circ \mathfrak{T}_{\omega}^{-1} \circ d\Gamma(\check{\iota}_0)^{-1}, \quad \widetilde{\Gamma}(\psi, v, w, \zeta) := (\Gamma(\psi, v, w), \zeta) \quad (5.51)$$

is an approximate right inverse for

$$\begin{aligned} d_{\iota, \zeta} F_{\omega}(\iota) &\stackrel{(5.25)}{=} d_{\iota, \zeta} F_{\omega}(\iota_{\text{iso}}) + G_1 \\ &\stackrel{(5.27)}{=} d\Gamma(\check{\iota}_0)(\omega \cdot \partial_{\varphi} - d_{\iota, \zeta} X_{K_{\varepsilon, \zeta}}(\check{\iota}_0)) d\widetilde{\Gamma}(\check{\iota}_0)^{-1} + G_2 + G_1 \\ &\stackrel{(5.31)}{=} d\Gamma(\check{\iota}_0) \mathfrak{T}_{\omega} d\widetilde{\Gamma}(\check{\iota}_0)^{-1} + d\Gamma(\check{\iota}_0) G_3 d\widetilde{\Gamma}(\check{\iota}_0)^{-1} + G_2 + G_1. \end{aligned} \quad (5.52)$$

It is convenient to introduce the norm $\|(\psi, v, W, \zeta)\|_{s, \sigma}^{\gamma\text{lip}} := \max\{\|(\psi, v, W)\|_{s, \sigma}^{\gamma\text{lip}}, |\zeta|^{\gamma\text{lip}}\}$.

Theorem 5.2. (Approximate right inverse) *For any constant $C > 0$, there exist $\delta_1 = \delta_1(|S|, \tau, s_*, C)$ with $0 < \delta_1 < 1$ and a positive integer $\mu_1 = \mu_1(|S|, \tau) \in \mathbb{Z}_{\geq 1}$ with $\delta_1 < \delta_0$, $\mu_1 > \mu_0$ and δ_0, μ_0 given as in Theorem 5.1, such that whenever*

$$\|\iota\|_{s_0+\mu_1}^{\gamma\text{lip}} \leq C\varepsilon\gamma^{-2}, \quad \|F_{\omega}(\iota, \zeta)\|_{s_0+\mu_1, \sigma-2}^{\gamma\text{lip}} \leq C\varepsilon, \quad \varepsilon\gamma^{-4} \leq \delta_1, \quad (5.53)$$

then the family of operators $\mathbf{T} = (\mathbf{T}_{\omega})_{\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)}$ with $\Omega_{\text{Mel}}^{2\gamma}(\iota) \equiv \Omega_{\text{Mel}}^{2\gamma}(\iota; \Omega_0(\iota))$ as in Theorem 5.1 has the following properties: for any $g := (g_1, g_2, g_3) \in H^{s+\mu_1}(\mathbb{T}^S, \mathbb{R}^S \times \mathbb{R}^S \times h_{\perp}^{\sigma-2} \times h_{\perp}^{\sigma-2})$ with $s_0 \leq s \leq s_* - \mu_1$, the operator \mathbf{T} defined in (5.51) satisfies

$$\|\mathbf{T}g\|_{s, \sigma}^{\gamma\text{lip}} \leq_s \gamma^{-2} (\|g\|_{s+\mu_1, \sigma-2}^{\gamma\text{lip}} + \|\iota\|_{s+\mu_1}^{\gamma\text{lip}} \|g\|_{s_0+\mu_1, \sigma-2}^{\gamma\text{lip}}). \quad (5.54)$$

Furthermore \mathbf{T}_{ω} is an approximate right inverse of $d_{\iota, \zeta} F_{\omega}(\iota)$, namely

$$\begin{aligned} &\|(d_{\iota, \zeta} F_{\omega}(\iota) \circ \mathbf{T}_{\omega} - \text{Id})g\|_{s, \sigma-2}^{\gamma\text{lip}} \\ &\leq_s \gamma^{-3} \left(\|F_{\omega}(\iota, \zeta)\|_{s_0+\mu_1, \sigma-2}^{\gamma\text{lip}} \|g\|_{s+\mu_1, \sigma-2}^{\gamma\text{lip}} + \|F_{\omega}(\iota, \zeta)\|_{s+\mu_1, \sigma-2}^{\gamma\text{lip}} \|g\|_{s_0+\mu_1, \sigma-2}^{\gamma\text{lip}} \right. \\ &\quad \left. + \|F_{\omega}(\iota, \zeta)\|_{s_0+\mu_1, \sigma-2}^{\gamma\text{lip}} \|\iota\|_{s+\mu_1}^{\gamma\text{lip}} \|g\|_{s_0+\mu_1, \sigma-2}^{\gamma\text{lip}} \right). \end{aligned} \quad (5.55)$$

Proof. The tame estimate (5.54) follows from the definition (5.51) of \mathbf{T}_{ω} , the estimate of $\mathfrak{T}_{\omega}^{-1}$ of Corollary 5.2, and the estimates of $d\Gamma(\check{\iota}_0)$, $d\Gamma(\check{\iota}_0)^{-1}$ of Lemma 5.4.

The estimate (5.55) can be obtained as follows: using the formula (5.52) for $d_{\iota, \zeta} F_{\omega}(\iota)$ and the definition (5.51) of \mathbf{T}_{ω} , one sees that $d_{\iota, \zeta} F_{\omega}(\iota) \circ \mathbf{T}_{\omega} - \text{Id}$ is the sum of the three terms $d\Gamma(\check{\iota}_0) G_3 \mathfrak{T}_{\omega}^{-1} d\widetilde{\Gamma}(\check{\iota}_0)^{-1}$, $G_2 d\widetilde{\Gamma}(\check{\iota}_0) \mathfrak{T}_{\omega}^{-1} d\Gamma(\check{\iota}_0)^{-1}$, and $G_1 d\widetilde{\Gamma}(\check{\iota}_0) \mathfrak{T}_{\omega}^{-1} d\Gamma(\check{\iota}_0)^{-1}$, which are estimated separately, combining the estimates of G_1 , G_2 , and G_3 of Lemma 5.6, Lemma 5.7, and, respectively, Lemma 5.8 with the estimate of $\mathfrak{T}_{\omega}^{-1}$ of Corollary 5.2, and the estimates of $d\Gamma(\check{\iota}_0)$, $d\Gamma(\check{\iota}_0)^{-1}$ of Lemma 5.4.

The integer $\mu_1 > \mu_0$, and the constant $0 < \delta_1 < \delta_0$ are chosen in such way that the lemmas used to derive the estimates (5.54), (5.55) apply. \square

6 Reduction of \mathfrak{L}_ω . Part 1

For proving Theorem 5.1 it is useful to express the Hamiltonian operator \mathfrak{L}_ω , introduced in (5.35), in terms of the Hamiltonian H_ε rather than $K_\varepsilon = H_\varepsilon \circ \Gamma$ defined in (5.22). By (5.35), (3.24) and (3.22) we have

$$\mathfrak{L}_\omega = \omega \cdot \partial_\varphi \mathbb{I}_2 + J \left(\frac{\partial_w \nabla_{\bar{w}} K_\varepsilon}{\partial_{\bar{w}} \nabla_{\bar{w}} K_\varepsilon} \quad \frac{\partial_{\bar{w}} \nabla_{\bar{w}} K_\varepsilon}{\partial_w \nabla_{\bar{w}} K_\varepsilon} \right) \circ \check{\iota}_0, \quad J = \begin{pmatrix} i \text{Id}_\perp & 0 \\ 0 & -i \text{Id}_\perp \end{pmatrix}. \quad (6.1)$$

Taking into account the definition of Γ in (5.16), (5.17), (5.18) one computes

$$\nabla_{\bar{w}} K_\varepsilon = \nabla_{\bar{z}} H_\varepsilon \circ \Gamma + Y_{\bar{w}}^t \nabla_y H_\varepsilon \circ \Gamma, \quad \partial_w \nabla_{\bar{w}} K_\varepsilon = \partial_z \nabla_{\bar{z}} H_\varepsilon \circ \Gamma + R_1^\varepsilon \circ \Gamma, \quad (6.2)$$

where, by (5.18),

$$R_1^\varepsilon := \partial_y (\nabla_{\bar{z}} H_\varepsilon) Y_w + Y_{\bar{w}}^t \partial_z \nabla_y H_\varepsilon + Y_{\bar{w}}^t \partial_y (\nabla_y H_\varepsilon) Y_w. \quad (6.3)$$

Similarly, one has

$$\partial_{\bar{w}} \nabla_{\bar{w}} K_\varepsilon = \partial_{\bar{z}} \nabla_{\bar{z}} H_\varepsilon \circ \Gamma + R_2^\varepsilon \circ \Gamma, \quad (6.4)$$

where

$$R_2^\varepsilon := \partial_y (\nabla_{\bar{z}} H_\varepsilon) Y_{\bar{w}} + Y_{\bar{w}}^t \partial_z \nabla_y H_\varepsilon + Y_{\bar{w}}^t \partial_y (\nabla_y H_\varepsilon) Y_{\bar{w}}. \quad (6.5)$$

By (6.1) (6.2), (6.4) and since by (5.19), $\check{\iota}_{\text{iso}} = \Gamma \circ \check{\iota}_0$, we get

$$\mathfrak{L}_\omega = \omega \cdot \partial_\varphi \mathbb{I}_2 + J \mathfrak{A} + J \mathfrak{R}^\varepsilon \quad \text{where} \quad \mathfrak{A} := \begin{pmatrix} \partial_z \nabla_{\bar{z}} H_\varepsilon & \partial_{\bar{z}} \nabla_{\bar{z}} H_\varepsilon \\ \partial_{\bar{z}} \nabla_{\bar{z}} H_\varepsilon & \partial_z \nabla_{\bar{z}} H_\varepsilon \end{pmatrix} \circ \check{\iota}_{\text{iso}} \quad (6.6)$$

and

$$\mathfrak{R}^\varepsilon := \begin{pmatrix} \mathfrak{R}_1^\varepsilon & \mathfrak{R}_2^\varepsilon \\ \mathfrak{R}_2^\varepsilon & \mathfrak{R}_1^\varepsilon \end{pmatrix}, \quad \mathfrak{R}_1^\varepsilon := R_1^\varepsilon \circ \check{\iota}_{\text{iso}}, \quad \mathfrak{R}_2^\varepsilon := R_2^\varepsilon \circ \check{\iota}_{\text{iso}}. \quad (6.7)$$

According to Definition 3.2 $J \mathfrak{A}$ is Hamiltonian and since \mathfrak{L}_ω is also Hamiltonian so is $J \mathfrak{R}^\varepsilon$. We will show in Lemma 6.5 in Subsection 6.1 below that \mathfrak{R}^ε can be regarded as a remainder term in the reduction scheme for \mathfrak{L}_ω .

To reduce \mathfrak{L}_ω to a 2×2 block diagonal operator with φ -independent coefficients, we will use a KAM iteration scheme which requires to impose pertinent nonresonance conditions along the iteration. In view of the near resonance of the dNLS frequencies ω_k^{nls} and ω_{-k}^{nls} , this requires an asymptotic expansion of the eigenvalues of \mathfrak{L}_ω with a remainder term which decays in k . To this end, we perform in Subsections 6.2 - 6.4 three preliminary symplectic transformations which put \mathfrak{L}_ω into diagonal form with φ -independent coefficients up to a remainder, which is one smoothing and satisfies tame estimates. From a technical point of view, for proving the reduction scheme for the operator \mathfrak{L}_ω , stated in Theorem 7.1 in Section 7 below, it is convenient to use for operator valued maps $\varphi \mapsto \mathfrak{R}(\varphi) \in \mathcal{L}(h_\perp^{\sigma'} \times h_\perp^{\sigma'})$ the norm $|\mathfrak{R}|_{s, \sigma'}$ introduced in (2.9). We say that an operator of this type is one smoothing if $|\mathfrak{R} \mathfrak{D}|_{s, \sigma'} < \infty$. Here \mathfrak{D} is the operator introduced in (2.26).

By a slight abuse of terminology, we consider in the entire section operators such as \mathfrak{A} or \mathfrak{R}^ε with $\check{\iota}_{\text{iso}}$ in their definition replaced by an arbitrary torus embedding $\check{\iota} \equiv \check{\iota}_\omega$, of the type described at the beginning of Section 5. The estimates for \mathfrak{L}_ω are then obtained by applying the estimates, derived in this section, for $\check{\iota}$ given by $\check{\iota}_{\text{iso}}$ and using the estimates $\|\iota_{\text{iso}}\|_s^{\text{lip}} \leq_s \|\iota\|_{s+\mu}^{\text{lip}}$ and $\|d_\iota(\iota_{\text{iso}})[\hat{\iota}]\|_s \leq_s \|\hat{\iota}\|_{s+\mu} + \|\iota\|_{s+\mu} \|\hat{\iota}\|_{s_0+\mu}$ of Lemma 5.3. In the sequel, we always make the following smallness assumption, stated in (5.2),

$$\|\iota\|_{s_0+\mu_1}^{\text{lip}} \leq \varepsilon \gamma^{-2} \quad \text{with} \quad \varepsilon \gamma^{-4} \ll 1 \quad \text{and} \quad 0 < \gamma < 1. \quad (6.8)$$

6.1 Preliminary analysis of the operators \mathfrak{A} and \mathfrak{R}^ε

The aim of this subsection is to identify the main part of the operator \mathfrak{A} defined in (6.6) and to show that the remainder as well as the operator \mathfrak{R}^ε in (6.6) are one smoothing and satisfy tame estimates.

First note that since $H_\varepsilon = H^{nls} + \varepsilon P$ (cf (1.18)), the operator \mathfrak{A} can be written as $\mathfrak{A} = \mathfrak{S}^{nls} + \varepsilon \mathfrak{S}^P$ where

$$\mathfrak{S}^{nls} := \begin{pmatrix} \partial_z \nabla_{\bar{z}} H^{nls} & \partial_{\bar{z}} \nabla_{\bar{z}} H^{nls} \\ \partial_{\bar{z}} \nabla_{\bar{z}} H^{nls} & \partial_z \nabla_{\bar{z}} H^{nls} \end{pmatrix} \circ \check{\iota} \quad \mathfrak{S}^P = \begin{pmatrix} \partial_z \nabla_{\bar{z}} P & \partial_{\bar{z}} \nabla_{\bar{z}} P \\ \partial_{\bar{z}} \nabla_{\bar{z}} P & \partial_z \nabla_{\bar{z}} P \end{pmatrix} \circ \check{\iota}. \quad (6.9)$$

The operators \mathfrak{S}^{nls} , \mathfrak{S}^P , and \mathfrak{R}^ε are analyzed separately.

Analysis of \mathfrak{S}^{nls} . Recall that $H^{nls} = H^{nls}(\xi + y, z\bar{z})$ with $z\bar{z} := (z_n \bar{z}_n)_{n \in S^\perp}$, yielding

$$\nabla_{\bar{z}} H^{nls} \circ \check{\iota} = ((\omega_k^{nls} z_k) \circ \check{\iota})_{k \in S^\perp}$$

with $\omega_k^{nls} = \partial_{I_k} H^{nls}$. To simplify notation, we will drop $\check{\iota}$ whenever the context permits. In particular, we will often write I for $I \circ \check{\iota}$ and ω_k^{nls} for $\omega_k^{nls}(I \circ \check{\iota})$. Then we have

$$\partial_z \nabla_{\bar{z}} H^{nls} = \text{diag}_{k \in S^\perp} (\omega_k^{nls}) + R_1^{nls}, \quad \partial_{\bar{z}} \nabla_{\bar{z}} H^{nls} = R_2^{nls} \quad (6.10)$$

where R_1^{nls}, R_2^{nls} are the operators of h_\perp^σ with matrix coefficients (cf (2.8))

$$(R_1^{nls})_k^j := (\partial_{I_j} \omega_k^{nls}) z_k \bar{z}_j, \quad (R_2^{nls})_k^j := (\partial_{I_j} \omega_k^{nls}) z_k z_j, \quad \forall j, k \in S^\perp. \quad (6.11)$$

By (6.9), (6.10), and in view of the asymptotics $\omega_k^{nls} = 4\pi^2 k^2 + O(1)$ of Theorem 3.2 we write

$$\mathfrak{S}^{nls} = D^2 \mathbb{I}_2 + \Omega^{nls} \mathbb{I}_2 + \mathfrak{R}^{nls}, \quad \mathfrak{R}^{nls} := \begin{pmatrix} \mathfrak{R}_1^{nls} & \mathfrak{R}_2^{nls} \\ \mathfrak{R}_2^{nls} & \mathfrak{R}_1^{nls} \end{pmatrix}, \quad \mathfrak{R}_a^{nls} = R_a^{nls} \circ \check{\iota}, \quad a = 1, 2, \quad (6.12)$$

where D is the diagonal operator defined in (2.11) and

$$\Omega^{nls} := \text{diag}_{k \in S^\perp} (\omega_k^{nls} - 4\pi^2 k^2). \quad (6.13)$$

We claim that $D^2 \mathbb{I}_2 + \Omega^{nls} \mathbb{I}_2$ is the main part of \mathfrak{S}^{nls} , meaning that \mathfrak{R}^{nls} is a (small) one smoothing operator. More precisely the following estimates hold. We recall that throughout the paper, we assume that $\sigma \geq 4$, if not stated otherwise.

Lemma 6.1. (Estimates for Ω^{nls} and \mathfrak{R}^{nls}) *Let $s \geq s_0$. Then the following estimates hold:*

(i) *For any $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$,*

$$|\Omega^{nls}|_{s, \sigma'} \leq_s 1 + \|\iota\|_{s+2s_0}, \quad |\Omega^{nls}|_{s, \sigma'}^{\gamma \text{lip}} \leq_s 1 + \|\iota\|_{s+2s_0}^{\gamma \text{lip}}. \quad (6.14)$$

(ii) *The remainder \mathfrak{R}^{nls} defined in (6.12) satisfies the estimates*

$$|\mathfrak{R}^{nls} \mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad |\mathfrak{R}^{nls} \mathfrak{D}|_{s, \sigma-1}^{\gamma \text{lip}} \leq_s \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}^{\gamma \text{lip}}, \quad (6.15)$$

where \mathfrak{D} is defined in (2.26).

Proof. (i) We now prove the first estimate in (6.14). As Ω^{nls} is a diagonal operator it suffices to prove the claimed estimate for $\sigma' = \sigma$. By Theorem 3.2, the dNLS frequencies admit the asymptotics

$$\omega_k^{nls}(I) = 4\pi^2 k^2 + 4 \sum_{j \in \mathbb{Z}} I_j + \frac{r_k(I)}{k}$$

where $(r_k)_{k \in \mathbb{Z}} : \ell_+^{1,4}(\mathbb{Z}, \mathbb{R}) \rightarrow \ell^\infty(\mathbb{Z}, \mathbb{R})$ is real analytic. Accordingly we decompose Ω^{nls} , defined in (6.13), as

$$\Omega^{nls} = \left(4 \sum_{j \in \mathbb{Z}} I_j \right) \text{Id}_\perp + \text{diag}_{k \in S^\perp} \frac{r_k(I)}{k} \quad (6.16)$$

and estimate the norms of the latter two operators separately. To estimate $|(\sum_{j \in \mathbb{Z}} I_j(\varphi)) \text{Id}_\perp|_{s, \sigma}$ we write

$$\sum_{j \in \mathbb{Z}} I_j(\varphi) = \left(\sum_{j \in S} \xi_j \right) \text{Id}_\perp + g(\varphi) \text{Id}_\perp \quad \text{where} \quad g(\varphi) := \sum_{j \in S} y_j(\varphi) + \sum_{j \in S^\perp} z_j(\varphi) \bar{z}_j(\varphi). \quad (6.17)$$

By the definition (2.9) of the operator norm $|\cdot|_{s, \sigma}$,

$$|g \text{Id}_\perp|_{s, \sigma} = \left(\sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} \|\hat{g}(\ell) \text{Id}_\perp\|_{\mathcal{L}(h_\perp^\sigma)}^2 \right)^{1/2} = \left(\sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} |\hat{g}(\ell)|^2 \right)^{1/2} = \|g\|_s \quad (6.18)$$

where, for brevity, we set $\|g\|_s := \|g\|_{H^s(\mathbb{T}^S, \mathbb{C})}$. By (6.17), using Lemma 2.7 and the Cauchy-Schwartz inequality, we estimate

$$\|g\|_s \leq_s \|y\|_s + \sum_{j \in S^\perp} \|z_j \bar{z}_j\|_s \leq_s \|y\|_s + \sum_{j \in S^\perp} \|z_j\|_{s_0} \|\bar{z}_j\|_s \leq_s \|y\|_s + \|z\|_{s_0, \sigma} \|z\|_{s, \sigma} \leq_s \|\iota\|_s.$$

In conclusion

$$\left| \left(\sum_{j \in \mathbb{Z}} I_j \right) \text{Id}_\perp \right|_{s, \sigma} \leq_s |\xi| + \|g\|_s \leq_s |\xi| + \|\iota\|_s \leq_s 1 + \|\iota\|_s. \quad (6.19)$$

Towards the second operator on the right hand side of (6.16), note that the operator norm of the Fourier coefficient $\hat{A}(\ell)$, $\ell \in \mathbb{Z}^S$, of the map $\varphi \rightarrow A(\varphi) := \text{diag}_{k \in S^\perp} \frac{1}{k} (r_k \circ I)(\varphi)$ is

$$\|\hat{A}(\ell)\|_{\mathcal{L}(h_\perp^\sigma)} = \sup_{k \in S^\perp} \frac{1}{|k|} |(\widehat{r_k \circ I})(\ell)|$$

and hence, recalling the definition (2.9) of the operator norm $|\cdot|_{s, \sigma}$,

$$|A|_{s, \sigma}^2 = \sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} \sup_{k \in S^\perp} \frac{1}{k^2} |(\widehat{r_k \circ I})(\ell)|^2 \leq \sum_{k \in S^\perp} \frac{1}{k^2} \sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} |(\widehat{r_k \circ I})(\ell)|^2 = \sum_{k \in S^\perp} \frac{1}{k^2} \|r_k \circ I\|_s^2. \quad (6.20)$$

By Theorem 3.2, the map $(r_k)_{k \in S^\perp} : \ell^{1,4} \rightarrow \ell_\perp^\infty$ is real analytic and there exists a neighborhood $V \subset \ell^{1,4}$ of $(\Pi + U_0) \times \{0\}$ and $C > 0$ such that $\sup_{I \in V} |r_k(I)| \leq C$, $\forall k \in S^\perp$. Since for any $\xi \in \Pi$, the map

$$B_\sigma(0, 0) \subseteq \mathbb{R}^S \times h_\perp^\sigma \rightarrow V, \quad (y, z) \mapsto (\xi + y, z\bar{z}) \in V$$

is real analytic in a sufficiently small neighborhood of $(0, 0)$, $B_\sigma(0, 0) \subseteq \mathbb{R}^S \times h_\perp^\sigma$ (see the proof of Proposition 3.2), Lemma 2.11, applied to f given by the sequence $(r_k(\xi + y, z\bar{z}))_{k \in \mathbb{Z}}$ and $Y = \ell^\infty$ then yields

$$\|(r_k(\xi + y, z\bar{z}))_{k \in \mathbb{Z}}\|_{\mathcal{C}^s(\mathbb{T}^S, \ell^\infty)} \leq_s 1 + \|\iota\|_{\mathcal{C}^s(\mathbb{T}^S, M^\sigma)}. \quad (6.21)$$

As a consequence of (2.38), we get

$$\|r_k \circ I\|_s \leq_s 1 + \|\iota\|_{s+2s_0}, \quad \forall k \in S^\perp, \quad (6.22)$$

and, by (6.20), we conclude

$$|A|_{s, \sigma} \leq_s 1 + \|\iota\|_{s+2s_0}. \quad (6.23)$$

Combining (6.16) with (6.19) and (6.23), the first estimate of (6.14) follows. The second estimate of (6.14) is proved in a similar way.

(ii) Let us begin by proving the first estimate of (6.15). We only consider $\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle$ since the estimate for $\mathfrak{R}_2^{nls} \langle\langle D \rangle\rangle$ is done in the same way. We recall that $\langle\langle D \rangle\rangle$ is the diagonal operator introduced in (2.12).

We write $\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle$ as the sum of its columns, namely

$$\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle = \sum_{j \in S^\perp} A_{(j)} \pi_j, \quad A_{(j)}(\varphi) := (z_k(\varphi) \langle j \rangle^2 f_{kj}(I(\varphi)) \bar{z}_j(\varphi) \langle\langle j \rangle\rangle)_{k \in S^\perp}, \quad (6.24)$$

where π_j denotes the projector

$$\pi_j : h_\perp^\sigma \rightarrow \mathbb{C}, \quad (w_n)_{n \in S^\perp} \rightarrow w_j, \quad (6.25)$$

and

$$f_{kj}(I) := \langle j \rangle^{-2} \partial_{I_j} \omega_k^{nls}(I), \quad I(\varphi) := (\xi + y(\varphi), I_\perp(\varphi)), \quad I_\perp := (z_k \bar{z}_k)_{k \in S^\perp}. \quad (6.26)$$

Then we have $|\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle|_{s, \sigma-1} \leq \sum_{j \in S^\perp} |A_{(j)} \pi_j|_{s, \sigma-1}$. Since by the definition (2.9) of the operator norm $|\cdot|_{s, \sigma-1}$

$$|A_{(j)} \pi_j|_{s, \sigma-1} = \left(\sum_{\ell \in \mathbb{Z}^S} \langle \ell \rangle^{2s} \|\hat{A}_{(j)}(\ell) \pi_j\|_{\mathcal{L}(h_\perp^{\sigma-1})}^2 \right)^{\frac{1}{2}}, \quad \|\hat{A}_{(j)}(\ell) \pi_j\|_{\mathcal{L}(h_\perp^{\sigma-1})} = \|\hat{A}_{(j)}(\ell)\|_{\sigma-1} \langle j \rangle^{-(\sigma-1)},$$

we have, by the property (2.7) of the $\|\cdot\|_s$ -norm

$$|A_{(j)}\pi_j|_{s,\sigma-1} = \langle j \rangle^{-(\sigma-1)} \|A_{(j)}\|_{s,\sigma-1} \leq \langle j \rangle^{-(\sigma-1)} \|A_{(j)}\|_{s,\sigma}. \quad (6.27)$$

We claim that

$$\|A_{(j)}\|_{s,\sigma} \leq_s \langle j \rangle^3 (\|\iota\|_{s+2s_0} \|z_j\|_{s_0} + \|\iota\|_{s_0} \|z_j\|_s). \quad (6.28)$$

Before proving (6.28) we complete the proof of the first estimate of (6.15). By (6.27) and (6.28), we get

$$\begin{aligned} |\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle|_{s,\sigma-1} &\leq_s \sum_{j \in S^\perp} \langle j \rangle^{4-\sigma} (\|\iota\|_{s+2s_0} \|z_j\|_{s_0} + \|\iota\|_{s_0} \|z_j\|_s) \\ &\leq_s \|\iota\|_{s+2s_0} \left(\sum_{j \in S^\perp} \langle j \rangle^{4-2\sigma} \|z_j\|_{s_0} \langle j \rangle^\sigma \right) + \|\iota\|_{s_0} \left(\sum_{j \in S^\perp} \langle j \rangle^{4-2\sigma} \|z_j\|_s \langle j \rangle^\sigma \right) \\ &\leq_s \|\iota\|_{s+2s_0} \|z\|_{s_0,\sigma} + \|\iota\|_{s_0} \|z\|_{s,\sigma} \end{aligned}$$

by applying the Cauchy-Schwartz inequality, using that $4(\sigma-2) > 1$. By the smallness assumption (6.8), the first estimate of (6.15) then follows. It remains to prove the estimate (6.28). By the definition (6.26) of f_{kj} and the estimates (3.38) one gets

$$\|f_{kj}(\xi + y, z\bar{z})\|_s \leq_s 1 + \|\iota\|_{s+2s_0}, \quad \forall j, k \in S^\perp, \quad \forall \xi \in \Pi. \quad (6.29)$$

We now can prove the estimate (6.28): recalling (2.7) and (6.24) we have

$$\begin{aligned} \|A_{(j)}\|_{s,\sigma}^2 &\leq_s \langle j \rangle^6 \sum_k \langle k \rangle^{2\sigma} \|z_k(f_{kj} \circ I)\bar{z}_j\|_s^2 \\ &\stackrel{(2.18)}{\leq_s} \langle j \rangle^6 \sum_k \langle k \rangle^{2\sigma} \left(\|z_k\|_s \|f_{kj} \circ I\|_{s_0} \|z_j\|_{s_0} + \|z_k\|_{s_0} \|f_{kj} \circ I\|_s \|z_j\|_{s_0} + \|z_k\|_{s_0} \|f_{kj} \circ I\|_{s_0} \|z_j\|_s \right)^2 \\ &\stackrel{(6.29),(6.8)}{\leq_s} \langle j \rangle^6 \sum_k \langle k \rangle^{2\sigma} \left(\|z_k\|_s \|z_j\|_{s_0} + \|z_k\|_{s_0} \|\iota\|_{s+2s_0} \|z_j\|_{s_0} + \|z_k\|_{s_0} \|z_j\|_s \right)^2 \\ &\stackrel{(2.7)}{\leq_s} \langle j \rangle^6 \left(\|z\|_{s,\sigma}^2 \|z_j\|_{s_0}^2 + \|z\|_{s_0,\sigma}^2 \|\iota\|_{s+2s_0}^2 \|z_j\|_{s_0}^2 + \|z\|_{s_0,\sigma}^2 \|z_j\|_s^2 \right). \end{aligned}$$

Using again the smallness assumptions (6.8), the claimed estimate (6.28) then follows. The second estimate in (6.15) can be proved in a similar way. \square

The next result is only needed in Section 9 for the proof of the measure estimates. Given two torus embeddings

$$\check{\iota}^{(a)}(\varphi) := (\varphi, 0, 0) + \iota^{(a)}(\varphi), \quad \iota^{(a)}(\varphi) = (\Theta^{(a)}(\varphi), y^{(a)}(\varphi), z^{(a)}(\varphi)), \quad a = 1, 2,$$

we write

$$\Delta_{12}\check{\iota} := \check{\iota}^{(1)} - \check{\iota}^{(2)}, \quad \Delta_{12}\iota := \iota^{(1)} - \iota^{(2)}, \quad \Delta_{12}z := z^{(1)} - z^{(2)}, \quad \dots \quad (6.30)$$

Note that $\Delta_{12}\check{\iota} = \Delta_{12}\iota$. Furthermore, introduce for $s \geq s_0$

$$\max_s(\iota) := \max\{\|\iota^{(1)}\|_s, \|\iota^{(2)}\|_s\}, \quad \max_s(z) := \max\{\|z^{(1)}\|_s, \|z^{(2)}\|_s\}, \quad \dots \quad (6.31)$$

Define $\Omega^{nls}(\check{\iota}^{(a)}) := \Omega^{nls}(I \circ \check{\iota}^{(a)})$, $a = 1, 2$, and use a similar notation for other operators.

Lemma 6.2. *Let $s \geq s_0$. Then for any torus embeddings $\check{\iota}^{(a)}(\varphi) := (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, satisfying (6.8), the following estimates hold:*

(i) *For any $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$, $\Delta_{12}\Omega^{nls} := \Omega^{nls}(\check{\iota}^{(1)}) - \Omega^{nls}(\check{\iota}^{(2)})$ satisfies the estimate*

$$|\Delta_{12}\Omega^{nls}|_{s,\sigma'} \leq_s \|\Delta_{12}\iota\|_s + \max_{s+2s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}.$$

(ii) *The operator $\Delta_{12}\mathfrak{R}^{nls} := \mathfrak{R}^{nls}(\check{\iota}^{(1)}) - \mathfrak{R}^{nls}(\check{\iota}^{(2)})$ satisfies the estimate*

$$|\Delta_{12}\mathfrak{R}^{nls}\mathfrak{D}|_{s,\sigma-1} \leq_s \varepsilon\gamma^{-2} \|\Delta_{12}\iota\|_s + \max_{s+2s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}.$$

Proof. (i) As Ω^{nls} is a diagonal operator it suffices to prove the claimed estimate for $\sigma' = \sigma$. Writing $I^{(a)} := (\xi + y^{(a)}, I_{\perp}^{(a)})$, $a = 1, 2$ and $\Delta_{12}I_j := I_j^{(1)} - I_j^{(2)}$, $j \in \mathbb{Z}$, one has, by (6.16),

$$\Omega^{nls}(i^{(1)}) - \Omega^{nls}(i^{(2)}) = \left(4 \sum_{j \in \mathbb{Z}} \Delta_{12}I_j\right) \text{Id}_{\perp} + \text{diag}_{k \in S^{\perp}} \frac{\Delta_{12}r_k(I)}{k}. \quad (6.32)$$

Since $\sum_{j \in \mathbb{Z}} \Delta_{12}I_j = \sum_{j \in S} \Delta_{12}y_j + \sum_{j \in S^{\perp}} \Delta_{12}I_j$, one gets, arguing as in (6.18), (6.19),

$$\begin{aligned} \left| \left(\sum_{j \in \mathbb{Z}} \Delta_{12}I_j \right) \text{Id}_{\perp} \right|_{s, \sigma} &\leq \left\| \sum_{j \in \mathbb{Z}} \Delta_{12}I_j \right\|_s \\ &\leq_s \sum_{j \in S} \|y_j^{(1)} - y_j^{(2)}\|_s + \sum_{j \in S^{\perp}} \|(z_j^{(1)} - z_j^{(2)})\bar{z}_j^{(1)}\|_s + \sum_{j \in S^{\perp}} \|z_j^{(2)}(\bar{z}_j^{(1)} - \bar{z}_j^{(2)})\|_s \\ &\stackrel{(6.8)}{\leq}_s \|\Delta_{12}\iota\|_s + \max_s(\iota) \|\Delta_{12}\iota\|_{s_0}. \end{aligned} \quad (6.33)$$

Now we estimate the second term on the right hand side of (6.32). The operator norm of the Fourier coefficient $\hat{A}(\ell)$, $\ell \in \mathbb{Z}^S$, of the map $\varphi \rightarrow A(\varphi) := \text{diag}_{k \in S^{\perp}} \frac{1}{k} \Delta_{12}r_k(\varphi)$ where $\Delta_{12}r_k := r_k(I^{(1)}) - r_k(I^{(2)})$ is

$$\|\hat{A}(\ell)\|_{\mathcal{L}(h_{\perp}^{\sigma})} = \sup_{k \in S^{\perp}} \frac{1}{|k|} |\widehat{\Delta_{12}r_k}(\ell)|$$

and hence, arguing as in (6.20)

$$|A|_{s, \sigma}^2 \leq \sum_{k \in S^{\perp}} \frac{1}{k^2} \|\Delta_{12}r_k(I)\|_s^2. \quad (6.34)$$

By the mean value theorem one has

$$\Delta_{12}r_k = \int_0^1 \partial_I r_k(I_t) dt \cdot \Delta_{12}I, \quad I_t := tI^{(1)} + (1-t)I^{(2)} \quad (6.35)$$

where

$$\partial_I r_k(I_t) \cdot \Delta_{12}I = \sum_{n \in \mathbb{Z}} \partial_{I_n} r_k(I_t) \Delta_{12}I_n. \quad (6.36)$$

Since by Theorem 3.2 item (ii), the map $(r_k)_{k \in S^{\perp}} : \ell^{1,4} \rightarrow \ell^{\infty}$ is real analytic there exists a neighborhood $V \subset \ell^{1,4}$ of $(\Pi + U_0) \times \{0\}$ such that

$$\sup_{k \in \mathbb{Z}} \sup_{I \in V} \|\partial_I r_k(I)\|_{(\ell^{1,4})^*} = \sup_{k \in \mathbb{Z}} \sup_{I \in V} \sup_{n \in \mathbb{Z}} \frac{|\partial_{I_n} r_k(I)|}{\langle n \rangle^4} \leq C. \quad (6.37)$$

(Here we used that the dual space of $\ell^{1,4}$ is $\ell^{\infty, -4}$.) Defining $p_{nk} := \langle n \rangle^{-4} \partial_{I_n} r_k$ we have, by Lemma 2.7,

$$\|\partial_I r_k(I_t) \cdot \Delta_{12}I\|_s \leq_s \sum_{n \in \mathbb{Z}} \|p_{nk} \circ I_t\|_s \langle n \rangle^4 \|\Delta_{12}I_n\|_{s_0} + \|p_{nk} \circ I_t\|_{s_0} \langle n \rangle^4 \|\Delta_{12}I_n\|_s. \quad (6.38)$$

Moreover, by (6.37), arguing as in the proof of the estimate (6.22), we get

$$\|p_{nk} \circ I_t\|_s \leq_s 1 + \max_{s+2s_0}(\iota). \quad (6.39)$$

Combining the estimates (6.35) - (6.39) with the smallness assumption (6.8) then yields

$$\begin{aligned} \|\Delta_{12}r_k\|_s &\leq_s \max_{s+2s_0}(\iota) \sum_{n \in \mathbb{Z}} \langle n \rangle^4 \|\Delta_{12}I_n\|_{s_0} + \sum_{n \in \mathbb{Z}} \langle n \rangle^4 \|\Delta_{12}I_n\|_s \\ &\leq_s \|\Delta_{12}y\|_s + \max_{s+2s_0}(\iota) \|\Delta_{12}y\|_{s_0} + \sum_{n \in S^{\perp}} \langle n \rangle^4 \|\Delta_{12}(z_n \bar{z}_n)\|_s + \max_{s+2s_0}(\iota) \sum_{n \in S^{\perp}} \langle n \rangle^4 \|\Delta_{12}(z_n \bar{z}_n)\|_{s_0}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{n \in S^\perp} \langle n \rangle^4 \|\Delta_{12}(z_n \bar{z}_n)\|_s &\leq_s \sum_{n \in S^\perp} \langle n \rangle^4 (\|\bar{z}_n^{(1)} \Delta_{12} z_n\|_s + \|z_n^{(2)} \Delta_{12} \bar{z}_n\|_s) \\ &\leq_s \sum_{n \in S^\perp} \langle n \rangle^4 (\|\Delta_{12} z_n\|_s \|z_n^{(1)}\|_{s_0} + \|\Delta_{12} z_n\|_{s_0} \|z_n^{(1)}\|_s + \|\Delta_{12} z_n\|_s \|z_n^{(2)}\|_{s_0} + \|\Delta_{12} z_n\|_{s_0} \|z_n^{(2)}\|_s) \end{aligned}$$

one then gets by Cauchy-Schwartz, the smallness assumption (6.8), and the assumption $\sigma \geq 4$

$$\sum_{n \in S^\perp} \langle n \rangle^4 \|\Delta_{12}(z_n \bar{z}_n)\|_s \leq_s \varepsilon \gamma^{-2} \|\Delta_{12} z\|_s + \max_s(z) \|\Delta_{12} z\|_{s_0}.$$

Altogether we proved that for any $k \in S^\perp$,

$$\|\Delta_{12} r_k\|_s \leq_s \|\Delta_{12} \ell\|_s + \max_{s+2s_0}(\ell) \|\Delta_{12} \ell\|_{s_0}, \quad (6.40)$$

implying, together with (6.34), that

$$|A|_{s,\sigma} \leq_s \|\Delta_{12} \ell\|_s + \max_{s+2s_0}(\ell) \|\Delta_{12} \ell\|_{s_0}.$$

Item (i) then follows in combination with (6.32), (6.33).

(ii) Since the claimed estimates for $\Delta_{12} \mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle$ and $\Delta_{12} \mathfrak{R}_2^{nls} \langle\langle D \rangle\rangle$ are obtained in the same way, we only consider $\Delta_{12} \mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle$. Recall that by (6.24), the operator $\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle$ can be written as

$$\mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle = \sum_{j \in S^\perp} A_{(j)} \pi_j, \quad A_{(j)}(\varphi) := (z_k(\varphi) \langle j \rangle^2 f_{kj}(I(\varphi))) \bar{z}_j(\varphi) \langle\langle j \rangle\rangle_{k \in S^\perp}$$

where π_j denotes the projector introduced in (6.25) and $f_{kj}(I)$ is defined in (6.26).

Then we have $|\Delta_{12} \mathfrak{R}_1^{nls} \langle\langle D \rangle\rangle|_{s,\sigma-1} \leq \sum_{j \in S^\perp} |\Delta_{12} A_{(j)} \pi_j|_{s,\sigma-1}$. Since

$$|\Delta_{12} A_{(j)} \pi_j|_{s,\sigma-1} = \left(\sum_{\ell} \langle \ell \rangle^{2s} \|\Delta_{12} \hat{A}_{(j)}(\ell) \pi_j\|_{\mathcal{L}(h_{\perp}^{\sigma-1})}^2 \right)^{\frac{1}{2}}, \quad \|\Delta_{12} \hat{A}_{(j)}(\ell) \pi_j\|_{\mathcal{L}(h_{\perp}^{\sigma-1})} = \|\Delta_{12} \hat{A}_{(j)}(\ell)\|_{\sigma-1} \langle j \rangle^{-(\sigma-1)}$$

one concludes in view of the property (2.7) of the $\|\cdot\|_s$ -norm that

$$|\Delta_{12} A_{(j)} \pi_j|_{s,\sigma-1} = \langle j \rangle^{-(\sigma-1)} \left(\sum_{\ell,k} \langle \ell \rangle^{2s} \langle k \rangle^{2(\sigma-1)} |\Delta_{12} \hat{A}_{(j),k}(\ell)|^2 \right)^{\frac{1}{2}} = \langle j \rangle^{-(\sigma-1)} \|\Delta_{12} A_{(j)}\|_{s,\sigma-1}. \quad (6.41)$$

To estimate $\|\Delta_{12} A_{(j)}\|_{s,\sigma-1}$, let $\Delta_{12} f_{kj} := f_{kj}(I^{(1)}) - f_{kj}(I^{(2)})$ and write $\Delta_{12} A_{(j)}$ as a telescoping sum,

$$\Delta_{12} A_{(j)} = B_{(j)} + C_{(j)} + D_{(j)} \quad (6.42)$$

where

$$\begin{aligned} B_{(j)} &:= (\langle j \rangle^2 z_k^{(1)} \bar{z}_j^{(1)} \langle\langle j \rangle\rangle \Delta_{12} f_{kj})_{k \in S^\perp}, \quad C_{(j)} := (\langle j \rangle^2 f_{kj}(I^{(2)}) \bar{z}_j^{(1)} \langle\langle j \rangle\rangle \Delta_{12} z_k)_{k \in S^\perp}, \\ D_{(j)} &:= (\langle j \rangle^2 f_{kj}(I^{(2)}) z_k^{(2)} \langle\langle j \rangle\rangle \Delta_{12} \bar{z}_j)_{k \in S^\perp}. \end{aligned}$$

We estimate the $\|\cdot\|_{s,\sigma-1}$ norm of the above three terms separately. Actually, we estimate the larger norm $\|\cdot\|_{s,\sigma}$ of these terms. One has

$$\begin{aligned} \|B_{(j)}\|_{s,\sigma}^2 &\leq_s \langle j \rangle^6 \sum_{k \in S^\perp} \langle k \rangle^{2\sigma} \|z_k^{(1)} \bar{z}_j^{(1)} \Delta_{12} f_{kj}\|_s^2 \\ &\leq_s \langle j \rangle^6 \sum_{k \in S^\perp} \langle k \rangle^{2\sigma} \left(\|\Delta_{12} f_{kj}\|_s^2 \|z_j^{(1)}\|_{s_0}^2 \|z_k^{(1)}\|_{s_0}^2 + \|\Delta_{12} f_{kj}\|_{s_0}^2 \|z_j^{(1)}\|_s^2 \|z_k^{(1)}\|_{s_0}^2 \right. \\ &\quad \left. + \|\Delta_{12} f_{kj}\|_{s_0}^2 \|z_j^{(1)}\|_{s_0}^2 \|z_k^{(1)}\|_s^2 \right). \end{aligned}$$

The term $\Delta_{12}f_{kj}$ can be estimated in the same way as $\Delta_{12}r_k$ of item (i), together with (3.38) of Proposition 3.2, obtaining

$$\|\Delta_{12}f_{kj}\|_s \leq_s \|\Delta_{12}\iota\|_s + \max_{s+2s_0}(\iota)\|\Delta_{12}\iota\|_{s_0}.$$

Hence by the smallness condition (6.8),

$$\begin{aligned} \|B_{(j)}\|_{s,\sigma}^2 &\leq_s \left(\|\Delta_{12}\iota\|_s^2 + \max_{s+2s_0}(\iota)^2 \|\Delta_{12}\iota\|_{s_0}^2 \right) \langle j \rangle^6 \|z_j^{(1)}\|_{s_0}^2 \sum_{k \in S^\perp} \langle k \rangle^{2\sigma} \|z_k^{(1)}\|_{s_0}^2 \\ &\quad + \langle j \rangle^6 \|z_j^{(1)}\|_s^2 \|\Delta_{12}\iota\|_{s_0}^2 \sum_{k \in S^\perp} \langle k \rangle^{2\sigma} \|z_k^{(1)}\|_{s_0}^2 + \langle j \rangle^6 \|z_j^{(1)}\|_{s_0}^2 \|\Delta_{12}\iota\|_{s_0}^2 \sum_{k \in S^\perp} \langle k \rangle^{2\sigma} \|z_k^{(1)}\|_s^2 \\ &\leq_s \left(\|\Delta_{12}\iota\|_s^2 + \max_{s+2s_0}(\iota)^2 \|\Delta_{12}\iota\|_{s_0}^2 \right) \langle j \rangle^6 \|z_j^{(1)}\|_{s_0}^2 \|z^{(1)}\|_{s_0,\sigma}^2 + \langle j \rangle^6 \|z_j^{(1)}\|_s^2 \|\Delta_{12}\iota\|_{s_0}^2 \|z^{(1)}\|_{s_0,\sigma}^2 \\ &\quad + \langle j \rangle^6 \|z_j^{(1)}\|_{s_0}^2 \|\Delta_{12}\iota\|_{s_0}^2 \|z^{(1)}\|_{s,\sigma}^2, \end{aligned}$$

implying together with (6.8) that

$$\|B_{(j)}\|_{s,\sigma} \leq_s \langle j \rangle^3 \left(\varepsilon\gamma^{-2} \|z_j^{(1)}\|_{s_0} \|\Delta_{12}\iota\|_s + (\|z_j^{(1)}\|_s + \max_{s+2s_0}(\iota) \|z_j^{(1)}\|_{s_0}) \|\Delta_{12}\iota\|_{s_0} \right). \quad (6.43)$$

Since by (6.29), $\|f_{kj} \circ I\|_s \leq_s 1 + \|\iota\|_{s+2s_0}$, one can prove in a similar way that

$$\|C_{(j)}\|_{s,\sigma} \leq_s \langle j \rangle^3 \left(\|z_j^{(1)}\|_{s_0} \|\Delta_{12}\iota\|_s + (\|z_j^{(1)}\|_s + \max_{s+2s_0}(\iota) \|z_j^{(1)}\|_{s_0}) \|\Delta_{12}\iota\|_{s_0} \right), \quad (6.44)$$

$$\|D_{(j)}\|_{s,\sigma} \leq_s \langle j \rangle^3 \left(\varepsilon\gamma^{-2} \|\Delta_{12}z_j\|_s + \max_{s+2s_0}(\iota) \|\Delta_{12}z_j\|_{s_0} \right). \quad (6.45)$$

