SIAM J. CONTROL OPTIM.
Vol. 42, No. 6, pp. 2078–2093

POLE PLACEMENT AND MATRIX EXTENSION PROBLEMS: A COMMON POINT OF VIEW*

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Dedicated to the memory of Meeyoung Kim, 1966–2001

Abstract. This paper studies a general inverse eigenvalue problem which generalizes many well-studied pole placement and matrix extension problems. It is shown that the problem corresponds geometrically to a so-called central projection from some projective variety. The degree of this variety represents the number of solutions the inverse problem has in the critical dimension. We are able to compute this degree in many instances, and we provide upper bounds in the general situation.

Key words. pole placement and inverse eigenvalue problems, matrix completion problems, Grassmann varieties, degree of a projective variety

AMS subject classifications. 14M15, 15A18, 93B55, 93B60

DOI. 10.1137/S0363012999354429

1. Introduction and motivational examples. Let $\mathbb{K}$ be an arbitrary field and consider matrices A of size $n \times n$ and matrices G, H of size $n \times m$. Let $Mat_{m\times n}$ be the vector space of all $m \times n$ matrices over $\mathbb{K}$, and let $\mathcal{L} \subset Mat_{m\times n}$ be a linear subspace of dimension d. This paper will be devoted to the following question.

Problem 1.1. Given an arbitrary monic polynomial $\varphi(s) \in \mathbb{K}[s]$ of degree n, is there a $F \in \mathcal{L}$ such that

$$\det[s(I + HF) - (A + GF)] = \varphi(s)? \tag{1.1}$$

We can think of the triple (A, G, H) representing a dynamical system in various ways, and we will say more about it in a moment. The set of monic polynomials of degree n can be identified with the vector space $\mathbb{K}^n$. If Problem 1.1 has a positive answer for all monic polynomials $\varphi(s) \in \mathbb{K}^n$ of degree n, then we will say that the system (A, G, H) is arbitrarily pole assignable in the class of feedback compensators $\mathcal{L}$. If for a generic set of monic polynomials $\varphi(s) \in \mathbb{K}^n$ of degree n Problem 1.1 has a positive answer, then we will say that system (A, G, H) is generically pole assignable in the class of feedback compensators $\mathcal{L}$.

Problem 1.1 covers a large set of "constrained" state and output pole placement problems. It also covers some important matrix extension problems. The following three examples will illustrate this.

Example 1.2 (constrained state feedback pole placement problem). Consider a linear system having the following form:

$$\dot{x} = Ax + Bu + H\dot{u}, \quad x \in \mathbb{C}^n,\ u \in \mathbb{C}^m. \tag{1.2}$$

*Received by the editors April 1, 1999; accepted for publication (in revised form) May 23, 2003; published electronically January 28, 2004.

http://www.siam.org/journals/sicon/42-6/35442.html

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If a state feedback law of the form $u = -Fx$ is applied, then the closed loop characteristic polynomial has the form $\det[s(I + HF) - (A - BF)]$ which is exactly the form of (1.1). Assume that not every feedback law of the form $u = -Fx$ can be applied, and that a valid feedback matrix must satisfy some linear constraints of its parameters, i.e., $F \in \mathcal{L}$ for some linear subspace $\mathcal{L} \subset Mat_{m\times n}$. Such constraints occur, e.g., if the ith input channel has to be kept zero all the time. Problem 1.1 covers such situations.

Of course if $H = 0$ and the feedback laws are unconstrained, then we simply deal with the classical state feedback problem, and it is well known that this problem always has a positive solution as soon as rank $\left[B, AB, \ \dots \ A^{n-1}B\right] = n$.

Example 1.3 (constrained output feedback pole placement problem). Consider the linear system

$$\dot{x} = Ax + Bu, \ y = Cx, \quad x \in \mathbb{C}^n, \ u \in \mathbb{C}^m, \text{ and } y \in \mathbb{C}^p. \tag{1.3}$$

The problem asks for a static feedback law $u = Ky$ such that the closed loop system has some desired closed loop characteristic polynomial. Once more assume that the feedback laws have to satisfy some linear constraints. For this we assume that $\mathcal{U} \subset Mat_{m\times p}$ is some linear subspace, and we do require that $K \in \mathcal{U}$. One immediately verifies that Problem 1.1 covers this situation if one chooses $H := 0$, $G := B$, and $\mathcal{L} := \{KC \mid K \in \mathcal{U}\}$.

For the unconstrained problem ($\mathcal{U} = Mat_{m\times p}$) the main result in this area of research was given by Brockett and Byrnes [2]. It states the following.

THEOREM 1.4. *If $\mathcal{U} = Mat_{m\times p}$ and if $n \le mp = \dim\mathcal{U}$, then for a generic set of matrices A, B, C, the system* (1.3) *is arbitrarily pole assignable. Moreover, if $n = mp$, then when counted with multiplicities there are exactly as many solutions as the degree of the complex Grassmann variety* $\mathrm{Grass}(m, \mathbb{C}^{m+p})$ *once embedded via the Plücker embedding.*

The importance of the matrix H becomes apparent if we deal with general proper systems of the form:

$$\dot{x} = Ax + Bu, \ y = Cx + Du, \quad x \in \mathbb{C}^n, \ u \in \mathbb{C}^m, \text{ and } y \in \mathbb{C}^p.$$

Assume C has rank p, and let C^+ be a right inverse of C, i.e., an $n \times p$ matrix such that $CC^+ = I_p$. Assume again that the feedback laws are constrained to some subspace $\mathcal{U} \subset Mat_{m\times p}$. If we apply the feedback law $u = Ky$, $K \in \mathcal{U}$, then we obtain the closed loop system:

$$\dot{x} = Ax + BKy, \ y = Cx + DKy, \quad x \in \mathbb{C}^n, \text{ and } y \in \mathbb{C}^p.$$

The closed loop characteristic polynomial is therefore computed as

$$\begin{aligned}
&\det \begin{bmatrix} sI_n - A & -BK \\ -C & I_p - DK \end{bmatrix} \\
&= \det(sI_n - A) \det\left(I_p - DK - C(sI_n - A)^{-1}BK\right) \\
&= \det(sI_n - A) \det\left(I_n - C^+DKC - (sI_n - A)^{-1}BKC\right) \\
&= \det\left((sI_n - A)(I_n - C^+DKC) - BKC\right) \\
&= \det\left(s(I_n - C^+DKC) - (A + (B - AC^+D)KC\right).
\end{aligned}$$

If one defines $H := -C^+D$, $G := (B - AC^+D)$, and $\mathcal{L} := \{KC \mid K \in \mathcal{U}\}$ one immediately verifies that the output feedback pole placement problem involving proper transfer functions is also covered by Problem 1.1 and that in this case it is of importance to have the matrix H in the formulation of Problem 1.1.

Example 1.5 (matrix extension problems). Let $m = n$, $H = 0$, and $G = I_n$. In this case Problem 1.1 asks for conditions which guarantee that the characteristic map

$$\chi_A : \mathcal{L} \longrightarrow \mathbb{K}^n, \quad F \longmapsto \det(sI + A + F), \tag{1.4}$$

is surjective or at least "generically" surjective. This general matrix extension problem itself contains many of the matrix completion problems as they were studied in [1, 4, 5, 7].

