The Hermann-Martin Curve

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Dedicated to Clyde F. Martin on Occasion of his 60th Birthday

Summary. Every linear system can be naturally identified with a rational curve in a Grassmann variety. The associated curve is often referred to as Hermann-Martin curve of the system.

This article explains this crucial link between systems theory and geometry. The geometric translation also provides important tools when studying control design problems. In a second part of the article it is shown how it is possible to tackle some important control design problems by geometric means.

Key words: Linear systems, Grassmann varieties, vector bundles over the projective line, Hermann-Martin curve, Grothendieck Quot scheme.

1 Introduction

In the late seventies Bob Hermann and Clyde Martin published a series of papers [14, 13, 19, 20] which showed a way how problems in linear systems theory can be translated into problems of algebraic geometry.

On the conceptual level this link provided a much deeper understanding for questions where topological properties of the class of linear systems played a role. The geometric understanding gave also tools at hand which helped to progress the research in several prominent problems like e.g. the static and dynamic pole placement problem.

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In Section 2 we will explain the link between systems theory and geometry provided by the Hermann-Martin curve. For this purpose we will have to review some basic notions in systems theory as well as some basic notions in algebraic geometry.

In Section 3 we will be concerned with topological properties of the set of linear systems having a fixed number of inputs, a fixed number of outputs and a fixed McMillan degree. An understanding of topological properties of this set has importance in the area of system identification and the area of robust controller design, to mention a few. The section will revisit two compactifications studied in the literature. Finally in Section 4 we will show how the geometric translation provided by the Hermann-Martin identification led to new results in the area of pole placement.

2 Linear systems and rational curves in Grassmannians

Consider an m-inputs, p-outputs linear system Σ_n having McMillan degree n. In state space form this system is governed by the equations:

$$\Sigma_n \ : \begin{cases} \dot{x} = Ax + \ Bu \\ y = Cx + Du. \end{cases} \tag{1}$$

In terms of the frequency domain system (1) has an associated *transfer function*

$$G(s) := C(sI - A)^{-1}B + D. \tag{2}$$

By definition $G(s)$ is a $p \times m$ matrix with rational entries. The concept of a transfer function can be defined for an arbitrary field $\mathbb{K}$ and we will work to a large degree in this general setting. Whenever some additional properties on the field are required we will say so. The transfer function $G(s)$ captures the input-output behavior of the linear system. We say the matrices A, B, C, D form a realization of the transfer function $G(s)$. When A, B forms a controllable pair of matrices (i.e. the matrix pencil $[sI - A\, B]$ is left prime) and if A, C forms an observable pair (i.e. the matrix pencil $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ is right prime), then we say that (1) forms a *minimal realization* of the transfer function (2). The size of the square matrix A in a minimal realization is called the McMillan degree of the transfer function $G(s)$.

Minimal realizations of proper transfer functions are unique in the following way: If $(\tilde{A}, \tilde{B})$ is a controllable pair and $(\tilde{A}, \tilde{C})$ is an observable pair with $G(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ then there is a unique invertible matrix S of size $n \times n$ such that

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (SAS^{-1}, SB, CS^{-1}, D). \tag{3}$$

We can identify the matrix S with an element of the general linear group Gl_n and we can view (3) as an orbit of a Gl_n action on a vector space. In this way we can view a linear system either through its transfer function $G(s)$ or as a Gl_n orbit in a vector space of dimension $n(n+m+p)+mp$.

For many applications it is very important to understand families of linear systems with a fixed number of inputs, fixed number of outputs and a fixed McMillan degree. Denote by $S^n_{p,m}$ the set of all proper $p \times m$ transfer functions having a fixed McMillan degree n. If the underlying base field are the real numbers Clark [3] showed in 1976 that $S^n_{p,m}$ has the structure of a smooth manifold of dimension $n(m+p)+mp$. Around the same time Hazewinkel [9] was able to show that $S^n_{p,m}$ has the structure of a quasi-affine variety, as soon as the base field $\mathbb{K}$ is algebraically closed. The basic proof techniques of Hazewinkel came from geometric invariant theory (GIT) and the interested reader is referred to [21, 36].