When combined, the above three estimates yield

$$\begin{aligned} |\Delta_{12}\mathfrak{R}_1^{nls}\langle\langle D \rangle\rangle|_{s,\sigma-1} &\leq \sum_{j \in S^\perp} |\Delta_{12}A_{(j)}\pi_j|_{s,\sigma-1} \stackrel{(6.41)}{\leq} \sum_{j \in S^\perp} \langle j \rangle^{-(\sigma-1)} \|\Delta_{12}A_{(j)}\|_{s,\sigma-1} \\ &\stackrel{(6.42)}{\leq} \sum_{j \in S^\perp} \langle j \rangle^{-(\sigma-1)} (\|B_{(j)}\|_{s,\sigma} + \|C_{(j)}\|_{s,\sigma} + \|D_{(j)}\|_{s,\sigma}) \\ &\stackrel{(6.43),(6.44),(6.45)}{\leq_s} \sum_{j \in S^\perp} \langle j \rangle^{4-\sigma} \left(\|z_j^{(1)}\|_{s_0} \|\Delta_{12}\iota\|_s + (\|z_j^{(1)}\|_s + \max_{s+2s_0}(\iota) \|z_j^{(1)}\|_{s_0}) \|\Delta_{12}\iota\|_{s_0} \right) \\ &\quad + \sum_{j \in S^\perp} \langle j \rangle^{4-\sigma} \left(\varepsilon\gamma^{-2} \|\Delta_{12}z_j\|_s + \max_{s+2s_0}(\iota) \|\Delta_{12}z_j\|_{s_0} \right). \end{aligned}$$

By the assumption $\sigma \geq 4$ and the smallness condition (6.8) the claimed estimate then follow. \square

Remark 6.1. *Arguing as in the proof of Lemma 6.2 (i), one can also obtain an estimate for $r_k(\xi + y, z\bar{z}) - r_k(\xi, 0)$, which we record for later reference: by the mean value theorem, one has*

$$r_k(\xi + y, z\bar{z}) - r_k(\xi, 0) = \int_0^1 \partial_{I_r} r_k(I_t) dt \cdot (y, z\bar{z}) \quad \text{with } I_t = (\xi, 0) + t(y, z\bar{z}), \quad z\bar{z} = (z_j \bar{z}_j)_{j \in S^\perp}.$$

By Theorem 3.2 (dNLS frequencies), and using (6.8), one has $\langle n \rangle^{-4} |\partial_{I_n} r_k(I_t)| \leq 1$. Then, from Lemma 2.11 (tame estimates for composition), it follows that $\|r_k(\xi + y, z\bar{z}) - r_k(\xi, 0)\|_s \leq_s \|\iota\|_{s+2s_0}$, using also (6.8). By similar arguments one can verify a corresponding bound for $\|r_k(\xi + y, z\bar{z}) - r_k(\xi, 0)\|_s^{\text{lip}}$. Under the same assumptions as in Lemma 6.1 one obtains in this way the estimate

$$\|r_k(\xi + y, z\bar{z}) - r_k(\xi, 0)\|_s^{\gamma \text{lip}} \leq_s \|\iota\|_{s+2s_0}^{\gamma \text{lip}}. \quad (6.46)$$

Analysis of \mathfrak{S}^P . In this paragraph it is convenient to denote by $\tilde{X}_{\mathcal{P}}$ the vector field obtained from the Hamiltonian vector field $-i\nabla_{\bar{u}}\mathcal{P}$ by adding its complex conjugate as a second component, $\tilde{X}_{\mathcal{P}} := (-i\nabla_{\bar{u}}\mathcal{P}, i\nabla_u\mathcal{P})$. We denote by \tilde{X}_P the Hamiltonian vector field $\tilde{X}_{\mathcal{P}}$, when expressed in Birkhoff coordinates,

$$\tilde{X}_P := (d\Phi \tilde{X}_{\mathcal{P}})|_{\Phi^{-1}}, \quad P = \mathcal{P} \circ \Phi^{-1}, \quad (6.47)$$

where $\Phi = \Phi^{nls}$ is the Birkhoff map of Theorem 3.1. Recall that F_{nls} denotes the version of the Fourier transform, introduced in (3.1). Denote its inverse by F_{nls}^{-1} . Using that by Theorem 3.1, $\Phi = F_{nls} + A^{nls}$ and $\Phi^{-1} = F_{nls}^{-1} + B^{nls}$, the differential of \tilde{X}_P can be computed as

$$d\tilde{X}_P = F_{nls} (d\tilde{X}_P)|_{\Phi^{-1}} F_{nls}^{-1} - J(T_1 + T_2 + T_3) \quad (6.48)$$

with

$$T_1 := JF_{nls} (d\tilde{X}_P)|_{\Phi^{-1}} dB^{nls}, \quad T_2 := J(dA^{nls} d\tilde{X}_P)|_{\Phi^{-1}} d\Phi^{-1}, \quad T_3 := J(d^2 A^{nls})|_{\Phi^{-1}} (d\Phi^{-1}(\cdot), (\tilde{X}_P)|_{\Phi^{-1}}).$$

By (1.5), one has $\tilde{X}_P = (-if(x, u), i\bar{f}(x, u))$ with $f(x, u(x)) = \partial_{\bar{z}} p|_{\zeta=u(x)}$ and hence the differential $d\tilde{X}_P$ of \tilde{X}_P is given by

$$d\tilde{X}_P = -J\mathcal{Q}, \quad \mathcal{Q} := \begin{pmatrix} \frac{\partial_u f}{\partial_{\bar{u}} f} & \frac{\partial_{\bar{u}} f}{\partial_u f} \end{pmatrix} = \begin{pmatrix} \frac{\partial_{\zeta} \partial_{\bar{\zeta}} p}{\partial_{\bar{\zeta}} \partial_{\zeta} p} & \frac{\partial_{\bar{\zeta}} \partial_{\zeta} p}{\partial_{\zeta} \partial_{\bar{\zeta}} p} \end{pmatrix} \Big|_{\zeta=u(x)}. \quad (6.49)$$

Since $\partial_{\zeta} \partial_{\bar{\zeta}} = \frac{1}{2}(\partial_{\zeta_1}^2 + \partial_{\zeta_2}^2)$, the function $\partial_{\zeta} \partial_{\bar{\zeta}} p$ is real valued whereas by a similar computation, $\partial_{\zeta} \partial_{\zeta} p$ is the complex conjugate of $\partial_{\bar{\zeta}} \partial_{\bar{\zeta}} p$. Thus, by (6.48) and since F_{nls} and J commute,

$$d\tilde{X}_P = -J(F_{nls} \mathcal{Q}|_{\Phi^{-1}} F_{nls}^{-1} + T_1 + T_2 + T_3). \quad (6.50)$$

We now evaluate $d\tilde{X}_P$ at the embedding $\check{\iota}(\varphi)$. In view of the definition (6.9) of \mathfrak{S}^P , (6.50) and (6.49) we get

$$\mathfrak{S}^P = \mathfrak{Q}_{\perp} + \mathfrak{R}^P, \quad \mathfrak{Q}_{\perp} := F_{nls}^{\perp} \begin{pmatrix} q_1 & q_2 \\ \bar{q}_2 & q_1 \end{pmatrix} F_{nls}^{-1}, \quad (6.51)$$

where $F_{nls}^{\perp}, F_{nls}^{-1}$ were introduced in (3.29) and

$$q_1 := (\partial_{\zeta} \partial_{\bar{\zeta}} p)|_{\zeta=\Phi^{-1}(\check{\iota})}, \quad q_2 := (\partial_{\zeta} \partial_{\zeta} p)|_{\zeta=\Phi^{-1}(\check{\iota})}, \quad \mathfrak{R}^P := \mathbb{I}_{\perp}((T_1 + T_2 + T_3) \circ \check{\iota}) \mathbb{I}_{\hookrightarrow}, \quad (6.52)$$

with \mathbb{I}_{\perp} denoting the projector and $\mathbb{I}_{\hookrightarrow}$ the standard inclusion introduced in (3.30). Above, in defining $\Phi^{-1}(\check{\iota})$ we have identified, by a slight abuse of terminology, the two components $(\theta(\varphi), y(\varphi))$ of $\check{\iota}(\varphi)$ with the Birkhoff coordinates $(z_j(\varphi))_{j \in S} := (\sqrt{\xi_j + y_j} e^{-i\theta_j})_{j \in S} \in \mathbb{C}^S$.

Lemma 6.3. (Estimates for q_1, q_2 , and \mathfrak{R}^P) For any $s_0 \leq s \leq s_* - 2s_0$ the following statements hold:

(i) The functions q_1, q_2 are in $H^s(\mathbb{T}^S, H^{\sigma}(\mathbb{T}_1))$, with q_1 real- and q_2 complex-valued. They satisfy

$$\|q_1\|_s, \|q_2\|_s \leq 1 + \|\iota\|_{s+s_0}, \quad \|q_1\|_s^{\text{lip}}, \|q_2\|_s^{\text{lip}} \leq 1 + \|\iota\|_{s+s_0}^{\text{lip}}. \quad (6.53)$$

(ii) The remainder \mathfrak{R}^P defined in (6.52) satisfies

$$|\mathfrak{R}^P \mathfrak{D}|_{s, \sigma-1} \leq 1 + \|\iota\|_{s+2s_0}, \quad |\mathfrak{R}^P \mathfrak{D}|_{s, \sigma-1}^{\text{lip}} \leq 1 + \|\iota\|_{s+2s_0}^{\text{lip}}. \quad (6.54)$$

Proof. (i) The bounds (6.53) follow by the definition (6.52) of q_1 and q_2 , the regularity assumption (1.6) of $\partial_{\bar{\zeta}} p$, and the tame estimates for the composition of maps of Lemma 2.11 in the case where $Y = \mathbb{C}$.

(ii) We now prove the first estimate in (6.54). According to Theorem 3.1, the maps A^{nls}, B^{nls} are real analytic and one smoothing: for any $\sigma' \in \mathbb{Z}_{\geq 2}$,

$$A^{nls} : H_r^{\sigma'-1} \rightarrow h_r^{\sigma'}, \quad B^{nls} : h_r^{\sigma'-1} \rightarrow H_r^{\sigma'}.$$

By Cauchy's theorem it then follows that

$$dA^{nls} : H_r^{\sigma'-1} \rightarrow \mathcal{L}(H_r^{\sigma'-1}, h_r^{\sigma'}), \quad dB^{nls} : h_r^{\sigma'-1} \rightarrow \mathcal{L}(h_r^{\sigma'-1}, H_r^{\sigma'}),$$

and $d^2 A^{nls} : H_r^{\sigma'-1} \rightarrow \mathcal{L}(H_r^{\sigma'-1} \times H_r^{\sigma'-1}, h_r^{\sigma'})$ are \mathcal{C}^{∞} -smooth maps. It follows that $T_1 \mathfrak{D}, T_2 \mathfrak{D}, T_3 \mathfrak{D}$ are maps from the phase space M^{σ} into $\mathcal{L}(h^{\sigma'})$ for $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$ which are as smooth as the second derivatives of p . We now apply the estimate (2.38) for the composite map $\varphi \mapsto \check{\iota}(\varphi) \mapsto T_j(\check{\iota}(\varphi))$, $j = 1, 2, 3$, which yields

$$|T_j \mathfrak{D} \circ \check{\iota}|_{s, \sigma-1} \leq 1 + \|\iota\|_{s+2s_0},$$

and hence (6.54) is proved. The second estimate in (6.54) is proved in a similar way. \square

Lemma 6.4. For any $s_0 \leq s \leq s_* - 2s_0$ and any torus embeddings $\check{\iota}^{(a)}(\varphi) := (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, satisfying (6.8), the following holds:

(i) The functions $\Delta_{12}q_1 := q_1(\check{\iota}^{(1)}) - q_1(\check{\iota}^{(2)})$ and $\Delta_{12}q_2 := q_2(\check{\iota}^{(1)}) - q_2(\check{\iota}^{(2)})$ satisfy the estimate

$$\|\Delta_{12}q_1\|_s, \|\Delta_{12}q_2\|_s \leq_s \|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}. \quad (6.55)$$

(ii) The difference of the remainders, $\Delta_{12}\mathfrak{R}^P := \mathfrak{R}^P(\check{\iota}^{(1)}) - \mathfrak{R}^P(\check{\iota}^{(2)})$, satisfies the estimate

$$\|\Delta_{12}\mathfrak{R}^P\mathcal{D}\|_{s,\sigma-1} \leq_s \|\Delta_{12}\iota\|_{s+2s_0} + \max_{s+2s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}.$$

Proof. Items (i) and (ii) follow from the definition (6.52), Lemma 2.11(ii) and Lemma 2.12(ii). \square

Analysis of \mathfrak{R}^ε . The operator \mathfrak{R}^ε , introduced in (6.7), is defined in terms of the operators $\mathfrak{R}_1^\varepsilon = R_1^\varepsilon \circ \check{\iota}$ and $\mathfrak{R}_2^\varepsilon = R_2^\varepsilon \circ \check{\iota}$, where according to (6.3), (6.5)

$$R_1^\varepsilon = \partial_y(\nabla_{\bar{z}}H_\varepsilon)Y_w + Y_w^t \partial_z \nabla_y H_\varepsilon + Y_w^t \partial_y(\nabla_y H_\varepsilon)Y_w, \quad R_2^\varepsilon = \partial_y(\nabla_{\bar{z}}H_\varepsilon)Y_w + Y_w^t \partial_z \nabla_y H_\varepsilon + Y_w^t \partial_y(\nabla_y H_\varepsilon)Y_w$$

and Y_w is defined in (5.18).

Lemma 6.5. (Estimate of \mathfrak{R}^ε) For any $s_0 \leq s \leq s_* - 2s_0$ one has

$$\|\mathfrak{R}^\varepsilon\mathcal{D}\|_{s,\sigma-1} \leq_s \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad \|\mathfrak{R}^\varepsilon\mathcal{D}\|_{s,\sigma-1}^{\text{lip}} \leq_s \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}^{\text{lip}}. \quad (6.56)$$

Proof. We now prove the first bound in (6.56). The various terms in $\mathfrak{R}_1^\varepsilon$ and $\mathfrak{R}_2^\varepsilon$ are estimated individually. Since these terms can be estimated in a similar way, let us concentrate on $(\partial_y \nabla_{\bar{z}} H_\varepsilon Y_w) \circ \check{\iota}$ only. Recall that by (5.18),

$$Y_w(\check{\iota}(\varphi)) := iB(\varphi)(\partial_\varphi \bar{z})^t(\varphi) : h_\perp^\sigma \rightarrow \mathbb{C}^S, \quad B(\varphi) := (\partial_\varphi \theta(\varphi))^{-t},$$

and, since $(\partial_\varphi \bar{z})^t = \sum_{m \in S^\perp} \partial_\varphi \bar{z}_m \pi_m$ where π_m is the projector defined in (6.25), we have

$$\partial_y(\nabla_{\bar{z}}H_\varepsilon)Y_w = i \sum_{m \in S^\perp} \sum_{j,k \in S} \partial_{y_j} \nabla_{\bar{z}} H_\varepsilon B_j^k \partial_{\varphi_k} \bar{z}_m \pi_m.$$

Clearly, recalling (2.12), one gets

$$|\partial_y(\nabla_{\bar{z}}H_\varepsilon)Y_w \langle\langle D \rangle\rangle \text{Id}_\perp|_{s,\sigma-1} \leq \sum_{m \in S^\perp} \sum_{j,k \in S} |\partial_{y_j} \nabla_{\bar{z}} H_\varepsilon B_j^k \partial_{\varphi_k} \bar{z}_m \langle\langle m \rangle\rangle \pi_m|_{s,\sigma-1}. \quad (6.57)$$

Arguing as in (6.27) one concludes that

$$\begin{aligned} |\partial_{y_j} \nabla_{\bar{z}} H_\varepsilon B_j^k \partial_{\varphi_k} \bar{z}_m \langle\langle m \rangle\rangle \pi_m|_{s,\sigma-1} &\leq_s \langle m \rangle^{-(\sigma-1)} \|\partial_{y_j} \nabla_{\bar{z}} H_\varepsilon B_j^k \partial_{\varphi_k} \bar{z}_m \langle\langle m \rangle\rangle\|_{s,\sigma-1} \\ &\leq_s \langle m \rangle^{-(2\sigma-2)} \|\partial_{y_j} \nabla_{\bar{z}} H_\varepsilon B_j^k \partial_{\varphi_k} (\langle m \rangle^\sigma \bar{z}_m)\|_{s,\sigma-1}. \end{aligned} \quad (6.58)$$

Since $B(\varphi) = (\partial_\varphi \theta(\varphi))^{-t}$ one has $\|B_j^k\|_s \leq_s 1 + \|\iota\|_{s+1}$. Furthermore, for any $m \in S^\perp$ and $k \in S$, $\|\partial_{\varphi_k} (\langle m \rangle^\sigma \bar{z}_m)\|_s \leq_s \|\iota\|_{s+1}$. Finally we analyze

$$\partial_y \nabla_{\bar{z}} H_\varepsilon = \partial_y \nabla_{\bar{z}} H^{nls} + \varepsilon \partial_y \nabla_{\bar{z}} P.$$

Note that $\partial_{y_j} \nabla_{\bar{z}} H^{nls} = (\partial_{y_j} \omega_n^{nls} z_n)_{n \in S^\perp}$. By (3.38), one has that

$$\sup_n \|\partial_{y_j} \omega_n^{nls}\|_s \leq_s 1 + \|\iota\|_{s+2s_0}, \quad \forall j \in S.$$

By the tame estimates for products of maps and the smallness assumption (6.8) one then concludes that

$$\|(\partial_{y_j} \omega_n^{nls} z_n)_{n \in S^\perp} B_j^k \partial_{\varphi_k} (\langle m \rangle^\sigma \bar{z}_m)\|_{s,\sigma} \leq_s \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad \forall j, k \in S, m \in S^\perp. \quad (6.59)$$

Next we consider $\partial_{y_j} \nabla_{\bar{z}} P$. By Proposition 3.3,

$$\|\partial_{y_j} \nabla_{\bar{z}} P \circ \check{\iota}\|_{s,\sigma} \leq_s 1 + \|\iota\|_{s+2s_0},$$

that, together with the smallness assumption (6.8), yields the estimate

$$\|(\partial_{y_j} \nabla_{\bar{z}} \varepsilon P \circ \check{\iota}) B_j^k \partial_{\varphi_k} (\langle m \rangle^\sigma \bar{z}_m)\|_{s,\sigma} \leq_s \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad \forall j, k \in S, m \in S^\perp. \quad (6.60)$$

Combining (6.57), (6.58), (6.59), (6.60) we get the claimed estimate for the term $\partial_y \nabla_{\bar{z}} H_\varepsilon Y_w$. The second estimate in (6.56) follows in a similar way. \square

Lemma 6.6. *For any $s_0 \leq s \leq s_* - 2s_0$ and any torus embeddings $\check{\iota}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, satisfying (6.8), the operator $\Delta_{12} \mathfrak{R}^\varepsilon := \mathfrak{R}^\varepsilon(\check{\iota}^{(1)}) - \mathfrak{R}^\varepsilon(\check{\iota}^{(2)})$ satisfies the estimate*

$$|\Delta_{12} \mathfrak{R}^\varepsilon \mathfrak{D}|_{s,\sigma-1} \leq_s \varepsilon \gamma^{-2} \|\Delta_{12} \iota\|_{s+2s_0} + \max_{s+2s_0}(\iota) \|\Delta_{12} \iota\|_{s_0}.$$

Proof. The claimed estimate can be deduced by arguing as in the proofs of Lemma 6.2 and Lemma 6.4. \square

We summarize the results obtained in this subsection as follows.

Proposition 6.1. *The Hamiltonian operator \mathfrak{L}_ω (cf (6.6)) can be decomposed as*

$$\mathfrak{L}_\omega = \omega \cdot \partial_\varphi \mathbb{I}_2 + J(D^2 \mathbb{I}_2 + \Omega^{nls} \mathbb{I}_2 + \varepsilon \mathfrak{Q}_\perp) + \mathfrak{R}_0, \quad \mathbb{I}_2 = \text{diag}(\text{Id}_\perp, \text{Id}_\perp), \quad (6.61)$$

where Ω^{nls} is defined in (6.13), \mathfrak{Q}_\perp in (6.51), and

$$\mathfrak{R}_0 := J\mathfrak{R}^\varepsilon + J\mathfrak{R}^{nls} + \varepsilon J\mathfrak{R}^P$$

with \mathfrak{R}^ε introduced in (6.7), \mathfrak{R}^{nls} in (6.12) and \mathfrak{R}^P in (6.52). The remainder \mathfrak{R}_0 is a linear Hamiltonian operator which is one smoothing and satisfies, for any $s_0 \leq s \leq s_* - 2s_0$,

$$|\mathfrak{R}_0 \mathfrak{D}|_{s,\sigma-1} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad |\mathfrak{R}_0 \mathfrak{D}|_{s,\sigma-1}^{\text{lip}} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}^{\text{lip}}. \quad (6.62)$$

Moreover if $\check{\iota}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, are two torus embeddings satisfying (6.8), then, $\Delta_{12} \mathfrak{R}_0 := \mathfrak{R}_0(\check{\iota}^{(1)}) - \mathfrak{R}_0(\check{\iota}^{(2)})$ satisfies the estimate

$$|\Delta_{12} \mathfrak{R}_0 \mathfrak{D}|_{s,\sigma-1} \leq_s \varepsilon \gamma^{-2} \|\Delta_{12} \iota\|_{s+2s_0} + \max_{s+2s_0}(\iota) \|\Delta_{12} \iota\|_{s_0}, \quad \forall s_0 \leq s \leq s_* - 2s_0. \quad (6.63)$$

Proof. Lemmata 6.1, 6.3, and 6.5 yield the estimate (6.62). Lemmata 6.2, 6.4, and 6.6 imply (6.63). \square

Note that the operator $\Omega^{nls} \mathbb{I}_2 : H^s(\mathbb{T}^S, h_\perp^{\sigma-1} \times h_\perp^{\sigma-1}) \rightarrow H^s(\mathbb{T}^S, h_\perp^{\sigma-1} \times h_\perp^{\sigma-1})$ is neither one smoothing nor small, whereas $\varepsilon \mathfrak{Q}_\perp$, which acts between the same spaces, is small but not one smoothing. In the subsequent sections we will introduce three linear symplectic transformations so that, when conjugated with these transformations, the operator $J(\Omega^{nls} \mathbb{I}_2 + \varepsilon \mathfrak{Q}_\perp)$ becomes a diagonal one with constant coefficients up to a one smoothing remainder. Note also that the leading part $J D^2 \mathbb{I}_2$ in \mathfrak{L}_ω is already a diagonal operator with constant coefficients.

6.2 First transformation

The purpose of the first transformation is to eliminate the off diagonal terms of \mathfrak{Q}_\perp in (6.61) up to a one smoothing remainder. The transformation is chosen to be the time 1-flow $\mathfrak{F}_1 : H^s(\mathbb{T}^S, h_\perp^{\sigma'} \times h_\perp^{\sigma'}) \rightarrow H^s(\mathbb{T}^S, h_\perp^{\sigma'} \times h_\perp^{\sigma'})$, $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$,

$$\mathfrak{F}_1 := \exp(-\varepsilon J F_{nls}^\perp A_1 F_{nls}^{-1}) = \mathbb{I}_2 - \varepsilon J F_{nls}^\perp A_1 F_{nls}^{-1} + \dots$$

of the linear vector field $-\varepsilon J F_{nls}^\perp A_1 F_{nls}^{-1}$ with A_1 of the form

$$A_1 = \begin{pmatrix} 0 & \langle\langle D \rangle\rangle^{-1} a_1 \langle\langle D \rangle\rangle^{-1} \\ \langle\langle D \rangle\rangle^{-1} \bar{a}_1 \langle\langle D \rangle\rangle^{-1} & 0 \end{pmatrix}, \quad \langle\langle D \rangle\rangle = (1 + D^2)^{\frac{1}{2}}, \quad D = \frac{1}{i} \partial_x. \quad (6.64)$$

By Lemma 3.2 the operator $JF_{nls}^\perp A_1 F_{nls}^{-1}$ is Hamiltonian and hence the flow Φ_1 symplectic (cf Definition 3.1). Note that for any $\varphi \in \mathbb{T}^S$, the operator $A_1(\varphi)$ is one smoothing (actually, it is even two smoothing) and the linear map $\Phi_1(\varphi)$ is invertible with inverse $\Phi_1^{-1}(\varphi) \equiv (\Phi_1(\varphi))^{-1}$ given by $\exp(\varepsilon JF_{nls}^\perp A_1(\varphi) F_{nls}^{-1})$. The form of the operator A_1 is chosen in such a way that the coefficients of the remainder R in (6.68) below involve only $\partial_x a_1$, and hence, by (6.69), $\partial_x q_2$.

The complex valued function $a_1 \equiv a_1(\varphi, x)$ will be chosen in such a way that the off-diagonal part in $\mathfrak{L}_1 := \Phi_1^{-1} \mathfrak{L}_\omega \Phi_1$ vanishes up to a one smoothing remainder. Note that the operators $\omega \cdot \partial_\varphi \mathbb{I}_2$, $JD^2 \mathbb{I}_2$, and $J\Omega^{nls} \mathbb{I}_2$ in $\mathfrak{L}_\omega = \omega \cdot \partial_\varphi \mathbb{I}_2 + JD^2 \mathbb{I}_2 + J\Omega^{nls} \mathbb{I}_2 + \varepsilon J\mathfrak{Q}_\perp + \mathfrak{R}_0$ are diagonal whereas (cf (6.51))

$$J\mathfrak{Q}_\perp = JF_{nls}^\perp \begin{pmatrix} q_1 & q_2 \\ \bar{q}_2 & q_1 \end{pmatrix} F_{nls}^{-1} \quad (6.65)$$

is not and \mathfrak{R}_0 is one smoothing. We then write $\mathfrak{L}_\omega \Phi_1$ in the form

$$\mathfrak{L}_\omega \Phi_1 = \Phi_1(\omega \cdot \partial_\varphi \mathbb{I}_2 + JD^2 \mathbb{I}_2 + J\Omega^{nls} \mathbb{I}_2) + \varepsilon J\mathfrak{Q}_\perp - \varepsilon [JD^2 \mathbb{I}_2, JF_{nls}^\perp A_1 F_{nls}^{-1}] + \mathfrak{R}^I \quad (6.66)$$

where $[\cdot, \cdot]$ denotes the commutator of operators and

$$\mathfrak{R}^I := (\omega \cdot \partial_\varphi)(\Phi_1 - \mathbb{I}_2) + [J\Omega^{nls} \mathbb{I}_2, \Phi_1 - \mathbb{I}_2] + \varepsilon J\mathfrak{Q}_\perp(\Phi_1 - \mathbb{I}_2) + \mathfrak{R}_0 \Phi_1 + [JD^2 \mathbb{I}_2, \Phi_1 - \mathbb{I}_2 + \varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1}]$$

collects operators which are one smoothing. We claim that the commutator $[JD^2 \mathbb{I}_2, JF_{nls}^\perp A_1 F_{nls}^{-1}]$ is a Hamiltonian operator of order zero. Indeed, since JD^2 commutes with J , F_{nls}^\perp and F_{nls}^{-1} , one has

$$[JD^2 \mathbb{I}_2, JF_{nls}^\perp A_1 F_{nls}^{-1}] = JF_{nls}^\perp [JD^2, A_1] F_{nls}^{-1}$$

and, recalling (6.64),

$$[JD^2, A_1] = i \begin{pmatrix} 0 & D^2 \langle\langle D \rangle\rangle^{-1} a_1 \langle\langle D \rangle\rangle^{-1} + \langle\langle D \rangle\rangle^{-1} a_1 \langle\langle D \rangle\rangle^{-1} D^2 \\ -D^2 \langle\langle D \rangle\rangle^{-1} \bar{a}_1 \langle\langle D \rangle\rangle^{-1} - \langle\langle D \rangle\rangle^{-1} \bar{a}_1 \langle\langle D \rangle\rangle^{-1} D^2 & 0 \end{pmatrix}.$$

Then, since $D^2 = \langle\langle D \rangle\rangle^2 - 1$, one has

$$[JD^2 \mathbb{I}_2, JF_{nls}^\perp A_1 F_{nls}^{-1}] = JF_{nls}^\perp \begin{pmatrix} 0 & 2ia_1 \\ -2i\bar{a}_1 & 0 \end{pmatrix} F_{nls}^{-1} - \mathfrak{R}^{II}, \quad \mathfrak{R}^{II} := F_{nls}^\perp \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix} F_{nls}^{-1} \quad (6.67)$$

where

$$R = 2\langle\langle D \rangle\rangle^{-1} a_1 \langle\langle D \rangle\rangle^{-1} - [a_1, \langle\langle D \rangle\rangle] \langle\langle D \rangle\rangle^{-1} - \langle\langle D \rangle\rangle^{-1} [\langle\langle D \rangle\rangle, a_1] = R^t. \quad (6.68)$$

Note that \mathfrak{R}^{II} is one smoothing, but its coefficients involve $\partial_x a_1 \in H^{\sigma-1}$. In view of (6.65), we choose

$$a_1 := -\frac{i}{2} q_2 \quad (6.69)$$

so that by (6.66), (6.67)

$$J\mathfrak{Q}_\perp - [JD^2 \mathbb{I}_2, JF_{nls}^\perp A_1 F_{nls}^{-1}] = JF_{nls}^\perp \begin{pmatrix} q_1 & 0 \\ 0 & q_1 \end{pmatrix} F_{nls}^{-1} + \mathfrak{R}^{II}. \quad (6.70)$$

Applying Φ_1^{-1} to the identity (6.66) and using (6.70) one gets

$$\mathfrak{L}_1 = \Phi_1^{-1} \mathfrak{L}_\omega \Phi_1 = \omega \cdot \partial_\varphi \mathbb{I}_2 + J(D^2 \mathbb{I}_2 + \Omega^{nls} \mathbb{I}_2 + \varepsilon F_{nls}^\perp q_1 F_{nls}^{-1}) + \mathfrak{R}_1, \quad (6.71)$$

where \mathfrak{R}_1 is the one smoothing operator

$$\mathfrak{R}_1 := \varepsilon(\Phi_1^{-1} - \mathbb{I}_2) JF_{nls}^\perp q_1 F_{nls}^{-1} + \Phi_1^{-1}(\mathfrak{R}^I + \varepsilon \mathfrak{R}^{II}). \quad (6.72)$$

Since Φ_1 is symplectic and \mathfrak{L}_ω is a linear Hamiltonian operator, Lemma 3.1 implies that also \mathfrak{L}_1 is Hamiltonian. Furthermore, the 0th order term of \mathfrak{L}_1 is given by $J(\Omega^{nls} \mathbb{I}_2 + \varepsilon F_{nls}^\perp q_1 F_{nls}^{-1})$ where Ω^{nls} is the φ -dependent diagonal operator defined in (6.13). As pointed out above, the operator \mathfrak{R}_1 is one smoothing, but its coefficients involve $\partial_x a_1$, i.e., they are maps with values in $h^{\sigma-1}$.

Lemma 6.7. (Estimates of A_1 , Φ_1 and \mathfrak{R}_1) For any $s_0 \leq s \leq s_* - 2s_0$ the following statements hold:

(i) For any $\varphi \in \mathbb{T}^S$ and $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2, \sigma - 3\}$, $A_1(\varphi) \in \mathcal{L}(H^{\sigma'-1}, H^{\sigma'})$ and

$$|JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s, \sigma'}, |JF_{nls}^\perp A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma'} \leq_s 1 + \|\iota\|_{s+s_0} \quad (6.73)$$

$$|JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s, \sigma'}^{\text{lip}}, |JF_{nls}^\perp A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma'}^{\text{lip}} \leq_s 1 + \|\iota\|_{s+s_0}^{\text{lip}}. \quad (6.74)$$

(ii) For any $\varphi \in \mathbb{T}^S$ and $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$, $\Phi_1(\varphi) \in \mathcal{L}(h_\perp^{\sigma'})$ and

$$\begin{aligned} |\Phi_1^{\pm 1} - \mathbb{I}_2|_{s, \sigma'}, |(\Phi_1^{\pm 1} - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma'} &\leq_s \varepsilon (1 + \|\iota\|_{s+s_0}) \\ |\Phi_1^{\pm 1} - \mathbb{I}_2|_{s, \sigma'}^{\text{lip}}, |(\Phi_1^{\pm 1} - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma'}^{\text{lip}} &\leq_s \varepsilon (1 + \|\iota\|_{s+s_0}^{\text{lip}}). \end{aligned}$$

(iii) \mathfrak{R}_1 is a linear Hamiltonian operator with $\mathfrak{R}_1(\varphi) \in \mathcal{L}(h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}, h_\perp^{\sigma-1} \times h_\perp^{\sigma-1})$ for any $\varphi \in \mathbb{T}^S$, and

$$|\mathfrak{R}_1 \mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad |\mathfrak{R}_1 \mathfrak{D}|_{s, \sigma-1}^{\text{lip}} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}^{\text{lip}}. \quad (6.75)$$

Proof. Since the proofs of the stated inequalities are similar for the range of values of σ' considered, we only treat the case $\sigma' = \sigma$.

(i) We begin by proving the estimate (6.73). In view of (2.26) and (6.64) we can write

$$JF_{nls}^\perp A_1 F_{nls}^{-1} = J \mathfrak{D}^{-1} F_{nls}^\perp \begin{pmatrix} 0 & a_1 \\ \bar{a}_1 & 0 \end{pmatrix} F_{nls}^{-1} \mathfrak{D}^{-1},$$

Since $|\mathfrak{D}^{-1}|_{s, \sigma} = \|\mathfrak{D}^{-1}\|_{\mathcal{L}(h^\sigma)} \leq 1$ one has $|JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s, \sigma} \leq |JF_{nls}^\perp A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma}$ and

$$|JF_{nls}^\perp A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma} \stackrel{\text{Lemma 2.4}}{\leq_s} \|a_1\|_s \stackrel{(6.69)}{\leq_s} \|q_2\|_s \stackrel{(6.53)}{\leq_s} 1 + \|\iota\|_{s+s_0}.$$

The estimate (6.74) is proved in a similar way.

(ii) By the smallness condition (6.8), the assumption of Lemma 2.10 is satisfied for the operator $\varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1}$ with ε sufficiently small, hence the claimed statement follows from this lemma and item (i).

(iii) We begin proving the first estimate in (6.75). The terms in $\mathfrak{R}_1 \mathfrak{D}$, with \mathfrak{R}_1 defined in (6.72) are estimated individually. The statement concerning $\mathfrak{R}_1(\varphi)$ can be verified in a straightforward way. Furthermore, the following estimates hold:

$$\begin{aligned} |\Phi_1^{\pm 1}|_{s, \sigma-1} &\stackrel{(ii)}{\leq_s} 1 + \varepsilon \|\iota\|_{s+s_0}, \quad |(\Phi_1^{\pm 1} - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma-1} \stackrel{(ii)}{\leq_s} \varepsilon (1 + \|\iota\|_{s+s_0}), \\ |\mathfrak{D}^{-1} JF_{nls}^\perp q_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma-1} &\leq_s |F_{nls}^\perp q_1 F_{nls}^{-1}|_{s, \sigma-2} \stackrel{\text{Lemma 2.4}}{\leq_s} \|q_1\|_s \stackrel{(6.53)}{\leq_s} 1 + \|\iota\|_{s+s_0}, \\ |(\omega \cdot \partial_\varphi)(\Phi_1 - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma-1} &\stackrel{\text{Def of } |\cdot|_{s, \sigma-1}}{\leq_s} |(\Phi_1 - \mathbb{I}_2) \mathfrak{D}|_{s+1, \sigma-1} \stackrel{(ii)}{\leq_s} \varepsilon (1 + \|\iota\|_{s+s_0+1}), \\ |J\Omega^{nls} \mathbb{I}_2|_{s, \sigma-1} &\stackrel{(6.14)}{\leq_s} 1 + \|\iota\|_{s+2s_0}, \quad |\mathfrak{V}_\perp|_{s, \sigma-1} \stackrel{\text{Lemma 2.4}}{\leq_s} \|q_1\|_s + \|q_2\|_s \stackrel{(6.53)}{\leq_s} 1 + \|\iota\|_{s+s_0}, \\ |\mathfrak{R}_0 \mathfrak{D}|_{s, \sigma-1} &\stackrel{(6.62)}{\leq_s} \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad |[q_2, \langle D \rangle]|_{s, \sigma-1} \stackrel{\text{Lemma 2.4}}{\leq_s} \|q_2\|_s \stackrel{(6.53)}{\leq_s} 1 + \|\iota\|_{s+s_0}, \\ \left| J \mathfrak{D}^2 \sum_{n \geq 2} \frac{1}{n!} (-\varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1})^n \mathfrak{D} \right|_{s, \sigma-1} &\stackrel{\text{Lemma 2.10}}{\leq_s} \varepsilon^2 |JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s, \sigma} |JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s_0, \sigma} \stackrel{(i)}{\leq_s} \varepsilon^2 (1 + \|\iota\|_{s+s_0}), \\ \left| \sum_{n \geq 2} \frac{1}{n!} (-\varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1})^n J \mathfrak{D}^3 \right|_{s, \sigma-1} &\stackrel{\text{Lemma 2.10}}{\leq_s} \varepsilon^2 |JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s, \sigma} |JF_{nls}^\perp A_1 F_{nls}^{-1}|_{s_0, \sigma} \stackrel{(i)}{\leq_s} \varepsilon^2 (1 + \|\iota\|_{s+s_0}). \end{aligned}$$

These estimates together with the tame estimate (2.21) for the composition of operator valued maps, allow to bound each term in $\mathfrak{R}_1 \mathfrak{D}$ by $\varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}$. The second estimate in (6.75) is proved in a similar way. \square

Lemma 6.8. For any $s_0 \leq s \leq s_* - 2s_0$ and any torus embeddings $\check{\iota}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, the following holds:

(i) For any $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2, \sigma - 3\}$, the operator $\Delta_{12}A_1 := A_1(\check{\iota}^{(1)}) - A_1(\check{\iota}^{(2)})$ satisfies

$$|JF_{nls}^\perp \Delta_{12}A_1 F_{nls}^{-1}|_{s, \sigma'}, \quad |JF_{nls}^\perp \Delta_{12}A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma'} \leq_s \|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}.$$

(ii) For any $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$, the operators $\Delta_{12}\Phi_1 := \Phi_1(\check{\iota}^{(1)}) - \Phi_1(\check{\iota}^{(2)})$ and $\Delta_{12}\Phi_1^{-1} := \Phi_1^{-1}(\check{\iota}^{(1)}) - \Phi_1^{-1}(\check{\iota}^{(2)})$ satisfy the estimate

$$|\Delta_{12}\Phi_1^{\pm 1}|_{s, \sigma'}, \quad |\Delta_{12}\Phi_1^{\pm 1} \mathfrak{D}|_{s, \sigma'} \leq_s \varepsilon (\|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{2s_0}).$$

(iii) The operator $\Delta_{12}\mathfrak{R}_1 := \mathfrak{R}_1(\check{\iota}^{(1)}) - \mathfrak{R}_1(\check{\iota}^{(2)})$ satisfies the estimate

$$|\Delta_{12}\mathfrak{R}_1 \mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon \gamma^{-2} \|\Delta_{12}\ell\|_{s+2s_0} + \max_{s+2s_0}(\ell) \|\Delta_{12}\ell\|_{3s_0}.$$

Proof. (i) Since the proofs of the stated inequalities are similar for the range of the values of σ' considered, we only treat the case $\sigma' = \sigma$. By the definition (6.64) of A_1 one has

$$JF_{nls}^\perp \Delta_{12}A_1 F_{nls}^{-1} = J\mathfrak{D}^{-1} F_{nls}^\perp \begin{pmatrix} 0 & \Delta_{12}a_1 \\ \Delta_{12}\bar{a}_1 & 0 \end{pmatrix} F_{nls}^{-1} \mathfrak{D}^{-1}.$$

Since $|\mathfrak{D}^{-1}|_{s, \sigma} = \|\mathfrak{D}^{-1}\|_{\mathcal{L}(h^\sigma)} \leq 1$ it then follows that

$$|JF_{nls}^\perp \Delta_{12}A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma} \stackrel{\text{Lemma 2.4}}{\leq_s} \|\Delta_{12}a_1\|_s \stackrel{(6.69)}{\leq_s} \|\Delta_{12}q_2\|_s \stackrel{(6.55)}{\leq_s} \|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}$$

and $|JF_{nls}^\perp \Delta_{12}A_1 F_{nls}^{-1}|_{s, \sigma} \leq |JF_{nls}^\perp \Delta_{12}A_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma}$, establishing the claimed estimates in the case $\sigma' = \sigma$.

(ii) The claimed estimate follows by Lemma 2.10 (v) and item (i).