The main result in the situation of Example 1.5 has been derived in [11]. It states the following.

THEOREM 1.6. *If the base field $\mathbb{K}$ is algebraically closed, then for a generic set of matrices $A \in Mat_{n\times n}$ the characteristic map* (1.4) *is dominant (generically surjective) if and only if*

1. $\dim \mathcal{L} \geq n$;
2. *there must be at least one element $L \in \mathcal{L}$ whose trace $tr(L) \neq 0$, i.e., $\mathcal{L} \not\subset sl_n$.*

The main results of this paper will show that if $\mathbb{K}$ is algebraically closed, then for a generic set of matrices (A, G, H) Problem 1.1 is solvable in a projective closure of $\mathcal{L}$ for every $\varphi(s)$ if and only if $\dim \mathcal{L} \geq n$ (Theorem 2.7), and if $\dim \mathcal{L} = n$, then there are at most $\min(m, n)^n$ solutions for each $\varphi(s)$ for each subspace $\mathcal{L}$ (Theorems 2.8, 4.3, and 4.8).

The paper is structured as follows: In section 2 we will introduce a natural compactification of the linear space $\mathcal{L}$ which we will denote by $\bar{\mathcal{L}}$. In order to prove the main theorems we will show that one has a characteristic map χ defined on a Zariski open set of the variety $\bar{\mathcal{L}}$. Geometrically χ describes a central projection from the variety $\bar{\mathcal{L}}$ to the projective space $\mathbb{P}^n$. As a consequence the number of solutions in the critical dimension, i.e., in the situation where $\dim \mathcal{L} = n$, is equal to $\deg \bar{\mathcal{L}}$ when counted with multiplicities and when some possible "infinite solutions" are taken into account. The results in section 2 generalize mathematical ideas which have been developed for the static pole placement problem by Brockett and Byrnes [2] and for the dynamic pole placement problem by Ravi, Rosenthal, and Wang [14], Rosenthal [15], and Rosenthal and Wang [16].

The degree of the variety $\bar{\mathcal{L}}$ is of crucial importance for the understanding of the characteristic map χ. In section 3 we compute the degree of $\bar{\mathcal{L}}$ in many special cases. As a corollary we will rediscover several matrix completion results as they were derived earlier in [3, 4, 5, 7].

In section 4 we will be concerned with the value of the "generic degree"; this is the largest possible degree a variety $\bar{\mathcal{L}}$ of a fixed dimension can have. We determine an upper bound for the generic degree in the case when $d = n$ and prove that this bound is reached when $m = n < 5$.

2. Compactification of the problem. The inverse eigenvalue problem formulated in Problem 1.1 describes an intersection problem in the linear variety $\mathcal{L}$. In order to invoke results from intersection theory [6] it is important to understand the intersection at the "boundary" of $\mathcal{L}$. What is needed is a good compactification of $\mathcal{L}$. It turns out that Problem 1.1 induces in a natural way a compactification, and we will explain this in what follows.

Recall [8, 9] that the Grassmannian $\mathrm{Grass}(m, \mathbb{K}^q)$, $m \leq q$, is the set of all m-dimensional subspaces in $\mathbb{K}^q$. For each $X \in \mathrm{Grass}(m, \mathbb{K}^q)$, let $X = \mathrm{span}\,\{\alpha_1, \ldots, \alpha_m\}$; then

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$$

is a vector in the exterior product $\wedge^m \mathbb{K}^q \cong \mathbb{K}^{\binom{q}{m}}$, and

$$\mathrm{span}\,\{\alpha_1, \ldots, \alpha_m\} = \mathrm{span}\,\{\beta_1, \ldots, \beta_m\}$$

if and only if

$$\alpha_1 \wedge \cdots \wedge \alpha_m = k(\beta_1 \wedge \cdots \wedge \beta_m)$$

for some $k \neq 0$ in $\mathbb{K}$. Therefore, by considering the components of $\alpha_1 \wedge \cdots \wedge \alpha_m$ as homogeneous coordinates of a point in $\mathbb{P}^{\binom{q}{m}-1}$, we have an embedding

$$\mathrm{Grass}(m, \mathbb{K}^q) \subset \mathbb{P}\left(\wedge^m \mathbb{K}^q\right) = \mathbb{P}^{\binom{q}{m}-1},$$

which is called Plücker embedding, and $\alpha_1 \wedge \cdots \wedge \alpha_m$ are called Plücker coordinates of $X \in \mathrm{Grass}(m, \mathbb{K}^q)$. If $\{\alpha_i\}$ are row vectors, then the Plücker coordinates of $\mathrm{span}\,\{\alpha_1, \ldots, \alpha_m\}$ are given by all the full size minors of the $m \times q$ matrix

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}.$$

The closed loop characteristic polynomial can be written as

$$\det\left[s(I + HF) - (A + GF)\right] = \det \begin{bmatrix} I_m & F \\ -sH + G & sI - A \end{bmatrix}. \tag{2.1}$$

Following an idea introduced by Brockett and Byrnes [2] for the static output pole placement problem we will identify $\mathrm{rowsp}\,[I_m\ F]$ with an element of $\mathrm{Grass}(m, \mathbb{K}^{m+n})$. In this way we have natural embeddings

$$\mathcal{L} \subset \mathrm{Grass}(m, \mathbb{K}^{m+n}) \subset \mathbb{P}\left(\wedge^m \mathbb{K}^{m+n}\right) = \mathbb{P}^N, \quad N = \binom{m+n}{m} - 1.$$

Definition 2.1. *Let $\bar{\mathcal{L}}$ be the projective closure of $\mathcal{L}$.*

By definition $\bar{\mathcal{L}}$ is a projective variety of dimension $\dim \bar{\mathcal{L}} = \dim \mathcal{L} = d$. The remainder of the paper will be devoted to a large extent to the study of this variety. In order to have a general idea of how the projective closure of $\mathcal{L}$ is defined, we start with an illustrative example.