The work of Martin and Hermann [20] provided a new avenue to understand the algebraic and topological properties of the set $S^n_{p,m}$. For simplicity assume that the base field constitutes the complex numbers $\mathbb{C}$. Denote by

$$\mathbb{P}^1_{\mathbb{C}} := \{\ell \subset \mathbb{C}^2 \mid \dim \ell = 1\} \hat{=} \{(x,1) \mid x \in \mathbb{C}\} \cup \{(1,0) =: \infty\}$$

the projective line over $\mathbb{C}$, i.e. the Riemann sphere and consider the Grassmann variety

$$\mathrm{Grass}(p, \mathbb{C}^{p+m}) := \{W \subset \mathbb{C}^n \mid \dim W = p\}$$

which parameterizes all p-dimensional linear subspace of the vector space $\mathbb{C}^n$. Martin and Hermann had the original idea to associate to each linear system a rational curve of genus zero inside $\mathrm{Grass}(p, \mathbb{C}^{p+m})$.

Definition 1 Let $G(s)$ be a $p \times m$ proper transfer function and consider the map

$$h:\ \mathbb{C} \longrightarrow \mathrm{Grass}(p, \mathbb{C}^{p+m}),\ \ s \mapsto \mathrm{rowspace}_{\mathbb{C}}[I_p\ G(s)]. \tag{4}$$

Then h is called the *Hermann-Martin map* associated to the transfer function $G(s)$.

The Hermann-Martin map is a rational map. As the target space is compact all poles are removable and the map extends therefore to a holomorphic map:

$$\hat{h}:\ \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathrm{Grass}(p, \mathbb{C}^{p+m}). \tag{5}$$

The image of the map $\hat{h}$ defines a curve of genus zero and degree n inside the Grassmannian $\mathrm{Grass}(p, \mathbb{C}^{p+m})$, sometimes referred to as the *Hermann-Martin curve* associated to the linear system $G(s)$.

Note that every holomorphic map from the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ to the Grassmannian is also rational. The following lemma is easily proved:

Lemma 2 *Let* $\hat{h}: \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathrm{Grass}(p, \mathbb{C}^{p+m})$ *be a rational map with* $\hat{h}(\infty) = \mathrm{rowspace}_{\mathbb{C}}[M_1 \;\; M_2]$ *where* M_1 *is a* $p \times p$ *invertible matrix. Then* $\hat{h}$ *is the Hermann-Martin map of some proper transfer function* $G(s)$*.*

Given a system Σ_n described by the equations (1). Then one defines the *observability indices* of Σ_n as the left Kronecker indices of the pencil $\begin{bmatrix} sI - A \\ C \end{bmatrix}$. See [15, page 413] for details. Similarly one defines the controllability indices of Σ_n as the right Kronecker indices of the pencil $[sI - A \mid B]$. In [20] Martin and Hermann were able to connect the observability indices of Σ_n to the Grothendieck indices of an associated vector bundle. In order to make this precise we recall Grothendieck's theorem about the classification of vector bundles over the Riemann sphere:

Theorem 3 ([7]). *If* ξ *is a holomorphic vector bundle over* $\mathbb{P}^1_{\mathbb{C}}$ *then* ξ *decomposes as a sum of line bundles:*

$$\xi = O(\nu_1) \oplus \cdots \oplus O(\nu_p),$$

where $\nu_1, \ldots, \nu_p$ *are the multiplicities of the line bundles. The nonnegative integers* $\nu_1, \ldots, \nu_p$ *depend up to order only on* ξ*.*

The indices $\nu_1, \ldots, \nu_p$ are sometimes referred to as the Grothendieck indices of ξ. The integer $\nu = \sum_{j=1}^{p} \nu_j$ is called the degree of ξ.

Remark 4 Theorem 3 was derived by Grothendieck using general results from the theory of holomorphic vector bundles like Serre duality and splitting theorems for subbundles. At the time he was not aware that his result is also a straight forward consequence of some results by Dedekind and Weber [4]. The interested reader will find more details in [28].

Remark 5 Grothendieck's theorem is valid over any base field and a short elementary proof was given by Hazewinkel and Martin [11].