(iii) The terms in $\Delta_{12}\mathfrak{R}_1 \mathfrak{D}$, with \mathfrak{R}_1 defined in (6.72), are estimated individually. The following estimates hold:

$$\begin{aligned} |\Delta_{12}\Phi_1^{\pm 1}|_{s, \sigma-1}, \quad & |\Delta_{12}\Phi_1^{\pm 1} \mathfrak{D}|_{s, \sigma'} \stackrel{(ii)}{\leq_s} \varepsilon (\|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{2s_0}), \\ |\mathfrak{D}^{-1} JF_{nls}^\perp \Delta_{12}q_1 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma-1} & \leq_s |F_{nls}^\perp \Delta_{12}q_1 F_{nls}^{-1}|_{s, \sigma-2} \stackrel{\text{Lemma 2.4}}{\leq_s} \|\Delta_{12}q_1\|_s \stackrel{(6.55)}{\leq_s} \|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}, \\ |(\omega \cdot \partial_\varphi)(\Delta_{12}\Phi_1) \mathfrak{D}|_{s, \sigma-1} & \stackrel{\text{Def of } |\cdot|_{s, \sigma-1}}{\leq_s} |\Delta_{12}\Phi_1 \mathfrak{D}|_{s+1, \sigma-1} \stackrel{(ii)}{\leq_s} \varepsilon (\|\Delta_{12}\ell\|_{s+s_0+1} + \max_{s+s_0+1}(\ell) \|\Delta_{12}\ell\|_{2s_0}), \\ |J\Delta_{12}\Omega^{nls} \mathbb{I}_2|_{s, \sigma-1} & \stackrel{\text{Lemma 6.2 (ii)}}{\leq_s} \|\Delta_{12}\ell\|_s + \max_{s+2s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}, \\ |\Delta_{12}\Omega_\perp|_{s, \sigma-1} & \stackrel{\text{Lemma 2.4}}{\leq_s} \|\Delta_{12}q_1\|_s + \|\Delta_{12}q_2\|_s \stackrel{(6.55)}{\leq_s} \|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}, \\ |\Delta_{12}\mathfrak{R}_0 \mathfrak{D}|_{s, \sigma-1} & \stackrel{(6.63)}{\leq_s} \varepsilon \gamma^{-2} \|\Delta_{12}\ell\|_{s+2s_0} + \max_{s+2s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}, \\ [|\Delta_{12}q_2, \langle D \rangle|]_{s, \sigma-1} & \stackrel{\text{Lemma 2.4}}{\leq_s} \|\Delta_{12}q_2\|_s \stackrel{(6.55)}{\leq_s} \|\Delta_{12}\ell\|_{s+s_0} + \max_{s+s_0}(\ell) \|\Delta_{12}\ell\|_{s_0}. \end{aligned}$$

Next we prove that

$$S_1, S_2 \leq_s \varepsilon^2 (\|\Delta_{12}\ell\|_{s+s_0} + \|\ell\|_{s+s_0} \|\Delta_{12}\ell\|_{s_0}) \quad (6.76)$$

where S_1 and S_2 are defined as follows

$$S_1 := \left| J\mathfrak{D}^2 \sum_{n \geq 2} \frac{1}{n!} \Delta_{12}(-\varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1})^n \mathfrak{D} \right|_{s, \sigma-1}, \quad S_2 := \left| \sum_{n \geq 2} \frac{1}{n!} \Delta_{12}(-\varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1})^n J\mathfrak{D}^3 \right|_{s, \sigma-1}.$$

Since the estimates for S_1 and S_2 can be proved in a similar fashion, we consider S_1 only. Let

$$B(\check{\iota}^{(a)}) := JF_{nls}^\perp A_1(\check{\iota}^{(a)}) F_{nls}^{-1}, \quad a = 1, 2, \quad \Delta_{12}B^n := B(\check{\iota}^{(1)})^n - B(\check{\iota}^{(2)})^n.$$

We then write $\Delta_{12}B^n$ with $n \geq 2$ as a telescoping sum,

$$\Delta_{12}B^n = (\Delta_{12}B)B(\tilde{\iota}^{(1)})^{n-1} + B(\tilde{\iota}^{(2)})(\Delta_{12}B)B(\tilde{\iota}^{(1)})^{n-2} + \cdots + B(\tilde{\iota}^{(2)})^{n-1}(\Delta_{12}B). \quad (6.77)$$

Each term $J\mathfrak{D}^2B(\tilde{\iota}^{(2)})^k(\Delta_{12}B)B(\tilde{\iota}^{(1)})^{n-k-1}\mathfrak{D}$, $0 \leq k \leq n-1$, is estimated individually. It turns out to be convenient to write the operator $B(\tilde{\iota}^{(a)})$ in the form

$$B(\tilde{\iota}^{(a)}) = \mathfrak{D}^{-1}E(\tilde{\iota}^{(a)})\mathfrak{D}^{-1}, \quad E(\tilde{\iota}^{(a)}) := JF_{nls}^\perp \begin{pmatrix} 0 & a_1(\tilde{\iota}^{(a)}) \\ \bar{a}_1(\tilde{\iota}^{(a)}) & 0 \end{pmatrix} F_{nls}^{-1},$$

so that $\Delta_{12}B = \mathfrak{D}^{-1}\Delta_{12}E\mathfrak{D}^{-1}$. Thus

$$J\mathfrak{D}(\Delta_{12}B)B(\tilde{\iota}^{(1)})^{n-1}\mathfrak{D} = J(\mathfrak{D}(\Delta_{12}E)\mathfrak{D}^{-1})(\mathfrak{D}^{-1}E(\tilde{\iota}^{(1)})\mathfrak{D}^{-1})^{n-2}(\mathfrak{D}^{-1}E(\tilde{\iota}^{(1)}))$$

and for any $1 \leq k \leq n-2$, $J\mathfrak{D}^2B(\tilde{\iota}^{(2)})^k(\Delta_{12}B)B(\tilde{\iota}^{(1)})^{n-k-1}\mathfrak{D}$ equals

$$J(\mathfrak{D}E(\tilde{\iota}^{(2)})\mathfrak{D}^{-1})(\mathfrak{D}^{-1}E(\tilde{\iota}^{(2)})\mathfrak{D}^{-1})^{k-1}(\mathfrak{D}^{-1}\Delta_{12}E\mathfrak{D}^{-1})(\mathfrak{D}^{-1}E(\tilde{\iota}^{(1)})\mathfrak{D}^{-1})^{n-k-2}(\mathfrak{D}^{-1}E(\tilde{\iota}^{(1)}))$$

whereas for $k = n-1$ one has

$$J\mathfrak{D}^2B(\tilde{\iota}^{(2)})^{n-1}(\Delta_{12}B)\mathfrak{D} = J(\mathfrak{D}E(\tilde{\iota}^{(2)})\mathfrak{D}^{-1})(\mathfrak{D}^{-1}E(\tilde{\iota}^{(2)})\mathfrak{D}^{-1})^{n-2}(\mathfrak{D}^{-1}\Delta_{12}E).$$

Note that

$$\begin{aligned} |\mathfrak{D}E(\tilde{\iota}^{(2)})\mathfrak{D}^{-1}|_{s,\sigma-1} &\stackrel{\text{Lemma 2.4}}{\leq_s} \|\mathfrak{D}\|_{\mathcal{L}(h_\perp^\sigma, h_\perp^{\sigma-1})} \|a_1(\tilde{\iota}^{(2)})\|_s \|\mathfrak{D}^{-1}\|_{\mathcal{L}(h_\perp^{\sigma-1}, h_\perp^\sigma)} \stackrel{(6.69), (6.53)}{\leq_s} 1 + \max_{s+s_0}(\iota), \\ |\mathfrak{D}^{-1}E(\tilde{\iota}^{(1)})|_{s,\sigma-1} &\stackrel{\text{Lemma 2.4}}{\leq_s} \|\mathfrak{D}^{-1}\|_{\mathcal{L}(h_\perp^{\sigma-1}, h_\perp^\sigma)} \|a_1(\tilde{\iota}^{(1)})\|_s \stackrel{(6.69), (6.53)}{\leq_s} 1 + \max_{s+s_0}(\iota), \end{aligned}$$

and that by the same arguments, $|\mathfrak{D}^{-1}E(\tilde{\iota}^{(a)})\mathfrak{D}^{-1}|_{s,\sigma-1}$, $a = 1, 2$, is also bounded by $1 + \max_{s+s_0}(\iota)$. Furthermore, again by Lemma 2.4, $|\mathfrak{D}\Delta_{12}E\mathfrak{D}^{-1}|_{s,\sigma-1}$ can be estimated by

$$\|\mathfrak{D}\|_{\mathcal{L}(h_\perp^\sigma, h_\perp^{\sigma-1})} \|\Delta_{12}a_1\|_s \|\mathfrak{D}^{-1}\|_{\mathcal{L}(h_\perp^{\sigma-1}, h_\perp^\sigma)} \stackrel{(6.69), (6.55)}{\leq_s} \|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}$$

and the same estimates hold for $|\mathfrak{D}^{-1}\Delta_{12}E\mathfrak{D}^{-1}|_{s,\sigma-1}$ and $|\mathfrak{D}^{-1}\Delta_{12}E|_{s,\sigma-1}$. By the tame estimate for the composition of operator valued maps (2.21) and the smallness condition (6.8) it then follows that for any $0 \leq k \leq n-1$,

$$|J\mathfrak{D}^2B(\tilde{\iota}^{(2)})^k(\Delta_{12}B)B(\tilde{\iota}^{(1)})^{n-k-1}\mathfrak{D}|_{s,\sigma-1} \leq C(s)^{n-1} (\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}).$$

In view of (6.77) this yields

$$|J\mathfrak{D}^2\Delta_{12}(JF_{nls}^\perp A_1 F_{nls}^{-1})^n \mathfrak{D}|_{s,\sigma-1} \leq nC(s)^{n-1} (\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0})$$

and leads to the claimed estimate (6.76),

$$\begin{aligned} S_1 &= \left| J\mathfrak{D}^2 \sum_{n \geq 2} \frac{1}{n!} \Delta_{12}(-\varepsilon JF_{nls}^\perp A_1 F_{nls}^{-1})^n \mathfrak{D} \right|_{s,\sigma-1} \leq \sum_{n \geq 2} \frac{nC(s)^{n-1} \varepsilon^n}{n!} (\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}) \\ &\leq_s \varepsilon^2 (\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}). \end{aligned}$$

The above estimates together with the estimates given in Lemma 6.7, the tame estimate (2.21) for the composition of operator valued maps, and the smallness assumption (6.8) allow to bound the $|\cdot|_{s,\sigma-1}$ norm of each term in $\Delta_{12}(\mathfrak{R}_1\mathfrak{D})$ by $\varepsilon\gamma^{-2}\|\Delta_{12}\iota\|_{s+2s_0} + \max_{s+2s_0}(\iota)\|\Delta_{12}\iota\|_{3s_0}$. Let us indicate how this bound is obtained by considering one specific term. Note that by the definition of \mathfrak{R}^I and the one of \mathfrak{R}_1 , $\mathfrak{R}^I\mathfrak{D}$ contains the operator $\mathfrak{F}_1^{-1}\mathfrak{R}_0\mathfrak{F}_1\mathfrak{D}$, which we write as $\mathfrak{F}_1^{-1}(\mathfrak{R}_0\mathfrak{D})(\mathfrak{D}^{-1}\mathfrak{F}_1\mathfrak{D})$. We then develop $\Delta_{12}(\mathfrak{F}_1^{-1}(\mathfrak{R}_0\mathfrak{D})(\mathfrak{D}^{-1}\mathfrak{F}_1\mathfrak{D}))$ in a telescoping sum, which among others contains the term $\mathfrak{F}_1^{-1}(\tilde{\iota}^{(2)})\Delta_{12}(\mathfrak{R}_0\mathfrak{D})(\mathfrak{D}^{-1}\mathfrak{F}_1(\tilde{\iota}^{(1)}\mathfrak{D}))$. By the tame

estimate (2.21) for the composition of operator valued maps, one then obtains a bound, given by a sum, which contains among other terms the following one

$$|\mathfrak{F}_1^{-1}(i^{(2)})|_{s,\sigma-1}|\Delta_{12}(\mathfrak{R}_0\mathfrak{D})|_{s_0,\sigma-1}|\mathfrak{D}^{-1}\mathfrak{F}_1(i^{(1)})\mathfrak{D}|_{s_0,\sigma-1}.$$

Then the estimate (6.63) for $|\Delta_{12}\mathfrak{R}_0\mathfrak{D}|_{s,\sigma-1}$, applied for s given by s_0 , yields

$$|\Delta_{12}\mathfrak{R}_0\mathfrak{D}|_{s_0,\sigma-1} \leq_s \varepsilon\gamma^{-2}\|\Delta_{12}\iota\|_{3s_0} + \max_{3s_0}(\iota)\|\Delta_{12}\iota\|_{s_0}.$$

Furthermore, by Lemma 6.7,

$$|\mathfrak{F}_1^{-1}(i^{(2)}) - \mathbb{I}_2|_{s,\sigma-1} \leq_s \varepsilon(1 + \|\iota^{(2)}\|_{s+s_0}) \quad \text{and} \quad |\mathfrak{D}^{-1}\mathfrak{F}_1(i^{(1)})\mathfrak{D}|_{s_0,\sigma-1} \leq_s 1.$$

Combining the above estimates, one concludes that

$$|\mathfrak{F}_1^{-1}(i^{(2)})|_{s,\sigma-1}|\Delta_{12}(\mathfrak{R}_0\mathfrak{D})|_{s_0,\sigma-1}|\mathfrak{D}^{-1}\mathfrak{F}_1(i^{(1)})\mathfrak{D}|_{s_0,\sigma-1} \leq_s \varepsilon\gamma^{-2}\|\Delta_{12}\iota\|_{s+2s_0} + \max_{s+2s_0}(\iota)\|\Delta_{12}\iota\|_{3s_0}.$$

All other terms are estimated in a similar fashion. \square

6.3 Second transformation

The purpose of the second transformation is to eliminate the space dependence of q_1 , appearing in the expression (6.71) for the operator \mathfrak{L}_1 , up to a one smoothing remainder. The transformation is chosen to be the time 1-flow $\mathfrak{F}_2 : H^s(\mathbb{T}^S, h_{\perp}^{\sigma'} \times h_{\perp}^{\sigma'}) \rightarrow H^s(\mathbb{T}^S, h_{\perp}^{\sigma'} \times h_{\perp}^{\sigma'})$, $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$,

$$\mathfrak{F}_2 := \exp(-\varepsilon JF_{nls}^{\perp}A_2F_{nls}^{-1}) = \mathbb{I}_2 - \varepsilon JF_{nls}^{\perp}A_2F_{nls}^{-1} + \dots$$

of the linear vector field $-\varepsilon JF_{nls}^{\perp}A_2F_{nls}^{-1}$ where

$$A_2 := \begin{pmatrix} D\langle\langle D \rangle\rangle^{-2}a_2 + a_2D\langle\langle D \rangle\rangle^{-2} & 0 \\ 0 & \overline{D}\langle\langle D \rangle\rangle^{-2}a_2 + a_2\overline{D}\langle\langle D \rangle\rangle^{-2} \end{pmatrix}. \quad (6.78)$$

Since we will chose $a_2(\varphi, x)$ to be real valued the operator $JF_{nls}^{\perp}A_2F_{nls}^{-1}$ is Hamiltonian (cf Lemma 3.2) and hence the flow \mathfrak{F}_2 symplectic. Furthermore we record that A_2 is one smoothing. We will choose $a_2 \equiv a_2(\varphi, x)$ in such a way that $\mathfrak{L}_2 := \mathfrak{F}_2^{-1}\mathfrak{L}_1\mathfrak{F}_2$ is x -independent up to a one smoothing remainder. To this end we write

$$\mathfrak{L}_1\mathfrak{F}_2 = \mathfrak{F}_2(\omega \cdot \partial_{\varphi} \mathbb{I}_2 + JD^2 \mathbb{I}_2 + J\Omega^{nls} \mathbb{I}_2) + \varepsilon JF_{nls}^{\perp}q_1 F_{nls}^{-1} - \varepsilon[JD^2 \mathbb{I}_2, JF_{nls}^{\perp}A_2F_{nls}^{-1}] + \mathfrak{R}^I \quad (6.79)$$

where

$$\mathfrak{R}^I := (\omega \cdot \partial_{\varphi})(\mathfrak{F}_2 - \mathbb{I}_2) + [J\Omega^{nls} \mathbb{I}_2, \mathfrak{F}_2 - \mathbb{I}_2] + \varepsilon JF_{nls}^{\perp}q_1 F_{nls}^{-1}(\mathfrak{F}_2 - \mathbb{I}_2) + \mathfrak{R}_1\mathfrak{F}_2 + [JD^2 \mathbb{I}_2, \mathfrak{F}_2 - \mathbb{I}_2 + \varepsilon JF_{nls}^{\perp}A_2F_{nls}^{-1}]$$

collects terms which are one smoothing. We now compute the commutator $[JD^2 \mathbb{I}_2, JF_{nls}^{\perp}A_2F_{nls}^{-1}]$.

Lemma 6.9. *The Hamiltonian operator $[JD^2 \mathbb{I}_2, JF_{nls}^{\perp}A_2F_{nls}^{-1}]$ can be expanded as*

$$[JD^2 \mathbb{I}_2, JF_{nls}^{\perp}A_2F_{nls}^{-1}] = 4JF_{nls}^{\perp}(\partial_x a_2)F_{nls}^{-1} - \mathfrak{R}^{II} \quad (6.80)$$

where \mathfrak{R}^{II} is the one smoothing operator given by

$$\mathfrak{R}^{II} := F_{nls}^{\perp} \text{diag}(R^{II}, \overline{R}^{II}) F_{nls}^{-1}, \quad (6.81)$$

$$R^{II} := (D\langle\langle D \rangle\rangle^{-2}(\partial_x^2 a_2) - (\partial_x^2 a_2)D\langle\langle D \rangle\rangle^{-2} + 2i\langle\langle D \rangle\rangle^{-2}(\partial_x a_2) + 2i(\partial_x a_2)\langle\langle D \rangle\rangle^{-2}). \quad (6.82)$$

Proof. Since JD^2 commutes with J , F_{nls}^\perp and F_{nls}^{-1} , we have

$$[JD^2 \mathbb{I}_2, JF_{nls}^\perp A_2 F_{nls}^{-1}] = JF_{nls}^\perp [JD^2, A_2] F_{nls}^{-1}.$$

By the definition of J in (6.1) and of A_2 in (6.78) the operator $[JD^2, A_2]$ is diagonal and with first component given by

$$[iD^2, (\langle\langle D \rangle\rangle^{-2} D a_2 + a_2 D \langle\langle D \rangle\rangle^{-2})] = T_1 + T_2$$

where

$$T_1 = iD^2 \langle\langle D \rangle\rangle^{-2} D a_2 - i \langle\langle D \rangle\rangle^{-2} D a_2 D^2 \quad \text{and} \quad T_2 = iD^2 a_2 D \langle\langle D \rangle\rangle^{-2} - i a_2 D \langle\langle D \rangle\rangle^{-2} D^2.$$

Use that $iD = \partial_x$ and $D^2 \langle\langle D \rangle\rangle^{-2} = 1 - \langle\langle D \rangle\rangle^{-2}$ to conclude that

$$\begin{aligned} T_1 &= iD^2 \langle\langle D \rangle\rangle^{-2} D a_2 - i \langle\langle D \rangle\rangle^{-2} D^2 a_2 D + \langle\langle D \rangle\rangle^{-2} D (\partial_x a_2) D \\ &= 2 \langle\langle D \rangle\rangle^{-2} D^2 (\partial_x a_2) + i \langle\langle D \rangle\rangle^{-2} D (\partial_x^2 a_2) \\ &= 2 (\partial_x a_2) - 2 \langle\langle D \rangle\rangle^{-2} (\partial_x a_2) + i \langle\langle D \rangle\rangle^{-2} D (\partial_x^2 a_2). \end{aligned}$$

Similarly one has $T_2 = 2 (\partial_x a_2) - 2 (\partial_x a_2) \langle\langle D \rangle\rangle^{-2} - i (\partial_x^2 a_2) \langle\langle D \rangle\rangle^{-2} D$. Thus

$$i(T_1 + T_2) = 4i (\partial_x a_2) - (2i \langle\langle D \rangle\rangle^{-2} (\partial_x a_2) + \langle\langle D \rangle\rangle^{-2} D (\partial_x^2 a_2) + 2i (\partial_x a_2) \langle\langle D \rangle\rangle^{-2} - (\partial_x^2 a_2) D \langle\langle D \rangle\rangle^{-2})$$

proving the lemma. \square

We choose a_2 so that $q_1 - 4\partial_x a_2$ is independent of x , i.e., $4\partial_x a_2 = q_1 - \text{av}(q_1)$ or

$$a_2 := \frac{1}{4} \partial_x^{-1} (q_1 - \text{av}(q_1)), \quad \text{av}(q_1) := \int_0^1 q_1 dx, \quad (6.83)$$

where the operator $\partial_x^{-1} : H^{\sigma'} \rightarrow H^{\sigma'+1}$ is defined by setting

$$\partial_x^{-1}(1) = 0, \quad \partial_x^{-1}(e^{i2\pi j x}) = \frac{1}{i2\pi j} e^{i2\pi j x} \quad \forall j \in \mathbb{Z} \setminus \{0\}.$$

Note that by (6.83) and Lemma 6.3, $a_2(\varphi, \cdot) \in H^{\sigma+1}$ for any $\varphi \in \mathbb{T}^S$. The remainder R^{II} , defined in (6.82), is given by

$$\frac{1}{4} \left(D \langle\langle D \rangle\rangle^{-2} (\partial_x q_1) - (\partial_x q_1) D \langle\langle D \rangle\rangle^{-2} + 2i \langle\langle D \rangle\rangle^{-2} (q_1 - \text{av}(q_1)) + 2i (q_1 - \text{av}(q_1)) \langle\langle D \rangle\rangle^{-2} \right) \quad (6.84)$$

and combining (6.80), (6.83) one has

$$JF_{nls}^\perp q_1 F_{nls}^{-1} - [JD^2 \mathbb{I}_2, JF_{nls}^\perp A_2 F_{nls}^{-1}] = JF_{nls}^\perp \text{av}(q_1) F_{nls}^{-1} + \mathfrak{R}^{II}.$$

By applying the inverse $\mathfrak{F}_2^{-1} = \exp(\varepsilon JF_{nls}^\perp A_2 F_{nls}^{-1})$ to (6.79), we get

$$\mathfrak{L}_2 = \mathfrak{F}_2^{-1} \mathfrak{L}_1 \mathfrak{F}_2 = \omega \cdot \partial_\varphi \mathbb{I}_2 + J(D^2 \mathbb{I}_2 + \Omega^{nls} \mathbb{I}_2 + \varepsilon \text{av}(q_1) \mathbb{I}_2) + \mathfrak{R}_2 \quad (6.85)$$

where \mathfrak{R}_2 is the one smoothing operator

$$\mathfrak{R}_2 := \varepsilon (\mathfrak{F}_2^{-1} - \mathbb{I}_2) J \text{av}(q_1) \mathbb{I}_2 + \mathfrak{F}_2^{-1} (\mathfrak{R}^I + \varepsilon \mathfrak{R}^{II}) \quad (6.86)$$

with \mathfrak{R}^I defined in (6.79) and \mathfrak{R}^{II} in (6.81). Since \mathfrak{F}_2 is symplectic and \mathfrak{L}_1 is a linear Hamiltonian operator, Lemma 3.1 implies that also \mathfrak{L}_2 is Hamiltonian. We point out that the 0th order term $(\Omega^{nls} + \varepsilon \text{av}(q_1)) \mathbb{I}_2$ in (6.85) is diagonal and x -independent, but still depends on φ . Note that the coefficients of the operator \mathfrak{R}_2 involve $\partial_x^2 a_2(\varphi, \cdot) \in H^{\sigma-1}$.

Using Lemma 6.7 to estimate the term $\mathfrak{R}_1 \mathfrak{F}_2$ in \mathfrak{R}^I and arguing as in the proof of Lemma 6.7, we get

Lemma 6.10. (Estimates of A_2 , Φ_2 and \mathfrak{R}_2) For any $s_0 \leq s \leq s_* - 2s_0$ the following statements hold:
(i) For any $\varphi \in \mathbb{T}^S$ and $\sigma' \in \{\sigma + 1, \sigma, \sigma - 1, \sigma - 2, \sigma - 3\}$, $A_2(\varphi) \in \mathcal{L}(H^{\sigma'-1}, H^{\sigma'})$ and

$$|JF_{nls}^\perp A_2 F_{nls}^{-1}|_{s, \sigma'}, |JF_{nls}^\perp A_2 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma'} \leq_s 1 + \|\iota\|_{s+s_0} \quad (6.87)$$

$$|JF_{nls}^\perp A_2 F_{nls}^{-1}|_{s, \sigma'}^{\text{lip}}, |JF_{nls}^\perp A_2 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma'}^{\text{lip}} \leq_s 1 + \|\iota\|_{s+s_0}^{\text{lip}}. \quad (6.88)$$

(ii) For any $\varphi \in \mathbb{T}^S$, $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$, $\Phi_2(\varphi) \in \mathcal{L}(h_\perp^{\sigma'} \times h_\perp^{\sigma'})$ and

$$\begin{aligned} |\Phi_2^{\pm 1} - \mathbb{I}_2|_{s, \sigma'}, |(\Phi_2^{\pm 1} - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma'} &\leq_s \varepsilon(1 + \|\iota\|_{s+s_0}) \\ |\Phi_2^{\pm 1} - \mathbb{I}_2|_{s, \sigma'}^{\text{lip}}, |(\Phi_2^{\pm 1} - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma'}^{\text{lip}} &\leq_s \varepsilon(1 + \|\iota\|_{s+s_0}^{\text{lip}}). \end{aligned}$$

(iii) \mathfrak{R}_2 is a linear Hamiltonian operator with $\mathfrak{R}_2(\varphi) \in \mathcal{L}(h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}, h_\perp^{\sigma-1} \times h_\perp^{\sigma-1})$ for any $\varphi \in \mathbb{T}^S$ and

$$|\mathfrak{R}_2 \mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}, \quad |\mathfrak{R}_2 \mathfrak{D}|_{s, \sigma-1}^{\text{lip}} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+2s_0}^{\text{lip}}. \quad (6.89)$$

Proof. (i) We begin proving (6.87). We consider the case $\sigma' = \sigma + 1$ only, since the other cases can be treated in a similar way. According to (6.78) we can write

$$JF_{nls}^\perp A_2 F_{nls}^{-1} = J \mathfrak{D}^{-2} F_{nls}^\perp \begin{pmatrix} Da_2 & 0 \\ 0 & -Da_2 \end{pmatrix} F_{nls}^{-1} + JF_{nls}^\perp \begin{pmatrix} a_2 D & 0 \\ 0 & -a_2 D \end{pmatrix} F_{nls}^{-1} \mathfrak{D}^{-2}$$

Since $|D \langle\langle D \rangle\rangle^{-2}|_{s, \sigma+1} \leq \| \langle\langle D \rangle\rangle^{-1} \|_{\mathcal{L}(h^{\sigma+1})} \leq 1$ one has $|JF_{nls}^\perp A_2 F_{nls}^{-1}|_{s, \sigma+1} \leq |JF_{nls}^\perp A_2 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma+1}$ and

$$|JF_{nls}^\perp A_2 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma+1} \stackrel{\text{Lemma 2.4}}{\leq_s} \|a_2\|_{s, \sigma+1} \stackrel{(6.83)}{\leq_s} \|q_1\|_{s, \sigma} \stackrel{(6.53)}{\leq_s} 1 + \|\iota\|_{s+s_0}.$$

The estimates (6.88) are proved in a similar way.

(ii) is proved in a similar way as item (ii) of Lemma 6.7.

(iii) We begin by proving the first estimate in (6.89). Note that the remainder \mathfrak{R}_2 introduced in (6.86),

$$\mathfrak{R}_2 = \varepsilon(\Phi_2^{-1} - \mathbb{I}_2) \text{av}(q_1) J + \Phi_2^{-1} (\mathfrak{R}^I + \varepsilon \mathfrak{R}^{II}),$$

is of the same form as the remainder \mathfrak{R}_1 in Lemma 6.7. Due to the definition (6.81) - (6.82) of \mathfrak{R}^{II} , the term $\varepsilon |\mathfrak{R}^{II} \mathfrak{D}|_{s, \sigma-1}$ can be estimated in the same way as the corresponding term of \mathfrak{R}_1 . Since, in contrast to A_1 , the operator A_2 is only one smoothing, the main difference for estimating $|\mathfrak{R}^I \mathfrak{D}|_{s, \sigma-1}$ concerns the term

$$[JD^2 \mathbb{I}_2, \Phi_2 - \mathbb{I}_2 + \varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1}].$$

Using that J and $F_{nls}^\perp A_2 F_{nls}^{-1}$ commute one has

$$\Phi_2 - \mathbb{I}_2 + \varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1} = -\frac{1}{2} \varepsilon^2 (F_{nls}^\perp A_2 F_{nls}^{-1})^2 + \sum_{n \geq 3} \frac{(-\varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1})^n}{n!}.$$

Using item (i) together with Lemma 2.10 (iv) we get

$$\left| JD^2 \mathbb{I}_2 \sum_{n \geq 3} \frac{(-\varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1})^n}{n!} \mathfrak{D} \right|_{s, \sigma-1}, \left| \sum_{n \geq 3} \frac{(-\varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1})^n}{n!} JD^2 \mathbb{I}_2 \mathfrak{D} \right|_{s, \sigma-1} \leq_s \varepsilon^3 (1 + \|\iota\|_{s+s_0}).$$

The estimate of the norm of the commutator $[JD^2 \mathbb{I}_2, (F_{nls}^\perp A_2 F_{nls}^{-1})^2] \mathfrak{D}$ requires more attention. Recalling (3.29) one has

$$[JD^2 \mathbb{I}_2, (F_{nls}^\perp A_2 F_{nls}^{-1})^2] = J[D^2 \mathbb{I}_2, (F_{nls}^\perp A_2 F_{nls}^{-1})^2] = JF_{nls}^\perp (D^2 A_2 \mathbb{I}_\perp A_2 - A_2 \mathbb{I}_\perp A_2 D^2) F_{nls}^{-1}.$$

The operator $A_2 \mathbb{I}_\perp A_2$ is of the form $\text{diag}(B, \bar{B})$ where, with the short hand notation $\Lambda := D \langle\langle D \rangle\rangle^{-2}$,

$$B := (\Lambda a_2 + a_2 \Lambda) \pi_\perp (\Lambda a_2 + a_2 \Lambda) = \Lambda a_2 \pi_\perp \Lambda a_2 + \Lambda a_2 \pi_\perp a_2 \Lambda + a_2 \Lambda^2 \pi_\perp a_2 + a_2 \Lambda \pi_\perp a_2 \Lambda. \quad (6.90)$$

Hence

$$- [JD^2\mathbb{I}_2, (F_{nls}^\perp A_2 F_{nls}^{-1})^2] = J[(F_{nls}^\perp A_2 F_{nls}^{-1})^2, D^2\mathbb{I}_2] = JF_{nls}^\perp \text{diag}([B, D^2], [\bar{B}, D^2])F_{nls}^{-1} \quad (6.91)$$

and the commutator $[B, D^2]$ is given by the sum $T_1 + T_2 + T_3 + T_4$ with

$$T_1 := [\Lambda a_2 \pi_\perp \Lambda a_2, D^2], \quad T_2 := [\Lambda a_2 \pi_\perp a_2 \Lambda, D^2], \quad T_3 := [a_2 \Lambda^2 \pi_\perp a_2, D^2], \quad T_4 := [a_2 \Lambda \pi_\perp a_2 \Lambda, D^2]. \quad (6.92)$$

The four operators are treated in the same way, so we consider T_1 only. Since $D^2 = -\partial_x^2$ one has

$$T_1 = \Lambda(\partial_x^2 a_2) \pi_\perp \Lambda a_2 + \Lambda a_2 \pi_\perp \Lambda(\partial_x^2 a_2) + 2\Lambda(\partial_x a_2) \pi_\perp \Lambda(\partial_x a_2) + 2i\Lambda(\partial_x a_2) \pi_\perp \Lambda a_2 D + 2i\Lambda a_2 \pi_\perp \Lambda(\partial_x a_2) D.$$

Since by (6.83)

$$\|a_2\|_{s, \sigma-1}, \quad \|\partial_x a_2\|_{s, \sigma-1}, \quad \|\partial_x^2 a_2\|_{s, \sigma-1} \leq_s \|q_1\|_s$$

it follows from Lemma 2.4 and the estimate $\|\Lambda\|_{\mathcal{L}(h^{\sigma'-1}, h^{\sigma'})} \ll 1$, valid for arbitrary σ' , that

$$|T_1 \langle\langle D \rangle\rangle|_{s, \sigma-1} \leq_s \|q_1\|_s \|q_1\|_{s_0} \stackrel{(6.53)}{\leq_s} 1 + \|\iota\|_{s+s_0}.$$

Since the operators T_2, T_3 , and T_4 can be estimated in the same way, one concludes that

$$|[JD^2\mathbb{I}_2, \varepsilon^2(F_{nls}^\perp A_2 F_{nls}^{-1})^2] \mathfrak{D}|_{s, \sigma-1} \leq \varepsilon^2(1 + \|\iota\|_{s+s_0}).$$

Altogether, this proves the first estimate in (6.89). The second estimate in (6.89) follows in a similar way. \square

Lemma 6.11. *For any $s_0 \leq s \leq s_* - 2s_0$ and any torus embeddings $\check{\iota}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, satisfying (6.8), the following estimates hold:*

(i) *For any $\sigma' \in \{\sigma+1, \sigma, \sigma-1, \sigma-2, \sigma-3\}$, the operator $\Delta_{12}A_2 := A_2(\check{\iota}^{(1)}) - A_2(\check{\iota}^{(2)})$ satisfies the estimates*

$$|JF_{nls}^\perp \Delta_{12}A_2 F_{nls}^{-1}|_{s, \sigma'}, \quad |JF_{nls}^\perp \Delta_{12}A_2 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma'} \leq_s \|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0},$$

(ii) *For any $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$, the operators $\Delta_{12}\Phi_2 := \Phi_2(\check{\iota}^{(1)}) - \Phi_2(\check{\iota}^{(2)})$ and $\Delta_{12}\Phi_2^{-1} := \Phi_2^{-1}(\check{\iota}^{(1)}) - \Phi_2^{-1}(\check{\iota}^{(2)})$ satisfy the estimate*

$$|\Delta_{12}\Phi_2^{\pm 1}|_{s, \sigma'}, \quad |(\Delta_{12}\Phi_2^{\pm 1}) \mathfrak{D}|_{s, \sigma'} \leq_s \varepsilon(\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{2s_0}),$$

(iii) *The operator $\Delta_{12}\mathfrak{R}_2 := \mathfrak{R}_2(\check{\iota}^{(1)}) - \mathfrak{R}_2(\check{\iota}^{(2)})$ satisfies the estimate*

$$|\Delta_{12}\mathfrak{R}_2 \mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon \gamma^{-2} \|\Delta_{12}\iota\|_{s+2s_0} + \max_{s+2s_0}(\iota) \|\Delta_{12}\iota\|_{3s_0}.$$

Proof. (i) We consider the case $\sigma' = \sigma + 1$ only, since the other cases can be treated in a similar way. According to the definition (6.78) we can write

$$JF_{nls}^\perp \Delta_{12}A_2 F_{nls}^{-1} = J\mathfrak{D}^{-2} F_{nls}^\perp \begin{pmatrix} D\Delta_{12}a_2 & 0 \\ 0 & -D\Delta_{12}a_2 \end{pmatrix} F_{nls}^{-1} + JF_{nls}^\perp \begin{pmatrix} \Delta_{12}a_2 D & 0 \\ 0 & -\Delta_{12}a_2 D \end{pmatrix} F_{nls}^{-1} \mathfrak{D}^{-2}$$

Since $|D\langle\langle D \rangle\rangle^{-2}|_{s, \sigma+1} \ll \|\langle\langle D \rangle\rangle^{-1}\|_{\mathcal{L}(h^{\sigma+1})} \ll 1$ one has

$$|JF_{nls}^\perp \Delta_{12}A_2 F_{nls}^{-1} \mathfrak{D}|_{s, \sigma+1} \stackrel{\text{Lemma 2.4}}{\leq_s} \|\Delta_{12}a_2\|_{s, \sigma+1} \stackrel{(6.83)}{\leq_s} \|\Delta_{12}q_1\|_{s, \sigma} \stackrel{(6.55)}{\leq_s} \|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota) \|\Delta_{12}\iota\|_{s_0}.$$

(ii) Follows by Lemma 2.10 (v) and item (i).

(iii) Note that the remainder \mathfrak{R}_2 introduced in (6.86),

$$\mathfrak{R}_2 = \varepsilon(\Phi_2^{-1} - \mathbb{I}_2) \text{av}(q_1) J + \Phi_2^{-1}(\mathfrak{R}^I + \varepsilon \mathfrak{R}^{II}),$$

is of the same form as the remainder \mathfrak{R}_1 in Lemma 6.7. Due to the definition (6.81) - (6.82) of \mathfrak{R}^{II} , the term $\varepsilon|\Delta_{12}\mathfrak{R}^{II} \mathfrak{D}|_{s, \sigma-1}$ can be estimated in the same way as the corresponding term of $\Delta_{12}\mathfrak{R}_1$. Since, in contrast

to A_1 , the operator A_2 is only one smoothing, the main difference for estimating $|\Delta_{12}\mathfrak{R}^f\mathfrak{D}|_{s,\sigma-1}$ concerns the operator

$$\Delta_{12}[JD^2\mathbb{I}_2, \Phi_2 - \mathbb{I}_2 + \varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1}]\mathfrak{D}.$$

Using that J and A_2 commute one has

$$\Delta_{12}\left(\Phi_2 - \mathbb{I}_2 + \varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1}\right) = -\frac{1}{2}\varepsilon^2\Delta_{12}\left(F_{nls}^\perp A_2 F_{nls}^{-1}\right)^2 + \sum_{n\geq 3} \frac{\Delta_{12}(-\varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1})^n}{n!}.$$

By the same arguments used for obtaining the estimate (6.76) in the proof of Lemma 6.8, one concludes from item (i) and Lemma 6.10(i),

$$S_1, S_2 \leq_s \varepsilon^3 (\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota)\|\Delta_{12}\iota\|_{s_0})$$

where

$$S_1 := \left| JD^2\mathbb{I}_2 \sum_{n\geq 3} \frac{\Delta_{12}(-\varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1})^n}{n!} \mathfrak{D} \right|_{s,\sigma-1}, \quad S_2 := \left| \sum_{n\geq 3} \frac{\Delta_{12}(-\varepsilon J F_{nls}^\perp A_2 F_{nls}^{-1})^n}{n!} JD^2\mathbb{I}_2 \mathfrak{D} \right|_{s,\sigma-1}.$$

The estimate of the norm of $-\frac{1}{2}\varepsilon^2[JD^2\mathbb{I}_2, \Delta_{12}(F_{nls}^\perp A_2 F_{nls}^{-1})^2]\mathfrak{D}$ requires more attention. By (6.91)

$$-[JD^2\mathbb{I}_2, \Delta_{12}(F_{nls}^\perp A_2 F_{nls}^{-1})^2] = J F_{nls}^\perp \text{diag}(\Delta_{12}[B, D^2], \Delta_{12}[\overline{B}, D^2]) F_{nls}^{-1}$$

where B is defined in (6.90) and $[B, D^2] = T_1 + T_2 + T_3 + T_4$ with T_1, T_2, T_3, T_4 defined in (6.92). Hence

$$\Delta_{12}[B, D^2] = \Delta_{12}T_1 + \Delta_{12}T_2 + \Delta_{12}T_3 + \Delta_{12}T_4.$$

The four terms are treated in the same way, so we consider $\Delta_{12}T_1$ only. Recall that

$$T_1 = \Lambda(\partial_x^2 a_2)\pi_\perp \Lambda a_2 + \Lambda a_2 \pi_\perp \Lambda(\partial_x^2 a_2) + 2\Lambda(\partial_x a_2)\pi_\perp \Lambda(\partial_x a_2) + 2i\Lambda(\partial_x a_2)\pi_\perp \Lambda a_2 D + 2i\Lambda a_2 \pi_\perp \Lambda(\partial_x a_2) D.$$

By (6.83) one has $\|a_2\|_{s,\sigma-1}, \|\partial_x a_2\|_{s,\sigma-1}, \|\partial_x^2 a_2\|_{s,\sigma-1} \leq_s \|q_1\|_s$, and

$$\|\Delta_{12}a_2\|_{s,\sigma-1}, \|\partial_x \Delta_{12}a_2\|_{s,\sigma-1}, \|\partial_x^2 \Delta_{12}a_2\|_{s,\sigma-1} \leq_s \|\Delta_{12}q_1\|_s.$$

It then follows from Lemma 2.4 and the estimate $\|\Lambda\|_{\mathcal{L}(h^{\sigma'-1}, h^{\sigma'})} \leq 1$ for σ' arbitrary, that

$$\begin{aligned} |\Delta_{12}T_1\langle\langle D \rangle\rangle|_{s,\sigma-1} &\leq_s \|\Delta_{12}q_1\|_s (\|q_1(\iota^{(1)})\|_{s_0} + \|q_1(\iota^{(2)})\|_{s_0}) + \|\Delta_{12}q_1\|_{s_0} (\|q_1(\iota^{(1)})\|_s + \|q_1(\iota^{(2)})\|_s) \\ &\stackrel{(6.53), (6.55)}{\leq_s} \|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota)\|\Delta_{12}\iota\|_{s_0}. \end{aligned}$$

Since the operators $\Delta_{12}T_2, \Delta_{12}T_3,$ and $\Delta_{12}T_4$ can be estimated in the same way, one concludes that

$$|[JD^2\mathbb{I}_2, \varepsilon^2\Delta_{12}(F_{nls}^\perp A_2 F_{nls}^{-1})^2]\mathfrak{D}|_{s,\sigma-1} \leq \varepsilon^2 (\|\Delta_{12}\iota\|_{s+s_0} + \max_{s+s_0}(\iota)\|\Delta_{12}\iota\|_{s_0}).$$

One then concludes the proof of item (iii) by arguing in the same way as at the end of the proof of item (iii) of Lemma 6.8. \square

6.4 Gauge transformation

Finally we eliminate the φ -dependence from $J(\Omega^{nls} + \varepsilon \text{av}(q_1))\mathbb{I}_2$ in (6.85) by a gauge transformation. More precisely, we conjugate \mathfrak{L}_2 with the symplectic map, given by the time 1-flow map

$$\Phi_3 := \exp\left(-\text{diag}(\beta_k)_{k \in S^\perp} J\right) = \text{diag}\left((e^{-i\beta_k})_{k \in S^\perp}, (e^{i\beta_k})_{k \in S^\perp}\right),$$

corresponding to the Hamiltonian $\sum_{k \in S^\perp} \beta_k(\varphi) z_k \bar{z}_k$ with $\beta_k = \beta_k(\varphi) \in \mathbb{R}$. The conjugated operator $\mathfrak{L}_3 := \Phi_3^{-1} \mathfrak{L}_2 \Phi_3$ is then given by

$$\mathfrak{L}_3 = \omega \cdot \partial_\varphi \mathbb{I}_2 - J \text{diag}_{k \in S^\perp} (\omega \cdot \partial_\varphi \beta_k) \mathbb{I}_2 + J(D^2 + \Omega^{nls} + \varepsilon \text{av}(q_1))\mathbb{I}_2 + \mathfrak{R}_3 \quad (6.93)$$

where $\mathfrak{R}_3 := \mathfrak{F}_3^{-1} \mathfrak{R}_2 \mathfrak{F}_3$. We choose the functions $\beta_k(\varphi)$, $k \in S^\perp$, so that

$$\omega \cdot \partial_\varphi \beta_k(\varphi) = \omega_k^{nls}(I(\varphi)) + \varepsilon \text{av}(q_1)(\varphi) - [[\omega_k^{nls} \circ I + \varepsilon q_1]], \quad \hat{\beta}_k(0) = 0, \quad (6.94)$$

where $[[g]]$ denotes the average in space and time of a function $g : \mathbb{T}^S \times \mathbb{T}_1 \rightarrow \mathbb{C}$,

$$[[g]] := \frac{1}{(2\pi)^{|S|}} \int_{\mathbb{T}^S \times \mathbb{T}_1} g(\varphi, x) d\varphi dx.$$

Since ω is assumed to be in $\Omega_0(\iota) \subset \Omega_{\gamma, \tau}$ it satisfies the diophantine condition (1.22) and by Lemma 2.2, the equations (6.94) have unique solutions. As a consequence by (6.93) and (6.13) we have

$$\mathfrak{L}_3 = \omega \cdot \partial_\varphi \mathbb{I}_2 + J(D^2 + [[\Omega^{nls}]] + \varepsilon[[q_1]])\mathbb{I}_2 + \mathfrak{R}_3, \quad \mathfrak{R}_3 = \mathfrak{F}_3^{-1} \mathfrak{R}_2 \mathfrak{F}_3 \quad (6.95)$$

where \mathfrak{R}_2 is defined in (6.86). By (6.13) one has $D^2 + [[\Omega^{nls}]] = \text{diag}_k([[\omega_k^{nls}]])_{k \in S^\perp}$.

Lemma 6.12. (Normal form of \mathfrak{L}_3) *The diagonal elements of $D^2 + [[\Omega^{nls}]] + \varepsilon[[q_1]]$ satisfy*

$$[[\omega_k^{nls}]] + \varepsilon[[q_1]] = \omega_k^{nls}(\xi, 0) + c_\varepsilon + \frac{1}{k} r_{k, \xi}, \quad k \in S^\perp, \quad (6.96)$$

where

$$|c_\varepsilon|^{\gamma \text{lip}}, |r_{k, \xi}|^{\gamma \text{lip}} \ll \varepsilon \gamma^{-2}. \quad (6.97)$$

Furthermore

$$|[[\omega_k^{nls}]] + \varepsilon[[q_1]]|^{\text{lip}} \ll 1. \quad (6.98)$$

Proof. Since by Theorem 3.2,

$$\omega_k^{nls} = 4\pi^2 k^2 + 4 \sum_{j \in \mathbb{Z}} I_j + \frac{r_k}{k}, \quad (r_k)_{k \in \mathbb{Z}} \in \ell^\infty,$$

we get (6.96) with

$$c_\varepsilon := \left[\left[4 \sum_{j \in S} y_j + 4 \sum_{j \in S^\perp} z_j \bar{z}_j + \varepsilon q_1 \right] \right] \quad \text{and} \quad r_{k, \xi} := \left[\left[r_k(\xi + y, z\bar{z}) - r_k(\xi, 0) \right] \right].$$

Since $[[[q_1]]]^{\gamma \text{lip}} \leq \|q_1\|_{s_0}^{\gamma \text{lip}}$ and $\|q_1\|_{s_0}^{\gamma \text{lip}} \stackrel{(6.53)}{\ll} 1 + \|\iota\|_{2s_0}$ it follows that $[[[q_1]]]^{\gamma \text{lip}} \stackrel{(6.8)}{\ll} 1$. Furthermore, by (6.19) and Lemma 6.2 (i), $[[[4 \sum_{j \in S} y_j + 4 \sum_{j \in S^\perp} z_j \bar{z}_j]]]^{\gamma \text{lip}} \stackrel{(6.8)}{\ll} \varepsilon \gamma^{-2}$. Similarly, $|r_{k, \xi}|^{\gamma \text{lip}} \leq \|r_k(\xi + y, z\bar{z}) - r_k(\xi, 0)\|_{s_0}^{\gamma \text{lip}}$ and hence by (6.46), $|r_{k, \xi}|^{\gamma \text{lip}} \ll \|\iota\|_{3s_0}^{\gamma \text{lip}}$. Altogether we thus have proved (6.97). The estimate (6.98) follows from (6.96), (6.97) since $\varepsilon \gamma^{-3} \leq 1$ and $\omega_k^{nls}(\xi(\omega), 0)$ is analytic and hence Lipschitz in ω . \square

Using the smallness assumption (6.8), we prove the following

Lemma 6.13. (Estimates of \mathfrak{F}_3 and \mathfrak{R}_3) *For any $s_0 \leq s \leq s_* - 4s_0 - \tau$, the following holds:*

(i) *For any $\varphi \in \mathbb{T}^S$ and $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$, $\mathfrak{F}_3(\varphi) \in \mathcal{L}(h_{\perp}^{\sigma'})$ and*

$$|\mathfrak{F}_3 - \mathbb{I}_2|_{s, \sigma'}, |\mathfrak{F}_3^{-1} - \mathbb{I}_2|_{s, \sigma'} \leq_s \gamma^{-1} (\varepsilon + \|\iota\|_{s+4s_0+\tau}) \quad (6.99)$$

$$|\mathfrak{F}_3^{\pm 1} - \mathbb{I}_2|_{s, \sigma'}^{\gamma \text{lip}} \leq_s \gamma^{-1} (\varepsilon + \|\iota\|_{s+4s_0+2\tau+1})^{\gamma \text{lip}}. \quad (6.100)$$

(ii) *\mathfrak{R}_3 is a linear Hamiltonian operator with $\mathfrak{R}_3(\varphi) \in \mathcal{L}(h_{\perp}^{\sigma-2} \times h_{\perp}^{\sigma-2}, h_{\perp}^{\sigma-1} \times h_{\perp}^{\sigma-1})$ for any $\varphi \in \mathbb{T}^S$ and*

$$|\mathfrak{R}_3 \mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+4s_0+\tau}, \quad |\mathfrak{R}_3 \mathfrak{D}|_{s, \sigma-1}^{\gamma \text{lip}} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+4s_0+2\tau+1}^{\gamma \text{lip}}. \quad (6.101)$$

Proof. (i) We begin by proving the estimate (6.99). We first estimate the right hand side of (6.94) which we rewrite as

$$\omega_k^{nls}(I(\varphi)) - \omega_k^{nls}(\xi, 0) - [[\omega_k^{nls} \circ I - \omega_k^{nls}(\xi, 0)]] + \varepsilon(\text{av}(q_1)(\varphi) - [[q_1]]),$$

where $I(\varphi) = (\xi + y(\varphi), z\bar{z}(\varphi))$. By (3.37)

$$\sup_{k \in S^\perp} \|\omega_k^{nls}(I) - \omega_k^{nls}(\xi, 0)\|_s \leq_s \|\iota\|_{s+2s_0}.$$

By Lemma 2.2, the solutions β_k of (6.94) satisfy

$$\sup_{k \in S^\perp} \|\beta_k\|_s \leq_s \gamma^{-1}(\|\iota\|_{s+2s_0+\tau} + \varepsilon\|\text{av}(q_1) - [[q_1]]\|_{s+\tau})$$

and since $\|\text{av}(q_1) - [[q_1]]\|_{s+\tau} \leq \|q_1\|_{s+\tau}$ and by (6.53), $\|q_1\|_{s+\tau} \leq_s 1 + \|\iota\|_{s+\tau+s_0}$ it then follows that

$$\sup_{k \in S^\perp} \|\beta_k\|_s \leq_s \gamma^{-1}(\varepsilon + \|\iota\|_{s+2s_0+\tau}).$$

Due to the fact that Φ_3 is diagonal we have, for $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$,

$$\|\Phi_3 - \mathbb{I}_2\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma'}))} = \sup_{k \in S^\perp} \|e^{i\beta_k} - 1\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathbb{C})} \leq_s \sup_{k \in S^\perp} \|\beta_k\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathbb{C})}$$

and since, by (2.10), $|\Phi_3 - \mathbb{I}_2|_{s, \sigma'} \leq_s \|\Phi_3 - \mathbb{I}_2\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma'}))}$ it then follows that

$$|\Phi_3 - \mathbb{I}_2|_{s, \sigma'} \leq_s \sup_{k \in S^\perp} \|\beta_k\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathbb{C})} \leq_s \sup_{k \in S^\perp} \|\beta_k\|_{s+2s_0} \leq_s \gamma^{-1}(\varepsilon + \|\iota\|_{s+\tau+4s_0}).$$

In the same way, one derives the claimed estimate for Φ_3^{-1} . The estimate (6.100) is proved in a similar way.