Example 2.2. Let $m = n = d = 3$, and let $\mathcal{L} \subset Mat_{3\times 3}$ be defined by

$$\mathcal{L} = \left\{ \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix} \right\}, \tag{2.2}$$

where $a,\ b,\ c \in \mathbb{K}$ are arbitrary elements. Then for fixed a, b, c

$$\mathrm{rowsp} \begin{bmatrix} 1 & 0 & 0 & a & b & 0 \\ 0 & 1 & 0 & c & a & b \\ 0 & 0 & 1 & 0 & c & a \end{bmatrix} \tag{2.3}$$

is a point in $\mathrm{Grass}(3,\mathbb{K}^6)$. Let z_{ijk} be the full size minor of (2.3) consisting of the ith, jth, kth columns. Then $\{z_{ijk}\}$ are the Plücker coordinates of $\mathrm{Grass}(3,\mathbb{K}^6)$ in $\mathbb{P}^{\binom{6}{3}-1}=\mathbb{P}^{19}$. $\mathcal{L}$ is defined by 6 linear equations of its entries. In terms of the Plücker coordinates, they become

$$\begin{aligned} &z_{234}=-z_{135}, \quad z_{234}=z_{126}, \quad z_{235}=-z_{136},\\ &z_{125}=-z_{134}, \quad z_{124}=0, \qquad z_{236}=0. \end{aligned} \tag{2.4}$$

$\mathcal{L}$ has 9 minors of size 2×2, but there are only 6 monomials of degree 2 of a,b,c:

$$a^2, b^2, c^2, ab, ac, bc.$$

So there are 3 linear relations among the 2×2 minors. In terms of the Plücker coordinates, they are

$$z_{146}=-z_{245}, \quad z_{345}=z_{156}, \quad z_{346}=-z_{256}. \tag{2.5}$$

The monomials $a^2, b^2, c^2, ab, ac, bc$ are not algebraically independent; they satisfy the relation

$$\operatorname{rank}\begin{bmatrix} a^2 & ab & ac\\ ab & b^2 & bc\\ ac & bc & c^2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} a\\ b\\ c \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} \le 1; \tag{2.6}$$

i.e., all the 2×2 minors of (2.6) are zero, which induce 6 quadratic relations among the 2×2 minors of $\mathcal{L}$:

$$\begin{aligned} &z_{346}^2+z_{246}z_{356}=0, && z_{146}^2+z_{246}z_{145}=0,\\ &(z_{246}+z_{345})^2-z_{356}z_{145}=0, && z_{246}(z_{246}+z_{345})-z_{346}z_{146}=0,\\ &z_{346}(z_{246}+z_{345})+z_{356}z_{146}=0, && z_{146}(z_{246}+z_{345})+z_{145}z_{346}=0. \end{aligned} \tag{2.7}$$

Every point in $\mathrm{Grass}(3,\mathbb{K}^6)$ defined by $z_{123}=0$ and (2.4), (2.5), (2.7) is indeed a limit point of (2.3); therefore $\bar{\mathcal{L}}$ is defined by (2.4), (2.5), and (2.7) in $\mathrm{Grass}(3,\mathbb{K}^6)\subset\mathbb{P}^{19}$.

Note that every element in $\bar{\mathcal{L}}$ can be simply represented by a subspace of the form $\operatorname{rowsp}[F_1\ F_2]$, where the $m\times m$ matrix F_1 is not necessarily invertible. Row span $[F_1\ F_2]$ describes an element of $\mathcal{L}$ if and only if F_1 is invertible. Note that a characteristic equation is even defined if F_1 is singular unless the polynomial in (2.1) is the zero polynomial.

Let f_i, $i=0,\dots,N$, be the Plücker coordinates of $\operatorname{rowsp}[F_1\ F_2]$. In terms of the Plücker coordinates the characteristic equation can then be written as

$$\det\begin{bmatrix} F_1 & F_2\\ -sH+G & sI-A \end{bmatrix} = \sum_{i=0}^{N} f_i p_i(s), \tag{2.8}$$

where the $p_i(s)$ is the cofactor of f_i in the determinant (2.8).

Let $\mathcal{Z}\subset\mathbb{P}^N$ be the linear subspace defined by

$$\mathcal{Z}=\left\{ z\in\mathbb{P}^N \,|\, \sum_{i=0}^{N} p_i(s) z_i = 0 \right\}. \tag{2.9}$$

Following [14, 15, 18] we identify a closed loop characteristic polynomial $\varphi(s)$ with a point in $\mathbb{P}^n$. In analogy to the situation of the static pole placement problem considered in [2, 18] (compare also with [15, section 5]) one has a well-defined characteristic map

$$\begin{array}{rccc} \chi: & \bar{\mathcal{L}} - \mathcal{Z} & \longrightarrow & \mathbb{P}^n, \\ & \operatorname{rowsp}[F_1 \ F_2] & \longmapsto & \displaystyle\sum_{i=0}^{N} f_i p_i(s). \end{array} \tag{2.10}$$

It will turn out that surjectiveness of the map χ will imply the generic pole assignability of system (A, G, H) in the class of compensators $\mathcal{L}$.

Recall the notion of degree of a variety [10, Chapter I, section 7] and the notion of a central projection (see [17, Chapter I, section 4]). The geometric properties of the map χ are as follows.

THEOREM 2.3. *The map χ defines a central projection. In particular, if $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$ and $\dim \mathcal{L} = n$, then χ is surjective, and there are $\deg \bar{\mathcal{L}}$ many preimages (counted with multiplicity) for each point in $\mathbb{P}^n$, where $\deg \bar{\mathcal{L}}$ is the degree of the projective variety $\bar{\mathcal{L}}$ in $\mathbb{P}^N$.*

The proof for this theorem is identical to the one given in [14, 18]. In the algebraic geometry literature (see, e.g., [9, 13, 17]) χ is sometimes referred to as a *projection of $\mathcal{L}$ from the center $\mathcal{Z}$ to $\mathbb{P}^n$*, and $\mathcal{Z} \cap \bar{\mathcal{L}}$ is sometimes referred to as the *base locus*. Of course the interesting part of the theorem occurs when $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$ since in this situation very specific information on the number of solutions is provided. If $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$ and $\dim \mathcal{L} = n$, then one says that χ describes a *finite morphism* from the projective variety $\bar{\mathcal{L}}$ onto the projective space $\mathbb{P}^n$.

In analogy to the situation of the static pole placement problem [2, 18] and the dynamic pole placement problem [15] we introduce a definition for this important situation.

DEFINITION 2.4. *A particular system (A, G, H) is called* $\mathcal{L}$-nondegenerate *if $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$. A system which is not $\mathcal{L}$-nondegenerate will be called $\mathcal{L}$-degenerate.*

In general it will always happen that certain systems A, G, H are $\mathcal{L}$-degenerate. We first make a definition.

DEFINITION 2.5. *Let X be an arbitrary (affine or projective) variety. A subset $S \subset X$ is called a* generic set *of X if it contains a nonempty Zariski open set of X.*

The next theorem shows that if the dimension of $\mathcal{L}$ is not too large, then the set of systems A, G, H which are $\mathcal{L}$-degenerate are contained in a proper algebraic subset when viewed as a subset in the vector space $\mathbb{K}^{(n^2+2mn)}$.

LEMMA 2.6. *Assume the base field $\mathbb{K}$ is algebraically closed. If $\dim \mathcal{L} > n$, then every system A, G, H is $\mathcal{L}$-degenerate. If $\dim \mathcal{L} \leq n$, then a generic set of systems A, G, H is $\mathcal{L}$-nondegenerate.*

Proof. If $\dim \mathcal{L} > n$, then $\mathcal{Z} \cap \bar{\mathcal{L}}$ is nonempty by the (projective) dimension theorem (see, e.g., [10, Chapter I, Theorem 7.2]) and the fact that $\dim \mathcal{Z} \geq N - n - 1$.