The Grassmann manifold is equipped with a natural vector bundle called the universal bundle U. Let U^* be its dual. The following theorem is due to Martin and Hermann [20]:

Theorem 6. *Let* ξ *be the pull back of the bundle* U^* *under the holomorphic map* $\hat{h}$*. Then the Grothendieck indices* $\nu_1, \ldots, \nu_p$ *of* ξ *are up to order equal to the observability indices of the system* Σ_n*. Moreover the degree* $\nu = \sum_{j=1}^{p} \nu_j$ *of* ξ *is equal to the McMillan degree of* Σ_n*.*

One way to gain more insight into the connection between the Grothendieck indices of a vector bundle and the observability indices of a system is via the concept of minimal bases as introduced by Forney [6]. For this assume that the transfer function $G(s) = C(sI - a)^{-1}B + D$ has a left coprime factorization $G(s) = D^{-1}(s)N(s)$. Without loss of generality we can assume that the rows of $[D(s) \mid N(s)]$ form a minimal polynomial basis of the rational vector space $\text{rowspace}_{\mathbb{C}(s)}[I_p\ G(s)]$ having ordered Forney indices $\nu_1 \geq \cdots \geq \nu_p$ and total degree $n = \sum_{j=1}^{p} \nu_j$. Then one has [6]:

Theorem 7. *1.* $\det D(s)$ *is equal to the characteristic polynomial of the transfer function* $G(s)$*, in particular* $n = \deg \det D(s)$ *is equal to the McMillan degree of* Σ_n*.*
2. The indices ν_j *are equal to the observability indices of* $G(s)$*.*

Remark 8 Since the observability indices are also equal to the Grothendieck indices it follows that under suitable translation the Forney indices of the polynomial matrix $[D(s) \mid N(s)]$ are also equal to the Grothendieck indices.

3 Compactification of the set of linear systems having fixed McMillan degree

For many problems in linear systems theory such as questions of robustness and problems in identification theory it is very important to have an understanding of the topology of the set $S_{p,m}^n$ of systems having m inputs, p outputs and McMillan degree n. We already mentioned that $S_{p,m}^n$ has both the structure of a manifold and the structure of a quasi-affine variety [3, 9]. Further topological properties of $S_{p,m}^n$ have been derived over the years and we refer the reader e.g. to [12].

To understand degeneration phenomena of systems [10] or to understand the pole placement problem, a compactification of this space is desirable as well. The Hermann-Martin identification gives a natural way to achieve both these goals. We will explain this procedure for a general base field $\mathbb{K}$.

Denote by $\text{Rat}_n(\mathbb{P}^1, \mathbb{P}^N)$ the set of rational maps from the projective line $\mathbb{P}^1_{\mathbb{K}}$ to the projective space $\mathbb{P}^N_{\mathbb{K}}$ having degree n. Every element of $\varphi \in \text{Rat}_n(\mathbb{P}^1, \mathbb{P}^N)$ can be described through:

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^N,\ (s,t) \longmapsto (a_0(s,t), \ldots, a_N(s,t)), \tag{6}$$

where $a_i(x,y) \in \mathbb{K}[x,y], i = 0, \ldots, N$ are homogeneous polynomials of degree n. The description (6) is unique up to a nonzero constant factor $c \in \mathbb{K}^*$. In this way we can view an element of $\text{Rat}_n(\mathbb{P}^1, \mathbb{P}^N)$ as a point in the projective space

$$\mathbb{P}(\mathbb{K}^{n+1} \otimes \mathbb{K}^{N+1}) = \mathbb{P}_{\mathbb{K}}^{nN+n+N}.$$

Since we can view a linear system as a rational map from the projective line to some Grassmannian and since a Grassmannian can be naturally seen as a subset of a projective space via the Plücker embedding we can view a linear system ultimately as a point of a projective space. This gives raise to an embedding of $S^n_{p,m}$ into a projective space as the following sequence of maps makes this precise:

$$
\begin{aligned}
S^n_{p,m} &\stackrel{\text{Her.}-\text{Mar.}}{\longrightarrow} \mathrm{Rat}_n(\mathbb{P}^1, \mathrm{Grass}(p, m+p)) \\
&\stackrel{\text{Plücker}}{\longrightarrow} \mathrm{Rat}_n(\mathbb{P}^1, \mathbb{P}^1(\wedge^p \mathbb{K}^{m+p})) \\
&\stackrel{\tau}{\longrightarrow} \mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p}).
\end{aligned} \tag{7}
$$

Definition 9 Denote by $K^n_{p,m}$ the Zariski closure of the image of $S^n_{p,m}$ inside the projective space $\mathbb{P}(\mathbb{K}^{n+1} \otimes \wedge^p \mathbb{K}^{m+p})$.