(ii) Since Φ_3 is diagonal it commutes with \mathfrak{D} and hence $\mathfrak{R}_3 \mathfrak{D} = \Phi_3^{-1}(\mathfrak{R}_2 \mathfrak{D})\Phi_3$. The first estimate in (6.101) then follows from (i), Lemma 6.10 (iii), and the tame estimate of Lemma 2.8 for operator valued maps. The second estimate in (6.101) is proved in a similar way. \square

Lemma 6.14. *For any torus embeddings $\check{\iota}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, satisfying (6.8) and any $s_0 \leq s \leq s_* - 4s_0 - \tau$, the following estimates hold:*

(i) *For any $\sigma' \in \{\sigma, \sigma-1, \sigma-2\}$, the operators $\Delta_{12}\Phi_3 := \Phi_3(\check{\iota}^{(1)}) - \Phi_3(\check{\iota}^{(2)})$ and $\Delta_{12}\Phi_3^{-1} := \Phi_3^{-1}(\check{\iota}^{(1)}) - \Phi_3^{-1}(\check{\iota}^{(2)})$ satisfy*

$$|\Delta_{12}\Phi_3^{\pm 1}|_{s, \sigma'} \leq_s \gamma^{-1}(\|\Delta_{12}\iota\|_{s+4s_0+\tau} + \max_{s+4s_0+\tau}(\iota)\|\Delta_{12}\iota\|_{s_0}).$$

(ii) *The operator $\Delta_{12}\mathfrak{R}_3 := \mathfrak{R}_3(\check{\iota}^{(1)}) - \mathfrak{R}_3(\check{\iota}^{(2)})$ satisfies the estimate*

$$|\Delta_{12}\mathfrak{R}_3\mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon\gamma^{-2}\|\Delta_{12}\iota\|_{s+4s_0+\tau} + \max_{s+4s_0+\tau}(\iota)\|\Delta_{12}\iota\|_{5s_0+\tau}. \quad (6.102)$$

Proof. (i) Note that $\Delta_{12}\beta_k := \beta_k^{(1)} - \beta_k^{(2)}$ with $\beta_k^{(a)} \equiv \beta_k(\iota^{(a)})$, $a = 1, 2$, satisfies the equation

$$\omega \cdot \partial_\varphi \Delta_{12}\beta_k = \Delta_{12}(\omega_k^{nls}(I(\varphi)) - [[\omega_k^{nls} \circ I]]) + \varepsilon(\text{av}(q_1)(\varphi) - [[q_1]]). \quad (6.103)$$

Using the same strategy developed in the proof of Lemma 6.2 to obtain the estimate (6.40), we get with $I^{(a)}(\varphi) := (\xi + y^{(a)}(\varphi), z^{(a)}\bar{z}^{(a)}(\varphi))$, $a = 1, 2$,

$$\|\Delta_{12}(\omega_k^{nls} \circ I)\|_s = \|\omega_k^{nls} \circ I^{(1)} - \omega_k^{nls} \circ I^{(2)}\|_s \stackrel{(3.38)}{\leq_s} \|\Delta_{12}\iota\|_s + \max_{s+2s_0}(\iota)\|\Delta_{12}\iota\|_{s_0}.$$

Since $\|\Delta_{12}(\text{av}(q_1) - [[q_1]])\|_s \leq \|\Delta_{12}q_1\|_s$, it then follows from (6.55) that it can be bounded in the same way as $\|\Delta_{12}(\omega_k^{nls} \circ I)\|_s$. Hence by (6.103) and Lemma 2.2, $\Delta_{12}\beta_k$ satisfies

$$\|\Delta_{12}\beta_k\|_s \leq_s \gamma^{-1}(\|\Delta_{12}\iota\|_{s+\tau} + \max_{s+2s_0+\tau}(\iota)\|\Delta_{12}\iota\|_{s_0}). \quad (6.104)$$

Since Φ_3 is diagonal, so is $\Delta_{12}\Phi_3$ and we have for any $\sigma' \in \{\sigma, \sigma - 1, \sigma - 2\}$,

$$\|\Delta_{12}\Phi_3\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma'}))} = \sup_k \|\Delta_{12}e^{i\beta_k}\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathbb{C})}.$$

Using that, by (2.10) $|\Delta_{12}\Phi_3|_{s, \sigma'} \leq_s \|\Delta_{12}\Phi_3\|_{\mathcal{C}^{s+s_0}(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma'}))}$ it then follows from (6.104) that

$$|\Delta_{12}\Phi_3|_{s, \sigma'} \leq_s \gamma^{-1} (\|\Delta_{12}\ell\|_{s+4s_0+\tau} + \max_{s+4s_0+\tau}(\ell) \|\Delta_{12}\ell\|_{s_0}).$$

In the same way one derives the claimed estimate for $\Delta_{12}\Phi_3^{-1}$. This proves item (i). Concerning item (ii), the claimed estimate follows from Lemma 6.10(iii), Lemma 6.11(iii), Lemma 6.13(i), and item (i) by using the tame estimate of Lemma 2.8 and the smallness assumption $\varepsilon\gamma^{-4} \ll 1$. \square

Remark 6.2. Taking into account the asymptotics of the dNLS frequencies (3.8), as an alternative, one can choose a simpler gauge transformation by defining $\beta_k(\varphi) := \beta(\varphi)$, $\forall k$, with $\beta(\varphi)$ the solution of

$$\omega \cdot \partial_{\varphi} \beta(\varphi) = c_0(\varphi) - [[c_0]], \quad c_0(\varphi) := 4 \sum_{j \in S} y_j(\varphi) + 4 \sum_{j \in S^{\perp}} z_j(\varphi) \bar{z}_j(\varphi) + \varepsilon \text{av}(q_1)(\check{i}(\varphi)).$$

In this case, there are additional φ -dependent diagonal terms of size $O(\varepsilon\gamma^{-2}/k)$.

The operator \mathfrak{L}_3 in (6.95) is now in diagonal form up to a one smoothing remainder of small norm. More precisely, the k -th diagonal component of $\mathfrak{L}_3(\widehat{z}, \widehat{w})$ is of the form

$$\omega \cdot \partial_{\varphi} \widehat{z}_k + i([\omega_k^{nls}] + \varepsilon[[q_1]]) \widehat{z}_k + \dots$$

In the subsequent section we will block diagonalize the remainder in \mathfrak{L}_3 by a KAM-reduction scheme.

7 Reduction of \mathfrak{L}_{ω} . Part 2

In this section we reduce the linear Hamiltonian operator \mathfrak{L}_3 , defined in (6.95), by means of a KAM iteration scheme. Recall that \mathfrak{L}_3 is an operator from $H^s(\mathbb{T}^S, h_{\perp}^{\sigma} \times h_{\perp}^{\sigma})$ into $H^{s-1}(\mathbb{T}^S, h_{\perp}^{\sigma-2} \times h_{\perp}^{\sigma-2})$ for any $s_0 \leq s \leq s_* - \bar{\mu}$, where

$$\bar{\mu} := 4s_0 + 2\tau + 1. \quad (7.1)$$

To describe the reduction scheme, it is convenient to denote \mathfrak{L}_3 by \mathbf{L}_0 and write

$$\mathbf{L}_0 = \omega \cdot \partial_{\varphi} \mathbb{I}_2 + \mathbf{N}_0 + \mathbf{R}_0 \quad (7.2)$$

where

$$\mathbf{N}_0 := J \begin{pmatrix} \mathbf{N}_0^{(1)} & 0 \\ 0 & \mathbf{N}_0^{(1)} \end{pmatrix}, \quad \mathbf{N}_0^{(1)} := \text{diag}_{k \in S^{\perp}}([\omega_k^{nls}] + \varepsilon[[q_1]]), \quad \mathbf{R}_0 := \mathfrak{R}_3, \quad (7.3)$$

with the normal form \mathbf{N}_0 described in Lemma 6.12 and \mathfrak{R}_3 given by (6.95). We recall that \mathbf{R}_0 is one smoothing (meaning that $\mathbf{R}_0 \mathfrak{D} \in H^s(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma-1} \times h_{\perp}^{\sigma-1}))$) and satisfies the estimate (cf (6.101))

$$|\mathbf{R}_0 \mathfrak{D}|_{s, \sigma-1}^{\gamma \text{lip}} \leq_s \varepsilon + \varepsilon \gamma^{-2} \|\ell\|_{s+\bar{\mu}}^{\gamma \text{lip}}, \quad \forall s_0 \leq s \leq s_* - \bar{\mu}. \quad (7.4)$$

The linear Hamiltonian operators \mathbf{L}_0 , \mathbf{N}_0 , \mathbf{R}_0 depend on the torus embedding $\check{i} \equiv \check{i}_{\omega} : \mathbb{T}^S \rightarrow M^{\sigma}$, satisfying the smallness assumption (6.8), with $\omega \in \Omega_o(\ell)$. Here

$$\Omega_o(\ell) \subset \Omega_{\gamma, \tau} \subset \Omega, \quad 0 < \gamma < 1, \quad (7.5)$$

and $\Omega_{\gamma, \tau}$ denotes the set of diophantine frequencies (1.22).

7.1 KAM reduction scheme for \mathbf{L}_0

In view of the near resonances of the dNLS frequencies ω_k^{nls} , ω_{-k}^{nls} , we group the coordinates z_{-k} and z_k together. Our aim is to reduce \mathbf{L}_0 to a 2×2 block diagonal operator with φ -independent coefficients, referred to as its *normal form*. Accordingly, a complex linear operator A in $\mathcal{L}(h_{\perp}^{\sigma'})$ with matrix representation $(A_j^k)_{j,k \in S^{\perp}}$, $A_j^k \in \mathbb{C}$ for all $j, k \in S^{\perp}$, (cf (2.8)) is written as a matrix of 2×2 matrices $([A]_j^k)_{j,k \in S_{\perp}^{\perp}}$ where

$$[A]_j^k := \begin{pmatrix} A_{-j}^{-k} & A_{-j}^k \\ A_j^{-k} & A_j^k \end{pmatrix}, \quad j, k \in S_{\perp}^{\perp} := S^{\perp} \cap \mathbb{N}.$$

We denote by $\|\cdot\|$ the operator norm of these 2×2 matrices. Actually any other norm could be used as well. We say that A is a 2×2 block diagonal operator if $[A]_j^k = 0$ for any $j, k \in S_{\perp}^{\perp}$ with $j \neq k$. Let $N_0 > 0$ be given and define

$$N_{-1} := 1, \quad N_{\nu} := N_0^{\chi^{\nu}} \quad \forall \nu \geq 1, \quad \chi := 3/2. \quad (7.6)$$

Note that $N_{\nu+1} = N_{\nu}^{\frac{3}{2}}$ for any $\nu \geq 0$. Along the iteration scheme, we shall consider the following decreasing sequence $(\Omega_{\nu}^{\gamma}(\iota))_{\nu \geq 0}$ of subsets of frequencies

$$\Omega_0^{\gamma}(\iota) := \Omega_o(\iota) \subset \Omega_{\gamma, \tau}, \quad \Omega_{\nu}^{\gamma}(\iota) := \{\omega \in \Omega_{\nu-1}^{\gamma}(\iota) : (7.29) - (7.30) \text{ hold}\}, \quad \nu \geq 1. \quad (7.7)$$

We point out that the conditions (7.29)-(7.30) also involve an exponent $\tau > |S|$ and that set $\Omega_{\gamma, \tau}$ is defined in (1.22). We introduce the following constants α, β , which appear in the exponents of the Sobolev spaces in the iterative scheme,

$$\alpha \equiv \alpha(\tau) := 6\tau + 4, \quad \beta \equiv \beta(\tau) := \alpha + 1. \quad (7.8)$$

In addition we require that

$$s_0 + \beta + \bar{\mu} \leq s_* \quad (7.9)$$

where $\bar{\mu}$ is given by (7.1).

Theorem 7.1. (Reduction scheme for \mathbf{L}_0) *There exists $N_0 = N_0(\tau, |S|, s_*) \in \mathbb{N}$ such that, if*

$$\gamma^{-1} N_0^{C_0} |\mathbf{R}_0 \mathfrak{D}|_{s_0 + \beta, \sigma - 1}^{\gamma \text{lip}} \leq 1, \quad C_0 := 2\tau + 2 + \alpha \quad (7.10)$$

then for any $\nu \geq 1$, the following statements hold:

(S1) $_{\nu}$ *For any $\omega \in \Omega_{\nu}^{\gamma}(\iota)$ there exists a symplectic transformation $\Phi_{\nu-1} := \exp(-\Psi_{\nu-1})$ such that for any $\varphi \in \mathbb{T}^S$, $\Phi_{\nu-1}(\varphi) \in \mathcal{L}(h_{\perp}^{\sigma'} \times h_{\perp}^{\sigma'})$, $\sigma' \in \{\sigma - 2, \sigma - 1, \sigma\}$, $\Psi_{\nu-1}$ is a linear Hamiltonian vector field satisfying for any $s \in [s_0, s_* - \bar{\mu} - \beta]$ the estimates*

$$|\Psi_{\nu-1}|_{s, \sigma}^{\gamma \text{lip}}, \quad |\Psi_{\nu-1} \mathfrak{D}|_{s, \sigma - 1}^{\gamma \text{lip}} \leq \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s + \beta, \sigma - 1}^{\gamma \text{lip}} N_{\nu-1}^{2\tau+1} N_{\nu-2}^{-\alpha}, \quad (7.11)$$

and

$$\mathbf{L}_{\nu} := \Phi_{\nu-1}^{-1} \mathbf{L}_{\nu-1} \Phi_{\nu-1} = \omega \cdot \partial_{\varphi} \mathbb{I}_2 + \mathbf{N}_{\nu} + \mathbf{R}_{\nu} \quad (7.12)$$

where \mathbf{N}_{ν} and \mathbf{R}_{ν} have the following properties: \mathbf{N}_{ν} is in normal form, i.e., \mathbf{N}_{ν} is a φ -independent 2×2 block diagonal operator,

$$\mathbf{N}_{\nu} = J \begin{pmatrix} \mathbf{N}_{\nu}^{(1)} & 0 \\ 0 & \overline{\mathbf{N}}_{\nu}^{(1)} \end{pmatrix}, \quad \mathbf{N}_{\nu}^{(1)} = \text{diag}_{k \in S_{\perp}^{\perp}} [\mathbf{N}_{\nu}^{(1)}]_k^k, \quad (7.13)$$

where for any $k \in S_{\perp}^{\perp}$, $[\mathbf{N}_{\nu}^{(1)}]_k^k \in \mathbb{C}^{2 \times 2}$ is self-adjoint

$$(\mathbf{N}_{\nu}^{(1)})_{-k}^{-k}, \quad (\mathbf{N}_{\nu}^{(1)})_k^k \in \mathbb{R}, \quad (\mathbf{N}_{\nu}^{(1)})_{-k}^k = (\overline{\mathbf{N}}_{\nu}^{(1)})_k^{-k} \in \mathbb{C} \quad (7.14)$$

and satisfies

$$\|[\mathbf{N}_{\nu}^{(1)} - \mathbf{N}_{\nu-1}^{(1)}]_k^k\|^{\gamma \text{lip}} \leq |\mathbf{R}_{\nu-1} \mathfrak{D}|_{s_0, \sigma - 1}^{\gamma \text{lip}} k^{-1}, \quad \|[\mathbf{N}_{\nu}^{(1)}]_k^k\|^{\text{lip}} \leq 1. \quad (7.15)$$

The remainder \mathbf{R}_ν in (7.12) is a linear Hamiltonian operator

$$\mathbf{R}_\nu = J \begin{pmatrix} \mathbf{R}_\nu^{(1)} & \mathbf{R}_\nu^{(2)} \\ \overline{\mathbf{R}}_\nu^{(2)} & \overline{\mathbf{R}}_\nu^{(1)} \end{pmatrix}, \quad \mathbf{R}_\nu^{(1)} = (\mathbf{R}_\nu^{(1)})^*, \quad \mathbf{R}_\nu^{(2)} = (\mathbf{R}_\nu^{(2)})^t \quad (7.16)$$

satisfying for any $s \in [s_0, s_* - \bar{\mu} - \beta]$ the following estimates

$$|\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}^{\gamma_{\text{lip}}} \leq |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1}^{\gamma_{\text{lip}}} N_{\nu-1}^{-\alpha}, \quad |\mathbf{R}_\nu \mathfrak{D}|_{s+\beta, \sigma-1}^{\gamma_{\text{lip}}} \leq |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1}^{\gamma_{\text{lip}}} N_{\nu-1}. \quad (7.17)$$

In $(\mathbf{S1})_\nu$, all the Lipschitz norms are computed on the set $\Omega'_\nu(\iota)$.

$(\mathbf{S2})_\nu$ For any $k \in S_+^\perp$, there exists a Lipschitz extension $[\tilde{\mathbf{N}}_\nu^{(1)}]_k^k$ of $[\mathbf{N}_\nu^{(1)}]_k^k$ to the set $\Omega_o(\iota)$, which is self-adjoint and satisfies the estimate

$$\|[\tilde{\mathbf{N}}_\nu^{(1)}]_k^k - [\tilde{\mathbf{N}}_{\nu-1}^{(1)}]_k^k\|^{\gamma_{\text{lip}}} \ll |\mathbf{R}_{\nu-1} \mathfrak{D}|_{s_0, \sigma-1}^{\gamma_{\text{lip}}} k^{-1}, \quad (7.18)$$

where we set $[\tilde{\mathbf{N}}_0^{(1)}]_k^k = [\mathbf{N}_0^{(1)}]_k^k$.

Theorem 7.1 is proved in Section 7.4. In the subsequent two sections we establish some auxiliary results.

7.2 2×2 block representation of operators

Let us write an element $z = (z_k)_{k \in S^\perp}$ in $h_{\perp}^{\sigma'}$ as a sequence of vectors

$$z = (\vec{z}_k)_{k \in S_+^\perp}, \quad \vec{z}_k := (z_{-k}, z_k), \quad S_+^\perp = S^\perp \cap \mathbb{N}.$$

Its Sobolev norm is thus

$$\|z\|_{\sigma'}^2 = \sum_{k \in S^\perp} |z_k|^2 \langle k \rangle^{2\sigma'} = \sum_{k \in S_+^\perp} |\vec{z}_k|^2 \langle k \rangle^{2\sigma'}.$$

For each complex linear operator $A \in \mathcal{L}(h_{\perp}^{\sigma'})$ and $z = (\vec{z}_k)_{k \in S_+^\perp} \in h_{\perp}^{\sigma'}$, $Az = (\vec{Az})_{j \in S_+^\perp}$ with

$$(\vec{Az})_j = \sum_{m \in S_+^\perp} [A]_j^m \vec{z}_m.$$

Furthermore, we denote by A^{diag} the linear operator obtained from A by setting for any $j, k \in S_+^\perp$

$$[A^{\text{diag}}]_j^k = [A]_j^k \quad \text{if } j = k, \quad [A^{\text{diag}}]_j^k = 0 \quad \text{if } j \neq k. \quad (7.19)$$

Lemma 7.1. *Let $A \in \mathcal{L}(h_{\perp}^{\sigma'})$ with $\sigma' \leq \sigma$. Then the following holds:*

- (i) $A^{\text{diag}} \in \mathcal{L}(h_{\perp}^{\sigma'})$ and $\|A^{\text{diag}}\|_{\mathcal{L}(h_{\perp}^{\sigma'})} \ll \|A\|_{\mathcal{L}(h_{\perp}^{\sigma'})}$;
- (ii) $\sum_{j \in S_+^\perp} \|[A]_j^k\|^2 \langle j \rangle^{2\sigma'} \ll \|A\|_{\mathcal{L}(h_{\perp}^{\sigma'})}^2 \langle k \rangle^{2\sigma'}, \forall k \in S_+^\perp$;
- (iii) for any $(\vec{h}_k)_{k \in S_+^\perp} \in h_{\perp}^{\sigma'}$,

$$\sum_{j \in S_+^\perp} \left(\sum_{k \neq j} \frac{\|[A]_j^k\| \|\vec{h}_k\|}{|j-k|} \right)^2 \langle j \rangle^{2\sigma'} \ll \|A\|_{\mathcal{L}(h_{\perp}^{\sigma'})}^2 \|h\|_{\sigma'}^2.$$

Proof. (i) The estimate holds, since each matrix element of $[A]_j^j \in \mathbb{C}^{2 \times 2}$, $j \in S_+^\perp$, is bounded by $\|A\|_{\mathcal{L}(h_{\perp}^{\sigma'})}$.

(ii) By the definition of the operator norm, for any $h \in h_{\perp}^{\sigma'}$ one has

$$\|Ah\|_{\sigma'}^2 = \sum_{j \in S_+^\perp} \left\| \sum_{m \in S_+^\perp} [A]_j^m \vec{h}_m \right\|^2 \langle j \rangle^{2\sigma'} \leq \|A\|_{\mathcal{L}(h_{\perp}^{\sigma'})}^2 \|h\|_{\sigma'}^2.$$

For the sequence $h = (\vec{h}_k \delta_{k,m})_{m \in S_{\pm}^{\perp}}$ (with $\delta_{k,m} = 0$ for $m \neq k$ and $\delta_{k,k} = 1$), we find

$$\sum_{j \in S_{\pm}^{\perp}} \|[A]_j^k \vec{h}_k\|^2 \langle j \rangle^{2\sigma'} \ll \|A\|_{\mathcal{L}(h_{\pm}^{\sigma'})}^2 |\vec{h}_k|^2 \langle k \rangle^{2\sigma'}.$$

By choosing $\vec{h}_k = (1, 0)$ and $\vec{h}_k = (0, 1)$, respectively, one gets

$$\sum_{j \in S_{\pm}^{\perp}} \left\| \begin{pmatrix} A_{-j}^{-k} \\ A_j^{-k} \end{pmatrix} \right\|^2 \langle j \rangle^{2\sigma'}, \quad \sum_{j \in S_{\pm}^{\perp}} \left\| \begin{pmatrix} A_{-j}^k \\ A_j^k \end{pmatrix} \right\|^2 \langle j \rangle^{2\sigma'} \ll \|A\|_{\mathcal{L}(h_{\pm}^{\sigma'})}^2 \langle k \rangle^{2\sigma'}.$$

Since $\|[A]_j^k\|$ is bounded by $|A_{-j}^{-k}|^2 + |A_j^{-k}|^2 + |A_{-j}^k|^2 + |A_j^k|^2$, item (ii) follows.

(iii) Using the Cauchy Schwartz inequality one has

$$\begin{aligned} \sum_{j \in S_{\pm}^{\perp}} \left(\sum_{k \neq j} \frac{\|[A]_j^k\| \|\vec{h}_k\|}{|j-k|} \right)^2 \langle j \rangle^{2\sigma'} &\leq \sum_{j \in S_{\pm}^{\perp}} \left(\sum_{k \in S_{\pm}^{\perp}} \|[A]_j^k\|^2 \|\vec{h}_k\|^2 \langle j \rangle^{2\sigma'} \right) \left(\sum_{k \neq j} \frac{1}{|j-k|^2} \right) \\ &\ll \sum_{j \in S_{\pm}^{\perp}} \sum_{k \in S_{\pm}^{\perp}} \|[A]_j^k\|^2 \|\vec{h}_k\|^2 \langle j \rangle^{2\sigma'} \ll \sum_{k \in S_{\pm}^{\perp}} \|\vec{h}_k\|^2 \sum_{j \in S_{\pm}^{\perp}} \|[A]_j^k\|^2 \langle j \rangle^{2\sigma'} \\ &\stackrel{(ii)}{\ll} \sum_{k \in S_{\pm}^{\perp}} \|\vec{h}_k\|^2 \|A\|_{\mathcal{L}(h_{\pm}^{\sigma'})}^2 \langle k \rangle^{2\sigma'} = \|A\|_{\mathcal{L}(h_{\pm}^{\sigma'})}^2 \|h\|_{\sigma'}^2, \end{aligned}$$

establishing the claimed estimate. \square

Let us denote by $\mathbb{C}^{2 \times 2}$ the 4-dimensional Hilbert space of the complex 2×2 matrices equipped with the inner product given for any $X, Y \in \mathbb{C}^{2 \times 2}$ by

$$\langle X, Y \rangle := \text{Tr}(XY^*), \quad Y^* = \overline{Y}^t. \quad (7.20)$$

For any $A \in \mathbb{C}^{2 \times 2}$, denote by $M_L(A)$, $M_R(A)$ the linear operators on $\mathbb{C}^{2 \times 2}$, defined for any $X \in \mathbb{C}^{2 \times 2}$ as left respectively right multiplication by A ,

$$M_L(A)X := AX, \quad M_R(A)X := XA.$$

For what follows it is convenient to associate to arbitrary vectors $v, w \in \mathbb{C}^2$ the 2×2 matrix $(v \ w)$ defined as

$$(v \ w) := \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}, \quad \text{where } v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Furthermore, for any $A \in \mathbb{C}^{2 \times 2}$ denote by $\text{spec}(A)$ the spectrum of A and recall that $\text{spec}(A) = \text{spec}(A^t)$.

Lemma 7.2. (i) Let $A \in \mathbb{C}^{2 \times 2}$. Then any $\lambda \in \text{spec}(A)$ is an eigenvalue of the operators $M_L(A)$ and $M_R(A)$. More precisely for any $v, w \in \mathbb{C}^2$, with $Av = \lambda v$ and $A^t w = \lambda w$, one has for any $\alpha, \beta \in \mathbb{C}$,

$$M_L(A)(\alpha v \ \beta v) = \lambda(\alpha v \ \beta v), \quad M_R(A)(\alpha w \ \beta w)^t = \lambda(\alpha w \ \beta w)^t.$$

(ii) For any $A, B \in \mathbb{C}^{2 \times 2}$, $\lambda \in \text{spec}(A)$, $\mu \in \text{spec}(B)$ and for any $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ in \mathbb{C}^2 with $Av = \lambda v$, $B^t w = \mu w$, $\lambda \pm \mu$ is an eigenvalue of $M_L(A) \pm M_R(B)$, namely

$$(M_L(A) \pm M_R(B))(w_1 v \ w_2 v) = (\lambda \pm \mu)(w_1 v \ w_2 v).$$

(iii) Let $A \in \mathbb{C}^{2 \times 2}$ be self-adjoint. Then $M_L(A)$ and $M_R(A)$ are self-adjoint operators on $\mathbb{C}^{2 \times 2}$ with respect to the scalar product defined in (7.20).

Proof. (i) One has

$$M_R(A)(\alpha w \ \beta w)^t = (\alpha w \ \beta w)^t A = (A^t(\alpha w \ \beta w))^t = \lambda(\alpha w \ \beta w)^t.$$

Similarly one proves $M_L(A)(\alpha v \ \beta v) = \lambda(\alpha v \ \beta v)$.

(ii) By item (i) one has

$$M_L(A)(w_1 v \ w_2 v) = \lambda(w_1 v \ w_2 v)$$

and using that $(w_1 v \ w_2 v)^t = (v_1 w \ v_2 w)$

$$M_R(B)(w_1 v \ w_2 v) = (w_1 v \ w_2 v)B = (B^t(w_1 v \ w_2 v))^t = (B^t(v_1 w \ v_2 w))^t = \mu(v_1 w \ v_2 w)^t = \mu(w_1 v \ w_2 v).$$

Altogether this proves item (ii).

(iii) For any $X, Y \in \mathbb{C}^{2 \times 2}$

$$\langle M_L(A)X, Y \rangle \stackrel{(7.20)}{=} \text{Tr}(AXY^*) = \text{Tr}(XY^*A) \stackrel{A=A^*}{=} \text{Tr}(X(AY)^*) = \langle X, M_L(A)Y \rangle.$$

The self-adjointness of $M_R(A)$ is verified similarly. \square

7.3 Homological equation

We now show how, at the ν th step of the KAM iteration scheme, described in Theorem 7.1, one constructs a symplectic transformation

$$\Phi_\nu := \exp(-\Psi_\nu) = \mathbb{I}_2 - \Psi_\nu + \dots$$

so that $\mathbf{L}_{\nu+1} = \Phi_\nu^{-1} \mathbf{L}_\nu \Phi_\nu$ has the desired properties. Recall that for any $\nu \geq 0$, \mathbf{L}_ν is of the form (7.12), $\mathbf{L}_\nu = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_\nu + \mathbf{R}_\nu$, and Ψ_ν is required to be a linear Hamiltonian vector field acting on $h_\perp^\sigma \times h_\perp^\sigma$,

$$\Psi_\nu = J \begin{pmatrix} \Psi_\nu^{(1)} & \Psi_\nu^{(2)} \\ \overline{\Psi}_\nu^{(2)} & \overline{\Psi}_\nu^{(1)} \end{pmatrix}, \quad \Psi_\nu^{(1)} = (\Psi_\nu^{(1)})^*, \quad \Psi_\nu^{(2)} = (\Psi_\nu^{(2)})^t. \quad (7.21)$$

The map Ψ_ν will be chosen to be a trigonometric polynomial in φ ,

$$\Psi_\nu(\varphi) = \sum_{\ell \in \mathbb{Z}^S, |\ell| \leq N_\nu} \hat{\Psi}_\nu(\ell) e^{i\ell \cdot \varphi}, \quad \hat{\Psi}_\nu(\ell) \in \mathcal{L}(h_\perp^{\sigma'} \times h_\perp^{\sigma'}), \quad \sigma' \in \{\sigma - 2, \sigma - 1, \sigma\}. \quad (7.22)$$

With Π_{N_ν} denoting the projector introduced in (2.15), and $\Pi_{N_\nu}^\perp = \text{Id} - \Pi_{N_\nu}$ we write

$$\mathbf{L}_\nu \Phi_\nu = \Phi_\nu(\omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_\nu) + (-\omega \cdot \partial_\varphi \Psi_\nu - [\mathbf{N}_\nu, \Psi_\nu] + \Pi_{N_\nu} \mathbf{R}_\nu) + \tilde{\mathbf{R}}_\nu, \quad (7.23)$$

where

$$\tilde{\mathbf{R}}_\nu := (\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 + \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 + \Psi_\nu] + \Pi_{N_\nu}^\perp \mathbf{R}_\nu + \mathbf{R}_\nu(\Phi_\nu - \mathbb{I}_2). \quad (7.24)$$

We remark that in a non-analytic setup such as ours, it is necessary for the convergence of the KAM scheme, to consider in (7.23), the truncation $\Pi_{N_\nu} \mathbf{R}_\nu$ of the Fourier expansion of \mathbf{R}_ν .

We look for a solution of the *homological* equation

$$-(\omega \cdot \partial_\varphi) \Psi_\nu - [\mathbf{N}_\nu, \Psi_\nu] + \Pi_{N_\nu} \mathbf{R}_\nu = \mathbf{R}_\nu^{nf} \quad (7.25)$$

where \mathbf{R}_ν^{nf} is given by

$$\mathbf{R}_\nu^{nf} := J \begin{pmatrix} \mathbf{A}_\nu^{(1)} & 0 \\ 0 & \overline{\mathbf{A}}_\nu^{(1)} \end{pmatrix}, \quad \mathbf{A}_\nu^{(1)} := \hat{\mathbf{R}}_\nu^{(1)}(0)^{\text{diag}}. \quad (7.26)$$

We recall that $\hat{\mathbf{R}}_\nu^{(1)}(0)^{\text{diag}}$ is defined in (7.19) and $\hat{\mathbf{R}}_\nu^{(1)}(0)$ denotes the 0th Fourier coefficient of \mathbf{R}_ν ,

$$\hat{\mathbf{R}}_\nu^{(1)}(0) = \frac{1}{(2\pi)^{|S|}} \int_{\mathbb{T}^S} \mathbf{R}_\nu^{(1)}(\varphi) d\varphi.$$

By (7.16), $\mathbf{A}_\nu^{(1)} = (\mathbf{A}_\nu^{(1)})^*$. For any $\ell \in \mathbb{Z}^S$ and $j, k \in S_+^\perp$, let us introduce the following linear operators on the vector space $\mathbb{C}^{2 \times 2}$ of 2×2 matrices with complex coefficients,

$$L_\nu^+(\ell, j, k) \equiv L_\nu^+(\ell, j, k; \omega) := \omega \cdot \ell \operatorname{Id}_{\mathbb{C}^{2 \times 2}} + M_L([\mathbf{N}_\nu^{(1)}]_j^j) + M_R([\overline{\mathbf{N}}_\nu^{(1)}]_k^k) \quad (7.27)$$

$$L_\nu^-(\ell, j, k) \equiv L_\nu^-(\ell, j, k; \omega) := \omega \cdot \ell \operatorname{Id}_{\mathbb{C}^{2 \times 2}} + M_L([\mathbf{N}_\nu^{(1)}]_j^j) - M_R([\mathbf{N}_\nu^{(1)}]_k^k), \quad (7.28)$$

where $\operatorname{Id}_{\mathbb{C}^{2 \times 2}}$ denotes the identity operator on $\mathbb{C}^{2 \times 2}$. Note that apart from the sign, $L_\nu^-(\ell, j, k)$ differs from $L_\nu^+(\ell, j, k)$ since $L_\nu^-(\ell, j, k)$ involves the operator $M_R([\mathbf{N}_\nu^{(1)}]_k^k)$ rather than $M_R([\overline{\mathbf{N}}_\nu^{(1)}]_k^k)$.

Furthermore, let $\Omega_0^\gamma(\iota) := \Omega_o(\iota)$ (cf (5.1)), and for any $\nu \geq 0$, let $\Omega_{\nu+1}^\gamma(\iota)$ be the subset of $\Omega_\nu^\gamma(\iota)$, consisting of all $\omega \in \Omega_\nu^\gamma(\iota)$ satisfying the so-called second order Melnikov conditions:

$(\mathbf{M}_{+, \gamma}^{II})_{\nu+1} \forall \ell \in \mathbb{Z}^S, |\ell| \leq N_\nu, \forall j, k \in S_+^\perp$, the operator $L_\nu^+(\ell, j, k; \omega)$ is invertible and

$$\left\| L_\nu^+(\ell, j, k; \omega)^{-1} \right\| \leq \frac{\langle \ell \rangle^\tau}{\gamma \langle j^2 + k^2 \rangle} \quad (7.29)$$

$(\mathbf{M}_{-, \gamma}^{II})_{\nu+1} \forall \ell \in \mathbb{Z}^S, |\ell| \leq N_\nu, \forall j, k \in S_+^\perp$ with $(\ell, j, k) \neq (0, j, j)$, the operator $L_\nu^-(\ell, j, k; \omega)$ is invertible and

$$\left\| L_\nu^-(\ell, j, k; \omega)^{-1} \right\| \leq \frac{\langle \ell \rangle^\tau}{\gamma \langle j^2 - k^2 \rangle}. \quad (7.30)$$

Since $[\mathbf{N}_\nu^{(1)}]_j^j$ is self-adjoint it follows from Lemma 7.2 (iii) that $L_\nu^\pm(\ell, j, k)$ are self-adjoint operators on $\mathbb{C}^{2 \times 2}$ for any $\ell \in \mathbb{Z}^S$ and $j, k \in S_+^\perp$. Therefore conditions (7.29), (7.30) are lower bounds for the modulus of the eigenvalues of $L_\nu^\pm(\ell, j, k)$. Note that by Lemma 7.2 (ii), the operator $L_\nu^-(0, j, j)$ has a zero eigenvalue, hence condition (7.30) is violated for $(\ell, j, k) = (0, j, j)$.

In the next lemma Condition (7.29) will be used to reduce $\mathbf{R}_\nu^{(2)}$, whereas (7.30) will be used for $\mathbf{R}_\nu^{(1)}$.

Lemma 7.3. (Homological equation) *For any $\omega \in \Omega_{\nu+1}^\gamma(\iota)$ there exists a unique solution Ψ_ν of the form (7.21) of the homological equation (7.25) with the normalization $[\hat{\Psi}_\nu^{(1)}(0)]_j^j = 0, j \in S_+^\perp$. For any $s_0 \leq s \leq s_* - \bar{\mu}$, the map Ψ_ν satisfies the following estimates*

$$|\Psi_\nu|_{s, \sigma}, \quad |\Psi_\nu \mathfrak{D}|_{s, \sigma-1} \leq \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1} N_\nu^\tau \quad (7.31)$$

$$|\Psi_\nu|_{s, \sigma}^{\gamma \text{lip}}, \quad |\Psi_\nu \mathfrak{D}|_{s, \sigma-1}^{\gamma \text{lip}} \leq \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}^{\gamma \text{lip}} N_\nu^{2\tau+1}. \quad (7.32)$$

As a consequence $\Psi_\nu \in H^s(\mathbb{T}^S, \mathcal{L}(h_{\perp}^{\sigma-2}))$ and

$$|\Psi_\nu|_{s, \sigma-2}^{\gamma \text{lip}} \leq \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}^{\gamma \text{lip}} N_\nu^{2\tau+1} \quad (7.33)$$

Proof. To simplify notations in this proof, we frequently drop the index ν in $N_\nu, \Psi_\nu, \mathbf{R}_\nu$ and simply write N, Ψ, \mathbf{R} instead. For any ω in $\Omega_\nu^\gamma(\iota)$, the homological equation (7.25), when expressed in Fourier coefficients, reads

$$i\omega \cdot \ell \hat{\Psi}(\ell) + [\mathbf{N}, \hat{\Psi}(\ell)] = \hat{\mathbf{R}}(\ell) - \hat{\mathbf{R}}^{nf}(\ell), \quad \forall \ell \in \mathbb{Z}^S, |\ell| \leq N.$$

In view of (7.22) it suffices to consider the equations for the components $\hat{\Psi}^{(1)}(\ell)$ and $\hat{\Psi}^{(2)}(\ell)$ with $|\ell| \leq N$,

$$\begin{aligned} \omega \cdot \ell \hat{\Psi}^{(2)}(\ell) + \mathbf{N}^{(1)} \hat{\Psi}^{(2)}(\ell) + \hat{\Psi}^{(2)}(\ell) \overline{\mathbf{N}}^{(1)} &= -i \hat{\mathbf{R}}^{(2)}(\ell), \\ \omega \cdot \ell \hat{\Psi}^{(1)}(\ell) + \mathbf{N}^{(1)} \hat{\Psi}^{(1)}(\ell) - \hat{\Psi}^{(1)}(\ell) \mathbf{N}^{(1)} &= -i \hat{\mathbf{R}}^{(1)}(\ell) + i \hat{\mathbf{R}}^{(1)}(0) \operatorname{diag} \delta_{0, \ell} \end{aligned}$$

where $\delta_{0, \ell} = 0$ for $\ell \neq 0$ and $\delta_{0, 0} = 1$. Taking into account that $[\hat{\Psi}^{(1)}(0)]_k^k = 0$ by the chosen normalization, the following equations then need to be solved ($|\ell| \leq N, j, k \in S_+^\perp$)

$$\begin{aligned} \omega \cdot \ell [\hat{\Psi}^{(2)}(\ell)]_j^k + [\mathbf{N}^{(1)}]_j^j [\hat{\Psi}^{(2)}(\ell)]_j^k + [\hat{\Psi}^{(2)}(\ell)]_j^k [\overline{\mathbf{N}}^{(1)}]_k^k &= -i [\hat{\mathbf{R}}^{(2)}(\ell)]_j^k, \quad \forall (\ell, j, k), \\ \omega \cdot \ell [\hat{\Psi}^{(1)}(\ell)]_j^k + [\mathbf{N}^{(1)}]_j^j [\hat{\Psi}^{(1)}(\ell)]_j^k - [\hat{\Psi}^{(1)}(\ell)]_j^k [\mathbf{N}^{(1)}]_k^k &= -i [\hat{\mathbf{R}}^{(1)}(\ell)]_j^k, \quad \forall (\ell, j, k) \neq (0, j, j). \end{aligned}$$

For any $\omega \in \Omega_{\nu+1}^\gamma(\ell)$, these equations admit unique solutions. We have

$$[\hat{\Psi}^{(2)}(\ell)]_j^k = -iL^+(\ell, j, k)^{-1}[\hat{\mathbf{R}}^{(2)}(\ell)]_j^k, \quad \forall \ell \in \mathbb{Z}^S, \quad |\ell| \leq N, \quad j, k \in S_+^\perp, \quad (7.34)$$

$$[\hat{\Psi}^{(1)}(\ell)]_j^k = -iL^-(\ell, j, k)^{-1}[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k, \quad \forall \ell \in \mathbb{Z}^S, \quad |\ell| \leq N, \quad (\ell, j, k) \neq (0, j, j). \quad (7.35)$$

The remaining Fourier coefficients of $\Psi^{(1)}$ and $\Psi^{(2)}$ are set equal to 0. By (7.29), (7.30) we deduce

$$\|[\hat{\Psi}^{(2)}(\ell)]_j^k\| \leq \frac{N^\tau}{\gamma\langle j^2 + k^2 \rangle} \|[\hat{\mathbf{R}}^{(2)}(\ell)]_j^k\|, \quad \|[\hat{\Psi}^{(1)}(\ell)]_j^k\| \leq \frac{N^\tau}{\gamma\langle j^2 - k^2 \rangle} \|[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k\|.$$

Estimate for $|\Psi\mathcal{D}|_{s, \sigma-1}$: In view of the definition operator norm (2.9), we need to estimate $\|\hat{\Psi}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\sigma-1}$. For any $h \in h_\perp^\sigma$ we have

$$\begin{aligned} \|(\hat{\Psi}^{(1)}(\ell)\langle\langle D \rangle\rangle)h\|_{\sigma-1}^2 &\leq \sum_{j \in S_+^\perp} \left(\sum_{k \in S_+^\perp} \|[\hat{\Psi}^{(1)}(\ell)]_j^k\| \langle\langle k \rangle\rangle |(h_{-k}, h_k)| \right)^2 \langle j \rangle^{2(\sigma-1)} \\ &\leq N^{2\tau} \gamma^{-2} \sum_{j \in S_+^\perp} \left(\|[\hat{\mathbf{R}}^{(1)}(\ell)]_j^j\| j |(h_{-j}, h_j)| + \sum_{k \in S_+^\perp, k \neq j} \frac{\|[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k\|}{|j-k|} \frac{k}{j+k} |(h_{-k}, h_k)| \right)^2 \langle j \rangle^{2(\sigma-1)}. \end{aligned}$$

Since

$$\sum_{j \in S_+^\perp} \|[\hat{\mathbf{R}}^{(1)}(\ell)\langle\langle D \rangle\rangle]_j^j\|^2 |(h_{-j}, h_j)|^2 \langle j \rangle^{2(\sigma-1)} \stackrel{\text{Lemma 7.1 (i)}}{\leq} \|\hat{\mathbf{R}}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}^2 \|h\|_{\sigma-1}^2$$

and

$$\sum_{j \in S_+^\perp} \left(\sum_{k \in S_+^\perp, k \neq j} \frac{\|[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k\|}{|j-k|} |(h_{-k}, h_k)| \right)^2 \langle j \rangle^{2(\sigma-1)} \stackrel{\text{Lemma 7.1 (iii)}}{\leq} \|\hat{\mathbf{R}}^{(1)}(\ell)\|_{\mathcal{L}(h_\perp^{\sigma-1})}^2 \|h\|_{\sigma-1}^2,$$

one sees that

$$\|\hat{\Psi}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})} \leq N^\tau \gamma^{-1} \|\hat{\mathbf{R}}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}.$$

A similar bound holds for $\hat{\Psi}^{(2)}(\ell)$, hence in view of the definition of the operator norm (2.9)

$$|\Psi\mathcal{D}|_{s, \sigma-1} \leq N^\tau \gamma^{-1} |\mathbf{R}\mathcal{D}|_{s, \sigma-1}.$$

Estimate for $|\Psi|_{s, \sigma}$: Since

$$\begin{aligned} &\sum_{j \in S_+^\perp} \left(\sum_{k \in S_+^\perp} \|[\hat{\Psi}^{(1)}(\ell)]_j^k\| |(h_{-k}, h_k)| \right)^2 \langle j \rangle^{2\sigma} \\ &\leq N^{2\tau} \gamma^{-2} \sum_{j \in S_+^\perp} \left(\|[\hat{\mathbf{R}}^{(1)}(\ell)]_j^j\| j |(h_{-j}, h_j)| + \sum_{k \in S_+^\perp, k \neq j} \frac{\|[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k\|}{|j-k|} \frac{j}{j+k} |(h_{-k}, h_k)| \right)^2 \langle j \rangle^{2(\sigma-1)}, \end{aligned}$$

the previous arguments yield

$$\|\hat{\Psi}^{(1)}(\ell)\|_{\mathcal{L}(h_\perp^\sigma)} \leq N^\tau \gamma^{-1} \|\hat{\mathbf{R}}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}.$$

Similar estimates also hold for $\hat{\Psi}^{(2)}(\ell)$ and hence $|\Psi|_{s, \sigma} \leq N^\tau \gamma^{-1} |\mathbf{R}\mathcal{D}|_{s, \sigma-1}$.

Estimate for $|\Psi\mathcal{D}|_{s, \sigma-1}^{\text{lip}}$: Let us first estimate $|\Psi^{(1)}\langle\langle D \rangle\rangle|_{s, \sigma-1}^{\text{lip}}$. For any $\omega_1, \omega_2 \in \Omega_{\nu+1}^\gamma(\ell)$ one has

$$L^-(\ell, j, k; \omega_1)^{-1} - L^-(\ell, j, k; \omega_2)^{-1} = L^-(\ell, j, k; \omega_2)^{-1} (L^-(\ell, j, k; \omega_2) - L^-(\ell, j, k; \omega_1)) L^-(\ell, j, k; \omega_1)^{-1}$$

with $L^-(\ell, j, k; \omega_2) - L^-(\ell, j, k; \omega_1)$ given by

$$(\omega_2 - \omega_1) \cdot \ell + M_L([\mathbf{N}^{(1)}(\omega_1) - \mathbf{N}^{(1)}(\omega_2)]_j^j) - M_R([\mathbf{N}^{(1)}(\omega_1) - \mathbf{N}^{(1)}(\omega_2)]_k^k).$$

Since by (7.15), $\|[\mathbf{N}^{(1)}]_j^j\|^{\text{lip}} \ll 1$ for any $j \in S_\perp^\perp$, we get

$$\|L^-(\ell, j, k; \omega_2) - L^-(\ell, j, k; \omega_1)\| \ll \langle \ell \rangle |\omega_1 - \omega_2| \ll N |\omega_1 - \omega_2|, \quad \forall \ell \in \mathbb{Z}^S \text{ with } |\ell| \leq N.$$

This together with (7.30) yields

$$\|L^-(\ell, j, k; \omega_1)^{-1} - L^-(\ell, j, k; \omega_2)^{-1}\| \ll \frac{N^{2\tau+1}}{\gamma^2(j^2 - k^2)^2} |\omega_1 - \omega_2|.$$

Arguing as in the proof of the estimate for $\|\hat{\Psi}^{(1)}(\ell)\langle D \rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}$, we get that for any $\ell \in \mathbb{Z}^S$, $|\ell| \leq N$,

$$\begin{aligned} \|(\hat{\Psi}^{(1)}(\ell; \omega_1) - \hat{\Psi}^{(1)}(\ell; \omega_2))\langle D \rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})} &\ll N^\tau \gamma^{-1} \|(\hat{\mathbf{R}}^{(1)}(\ell; \omega_1) - \hat{\mathbf{R}}^{(1)}(\ell; \omega_2))\langle D \rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})} \\ &\quad + N^{2\tau+1} \gamma^{-2} |\omega_1 - \omega_2| \|\hat{\mathbf{R}}^{(1)}(\ell; \omega_2)\|_{\mathcal{L}(h_\perp^{\sigma-1})} \end{aligned}$$

which in view of the definition of the norm $|\cdot|_{s, \sigma'}^{\text{lip}} = |\cdot|_{s, \sigma'}^{\text{sup}} + \gamma |\cdot|_{s, \sigma'}^{\text{lip}}$, implies that

$$\|\hat{\Psi}^{(1)}(\ell)\langle D \rangle\|_{s, \sigma-1}^{\text{lip}} \ll N^\tau \gamma^{-2} \cdot \gamma \|\hat{\mathbf{R}}^{(1)}(\ell)\langle D \rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}^{\text{lip}} + N^{2\tau+1} \gamma^{-2} \|\hat{\mathbf{R}}^{(1)}(\ell)\|_{\mathcal{L}(h_\perp^{\sigma-1})}^{\text{sup}} \ll N^{2\tau+1} \gamma^{-2} \|\hat{\mathbf{R}}^{(1)}(\ell)\langle D \rangle\|_{s, \sigma-1}^{\text{lip}}.$$

In the same way one proves the corresponding estimate for $|\hat{\Psi}^{(2)}(\ell)\langle D \rangle\|_{s, \sigma-1}^{\text{lip}}$, yielding altogether

$$\gamma |\Psi \mathfrak{D}|_{s, \sigma-1}^{\text{lip}} \ll N^{2\tau+1} \gamma^{-1} |\mathbf{R} \mathfrak{D}|_{s, \sigma-1}^{\text{lip}}.$$

Estimate for $|\Psi|_{s, \sigma}^{\text{lip}}$: In the same way one shows that $\gamma |\Psi|_{s, \sigma}^{\text{lip}} \ll N^{2\tau+1} \gamma^{-1} |\mathbf{R} \mathfrak{D}|_{s, \sigma-1}^{\text{lip}}$.