Assume now that $\dim \mathcal{L} \leq n$. Consider

$$\det \begin{bmatrix} I_m & F \\ -sH+G & sE-A \end{bmatrix}, \tag{2.11}$$

and identify the set of matrices E, A, G, H with the vector space $\mathbb{K}^{2n(m+n)}$. In analogy to the proof of [15, Lemma 5.3] we compute the dimension of the coincidence set

$$\mathcal{S} := \left\{ (F_1, F_2;\ E, A, G, H) \in \bar{\mathcal{L}} \times \mathbb{K}^{2n(m+n)} \mid \det \begin{bmatrix} F_1 & F_2 \\ -sH+G & sE-A \end{bmatrix} = 0 \right\}.$$

Using the same arguments as in [15] one computes

$$\dim \mathcal{S} = \dim \bar{\mathcal{L}} + 2n(m+n) - n - 1.$$

Since $\bar{\mathcal{L}}$ is projective the projection onto the second factor (namely, $\mathbb{K}^{2n(m+n)}$) is an algebraic set by the main theorem of elimination theory (see, e.g., [13]). This projection can result in an algebraic set of dimension at most $\dim \mathcal{S} < 2n(m+n)$. So for generic matrices E, A, G, H, we have $\det E \neq 0$ and

$$\det \begin{bmatrix} F_1 & F_2 \\ -sH+G & sE-A \end{bmatrix} \neq 0$$

for all $[F_1 \quad F_2]$ in $\bar{\mathcal{L}}$. For such matrices $\{E, A, G, H\}$, the systems $\hat{A} = E^{-1}A, \hat{G} = E^{-1}G, \hat{H} = E^{-1}H$ are $\mathcal{L}$-nondegenerate, and the claim therefore follows. □

We are now in a position to state one of the main theorems of this paper.

THEOREM 2.7. *Assume the base field $\mathbb{K}$ is algebraically closed. Let $\mathcal{L} \subset Mat_{m\times n}$ be a fixed subspace. Then the map χ introduced in* (2.10) *is surjective for a generic set of matrices A, G, H if and only if $\dim \mathcal{L} \geq n$. If $\dim \mathcal{L} = n$, then for a generic set of matrices A, G, H the intersection $\mathcal{Z} \cap \bar{\mathcal{L}} = \emptyset$ and there are $\deg \bar{\mathcal{L}}$ many preimages (counted with multiplicity) for each point in $\mathbb{P}^n$.*

Proof. If $\dim \mathcal{L} < n$, then a simple dimension argument shows that χ cannot be surjective. The result for $\dim \mathcal{L} = n$ follows from Lemma 2.6 and Theorem 2.3. If $\dim \mathcal{L} > n$, we can always choose a subspace of $\mathcal{L}$ of dimension n. It follows that χ is surjective as soon as $\dim \mathcal{L} \geq n$. □

Theorem 2.7 gives a partial answer to Problem 1.1. The following result makes this clear.

THEOREM 2.8. *Let $\mathcal{L} \subset Mat_{m\times n}$ be a fixed subspace, and assume that $\mathbb{K}$ is algebraically closed. If $\dim \mathcal{L} < n$, then for almost all monic polynomials $\varphi(s) \in \mathbb{K}[s]$ of degree n there does not exist a $F \in \mathcal{L}$ such that* (1.1) *holds true.*

If $\dim \mathcal{L} \geq n$, then for a generic set of matrices A, G, H and a generic set of monic polynomials $\varphi(s)$ of degree n Problem 1.1 *has a solution.*

Finally, if $\dim \mathcal{L} = n$, then for a generic set of matrices A, G, H and a generic set of polynomials the number of solutions is always finite, and when counted with multiplicities there are exactly $\deg \bar{\mathcal{L}}$ solutions.

Proof. The only statement which does not immediately follow from Theorem 2.7 is the claim about the number of solutions in the critical dimension. We therefore assume that $\dim \mathcal{L} = n$. Consider the characteristic map χ introduced in (2.10). According to Theorem 2.7, χ is a finite morphism of degree $\bar{\mathcal{L}}$. For a generic set of polynomials $\varphi(s) \in \mathbb{K}^n \subset \mathbb{P}^n$ the inverse image $\chi^{-1}(\varphi(s))$ contains $\deg\left(\bar{\mathcal{L}}\right)$ different solutions, and all these solutions are contained in $\mathcal{L} \subset \bar{\mathcal{L}}$. □

Theorems 2.7 and 2.8 assume that the field is algebraically closed. For general fields it is often possible to deduce some results by considering the corresponding question over the algebraic closure. The following result is of this sort.

COROLLARY 2.9. *If the degree of the variety $\bar{\mathcal{L}}$ defined over the complex numbers $\mathbb{C}$ is odd and if $\dim \mathcal{L} \geq n$, then χ is also surjective over the real numbers $\mathbb{R}$ for a generic set of real matrices A, G, H.*

Proof. We will view a triple of complex matrices A, G, H as a point in $\mathbb{C}^{n(2m+n)}$. If $d := \dim \mathcal{L} = n$, then χ represents a finite morphism for a generic set of complex matrices A, G, H by Theorem 2.7. Since the subset of real matrices inside $\mathbb{C}^{n(2m+n)}$ is not contained in any algebraic set, χ is even a finite morphism for a generic set of real matrices A, G, H. Over the complex numbers the inverse image $\chi^{-1}(y) \subset \bar{\mathcal{L}}$

represents a finite set of complex conjugate points for every real point $y \in \mathbb{P}^n$; in particular, $\chi^{-1}(y)$ contains a real point for a generic set of real matrices A, G, H.

If $d < n$ we follow the reasoning in the proof of [14, Theorem 2.14]: for a generic set of real matrices A, G, H one has

$$\dim \mathcal{Z} \cap \bar{\mathcal{L}} = \dim \mathcal{L} - n - 1 = d - n - 1.$$

If this dimension formula holds choose a subspace $H \subset \mathbb{P}^N$ having codimension $d-n$ inside $\mathbb{P}^N$ and having the property that

$$\bar{\mathcal{L}} \cap \mathcal{Z} \cap H = \emptyset. \tag{2.12}$$

Such a subspace H exists by [13, Corollary (2.29)]. Let $\pi_1 : \bar{\mathcal{L}} \to \mathbb{P}^d$ be the central projection with center $\mathcal{Z} \cap H$, and let $\pi_2 : \mathbb{P}^d - \pi_1(\mathcal{Z}) \to \mathbb{P}^n$ be the central projection with center $\pi_1(\mathcal{Z})$. Then π_1 is a finite morphism which is surjective over $\mathbb{C}$. π_2 is a linear map, it is surjective as well, and

$$\chi = \pi_2 \circ \pi_1.$$

If the degree of $\bar{\mathcal{L}}$ is odd, then π_1 is surjective over the reals, and the claim follows. □

3. The degree of $\bar{\mathcal{L}}$ in some special situations. From Theorems 2.7 and 2.8 it became clear that $\deg \bar{\mathcal{L}}$ is equal to the number of solutions for Problem 1.1 in the critical dimension, at least generically. In this and in the next section we will compute $\deg \bar{\mathcal{L}}$ in many situations.