In [26, 27] it has been shown:

Theorem 10. *For any base field $\mathbb{K}$, $K^n_{p,m}$ is a projective variety of dimension $n(m+p)+mp$ containing the set $S^n_{p,m}$ of $p \times m$ proper transfer functions of McMillan degree n as a Zariski dense subset.*

The space $K^n_{p,m}$ has also been studied in the area of conformal quantum field theory. For this reason Sottile [33, 34] calls the variety $K^n_{p,m}$ the *quantum Grassmannian.* In [1] this variety is also called the *Uhlenbeck compactification.* When $\min(m,p) = 1$ then $K^n_{p,m}$ represents simply a projective space. In general $K^n_{p,m}$ is however a singular variety [27].

On the side of $K^n_{p,m}$ there is a second well studied compactification of $S^n_{p,m}$ due to Grothendieck. In [8] Grothendieck showed that the set $\mathcal{Q}^n_{p,m}$ parameterizing all quotient sheaves $\mathcal{B}$ of $\mathbb{K}^{m+p} \otimes \mathcal{O}_{\mathbb{P}}$ having rank m, degree n and Hilbert polynomial $\chi(\mathcal{B}(x)) = px + p + n$ has naturally the structure of a scheme. In the algebraic geometry literature $\mathcal{Q}^n_{p,m}$ is usually referred as a *Quot scheme.* For the particular Quot scheme $\mathcal{Q}^n_{p,m}$ under consideration Strømme [35] showed that $\mathcal{Q}^n_{p,m}$ has the structure of a smooth projective variety of dimension $n(m+p)+mp$. Furthermore $\mathcal{Q}^n_{p,m}$ compactifies the space $\mathrm{Rat}_n(\mathbb{P}^1, \mathrm{Grass}(p, m+p))$ of all rational maps of degree n from the projective line to the Grassmannian $\mathrm{Grass}(p, m+p)$.

The fact that Grothendieck's Quot scheme $\mathcal{Q}^n_{p,m}$ has a relevance in linear systems theory was first recognized by Lomadze [18]. The author in collaboration with Ravi was able to give a direct systems theoretic interpretation of $\mathcal{Q}^n_{p,m}$ in terms of matrix pencils and polynomial matrices. We follow here the original description in [22, 23].

Let $\mathbb{K}$ be an arbitrary field and consider a $p \times (m+p)$ polynomial matrix

$$
P(s,t) := \begin{pmatrix} f_{11}(s,t) & f_{12}(s,t) & \dots & f_{1,m+p}(s,t) \\ f_{21}(s,t) & f_{22}(s,t) & \dots & f_{2,m+p}(s,t) \\ \vdots & \vdots & & \vdots \\ f_{p1}(s,t) & f_{p2}(s,t) & \dots & f_{p,m+p}(s,t) \end{pmatrix}. \tag{8}
$$

We say $P(s,t)$ is homogeneous if each element $f_{ij}(s,t) \in \mathbb{K}[s,t]$ is a homogeneous polynomial of degree ν_i. We say two homogeneous matrices $P(s,t)$

and $\tilde{P}(s,t)$ are equivalent if after a possible permutation they have the same row-degrees and if there is a unimodular matrix $U(s,t)$ such that $P = U\tilde{P}$. Using this equivalence relation we define:

Definition 11 An equivalence class of full rank homogeneous polynomial matrices $P(s,t)$ will be called a *homogeneous autoregressive system.* The McMillan degree of a homogeneous autoregressive system is defined as the sum of the row degrees, i.e. through $n := \sum_{i=1}^{p} \nu_i$.