Combining the four estimates above then proves (7.32).

Estimate of $|\Psi|_{s, \sigma-2}^{\text{lip}}$: Since $\mathfrak{D} : h_\perp^{\sigma-1} \times h_\perp^{\sigma-1} \rightarrow h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}$ is a linear isomorphism, it follows from (7.32) that for any $\ell \in \mathbb{Z}^S$, $\hat{\Psi}(\ell) \in \mathcal{L}(h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ and that the claimed estimate (7.33) holds. \square

7.4 Proof of Theorem 7.1

Proof of $(\mathbf{S1})_\nu$: We prove $(\mathbf{S1})_\nu$ by induction with respect to $\nu \geq 1$. In view of the smallness assumption (7.10), the proof of $(\mathbf{S1})_1$ and the one of the inductive step are similar, hence we only consider the latter one: Assuming that $(\mathbf{S1})_\nu$ is true for a given $\nu \geq 1$, it is to prove that $(\mathbf{S1})_{\nu+1}$ holds. To simplify notations we write $|\cdot|_{s, \sigma-1}$ instead of $|\cdot|_{s, \sigma-1}^{\text{lip}}$. By Lemma 7.3, for any $\omega \in \Omega_{\nu+1}^\gamma(\iota)$, there exists a solution Ψ_ν of the homological equation (7.25) of the form (7.21), which by (7.32) satisfies for any $s_0 \leq s \leq s_* - \bar{\mu}$

$$|\Psi_\nu|_{s, \sigma}, |\Psi_\nu \mathfrak{D}|_{s, \sigma-1} \stackrel{(7.32)}{\ll} N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}. \quad (7.36)$$

By the induction hypothesis, (7.17) holds for any $s_0 \leq s \leq s_* - \bar{\mu} - \beta$ and hence

$$|\Psi_\nu|_{s, \sigma}, |\Psi_\nu \mathfrak{D}|_{s, \sigma-1} \ll N_\nu^{2\tau+1} N_{\nu-1}^{-\alpha} \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1} \quad (7.37)$$

which is the estimate (7.11) at the inductive step $\nu+1$. It follows that for any $\varphi \in \mathbb{T}^S$, $\Phi_\nu(\varphi) = \exp(-\Psi_\nu(\varphi))$ is bounded and invertible when viewed as an operator on $h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}$. Furthermore, in view of the definition (7.6) of N_ν and (7.8) of $\alpha \equiv \alpha(\tau)$ and by the assumption $\tau \geq |S|+1$, it also follows that for any $s_0 \leq s \leq s_* - \beta$, $\Phi_\nu^{\pm 1} = \exp(\mp \Psi_\nu)$ are maps in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}))$ and $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^\sigma \times h_\perp^\sigma))$. By (7.23) and (7.25) one has

$$\mathbf{L}_{\nu+1} = \Phi_\nu^{-1} \mathbf{L}_\nu \Phi_\nu = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_{\nu+1} + \mathbf{R}_{\nu+1}$$

where

$$\mathbf{N}_{\nu+1} := \mathbf{N}_\nu + \mathbf{R}_\nu^{nf}, \quad \mathbf{R}_{\nu+1} := \Phi_\nu^{-1} \tilde{\mathbf{R}}_\nu + (\Phi_\nu^{-1} - \mathbb{I}_2) \mathbf{R}_\nu^{nf} \quad (7.38)$$

and $\tilde{\mathbf{R}}_\nu$ is defined in (7.24). By construction, $\mathbf{N}_{\nu+1}$ is of the form (7.13)-(7.14). In particular by (7.26), $[\mathbf{N}_{\nu+1}^{(1)} - \mathbf{N}_\nu^{(1)}]_k^k = [\hat{\mathbf{R}}_\nu^{(1)}(0)]_k^k$ for any $k \in S_\perp^\perp$ and hence

$$\|[\mathbf{N}_{\nu+1}^{(1)} - \mathbf{N}_\nu^{(1)}]_k^k\|^{\text{lip}} \ll |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1}^{\text{lip}} k^{-1}, \quad (7.39)$$

establishing the first estimate of (7.15) at the inductive step $\nu + 1$. To prove the second estimate write $[\mathbf{N}_{\nu+1}^{(1)}]_j^j = [\mathbf{N}_0^{(1)}]_j^j + \sum_{n=1}^{\nu+1} [\mathbf{N}_n^{(1)} - \mathbf{N}_{n-1}^{(1)}]_j^j$ as a telescoping sum, and use the estimates

$$\|[\mathbf{N}_0^{(1)}]_j^j\|^{\text{lip}} \stackrel{(7.3), (6.98)}{\ll} 1, \quad \forall j \in S^\perp, \quad (7.40)$$

$\|[\mathbf{N}_n^{(1)} - \mathbf{N}_{n-1}^{(1)}]_j^j\|^{\gamma \text{lip}} \ll |\mathbf{R}_{n-1} \mathfrak{D}|_{s_0, \sigma-1}^{\gamma \text{lip}} j^{-1}$ (by (7.15)), and $|\mathbf{R}_{n-1} \mathfrak{D}|_{s_0, \sigma-1}^{\gamma \text{lip}} \leq |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1}^{\gamma \text{lip}} N_{n-2}^{-\alpha}$ (by (7.17)) to conclude that $\|[\mathbf{N}_{\nu+1}^{(1)}]_j^j\|^{\text{lip}} \ll 1 + \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1}^{\gamma \text{lip}} \stackrel{(7.10)}{\ll} 1$.

Since by Lemma 3.1, $\mathbf{L}_{\nu+1}$ is a linear Hamiltonian operator, so is $\mathbf{R}_{\nu+1}$ and hence has the form (7.16). It remains to verify the claimed estimate (7.17) for $\mathbf{R}_{\nu+1}$. To this end, we first need to establish estimates for $\Phi_\nu^{\pm 1}$ which we derive from Lemma 2.10. Indeed, one has

$$\begin{aligned} |(\Phi_\nu^{\pm 1} - \mathbb{I}_2) \mathfrak{D}|_{s, \sigma-1} &\stackrel{\text{Lemma 2.10 (ii)}}{\leq_s} |\Psi_\nu \mathfrak{D}|_{s, \sigma-1} \stackrel{(7.36)}{\leq_s} N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}, \\ |\Phi_\nu^{\pm 1} - \mathbb{I}_2|_{s, \sigma} &\stackrel{\text{Lemma 2.10 (i)}}{\leq_s} |\Psi_\nu|_{s, \sigma} \stackrel{(7.36)}{\leq_s} N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}. \end{aligned} \quad (7.41)$$

We now estimate $\mathbf{R}_{\nu+1} = \Phi_\nu^{-1} \tilde{\mathbf{R}}_\nu + (\Phi_\nu^{-1} - \mathbb{I}_2) \mathbf{R}_\nu^{nf}$ where we recall that

$$\tilde{\mathbf{R}}_\nu := (\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 - \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 - \Psi_\nu] + (\Pi_{N_\nu} \mathbf{R}_\nu)(\Phi_\nu - \mathbb{I}_2) + (\Pi_{N_\nu}^\perp \mathbf{R}_\nu) \Phi_\nu.$$

The terms in $\mathbf{R}_{\nu+1}$ are estimated individually. One has

$$(\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 - \Psi_\nu) = \sum_{n \geq 2} (-1)^n \frac{(\omega \cdot \partial_\varphi)(\Psi_\nu^n)}{n!}, \quad (\omega \cdot \partial_\varphi)(\Psi_\nu^n) = \sum_{n_1+n_2+1=n} \Psi_\nu^{n_1} (\omega \cdot \partial_\varphi \Psi_\nu) \Psi_\nu^{n_2}, \quad \forall n \geq 2.$$

Furthermore writing

$$[\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 - \Psi_\nu] = \sum_{n \geq 2} (-1)^n \frac{[\mathbf{N}_\nu, \Psi_\nu^n]}{n!},$$

and using that by the homological equation (7.25), $[\mathbf{N}_\nu, \Psi_\nu^n] = \sum_{n_1+n_2+1=n} \Psi_\nu^{n_1} [\mathbf{N}_\nu, \Psi_\nu] \Psi_\nu^{n_2}$ equals

$$- \sum_{n_1+n_2+1=n} \Psi_\nu^{n_1} (\omega \cdot \partial_\varphi \Psi_\nu) \Psi_\nu^{n_2} + \sum_{n_1+n_2+1=n} \Psi_\nu^{n_1} (\Pi_{N_\nu} \mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^{n_2},$$

one obtains altogether

$$(\omega \cdot \partial_\varphi)(\Psi_\nu^n) + [\mathbf{N}_\nu, \Psi_\nu^n] = \sum_{n_1+n_2+1=n} \Psi_\nu^{n_1} (\Pi_{N_\nu} \mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^{n_2}. \quad (7.42)$$

Choosing $C(s) > 2C_{op}(s)$ large enough with $C_{op}(s)$ as in Lemma 2.10 we get for any $n \geq 2$,

$$\begin{aligned} |((\omega \cdot \partial_\varphi)(\Psi_\nu^n) + [\mathbf{N}_\nu, \Psi_\nu^n]) \mathfrak{D}|_{s, \sigma-1} &\stackrel{(2.23)}{\leq} n(C(s) |\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1})^{n-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1} \\ &+ n(n-1)(C(s) |\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1})^{n-2} C(s) |\Psi_\nu \mathfrak{D}|_{s, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} \\ &\stackrel{(7.36)}{\leq} n^2 C(s)^{n-1} (|\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1})^{n-2} N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}. \end{aligned}$$

Choosing $N_0 = N_0(s_*, \tau, |S|) > 0$ in (7.6) large enough so that

$$|\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1} \stackrel{(7.37)}{\ll} N_\nu^{2\tau+1} N_{\nu-1}^{-\alpha} \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} \stackrel{(7.8), (7.10)}{\leq} 1 \quad (7.43)$$

one then obtains

$$|((\omega \cdot \partial_\varphi)(\Psi_\nu^n) + [\mathbf{N}_\nu, \Psi_\nu^n]) \mathfrak{D}|_{s, \sigma-1} \stackrel{(7.43)}{\leq} n^2 C(s)^{n-1} N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}$$

which implies

$$|((\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 - \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 - \Psi_\nu])\mathfrak{D}|_{s, \sigma-1} \leq_s N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}.$$

Furthermore, by (2.21) and (7.41) one has

$$|(\Pi_{N_\nu} \mathbf{R}_\nu)(\Phi_\nu - \mathbb{I}_2)\mathfrak{D}|_{s, \sigma-1}, \quad |(\Phi_\nu^{-1} - \mathbb{I}_2)\mathbf{R}_\nu^{nf}\mathfrak{D}|_{s, \sigma-1} \leq_s N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1},$$

yielding, with $\Phi_\nu = \mathbb{I}_2 + (\Phi_\nu - \mathbb{I}_2)$,

$$|(\Pi_{N_\nu}^\perp \mathbf{R}_\nu)\Phi_\nu \mathfrak{D}|_{s, \sigma-1} \leq_s |(\Pi_{N_\nu}^\perp \mathbf{R}_\nu)\mathfrak{D}|_{s, \sigma-1} + N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1}.$$

Combining the estimates above with the estimate $|\Psi_\nu|_{s, \sigma-1} \stackrel{(7.36)}{\leq} N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}$ and using again (2.21) and the smallness assumption (7.10) one then gets

$$|\mathbf{R}_{\nu+1} \mathfrak{D}|_{s, \sigma-1} \leq_s |(\Pi_{N_\nu}^\perp \mathbf{R}_\nu)\mathfrak{D}|_{s, \sigma-1} + N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1}, \quad (7.44)$$

which by the induction hypothesis leads to

$$\begin{aligned} |\mathbf{R}_{\nu+1} \mathfrak{D}|_{s, \sigma-1} &\stackrel{(2.16)}{\leq_s} N_\nu^{-\beta} |\mathbf{R}_\nu \mathfrak{D}|_{s+\beta, \sigma-1} + N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} \\ &\stackrel{(7.17)}{\leq} C(s) (N_\nu^{-\beta} N_{\nu-1} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1} + N_\nu^{2\tau+1} \gamma^{-1} N_{\nu-1}^{-2\alpha} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1}). \end{aligned} \quad (7.45)$$

In order to insure that $|\mathbf{R}_{\nu+1} \mathfrak{D}|_{s, \sigma-1}$ can be bounded by $|\mathbf{R}_0 \mathfrak{D}|_{s, \sigma-1} N_\nu^{-\alpha}$ we need that for any $\nu \geq 0$

$$C(s) N_\nu^{-\beta} N_{\nu-1} N_\nu^\alpha \leq 1/2 \quad \text{and} \quad C(s) N_\nu^{2\tau+1} N_{\nu-1}^{-2\alpha} N_\nu^\alpha |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} \gamma^{-1} \leq 1/2.$$

The latter conditions are fulfilled since by (7.8) $\beta = \alpha + 1$, $\alpha = 6\tau + 4$ and by (7.10), $N_0^{C_0} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} \gamma^{-1} \leq 1$, with $C_0 = 2\tau + 2 + \alpha$, taking N_0 large enough. Thus the first inequality of (7.17) at the inductive step $\nu + 1$ is verified. By (7.44), applied for $s + \beta$ with $s_0 \leq s \leq s_* - \bar{\mu} - \beta$, we get

$$|\mathbf{R}_{\nu+1} \mathfrak{D}|_{s+\beta, \sigma-1} \leq_{s+\beta} |\mathbf{R}_\nu \mathfrak{D}|_{s+\beta, \sigma-1} + N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu \mathfrak{D}|_{s+\beta, \sigma-1} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1}. \quad (7.46)$$

Then (7.46), (7.17), (7.10), (7.8) imply the inequality

$$|\mathbf{R}_{\nu+1} \mathfrak{D}|_{s+\beta, \sigma-1} \leq_{s+\beta} |\mathbf{R}_\nu \mathfrak{D}|_{s+\beta, \sigma-1},$$

whence by the induction hypothesis (7.17) we get

$$|\mathbf{R}_{\nu+1} \mathfrak{D}|_{s+\beta, \sigma-1} \leq N_\nu |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1}$$

for $N_0 = N_0(s_*, \tau, S) > 0$ in (7.10) large enough, which is the second inequality of (7.17) at the step $\nu + 1$.

Proof of (S2) $_{\nu+1}$: For any $k \in S_+^\perp$

$$\|[\mathbf{N}_{\nu+1}^{(1)}]_k^k - [\mathbf{N}_\nu^{(1)}]_k^k\|^{\gamma_{\text{lip}}} \stackrel{(7.39)}{\leq} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} k^{-1} \stackrel{(7.17)}{\leq} N_{\nu-1}^{-\alpha} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} k^{-1} \quad (7.47)$$

where the Lipschitz seminorm is computed on $\Omega_{\nu+1}^\gamma(\iota)$. By Lemma M.5 in [23] and its proof, the matrix elements of $[\mathbf{N}_\nu^\Delta]_k^k := [\mathbf{N}_{\nu+1}^{(1)}]_k^k - [\mathbf{N}_\nu^{(1)}]_k^k$ can be extended to all of $\Omega_\sigma(\iota)$ so that the extension $[\tilde{\mathbf{N}}_\nu^\Delta]_k^k$ of $[\mathbf{N}_\nu^\Delta]_k^k$ is Lipschitz, self-adjoint and satisfies the estimate (7.47). (S2) $_{\nu+1}$ then follows by setting

$$[\tilde{\mathbf{N}}_{\nu+1}^{(1)}]_k^k := [\tilde{\mathbf{N}}_\nu^{(1)}]_k^k + [\tilde{\mathbf{N}}_\nu^\Delta]_k^k.$$

This concludes the proof of Theorem 7.1.

7.5 2×2 block diagonalization of \mathbf{L}_0

In this subsection we study the limit of the sequence of operators \mathbf{L}_ν , introduced in Theorem 7.1, and show that it is the 2×2 block diagonalization of \mathbf{L}_0 . Recall that, for any $k \in S_+^\perp$, the 2×2 matrices $[\tilde{\mathbf{N}}_\nu^{(1)}]_k^k$, $\nu \geq 1$, were introduced in $(\mathbf{S2})_\nu$ of Theorem 7.1 and that $[\tilde{\mathbf{N}}_0^{(1)}]_k^k$ is given by $[\mathbf{N}_0^{(1)}]_k^k$.

Lemma 7.4. *Assume that (7.10) holds. Then for any $k \in S_+^\perp$, the sequence $([\tilde{\mathbf{N}}_\nu^{(1)}]_k^k)_{\nu \geq 0}$ converges in the norm $\|\cdot\|^{\gamma\text{lip}}$ to a φ -independent 2×2 matrix $[\mathbf{N}_\infty^{(1)}]_k^k$. The limit $[\mathbf{N}_\infty^{(1)}]_k^k$ is self-adjoint and satisfies the estimate*

$$\|[\mathbf{N}_\infty^{(1)}]_k^k - [\tilde{\mathbf{N}}_\nu^{(1)}]_k^k\|^{\gamma\text{lip}} \leq N_{\nu-1}^{-\alpha} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} k^{-1}, \quad \forall \nu \geq 0. \quad (7.48)$$

Proof. Note that for any $k \in S_+^\perp$ and any $\nu \geq 0$

$$\begin{aligned} \sum_{n \geq \nu+1} \|[\tilde{\mathbf{N}}_n^{(1)}]_k^k - [\tilde{\mathbf{N}}_{n-1}^{(1)}]_k^k\|^{\gamma\text{lip}} &\stackrel{(7.18)}{\leq} \sum_{n \geq \nu+1} |\mathbf{R}_{n-1} \mathfrak{D}|_{s_0, \sigma-1}^{\gamma\text{lip}} k^{-1} \\ &\stackrel{(7.17)}{\leq} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1}^{\gamma\text{lip}} k^{-1} \sum_{n \geq \nu+1} N_{n-2}^{-\alpha} \stackrel{(7.6), (7.8)}{\leq} N_{\nu-1}^{-\alpha} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1}^{\gamma\text{lip}} k^{-1}. \end{aligned}$$

Hence the sequence $[\tilde{\mathbf{N}}_\nu^{(1)}]_k^k$ has a limit, denoted by $[\mathbf{N}_\infty^{(1)}]_k^k$, and (7.48) holds. Since $[\mathbf{N}_0^{(1)}]_k^k$ (by (7.3)) and $[\tilde{\mathbf{N}}_\nu^{(1)}]_k^k$ (by $(\mathbf{S2})_\nu$) are self-adjoint so is $[\mathbf{N}_\infty^{(1)}]_k^k$. \square

In Theorem 7.2 below we prove that \mathbf{L}_0 is conjugated to the normal form Hamiltonian operator

$$\mathbf{L}_\infty(\omega) := \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_\infty(\omega) \quad (7.49)$$

where

$$\mathbf{N}_\infty := J \begin{pmatrix} \mathbf{N}_\infty^{(1)} & 0 \\ 0 & \overline{\mathbf{N}}_\infty^{(1)} \end{pmatrix}, \quad \mathbf{N}_\infty^{(1)} := \text{diag}_{k \in S_+^\perp} [\mathbf{N}_\infty^{(1)}]_k^k. \quad (7.50)$$

To this end we study the compositions of the symplectic transformations Φ_ν , $\nu \geq 0$, introduced in $(\mathbf{S1})_\nu$ of Theorem 7.1. For any $\nu \geq 0$, we define

$$\tilde{\Phi}_\nu := \Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_\nu.$$

Lemma 7.5. (Composition of Φ_ν) *Assume that (7.10) holds with $N_0 = N_0(s_*, \tau, |S|) > 0$ sufficiently large. Then on the set $\cap_{\nu \geq 0} \Omega_\nu^\gamma(\iota)$, the sequence of symplectic transformations $\tilde{\Phi}_\nu$ converges to an invertible map Φ_∞ in the norm $|\cdot|_{s, \sigma'}^{\gamma\text{lip}}$ for $\sigma' = \sigma, \sigma - 2$ and $s \in [s_0, s_* - \bar{\mu} - \beta]$. Moreover $\Phi_\infty, \Phi_\infty^{-1}$ are symplectic and satisfy the estimates*

$$|\Phi_\infty^{\pm 1} - \mathbb{I}_2|_{s, \sigma-2}^{\gamma\text{lip}}, \quad |\Phi_\infty^{\pm 1} - \mathbb{I}_2|_{s, \sigma}^{\gamma\text{lip}} \leq s \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1}^{\gamma\text{lip}}.$$

Proof. To simplify notations we write $|\cdot|_{s, \sigma-1}$ instead of $|\cdot|_{s, \sigma-1}^{\gamma\text{lip}}$. For any $\nu \geq 0$, write

$$\Phi_\nu = \mathbb{I}_2 + \Psi_\nu^\Sigma, \quad \Psi_\nu^\Sigma := \sum_{n \geq 1} \frac{\Psi_\nu^n}{n!}.$$

By (7.11) and the smallness condition (7.10), as specified in (7.43), we get $C(s_*) |\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1} \leq 1$, where $C(s)$ denotes the same constant as in (7.43). Hence, for any $s \in [s_0, s_* - \beta]$, we obtain

$$|\Psi_\nu^\Sigma \mathfrak{D}|_{s, \sigma-1} \stackrel{\text{Lemma 2.10}}{\leq_s} |\Psi_\nu \mathfrak{D}|_{s, \sigma-1} \stackrel{(7.11)}{\leq} \varepsilon_\nu(s), \quad \varepsilon_\nu(s) := K(s) \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1} N_\nu^{2\tau+1} N_{\nu-1}^{-\alpha} \quad (7.51)$$

for some constant $K(s) \geq C(s)$, chosen to be increasing in s . In particular one has

$$|\Phi_\nu - \mathbb{I}_2|_{s, \sigma-1} \leq \varepsilon_\nu(s). \quad (7.52)$$

We claim that for any $\nu \geq 0$ and $s \in [s_0, s_* - \beta]$,

$$|\tilde{\Phi}_\nu - \mathbb{I}_2|_{s, \sigma-1} \leq 2\varepsilon_0(s). \quad (7.53)$$

To prove it we argue by induction. For $\nu = 0$, inequality (7.53) follows from (7.52) since $\tilde{\Phi}_0 = \Phi_0$. To prove the inductive step from ν to $\nu + 1$, we write $\tilde{\Phi}_{\nu+1} - \mathbb{I}_2$ as a telescoping sum

$$\tilde{\Phi}_{\nu+1} - \mathbb{I}_2 = \sum_{k=0}^{\nu} (\tilde{\Phi}_{k+1} - \tilde{\Phi}_k) + \tilde{\Phi}_0 - \mathbb{I}_2. \quad (7.54)$$

Using that

$$\tilde{\Phi}_{k+1} - \tilde{\Phi}_k = (\tilde{\Phi}_k - \mathbb{I}_2)(\Phi_{k+1} - \mathbb{I}_2) + \Phi_{k+1} - \mathbb{I}_2,$$

one has by Lemma 2.8 and by (7.52)

$$|\tilde{\Phi}_{k+1} - \tilde{\Phi}_k|_{s, \sigma-1} \leq C_{op}(s) |\tilde{\Phi}_k - \mathbb{I}_2|_{s_0, \sigma-1} \varepsilon_{k+1}(s) + C_{op}(s) |\tilde{\Phi}_k - \mathbb{I}_2|_{s, \sigma-1} \varepsilon_{k+1}(s_0) + \varepsilon_{k+1}(s).$$

By the induction hypothesis, $|\tilde{\Phi}_k - \mathbb{I}_2|_{s, \sigma-1} \leq 2\varepsilon_0(s)$. Since by (7.51) $2\varepsilon_0(s)\varepsilon_{k+1}(s_0) = 2\varepsilon_0(s_0)\varepsilon_{k+1}(s)$ one sees that $|\tilde{\Phi}_k - \mathbb{I}_2|_{s, \sigma-1} \varepsilon_{k+1}(s_0) \leq 2\varepsilon_0(s_0)\varepsilon_{k+1}(s)$, yielding with $C(s) = 2C_{op}(s)$ altogether

$$|\tilde{\Phi}_{k+1} - \tilde{\Phi}_k|_{s, \sigma-1} \leq (2C(s)\varepsilon_0(s_0) + 1)\varepsilon_{k+1}(s).$$

Substituting this estimate into (7.54) leads to

$$|\tilde{\Phi}_{\nu+1} - \mathbb{I}_2|_{s, \sigma-1} \leq (2C(s)\varepsilon_0(s_0) + 1) \sum_{k=0}^{\nu} \varepsilon_{k+1}(s) + \varepsilon_0(s).$$

With N_0 in (7.11) chosen large enough, it follows that $|\tilde{\Phi}_{\nu+1} - \mathbb{I}_2|_{s, \sigma-1} \leq 2\varepsilon_0(s)$ and hence (7.53) is established. Finally for all $\nu_2 > \nu_1 > 0$

$$\begin{aligned} |(\tilde{\Phi}_{\nu_2} - \tilde{\Phi}_{\nu_1})\mathfrak{D}|_{s, \sigma-1} &\leq \sum_{\nu=\nu_1}^{\nu_2-1} |(\tilde{\Phi}_{\nu+1} - \tilde{\Phi}_\nu)\mathfrak{D}|_{s, \sigma-1} \\ &= \sum_{\nu=\nu_1}^{\nu_2-1} |\tilde{\Phi}_\nu \Psi_{\nu+1}^\Sigma \mathfrak{D}|_{s, \sigma-1} \stackrel{(2.22)}{\leq} \sum_{\nu=\nu_1}^{\nu_2-1} \left(|\tilde{\Phi}_\nu|_{s, \sigma-1} |\Psi_{\nu+1}^\Sigma \mathfrak{D}|_{s_0, \sigma-1} + |\tilde{\Phi}_\nu|_{s_0, \sigma-1} |\Psi_{\nu+1}^\Sigma \mathfrak{D}|_{s, \sigma-1} \right) \\ &\stackrel{(7.51), (7.52)}{\leq} \sum_{\nu=\nu_1}^{\nu_2-1} \left((1 + 2\varepsilon_0(s))\varepsilon_{\nu+1}(s_0) + (1 + 2\varepsilon_0(s_0))\varepsilon_{\nu+1}(s) \right). \end{aligned}$$

Using again $\varepsilon_0(s)\varepsilon_{\nu+1}(s_0) = \varepsilon_0(s_0)\varepsilon_{\nu+1}(s)$, it then follows from the smallness assumption (7.10) that

$$|(\tilde{\Phi}_{\nu_2} - \tilde{\Phi}_{\nu_1})\mathfrak{D}|_{s, \sigma-1} \leq_s \varepsilon_{\nu_1}(s) \leq_s \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta, \sigma-1} N_{\nu_1}^{2\tau+1} N_{\nu_1-1}^{-\alpha}$$

Therefore the sequence $((\tilde{\Phi}_\nu - \mathbb{I}_2)\mathfrak{D})_{\nu \geq 0}$ is a Cauchy sequence with respect to the norm $|\cdot|_{s, \sigma-1}$ and hence converges in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^{\sigma-1} \times h_\perp^{\sigma-1}))$. It then follows that $(\tilde{\Phi}_\nu)_{\nu \geq 0}$ is a Cauchy sequence in the space $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}))$ and hence has a limit $\tilde{\Phi}_\infty$ in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^{\sigma-2} \times h_\perp^{\sigma-2}))$. Since $\tilde{\Phi}_\nu^{-1} = \exp(\Psi_\nu)$, one can show by the same arguments that the sequence $(\tilde{\Phi}_\nu^{-1})_{\nu \geq 0}$ satisfies the same bounds. Since $\tilde{\Phi}_\nu \tilde{\Phi}_\nu^{-1} = \mathbb{I}_2$ for all $\nu \geq 0$, the limit of $(\tilde{\Phi}_\nu^{-1})_{\nu \geq 0}$ is equal to $\tilde{\Phi}_\infty^{-1}$. By the same arguments one shows that $(\tilde{\Phi}_\nu^{\pm 1})_{\nu \geq 0}$ is a Cauchy sequence in $H^s(\mathbb{T}^S, \mathcal{L}(h_\perp^\sigma \times h_\perp^\sigma))$ and hence it also converges in this space to (the restriction of) $\tilde{\Phi}_\infty^{\pm 1}$. By Theorem 7.1, the maps $\tilde{\Phi}_\nu$ are symplectic for any $\nu \geq 0$ and hence by the characterization (3.18) of symplectic maps, so are $\tilde{\Phi}_\nu$ and in turn $\tilde{\Phi}_\infty^{\pm 1}$. \square

For any $\ell \in \mathbb{Z}^S$, $j, k \in S_+^\perp$ and $\omega \in \Omega_o(\iota)$, we define

$$L_\infty^+(\ell, j, k) \equiv L_\infty^+(\ell, j, k; \omega) := \omega \cdot \ell \operatorname{Id}_{\mathbb{C}^{2 \times 2}} + M_L([\mathbf{N}_\infty^{(1)}]_j^j) + M_R([\overline{\mathbf{N}}_\infty^{(1)}]_k^k) \quad (7.55)$$

$$L_\infty^-(\ell, j, k) \equiv L_\infty^-(\ell, j, k; \omega) := \omega \cdot \ell \operatorname{Id}_{\mathbb{C}^{2 \times 2}} + M_L([\mathbf{N}_\infty^{(1)}]_j^j) - M_R([\mathbf{N}_\infty^{(1)}]_k^k) \quad (7.56)$$

and the set

$$\Omega_\infty^{2\gamma}(\iota) := \{\omega \in \Omega_o(\iota) : (\mathbf{M}_{+,2\gamma}^{II})_\infty, (\mathbf{M}_{-,2\gamma}^{II})_\infty \text{ hold}\} \quad (7.57)$$

where $(\mathbf{M}_{+,2\gamma}^{II})_\infty, (\mathbf{M}_{-,2\gamma}^{II})_\infty$ are the following second order Melnikov conditions:

$(\mathbf{M}_{+,2\gamma}^{II})_\infty$ For any $\ell \in \mathbb{Z}^S$, $j, k \in S_+^\perp$, the operator $L_\infty^+(\ell, j, k; \omega)$ is invertible and

$$\|L_\infty^+(\ell, j, k; \omega)^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma \langle j^2 + k^2 \rangle}. \quad (7.58)$$

$(\mathbf{M}_{-,2\gamma}^{II})_\infty$ For any $\ell \in \mathbb{Z}^S$, $j, k \in S_+^\perp$ with $(\ell, j, k) \neq (0, j, j)$, the operator $L_\infty^-(\ell, j, k; \omega)$ is invertible and

$$\|L_\infty^-(\ell, j, k; \omega)^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma \langle j^2 - k^2 \rangle}. \quad (7.59)$$

We remark that the superindex 2γ in $\Omega_\infty^{2\gamma}(\iota)$ stands for the factor 2γ in the denominator of the bounds in (7.58) and (7.59). The set can be localized as follows:

Lemma 7.6. *If (7.10) holds, with $N_0 = N_0(s_*, \tau, |S|) > 0$ sufficiently large, then $\Omega_\infty^{2\gamma}(\iota) \subseteq \cap_{\nu \geq 0} \Omega_\nu^\gamma(\iota)$.*

Proof. Note that by the definition (7.7), $(\Omega_\nu^\gamma(\iota))_{\nu \geq 0}$ is a decreasing sequence. Hence it suffices to show that for any $\nu \geq 0$, $\Omega_\infty^{2\gamma}(\iota) \subseteq \Omega_\nu^\gamma(\iota)$. We argue by induction. Since $\Omega_0^\gamma(\iota) = \Omega_o(\iota)$ by (7.7), it follows from the definition (7.57) that $\Omega_\infty^{2\gamma}(\iota) \subseteq \Omega_0^\gamma(\iota)$. To prove the inductive step from ν to $\nu + 1$ we have to verify that $\Omega_\infty^{2\gamma}(\iota) \subseteq \Omega_{\nu+1}^\gamma(\iota)$. Let $\omega \in \Omega_\infty^{2\gamma}(\iota)$. By the induction hypothesis we know that $\omega \in \Omega_\nu^\gamma(\iota)$. Theorem 7.1 then implies that the 2×2 matrices $[\mathbf{N}_\nu^{(1)}(\omega)]_k^k$, $k \in S_+^\perp$, are well defined and that $[\mathbf{N}_\nu^{(1)}(\omega)]_k^k = [\tilde{\mathbf{N}}_\nu^{(1)}(\omega)]_k^k$. By the definitions (7.27) and (7.28), also the matrices $L_\nu^\pm(\ell, j, k; \omega)$ are well defined. Since $\omega \in \Omega_\infty^{2\gamma}(\iota)$, $L_\infty^-(\ell, j, k; \omega)$ is invertible and we may write

$$L_\nu^-(\ell, j, k; \omega) = L_\infty^-(\ell, j, k; \omega) + L_\Delta^-(\ell, j, k; \omega) = L_\infty^-(\ell, j, k; \omega) (\operatorname{Id}_{\mathbb{C}^{2 \times 2}} + L_\infty^-(\ell, j, k; \omega)^{-1} L_\Delta^-(\ell, j, k; \omega))$$

where

$$L_\Delta^-(\ell, j, k; \omega) := M_L([\mathbf{N}_\nu^{(1)}(\omega) - \mathbf{N}_\infty^{(1)}(\omega)]_j^j) - M_R([\mathbf{N}_\nu^{(1)}(\omega) - \mathbf{N}_\infty^{(1)}(\omega)]_k^k).$$

By the estimate (7.48)

$$\|L_\Delta^-(\ell, j, k; \omega)\| \leq N_{\nu-1}^{-\alpha} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} k^{-1}.$$

By (7.59) it then follows that for any $|\ell| \leq N_\nu$ and $j, k \in S_+^\perp$, with $(\ell, j, k) \neq (0, j, j)$

$$\|L_\infty^-(\ell, j, k; \omega)^{-1} L_\Delta^-(\ell, j, k; \omega)\| \leq C \frac{N_\nu^\tau N_{\nu-1}^{-\alpha}}{2\gamma \langle j^2 - k^2 \rangle} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} \stackrel{(7.8), (7.10)}{\leq} \frac{1}{2}, \quad (7.60)$$

with $N_0 > 0$ in (7.10) large enough. Hence the 2×2 matrix $L_\nu^-(\ell, j, k; \omega)$ is invertible, with inverse given by a Neumann series. For all $|\ell| \leq N_\nu$, $j, k \in S_+^\perp$ with $(\ell, j, k) \neq (0, j, j)$

$$\|L_\nu^-(\ell, j, k; \omega)^{-1}\| \leq \frac{\|L_\infty^-(\ell, j, k; \omega)^{-1}\|}{1 - \|L_\infty^-(\ell, j, k; \omega)^{-1} L_\Delta^-(\ell, j, k; \omega)\|} \stackrel{(7.60)}{\leq} 2 \|L_\infty^-(\ell, j, k; \omega)^{-1}\| \stackrel{(7.59)}{\leq} \frac{\langle \ell \rangle^\tau}{\gamma \langle j^2 - k^2 \rangle}.$$

By similar arguments, one can prove that, for any $|\ell| \leq N_\nu$ and $j, k \in S_+^\perp$

$$\|L_\nu^+(\ell, j, k; \omega)^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{\gamma \langle j^2 + k^2 \rangle}.$$

Hence, by the definition (7.7), $\omega \in \Omega_{\nu+1}^\gamma(\iota)$ and the inductive step is proved. \square

As advertised we now prove that \mathbf{L}_0 is conjugated to the normal form Hamiltonian operator \mathbf{L}_∞ :

Theorem 7.2. (2×2 diagonalization of \mathbf{L}_0) *There exists $0 < \delta \equiv \delta(|S|, \tau, s_*) < 1$ such that for any $\iota : \mathbb{T}^S \times \Omega_o(\iota) \rightarrow M^\sigma$ with*

$$\|\iota\|_{s_0+\bar{\mu}+\beta}^{\gamma\text{lip}} \leq C\varepsilon\gamma^{-2}, \quad \varepsilon\gamma^{-4} \leq \delta, \quad (7.61)$$

where $\bar{\mu}$ is given as in (7.1), and β as in (7.8), the following holds:

(i) For any $\omega \in \Omega_\infty^{2\gamma}(\iota)$ and $s \in [s_0, s_* - \bar{\mu} - \beta]$, the transformations $\Phi_\infty, \Phi_\infty^{-1}$ satisfy the estimates

$$|\Phi_\infty^{\pm 1} - \mathbb{I}_2|_{s,\sigma}^{\gamma\text{lip}}, \quad |\Phi_\infty^{\pm 1} - \mathbb{I}_2|_{s,\sigma-2}^{\gamma\text{lip}} \leq_s \gamma^{-1}(\varepsilon + \varepsilon\gamma^{-2}\|\iota\|_{s+\bar{\mu}+\beta}^{\gamma\text{lip}}). \quad (7.62)$$

(ii) For any $\omega \in \Omega_\infty^{2\gamma}(\iota)$ and any $s \in [s_0 + 1, s_* - \bar{\mu} - \beta]$, the Hamiltonian operator

$$\mathbf{L}_0(\omega) : H^s(\mathbb{T}^S, h_\perp^\sigma \times h_\perp^\sigma) \rightarrow H^{s-1}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$$

in (7.2) is conjugated to the normal form Hamiltonian operator $\mathbf{L}_\infty(\omega)$ in (7.49) by $\Phi_\infty(\omega)$,

$$\mathbf{L}_\infty(\omega) = \Phi_\infty^{-1}(\omega)\mathbf{L}_0(\omega)\Phi_\infty(\omega). \quad (7.63)$$

(iii) For any $k \in S_+^\perp$, the two eigenvalues of $[\mathbf{N}_\infty^{(1)}]_k^k$ are real and of the form

$$\omega_{-k}^{nls}(\xi, 0) + c_\varepsilon + \frac{r_{\xi,\varepsilon}^{(-)}(k)}{k} = 4\pi^2 k^2 + c_{\xi,\varepsilon} + \frac{\rho_{\xi,\varepsilon}^{(-)}(k)}{k}, \quad (7.64)$$

$$\omega_k^{nls}(\xi, 0) + c_\varepsilon + \frac{r_{\xi,\varepsilon}^{(+)}(k)}{k} = 4\pi^2 k^2 + c_{\xi,\varepsilon} + \frac{\rho_{\xi,\varepsilon}^{(+)}(k)}{k} \quad (7.65)$$

where

$$|c_\varepsilon|^{\text{sup}} = O(\varepsilon\gamma^{-2}), \quad |r_{\xi,\varepsilon}^{(\pm)}(k)|^{\text{sup}} = O(\varepsilon\gamma^{-2}), \quad |c_{\xi,\varepsilon}|^{\text{sup}} = O(1), \quad \sup_{k \in S_+^\perp} |\rho_{\xi,\varepsilon}^{(\pm)}(k)|^{\text{sup}} = O(1). \quad (7.66)$$

When listed according to size, they are denoted by $\lambda_k^{(\pm)}$, i.e. $\lambda_k^{(-)} \leq \lambda_k^{(+)}$. Then $\lambda_k^{(\pm)}$ are Lipschitz continuous and satisfy

$$\sup_{k \in S_+^\perp} |\lambda_k^{(\pm)}|^{\text{lip}} = O(1). \quad (7.67)$$

Proof. By the estimate (7.4), we get

$$|\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta}^{\gamma\text{lip}} \leq_{s_0+\beta} \varepsilon + \varepsilon\gamma^{-2}\|\iota\|_{s_0+\bar{\mu}+\beta}^{\gamma\text{lip}} \stackrel{(7.61)}{\leq_{s_0+\beta}} \varepsilon. \quad (7.68)$$

This together with the smallness condition (7.61) implies that the smallness condition (7.10) of Theorem 7.1 holds once δ_0 is chosen so that $\delta_0 \leq_{s_*} N_0^{-C_0}$ (recall (7.9)). We now prove items (i) and (ii).

(i) Since $\Omega_\infty^{2\gamma}(\iota) \stackrel{\text{Lemma 7.6}}{\subseteq} \cap_{\nu \geq 0} \Omega_\nu^\gamma(\iota)$, Lemma 7.5 implies that

$$|\Phi_\infty^{\pm 1} - \mathbb{I}_2|_{s,\sigma}^{\gamma\text{lip}}, \quad |\Phi_\infty^{\pm 1} - \mathbb{I}_2|_{s,\sigma-2}^{\gamma\text{lip}} \leq_s \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s+\beta,\sigma-1}^{\gamma\text{lip}}.$$

Furthermore by (7.4), the operator \mathbf{R}_0 in (7.2) satisfies

$$|\mathbf{R}_0 \mathfrak{D}|_{s+\beta,\sigma-1}^{\gamma\text{lip}} \leq_{s+\beta} \varepsilon + \varepsilon\gamma^{-2}\|\iota\|_{s+\bar{\mu}+\beta}^{\gamma\text{lip}}, \quad (7.69)$$

yielding the claimed estimates (7.62).

(ii) By (7.12), we get

$$\mathbf{L}_\nu = \tilde{\Phi}_{\nu-1}^{-1} \mathbf{L}_0 \tilde{\Phi}_{\nu-1} = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_\nu + \mathbf{R}_\nu, \quad \tilde{\Phi}_\nu = \Phi_0 \circ \dots \circ \Phi_\nu. \quad (7.70)$$

Since $|\mathbf{N}_\infty^{(1)} - \mathbf{N}_\nu^{(1)}|_{\sigma-2}^{\gamma\text{lip}} \leq |(\mathbf{N}_\infty^{(1)} - \mathbf{N}_\nu^{(1)})\mathfrak{D}|_{\sigma-1}^{\gamma\text{lip}} \ll \sup_{k \in S_\pm^+} \|[\mathbf{N}_\infty^{(1)} - \mathbf{N}_\nu^{(1)}]_k^k\|_{\gamma\text{lip}}$ one has

$$|\mathbf{N}_\infty^{(1)} - \mathbf{N}_\nu^{(1)}|_{\sigma-2}^{\gamma\text{lip}} \stackrel{(7.48), (7.68)}{\leq_{s_0+\beta}} N_{\nu-1}^{-\alpha} \varepsilon \xrightarrow{\nu \rightarrow \pm\infty} 0$$

and for any $s \in [s_0, s_* - \bar{\mu} - \beta]$

$$|\mathbf{R}_\nu|_{s, \sigma-2}^{\gamma\text{lip}} \ll |\mathbf{R}_\nu \mathfrak{D}|_{s, \sigma-1}^{\gamma\text{lip}} \stackrel{(7.17), (7.69)}{\ll} N_{\nu-1}^{-\alpha} (\varepsilon + \varepsilon \gamma^{-2} \|\iota\|_{s+\bar{\mu}+\beta}^{\gamma\text{lip}}) \xrightarrow{\nu \rightarrow \pm\infty} 0.$$

Hence $\mathbf{L}_\nu - \mathbf{L}_\infty \xrightarrow{\nu \rightarrow \pm\infty} 0$ with respect to the norm $|\cdot|_{s, \sigma-2}^{\gamma\text{lip}}$ and $\mathbf{L}_\nu \xrightarrow{\nu \rightarrow \pm\infty} \mathbf{L}_\infty$ in the space of linear, bounded operators from $H^s(\mathbb{T}^S, h_\perp^\sigma \times h_\perp^\sigma) \rightarrow H^{s-1}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$. Since by Lemma 7.5, $\tilde{\Phi}_\nu \xrightarrow{\nu \rightarrow \pm\infty} \Phi_\infty$ in the norm $|\cdot|_{s, \sigma}^{\gamma\text{lip}}$ and similarly, $\tilde{\Phi}_\nu^{-1} \xrightarrow{\nu \rightarrow \pm\infty} \Phi_\infty^{-1}$ in the norm $|\cdot|_{s-1, \sigma-2}^{\gamma\text{lip}}$ for any $s_0 + 1 \leq s \leq s_* - \bar{\mu} - \beta$, formula (7.63) follows by passing to the limit in (7.70).

(iii) *Proof of formula (7.64)-(7.66):* We write $[\mathbf{N}_\infty^{(1)}]_k^k = [\mathbf{N}_0^{(1)}]_k^k + [\mathbf{N}_\infty^{(1)} - \mathbf{N}_0^{(1)}]_k^k$ and note that

$$\|[\mathbf{N}_\infty^{(1)}]_k^k - [\mathbf{N}_0^{(1)}]_k^k\|_{\gamma\text{lip}} \stackrel{(7.48)}{\ll} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} k^{-1} \stackrel{(7.68)}{\ll} \varepsilon k^{-1}. \quad (7.71)$$

By (7.3), (6.96), the matrix $[\mathbf{N}_0^{(1)}]_k^k$ is diagonal and its entries are given by

$$\omega_{-k}^{nls}(\xi, 0) + c_\varepsilon + \frac{1}{-k} r_{-k, \xi}, \quad \omega_k^{nls}(\xi, 0) + c_\varepsilon + \frac{1}{k} r_{k, \xi}, \quad |c_\varepsilon|^{\gamma\text{lip}}, \quad \sup_{k \in S_\pm^+} |r_{\pm k, \xi}|^{\gamma\text{lip}} \stackrel{(6.97)}{=} O(\varepsilon \gamma^{-2}). \quad (7.72)$$

By standard perturbation theory for the eigenvalues of self-adjoint 2×2 matrices, the estimates (7.71) and (7.72) imply that the eigenvalues of $[\mathbf{N}_\infty^{(1)}]_k^k$ are given by the left hand side of the identities (7.64)-(7.65) with estimates $|c_\varepsilon|^{\text{sup}} = O(\varepsilon \gamma^{-2})$, $|r_{\xi, \varepsilon}^{(\pm)}(k)|^{\text{sup}} = O(\varepsilon \gamma^{-2})$, cf (7.66). The right hand side of the identities (7.64)-(7.65) are obtained by expanding $\omega_{\pm k}^{nls}(\xi, 0)$ by Theorem 3.2 item (ii).