For the rest of the paper we will assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. We will show in a moment that the compactification $\bar{\mathcal{L}}$ is, in many cases, isomorphic to the product of some Schubert varieties. This will allow us to compute the degree of $\bar{\mathcal{L}} \subset \mathbb{P}^N$ in these cases.

For the convenience of the reader we summarize the basic notions. More details can be found in [12, 16] and [6, Chapter 14].

Consider a flag of linear subspaces

$$\mathcal{F} : \ \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{m+n} = \mathbb{K}^{m+n},$$

where we assume that $\dim V_q = q$ for $q = 1, \ldots, m+n$. Let $\nu = (\nu_1, \ldots, \nu_m)$ be an ordered index set satisfying

$$1 \le \nu_1 < \cdots < \nu_m \le m+n.$$

With respect to the flag $\mathcal{F}$ one defines the Schubert variety

$$S(\nu_1, \ldots, \nu_m) := \{W \in \mathrm{Grass}(m, \mathbb{K}^{m+n}) \mid \dim(W \bigcap V_{\nu_k}) \ge k \ \text{ for } k = 1, \ldots, m\} \tag{3.1}$$

and the Schubert cell

$$C(\nu_1, \ldots, \nu_m) := \{W \in S(\nu_1, \ldots, \nu_m) \mid \dim(W \bigcap V_{\nu_k - 1}) = k - 1 \text{ for } k = 1, \ldots, m\}. \tag{3.2}$$

The closure of the Schubert cell $C(\nu_1, \ldots, \nu_m)$ inside the variety $\mathrm{Grass}(m, \mathbb{K}^{m+n}) \subset \mathbb{P}^N$ is equal to the Schubert variety $S(\nu_1, \ldots, \nu_m)$. By definition, $S(\nu_1, \ldots, \nu_m)$ is

a projective variety. There is a well-known formula for the degree of a Schubert variety [12, Chapter XIV, section 6, equation (7)]:

$$\deg S(\nu_1,\dots,\nu_k) = \left(\sum_i(\nu_i - i)\right)!\frac{\prod_{j>i}(\nu_j-\nu_i)}{\prod_i(\nu_i-1)!}.$$

Let $\mathcal{B} := \{v_1,\dots,v_{m+n}\} \subset \mathbb{K}^{m+n}$ be a basis which is compatible with the flag $\mathcal{F}$. In other words this basis has the property that $V_i = \operatorname{span}(v_1,\dots,v_i)$. With respect to the basis $\mathcal{B}$ one can represent the Schubert cell $C(\nu_1,\dots,\nu_m)$ as the set of all m-dimensional subspaces in $\mathbb{K}^{m+n}$ which are the rowspaces of a matrix of the form

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & \cdots & * & \cdots & * & 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.3}$$

where the 1's are in the columns $\nu_1,\dots,\nu_m$.

The cell $C(\nu_1,\dots,\nu_m)$ is isomorphic to $\mathbb{K}^d$, where $d = \sum_{i=0}^m(\nu_i - i)$. In particular, the cell $C(\nu_1,\dots,\nu_m)$ is isomorphic to every subspace $\mathcal{L} \subset Mat_{m\times n}$ having dimension $\dim \mathcal{L} = d$. In general it is not true that the closures $S(\nu_1,\dots,\nu_m) \subset \mathbb{P}^N$ and $\bar{\mathcal{L}} \subset \mathbb{P}^N$ are isomorphic. This happens, however, in the following situation.

Let $E_{i,j}$ be the $m \times n$ matrix whose i,j-entry is 1 and all the other entries are 0. Let $\mu = (\mu_1,\dots,\mu_m)$ be an ordered index set satisfying

$$0 \le \mu_1 \le \cdots \le \mu_m \le n.$$

Definition 3.1. *$\mathcal{L} \subset Mat_{m\times n}$ is called a* lower left filled linear space *of type μ if $\mathcal{L}$ is spanned by the matrices*

$$E_{i,j} \quad \textit{for} \quad j \le \mu_i,\ i = 1,\dots,m.$$

Lemma 3.2. *If $\mathcal{L} \subset Mat_{m\times n}$ is a lower left filled linear space of type μ, then $\bar{\mathcal{L}}$ is isomorphic to the Schubert variety $S(\mu_1+1, \mu_2+2, \dots, \mu_m+m)$.*

Proof. Let $\nu_i := \mu_i + 1$, $i = 1,\dots,m$. There is a fixed $(m+n)\times(m+n)$ permutation matrix P such that the set

$$\{[I_m \quad F]\,P \mid F \in \mathcal{L}\} \subset Mat_{m\times(m+n)}$$

is equal to the cell $C(\nu_1,\dots,\nu_m)$ described in (3.3). The linear transformation $P \in Gl_{m+n}$ extends to a linear transformation in $\mathbb{P}(\wedge^m\mathbb{K}^{m+n}) = \mathbb{P}^N$, and this linear transformation maps $\bar{\mathcal{L}}$ isomorphically onto $S(\nu_1,\dots,\nu_m)$. ∎

The proof of the lemma shows in particular that permutations of the columns inside $Mat_{m\times n}$ result in isomorphic compactifications. The following lemma shows that a broader range of transformations do not change the topological properties of the compactification.

Lemma 3.3. *Assume there are subspaces $\mathcal{L}_1, \mathcal{L}_2 \subset Mat_{m\times n}$. If there are linear transformations $S \in Gl_m$ and $T \in Gl_n$ such that $\mathcal{L}_2 = S\mathcal{L}_1T^{-1}$, then there exists an automorphism of $\mathbb{P}^N$ which maps the compactification $\overline{\mathcal{L}_1}$ isomorphically onto the compactification $\overline{\mathcal{L}_2}$.*

Proof.

$$\begin{bmatrix} I_m & S\mathcal{L}_1T^{-1} \end{bmatrix} = S\begin{bmatrix} I_m & \mathcal{L}_1 \end{bmatrix}\begin{bmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}.$$

The matrix to the right, an element of GL_{m+n}, induces a linear transformation on the projective space $\mathbb{P}(\wedge^m \mathbb{K}^{m+n}) = \mathbb{P}^N$ which maps $\overline{\mathcal{L}_1}$ onto $\overline{\mathcal{L}_2}$. □

THEOREM 3.4. *Assume there are linear transformations $S \in Gl_m$ and $T \in Gl_n$ such that*

$$S\mathcal{L}T^{-1} = \begin{pmatrix} \mathcal{L}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{L}_k \end{pmatrix},$$

where each $\mathcal{L}_l, l = 1, \ldots, k$, is the space of $m_l \times n_l$ lower left filled matrices of type μ^l:

$$0 \le \mu_1^l \le \cdots \le \mu_{m_l}^l \le n_l.$$

Then $\bar{\mathcal{L}}$ is isomorphic to the product of Schubert varieties

$$S(\mu_1^1 + 1, \mu_2^1 + 2, \ldots, \mu_{m_1}^1 + m_1) \times \cdots \times S(\mu_1^k + 1, \mu_2^k + 2, \ldots, \mu_{m_k}^k + m_k)$$

and

$$\deg \bar{\mathcal{L}} = \left(\sum_{i,l} \mu_i^l \right)! \frac{\prod_{i,l_r > l_s} \left(\mu_i^{l_r} + l_r - \mu_i^{l_s} - l_s \right)}{\prod_{i,l} (\mu_i^l + l - 1)!}.$$

Proof. The closure of $[I_{m_l} \;\; \mathcal{L}_l]$ in the Grassmann variety $\mathrm{Grass}(m_l, \mathbb{K}^{m_l+n_l})$ is the Schubert variety $S(\mu_1^l + 1, \ldots, \mu_{m_l}^l + m_l)$, and $\bar{\mathcal{L}}$ is a product of Schubert varieties.