The main theorem of [22] states:

Theorem 12. *The set of $p \times (m+p)$ homogeneous autoregressive systems of degree n is in bijective correspondence to the points of the Grothendieck Quot scheme $\mathcal{Q}_{p,m}^n$. The set $S_{p,m}^n$ of proper transfer functions can be viewed as a Zariski open subset of $\mathcal{Q}_{p,m}^n$. In particular we can view $\mathcal{Q}_{p,m}^n$ as a smooth compactification of $S_{p,m}^n$.*

In the sequel we will elaborate on the connection of homogeneous autoregressive systems to algebraic geometry and to systems theory.

The connection to algebraic geometry can be seen in the following way: If the row degrees of the $p \times (m+p)$ matrix $P(s,t)$ are ν_i then there is a short exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{p} \mathcal{O}_{\mathbb{P}}(-\nu_i) \xrightarrow{P(s,t)} \mathbb{K}^{m+p} \bigotimes \mathcal{O}_{\mathbb{P}} \xrightarrow{\Phi} \mathcal{B} \longrightarrow 0. \tag{9}$$

In this way every homogeneous autoregressive system defines in a natural way a quotient sheaf $\mathcal{B}$.

There is also a direct connection to Hermann-Martin maps. When $m, p > 0$ then every $p \times (m+p)$ matrix $P(s,t)$ defines a rational map:

$$\hat{h} : \mathbb{P}_{\mathbb{K}}^1 \longrightarrow \mathrm{Grass}(p, \mathbb{K}^{p+m}), \quad (s,t) \longmapsto \mathrm{rowspace}_{\mathbb{K}} P(s,t). \tag{10}$$

In case that $P(s,t)$ is left prime then the morphism $\hat{h}$ has no poles and therefore is a regular map.

Having introduced a smooth compactification of the set $S_{p,m}^n$ which parameterizes all m-input, p-output systems of McMillan degree n it is of course an interesting question if one can give a systems theoretic interpretation for the systems added in the compactification process. This can indeed be done.

We will need the notion of generalized state space systems as studied by Kuijper and Schumacher [17]. For this let G, F be matrices of size $n \times (m+n)$ and let H be a matrix of size $(m+p) \times (m+n)$. The matrices (G, F, H) describe over any field a discrete time linear system through:

$$Gz(t+1) = Fz(t), \quad w(t) = Hz(t). \tag{11}$$

In this representation $z(t) \in Z \simeq \mathbb{K}^{m+n}$ describes the set of "internal variables" and $w(t) \in \mathbb{K}^{m+p}$ describes the set of "external variables". The matrices G, F are linear maps from the space of internal variables $Z \simeq \mathbb{K}^{m+n}$ to the state space $X \simeq \mathbb{K}^n$. Corresponding to change of coordinates in X and Z one has a natural equivalence among pencil representations:

$$(G, F, H) \;\sim\; (SGT^{-1}, SFT^{-1}, HT^{-1}). \tag{12}$$

In above equivalence it is assumed that $S \in Gl_n$ and $T \in Gl_{m+n}$. The realization (11) reduces to the familiar A, B, C, D representation (1) as soon as there is an invertible matrix T such that:

$$GT^{-1} = \begin{bmatrix} I \; 0 \end{bmatrix}, \quad FT^{-1} = \begin{bmatrix} A \; B \end{bmatrix}, \quad HT^{-1} = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}. \tag{13}$$

If there is no invertible matrix T to do such a transformation then (11) does not describe an input output system under the natural partitioning $w = \begin{bmatrix} u \\ y \end{bmatrix}$.

The main theorem of [23] states:

Theorem 13. *Let X be the set of all matrix triples (G, F, H) where G, F are matrices of size $n \times (m + n)$ and H is a matrix of size $(m + p) \times (m + n)$. Then the stable points in the sense of GIT [21] under the $Gl_n \times Gl_{m+n}$ action induced by (12) are given by the conditions:*

1. *$[sG - tF]$ has full row rank n.*
2. *$\begin{bmatrix} sG - tF \\ H \end{bmatrix}$ has full column rank $m + n$ for all $(s, t) \in \mathbb{K}^2 \setminus \{(0, 0)\}$.*

Finally the geometric quotient of the stable points modulo the group action is equal to the variety $\mathcal{Q}^n_{p,m}$.