Proof of formula (7.67): The eigenvalues $\lambda_k^{(\pm)}(\omega)$ of the matrix $[\mathbf{N}_\infty^{(1)}]_k^k(\omega)$ are Lipschitz continuous functions of the matrices

$$|\lambda_k^\pm(\omega_2) - \lambda_k^\pm(\omega_1)| \ll \|[\mathbf{N}_\infty^{(1)}]_k^k(\omega_2) - [\mathbf{N}_\infty^{(1)}]_k^k(\omega_1)\| \ll |\omega_2 - \omega_1|$$

by (7.71), (7.72) and Theorem 3.2 item (ii). \square

7.6 Proof of Theorem 5.1

By Theorem 7.2, the normal form Hamiltonian operator $\mathbf{L}_\infty(\omega) = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_\infty(\omega)$ is a φ -independent 2×2 block diagonal operator for any ω in $\Omega_\infty^{2\gamma}(\iota)$, which is defined in (7.57). Furthermore, the operator \mathbf{L}_∞ is conjugated to \mathfrak{L}_ω introduced in (5.35) by the composition of the symplectic transformations Φ_1 , Φ_2 , Φ_3 (Section 6), and Φ_∞ (Section 7.5),

$$\mathfrak{L}_\omega = \Phi_1 \Phi_2 \Phi_3 \Phi_\infty \mathbf{L}_\infty \Phi_\infty^{-1} \Phi_3^{-1} \Phi_2^{-1} \Phi_1^{-1}. \quad (7.73)$$

This representation allows to prove Theorem 5.1. To this end, introduce

$$\Omega_{\text{Mel}}^{2\gamma}(\iota) := \{\omega \in \Omega_\infty^{2\gamma}(\iota) : \omega \text{ satisfies } (\mathbf{M}_{2\gamma}^I)_\infty\}, \quad (7.74)$$

where $(\mathbf{M}_{2\gamma}^I)_\infty$ is the following first order Melnikov condition:

$(\mathbf{M}_{2\gamma}^I)_\infty$ For any $\ell \in \mathbb{Z}^S$, $j \in S_\pm^\perp$, the operator $\omega \cdot \ell \text{Id}_2 + [\mathbf{N}_\infty^{(1)}]_j^j$ is invertible and

$$\|(\omega \cdot \ell \text{Id}_2 + [\mathbf{N}_\infty^{(1)}]_j^j)^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma j^2}. \quad (7.75)$$

Before proving Theorem 5.1, we need to establish the following

Lemma 7.7. (Estimate of \mathbf{L}_∞^{-1}) For any $\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$ and $g \in H^{s+\tau}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$ the linear equation $\mathbf{L}_\infty(\omega)h = g$ has a unique solution h in $H^s(\mathbb{T}^S, h_\perp^\sigma \times h_\perp^\sigma)$, denoted by $\mathbf{L}_\infty^{-1}g$. Moreover, if g is a Lipschitz family in $H^{s+2\tau+1}(\mathbb{T}^S, h_\perp^{\sigma-2} \times h_\perp^{\sigma-2})$,

$$\|\mathbf{L}_\infty^{-1}g\|_{s,\sigma}^{\text{lip}} \leq \gamma^{-1} \|g\|_{s+2\tau+1,\sigma-2}^{\text{lip}}. \quad (7.76)$$

Proof. By (7.49), the normal form Hamiltonian operator \mathbf{L}_∞ can be written as

$$\mathbf{L}_\infty = \begin{pmatrix} \mathbf{L}_\infty^{(1)} & 0 \\ 0 & \bar{\mathbf{L}}_\infty^{(1)} \end{pmatrix}, \quad \mathbf{L}_\infty^{(1)} := \omega \cdot \partial_\varphi \mathbf{I}_2 + i\mathbf{N}_\infty^{(1)}, \quad \mathbf{N}_\infty^{(1)} := \text{diag}_{j \in S_\pm^+} [\mathbf{N}_\infty^{(1)}]_j^j.$$

It thus suffices to study the operator $\mathbf{L}_\infty^{(1)}$. For any $\omega \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$ and $g \in H^{s+\tau}(\mathbb{T}^S, h_\perp^{\sigma-2})$, one has by (7.75)

$$(\mathbf{L}_\infty^{(1)})^{-1}g = \sum_{\ell \in \mathbb{Z}^S} \left(\mathbf{A}_\infty(\ell, j)^{-1} \begin{pmatrix} \hat{g}_{-j}(\ell) \\ \hat{g}_j(\ell) \end{pmatrix} \right)_{j \in S_\pm^+} e^{i\ell \cdot \varphi}, \quad \mathbf{A}_\infty(\ell, j) \equiv [\mathbf{A}_\infty(\ell)]_j^j := i \left(\omega \cdot \ell \text{Id}_2 + [\mathbf{N}_\infty^{(1)}]_j^j \right).$$

In view of Lemma 7.1 (i) and (7.75) one then obtains

$$\|(\mathbf{L}_\infty^{(1)})^{-1}g\|_{s,\sigma} \leq \gamma^{-1} \|g\|_{s+\tau,\sigma-2}. \quad (7.77)$$

Concerning the Lipschitz seminorm, given any $\omega_1, \omega_2 \in \Omega_{\text{Mel}}^{2\gamma}(\iota)$, write $(\mathbf{L}_\infty^{(1)}(\omega_1))^{-1}g_{\omega_1} - (\mathbf{L}_\infty^{(1)}(\omega_2))^{-1}g_{\omega_2}$ as

$$(\mathbf{L}_\infty^{(1)}(\omega_1))^{-1}(g_{\omega_1} - g_{\omega_2}) + ((\mathbf{L}_\infty^{(1)}(\omega_1))^{-1} - (\mathbf{L}_\infty^{(1)}(\omega_2))^{-1})g_{\omega_2}. \quad (7.78)$$

The latter two terms are estimated individually: by (7.77), the first term satisfies the estimate

$$\|(\mathbf{L}_\infty^{(1)}(\omega_1))^{-1}(g_{\omega_1} - g_{\omega_2})\|_{s,\sigma} \leq \gamma^{-1} \|g\|_{s+\tau,\sigma-2}^{\text{lip}} |\omega_1 - \omega_2| \quad (7.79)$$

whereas the term $((\mathbf{L}_\infty^{(1)}(\omega_1))^{-1} - (\mathbf{L}_\infty^{(1)}(\omega_2))^{-1})g_{\omega_2}$ equals

$$\sum_{\ell \in \mathbb{Z}^S} \left((A_\infty(\ell, j; \omega_1)^{-1} - A_\infty(\ell, j; \omega_2)^{-1}) \begin{pmatrix} \hat{g}_{-j}(\ell; \omega_2) \\ \hat{g}_j(\ell; \omega_2) \end{pmatrix} \right)_{j \in S_\pm^+} e^{i\ell \cdot \varphi}. \quad (7.80)$$

Since

$$A_\infty(\ell, j; \omega_1)^{-1} - A_\infty(\ell, j; \omega_2)^{-1} = A_\infty(\ell, j; \omega_2)^{-1} (A_\infty(\ell, j; \omega_2) - A_\infty(\ell, j; \omega_1)) A_\infty(\ell, j; \omega_1)^{-1},$$

we have

$$\|A_\infty(\ell, j; \omega_1)^{-1} - A_\infty(\ell, j; \omega_2)^{-1}\| \stackrel{(7.75)}{\leq} \frac{\langle \ell \rangle^{2\tau}}{\gamma^2 j^4} \|A_\infty(\ell, j; \omega_2) - A_\infty(\ell, j; \omega_1)\| \quad (7.81)$$

with $\|A_\infty(\ell, j; \omega_2) - A_\infty(\ell, j; \omega_1)\| \leq |\omega_2 - \omega_1| |\ell| + \|[\mathbf{N}_\infty^{(1)}(\omega_2) - \mathbf{N}_\infty^{(1)}(\omega_1)]_j^j\|$. Since $\|[\mathbf{N}_\infty^{(1)}(\omega_2) - \mathbf{N}_\infty^{(1)}(\omega_1)]_j^j\|$ is bounded by

$$\|[\mathbf{N}_\infty^{(1)}(\omega_2) - \mathbf{N}_0^{(1)}(\omega_2)]_j^j\| + \|[\mathbf{N}_0^{(1)}(\omega_2) - \mathbf{N}_0^{(1)}(\omega_1)]_j^j\| + \|[\mathbf{N}_0^{(1)}(\omega_1) - \mathbf{N}_\infty^{(1)}(\omega_1)]_j^j\|$$

and

$$\|[\mathbf{N}_\infty^{(1)} - \mathbf{N}_0^{(1)}]_j^j\|^{\text{lip}} \leq \gamma^{-1} \|[\mathbf{N}_\infty^{(1)} - \mathbf{N}_0^{(1)}]_j^j\|^{\text{lip}} \stackrel{(7.48)}{\leq} \gamma^{-1} |\mathbf{R}_0 \mathfrak{D}|_{s_0+\beta, \sigma-1} j^{-1} \stackrel{(7.68)}{\leq} \frac{\varepsilon \gamma^{-1} \leq 1}{\leq} 1$$

one concludes that

$$\|[\mathbf{N}_\infty^{(1)}(\omega_2) - \mathbf{N}_\infty^{(1)}(\omega_1)]_j^j\| \leq |\omega_1 - \omega_2| + \|[\mathbf{N}_0^{(1)}(\omega_2) - \mathbf{N}_0^{(1)}(\omega_1)]_j^j\| + |\omega_1 - \omega_2| \stackrel{(7.3), (7.40)}{\leq} |\omega_1 - \omega_2|.$$

We thus have proved that

$$\|A_\infty(\ell, j; \omega_2) - A_\infty(\ell, j; \omega_1)\| \ll |\omega_2 - \omega_1| \langle \ell \rangle$$

and hence (7.81), (7.6) imply that

$$\|A_\infty(\ell, j; \omega_1)^{-1} - A_\infty(\ell, j; \omega_2)^{-1}\| \ll \frac{\langle \ell \rangle^{2\tau+1}}{\gamma^2 j^4} |\omega_1 - \omega_2|.$$

Applying this estimate to (7.80), one sees that

$$\|(\mathbf{L}_\infty^{(1)}(\omega_1))^{-1} - (\mathbf{L}_\infty^{(1)}(\omega_2))^{-1}\| g_{\omega_2} \Big|_{s,\sigma} \ll \gamma^{-2} \|g\|_{s+2\tau+1, \sigma-2}. \quad (7.82)$$

Combining (7.78), (7.79), and (7.82) leads to

$$\|(\mathbf{L}_\infty^{(1)})^{-1} g\|_{s,\sigma}^{\text{lip}} \ll \gamma^{-1} \|g\|_{s+\tau, \sigma-2}^{\text{lip}} + \gamma^{-2} \|g\|_{s+2\tau+1, \sigma-2}$$

which, together with (7.77), proves (7.76). \square

Proof of Theorem 5.1. By Lemmata 6.7, 6.10, 6.13, Theorem 7.2, and the smallness condition $\varepsilon\gamma^{-4} \leq 1$ one gets

$$|\Phi_j|_{s,\sigma}^{\gamma \text{lip}}, |\Phi_\infty|_{s,\sigma}^{\gamma \text{lip}} \leq_s 1 + \varepsilon\gamma^{-3} \|\iota\|_{s+\bar{\mu}+\beta}^{\gamma \text{lip}} \leq_s 1 + \|\iota\|_{s+\bar{\mu}+\beta}^{\gamma \text{lip}}, \quad \forall j \in \{1, 2, 3\}, \quad (7.83)$$

implying together with (5.2) that

$$|\Phi_j|_{s_0,\sigma}^{\gamma \text{lip}}, |\Phi_\infty|_{s_0,\sigma}^{\gamma \text{lip}} \leq 1, \quad \forall j \in \{1, 2, 3\}.$$

It then follows by Lemma 2.9 that

$$\begin{aligned} \|\Phi_1 \Phi_2 \Phi_3 \Phi_\infty \mathbf{L}_\infty^{-1} g\|_{s,\sigma}^{\gamma \text{lip}} &\stackrel{(7.83)}{\leq_s} \|\mathbf{L}_\infty^{-1} g\|_s^{\gamma \text{lip}} + \|\iota\|_{s+\bar{\mu}+\beta}^{\gamma \text{lip}} \|\mathbf{L}_\infty^{-1} g\|_{s_0}^{\gamma \text{lip}} \\ &\stackrel{(7.76)}{\leq_s} \gamma^{-1} (\|g\|_{s+2\tau+1, \sigma-2}^{\gamma \text{lip}} + \|\iota\|_{s+\bar{\mu}+\beta}^{\gamma \text{lip}} \|g\|_{s_0+2\tau+1, \sigma-2}^{\gamma \text{lip}}). \end{aligned}$$

Similarly one has

$$\|\Phi_\infty^{-1} \Phi_3^{-1} \Phi_2^{-1} \Phi_1^{-1} g\|_{s+2\tau+1, \sigma-2}^{\gamma \text{lip}} \leq_s \|g\|_{s+2\tau+1, \sigma-2}^{\gamma \text{lip}} + \|\iota\|_{s+\bar{\mu}+\beta+2\tau+1}^{\gamma \text{lip}} \|g\|_{s_0+2\tau+1, \sigma-2}^{\gamma \text{lip}}.$$

Combining the above estimates yield

$$\|\Phi_1 \Phi_2 \Phi_3 \Phi_\infty \mathbf{L}_\infty^{-1} \Phi_\infty^{-1} \Phi_3^{-1} \Phi_2^{-1} \Phi_1^{-1} g\|_{s,\sigma}^{\gamma \text{lip}} \leq_s \gamma^{-1} (\|g\|_{s+2\tau+1, \sigma-2}^{\gamma \text{lip}} + \|\iota\|_{s+\bar{\mu}+\beta+2\tau+1}^{\gamma \text{lip}} \|g\|_{s_0+2\tau+1, \sigma-2}^{\gamma \text{lip}}),$$

which, recalling (7.73), is the estimate (5.39) of Theorem 5.1, with

$$\mu_0 := \bar{\mu} + \beta + 2\tau + 1 \stackrel{(7.1), (7.8)}{=} 4s_0 + 10\tau + 7. \quad (7.84)$$

7.7 Variation with respect to ι

In this section we provide estimates for the variation of the 2×2 matrices $[\mathbf{N}_\nu^{(1)}]_k^k$, introduced in Theorem 7.1, with respect to ι . They are required in Section 9 for obtaining the measure estimate of Theorem 4.1. To prove them, we also need such estimates for the remainder terms \mathbf{R}_ν , $\nu \geq 0$, of Theorem 7.1.

Theorem 7.3. *Let $\check{\iota}^{(a)}(\varphi) = (\varphi, 0, 0) + \iota^{(a)}(\varphi)$, $a = 1, 2$, be two Lipschitz families of torus embeddings with $\check{\iota}^{(a)} \equiv \check{\iota}_\omega^{(a)}$ defined on $\Omega_o(\iota^{(a)})$ where $\Omega_o(\iota^{(2)}) \subseteq \Omega_o(\iota^{(1)})$ with $\Omega_o(\iota^{(1)}) \subseteq \Omega_{2\gamma, \tau}$ for some given $0 < \gamma < 1/2$. Furthermore we assume that $\iota^{(1)}$ and $\iota^{(2)}$ satisfy the smallness condition (7.61) (with 2γ). Then the following statements hold:*

(S1) $_{\nu}$ There exists a constant $C_{\text{var}} = C_{\text{var}}(\tau, |S|) > 0$ so that for any $\nu \geq 0$ and any $\gamma/2 \leq \gamma_1, \gamma_2 \leq 2\gamma$, the operator $\Delta_{12}\mathbf{R}_{\nu} := \mathbf{R}_{\nu}(\check{\iota}^{(1)}) - \mathbf{R}_{\nu}(\check{\iota}^{(2)})$, defined for $\omega \in \Omega_{\nu}^{\gamma_1}(\iota^{(1)}) \cap \Omega_{\nu}^{\gamma_2}(\iota^{(2)})$ (with $\Omega_{\nu}^{\gamma_a}(\iota^{(a)})$ as in (7.7)) satisfies

$$|\Delta_{12}\mathbf{R}_{\nu}\mathfrak{D}|_{s_0, \sigma-1} \leq C_{\text{var}}N_{\nu-1}^{-\alpha} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta}, \quad |\Delta_{12}\mathbf{R}_{\nu}\mathfrak{D}|_{s_0+\beta, \sigma-1} \leq C_{\text{var}}N_{\nu-1} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} \quad (7.85)$$

where $\bar{\mu}$, N_{ν} , and α , β are given in (7.1), (7.6), and (7.8), respectively. Moreover, for any $k \in S_{\mp}^{\perp}$ one has

$$\|\Delta_{12}[\mathbf{N}_{\nu}^{(1)}]_k^k\| \leq \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} \quad (7.86)$$

and, in case $\nu \geq 1$,

$$\|\Delta_{12}([\mathbf{N}_{\nu}^{(1)} - \mathbf{N}_{\nu-1}^{(1)}]_k^k)\| \leq |\Delta_{12}\mathbf{R}_{\nu-1}\mathfrak{D}|_{s_0, \sigma-1} k^{-1}. \quad (7.87)$$

(S2) $_{\nu}$ There exists a constant $C'_{\text{var}} = C'_{\text{var}}(\tau, |S|) > 0$ so that for any given $0 < \rho \leq \gamma/2$,

$$C'_{\text{var}}N_{\nu-1}^{\tau} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta}^{\sup} \leq \rho \implies \Omega_{\nu}^{\gamma}(\iota^{(1)}) \cap \Omega_o(\iota^{(2)}) \subseteq \Omega_{\nu}^{\gamma-\rho}(\iota^{(2)}). \quad (7.88)$$

Proof. We argue by induction. First let us prove (S1) $_0$ and (S2) $_0$. Concerning (S1) $_0$, note that by (6.102), the operator $\mathbf{R}_0 = \mathfrak{R}_3$ satisfies for any $\omega \in \Omega_o(\iota^{(2)}) (= \Omega_o(\iota^{(1)}))$

$$|\Delta_{12}\mathbf{R}_0\mathfrak{D}|_{s_0+\beta, \sigma-1} \leq \varepsilon\gamma^{-2} \|\Delta_{12}\iota\|_{s_0+\beta+4s_0+\tau} + \max_{s_0+\beta+4s_0+\tau}(\iota) \|\Delta_{12}\iota\|_{5s_0+\tau},$$

implying that

$$|\Delta_{12}\mathbf{R}_0\mathfrak{D}|_{s_0+\beta, \sigma-1} \stackrel{(7.1)}{\leq} (\varepsilon\gamma^{-2} + \max_{s_0+\bar{\mu}+\beta}(\iota)) \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} \stackrel{(7.61)}{\leq} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta}.$$

Since $N_{-1} = 1$, the estimates (7.85) for $\nu = 0$ then follow by choosing $C_{\text{var}}(\tau, |S|) > 0$ large enough. Concerning the estimate (7.86) for $\nu = 0$ recall that by (7.3), the matrix element $(\mathbf{N}_0^{(1)})_k^k$, $k \in S^{\perp}$, is given by $[[\omega_k^{nls}]] + \varepsilon[[q_1]] = 4\pi^2k^2 + [[\Omega_k^{nls}]] + \varepsilon[[q_1]]$. By the estimates of $\Delta_{12}\Omega^{nls}$ and $\Delta_{12}q_1$ in Lemma 6.2 (i) and, respectively, Lemma 6.4 (i) (valid uniformly on $\Omega_o(\iota^{(2)})$) and using the smallness condition (7.61), one concludes that for any $k \in S_{\mp}^{\perp}$

$$\|\Delta_{12}[\mathbf{N}_0^{(1)}]_k^k\| \leq \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta},$$

which is the estimate (7.86) for $\nu = 0$. Clearly, (S2) $_0$ holds for any choice of C'_{var} since by assumption, $\Omega_o(\iota^{(2)}) \subseteq \Omega_o(\iota^{(1)})$ and by (7.7), $\Omega_0^{\gamma}(\iota^{(a)}) = \Omega_o(\iota^{(a)})$, $a = 1, 2$, implying that $\Omega_0^{\gamma}(\iota^{(1)}) \cap \Omega_o(\iota^{(2)}) = \Omega_o(\iota^{(2)})$.

Let us now prove the inductive step from ν to $\nu + 1$. We assume that (S1) $_{\nu}$, (S2) $_{\nu}$ hold and begin by showing (S1) $_{\nu+1}$. Since the torus embeddings $\check{\iota}^{(1)}$, $\check{\iota}^{(2)}$ satisfy (7.61), it follows from (7.4) that the operators $\mathbf{R}_0(\check{\iota}^{(a)})$, $a = 1, 2$, satisfy

$$|\mathbf{R}_0(\check{\iota}^{(a)})\mathfrak{D}|_{s_0+\beta, \sigma-1} \leq \varepsilon\gamma^{-2}. \quad (7.89)$$

In particular, the condition (7.10) of Theorem 7.1 holds and hence (7.17), combined with (7.89), yields

$$|\mathbf{R}_{\nu}(\check{\iota}^{(a)})\mathfrak{D}|_{s_0, \sigma-1} \leq \varepsilon\gamma^{-2}N_{\nu-1}^{-\alpha}, \quad |\mathbf{R}_{\nu}(\check{\iota}^{(a)})\mathfrak{D}|_{s_0+\beta, \sigma-1} \leq \varepsilon\gamma^{-2}N_{\nu-1}, \quad a = 1, 2. \quad (7.90)$$

We have to estimate $\Delta_{12}\mathbf{R}_{\nu+1}$, which according to (7.38) is given by

$$\Delta_{12}\mathbf{R}_{\nu+1} = \Delta_{12}(\Phi_{\nu}^{-1}\tilde{\mathbf{R}}_{\nu}) + \Delta_{12}((\Phi_{\nu}^{-1} - \mathbb{I}_2)\mathbf{R}_{\nu}^{nf}) \quad (7.91)$$

where by (7.24)

$$\tilde{\mathbf{R}}_{\nu} = \Pi_{N_{\nu}}^{\perp} \mathbf{R}_{\nu} + (\omega \cdot \partial_{\varphi})(\Phi_{\nu} - \mathbb{I}_2 + \Psi_{\nu}) + [\mathbf{N}_{\nu}, \Phi_{\nu} - \mathbb{I}_2 + \Psi_{\nu}] + \mathbf{R}_{\nu}(\Phi_{\nu} - \mathbb{I}_2). \quad (7.92)$$

We first need to estimate $\Delta_{12}\Psi_{\nu} = \Psi_{\nu}(\check{\iota}^{(1)}) - \Psi_{\nu}(\check{\iota}^{(2)})$ where $\Psi_{\nu}(\check{\iota}^{(a)})$, $a = 1, 2$, are the solutions of the homological equation (7.25) with $\mathbf{R}_{\nu} = \mathbf{R}_{\nu}(\check{\iota}^{(a)})$:

Lemma 7.8. For $s = s_0$ and $s = s_0 + \beta$, the norms $|\Delta_{12}\Psi_\nu\mathfrak{D}|_{s,\sigma-1}$, $|\Delta_{12}\Psi_\nu|_{s,\sigma}$, and $|\Delta_{12}\Psi_\nu|_{s,\sigma-2}$ are \ll bounded for any $\nu \geq 0$ by

$$N_\nu^{2\tau} \left(\gamma^{-2} |\mathbf{R}_\nu(\check{\iota}^{(1)})\mathfrak{D}|_{s,\sigma-1} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} + \gamma^{-2} |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s,\sigma-1} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} + \gamma^{-1} |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s,\sigma-1} \right).$$

Proof. To simplify notations, we drop the index ν in this proof. Since Ψ_ν is of the form (7.21), it suffices to prove the estimates corresponding to the claimed ones for the operators $\Delta_{12}\Psi^{(1)}\langle\langle D \rangle\rangle$ and $\Delta_{12}\Psi^{(2)}\langle\langle D \rangle\rangle$. The estimates for these two operators can be shown in the same way and hence we consider $\Delta_{12}\Psi^{(1)}\langle\langle D \rangle\rangle$ only. Evaluating (7.35) at $\iota^{(a)}$, one has for any $j, k \in S_\perp^\pm$ and any ω in $\Omega_{\nu+1}^{\gamma_a}(\check{\iota}^{(a)})$,

$$[\hat{\Psi}^{(1)}(\ell)]_j^k = -iL^-(\ell, j, k)^{-1} [\hat{\mathbf{R}}^{(1)}(\ell)]_j^k, \quad \forall \ell \in \mathbb{Z}^S, \quad |\ell| \leq N, \quad (\ell, j, k) \neq (0, j, j)$$

and hence for any $\omega \in \Omega_{\nu+1}^{\gamma_1}(\check{\iota}^{(1)}) \cap \Omega_{\nu+1}^{\gamma_2}(\check{\iota}^{(2)})$,

$$\Delta_{12}[\hat{\Psi}^{(1)}(\ell)]_j^k = -i(\Delta_{12}L^-(\ell, j, k)^{-1}) [\hat{\mathbf{R}}^{(1)}(\ell; \check{\iota}^{(1)})]_j^k - iL^-(\ell, j, k; \check{\iota}^{(2)})^{-1} (\Delta_{12}[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k). \quad (7.93)$$

Together with

$$\Delta_{12}L^-(\ell, j, k)^{-1} = -L^-(\ell, j, k; \check{\iota}^{(2)})^{-1} \Delta_{12}L^-(\ell, j, k) L^-(\ell, j, k; \check{\iota}^{(1)})^{-1},$$

the definition (7.28) of $L^-(\ell, j, k)$ implies that

$$\Delta_{12}L^-(\ell, j, k) = M_L(\Delta_{12}[\mathbf{N}^{(1)}]_j^j) - M_R(\Delta_{12}[\mathbf{N}^{(1)}]_k^k).$$

By the induction hypothesis, estimate (7.86) holds and hence $\|\Delta_{12}L^-(\ell, j, k)\| \ll \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta}$. This together with (7.30) then yields

$$\|\Delta_{12}L^-(\ell, j, k)^{-1}\| \ll \frac{N^{2\tau}}{\gamma_1\gamma_2(j^2 - k^2)^2} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta}.$$

Hence (7.93) implies that for any $\ell \in \mathbb{Z}^S$, $|\ell| \leq N$, and $j, k \in S_\perp^\pm$,

$$\|\Delta_{12}[\hat{\Psi}^{(1)}(\ell)]_j^k\| \ll \frac{N^{2\tau}}{\gamma_1\gamma_2(j^2 - k^2)^2} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} \|\hat{\mathbf{R}}^{(1)}(\ell; \check{\iota}^{(1)})\|_j^k + \frac{N^\tau}{\gamma_2(j^2 - k^2)} \|\Delta_{12}[\hat{\mathbf{R}}^{(1)}(\ell)]_j^k\|.$$

Arguing as in the proof of Lemma 7.3 for deriving the estimate of $\|\hat{\Psi}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}$ and using the assumption $\gamma_1, \gamma_2 \geq \gamma/2$, one sees that for any $\ell \in \mathbb{Z}^S$, $|\ell| \leq N$,

$$\|\Delta_{12}\hat{\Psi}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})} \ll N^{2\tau}\gamma^{-2} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} \|\hat{\mathbf{R}}^{(1)}(\ell; \check{\iota}^{(2)})\|_{\mathcal{L}(h_\perp^{\sigma-1})} + N^\tau\gamma^{-1} \|\Delta_{12}\hat{\mathbf{R}}^{(1)}(\ell)\langle\langle D \rangle\rangle\|_{\mathcal{L}(h_\perp^{\sigma-1})}$$

which implies that $|\Delta_{12}\Psi^{(1)}\langle\langle D \rangle\rangle|_{s,\sigma-1}$ satisfies the claimed estimate. The one for $|\Delta_{12}\Psi^{(1)}|_{s,\sigma}$ follows by similar arguments. Finally, the estimate for $|\Delta_{12}\Psi^{(1)}\langle\langle D \rangle\rangle|_{s,\sigma-1}$ implies the claimed one for $|\Delta_{12}\Psi^{(1)}|_{s,\sigma-2}$ since $|\Delta_{12}\Psi^{(1)}|_{s,\sigma-2} \leq |\Delta_{12}\Psi^{(1)}\langle\langle D \rangle\rangle|_{s,\sigma-1}$. \square

We estimate each term in the expression (7.91) for $\Delta_{12}\mathbf{R}_{\nu+1}$ individually. For convenience, introduce

$$R_\nu(s) := \max\{|\mathbf{R}_\nu(\check{\iota}^{(1)})\mathfrak{D}|_{s,\sigma-1}, |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s,\sigma-1}\}, \quad s = s_0, s_0 + \beta.$$

By Lemma 7.8 and then using the induction hypothesis, one sees that

$$\begin{aligned} |\Delta_{12}\Psi_\nu\mathfrak{D}|_{s_0,\sigma-1} &\ll N_\nu^{2\tau} (\gamma^{-2} R_\nu(s_0) \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} + \gamma^{-1} |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0,\sigma-1}) \\ &\stackrel{(7.90), (7.85), \varepsilon\gamma^{-1} \leq 1}{\ll} N_\nu^{2\tau} N_{\nu-1}^{-\alpha} \gamma^{-1} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} \end{aligned} \quad (7.94)$$

and

$$\begin{aligned} |\Delta_{12}\Psi_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} &\ll N_\nu^{2\tau} (\gamma^{-2} R_\nu(s_0 + \beta) \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta} + \gamma^{-1} |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1}) \\ &\stackrel{(7.90), (7.85), \varepsilon\gamma^{-1} \leq 1}{\ll} N_\nu^{2\tau} N_{\nu-1} \gamma^{-1} \|\Delta_{12}\iota\|_{s_0+\bar{\mu}+\beta}. \end{aligned} \quad (7.95)$$

By Lemma 7.3, the operators $\Psi_\nu(\check{\iota}^{(a)})$, $a = 1, 2$, satisfy the estimates

$$|\Psi_\nu(\check{\iota}^{(a)})\mathfrak{D}|_{s,\sigma-1}, |\Psi_\nu(\check{\iota}^{(a)})|_{s,\sigma}, |\Psi_\nu(\check{\iota}^{(a)})|_{s,\sigma-2} \leq N_\nu^\tau \gamma^{-1} R_\nu(s), \quad s = s_0, s_0 + \beta. \quad (7.96)$$

Taking into account that

$$N_\nu^\tau \gamma^{-1} R_\nu(s_0) \stackrel{(7.90)}{\leq} N_\nu^\tau N_{\nu-1}^{-\alpha} \varepsilon \gamma^{-3} \stackrel{(7.8),(7.61)}{\leq} 1, \quad (7.97)$$

one then concludes from (2.30) and (7.96) that

$$|\Delta_{12}\Phi_\nu^{\pm 1}\mathfrak{D}|_{s_0,\sigma-1} \leq |\Delta_{12}\Psi_\nu\mathfrak{D}|_{s_0,\sigma-1} \stackrel{(7.94)}{\leq} N_\nu^{2\tau} N_{\nu-1}^{-\alpha} \gamma^{-1} \|\Delta_{12}\ell\|_{s_0+\bar{\mu}+\beta} \quad (7.98)$$

and

$$\begin{aligned} |\Delta_{12}\Phi_\nu^{\pm 1}\mathfrak{D}|_{s_0+\beta,\sigma-1} &\leq |\Delta_{12}\Psi_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} + (|\Psi_\nu(\check{\iota}^{(1)})\mathfrak{D}|_{s_0+\beta,\sigma-1} + |\Psi_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0+\beta,\sigma-1})|\Delta_{12}\Psi_\nu\mathfrak{D}|_{s_0,\sigma-1} \\ &\stackrel{(7.94),(7.95),(7.96)}{\leq} N_\nu^{2\tau} N_{\nu-1} \gamma^{-1} \|\Delta_{12}\ell\|_{s_0+\bar{\mu}+\beta} + N_\nu^\tau \gamma^{-1} R_\nu(s_0 + \beta) N_\nu^{2\tau} N_{\nu-1}^{-\alpha} \gamma^{-1} \|\Delta_{12}\ell\|_{s_0+\bar{\mu}+\beta} \\ &\stackrel{(7.90),(7.8),\varepsilon\gamma^{-3}\leq 1}{\leq} N_\nu^{2\tau} N_{\nu-1} \gamma^{-1} \|\Delta_{12}\ell\|_{s_0+\bar{\mu}+\beta}. \end{aligned} \quad (7.99)$$

Estimate of $\Delta_{12}\tilde{\mathbf{R}}_\nu$: We begin by estimating the term $\Delta_{12}(\mathbf{R}_\nu(\Phi_\nu - \mathbb{I}_2))$ in $\Delta_{12}\tilde{\mathbf{R}}_\nu$ (cf (7.92)):

$$\begin{aligned} |\Delta_{12}(\mathbf{R}_\nu(\Phi_\nu - \mathbb{I}_2))\mathfrak{D}|_{s_0,\sigma-1} &\leq |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0,\sigma-1} |(\Phi_\nu(\check{\iota}^{(1)}) - \mathbb{I}_2)\mathfrak{D}|_{s_0,\sigma-1} + |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0,\sigma-1} |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0,\sigma-1} \\ &\stackrel{(2.28)}{\leq} |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0,\sigma-1} |\Psi_\nu(\check{\iota}^{(1)})\mathfrak{D}|_{s_0,\sigma-1} + |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0,\sigma-1} |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0,\sigma-1}. \end{aligned}$$

Using the induction hypothesis one sees that

$$|\Delta_{12}(\mathbf{R}_\nu(\Phi_\nu - \mathbb{I}_2))\mathfrak{D}|_{s_0,\sigma-1} \stackrel{(7.98),(7.96),(7.90),(7.85)}{\leq} N_\nu^{2\tau} N_{\nu-1}^{-2\alpha} \varepsilon \gamma^{-3} \|\Delta_{12}\ell\|_{s_0+\bar{\mu}+\beta}. \quad (7.100)$$

Similarly, $|\Delta_{12}(\mathbf{R}_\nu(\Phi_\nu - \mathbb{I}_2))\mathfrak{D}|_{s_0+\beta,\sigma-1}$ is \leq bounded by

$$\begin{aligned} &|\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} |(\Phi_\nu(\check{\iota}^{(1)}) - \mathbb{I}_2)\mathfrak{D}|_{s_0,\sigma-1} + |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0,\sigma-1} |(\Phi_\nu(\check{\iota}^{(1)}) - \mathbb{I}_2)\mathfrak{D}|_{s_0+\beta,\sigma-1} \\ &+ |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0+\beta,\sigma-1} |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0,\sigma-1} + |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0,\sigma-1} |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} \\ &\stackrel{(2.28)}{\leq} |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} |\Psi_\nu(\check{\iota}^{(1)})\mathfrak{D}|_{s_0,\sigma-1} + |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0,\sigma-1} |\Psi_\nu(\check{\iota}^{(1)})|_{s_0+\beta,\sigma-1} \\ &+ |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0+\beta,\sigma-1} |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0,\sigma-1} + |\mathbf{R}_\nu(\check{\iota}^{(2)})\mathfrak{D}|_{s_0,\sigma-1} |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} \end{aligned}$$

which by (7.96) is \leq bounded by

$$\begin{aligned} &|\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1} N_\nu^\tau \gamma^{-1} R_\nu(s_0) + |\Delta_{12}\mathbf{R}_\nu\mathfrak{D}|_{s_0,\sigma-1} N_\nu^\tau \gamma^{-1} R_\nu(s_0 + \beta) \\ &+ R_\nu(s_0 + \beta) |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0,\sigma-1} + R_\nu(s_0) |\Delta_{12}\Phi_\nu\mathfrak{D}|_{s_0+\beta,\sigma-1}. \end{aligned}$$

Again using the induction hypothesis, one then obtains by (7.98), (7.99), (7.97), (7.90), (7.85)

$$|\Delta_{12}(\mathbf{R}_\nu(\Phi_\nu - \mathbb{I}_2))\mathfrak{D}|_{s_0+\beta,\sigma-1} \leq N_{\nu-1} \|\Delta_{12}\ell\|_{s_0+\bar{\mu}+\beta}. \quad (7.101)$$

Next we estimate the term $\Delta_{12}(\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 + \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 + \Psi_\nu]$ in $\Delta_{12}\tilde{\mathbf{R}}_\nu$. Since $\Phi_\nu = \exp(-\Psi_\nu)$, one has

$$(\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 + \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 + \Psi_\nu] = \sum_{n \geq 2} (-1)^n \frac{(\omega \cdot \partial_\varphi)(\Psi_\nu^n) + [\mathbf{N}_\nu, \Psi_\nu^n]}{n!} \quad (7.102)$$

where by (7.42)

$$(\omega \cdot \partial_\varphi)(\Psi_\nu^n) + [\mathbf{N}_\nu, \Psi_\nu^n] = \sum_{n_1+n_2+1=n} \Psi_\nu^{n_1} (\Pi_{N_\nu} \mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^{n_2}. \quad (7.103)$$

Iterating the tame estimates (2.21) for the composition of operator valued maps one sees that for any i, k with $i + k + 1 = n$ (≥ 2), $|\Delta_{12}(\Psi_\nu^i(\Pi_{N_\nu} \mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^k) \mathfrak{D}|_{s_0, \sigma-1}$ is bounded by

$$(C' |\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1})^{n-1} |\Delta_{12} \mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} + (n-1) C' (C' |\Psi_\nu \mathfrak{D}|_{s_0, \sigma-1})^{n-2} |\mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} |\Delta_{12} \Psi_\nu \mathfrak{D}|_{s_0, \sigma-1}$$

where $C' \equiv C'(s_0) := 2C_{op}(s_0)$ with $C_{op}(s)$ as in (2.21). Using (7.96), (7.94) and increasing C' if necessary, one sees that the latter expression is bounded by

$$\begin{aligned} & (C' N_\nu^\tau \gamma^{-1} R_\nu(s_0))^{n-1} |\Delta_{12} \mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} + (n-1) C' (C' N_\nu^\tau \gamma^{-1} R_\nu(s_0))^{n-2} R_\nu(s_0) \gamma^{-1} N_\nu^{2\tau} N_\nu^{-\alpha} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \\ & \stackrel{(7.85), (7.90)}{\leq} n C^{n-1} (N_\nu^\tau N_\nu^{-\alpha} \varepsilon \gamma^{-3})^{n-2} N_\nu^{2\tau} N_\nu^{-2\alpha} \varepsilon \gamma^{-3} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \end{aligned}$$

with $C \equiv C(s_0) > C'$ chosen sufficiently large. Together with (7.97) this then implies that

$$|\Delta_{12}(\Psi_\nu^i(\Pi_{N_\nu} \mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^k) \mathfrak{D}|_{s_0, \sigma-1} \leq n C(s_0)^{n-1} N_\nu^{2\tau} N_\nu^{-2\alpha} \varepsilon \gamma^{-3} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.104)$$

Similarly, using (7.90), the induction hypothesis (7.85), and (7.94), (7.95), (7.96), one sees that for $C(s_0 + \beta) > 2C_{op}(s_0 + \beta)$ sufficiently large and any i, k with $i + k + 1 = n$ (≥ 2), $|\Delta_{12}(\Psi_\nu^i(\mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^k) \mathfrak{D}|_{s_0 + \beta, \sigma-1}$ is bounded by

$$n^2 C(s_0 + \beta)^{n-1} (N_\nu^\tau N_\nu^{-\alpha} \varepsilon \gamma^{-3})^{n-2} N_\nu^{2\tau} N_\nu^{-\alpha} \varepsilon \gamma^{-3} N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}$$

yielding

$$|\Delta_{12}(\Psi_\nu^i(\mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^k) \mathfrak{D}|_{s_0 + \beta, \sigma-1} \stackrel{(7.8), (7.61)}{\leq} n^2 C(s_0 + \beta)^{n-1} N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.105)$$

Hence by (7.102)

$$\begin{aligned} & |((\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 + \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 + \Psi_\nu]) \mathfrak{D}|_{s_0, \sigma-1} \stackrel{(7.103)}{\leq} \sum_{n \geq 2} \frac{1}{n!} \sum_{i+k+1=n} |\Delta_{12}(\Psi_\nu^i(\mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^k) \mathfrak{D}|_{s_0, \sigma-1} \\ & \stackrel{(7.104)}{\leq} N_\nu^{2\tau} N_\nu^{-2\alpha} \varepsilon \gamma^{-3} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \sum_{n \geq 2} \frac{C(s_0)^{n-1}}{(n-2)!} \leq N_\nu^{2\tau} N_\nu^{-2\alpha} \varepsilon \gamma^{-3} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}. \end{aligned} \quad (7.106)$$

Similarly, $|((\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 + \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 + \Psi_\nu]) \mathfrak{D}|_{s_0 + \beta, \sigma-1}$ is bounded by

$$\sum_{n \geq 2} \frac{1}{n!} \sum_{i+k+1=n} |\Delta_{12}(\Psi_\nu^i(\mathbf{R}_\nu - \mathbf{R}_\nu^{nf}) \Psi_\nu^k) \mathfrak{D}|_{s_0 + \beta, \sigma-1} \stackrel{(7.105)}{\leq} N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \sum_{n \geq 2} n \frac{C(s_0 + \beta)^{n-1}}{(n-2)!}$$

leading to the estimate

$$|((\omega \cdot \partial_\varphi)(\Phi_\nu - \mathbb{I}_2 + \Psi_\nu) + [\mathbf{N}_\nu, \Phi_\nu - \mathbb{I}_2 + \Psi_\nu]) \mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.107)$$

Finally, the term $\Delta_{12} \Pi_{N_\nu}^\perp \mathbf{R}_\nu = \Pi_{N_\nu}^\perp \Delta_{12} \mathbf{R}_\nu$ in $\Delta_{12} \tilde{\mathbf{R}}_\nu$ (cf (7.92)) can be estimated as

$$|\Pi_{N_\nu}^\perp \Delta_{12} \mathbf{R}_\nu \mathfrak{D}|_{s_0, \sigma-1} \stackrel{(2.16)}{\leq} N_\nu^{-\beta} |\Delta_{12} \mathbf{R}_\nu \mathfrak{D}|_{s_0 + \beta, \sigma-1} \stackrel{(7.85)}{\leq} N_\nu^{-\beta} N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \quad (7.108)$$

and

$$|\Pi_{N_\nu}^\perp \Delta_{12} \mathbf{R}_\nu \mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq |\Delta_{12} \mathbf{R}_\nu \mathfrak{D}|_{s_0 + \beta, \sigma-1} \stackrel{(7.85)}{\leq} N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.109)$$

Combining the estimates (7.100), (7.106), and (7.108) we get

$$|\Delta_{12} \tilde{\mathbf{R}}_\nu \mathfrak{D}|_{s_0, \sigma-1} \leq (N_\nu^{-1} N_\nu^{-\beta} + N_\nu^{2\tau} N_\nu^{-2\alpha} \varepsilon \gamma^{-3}) \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}, \quad (7.110)$$

whereas (7.101), (7.107), and (7.109) lead to

$$|\Delta_{12} \tilde{\mathbf{R}}_\nu \mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq N_\nu^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.111)$$

Estimate of $\Delta_{12}\mathbf{R}_{\nu+1}$: Arguing as in (7.100), (7.101), we get

$$|\Delta_{12}((\Phi_\nu^{-1} - \mathbb{I}_2)\mathbf{R}_\nu^{nf})\mathfrak{D}|_{s_0, \sigma-1} \leq N_\nu^{2\tau} N_{\nu-1}^{-2\alpha} \varepsilon \gamma^{-3} \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}, \quad (7.112)$$

$$|\Delta_{12}((\Phi_\nu^{-1} - \mathbb{I}_2)\mathbf{R}_\nu^{nf})\mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq N_{\nu-1} \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.113)$$

Moreover, by the arguments in the proof of $(\mathbf{S1})_\nu$ in Section 7.4, the operators $\tilde{\mathbf{R}}_\nu(\check{\iota}^{(a)})$, $a = 1, 2$, satisfy

$$|\tilde{\mathbf{R}}_\nu\mathfrak{D}|_{s, \sigma-1} \leq s |\Pi_{N_\nu}^\perp \mathbf{R}_\nu\mathfrak{D}|_{s, \sigma-1} + N_\nu^{2\tau+1} \gamma^{-1} |\mathbf{R}_\nu\mathfrak{D}|_{s, \sigma-1} |\mathbf{R}_\nu\mathfrak{D}|_{s_0, \sigma-1}.$$

Since $|\Pi_{N_\nu}^\perp \mathbf{R}_\nu\mathfrak{D}|_{s_0, \sigma-1} \leq N_\nu^{-\beta} |\Pi_{N_\nu}^\perp \mathbf{R}_\nu\mathfrak{D}|_{s_0 + \beta, \sigma-1}$ one concludes from (7.90) together with (7.8), (7.61) that

$$|\tilde{\mathbf{R}}_\nu(\check{\iota}^{(a)})\mathfrak{D}|_{s_0, \sigma-1} \leq s N_{\nu-1} N_\nu^{-\beta} \varepsilon \gamma^{-2} + N_\nu^{2\tau+1} N_{\nu-1}^{-2\alpha} \varepsilon \gamma^{-1}, \quad |\tilde{\mathbf{R}}_\nu(\check{\iota}^{(a)})\mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq N_{\nu-1} \varepsilon \gamma^{-2}. \quad (7.114)$$

Recalling that for $a = 1, 2$,

$$\begin{aligned} |(\Phi_\nu^{-1}(\check{\iota}^{(a)}) - \mathbb{I}_2)\mathfrak{D}|_{s_0, \sigma-1} &\stackrel{(2.28)}{\leq} |\Psi_\nu(\check{\iota}^{(a)})\mathfrak{D}|_{s_0, \sigma-1} \stackrel{(7.96), (7.90)}{\leq} N_\nu^\tau N_{\nu-1}^{-\alpha} \varepsilon \gamma^{-3}, \\ |(\Phi_\nu^{-1}(\check{\iota}^{(a)}) - \mathbb{I}_2)\mathfrak{D}|_{s_0 + \beta, \sigma-1} &\stackrel{(2.28)}{\leq} |\Psi_\nu(\check{\iota}^{(a)})\mathfrak{D}|_{s_0 + \beta, \sigma-1} \stackrel{(7.96), (7.90)}{\leq} N_\nu^\tau N_{\nu-1} \varepsilon \gamma^{-3}, \end{aligned}$$

and using (7.98), (7.99), (7.110), (7.111), (7.114), $\varepsilon \gamma^{-3} \leq 1$ (cf (7.61)) one sees that

$$|\Delta_{12}(\Phi_\nu^{-1} \tilde{\mathbf{R}}_\nu)\mathfrak{D}|_{s_0, \sigma-1} \leq (N_{\nu-1} N_\nu^{-\beta} + N_\nu^{2\tau+1} N_{\nu-1}^{-2\alpha} \varepsilon \gamma^{-3}) \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}, \quad (7.115)$$

$$|\Delta_{12}(\Phi_\nu^{-1} \tilde{\mathbf{R}}_\nu)\mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq N_{\nu-1} \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}. \quad (7.116)$$

By (7.91),

$$|\Delta_{12}\mathbf{R}_{\nu+1}\mathfrak{D}|_{s_0, \sigma-1} \stackrel{(7.112), (7.115)}{\leq} C(\tau, |S|) (N_{\nu-1} N_\nu^{-\beta} + N_\nu^{2\tau+1} N_{\nu-1}^{-2\alpha} \varepsilon \gamma^{-3}) \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}$$

for some constant $C(\tau, |S|) > 0$. Hence one has

$$|\Delta_{12}\mathbf{R}_{\nu+1}\mathfrak{D}|_{s_0, \sigma-1} \leq C_{\text{var}} N_\nu^{-\alpha} \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}$$

provided that C_{var} can be chosen such that for any $\nu \geq 0$,

$$C(\tau, |S|) N_{\nu-1} N_\nu^{-\beta} N_\nu^\alpha \leq C_{\text{var}}/2 \quad \text{and} \quad \tilde{C}(\tau, |S|) N_\nu^{2\tau+1} N_\nu^\alpha N_{\nu-1}^{-2\alpha} \varepsilon \gamma^{-3} \leq C_{\text{var}}/2.$$

In view of (7.8), (7.61) this is possible by choosing N_0 large enough. Furthermore,

$$|\Delta_{12}\mathbf{R}_{\nu+1}\mathfrak{D}|_{s_0 + \beta, \sigma-1} \stackrel{(7.113), (7.116)}{\leq} \tilde{C}(\tau, |S|) N_{\nu-1} \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta},$$

for some constant $\tilde{C}(\tau, |S|) > 0$, implying that by increasing N_0 , if necessary,

$$|\Delta_{12}\mathbf{R}_{\nu+1}\mathfrak{D}|_{s_0 + \beta, \sigma-1} \leq C_{\text{var}} N_\nu \|\Delta_{12}\iota\|_{s_0 + \bar{\mu} + \beta}.$$

This establishes (7.85) at the inductive step $\nu + 1$. Since for any $k \in S_+^\perp$, $[\mathbf{N}_{\nu+1}^{(1)} - \mathbf{N}_\nu]_k^k = [\hat{\mathbf{R}}_\nu^{(1)}(0)]_k^k$ (see (7.26)) the estimate (7.87) follows directly from (7.85) and implies (7.86) by a telescopic argument, using the estimate (7.86) in the case $\nu = 0$, established at the beginning of the proof.