The degree formula of a product of projective varieties under the Segre embedding [19, Proposition 2.1] is given by

$$\deg Z_1 \times \cdots \times Z_k = \frac{(\sum_i \dim Z_i)!}{\prod_i (\dim Z_i)!} \prod_i \deg Z_i.$$

Combining these formulas gives the result. □

COROLLARY 3.5. *When $\mu_1^1 = \cdots = \mu_{m_1}^1 = n_1$ and $\mu_i^l = 0$ for $l > 1$, then the compactification $\bar{\mathcal{L}} = \mathrm{Grass}(m_1, \mathbb{K}^{m_1+n_1})$ and its degree is*

$$\frac{(m_1 n_1)! 1! 2! \cdots (m_1 - 1)!}{n_1! (n_1 + 1)! \cdots (n_1 + m_1 - 1)!}.$$

Using Lemma 3.3 and Corollary 3.5 we can deduce Theorem 1.4, the result of Brockett and Byrnes. For this assume that $H = 0$, $G = I$, and $\mathcal{L} = \{BFC \mid F \in Mat_{m\times p}\}$. Without loss of generality we can assume that B, C have full rank, $\operatorname{rank} B = m$, and $\operatorname{rank} C = p$. (Theorem 1.4 assumes genericity!) There are invertible matrices S, T such that $SB = \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right]$ and $CT^{-1} = \left[\begin{smallmatrix} I_p \\ 0 \end{smallmatrix}\right]$. It follows that

$$S\mathcal{L}T^{-1} = \left\{ \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \in Mat_{n\times n} \mid F \in Mat_{m\times p} \right\}.$$

According to Lemma 3.3 and Corollary 3.5 the compactification is isomorphic to the Grassmannian $\mathrm{Grass}(m, \mathbb{K}^{m+p})$ as predicted by Theorem 1.4. In order to fully prove

Theorem 1.4 it remains to be shown that for a generic set of matrices $A \in Mat_{n\times n}$ the system is $\mathcal{L}$-nondegenerate as soon as $n = mp$.

COROLLARY 3.6. *When $m_l = n_l = 1$ and $\mu_1^l = 1$ for all l, then $\bar{\mathcal{L}} = \prod_{i=1}^{n} \mathbb{P}^1 = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ and its degree is*

$$n!.$$

Corollary 3.6 covers a result first studied by Friedland [4, 5]. Indeed the subspaces $\mathcal{L} \subset Mat_{n\times n}$ correspond in this case exactly to the set of diagonal matrices. By Theorem 2.7 we know that for a generic set of matrices A, G, H the characteristic map χ is a finite morphism of mapping degree $n!$. Friedland [4, 5] and Byrnes and Wang [3] did show that the set of all matrices of the form $A, I_n, 0$ belongs to this generic set. We therefore have the following result.

THEOREM 3.7 (see [3, 4, 5]). *Let $\mathcal{L} \subset Mat_{n\times n}$ be the set of all diagonal matrices defined over an algebraically closed field $\mathbb{K}$. If $A \in Mat_{n\times n}$ is an arbitrary matrix and $\varphi \in \mathbb{K}[s]$ is an arbitrary monic polynomial of degree n, then there are exactly $n!$ diagonal matrices $F \in \mathcal{L}$ (when counted with multiplicity) such that the matrix $A + F$ has characteristic polynomial $\varphi(s)$.*

4. The degree of $\bar{\mathcal{L}}$ in the generic situation. In the previous section we computed the degree of the variety $\bar{\mathcal{L}}$ in many special cases. The set of all subspaces $\mathcal{L} \subset Mat_{m\times n}$ having the property that $\dim \mathcal{L} = d$ can be identified with the Grassmannian variety $\mathrm{Grass}(d, \mathbb{K}^{mn})$. The degree in our concern attains its maximal value on a Zariski open subset of $\mathrm{Grass}(d, \mathbb{K}^{mn})$. This largest possible degree is sometimes referred to as the *generic degree*. In other words, the generic degree is obtained by algebraic perturbation of the subvariety $\mathcal{L}$ in the ambient space, $\mathbb{K}^{mn}$. In this section we determine an upper bound of the generic degree in the case when $d = n$ and prove that this upper is reached when $m = n < 5$. Let $\mathbb{K}$ be an algebraically closed field.

LEMMA 4.1. *Let $H_1, \ldots, H_k$, $k \le n$, be hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, \ldots, d_k$, respectively, and let $Z_1, \ldots, Z_m$ be the irreducible components of $\bigcap_{i=1}^{k} H_i$ (not necessarily the same dimensions). Then*

$$\sum_{j=1}^{m} \deg Z_j \le \prod_{i=1}^{k} d_i.$$

Proof. It is sufficient to prove that for any l-dimensional projective variety Z of degree d, $l > 0$, and for any hypersurface H of degree q, the sum of the degrees of the irreducible components of $Z \cap H$ is at most dq.

If $Z \subset H$, then $Z \cap H = Z$ and $\deg Z = d \le dq$. On the other hand, if $Z \not\subset H$, then each irreducible component of $Z \cap H$ has dimension $l - 1$ (see the proof of [10, Chapter 1, Proposition 7.1]), and by Bézout's theorem [9, Theorem 18.4, p. 228], the sum of the degrees of the irreducible components of $Z \cap H$ is dq counted with multiplicity. ∎

LEMMA 4.2. *Let $p_i(x) \in \mathbb{K}[x_1, \ldots, x_n]$ be polynomials of degree d_i, $i = 1, \ldots, k$, $k \le n$. Then the number of irreducible components of*

$$\{x \in \mathbb{K}^n \mid p_i(x) = 0, i = 1, \ldots, k\}$$

is at most $\prod_{i=1}^{k} d_i$.