We would like to conclude this section with the remark that generalized first order representations of the form (11) as well as Grothendieck's Quot scheme $\mathcal{Q}^n_{p,m}$ are well defined when $m = 0$. In systems theoretic terms we are then dealing with observable A, C systems having no inputs. When $m = 0$ the Hermann-Martin map (10) is however very degenerate and a sheaf theoretic interpretation is required to distinguish among the different points of $\mathcal{Q}^n_{p,0}$.

4 Results on pole placement by geometric methods

An area where algebraic geometric methods were very successfully applied in systems theory are the different questions of pole placement and stabilization of linear system. Crucial for the solution of these problems was the understanding of the manifold $S^n_{p,m}$ and its compactifications $K^n_{p,m}$ and $\mathcal{Q}^n_{p,m}$. In this section we will explain these results.

Consider a strictly proper linear system Σ_n of McMillan degree n:

$$\Sigma_n \ : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{14}$$

A dynamic compensator of order q is a linear system of degree q, having the following state space representation:

$$\Sigma_q \ : \begin{cases} \dot{z} = Fz + Gy \\ u = Hz + Ky \end{cases} \tag{15}$$

In this representation, z is a q–vector, which describes the state of the compensator. The special case of $q = 0$ corresponds to the case of static feedback. The overall system is described by:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} A + BKC & BH \\ GC & F \end{bmatrix}}_{M} \begin{bmatrix} x \\ z \end{bmatrix} \tag{16}$$

$$y = Cx \tag{17}$$

which is a linear system of McMillan degree $n + q$. Stability of the closed loop system depends on the location of the eigenvalues of the matrix M.

We will parameterize the eigenvalues of the matrix M through its characteristic polynomial $\chi_M(x) = \det(xI - M) \in \mathbb{K}[x]$. The polynomial $\chi_M(x)$ is a monic polynomial of degree $n + q$ and we can identify this polynomial with a point in the vector space $\mathbb{K}^{n+q}$. Similarly we can identify a tuple of matrices F, G, H, K with a point in the vector space $\mathbb{K}^{q(m+p+q)+mp}$. With this identification we define the *affine pole placement map* through:

$$\begin{aligned} \varphi : \quad \mathbb{K}^{q(m+p+q)+mp} &\longrightarrow \mathbb{K}^{n+q} \\ (F, G, H, K) &\longmapsto \det(xI - M). \end{aligned} \tag{18}$$

One says system (14) is arbitrary pole assignable with compensators of McMillan degree at most q as soon as the pole placement map (18) is surjective.

An important special case of the general question is the static pole placement problem. This is the situation when $q = 0$, i.e. the compensator (15) has the simple form $u = Ky$ and K is an $m \times p$ matrix.

In order to study this problem Hermann and Martin used the dominant morphism theorem of algebraic geometry to derive the result [14]:

Theorem 14. *Assume that the base field $\mathbb{K}$ is algebraically closed. Then for a generic set of systems Σ_n having the state space form* (14) *almost arbitrary pole placement by static compensators is possible if and only if $n \leq mp$.*

To derive this theorem it was probably the first time that deeper methods from algebraic geometry were used to tackle a problem in control systems design. The dominant morphism theorem does a 'local computation' and therefore cannot achieve results of full surjectivity.

A couple of years later Brockett and Byrnes studied the static pole placement problem by considering also the effects 'at the boundary' of the parameter space. They realized that it is best to compactify the compensator space via the Grassmann variety $\mathrm{Grass}(m, \mathbb{F}^{m+p})$. Note that this compactification simply consists of all degree 0 Hermann-Martin maps from the projective line to $\mathrm{Grass}(m, \mathbb{F}^{m+p})$! The main result of [2] states:

Theorem 15. *Assume that the base field $\mathbb{K}$ is algebraically closed. Then for a generic set of systems Σ_n having the state space form* (14) *arbitrary pole placement by static compensators is possible if and only if $n \leq mp$. Moreover when $n = mp$ the number of solutions (when counted with multiplicities) is exactly equal to the degree of the Grassmann variety:*

$$\deg \mathrm{Grass}(m, m+p) = \frac{1!2! \cdots (p-1)!(mp)!}{m!(m+1)! \cdots (m+p-1)!}. \tag{19}$$

In particular if $\deg \mathrm{Grass}(m, m+p)$ *is odd, pole assignment by real static feedback is possible.*

This was quite a surprising result. The degree of the Grassmannian as described in formula (19) was computed in the 19th century by Schubert [32]. At the time Schubert's computations were not generally accepted and Hilbert devoted his 15th problem to the Schubert calculus. The modern way to see Schubert's number (19) as the degree of a Grassmann variety has its origin in the 20th century and the interested reader will enjoy the article of Kleiman [16] in this regard.