Finally let us turn towards $(\mathbf{S2})_{\nu+1}$. Since by the definition (7.7), $\Omega_{\nu+1}^\gamma(\iota^{(1)}) \subseteq \Omega_\nu^\gamma(\iota^{(1)})$, by the induction hypothesis, $\Omega_\nu^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)}) \subseteq \Omega_\nu^{\gamma-\rho}(\iota^{(2)})$, and $\Omega_\nu^{\gamma-\rho}(\iota^{(2)}) \subseteq \Omega_o(\iota^{(2)})$, one has

$$\Omega_{\nu+1}^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)}) \subseteq \Omega_\nu^{\gamma-\rho}(\iota^{(2)}) \stackrel{0 < \rho < \gamma/2}{\subseteq} \Omega_\nu^{\gamma/2}(\iota^{(2)}).$$

By construction, for any $k \in S_+^\perp$, the 2×2 matrices $[\mathbf{N}_\nu^{(1)}(\iota^{(2)})]_k^k \equiv [\mathbf{N}_\nu^{(1)}(\omega, \iota^{(2)}(\omega))]_k^k$ are then defined for $\omega \in \Omega_{\nu+1}^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)})$ and hence by the definition (7.28), so are the operators $L_\nu^-(\ell, j, k; \iota^{(a)})$, $a = 1, 2$,

for any $\ell \in \mathbb{Z}^S$. Furthermore, if in addition, $|\ell| \leq N_\nu$ and $(\ell, j, k) \neq (0, j, j)$, then $L_\nu^-(\ell, j, k; \iota^{(1)})$ and $L_\nu^-(\ell, j, k; \iota^{(2)})$ are invertible for any $\omega \in \Omega_{\nu+1}^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)})$. Clearly, it follows from the definition (7.28) that

$$\begin{aligned} \|\Delta_{12} L_\nu^-(\ell, j, k)\| &\leq \|M_L(\Delta_{12}[\mathbf{N}_\nu^{(1)}]_k^k)\| + \|M_R(\Delta_{12}[\mathbf{N}_\nu^{(1)}]_j^j)\| \\ &\leq C_{\text{mult}} \sup_{\kappa \in S_\perp^+} \|\Delta_{12}[\mathbf{N}_\nu^{(1)}]_\kappa^\kappa\| \stackrel{(7.86)}{\leq} C_{\text{mult}} C_{\text{lip}} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \end{aligned} \quad (7.117)$$

where $C_{\text{mult}} > 0$ is an absolute constant related to the multiplication of 2×2 matrices and C_{lip} denotes the constant in (7.86), implying that for any $\kappa \in S_\perp^+$, $\|\Delta_{12}[\mathbf{N}_\nu^{(1)}]_\kappa^\kappa\| \leq C_{\text{lip}} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta}$. We then define $C'_{\text{var}} := C_{\text{mult}} C_{\text{lip}}$ and note that by assumption,

$$C'_{\text{var}} N_\nu^\tau \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \leq \rho. \quad (7.118)$$

It is to show that for any $\omega \in \Omega_{\nu+1}^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)})$, $L_\nu^-(\ell, j, k; \iota^{(2)}(\omega))$ is invertible and its inverse is bounded by $\frac{\langle \ell \rangle^\tau}{(\gamma - \rho) \langle j^2 - k^2 \rangle}$ (cf (7.30)). To this end we write $L_\nu^-(\ell, j, k; \iota^{(2)})$ in the form

$$L_\nu^-(\ell, j, k; \iota^{(2)}) = L_\nu^-(\ell, j, k; \iota^{(1)}) (\text{Id}_2 - L_\nu^-(\ell, j, k; \iota^{(1)})^{-1} \Delta_{12} L_\nu^-(\ell, j, k)) \quad (7.119)$$

where Id_2 denotes the 2×2 identity matrix. Since for any $\omega \in \Omega_{\nu+1}^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)})$

$$\begin{aligned} \|L_\nu^-(\ell, j, k; \iota^{(1)})^{-1} \Delta_{12} L_\nu^-(\ell, j, k)\| &\leq \|L_\nu^-(\ell, j, k; \iota^{(1)})^{-1}\| \|\Delta_{12} L_\nu^-(\ell, j, k)\| \\ &\stackrel{(7.117)}{\leq} C'_{\text{var}} \frac{\langle \ell \rangle^\tau}{\gamma \langle j^2 - k^2 \rangle} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \stackrel{|\ell| \leq N_\nu}{\leq} C'_{\text{var}} N_\nu^\tau \gamma^{-1} \|\Delta_{12} \iota\|_{s_0 + \bar{\mu} + \beta} \stackrel{(7.118)}{\leq} \rho \gamma^{-1} \end{aligned}$$

and $\rho \gamma^{-1} \leq 1/2$ it follows from (7.119) that $L_\nu^-(\ell, j, k; \iota^{(2)})$ is invertible by Neumann series and

$$\|L_\nu^-(\ell, j, k; \iota^{(2)})^{-1}\| \leq \frac{1}{1 - \rho \gamma^{-1}} \|L_\nu^-(\ell, j, k; \iota^{(1)})^{-1}\| \leq \frac{\gamma}{\gamma - \rho} \frac{\langle \ell \rangle^\tau}{\gamma \langle j^2 - k^2 \rangle} = \frac{\langle \ell \rangle^\tau}{(\gamma - \rho) \langle j^2 - k^2 \rangle}.$$

Using the same strategy, one can prove that for any $\omega \in \Omega_{\nu+1}^\gamma(\iota^{(1)}) \cap \Omega_o(\iota^{(2)})$, any $\ell \in \mathbb{Z}^S$ with $|\ell| \leq N_\nu$, and any $j, k \in S_\perp^+$, the operator $L_\nu^+(\ell, j, k; \iota^{(2)})$ is invertible and satisfies

$$\|L_\nu^+(\ell, j, k; \iota^{(2)})^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{(\gamma - \rho) \langle j^2 - k^2 \rangle}.$$

Altogether, we thus have verified **(S2)** $_{\nu+1}$. □

8 Nash-Moser iteration

In this section we prove Theorem 4.1 except for the measure estimate (4.1) which is proved in Section 9. Recall that in (2.14) we introduced the family of smoothing operators $(\Pi_t)_{t \geq 0}$ for the Sobolev spaces $H^s(\mathbb{T}^S, X)$. By a slight abuse of notation, we define, for $n \geq 0$,

$$\Pi_n \equiv \Pi_{N_n}, \quad \Pi_n^\perp = \text{Id} - \Pi_n, \quad N_n = N_0^{\chi^n}, \quad \chi = 3/2,$$

with $N_0 = N_0(|S|, \tau) > 0$ as is Theorem 8.1. By Lemma 2.5, the classical smoothing properties hold: for any $s \geq 0$, $k \geq 0$, and any Lipschitz family $\iota \equiv \iota_\omega \in H^s(\mathbb{T}^S, \mathbb{T}^S \times \mathbb{R}^S \times h_\perp^{\sigma'})$ with $\sigma' \leq \sigma$, we have

$$\|\Pi_n \iota\|_{s+k, \sigma'}^{\text{lip}} \leq N_n^k \|\iota\|_{s, \sigma'}^{\text{lip}}, \quad (8.1)$$

and for any Lipschitz family $\iota \equiv \iota_\omega \in H^{s+k}(\mathbb{T}^S, \mathbb{T}^S \times \mathbb{R}^S \times h_\perp^{\sigma'})$

$$\|\Pi_n^\perp \iota\|_{s, \sigma'}^{\text{lip}} \leq N_n^{-k} \|\iota\|_{s+k, \sigma'}^{\text{lip}}. \quad (8.2)$$

Furthermore, introduce for any $n \geq 0$

$$E_n := \{\varphi \mapsto \iota(\varphi) = (\Theta(\varphi), y(\varphi), z(\varphi)) : \Theta = \Pi_n \Theta, y = \Pi_n y \in U_0, z = \Pi_n z\} \subseteq \mathcal{C}^\infty(\mathbb{T}^S, M^\sigma), \quad E_{-1} := \{0\}$$

with $M^\sigma = \mathbb{T}^S \times U_0 \times h_\perp^\sigma$ introduced in (1.20). Recall that in Subsection 3.2, the differential of a possibly φ -dependent vector field on M^σ has been extended to a linear operator on $\mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma$ – see formula (3.14). This extension turned out to be useful in Sections 5 - 7 for the construction of an approximate right inverse of $d_{\iota, \zeta} F_\omega$. In the sequel, by a slight abuse of notation, we will identify a possibly φ -dependent vector $(\widehat{\theta}, \widehat{y}, \widehat{z}) \in \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma$ with the vector $(\widehat{\theta}, \widehat{y}, \widehat{z}, \overline{\widehat{z}}) \in \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma$. Define the constants

$$\eta_1 := 6\mu_1 + 1, \quad \alpha_1 := 2\mu_1 + \frac{2}{3}, \quad \kappa_1 := 6\mu_1 + 1, \quad \beta_1 := 12\mu_1 + 2 \quad (8.3)$$

where $\mu_1 = \mu_1(|S|, \tau) > 0$ is the integer of Theorem 5.2. Finally, for any $0 < \gamma < 1/2$, introduce

$$\gamma_n := \gamma(1 + 2^{-n}), \quad n \geq 0, \quad (8.4)$$

let $0 < \delta_1 < 1$ be as in Theorem 5.2, and recall that $\Omega_{\gamma, \tau}$ denotes the set of diophantine frequencies, introduced in (1.22). Let $N_{-1} := 1$.

Theorem 8.1. (Nash-Moser) *Assume that the perturbation f in (1.5) is $\mathcal{C}^{\sigma, s_*}$ -smooth with $s_* \geq s_0 + \beta_1 + \mu_1$ and let $\tau \geq 2|S| + 1$. Then there exist $0 < \delta_2 = \delta_2(|S|, \tau) \leq \delta_1 (< 1)$, $N_0 = N_0(|S|, \tau) > 0$, and $C_* \geq 1$ so that if $\varepsilon > 0$, $0 < \gamma < 1/4$ satisfy*

$$\varepsilon \gamma^{-4} < \delta_2, \quad (8.5)$$

then the following holds: for any $n \geq 0$, there exists a Lipschitz family $(\iota_{n+1}, \zeta_{n+1}) : \Omega_{n+1}^{\text{Mel}} \rightarrow E_n \times \mathbb{R}^S$ where

$$\Omega_{n+1}^{\text{Mel}} := \Omega_{\text{Mel}}^{2\gamma_n}(\iota_n) \quad (8.6)$$

with $\Omega_{\text{Mel}}^{2\gamma_n}(\iota_n)$ defined as in (7.74), (7.57) by choosing $\Omega_o(\iota_n)$ to be Ω_n^{Mel} in the case $n \geq 1$ whereas for $n = 0$

$$\Omega_o(\iota_0) \equiv \Omega_0^{\text{Mel}} := \Omega_{4\gamma, \tau} \quad \text{with} \quad (\iota_0, \zeta_0) := (0, 0) \quad (8.7)$$

so that the following estimates are valid for any $n \geq 0$:

(NM1) $_n$ (middle norms)

$$\|\iota_n\|_{s_0 + \mu_1}^{\gamma \text{lip}} \leq \varepsilon \gamma^{-2}, \quad \|F_\omega(\iota_n, \zeta_n)\|_{s_0 + \mu_1, \sigma - 2}^{\gamma \text{lip}} \leq \varepsilon. \quad (8.8)$$

The difference $\widehat{\iota}_n := \iota_n - \iota_{n-1}$ (with $\widehat{\iota}_0 := 0$) is defined on Ω_n^{Mel} and one has, in case $n \geq 1$,

$$\|\widehat{\iota}_n\|_{s_0 + \mu_1}^{\gamma \text{lip}} \leq \varepsilon \gamma^{-2} N_{n-1}^{-\alpha_1}. \quad (8.9)$$

(NM2) $_n$ (low norms) $\|F_\omega(\iota_n, \zeta_n)\|_{s_0, \sigma - 2}^{\gamma \text{lip}} \leq C_* \varepsilon N_{n-1}^{-\eta_1}$, $|\zeta_n|^{\gamma \text{lip}} \leq C_* \|F_\omega(\iota_n, \zeta_n)\|_{s_0, \sigma - 2}^{\gamma \text{lip}}$.

(NM3) $_n$ (high norms) $\|\iota_n\|_{s_0 + \beta_1}^{\gamma \text{lip}} \leq C_* \varepsilon \gamma^{-2} N_{n-1}^{\kappa_1}$, $\|F_\omega(\iota_n, \zeta_n)\|_{s_0 + \beta_1, \sigma - 2}^{\gamma \text{lip}} \leq C_* \varepsilon N_{n-1}^{\kappa_1}$.

In (NM1) $_n$ – (NM3) $_n$, the γlip norms are defined on Ω_n^{Mel} , namely $\|\cdot\|_s^{\gamma \text{lip}} = \|\cdot\|_{s, \Omega_n^{\text{Mel}}}^{\gamma \text{lip}}$.

Proof. The proof of Theorem 8.1 follows the scheme in [2]. Note however that in contrast to the setup in [2], the regularity in the space variable is fixed, meaning that σ in h_\perp^σ is kept unchanged along the iteration. The main ingredient for proving the claimed estimates are the tame estimates of the approximate right inverse \mathbf{T} of Theorem 5.2. To shorten notation, we write $\|\cdot\|$ for $\|\cdot\|_{s, \Omega_n^{\text{Mel}}}^{\gamma \text{lip}}$ in this proof.

Proof of (NM1) $_0$ – (NM3) $_0$: Since $\omega^{nls}(\xi, 0) = \omega$ (by the definition of $\xi = \xi(\omega)$) and $(\iota_0, \zeta_0) = (0, 0)$ (by definition) one has $X_{H^{nls}} \circ \check{\iota}_0 = (\omega^{nls}(\xi, 0), 0, 0)$ (cf (1.12)), and hence by the definition (4.4) of F_ω ,

$$F_\omega(\iota_0, \zeta_0) = -\varepsilon X_P \circ \check{\iota}_0$$

where X_P is the Hamiltonian vector field of the Hamiltonian P , expressed in the coordinates $(\theta, y, z) \in M^\sigma$. By (6.47) we have

$$\tilde{X}_P = (d\Phi \tilde{X}_{\mathcal{P}})|_{\Phi^{-1}}, \quad P = \mathcal{P} \circ \Phi^{-1}$$

where $\Phi = \Phi^{nls}$ is the Birkhoff map of Theorem 3.1 and \tilde{X}_P is obtained from $X_{\mathcal{P}}$ by expressing it in the Birkhoff coordinates $(w_n)_{n \in \mathbb{Z}}$ and then adding the complex conjugate as a second component. In this way one sees that for any $s_0 \leq s \leq s_* - 1$

$$\|X_P \circ \check{\iota}_0\|_{s, \sigma-2} \leq_s 1.$$

Altogether we proved that

$$\|F_\omega(t_0, \zeta_0)\|_{s, \sigma-2} \leq_s \varepsilon. \quad (8.10)$$

Since $N_{-1} = 1$ (by definition), one sees that the claimed estimates of $(NM1)_0 - (NM3)_0$ hold, once $C_* \equiv C_*(s_0 + \beta_1)$ is chosen large enough.

Proof of inductive step: Assume that $(NM1)_n - (NM3)_n$ hold for a given $n \geq 0$. Our task is to prove that $(NM1)_{n+1} - (NM3)_{n+1}$ hold as well. First we have to make sure that the smallness assumption (5.53) of Theorem 5.2 for (ι_n, ζ_n) is valid with $\Omega_o(\iota_n)$ given by Ω_n^{Mel} . Indeed, since (8.8) is satisfied by the induction hypothesis, (5.53) holds by choosing δ_2 in the statement of the theorem sufficiently small. Hence Theorem 5.2 applies to (ι_n, ζ_n) : by the definition of $\Omega_{n+1}^{\text{Mel}}$ in (8.6) there exists a family of operators $(\mathbf{T}_n(\omega))_{\omega \in \Omega_{n+1}^{\text{Mel}}}$ so that the estimates (5.54) hold,

$$\|\mathbf{T}_n g\|_{s, \sigma} \leq_s \gamma^{-2} (\|g\|_{s+\mu_1, \sigma-2} + \|\iota_n\|_{s+\mu_1} \|g\|_{s_0+\mu_1, \sigma-2}), \quad \forall s \in [s_0, s_0 + \beta_1], \quad (8.11)$$

implying together with (8.8) and (8.5) that

$$\|\mathbf{T}_n g\|_{s_0, \sigma} \leq_{s_0} \gamma^{-2} \|g\|_{s_0+\mu_1, \sigma-2}. \quad (8.12)$$

Furthermore, denoting by L_n the differential $d_{\iota, \zeta} F_\omega(\iota_n, \zeta_n)$, one has by (5.55) for any s in $[s_0, s_0 + \beta_1]$,

$$\begin{aligned} \|(L_n \circ \mathbf{T}_n - \text{Id})g\|_{s, \sigma-2} &\leq_s \gamma^{-3} \|F_\omega(\iota_n, \zeta_n)\|_{s_0+\mu_1, \sigma-2} \|g\|_{s+\mu_1, \sigma-2} + \\ &\gamma^{-3} \|F_\omega(\iota_n, \zeta_n)\|_{s+\mu_1, \sigma-2} \|g\|_{s_0+\mu_1, \sigma-2} + \gamma^{-3} \|\iota_n\|_{s+\mu_1} \|F_\omega(\iota_n, \zeta_n)\|_{s_0+\mu_1, \sigma-2} \|g\|_{s_0+\mu_1, \sigma-2}. \end{aligned} \quad (8.13)$$

For $s = s_0$, this yields $\|(L_n \circ \mathbf{T}_n - \text{Id})g\|_{s_0, \sigma-2} \leq_{s_0} \gamma^{-3} \|F_\omega(\iota_n, \zeta_n)\|_{s_0+\mu_1, \sigma-2} \|g\|_{s_0+\mu_1, \sigma-2}$. Using that

$$\begin{aligned} \|F_\omega(\iota_n, \zeta_n)\|_{s_0+\mu_1, \sigma-2} &\leq_s \|\Pi_n F_\omega(\iota_n, \zeta_n)\|_{s_0+\mu_1, \sigma-2} + \|\Pi_n^\perp F_\omega(\iota_n, \zeta_n)\|_{s_0+\mu_1, \sigma-2} \\ &\stackrel{(8.1), (8.2)}{\leq} N_n^{\mu_1} \|F_\omega(\iota_n, \zeta_n)\|_{s_0, \sigma-2} + N_n^{\mu_1 - \beta_1} \|F_\omega(\iota_n, \zeta_n)\|_{s_0+\beta_1, \sigma-2} \end{aligned} \quad (8.14)$$

the above estimate then leads to

$$\begin{aligned} \|(L_n \circ \mathbf{T}_n - \text{Id})g\|_{s_0, \sigma-2} &\leq_{s_0} N_n^{\mu_1} \gamma^{-3} \|F_\omega(\iota_n, \zeta_n)\|_{s_0, \sigma-2} \|g\|_{s_0+\mu_1, \sigma-2} \\ &+ N_n^{\mu_1 - \beta_1} \gamma^{-3} \|F_\omega(\iota_n, \zeta_n)\|_{s_0+\beta_1, \sigma-2} \|g\|_{s_0+\mu_1, \sigma-2}. \end{aligned} \quad (8.15)$$

For convenience we define $\mathcal{S}_n := (\iota_n, \zeta_n)$. As advertised at the beginning of this section, we identify the vectors $(\hat{\theta}, \hat{y}, \hat{z}) \in \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma$ and $(\hat{\theta}, \hat{y}, \hat{z}, \hat{z}) \in \mathbb{R}^S \times \mathbb{R}^S \times h_\perp^\sigma \times h_\perp^\sigma$. With this convention the Taylor expansion up to order 1 of F_ω at \mathcal{S}_n , reads

$$F_\omega(\mathcal{S}_n + \hat{\mathcal{S}}) = F_\omega(\mathcal{S}_n) + L_n \hat{\mathcal{S}} + Q(\mathcal{S}_n, \hat{\mathcal{S}}),$$

where $\hat{\mathcal{S}} = (\hat{\iota}, \hat{\zeta})$ is assumed to be a sufficiently small element in $E_n \times \mathbb{R}^S$ and $Q(\mathcal{S}_n, \hat{\mathcal{S}})$ denotes the Taylor remainder term. By the Newton-Nash-Moser iteration scheme, we define \mathcal{S}_{n+1} as $\mathcal{S}_n + \hat{\mathcal{S}}_{n+1}$ with $\hat{\mathcal{S}}_{n+1} := (\hat{\iota}_{n+1}, \hat{\zeta}_{n+1})$ chosen to be an approximate solution of the equation $F_\omega(\mathcal{S}_n) + L_n \hat{\mathcal{S}} = 0$. More precisely, we define \mathcal{S}_{n+1} on $\Omega_{n+1}^{\text{Mel}}$ by

$$\mathcal{S}_{n+1} := \mathcal{S}_n + \hat{\mathcal{S}}_{n+1}, \quad \hat{\mathcal{S}}_{n+1} := -\tilde{\Pi}_n \mathbf{T}_n \Pi_n F_\omega(\mathcal{S}_n) \quad (8.16)$$

where $\tilde{\Pi}_n(\iota, \zeta) := (\Pi_n \iota, \zeta)$. Arguing as above and using the induction hypothesis, one verifies that \mathcal{S}_{n+1} and $\hat{\mathcal{S}}_{n+1}$ are in $E_n \times \mathbb{R}^S$. (We choose C_1, N_0 sufficiently large and δ_2 sufficiently small.) Then

$$F_\omega(\mathcal{S}_{n+1}) = F_\omega(\mathcal{S}_n) + L_n \hat{\mathcal{S}}_{n+1} + Q_n, \quad Q_n := Q(\mathcal{S}_n, \hat{\mathcal{S}}_{n+1}). \quad (8.17)$$

Upon substituting the expression for $\hat{\mathcal{S}}_{n+1}$ in (8.16) and writing $\tilde{\Pi}_n$ as $\text{Id} - \tilde{\Pi}_n^\perp$ with $\tilde{\Pi}_n^\perp(\iota, \zeta) := (\Pi_n^\perp \iota, 0)$, the identity (8.17) reads

$$F_\omega(\mathcal{S}_{n+1}) = F_\omega(\mathcal{S}_n) - L_n \mathbf{T}_n \Pi_n F_\omega(\mathcal{S}_n) + L_n \tilde{\Pi}_n^\perp \mathbf{T}_n \Pi_n F_\omega(\mathcal{S}_n) + Q_n.$$

The first two terms in the latter expression are split up by applying $\text{Id} = \Pi_n + \Pi_n^\perp$, yielding

$$F_\omega(\mathcal{S}_{n+1}) = \Pi_n^\perp F_\omega(\mathcal{S}_n) + R_n + Q'_n + Q_n \quad (8.18)$$

where

$$R_n := (L_n \tilde{\Pi}_n^\perp - \Pi_n^\perp L_n) \mathbf{T}_n \Pi_n F_\omega(\mathcal{S}_n), \quad Q'_n := -\Pi_n (L_n \mathbf{T}_n - \text{Id}) \Pi_n F_\omega(\mathcal{S}_n). \quad (8.19)$$

We estimate the terms $Q_n, Q'_n,$ and R_n separately.

Estimate of Q_n : By (4.4), ζ_n appears linearly in $F_\omega(\mathcal{S}_n)$, hence for any $\hat{\mathcal{S}} = (\hat{\iota}, \hat{\zeta}) \in E_n \times \mathbb{R}^S$, $Q(\mathcal{S}_n, \hat{\mathcal{S}})$ is independent of ζ_n and $\hat{\zeta}$. By Lemmata 3.3, 3.4 and using (8.1), (8.8) we conclude that

$$\|Q(\mathcal{S}_n, \hat{\mathcal{S}})\|_{s, \sigma-2} \leq_s \|\hat{\iota}\|_s \|\hat{\zeta}\|_{s_0} + \|\iota_n\|_{s+2s_0} \|\hat{\zeta}\|_{s_0}^2, \quad \forall s \in [s_0, s_0 + \beta_1], \quad (8.20)$$

$$\|Q(\mathcal{S}_n, \hat{\mathcal{S}})\|_{s_0, \sigma-2} \leq_{s_0} \|\hat{\iota}\|_{s_0}^2. \quad (8.21)$$

By the definition of $\hat{\mathcal{S}}_{n+1}$ in (8.16), one gets by using first (8.1) and then (8.11) together with (8.8), 8.5,

$$\|\hat{\iota}_{n+1}\|_{s_0+\beta_1} \leq N_n^{\mu_1} \|\hat{\iota}_{n+1}\|_{s_0+\beta_1-\mu_1} \leq_{s_0+\beta_1} N_n^{\mu_1} (\gamma^{-2} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \|\iota_n\|_{s_0+\beta_1}), \quad (8.22)$$

and similarly,

$$\|\hat{\iota}_{n+1}\|_{s_0} \stackrel{(8.12)}{\leq} \gamma^{-2} \|\Pi_n F_\omega(\mathcal{S}_n)\|_{s_0+\mu_1, \sigma-2} \stackrel{(8.1)}{\leq} \gamma^{-2} N_n^{\mu_1} \|F_\omega(\mathcal{S}_n)\|_{s_0, \sigma-2} \quad \text{and} \quad \|\hat{\iota}_{n+1}\|_{s_0} \stackrel{(8.8)}{\leq} \varepsilon \gamma^{-2}. \quad (8.23)$$

Hence the term Q_n , defined in (8.17), satisfies by (8.21) and (8.23)

$$\|Q_n\|_{s_0, \sigma-2} \leq_{s_0} \gamma^{-4} N_n^{2\mu_1} \|F_\omega(\mathcal{S}_n)\|_{s_0, \sigma-2}^2 \quad (8.24)$$

and by (8.20), (8.22), (8.23) together with (8.8)

$$\|Q_n\|_{s_0+\beta_1, \sigma-2} \leq_{s_0+\beta_1} N_n^{\mu_1} \varepsilon \gamma^{-2} (\gamma^{-2} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \|\iota_n\|_{s_0+\beta_1}). \quad (8.25)$$

Estimate of Q'_n : Using (8.15) and, respectively, (8.1), (8.13), together with (8.3), (8.8) one verifies that

$$\|Q'_n\|_{s_0, \sigma-2} \leq_{s_0} N_n^{2\mu_1} \gamma^{-3} (\|F_\omega(\mathcal{S}_n)\|_{s_0, \sigma-2} + N_n^{-\beta_1} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2}) \|F_\omega(\mathcal{S}_n)\|_{s_0, \sigma-2}, \quad (8.26)$$

$$\|Q'_n\|_{s_0+\beta_1, \sigma-2} \leq N_n^{\mu_1} \|Q'_n\|_{s_0+\beta_1-\mu_1, \sigma-2} \leq_{s_0+\beta_1} N_n^{\mu_1} \varepsilon \gamma^{-3} (\|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \varepsilon \|\iota_n\|_{s_0+\beta_1}). \quad (8.27)$$

Estimate of R_n : In a first step we estimate the operator $L_n \tilde{\Pi}_n^\perp - \Pi_n^\perp L_n$. For $\hat{\mathcal{S}} := (\hat{\iota}, \hat{\zeta})$ we have

$$\begin{aligned} L_n \hat{\mathcal{S}} &= \omega \cdot \partial_\varphi \hat{\iota} - d_\iota X_{H_\varepsilon}(\iota_n) [\hat{\zeta}] + (0, \hat{\zeta}, 0, 0) \\ &= \omega \cdot \partial_\varphi \hat{\iota} - d_\iota X_{H^{nl_s}}(\iota_n) [\hat{\zeta}] - \varepsilon d_\iota X_P(\iota_n) [\hat{\zeta}] + (0, \hat{\zeta}, 0, 0). \end{aligned} \quad (8.28)$$

Writing $d_\iota X_{H^{nl_s}}(\iota_n) = d_\iota X_{H^{nl_s}}(\iota_0) + (d_\iota X_{H^{nl_s}}(\iota_n) - d_\iota X_{H^{nl_s}}(\iota_0))$ we get

$$L_n \hat{\mathcal{S}} = L_n^I \hat{\mathcal{S}} + L_n^{II} \hat{\mathcal{S}} + (0, \hat{\zeta}, 0, 0)$$

where

$$L_n^I \widehat{\mathcal{S}} := \omega \cdot \partial_{\varphi} \widehat{\iota} - d_\iota X_{H^{nls}}(\iota_0)[\widehat{\iota}], \quad L_n^{II} \widehat{\mathcal{S}} := (d_\iota X_{H^{nls}}(\iota_n) - d_\iota X_{H^{nls}}(\iota_0))[\widehat{\iota}] + \varepsilon d_\iota X_P(\iota_n)[\widehat{\iota}].$$

Since

$$d_\iota X_{H^{nls}}(\iota_0)[\widehat{\iota}] = \left(\left(\sum_{k \in S} \partial_{I_k} \omega_n(\xi, 0) \widehat{y}_k \right)_{n \in S}, 0, -i(\omega_n(\xi, 0) \widehat{z}_n)_{n \in S^\perp}, i(\omega_n(\xi, 0) \widehat{z}_n)_{n \in S^\perp} \right),$$

the 'commutator' $L_n^I \widetilde{\Pi}_n^\perp - \Pi_n^\perp L_n^I$ vanishes, implying that

$$L_n \widetilde{\Pi}_n^\perp - \Pi_n^\perp L_n = L_n^{II} \widetilde{\Pi}_n^\perp - \Pi_n^\perp L_n^{II}.$$

Using Proposition 3.3, Corollary 3.1, the smallness condition (8.8), and the smoothing properties (8.1), (8.2), it follows that for any $\widehat{\mathcal{S}}$ in $E_n \times \mathbb{R}^S$

$$\|(L_n \widetilde{\Pi}_n^\perp - \Pi_n^\perp L_n) \widehat{\mathcal{S}}\|_{s_0, \sigma-2} \leq_{s_0+\beta_1} N_n^{-\beta_1+\mu_1} (\varepsilon \gamma^{-2} \|\widehat{\iota}\|_{s_0+\beta_1} + \|\iota_n\|_{s_0+\beta_1} \|\widehat{\iota}\|_{s_0}), \quad (8.29)$$

$$\|(L_n \widetilde{\Pi}_n^\perp - \Pi_n^\perp L_n) \widehat{\mathcal{S}}\|_{s_0+\beta_1, \sigma-2} \leq_{s_0+\beta_1} N_n^{\mu_1} (\varepsilon \gamma^{-2} \|\widehat{\iota}\|_{s_0+\beta_1} + \|\iota_n\|_{s_0+\beta_1} \|\widehat{\iota}\|_{s_0}). \quad (8.30)$$

Hence, applying (8.11), (8.29), (8.30), (8.5), (8.8), (8.1), the term R_n defined in (8.19) satisfies

$$\|R_n\|_{s_0, \sigma-2} \leq_{s_0+\beta_1} N_n^{2\mu_1-\beta_1} (\varepsilon \gamma^{-4} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \varepsilon \gamma^{-2} \|\iota_n\|_{s_0+\beta_1}), \quad (8.31)$$

$$\|R_n\|_{s_0+\beta_1, \sigma-2} \leq_{s_0+\beta_1} N_n^{2\mu_1} (\varepsilon \gamma^{-4} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \varepsilon \gamma^{-2} \|\iota_n\|_{s_0+\beta_1}). \quad (8.32)$$

Estimate of $F_\omega(\mathcal{S}_{n+1})$: By the identity (8.18) and the estimates (8.25), (8.24), (8.27), (8.26), (8.31), (8.32), (8.5), (8.8), we get

$$\|F_\omega(\mathcal{S}_{n+1})\|_{s_0, \sigma-2} \leq_{s_0+\beta_1} N_n^{2\mu_1-\beta_1} (\|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \varepsilon \gamma^{-2} \|\iota_n\|_{s_0+\beta_1}) + N_n^{2\mu_1} \gamma^{-4} \|F_\omega(\mathcal{S}_n)\|_{s_0, \sigma-2}^2, \quad (8.33)$$

$$\|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\beta_1, \sigma-2} \leq_{s_0+\beta_1} N_n^{2\mu_1} (\|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + \varepsilon \gamma^{-2} \|\iota_n\|_{s_0+\beta_1}). \quad (8.34)$$

Estimate of ι_{n+1} : Using (8.22) the term $\iota_{n+1} = \iota_n + \widehat{\iota}_{n+1}$ can be estimated as follows:

$$\|\iota_{n+1}\|_{s_0+\beta_1} \leq_{s_0+\beta_1} \|\iota_n\|_{s_0+\beta_1} + \|\widehat{\iota}_{n+1}\|_{s_0+\beta_1} \leq_{s_0+\beta_1} N_n^{\mu_1} (\|\iota_n\|_{s_0+\beta_1} + \gamma^{-2} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1}). \quad (8.35)$$

Proof of $(NM3)_{n+1}$: By (8.34), $(NM3)_n$ we have

$$\begin{aligned} \|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\beta_1} &\leq_{s_0+\beta_1} N_n^{2\mu_1} \|F_\omega(\mathcal{S}_n)\|_{s_0+\beta_1, \sigma-2} + N_n^{2\mu_1} \varepsilon \gamma^{-2} \|\iota_n\|_{s_0+\beta_1} \\ &\leq_{s_0+\beta_1} N_n^{2\mu_1} C_* \varepsilon N_n^{\kappa_1} + \varepsilon \gamma^{-2} N_n^{2\mu_1} C_* \varepsilon \gamma^{-2} N_n^{\kappa_1} \stackrel{\varepsilon \gamma^{-4} \leq 1}{\leq} C(s_0 + \beta_1) C_* \varepsilon N_n^{2\mu_1} N_n^{\kappa_1}. \end{aligned} \quad (8.36)$$

Hence $\|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\beta_1} \leq C_* \varepsilon N_n^{\kappa_1}$ provided that

$$N_j^{\kappa_1-2\mu_1} N_{j-1}^{-\kappa_1} \geq C(s_0 + \beta_1), \quad \forall j \geq 0,$$

which is satisfied by choosing κ_1 as in (8.3) and N_0 sufficiently large. The bound for $\|\iota_{n+1}\|_{s_0+\beta_1}$ is proved similarly, hence $(NM3)_{n+1}$ is established.

Proof of $(NM2)_{n+1}$: By (8.33), $(NM2)_n$, $(NM3)_n$, and $\varepsilon \gamma^{-4} \leq 1$ (cf (8.5)), one has

$$\|F_\omega(\mathcal{S}_{n+1})\|_{s_0, \sigma-2} \leq C(s_0 + \beta_1) (N_n^{2\mu_1-\beta_1} N_{n-1}^{\kappa_1} C_* \varepsilon + N_n^{2\mu_1} N_{n-1}^{-2\eta_1} C_*^2 \varepsilon^2 \gamma^{-4}).$$

Hence $\|F_\omega(\mathcal{S}_{n+1})\|_{s_0, \sigma-2} \leq C_* \varepsilon N_n^{-\eta_1}$ provided that

$$C(s_0 + \beta_1) N_j^{2\mu_1+\eta_1-\beta_1} N_{j-1}^{\kappa_1} \leq \frac{1}{2}, \quad C(s_0 + \beta_1) C_* N_j^{2\mu_1+\eta_1} N_{j-1}^{-2\eta_1} \varepsilon \gamma^{-4} \leq \frac{1}{2}, \quad \forall j \geq 0.$$

The latter conditions are fulfilled by choosing η_1, β_1 as in (8.3), N_0 sufficiently large and δ_2 in (8.5) sufficiently small. Moreover, the claimed estimate for ζ_n follows from Lemma 5.1 (no induction needed). Altogether, this establishes $(NM2)_{n+1}$.

Proof of estimate (8.9): The bound (8.9) for $\widehat{\iota}_1$ follows by (8.16) and (8.11) (for $s = s_0 + \mu_1$) together with the estimate $\|F_\omega(\mathcal{S}_0)\|_{s_0+2\mu_1, \sigma-2} \leq_{s_0+2\mu_1} \varepsilon$ of (8.10). Similarly, the bound (8.9) for $\widehat{\iota}_{n+1}$ is obtained from (8.16) and (8.11) (cf (8.22)), using (8.1) and (8.3).

Proof of estimate (8.8): It remains to prove the inductive step from n to $n+1$ of (8.8). We have

$$\|\iota_{n+1}\|_{s_0+\mu_1} \leq \sum_{k=1}^{n+1} \|\widehat{\iota}_k\|_{s_0+\mu_1} \ll \varepsilon \gamma^{-2} \sum_{k \geq 1} N_{k-1}^{-\alpha_1} \ll \varepsilon \gamma^{-2}.$$

Finally, to prove the claimed estimate for $\|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\mu_1, \sigma-2}$ we write $F_\omega(\mathcal{S}_{n+1})$ as a sum, $\Pi_n F_\omega(\mathcal{S}_{n+1}) + \Pi_n^\perp F_\omega(\mathcal{S}_{n+1})$, and then use (8.1) to get

$$\|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\mu_1, \sigma-2} \leq N_n^{\mu_1} \|F_\omega(\mathcal{S}_{n+1})\|_{s_0, \sigma-2} + N_n^{\mu_1 - \beta_1} \|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\beta_1, \sigma-2}.$$

By $(NM2)_{n+1}$, $(NM3)_{n+1}$, and (8.3) it then follows that

$$\|F_\omega(\mathcal{S}_{n+1})\|_{s_0+\mu_1, \sigma-2} \leq C_* \varepsilon N_n^{\mu_1 - \eta_1} + C_* \varepsilon N_n^{\mu_1 - \beta_1 + \kappa_1} \ll \varepsilon,$$

which is the second inequality in (8.8) at the step $n+1$. This finishes the proof of the inductive step. \square

Theorem 8.1 leads in a straightforward way to a proof of Theorem 4.1, except for the measure estimate (4.1) which is proved in Section 9. By $(NM1)_n$ the sequence $(\iota_n(\cdot; \omega))_{n \geq 0}$ converges to ι_ω in the norm $\|\cdot\|_{s_0+\mu_1}^{\text{lip}}$, while $(NM2)_n$ implies that $F_\omega(\iota_n, \zeta_n) \rightarrow 0$ and $\zeta_n \rightarrow 0$. Altogether it then follows that $F_\omega(\iota_\omega, 0) = 0$. The following corollary implies Theorem 4.1 with s_* chosen as in Theorem 8.1, μ_2 given by $\mu_1(|S|, \tau)$ with $\tau = 2|S| + 1$ (cf Section 9 for this choice of τ) and $0 < \varepsilon_0 < 1$ so that for some $0 < a < 1/4$, $\varepsilon_0^{1-4a} < \delta_2$ with δ_2 as in Theorem 8.1 (cf Theorem 9.1).

Corollary 8.1. (Invariant torus and linear stability) *Under the same assumptions as in Theorem 8.1, the sequence (ι_n, ζ_n) converges in the norm $\|\cdot\|_{s_0+\mu_1}^{\text{lip}}$ on the set*

$$\Omega_\infty^{\text{Mel}} := \bigcap_{n \geq 0} \Omega_n^{\text{Mel}} \quad (8.37)$$

to $(\iota, 0)$ with $\iota \equiv \iota_\omega$, $\omega \in \Omega_\infty^{\text{Mel}}$, satisfying $F_\omega(\iota_\omega, 0) = 0$ and $\|\iota\|_{s_0+\mu_1}^{\text{lip}} \ll \varepsilon \gamma^{-2}$. The sets Ω_n^{Mel} are defined in (8.6). Furthermore, for any $\omega \in \Omega_\infty^{\text{Mel}}$, the torus $\check{\iota}_\omega(\mathbb{T}^S)$ is linearly stable in the sense of Lyapunov: linearizing the equation $\partial_t \check{\iota} - X_{H_\varepsilon}(\check{\iota}) = 0$ at the quasi-periodic solution $t \mapsto \iota_\omega(\omega t)$ in the coordinates provided in Section 5, one obtains

$$\begin{cases} \dot{\widehat{\psi}} = K_{2,0}(\omega t)[\widehat{v}] + K_{1,1}(\omega t)[\widehat{W}] \\ \dot{\widehat{v}} = 0 \\ \dot{\widehat{W}} = -\mathbb{J}_2 K_{0,2}(\omega t)[\widehat{W}] - \mathbb{J}_2 (K_{1,1}(\omega t))^t [\widehat{v}] \end{cases} \quad \mathbb{J}_2 := \mathfrak{i} \begin{pmatrix} 0 & \text{Id}_\perp \\ -\text{Id}_\perp & 0 \end{pmatrix}. \quad (8.38)$$

For any initial datum $(\widehat{v}_0, \widehat{W}_0)$ the solution of (8.38) satisfies

$$\widehat{v}(t) = \widehat{v}(0), \forall t \in \mathbb{R}, \quad \sup_{t \in \mathbb{R}} \|\widehat{W}(t, \cdot)\|_{h_\perp^\sigma \times h_\perp^\sigma} \ll \|\widehat{W}(0)\|_{h_\perp^\sigma \times h_\perp^\sigma} + |\widehat{v}_0|. \quad (8.39)$$

Proof. It remains to prove that $\check{\iota}_\omega(\mathbb{T}^S)$ is linearly stable for any $\omega \in \Omega_\infty^{\text{Mel}}$. By (5.26) and, since $F_\omega(\iota_\omega, 0) = 0$ implies that $G_2 = 0$ by Lemma 5.7, we have

$$d_{\iota, \zeta} F_\omega(\iota_{\text{iso}})[\widehat{v}, \widehat{\zeta}] = d\Gamma(\check{\iota}_0)(\omega \cdot \partial_\varphi - d_{\iota, \zeta} X_{K_{\varepsilon, \zeta}}(\check{\iota}_0))[d\Gamma(\check{\iota}_0)^{-1}[\widehat{v}, \widehat{\zeta}].$$

Since $\check{\iota}_\omega$ is an isotropic torus embedding it coincides with $\check{\iota}_{\text{iso}}$, constructed in Subsection 5.2 (cf (5.9), (5.6)). Furthermore recall that by (5.31), and since $G_3 = 0$ by Lemma 5.8, we have

$$\omega \cdot \partial_\varphi - d_{\iota, \zeta} X_{K_{\varepsilon, \zeta}}(\check{\iota}_0) = \mathfrak{F}_\omega$$

where \mathfrak{L}_ω , when expressed in the coordinates ψ, v, W , is given by

$$\mathfrak{L}_\omega[\widehat{v}, 0] = (\omega \cdot \partial_\varphi \widehat{\psi} - K_{2,0}(\varphi)[\widehat{v}] - K_{1,1}(\varphi)[\widehat{W}], \omega \cdot \partial_\varphi \widehat{v}, \omega \cdot \partial_\varphi \widehat{W} + \mathbb{J}_2 K_{1,1}(\varphi)^t[\widehat{v}] + \mathbb{J}_2 K_{0,2}(\varphi)[\widehat{W}]).$$

Then (8.38) follows. To prove (8.39) recall that the operator $\mathfrak{L}_\omega = \omega \cdot \partial_\varphi + \mathbb{J}_2 K_{0,2}(\varphi)$, introduced in (5.35), is conjugated to the φ -independent 2×2 block diagonal operator $\mathbf{L}_\infty(\omega) = \omega \cdot \partial_\varphi \mathbb{I}_2 + \mathbf{N}_\infty(\omega)$, defined in (7.49), (7.50),

$$\mathfrak{L}_\omega = \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3 \Phi_\infty \mathbf{L}_\infty \Phi_\infty^{-1} \mathfrak{F}_3^{-1} \mathfrak{F}_2^{-1} \mathfrak{F}_1^{-1},$$

by the composition of the symplectic transformations $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ (Section 6) and Φ_∞ (Subsection 7.5). The equation $\dot{\widehat{W}} = -\mathbb{J}_2 K_{0,2}(\omega t)[\widehat{W}] - \mathbb{J}_2 (K_{1,1}(\omega t))^t[\widehat{v}_0]$ then transforms into

$$\dot{\widehat{V}} = -\mathbf{N}_\infty(\omega) \widehat{V} - g_\infty(\omega t), \quad g_\infty(\omega t) := (\Phi_\infty(\omega t)^{-1} \circ \mathfrak{F}_3(\omega t)^{-1} \circ \mathfrak{F}_2(\omega t)^{-1} \circ \mathfrak{F}_1(\omega t)^{-1}) \mathbb{J}_2 (K_{1,1}(\omega t))^t[\widehat{v}_0]$$

where $\widehat{V}(t)$ is given by $(\Phi_\infty(\omega t)^{-1} \circ \mathfrak{F}_3(\omega t)^{-1} \circ \mathfrak{F}_2(\omega t)^{-1} \circ \mathfrak{F}_1(\omega t)^{-1}) \widehat{W}(t)$. Since the coordinate transformations $\mathfrak{F}_1(\omega t)^{-1}, \mathfrak{F}_2(\omega t)^{-1}, \mathfrak{F}_3(\omega t)^{-1}, \Phi_\infty(\omega t)^{-1} : h_\perp^\sigma \times h_\perp^\sigma \rightarrow h_\perp^\sigma \times h_\perp^\sigma$ (see Sections 6, 7) and the operator $(K_{1,1}(\omega t))^t : \mathbb{R}^S \rightarrow h_\perp^\sigma \times h_\perp^\sigma$ (see Lemma 5.10) are bounded, uniformly in t , one has

$$\sup_{t \in \mathbb{R}} \|g_\infty(\omega t)\|_{h_\perp^\sigma \times h_\perp^\sigma} \leq |\widehat{v}_0|.$$

By the definition of \mathbf{N}_∞ in (7.50) and the estimates provided by (7.64) - (7.66) in Theorem 7.2 it then follows by the method of the variation of constants that the solution of $\dot{\widehat{V}} = -\mathbf{N}_\infty \widehat{V} - g_\infty(\omega t)$ with initial datum \widehat{V}_0 satisfies

$$\sup_{t \in \mathbb{R}} \|\widehat{V}(t, \cdot)\|_{h_\perp^\sigma \times h_\perp^\sigma} \leq \|\widehat{V}_0\|_{h_\perp^\sigma \times h_\perp^\sigma} + |\widehat{v}_0|.$$

Finally, using that the coordinate transformations $\mathfrak{F}_1(\omega t), \mathfrak{F}_2(\omega t), \mathfrak{F}_3(\omega t), \Phi_\infty(\omega t)$ are bounded operators on $h_\perp^\sigma \times h_\perp^\sigma$, uniformly in t , (see Sections 6, 7), one concludes that the corresponding solution $\widehat{W}(t)$ of $\dot{\widehat{W}} = -\mathbb{J}_2 K_{0,2}(\omega t)[\widehat{W}] - \mathbb{J}_2 (K_{1,1}(\omega t))^t[\widehat{v}_0]$ satisfies (8.39). \square

Finally we prove the statement of Remark 4.1 saying that for most of the $\omega \in \Omega_\infty^{\text{Mel}}$, the distance of the embedded torus $\check{\iota}_\omega(\mathbb{T}^S)$ to the standard torus $\check{\iota}_0(\mathbb{T}^S)$ is of the order of $\varepsilon \gamma^{-1}$. To state our result more precisely, we introduce the first order Melnikov non resonance conditions for the unperturbed equation

$$\Omega_{\gamma, \tau}^{nls} := \left\{ \omega \in \Omega : |\omega \cdot \ell + \omega_k^{nls}(\xi(\omega), 0)| \geq \frac{\gamma k^2}{\langle \ell \rangle^\tau} \quad \forall (\ell, k) \in \mathbb{Z}^S \times S^\perp \right\}. \quad (8.40)$$

Arguing as in Section 9 (cf Lemmas 9.3, 9.4) one shows that $\text{meas}(\Omega \setminus \Omega_{\gamma, \tau}^{nls}) = O(\gamma)$. Then the following holds:

Corollary 8.2. (Size of perturbed torus) *For any $\omega \in \Omega_\infty^{\text{Mel}} \cap \Omega_{\gamma, \tau}^{nls}$, the torus embedding $\check{\iota}_\omega(\varphi) = (\theta(\varphi), y(\varphi), z(\varphi))$ of Corollary 8.1 satisfies*

$$\|y\|_{s_0}, \|z\|_{s_0, \sigma} \leq \varepsilon \gamma^{-1}.$$

Proof. The torus embedding $\check{\iota}(\varphi) = (\theta(\varphi), y(\varphi), z(\varphi))$ of Corollary 8.1 satisfies the equation $F_\omega(\iota, 0) = 0$. When written componentwise, the latter equation reads

$$\begin{cases} \omega \cdot \partial_\varphi \theta = \omega^{nls}(\xi + y, z\bar{z}) + \varepsilon \nabla_y P(\theta, y, z) \\ \omega \cdot \partial_\varphi y = -\varepsilon \nabla_\theta P(\theta, y, z) \\ i\omega \cdot \partial_\varphi z_k = \omega_k^{nls}(\xi + y, z\bar{z}) z_k + \varepsilon \partial_{z_k} P(\theta, y, z), \quad k \in S^\perp. \end{cases} \quad (8.41)$$

Furthermore, $\iota(\varphi) = (\Theta(\varphi), y(\varphi), z(\varphi))$ with $\Theta(\varphi) = \theta(\varphi) - \varphi$ can be estimated as follows

$$\|\iota\|_{s_0 + \mu_1} = \|\Theta\|_{s_0 + \mu_1} + \|y\|_{s_0 + \mu_1} + \|z\|_{s_0 + \mu_1, \sigma} \leq \varepsilon \gamma^{-2}$$

where μ_1 is the integer given in Theorem 5.2. Since μ_1 is larger than the integer μ_0 of Theorem 5.1 and $\mu_0 = 4s_0 + 10\tau + 7$ one has $\mu_1 \geq 2s_0 + \tau$, implying that

$$\|\iota\|_{s_0+2s_0+\tau} \ll \varepsilon\gamma^{-2}. \quad (8.42)$$

Estimate of $\|y\|_{s_0}$: Since $\omega \in \Omega_\infty^{\text{Mel}} \subset \Omega_{\gamma,\tau}$, the solution y of the equation $\omega \cdot \partial_\varphi y = -\varepsilon \nabla_\theta P(\theta, y, z)$,

$$y = -\varepsilon(\omega \cdot \partial_\varphi)^{-1} \nabla_\theta P(\theta, y, z),$$

can be estimated as follows

$$\|y\|_{s_0} \stackrel{\text{Lemma 2.2}}{\leq} \varepsilon\gamma^{-1} \|\nabla_\theta P(\theta, y, z)\|_{s_0+\tau} \stackrel{\text{Prop. 3.3 (i)}}{\ll} \varepsilon\gamma^{-1} (1 + \|\iota\|_{3s_0+\tau}) \stackrel{(8.42), (8.5)}{\ll} \varepsilon\gamma^{-1}.$$

Estimate of $\|z\|_{s_0,\sigma}$: For any $k \in S^\perp$ write $\omega_k^{nls}(\xi + y, z\bar{z}) = a_k^I + a_k^{II}$ where

$$a_k^I := \omega_k^{nls}(\xi, 0) \quad a_k^{II} := \omega_k^{nls}(\xi + y, z\bar{z}) - \omega_k^{nls}(\xi, 0) \quad (8.43)$$

and define the diagonal operators

$$A^I := \text{diag}_{k \in S^\perp} a_k^I, \quad A^{II} := \text{diag}_{k \in S^\perp} a_k^{II}. \quad (8.44)$$

The third equation in (8.41) can then be rewritten as

$$Bz = A^{II}z + \varepsilon \nabla_{\bar{z}} P(\theta, y, z), \quad B := i\omega \cdot \partial_\varphi \text{Id}_\perp - A^I. \quad (8.45)$$

Since by assumption $\omega \in \Omega_{\gamma,\tau}^{nls}$, the diagonal operator B is invertible and for any $g \in H^{s+\tau}(\mathbb{T}^S, h_\perp^{\sigma-2})$ one has $\|B^{-1}g\|_{s,\sigma} \leq \gamma^{-1} \|g\|_{s+\tau,\sigma-2}$. Furthermore, the identity (8.45) leads to

$$z = B^{-1}A^{II}z + \varepsilon B^{-1} \nabla_{\bar{z}} P(\theta, y, z). \quad (8.46)$$

The latter two terms are estimated individually:

$$\begin{aligned} \|B^{-1}A^{II}z\|_{s_0,\sigma} &\leq \gamma^{-1} \|A^{II}z\|_{s_0+\tau,\sigma} \stackrel{(8.43), (8.44), (3.37)}{\ll} \gamma^{-1} \|\iota\|_{3s_0+\tau,\sigma} \|z\|_{s_0+\tau,\sigma} \\ &\stackrel{(8.42)}{\ll} \varepsilon^2 \gamma^{-5} \ll (\varepsilon\gamma^{-1})(\varepsilon\gamma^{-4}) \stackrel{(8.5)}{\ll} \varepsilon\gamma^{-1}. \end{aligned} \quad (8.47)$$

The second term on the right hand side of (8.46) can be estimated as

$$\begin{aligned} \varepsilon \|B^{-1} \nabla_{\bar{z}} P(\theta, y, z)\|_{s_0,\sigma} &\leq \varepsilon\gamma^{-1} \|\nabla_{\bar{z}} P(\theta, y, z)\|_{s_0+\tau,\sigma} \stackrel{\text{Prop. 3.3 (i)}}{\ll} \varepsilon\gamma^{-1} (1 + \|\iota\|_{3s_0+\tau}) \\ &\stackrel{(8.42), (8.5)}{\ll} \varepsilon\gamma^{-1}. \end{aligned} \quad (8.48)$$

The identity (8.46) and the estimates (8.47), (8.48) then yield $\|z\|_{s_0,\sigma} \ll \varepsilon\gamma^{-1}$. \square

9 Measure estimate

The goal of this section is to prove the measure estimate of Theorem 4.1.