Proof. Let $\hat{p}_i(z)$ be the homogenization of $p(x)$, i.e.,

$$\hat{p}_i(z) = z_0^{d_i} p(z_1/z_0, \ldots, z_n/z_0).$$

Then $\hat{p}_i(z)$ defines a hypersurface of degree d_i in $\mathbb{P}^n$. Lemma 4.1 implies that the number of irreducible components of

$$\{z \in \mathbb{P}^n | \hat{p}_i(z) = 0, i = 1, \ldots, k\}$$

is at most $\prod_{i=1}^{k} d_i$. So is the number of irreducible components of

$$\{x \in \mathbb{K}^n \mid p_i(x) = 0, i = 1, \ldots, k\}. \qquad \square$$

THEOREM 4.3. *Let $\mathbb{K}$ be an algebraically closed field. The generic degree of $\bar{\mathcal{L}} \subset \mathbb{P}(\wedge^m \mathbb{K}^{m+n}) = \mathbb{P}^N$ is at most $\min(m,n)^n$. When $m = n$ the sharper bound $n(n-1)^{n-1}$ even holds.*

Proof. For a generic $(N-n)$-dimensional projective subspace $H \subset \mathbb{P}^N$, the intersection $H \cap \bar{\mathcal{L}}$ contains exactly $\deg \bar{\mathcal{L}}$ many points, and all of them are in $\mathcal{L}$. Let

$$\sum_{\nu} z_\nu (e_{\nu_1} \wedge \cdots \wedge e_{\nu_n}) \tag{4.1}$$

be the homogeneous coordinates of the points in $\wedge^m \mathbb{K}^{m+n}$, where $\{e_i\}$ is the standard basis of $\mathbb{K}^{m+n}$. Then each $(N-n)$-dimensional projective subspace H is defined by n linear equations in $\{z_\nu\}$. Let $\mathcal{L} = \text{span}\,\{F_1, \ldots, F_n\}$, and define

$$F(x) = x_1 F_1 + \cdots + x_n F_n. \tag{4.2}$$

Consider the Plücker coordinates z_ν for the row space of $[I_m,\ F(x)]$. There are now two cases. When $m \le n$, then the coordinates z_ν have degree at most m when viewed as polynomials of $\mathbb{K}[x_1, \ldots, x_n]$. Intersecting with the subspace H results in n polynomial equations in n variables having degree at most m. By the previous lemma the number of solutions is at most m^n.

When $m \ge n$, then the z_ν have degree at most n, and we get the upper bound n^n. When $m = n$ we get the sharpened upper bound as follows: In order to describe the generic plane H we can always choose a set of equations such that at most one of them contains the last Plücker coordinate $z_{(n+1,\ldots,2n)}$ which is equal to $\det F(x)$, a polynomial of degree n at most. All other polynomials z_ν have degree $n-1$ at most. Therefore $\deg \bar{\mathcal{L}}$ equals the number of solutions of one polynomial equation of degree at most n and $n-1$ of polynomial equations of degrees at most $n-1$. By Lemma 4.2, if the number of solutions is finite, then it is at most $n(n-1)^{n-1}$, and hence $\deg \bar{\mathcal{L}}$ is at most $n(n-1)^{n-1}$ when $m = n$. $\square$

The reader may wonder how good the bound of the degree in Theorem 4.3 is. It is readily shown that the bound is sharp when $\min(m,n) = 1$. In the following two pages we show that the degree is $n(n-1)^{n-1}$ for $m = n < 5$.

Note that the Plücker coordinate $z_\nu = z_{(\nu_1,\ldots,\nu_n)}$ defined by (4.1) is the full size minor consisting of the $\nu_1, \ldots, \nu_n$ columns of $[F_1,\ F_2] \in \text{Grass}(n, 2n)$. Define

$$|\nu| = \sum_{i=1}^{n} (\nu_i - i)$$

and partial order $\nu \le \mu$ if $\nu_i \le \mu_i$ for all i. Under the standard flag (i.e., the flag spanned by the ordered standard basis), the Schubert variety (3.1) is defined by

$$S(\nu) := S(\nu_1, \ldots, \nu_i) = \{z \in \text{Grass}(n, \mathbb{K}^{2n}) \mid z_\mu = 0 \text{ for } \mu \not\le \nu\},$$

and the Schubert cell (3.2) is defined by

$$C(\nu) := C(\nu_1, \ldots, \nu_i) = \{z \in \mathrm{Grass}(n, \mathbb{K}^{2n}) \mid z_\mu = 0 \text{ for } \mu \not\leq \nu \text{ and } z_\nu \neq 0\},$$

and $\dim S(\nu) = \dim C(\nu) = |\nu|$.

LEMMA 4.4. *Let*

$$f_k(z) = \sum_{|\nu|=k} z_\nu,$$

and let Z_k be the algebraic subset of $\mathrm{Grass}(n, \mathbb{K}^{2n})$ *defined by $f_{k+1}(z) = 0, f_{k+2}(z) = 0, \ldots, f_{n^2}(z) = 0$. Then*

$$Z_k = \bigcup_{|\nu|=k} S(\nu).$$

Proof. Clearly $\bigcup_{|\nu|=k} S(\nu) \subset Z_k$. Let $z \in Z_k$. If $z \notin \bigcup_{|\nu|=k} S(\nu)$, then z must be in a Schubert cell $C(\mu)$ for some μ with $|\mu| > k$. Then it implies that $z_\nu = 0$ for all ν such that $|\nu| = |\mu|$ and $\nu \neq \mu$. From one of the defining equations, $f_{|\mu|}(z) = 0$, we derive a contradiction: $z_\mu = 0$. Therefore $z \in \bigcup_{|\nu|=k} S(\nu)$ and $Z_k = \bigcup_{|\nu|=k} S(\nu)$. ∎

We call a coordinate z_ν type i if exactly i of the indices $\{\nu_1, \ldots, \nu_n\}$ are in the set $\{n+1, \ldots, 2n\}$. For example, $z_{(n+1,\ldots,2n)}$ is type n, and $z_{(1,3,\ldots,n,n+1)}$ is type 1. Let

$$\widehat{Mat_{n\times n}} = \{[I_n,\ F] \mid F \in Mat_{n\times n}\}.$$

Then $\widehat{Mat_{n\times n}}$ is an open set of $\mathrm{Grass}(n, \mathbb{K}^{2n})$ which is isomorphic to $Mat_{n\times n}$. In $\widehat{Mat_{n\times n}}$, a type k coordinate is a homogeneous polynomial of degree k of the entries of F.

LEMMA 4.5. *Let $k > 0$ and Z be a k-dimensional subvariety of* $\mathrm{Grass}(n, \mathbb{K}^{2n})$ *such that it is not completely contained in the hypersurface*

$$H_0 := \{z \in \mathrm{Grass}(n, \mathbb{K}^{2n}) | z_{(1,2,\ldots,n)} = 0\}.$$

Then each irreducible component of $Z \cap H$ has dimension $k-1$ for generic hypersurfaces H defined by linear equations of type 1 *coordinates.*

Proof. For any H, either $Z \subset H$ or each irreducible component of $Z \cap H$ has dimension $k-1$ (see the proof of [10, Chapter 1, Proposition 7.1]). Therefore we need only to show that $Z \not\subset H$ for generic H's. Since the set of all such H's form a Zariski open set, we need only to show that it is nonempty. Since $\dim Z > 0$ and $Z \not\subset H_0$, we can always find a point $[I_n,\ F] \in Z$ with $F \neq 0$. Let z_ν be the type 1 Plücker coordinate corresponding to a nonzero entry of F. Then Z is not contained in the hypersurface defined by $z_\nu = 0$. ∎

LEMMA 4.6 (see [5]). *Let $p_1(x), \ldots, p_n(x)$ be polynomials on $\mathbb{K}^n$ of degrees $d_1, \ldots, d_n$, respectively, and let $p_i^h(x)$ be the homogeneous part of $p_i(x)$ of the highest degree (degree d_i). If $p_1^h(x) = 0, \ldots, p_n^h(x) = 0$ have only zero solutions, then the system of polynomial equations*

$$\begin{aligned} p_1(x) &= b_1, \\ &\vdots \\ p_n(x) &= b_n \end{aligned}$$

has $\prod_{i=1}^n d_i$ solutions counted with multiplicity for any $(b_1, \ldots, b_n) \in \mathbb{K}^n$.