Both the results of Theorem 14 and Theorem 15 required that the base field is algebraically closed, e.g. the field of complex numbers. This is not surprising as some of the strongest results in algebraic geometry require that the base field is algebraically closed.

In 1992 Alex Wang adapted algebraic geometric methods for the study of the real Grassmannian to derive the following at the time very surprising result [38]:

Theorem 16. *For a generic set of real systems Σ_n having the state space form* (14) *arbitrary pole placement by static pole placement is possible as soon as $n < mp$.*

It was later realized that Wang's proof can be considerably simplified without requiring too deep results from algebraic geometry. The interested reader will find an elementary proof in [29].

In order to progress on the solution of the general pole placement problem with dynamic compensators it was necessary to come up with a suitable compactification of the space $S^q_{m,p}$, or equivalently the set of Hermann-Martin maps $\mathrm{Rat}_q(\mathbb{P}^1, \mathrm{Grass}(m, \mathbb{K}^{m+p}))$. As a generalization of the result by Brockett and Byrnes (Theorem 15), the author derived the following result [27]:

Theorem 17. *Assume that the base field $\mathbb{K}$ is algebraically closed. Then for a generic set of systems Σ_n having the state space form* (14) *arbitrary pole placement by dynamic compensators is possible if and only if*

$$n \leq q(m+p-1) + mp.$$

Moreover when $n = q(m+p-1)+mp$ the number of solutions (when counted with multiplicities) is exactly equal to the degree of the quantum Grassmannian $K^q_{m,p}$.

This theorem made it a challenge to compute the degree of the quantum Grassmannian and in this way to come up with a generalization of Schubert's famous formula (19). The result of this effort was:

Theorem 18 ([24, 25]). *The degree of the quantum Grassmannian $K^q_{m,p}$ is given by:*

$$(-1)^{q(m+1)}(mp + q(m+p))! \sum_{n_1+\cdots+n_m=q} \frac{\prod_{k<j}(j-k+(n_j-n_k)(m+p))}{\prod_{j=1}^{m}(p+j+n_j(m+p)-1)!} \tag{20}$$

The general pole placement problem over an arbitrary field $\mathbb{K}$ as described in this section is still not completely solved. Even for static compensators (when $q = 0$) and over the reals there is a gap of one degree of freedom in Wang's result. Eremenko and Gabrielov [5] have recently closed this gap for many cases but the gap still exists for infinite many cases. It would also be worthwhile to study the pole placement problem over other fields. E.g. convolutional codes can be viewed as linear systems over finite fields (see e.g. [30]) and the decoding problem seems to be closely connected to the problem of designing a linear observer.

5 Conclusion

In this paper we provided a survey about the Hermann-Martin curve, a crucial link between linear systems theory and algebraic geometry.

The Hermann-Martin curve provided a better understanding of the topology of the class of linear systems $S^n_{p,m}$ parameterizing the set of m-inputs, p-outputs system of McMillan degree n. The geometric point of view led to natural compactifications of the space $S^n_{p,m}$ and this was ultimately key in the progress on the static and dynamic pole placement problem.

It is our believe that a further investigation of the space $S^n_{p,m}$ would be very beneficial for many linear systems theory problems. E.g. the manifold

$S^n_{p,m}$ comes with some natural metrics which allows one to compute distances between linear systems. A first attempt in this direction was done in [31]. Another area where a topological understanding of the space $S^n_{p,m}$ is important is the design of stable numerical algorithms while solving control problems. As an example we mention the recent paper by Verschelde and Wang [37] where this issue stands out.

As all these remarks make it clear the translation from systems theoretic questions to geometric questions has been very fruitful in the past and we expect that further results will come out from this. A crucial starting point to explore this connection is the paper by Martin and Hermann [20].

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