Theorem 9.1. (Measure estimate) *Let $\tau := 2|S| + 1$. Assume the smallness condition (8.5) hold with ε, γ satisfying*

$$0 < \varepsilon^\alpha < \frac{1}{64}, \quad 0 < \alpha < 1/4, \quad \gamma = \varepsilon^\alpha. \quad (9.1)$$

Then there exists $0 < \mathfrak{b} \leq 1/2$ so that the set $\Omega_\varepsilon := \Omega_\infty^{\text{Mel}}$ (cf (8.37)), satisfies

$$\text{meas}(\Omega \setminus \Omega_\varepsilon) = O(\varepsilon^{\mathfrak{ab}}), \quad \text{as } \varepsilon \rightarrow 0. \quad (9.2)$$

The remaining part of this section is devoted to the proof of Theorem 9.1. We first choose

$$\gamma_* := \gamma^{1/2} = \varepsilon^{a/2}, \quad \tau_* := |S| + 1. \quad (9.3)$$

Note that, by (9.1), we have $8\gamma < \gamma_* < 1$. Then we consider the set of diophantine frequencies (cf (1.22))

$$\Omega_{\gamma_*, \tau_*} = \left\{ \omega \in \Omega : |\omega \cdot \ell| \geq \frac{\gamma_*}{|\ell|^{\tau_*}}, \quad \forall \ell \in \mathbb{Z}^S \setminus \{0\} \right\}. \quad (9.4)$$

To estimate the Lebesgue measure of the set $\Omega \setminus \Omega_\infty^{\text{Mel}}$, note that

$$\Omega \setminus \Omega_\infty^{\text{Mel}} \subseteq (\Omega \setminus \Omega_{\gamma_*, \tau_*}) \cup (\Omega_{\gamma_*, \tau_*} \cap \Omega \setminus \Omega_\infty^{\text{Mel}}). \quad (9.5)$$

Since Ω is compact and $\tau_* = |S| + 1$, one verifies by a standard estimate that

$$\text{meas}(\Omega \setminus \Omega_{\gamma_*, \tau_*}) = O(\gamma_*) \stackrel{(9.3)}{=} O(\varepsilon^{a/2}). \quad (9.6)$$

To deduce Theorem 9.1 it thus remains to prove that the measure of $(\Omega \setminus \Omega_\infty^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*}$ satisfies the estimate (9.2). Recall that by (8.37), $\Omega_\infty^{\text{Mel}} = \bigcap_{n \geq 0} \Omega_n^{\text{Mel}}$ where, according to (8.6)-(8.7), the sequence of subsets $(\Omega_n^{\text{Mel}})_{n \geq 0}$ is defined inductively by

$$\Omega_0^{\text{Mel}} = \Omega_{2\gamma_0, \tau}, \quad \text{and} \quad \Omega_{n+1}^{\text{Mel}} = \Omega_{\text{Mel}}^{2\gamma_n}(\iota_n), \quad n \geq 0. \quad (9.7)$$

Here $\gamma_n = \gamma(1 + 2^{-n})$ (hence $\gamma_0 = 2\gamma$) and $\Omega_{\text{Mel}}^{2\gamma_n}(\iota_n)$ is defined by (7.74), (7.57),

$$\Omega_{\text{Mel}}^{2\gamma_n}(\iota_n) = \left\{ \omega \in \Omega_n^{\text{Mel}} : (\mathbf{M}_{2\gamma_n}^I)_\infty, (\mathbf{M}_{+, 2\gamma_n}^{II})_\infty, (\mathbf{M}_{-, 2\gamma_n}^{II})_\infty \text{ hold} \right\}. \quad (9.8)$$

According to (7.75), (7.58), and (7.59) the Melnikov conditions $(\mathbf{M}_{2\gamma_n}^I)_\infty$, $(\mathbf{M}_{+, 2\gamma_n}^{II})_\infty$, and $(\mathbf{M}_{-, 2\gamma_n}^{II})_\infty$ for the Lipschitz family $\iota_n \equiv \iota_n(\cdot; \omega)$, $\omega \in \Omega_n^{\text{Mel}}$, are defined as follows:
 $(\mathbf{M}_{2\gamma_n}^I)_\infty$ For any $\ell \in \mathbb{Z}^S$, $j \in S_+^\perp$, the linear operator

$$A_\infty(\ell, j; \omega, \iota_n(\omega)) := \omega \cdot \ell \text{Id}_2 + [\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_j^j, \quad (9.9)$$

acting on the vector space \mathbb{C}^2 (cf Lemma 7.4), is invertible and

$$\|A_\infty(\ell, j; \omega, \iota_n(\omega))^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma_n \langle j \rangle^2}. \quad (9.10)$$

$(\mathbf{M}_{+, 2\gamma_n}^{II})_\infty$ For any $\ell \in \mathbb{Z}^S$, $j, k \in S_+^\perp$, the linear operator

$$L_\infty^+(\ell, j, k; \omega, \iota_n(\omega)) := \omega \cdot \ell \text{Id}_{\mathbb{C}^{2 \times 2}} + M_L([\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_j^j) + M_R([\overline{\mathbf{N}}_\infty^{(1)}(\omega, \iota_n(\omega))]_k^k), \quad (9.11)$$

acting on the vector space $\mathbb{C}^{2 \times 2}$ of 2×2 matrices (cf (7.56)), is invertible and

$$\|L_\infty^+(\ell, j, k; \omega, \iota_n(\omega))^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma_n \langle j^2 + k^2 \rangle}. \quad (9.12)$$

$(\mathbf{M}_{-, 2\gamma_n}^{II})_\infty$ For any $\ell \in \mathbb{Z}^S$, $j, k \in S_+^\perp$ with $(\ell, j, k) \neq (0, j, j)$, the linear operator

$$L_\infty^-(\ell, j, k; \omega, \iota_n(\omega)) := \omega \cdot \ell \text{Id}_{\mathbb{C}^{2 \times 2}} + M_L([\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_j^j) - M_R([\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_k^k), \quad (9.13)$$

acting on the vector space $\mathbb{C}^{2 \times 2}$ of 2×2 matrices (cf (7.55)), is invertible and

$$\|L_\infty^-(\ell, j, k; \omega, \iota_n(\omega))^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma_n \langle j^2 - k^2 \rangle}. \quad (9.14)$$

Since the sequence Ω_n^{Mel} , $n \geq 0$, is decreasing, $(\Omega \setminus \Omega_\infty^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*}$ can be written as a disjoint union,

$$(\Omega \setminus \Omega_\infty^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*} = \left((\Omega \setminus \Omega_0^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*} \right) \cap \left(\bigcup_{n \geq 0} (\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*} \right). \quad (9.15)$$

Since $\Omega_0^{\text{Mel}} = \Omega_{4\gamma, \tau}$, we have, by a standard estimate,

$$\text{meas}(\Omega \setminus \Omega_0^{\text{Mel}}) = O(\gamma). \quad (9.16)$$

To estimate the measure of $(\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*}$, write

$$(\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*} = \left(\bigcup_{\substack{\ell \in \mathbb{Z}^S \\ j \in S_+^\perp}} Q_{\ell j}(\iota_n) \right) \cup \left(\bigcup_{\substack{\ell \in \mathbb{Z}^S \\ j, k \in S_+^\perp}} R_{\ell j k}^+(\iota_n) \right) \cup \left(\bigcup_{\substack{\ell \in \mathbb{Z}^S, j, k \in S_+^\perp \\ (\ell, j, k) \neq (0, j, j)}} R_{\ell j k}^-(\iota_n) \right) \quad (9.17)$$

where, by (9.10), (9.12), (9.14), for any $\ell \in \mathbb{Z}^S$, j, k in S_+^\perp , and $n \geq 0$,

$$Q_{\ell j}(\iota_n) := \left\{ \omega \in \Omega_n^{\text{Mel}} \cap \Omega_{\gamma_*, \tau_*} : \text{either } A_\infty(\ell, j; \omega, \iota_n(\omega)) \text{ not invertible or } \right. \\ \left. A_\infty(\ell, j; \omega, \iota_n(\omega)) \text{ invertible and } \|A_\infty(\ell, j; \omega, \iota_n(\omega))^{-1}\| > \frac{\langle \ell \rangle^\tau}{2\gamma_n \langle j \rangle^2} \right\}, \quad (9.18)$$

$$R_{\ell j k}^+(\iota_n) := \left\{ \omega \in \Omega_n^{\text{Mel}} \cap \Omega_{\gamma_*, \tau_*} : \text{either } L_\infty^+(\ell, j, k; \omega, \iota_n(\omega)) \text{ not invertible or } \right. \\ \left. L_\infty^+(\ell, j, k; \omega, \iota_n(\omega)) \text{ invertible and } \|L_\infty^+(\ell, j, k; \omega, \iota_n(\omega))^{-1}\| > \frac{\langle \ell \rangle^\tau}{2\gamma_n \langle j^2 + k^2 \rangle} \right\}, \quad (9.19)$$

$$R_{\ell j k}^-(\iota_n) := \left\{ \omega \in \Omega_n^{\text{Mel}} \cap \Omega_{\gamma_*, \tau_*} : \text{either } L_\infty^-(\ell, j, k; \omega, \iota_n(\omega)) \text{ not invertible or } \right. \\ \left. L_\infty^-(\ell, j, k; \omega, \iota_n(\omega)) \text{ invertible and } \|L_\infty^-(\ell, j, k; \omega, \iota_n(\omega))^{-1}\| > \frac{\langle \ell \rangle^\tau}{2\gamma_n \langle j^2 - k^2 \rangle} \right\}. \quad (9.20)$$

Actually many of the subsets in (9.17) turn out to be empty due to the overlapping of Ω_n^{Mel} and $\Omega_{n+1}^{\text{Mel}}$. In order to show this we first prove that the eigenvalues of the normal form $\mathbf{N}_\infty^{(1)}$ (cf Lemma 7.4) evaluated at two consecutive approximate solutions $\check{\iota}_n, \check{\iota}_{n-1}$ are very close to each other.

Lemma 9.1. *For any $n \geq 1$,*

$$\sup_{j \in S_+^\perp} \left\| [\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_\infty^{(1)}(\iota_{n-1})]_j^j \right\| < \varepsilon \gamma^{-2} N_{n-1}^{-\alpha}, \quad \forall \omega \in \Omega_n^{\text{Mel}}, \quad (9.21)$$

where $\alpha = 6\tau + 4$ (cf (7.8)) and $[\mathbf{N}_\infty^{(1)}(\iota_n)]_j^j$ is a short for $[\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_j^j$.

Proof. We first task is to show that $(\mathbf{S2})_\nu$ of Theorem 7.3 with $(\nu, \gamma, \rho, \iota^{(1)}, \iota^{(2)})$ given by $(n, \gamma_{n-1}, \gamma 2^{-n}, \iota_{n-1}, \iota_n)$, applies. Since $\rho = \gamma 2^{-n} < \gamma_{n-1}/2$ and $\gamma_{n-1} - \rho = \gamma_n$ it means that

$$\Omega_\nu^{\gamma_{n-1}}(\iota_{n-1}) \cap \Omega_n^{\text{Mel}} \subseteq \Omega_\nu^{\gamma_n}(\iota_n), \quad \forall \nu \geq 0. \quad (9.22)$$

Since $n \geq 1$ one has by (9.7) $\Omega_n^{\text{Mel}} = \Omega_{\text{Mel}}^{2\gamma_{n-1}}(\iota_{n-1})$ and from (9.8) and Lemma 7.6 one concludes that

$$\Omega_{\text{Mel}}^{2\gamma_{n-1}}(\iota_{n-1}) \subseteq \Omega_\infty^{2\gamma_{n-1}}(\iota_{n-1}) \subseteq \cap_{\nu \geq 0} \Omega_\nu^{\gamma_{n-1}}(\iota_{n-1}).$$

In particular, one has $\Omega_n^{\text{Mel}} \subseteq \Omega_n^{\gamma_{n-1}}(\iota_{n-1})$ and hence for $\nu = n$, the inclusion (9.22) becomes

$$\Omega_n^{\text{Mel}} \subseteq \Omega_n^{\gamma_{n-1}}(\iota_{n-1}) \cap \Omega_n^{\gamma_n}(\iota_n). \quad (9.23)$$

To justify that $(\mathbf{S2})_\nu$ of Theorem 7.3 in the situation above applies it remains to verify the smallness condition in (7.88) of Theorem 7.3: To see it, recall that $\bar{\mu} = 4s_0 + 2\tau + 1$ (cf (7.1)), $\beta = 6\tau + 5$ (cf

(7.8)), $\mu_0 = 4s_0 + 10\tau + 7$ (cf remark after Theorem 5.1), and $\mu_0 < \mu_1$ (cf Theorem 5.2). Therefore $s_0 + \bar{\mu} + \beta < s_0 + \mu_0 < s_0 + \mu_1$ and in turn $\|\iota_n - \iota_{n-1}\|_{s_0 + \bar{\mu} + \beta} \leq \|\iota_n - \iota_{n-1}\|_{s_0 + \mu_1}$. Furthermore, by (8.9)

$$\|\iota_n - \iota_{n-1}\|_{s_0 + \mu_1} \leq N_{n-1}^{-\alpha_1} \varepsilon \gamma^{-2}.$$

Since $\alpha_1 = 2\mu_1 + 2/3 > \tau$ (cf (8.3)) one has $N_{n-1}^\tau N_{n-1}^{-\alpha_1} \leq 1$. Altogether we proved that for some $C' > 0$, $C'_{\text{var}} N_{n-1}^\tau \|\iota_n - \iota_{n-1}\|_{s_0 + \bar{\mu} + \beta} \leq C' \varepsilon \gamma^{-2}$ implying that

$$C'_{\text{var}} N_{n-1}^\tau \|\iota_n - \iota_{n-1}\|_{s_0 + \bar{\mu} + \beta} \leq \gamma 2^{-n} = \rho$$

for $\varepsilon \gamma^{-3}$ small enough. Hence the smallness condition in (7.88) is satisfied and therefore (9.23) holds.

Since by (9.23) $\Omega_n^{\text{Mel}} \subset \Omega_n^{\gamma^{n-1}}(\iota_{n-1}) \cap \Omega_n^{\gamma^n}(\iota_n)$ the 2×2 matrices $[\mathbf{N}_n^{(1)}(\iota_{n-1})]_j^j$ and $[\mathbf{N}_n^{(1)}(\iota_n)]_j^j$ are defined for any $\omega \in \Omega_n^{\text{Mel}}$, and by the estimate (7.86) of Theorem 7.3 with $\nu = n$ one has

$$\sup_{j \in S_+^\perp} \left\| [\mathbf{N}_n^{(1)}(\iota_n) - \mathbf{N}_n^{(1)}(\iota_{n-1})]_j^j \right\| \stackrel{(7.86)}{\leq} \|\iota_n - \iota_{n-1}\|_{s_0 + \bar{\mu} + \beta} \leq \|\iota_n - \iota_{n-1}\|_{s_0 + \mu_1}. \quad (9.24)$$

Moreover (7.48) (with $\nu = n$) and (7.68) imply that for any $j \in S_+^\perp$

$$\left\| [\mathbf{N}_\infty^{(1)}(\iota_{n-1}) - \mathbf{N}_n^{(1)}(\iota_{n-1})]_j^j \right\|, \quad \left\| [\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_n^{(1)}(\iota_n)]_j^j \right\| \leq \varepsilon N_{n-1}^{-\alpha}. \quad (9.25)$$

Since $\left\| [\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_\infty^{(1)}(\iota_{n-1})]_j^j \right\|$ is bounded by

$$\left\| [\mathbf{N}_n^{(1)}(\iota_n) - \mathbf{N}_n^{(1)}(\iota_{n-1})]_j^j \right\| + \left\| [\mathbf{N}_\infty^{(1)}(\iota_{n-1}) - \mathbf{N}_n^{(1)}(\iota_{n-1})]_j^j \right\| + \left\| [\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_n^{(1)}(\iota_n)]_j^j \right\|$$

one then concludes that for any $\omega \in \Omega_n^{\text{Mel}}$ and any $j \in S_+^\perp$,

$$\left\| [\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_\infty^{(1)}(\iota_{n-1})]_j^j \right\| \stackrel{(9.24), (9.25)}{\leq} \|\iota_n - \iota_{n-1}\|_{s_0 + \mu_1} + \varepsilon N_{n-1}^{-\alpha} \stackrel{(8.9)}{\leq} \varepsilon \gamma^{-2} N_{n-1}^{-\alpha}$$

where for the latter inequality we used that $\alpha_1 > \alpha$ since $\alpha_1 = 2\mu_1 + 2/3$ and $\mu_1 > \bar{\mu} + \alpha$ (cf (8.3), (7.8)). The claimed estimate (9.21) is thus established. \square

Lemma 9.2. For $\varepsilon \gamma^{-4}$ small enough one has for any $n \geq 1$, $\ell \in \mathbb{Z}^S$ with $|\ell| \leq N_{n-1}$, and $j, k \in S_+^\perp$,

$$Q_{\ell j}(\iota_n) = \emptyset, \quad R_{\ell j k}^+(\iota_n) = \emptyset, \quad (9.26)$$

and, if in addition $(\ell, j, k) \neq (0, j, j)$,

$$R_{\ell j k}^-(\iota_n) = \emptyset. \quad (9.27)$$

Proof. Since the proofs of the three stated inclusions are similar we only prove (9.27). For any $n \geq 1$, $\ell \in \mathbb{Z}^S$ with $|\ell| \leq N_{n-1}$, $j, k \in S_+^\perp$ with $(\ell, j, k) \neq (0, j, j)$, and $\omega \in \Omega_n^{\text{Mel}}$, the operator $L_\infty^-(\ell, j, k; \iota_{n-1})$ is invertible and hence we can write

$$L_\infty^-(\ell, j, k; \iota_n) = L_\infty^-(\ell, j, k; \iota_{n-1}) (\text{Id}_{\mathbb{C}^{2 \times 2}} + L_\infty^-(\ell, j, k; \iota_{n-1})^{-1} \Delta_\infty(j, k, n))$$

where

$$\Delta_\infty(j, k, n) := M_L([\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_\infty^{(1)}(\iota_{n-1})]_j^j) - M_R([\mathbf{N}_\infty^{(1)}(\iota_n) - \mathbf{N}_\infty^{(1)}(\iota_{n-1})]_k^k).$$

Since

$$\left\| L_\infty^-(\ell, j, k; \iota_{n-1})^{-1} \Delta_\infty(j, k, n) \right\| \stackrel{(9.14)}{\leq} \frac{\langle \ell \rangle^\tau}{2\gamma_{n-1} \langle j^2 - k^2 \rangle} \left\| \Delta_\infty(j, k, n) \right\| \stackrel{(9.21)}{\leq} C \varepsilon \gamma^{-3} \langle \ell \rangle^\tau N_{n-1}^{-\alpha}$$

and $|\ell| \leq N_{n-1}$ (by assumption), $\alpha > \tau$ (cf (7.8)) it follows that for $\varepsilon \gamma^{-3}$ small enough,

$$\left\| L_\infty^-(\ell, j, k; \iota_{n-1})^{-1} \Delta_\infty(j, k, n) \right\| \leq 1/2.$$

Therefore $L_\infty^-(\ell, j, k; \iota_n)$ is invertible by a Neumann series and

$$\|L_\infty^-(\ell, j, k; \iota_n)^{-1}\| \leq \|L_\infty^-(\ell, j, k; \iota_{n-1})^{-1}\| (1 + C\varepsilon\gamma^{-3}N_{n-1}^{\tau-\alpha}) \stackrel{(9.14)}{\leq} \frac{\langle \ell \rangle^\tau}{2\gamma_{n-1}\langle j^2 - k^2 \rangle} (1 + C\varepsilon\gamma^{-3}N_{n-1}^{\tau-\alpha}).$$

Choosing $\varepsilon\gamma^{-3}$ sufficiently small one achieves that $C\varepsilon\gamma^{-3}N_{n-1}^{\tau-\alpha} \leq \frac{1}{1+2^n}$ for any $n \geq 1$. Since by the definition of γ_n , $\frac{\gamma_{n-1}-\gamma_n}{\gamma_n} = \frac{1}{1+2^n}$ it then follows that

$$\|L_\infty^-(\ell, j, k; \iota_n)^{-1}\| \leq \frac{\langle \ell \rangle^\tau}{2\gamma_n\langle j^2 - k^2 \rangle}.$$

Hence, recalling (9.20), we have proved that $R_{\ell j k}^-(\iota_n) = \emptyset$. \square

As an immediate consequence of Lemma 9.2, one gets the following

Corollary 9.1. *For any $n \geq 1$,*

$$(\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*} \stackrel{(9.17)}{=} \left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j \in S_+^\perp}} Q_{\ell j}(\iota_n) \right) \cup \left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j, k \in S_+^\perp}} R_{\ell j k}^+(\iota_n) \right) \cup \left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j, k \in S_+^\perp \\ (\ell, j, k) \neq (0, j, j)}} R_{\ell j k}^-(\iota_n) \right). \quad (9.28)$$

Proof. By definition, $R_{\ell j k}^\pm(\iota_n)$, $Q_{\ell j}(\iota_n) \subset \Omega_n^{\text{Mel}}$ and, by (9.26), for any $\ell \in \mathbb{Z}^S$ with $|\ell| \leq N_{n-1}$, one has $R_{\ell j k}^\pm(\iota_n) \subseteq R_{\ell j k}^\pm(\iota_{n-1})$ and $Q_{\ell j}(\iota_n) \subseteq Q_{\ell j}(\iota_{n-1})$. By definition, one also has $R_{\ell j k}^\pm(\iota_{n-1}) \cap \Omega_n^{\text{Mel}}$ and $Q_{\ell j}(\iota_{n-1}) \cap \Omega_n^{\text{Mel}}$ are empty sets. As a consequence, for any ℓ with $|\ell| \leq N_{n-1}$, $R_{\ell j k}^\pm(\iota_n)$, $Q_{\ell j}(\iota_n) = \emptyset$. \square

The next lemma is the core of the measure estimates. To prove (iv) the key ingredients are the asymptotic expansion of the dNLS frequencies of Theorem 3.2 (ii) and the one of the eigenvalues of the normal form $\mathbf{N}_\infty^{(1)}$ up to order -1 , obtained in (7.64)-(7.66).

Lemma 9.3. *For any $n \geq 0$, $\ell \in \mathbb{Z}^S$, and $j, k \in S_+^\perp$, the following statements hold:*

(i) *If $Q_{\ell j}(\iota_n) \neq \emptyset$, then $j^2 \ll \langle \ell \rangle$. (ii) If $R_{\ell j k}^+(\iota_n) \neq \emptyset$, then $|j^2 + k^2| \ll \langle \ell \rangle$.*

(iii) *If $R_{\ell j k}^-(\iota_n) \neq \emptyset$ and $j \neq k$ then $|j^2 - k^2| \ll \langle \ell \rangle$. (iv) If $R_{\ell j j}^-(\iota_n) \neq \emptyset$ and $\ell \neq 0$ then $|j| \ll \gamma_*^{-1} \langle \ell \rangle^{\tau_*}$.*

As a consequence, for any $C > 0$ there are finitely many triples $(\ell, j, k) \neq (0, j, j)$ with $|\ell| \leq C$ and $j, k \in S_+^\perp$ so that at least one of the sets $Q_{\ell j}(\iota_n)$, $R_{\ell j k}^+(\iota_n)$, or $R_{\ell j k}^-(\iota_n)$ is nonempty.

Proof. We prove item (iii) and (iv) in detail. Items (i) and (ii) follow by similar, but simpler arguments as a less precise asymptotic expansion suffices. Since the operator $L_\infty^-(\ell, j, k) \in \mathcal{L}(\mathbb{C}^{2 \times 2})$, defined in (9.13), is self-adjoint, the norm of $L_\infty^-(\ell, j, k)^{-1}$ (when it exists) is given by the inverse of the minimum modulus of the four eigenvalues of $L_\infty^-(\ell, j, k)$. By Lemma 7.2, these eigenvalues are given by

$$\omega \cdot \ell + \lambda_j^{(a)}(\omega) - \lambda_k^{(b)}(\omega), \quad a, b \in \{+, -\},$$

where for any $\kappa \in S_+^\perp$, $\lambda_\kappa^{(+)}(\omega)$, $\lambda_\kappa^{(-)}(\omega)$ denote the two eigenvalues of the matrix $[\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_\kappa^\kappa \in \mathbb{C}^{2 \times 2}$. By the definition (9.20), $R_{\ell j k}^-(\iota_n)$ thus reads

$$R_{\ell j k}^-(\iota_n) = \left\{ \omega \in \Omega_n^{\text{Mel}} \cap \Omega_{\gamma_*, \tau_*} : \exists a, b \in \{+, -\} \text{ with } |\omega \cdot \ell + \lambda_j^{(a)}(\omega) - \lambda_k^{(b)}(\omega)| < \frac{2\gamma_n \langle j^2 - k^2 \rangle}{\langle \ell \rangle^\tau} \right\}. \quad (9.29)$$

By item (iii) of Theorem 7.2, we have for $a \in \{+, -\}$

$$\lambda_\kappa^{(a)} = 4\pi^2 \kappa^2 + c_{\varepsilon, \xi} + \frac{\rho_{\xi, \varepsilon}^{(a)}(\kappa)}{\kappa}, \quad |c_{\varepsilon, \xi}| = O(1), \quad \sup_{\kappa \in S_+^\perp} |\rho_{\xi, \varepsilon}^{(a)}(\kappa)| = O(1). \quad (9.30)$$

Case $j \neq k$: Assume that $R_{\ell j k}^-(\iota_n) \neq \emptyset$. By (9.29), given $\omega \in R_{\ell j k}^-(\iota_n)$ there exist $a, b \in \{+, -\}$ so that

$$|\lambda_j^{(a)}(\omega) - \lambda_k^{(b)}(\omega)| < \frac{2\gamma_n |j^2 - k^2|}{\langle \ell \rangle^\tau} + |\omega| |\ell|. \quad (9.31)$$

On the other hand, by (9.30), one sees that

$$|\lambda_j^{(a)}(\omega) - \lambda_k^{(b)}(\omega)| \geq |j^2 - k^2| - C' \quad (9.32)$$

for some constant $C' > 0$. Hence (9.31) and (9.32) imply that

$$|\omega||\ell| + C' \geq \left(1 - \frac{2\gamma_n}{\langle \ell \rangle^\tau}\right) |j^2 - k^2| \geq (1 - 2\gamma_n) |j^2 - k^2| \geq \frac{1}{2} |j^2 - k^2|$$

taking γ in $\gamma_n = \gamma(1 + 2^{-n})$ so small that $\gamma_n \leq 1/4$. One concludes that $|j^2 - k^2| \ll \langle \ell \rangle$ and item (iii) is proved.

Case $j = k$, $\ell \neq 0$: Assume that $R_{\ell jj}^-(\iota_n) \neq \emptyset$. By (9.29), given $\omega \in R_{\ell jj}^-(\iota_n)$, there exist $a, b \in \{+, -\}$ so that

$$|\omega \cdot \ell + \lambda_j^{(a)}(\omega) - \lambda_j^{(b)}(\omega)| < \frac{2\gamma_n}{\langle \ell \rangle^\tau}. \quad (9.33)$$

Assume that $a = b$. By (9.33) and since $\omega \in \Omega_{\gamma_*, \tau_*}$ (see (9.4)) one has

$$\frac{2\gamma_n}{\langle \ell \rangle^\tau} > |\omega \cdot \ell| \geq \frac{\gamma_*}{\langle \ell \rangle^{\tau_*}} > \frac{2\gamma_n}{\langle \ell \rangle^\tau}$$

since $\gamma_* > 8\gamma \geq 2\gamma_n$ and $\tau > \tau_*$. The assumption $a = b$ thus yields a contradiction. Hence $a \neq b$. Using the asymptotics (9.30), we get that, for some constant $C' > 0$,

$$|\omega \cdot \ell + \lambda_j^{(a)}(\omega) - \lambda_j^{(b)}(\omega)| \geq |\omega \cdot \ell| - \frac{C'}{|j|} \stackrel{(9.4)}{\geq} \frac{\gamma_*}{\langle \ell \rangle^{\tau_*}} - \frac{C'}{|j|}, \quad (9.34)$$

which, together with (9.33) and $\tau > \tau_*$, implies that

$$\frac{C'}{|j|} \geq \frac{\gamma_* - 2\gamma_n}{\langle \ell \rangle^{\tau_*}} \geq \frac{\gamma_*}{2\langle \ell \rangle^{\tau_*}}$$

because $\gamma_n \leq 2\gamma$ and $8\gamma < \gamma_*$. The claimed inequality $|j| \ll \gamma_*^{-1} \langle \ell \rangle^{\tau_*}$ of item (iv) is proved. \square

Combining Corollary 9.1 and Lemma 9.3, one sees that there exists a constant $C_* > 0$ so that the identity (9.28) for $(\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*}$ with $n \geq 1$ becomes

$$\left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j \in S_+^\perp \\ |j| \leq C_* |\ell|^{1/2}}} Q_{\ell j}(\iota_n) \right) \cup \left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j, k \in S_+^\perp \\ j^2 + k^2 \leq C_* |\ell|}} R_{\ell jk}^+(\iota_n) \right) \cup \left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j, k \in S_+^\perp, j \neq k \\ |j^2 - k^2| \leq C_* |\ell|}} R_{\ell jk}^-(\iota_n) \right) \cup \left(\bigcup_{\substack{|\ell| > N_{n-1} \\ j \in S_+^\perp \\ |j| \leq C_* \gamma_*^{-1} |\ell|^{\tau_*}}} R_{\ell jj}^-(\iota_n) \right). \quad (9.35)$$

The measures of these resonant sets are now estimated individually:

Lemma 9.4. *There exists a constant $\tilde{C} > 0$ so that for any $n \geq 0$, $j, k \in S_+^\perp$, and $\ell \in \mathbb{Z}^S$ with $|\ell| \geq \tilde{C}$ the following holds: (i) $\text{meas}(Q_{\ell j}(\iota_n)) \ll \gamma \langle j \rangle^2 \langle \ell \rangle^{-\tau-1}$; (ii) $\text{meas}(R_{\ell jk}^+(\iota_n)) \ll \gamma \langle j^2 + k^2 \rangle \langle \ell \rangle^{-\tau-1}$; (iii) $\text{meas}(R_{\ell jk}^-(\iota_n)) \ll \gamma \langle j^2 - k^2 \rangle \langle \ell \rangle^{-\tau-1}$.*

Proof. Since the proofs of the three items are similar, we only prove item (iii). Assume that $j, k \in S_+^\perp$ and $\ell \in \mathbb{Z}^S$ with $\ell \neq 0$. Consider the straight line in Ω of the form

$$\omega(s) = s \frac{\ell}{|\ell|} + v, \quad v \cdot \ell = 0$$

where s is a real parameter of appropriate range. The four eigenvalues of the operator $L_\infty^-(\ell, j, k; s \frac{\ell}{|\ell|} + v)$ in $\mathcal{L}(\mathbb{C}^{2 \times 2})$ are given by $\phi_{a,b}(s) := |\ell|s + \tilde{\lambda}_j^{(a)}(s) - \tilde{\lambda}_k^{(b)}(s)$ where $a, b \in \{+, -\}$ and

$$\tilde{\lambda}_\kappa^{(a)}(s) := \lambda_\kappa^{(a)}\left(s \frac{\ell}{|\ell|} + v\right), \quad a \in \{+, -\}, \quad \kappa \in \{j, k\}.$$

Recall that $\lambda_\kappa^{(-)}(\omega)$, $\lambda_\kappa^{(+)}(\omega)$ denote the two eigenvalues of $[\mathbf{N}_\infty^{(1)}(\omega, \iota_n(\omega))]_\kappa^\kappa$ (cf (9.30)), listed according to their size, $\lambda_\kappa^{(-)}(\omega) \leq \lambda_\kappa^{(+)}(\omega)$. By (7.67), they are Lipschitz continuous and, for any $\kappa \in S^\perp$, $a \in \{+, -\}$,

$$|\tilde{\lambda}_\kappa^{(a)}(s)|^{\text{lip}} \leq 1.$$

Hence for any $a, b \in \{+, -\}$, $\phi_{a,b}(s)$ satisfies the estimate $|\phi_{a,b}(s_1) - \phi_{a,b}(s_2)| \geq (|\ell| - C')|s_1 - s_2|$ for some constant $C' > 0$. Setting $\tilde{C} := 2C'$ it then follows that for any $\ell \in \mathbb{Z}^S$ with $|\ell| \geq \tilde{C}$,

$$|\phi_{a,b}(s_1) - \phi_{a,b}(s_2)| \geq \frac{|\ell|}{2}|s_1 - s_2|.$$

Since Ω is compact and by (9.29)

$$\left\{s \in \mathbb{R} : s \frac{\ell}{|\ell|} + v \in R_{\ell j k}^-(\iota_n)\right\} = \left\{s \in \mathbb{R} : \exists a, b \in \{+, -\} \text{ with } |\phi_{a,b}(s)| < \frac{2\gamma_n \langle j^2 - k^2 \rangle}{\langle \ell \rangle^\tau}\right\}$$

one sees by a standard argument that

$$\text{meas}\left(\left\{s \in \mathbb{R} : s \frac{\ell}{|\ell|} + v \in R_{\ell j k}^-(\iota_n)\right\}\right) \leq \frac{\gamma \langle j^2 - k^2 \rangle}{\langle \ell \rangle^{\tau+1}}$$

which then yields item (iii) using Fubini's theorem. \square

By choosing $N_0 \geq \tilde{C}$, where \tilde{C} is the constant given in Lemma 9.4, we have estimated in the latter lemma the measures of all the resonant sets appearing in (9.35), which will allow us to derive measure estimates of $\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}$ for any $n \geq 1$. In view of (9.15), it then remains to estimate the measure of $\Omega_0^{\text{Mel}} \setminus \Omega_1^{\text{Mel}}$. Hence taking into account (9.17) and Lemma 9.4 we need to estimate the measures of $Q_{\ell j}(\iota_0)$, $R_{\ell j k}^+(\iota_0)$, $R_{\ell j k}^-(\iota_0)$ for any $\ell \in \mathbb{Z}^S$ with $|\ell| \leq \tilde{C}$. We use the analyticity of the dNLS frequencies to obtain the following:

Lemma 9.5. *There exists $\mathbf{b}' \in (0, 1]$ so that for any $j, k \in S_+^\perp$ and $\ell \in \mathbb{Z}^S$ with $|\ell| \leq \tilde{C}$ (with \tilde{C} as in Lemma 9.4) the following statements hold: (i) $\text{meas}(Q_{\ell j}(\iota_0)) = O(\gamma^{\mathbf{b}'})$; (ii) $\text{meas}(R_{\ell j k}^+(\iota_0)) = O(\gamma^{\mathbf{b}'})$; (iii) if in addition $(\ell, j, k) \neq (0, j, j)$ then $\text{meas}(R_{\ell j k}^-(\iota_0)) = O(\gamma^{\mathbf{b}'})$.*

Proof. Since the proofs of the three items are similar, we only consider item (iii). By Lemma 9.3 there are finitely many triples $(\ell, j, k) \neq (0, j, j)$ in $\mathbb{Z}^S \times S_+^\perp \times S_+^\perp$ with $|\ell| \leq \tilde{C}$ so that $R_{\ell j k}^-(\iota_0) \neq \emptyset$. For these finitely many triples it follows from the definition (9.29) and (7.64)-(7.66) that there exists $C' > 0$ so that when choosing $\varepsilon \gamma^{-3}$ small enough

$$R_{\ell j k}^-(\iota_0) \subseteq \bigcup_{a,b \in \{+, -\}} \left\{\omega \in \Omega_0^{\text{Mel}} \cap \Omega_{\gamma^*, \tau^*} : |\omega \cdot \ell + \omega_{aj}^{nls}(\xi, 0) - \omega_{bk}^{nls}(\xi, 0)| < C' \gamma\right\}.$$

By Theorem 3.2, $\omega \mapsto \xi(\omega)$, being the inverse map of $\xi \mapsto (\omega_\kappa^{nls}(\xi, 0))_{\kappa \in S}$, is analytic as are the maps

$$\omega \mapsto \omega \cdot \ell + \omega_{aj}^{nls}(\xi(\omega), 0) - \omega_{bk}^{nls}(\xi(\omega), 0)$$

are analytic. By Proposition 3.1, none of these maps vanishes identically. The claimed estimate of item (iii) then follows by the Weierstrass preparation theorem as used for instance in [7, Proposition 3.1]. \square

Lemma 9.4 and Lemma 9.5 are now used to prove measure estimates of $(\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma^*, \tau^*}$ for any $n \geq 0$.

Lemma 9.6. *The following estimates hold:*

$$\text{meas}\left((\Omega_0^{\text{Mel}} \setminus \Omega_1^{\text{Mel}}) \cap \Omega_{\gamma^*, \tau^*}\right) = O(\gamma^{\mathbf{b}'}), \quad \text{meas}\left((\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma^*, \tau^*}\right) = O(\gamma \gamma_*^{-1} N_{n-1}^{-1}), \quad \forall n \geq 1.$$

Proof. To estimate $\text{meas}((\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*})$ for $n \geq 1$, note that by (9.35) and Lemma 9.4, it is \leq bounded by

$$\begin{aligned} & \sum_{\substack{|\ell| > N_{n-1} \\ j \in S_+^\perp \\ |j| \leq C_* \langle \ell \rangle^{\frac{1}{2}}}} \frac{\gamma \langle j \rangle^2}{\langle \ell \rangle^{\tau+1}} + \sum_{\substack{|\ell| > N_{n-1} \\ j, k \in S_+^\perp \\ j^2 + k^2 \leq C_* \langle \ell \rangle}} \frac{\gamma \langle j^2 + k^2 \rangle}{\langle \ell \rangle^{\tau+1}} + \sum_{\substack{|\ell| > N_{n-1} \\ j, k \in S_+^\perp, j \neq k \\ |j^2 - k^2| \leq C_* \langle \ell \rangle}} \frac{\gamma \langle j^2 - k^2 \rangle}{\langle \ell \rangle^{\tau+1}} + \sum_{\substack{|\ell| > N_{n-1} \\ j \in S_+^\perp \\ |j| \leq C_* \gamma_*^{-1} \langle \ell \rangle^{\tau_*}}} \frac{\gamma}{\langle \ell \rangle^{\tau+1}} \\ & \leq \gamma \sum_{|\ell| > N_{n-1}} \frac{1}{\langle \ell \rangle^{\tau - \frac{1}{2}}} + \gamma \sum_{|\ell| > N_{n-1}} \frac{1}{\langle \ell \rangle^{\tau-1}} + \gamma \sum_{|\ell| > N_{n-1}} \frac{1}{\langle \ell \rangle^{\tau-1}} + \gamma \gamma_*^{-1} \sum_{|\ell| > N_{n-1}} \frac{1}{\langle \ell \rangle^{\tau+1-\tau_*}}. \end{aligned}$$

Since by definition, $\tau = 2|S| + 1$ and $\tau_* = |S| + 1$ (cf (9.3)), one has $\tau + 1 - \tau_* = |S| + 1$, yielding the estimate

$$\text{meas}((\Omega_n^{\text{Mel}} \setminus \Omega_{n+1}^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*}) \leq \gamma \gamma_*^{-1} \sum_{|\ell| > N_{n-1}} \frac{1}{\langle \ell \rangle^{\tau+1-\tau_*}} \leq \gamma \gamma_*^{-1} \frac{1}{N_{n-1}}.$$

The estimate of $\text{meas}((\Omega_0^{\text{Mel}} \setminus \Omega_1^{\text{Mel}}) \cap \Omega_{\gamma_*, \tau_*})$ follows by similar arguments, using in addition Lemma 9.5. \square

Proof of Theorem 9.1: By (9.5), (9.6), (9.16) and Lemma 9.6 one has that

$$\text{meas}(\Omega \setminus \Omega_\infty^{\text{Mel}}) \leq O(\gamma_*) + O(\gamma) + O(\gamma^{b'}) + O(\gamma \gamma_*^{-1}) \sum_{n \geq 1} \frac{1}{N_{n-1}} \leq O(\gamma^{b'}) + O(\gamma_*) + O(\gamma_*^{-1} \gamma).$$

Thanks to our choice of γ_* in (9.3) and $\gamma = \varepsilon^a$, we have $\gamma_* = \gamma_*^{-1} \gamma = \varepsilon^{a/2}$ and (9.2) then follows with $\mathfrak{b} := \min\{b', 1/2\}$.

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