Proof. Let $(1, x_1, \ldots, x_n) = (1, z_1/z_0, \ldots, z_n/z_0)$. Then $\mathbb{K}^n$ can be considered as an open subset of $\mathbb{P}^n$ defined by $z_0 \neq 0$. Let $h_i(z)$ be the homogenization of $p_i(x) - b_i$; i.e., $h_i(z) = z_0^{d_i}(p_i(z_1/z_0, \ldots, z_n/z_0) - b_i)$. $h_i(z)$ defines a hypersurface H_i of degree d_i in $\mathbb{P}^n$. Let H_0 be the hyperplane in $\mathbb{P}^n$ defined by $z_0 = 0$. The condition implies that the system of equations $z_0 = 0$ and $h_i(z) = 0$, for $i = 1, \ldots, n$, has only zero solution $z_i = 0$, $i = 0, \ldots, n$; i.e., in $\mathbb{P}^n$, $\cap_{i=0}^n H_i = \emptyset$. By the projective dimension theorem, $\dim \cap_{i=1}^n H_i = 0$, and by Bézout's theorem, $\cap_{i=1}^n H_i$ contains exactly $\prod_{i=1}^n d_i$ points counted with multiplicity and again by the given condition, all of them are in $\mathbb{K}^n$. ∎

THEOREM 4.7. *The generic degree of $\bar{\mathcal{L}}$ is $n(n-1)^{n-1}$ for $m = n < 5$.*

Proof. By Theorem 4.3, we need only to find one $\mathcal{L}$ such that $\deg \bar{\mathcal{L}} = n(n-1)^{n-1}$. From Lemma 4.4 we know that

$$\{z \in \mathrm{Grass}(n, \mathbb{K}^{2n}) \mid f_{n^2}(z) = 0, f_{n^2-1}(z) = 0, \ldots, f_{n^2-n+1}(z) = 0\} = \bigcup_{|\nu|=n^2-n} S(\nu).$$

By repeatedly using Lemma 4.5 we can find $n^2 - n$ linear equations

$$l_1(z) = 0, \ldots, l_{n^2-n}(z) = 0$$

of type 1 coordinates such that each irreducible component of

$$Z := \left\{ z \in \bigcup_{|\nu|=n^2-n} S(\nu) \mid l_1(z) = 0, \ldots, l_{n^2-n}(z) = 0 \right\}$$

either is contained completely in H_0 or has dimension 0.

Note that for $n < 5$, $f_{n^2-1}(z) = 0, \ldots, f_{n^2-n+1}(z) = 0$ are linear equations of type $n-1$ coordinates, and in $\widehat{Mat_{n\times n}} = \{[I_n,\ F]\}$ they are homogeneous equations of degree $n-1$ of the entries of F. Therefore if $[I_n,\ F]$ is in Z, then $[I_n,\ tF]$ are also in Z for all t. Since $\dim Z \cap \widehat{Mat_{n\times n}} = 0$, we must have

$$\dim Z \cap \widehat{Mat_{n\times n}} = \{[I_n,\ 0]\}. \tag{4.3}$$

On $\widehat{Mat_{n\times n}}$, $l_1(z) = 0, \ldots, l_{n^2-n}(z) = 0$ are linear equations of the entries of matrices $F \in Mat_{n\times n}$. Let $\mathcal{L}$ be the n-dimensional linear subspace of $Mat_{n\times n}$ defined by these linear equations, and let $F(x)$ be defined as in (4.2). Then in terms of x, the equation $0 = f_{n^2}(z) = \det F(x)$ is a homogeneous equation of degree n, and the equations $f_{n^2-1}(z) = 0, \ldots, f_{n^2-n+1}(z) = 0$ are homogeneous equations of degree $n-1$, and (4.3) implies that the system of these equations has only zero solution. Therefore, by Lemma 4.6, the system $f_{n^2}(z) = b_1, \ldots, f_{n^2-n+1}(z) = b_n$ has $n(n-1)^{n-1}$ solutions counted with multiplicity in $\mathcal{L}$, which means that the linear system of the Plücker coordinates

$$\begin{aligned} f_{n^2}(z) &= b_1 z_{(1,2,\ldots,n)}, \\ &\vdots \\ f_{n^2-n+1}(z) &= b_n z_{(1,2,\ldots,n)} \end{aligned}$$

has $n(n-1)^{n-1}$ solutions in $\bar{\mathcal{L}}$ counted with multiplicity, i.e., $\deg \bar{\mathcal{L}} = n(n-1)^{n-1}$. ∎

Theorem 2.8 showed the existence of solutions over an algebraically closed field when $d \geq n$. In the critical dimension ($d = n$) we already know that the number of

solutions is finite generically. Theorem 4.3 gives the following upper bound for the number of solutions in Problem 1.1.

THEOREM 4.8. *Let $\mathbb{K}$ be an arbitrary field. Let $\mathcal{L} \subset Mat_{m\times n}$ be a subspace of dimension n, let A, G, H be a "generic set of matrices," and let $\varphi(s) \in \mathbb{K}^n$ be an arbitrary monic polynomial $\varphi(s) \in \mathbb{K}[s]$ of degree n. Then there exist at most $\min(m,n)^n$ different feedback laws $F \in \mathcal{L}$ such that (1.1) holds. If, in addition, $m = n$, then the number of solutions is bounded by $n(n-1)^{n-1}$.*

Proof. Consider the problem over the algebraic closure $\bar{\mathbb{K}}$ of $\mathbb{K}$. By Theorem 2.7 there are for every monic polynomial $\varphi(s)$ of degree n exactly $\deg \bar{\mathcal{L}}$ many feedback laws $[F_1\ F_2] \in \bar{\mathcal{L}}$ (when counted with multiplicity) such that

$$\det \begin{bmatrix} F_1 & F_2 \\ -sH+G & sI-A \end{bmatrix} = \varphi(s).$$

Therefore, there are at most $\deg \bar{\mathcal{L}}$ many feedback laws $F \in \mathcal{L}$ whose coefficients are in the base field $\mathbb{K}$. $\deg \bar{\mathcal{L}}$ is always less than the generic degree. By Theorem 4.3 the generic degree is at most $\min(m,n)^n$ (respectively, $n(n-1)^{n-1}$ when $m = n$). ∎

Acknowledgment. We started to work on this project in 1998. Meeyoung was very enthusiastic about this research since it brought together nontrivial ideas from algebraic geometry and systems theory. She hoped that it would be the start of a longer research project. In June 2001, Meeyoung died in a tragic drowning accident in the Mediterranean Sea at the young age of 35. Her passing leaves everybody who knew her saddened, and we feel a deep sense of loss.

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